On the Theory and Experimental Results of a Sensitive Fiber Optic Laser Accelerometer

Fall 1983

Steven C. Furlong
University of Central Florida

Find similar works at: http://stars.library.ucf.edu/rtd

University of Central Florida Libraries http://library.ucf.edu

Part of the Engineering Commons

STARS Citation

http://stars.library.ucf.edu/rtd/681

This Masters Thesis (Open Access) is brought to you for free and open access by STARS. It has been accepted for inclusion in Retrospective Theses and Dissertations by an authorized administrator of STARS. For more information, please contact lee.dotson@ucf.edu.
ON THE THEORY AND EXPERIMENTAL RESULTS
OF A SENSITIVE FIBER OPTIC LASER ACCELEROMETER

BY

STEVEN CRAIG FURLONG
B.S., University of West Florida, 1979

RESEARCH REPORT

Submitted in partial fulfillment of the requirements
for the degree of Master of Science
in the Graduate Studies Program of the College of Engineering
University of Central Florida
Orlando, Florida

Fall Term
1983
The author wishes to express his appreciation to the many people who contributed toward the preparation of this research paper. A special thanks goes to the Orlando division of the Martin Marietta Aerospace Corporation which funded this research. In particular, I am indebted to Mr. Jim Martin for his valuable insight and to Dr. Jim Murphy who formulated the solution to the cylinder deformation problem. Gratitude is also extended to Mr. Jeff Bush who made possible all the experimentation discussed in this paper. Barbara Rubio is thanked for her meticulous work in typing the manuscript—which frequently required taming an onery word processor. Also, Dr. Ronald Phillips deserves special recognition for his guidance and unwavering support in overseeing this work. And finally, I dedicate this paper to my wife Mary, for her constant love and encouragement, without which this paper could not have been written.
# TABLE OF CONTENTS

Chapter

I. Introduction and Scope of Paper ........................................... 1

II. Mechanical Considerations .............................................. 3

- Deformation of Right Circular Cylinders
  - Under Constant Axial Acceleration .................................... 3

- The Undeformed Fiber Length ........................................... 11
- The Deformed Fiber Length ............................................. 17

III. Analysis of the Proposed Sensor ....................................... 22

- Scale Factor ......................................................................... 22
- Mass Loading Effect ......................................................... 26

IV. Experimental Results ...................................................... 28

- Frequency Response and Scale Factor Linearity ..................... 28

V. Concluding Remarks .......................................................... 36

Appendix A: Solution of the Reduced System
  Equations ............................................................................ 37

Appendix B: The Deformed Fiber Length ................................. 41

References .............................................................................. 46
CHAPTER I

INTRODUCTION AND SCOPE OF PAPER

In this paper we will investigate a fiber optic accelerometer which incorporates as its transducer an optical fiber tightly wrapped along the lateral surface of a solid right circular cylinder as shown in Figure 1.

![Fiber-wrapped cylinder](image)

**Figure 1. Fiber-wrapped cylinder.**

When the cylinder experiences an axial acceleration, it undergoes a deformation in both the radial and longitudinal directions. Under a prescribed set of boundary conditions a solution to the deformation problem will be formulated. The cylinder deformation will in turn alter the physical length of the optical fiber. Using the results of the deformation analysis, an explicit relationship will be derived relating the change in length of the optical
of the optical fiber to the bulk properties of the cylinder material (Young’s Modulus, Poisson’s ratio, etc.) and the magnitude of the axial acceleration.

It is proposed that this change in length of the optical fiber be detected interferometrically. A detailed explanation of the detection technique can be found in this document under the heading, "Experimental Results." For now, it suffices to say that a change in the physical length of the optical fiber will alter the phase relationship of an optical wave traversing the fiber, and the detection technique centers around measuring this induced phase shift. Also, the mechanical deformation of the fiber alters certain optical properties which in turn affect the phase; this interaction between mechanical deformation (strain) and optical properties of the fiber will be analyzed for the proposed configuration of the sensor.

And finally, with the theory having been developed, a working prototype of the fiber optic accelerometer has been fabricated and tested to compare theoretical predictions versus experimental results.
CHAPTER II
MECHANICAL CONSIDERATIONS

Deformation of Right Circular Cylinders Under Constant Axial Acceleration

In this section we investigate the elastic deformation of a right circular cylinder under constant axial acceleration. In general, a body is called elastic if it possesses the property of recovering its original shape when the forces causing the deformation are removed. This elastic property of material media is shared by all substances provided that the deformations do not exceed certain limits determined by the individual constituents of the body. Therefore, the assumption of elastic deformation does not degrade the "exactness" of the solution to be found; it will, however, put an upper bound on the range over which the solution is valid.

What follows is a compendium of a general analysis on the deformation of elastic bodies which has been relegated to Appendix A. The general technique and notation used throughout follow closely that of I.S. Sokolnikoff.¹

We now consider the elastic deformation of a homogeneous isotropic right circular cylinder under constant axial acceleration. The undeformed cylinder is situated on the positive side of and
attached to a surface in the \( z = 0 \) plane of a cylindrical coordinate system so as to form a body of revolution about the \( z \) axis, as shown in Figure 2.

![Diagram of an undeformed cylinder](image)

**Figure 2.** The undeformed cylinder.

Throughout this discussion let \( \mathbf{R} \) be the position vector of an arbitrary point, \((r, \theta, z)\), of the cylinder in the undeformed state. Also, let \( \mathbf{S} \) represent a displacement vector which originates at the arbitrary point, \((r, \theta, z)\), in the undeformed state and terminates on the same point in the deformed state. In other words, the position vector after deformation becomes \( \mathbf{R} + \mathbf{S} \). The vectors \( \mathbf{R} \) and \( \mathbf{S} \) are depicted in Figure 3, where the dashed lines represent an assumed shape of cylinder after deformation.
The components of the displacement vector, $\xi$, will be designated, in cylindrical coordinates, as $(u,v,w)$, where it is implicit that

\begin{align*}
u &= u(r, \theta, z), \\
v &= v(r, \theta, z), \\
w &= w(r, \theta, z).
\end{align*}

Therefore, if the coordinates of a point in the undeformed body are $(r, \theta, z)$, then the coordinates of the same point after deformation become $(r + u, \theta + v, z + w)$. So, the problem at hand becomes finding suitable expressions for the scalar functions $u$, $v$, and $w$.

The most general statement that the theory of elasticity makes concerning this problem is that the scalar functions $u$, $v$, and $w$ must satisfy the system of equations shown in Table 1. There, expressions (2.1) thru (2.3) are the equations of equilibrium and (2.4) thru (2.9) are the stress-strain relations—both written in cylindrical coordinates.
Table 1

The System Equations

\[ \frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = -F_r \]  

(2.1)

\[ \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2}{r} \tau_{r\theta} = -F_\theta \]  

(2.2)

\[ \frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} + \frac{1}{r} \tau_{rz} = -F_z \]  

(2.3)

\[ \frac{\partial u}{\partial r} = \frac{1}{E} \left( \tau_{rr} - \sigma (\tau_{\theta\theta} + \tau_{zz}) \right) \]  

(2.4)

\[ \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r} = \frac{1}{E} \left( \tau_{r\theta} - \sigma (\tau_{rr} + \tau_{zz}) \right) \]  

(2.5)

\[ \frac{\partial w}{\partial z} = \frac{1}{E} \left( \tau_{zz} - \sigma (\tau_{\theta\theta} + \tau_{rr}) \right) \]  

(2.6)

\[ \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} = \frac{2(1+\sigma)}{E} \tau_{r\theta} \]  

(2.7)

\[ \frac{\partial v}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial \theta} = \frac{2(1+\sigma)}{E} \tau_{\theta z} \]  

(2.8)

\[ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = \frac{2(1+\sigma)}{E} \tau_{rz} \]  

(2.9)
The E is Young's modulus, \( \sigma \) is Poisson's ratio, and the body force vector per unit volume, \( \dot{F} \), has as its components,

\[
\dot{F} = F_r \dot{\hat{a}}_r + F_\theta \dot{\hat{a}}_\theta + F_z \dot{\hat{a}}_z ,
\]

with \( \dot{\hat{a}}_i \) representing a unit vector in the \( i^{th} \) direction. The nine scalar quantities \( \tau_{ij} \) are the components of the stress tensor in cylindrical coordinates.

The general form of the boundary conditions imposed on the surface of the cylinder can be stated in two parts. They are as follows:

I. The components of the vectors,

\[
(\tau_{rr}, \tau_{r\theta}, \tau_{rz}),
\]

and

\[
(\tau_{\theta r}, \tau_{\theta \theta}, \tau_{\theta z}), \quad \text{and}
\]

\[
(\tau_{z r}, \tau_{z \theta}, \tau_{zz}),
\]

must vanish along a normal to the surface at all points on the surface which are unattached and free to deform.
II. \( \tau_{rz}, \tau_{\theta z}, \) and \( \tau_{zz} \) are prescribed boundary functions on the attaching surface at \( z=0. \)

A solution to the system equations and the associated boundary conditions will now be developed for the proposed configuration of the sensor. Fortunately, there are symmetry arguments which can be made apriori which greatly reduce the formidable nature of the problem:

The undeformed cylinder is a body of revolution about the \( z \) axis, i.e., it possesses symmetry about the \( z \) axis. Since we are seeking a solution to the case of axial acceleration along this axis of symmetry, it is only natural to assume that a homogeneous isotropic cylinder will deform in such a manner to conserve this symmetry. This will occur if the deformation has no dependence in the \( \theta \) direction, which results in \( v=0 \) (recall that \( v \) is the \( \theta \) component of the displacement vector) and a reduction in complexity of the radial and longitudinal components of the displacement vector, i.e.,

\[
\begin{align*}
\mathbf{u}(r,\theta,z) & = \mathbf{u}(r,z), \\
\mathbf{w}(r,\theta,z) & = \mathbf{w}(r,z).
\end{align*}
\]

Also, this independence of solution in the variable \( \theta \) requires that the partial derivative with respect to \( \theta \), for all tensor and scalar functions in the system equations, must vanish, i.e.,
\[
\frac{\partial}{\partial \theta} (\tau_{ij}) = 0, \text{ for all } i \text{ and } j
\]

and

\[
\frac{\partial}{\partial \theta} (u) = \frac{\partial}{\partial \theta} (w) = 0.
\]

Therefore, with \( v = 0 \) and \( \frac{3u}{\partial \theta} = 0 \), system equation (2.7) predicts that \( \tau_{\theta \theta} \) must vanish since \( \sigma \) is by definition a non-negative quantity. A similar argument based on system equation (2.8) results in \( \tau_{\theta z} = 0 \). And finally, with the sensor undergoing a constant acceleration in the positive z direction, the components of the body force vector per unit volume, \( F \), become

\[
\begin{align*}
F_r &= 0, \\
F_\theta &= 0, \\
F_z &= -\rho a_z,
\end{align*}
\]

where \( \rho \) is the density of the cylinder material, and \( a_z \) is the component of acceleration in the z direction. To simplify the notation, the subscript \( z \) (in \( a_z \)) will be omitted hereafter and it will be understood that we are considering the acceleration to be along the z axis. Under these conditions, the system equations reduce to
These equations are subject to the same boundary conditions previously outlined.

A solution to the above system of equations has been found. Those who wish to follow the detailed maneuvers which lead to a solution are referred to Appendix A. The relevant portion of the solution is, of course, the finding of scalar functions, $u,v,$ and $w,$ which define the displacement vector. Those results are as follows:

$$u = -\frac{\rho\sigma a}{E} (z - \ell_0), \quad (2.10)$$

$$v = 0, \quad (2.11)$$

$$w = \frac{pa}{2E} \left( (z - \ell_0)^2 - \ell_0^2 + \sigma r \right), \quad (2.12)$$
where \( \rho = \) density of the cylinder material,

\( \sigma = \) Poisson's ratio for the cylinder material,

\( E = \) Young's modulus for the cylinder material,

\( l_0 = \) the length of the undeformed right circular cylinder.

The above results will be put to use in the next section. There, the problem of finding the change in physical length of a fiber optic cable (see Figure 1) due to cylinder deformation will be solved.

**The Undeformed Fiber Length**

In this section we consider the problem of finding the length of a fiber optic cable wrapped around the undeformed right circular cylinder shown in Figure 4.

![Figure 4. Cylinder orientation.](image-url)
It will be assumed that the fiber is wrapped over the entire length of the cylinder and that the spacing between successive loops, $\Delta$, is negligible. Under these conditions, the fiber will form a helix along the lateral surface of the cylinder. Now, consider the following set of parametric equations,

\[
\begin{align*}
    r(t) &= r_c, \\
    \theta(t) &= 2\pi t, \\
    z(t) &= r_f + 2r_f t,
\end{align*}
\]

where $r_c =$ radius of the cylinder, $r_f =$ radius of the fiber optic cable.

This set of parametric equations, in the parameter $t$, defines a helix winding "up" the lateral surface of the cylinder, shown in Figure 5 for $t > 0$, increasing monotonically.

![Figure 5. Parametric representation of one fiber loop.](image)
In order that the finite radius of the fiber cable, \( r_f \), does not somehow get overlooked, Figure 6 below depicts what is seen when viewing the surface of the cylinder in the \( xz \) plane with \( y = 0 \).

![Figure 6. Cross-sectional view of the fiber-wrapped cylinder](image)

If the reader will now reflect upon the form of the parametric equations and Figures 5 and 6, it should become apparent that the smooth curve defined by the parametric equations (2.13).

\[
\begin{align*}
    r(t) &= r_c, \\
    \theta(t) &= 2\pi t, \\
    z(t) &= r_f + 2r_f t,
\end{align*}
\]  

(2.13)

follows the point of contact between the cylinder and the fiber cable as the fiber cable winds along the lateral surface of the cylinder.
Therefore, finding the length of the fiber wrapped along the lateral surface of the cylinder is equivalent to finding the arc length of the smooth curve defined above, and as such, the powerful techniques of the vector calculus can be employed.

In general, the arc length measured along a smooth curve from some initial point $\mathbf{R}(t_1)$ to an arbitrary terminal point $\mathbf{R}(t)$ is given by

$$L(t) = \int_{t_1}^{t} \sqrt{\left(\frac{d\mathbf{R}(t)}{dt}\right)^2} \, dt, \quad t > t_1 \quad (2.14)$$

Figure 7. Position vectors along a smooth curve.
In practice, one always invokes the fundamental theorem of the calculus which requires that

\[ \frac{d}{dt} [L(t)] = \left| \frac{d\mathbf{R}}{dt} \right|, \]

or, using standard abbreviated notation

\[ \dot{L}(t) = |\dot{\mathbf{R}}|. \]

Now, in cylindrical coordinate it can be shown that

\[ \dot{L}(t) = \left\{ \left( \dot{r}(t) \right)^2 + \left( r(t) \dot{\theta}(t) \right)^2 + \left( \dot{z}(t) \right)^2 \right\}^{1/2}, \]

and, using the above results, Equation (2.14) becomes

\[ L(t) = \int_{t_0}^{t_1} \left\{ \left( \dot{r}(t) \right)^2 + \left( r(t) \dot{\theta}(t) \right)^2 + \left( \dot{z}(t) \right)^2 \right\}^{1/2} dt. \quad (2.15) \]

This is the general expression we will use to calculate the undeformed fiber length. By inspection of Figure 5 and the parametric equations (2.13), as the parameter \( t \) varies continuously from \( t=0 \) to \( t=1 \) the curve (and the fiber) traverses one loop around the cylinder. If we introduce the notation that \( L_1 \) represents the arc length of one loop, it follows from (2.13) that

\[ L_1 = \int_{t=0}^{t=1} \left\{ \left( \dot{r}(t) \right)^2 + \left( r(t) \dot{\theta}(t) \right)^2 + \left( \dot{z}(t) \right)^2 \right\}^{1/2} dt. \quad (2.16) \]
Note that $L_1$ is not a function of $t$, which merely reflects the fact that the arc length must be independent of the parametrization used to describe the curve. Similarly, if we let $L_n$ represent the arc length of $n$ loops around the undeformed cylinder, it immediately follows from (2.16) that

$$L_n = \int_{t=0}^{n} \left\{ \left( r(t) \right)^2 + \left( r(t) \theta(t) \right)^2 + \left( z(t) \right)^2 \right\}^{1/2} dt.$$  

(2.17)

Therefore, for our chosen form of parametrization, insertion of (2.13) into (2.17) yields

$$L_n = \int_{t=0}^{\pi} \left\{ \left( \rho \right)^2 + \left( r_c \right)^2 + \left( 2r_f \right)^2 \right\}^{1/2} dt,$$

$$L_n = \left\{ (2\pi r_c)^2 + (2r_f)^2 \right\}^{1/2} n.$$  

(2.18)

and, if the fiber cable is wrapped over the entire length of the cylinder, the number of resulting loops become

$$n = \frac{\lambda_o}{2r_f},$$  

(2.19)

where $\lambda_o$ = the undeformed length of the cylinder, $r_f$ = the radius of the fiber cable.
Therefore, it immediately follows from (2.18) and (2.19) that the length of the fiber optic cable wrapped along the entire length of the undeformed cylinder, \( L_{\text{undeformed}} \), becomes

\[
L_{\text{undeformed}} = \left\{ \left( 2\pi r_c \right)^2 + \left( 2r_f \right)^2 \right\}^{1/2} \frac{t_0}{2r_f}.
\]

where \( r_c \) = the radius of the undeformed cylinder.

The Deformed Fiber Length

In this section we consider the problem of finding the length of a fiber optic cable wrapped around a right circular cylinder which, under constant axial acceleration, deforms in accordance with Equations (2.10) thru (2.12). We found earlier that the components of the displacement vector which characterize the cylinder deformation are

\[
u = \frac{-\rho_0 r_a}{E} \left( z - l_0 \right), \tag{2.10}\]

\[
v = 0, \tag{2.11}\]

\[
w = \frac{\rho a}{2E} \left( \left( z - l_0 \right)^2 - l_0^2 + \sigma r^2 \right), \tag{2.12}\]

where, \( u \), \( v \), and \( w \) are respectively the radial, azimuthal, and longitudinal components in cylindrical coordinates. Also, we found
that a suitable parametrization of a fiber optic cable wrapped around the same cylinder in the undeformed state is given by

\[ r(t) = r_c, \]

\[ \theta(t) = 2\pi t, \] \hspace{1cm} (2.13)

\[ z(t) = r_f + 2r_f t. \]

At this point it would be advantageous if we could find a suitable combination between the components of the displacement vector and the system of equations, (2.13), so as to generate a parametrization of the fiber wrapped around the cylinder in the deformed state. With this accomplished, the length of the deformed fiber could be calculated via the same techniques we used in the previous section to find the undeformed length, i.e., performing the appropriate line integral along the curve defined by the parametrization. Before plunging into this analysis, it seems appropriate to at least categorize the concepts that will be used to avoid later confusion. Therefore, in the analysis that follows, the convention regarding the coordinate systems is as follows:

1. The unprimed coordinates represent the components of a position vector to a point of contact between the optical fiber and the lateral surface of the cylinder in the undeformed state.
2. The primed coordinates represent the components of a position vector to the same point on the lateral surface after deformation.

3. In equations (2.10) thru (2.12) by letting \( r = r_c \), the radius of the undeformed cylinder, a suitable transformation from the unprimed coordinates to the primed coordinates can be found and is shown below:

\[
\begin{align*}
\dot{r}(t) &= r(t) + u(r_c, z(t)), \\
\dot{\theta}(t) &= \theta(t), \\
\dot{z}(t) &= z(t) + w(r_c, z(t)).
\end{align*}
\]

Note that \( u \) and \( w \) are composite functions of the parameter \( t \) thru the \( z \) position coordinate. Or, by using (2.13) and the explicit relationships for \( u \) and \( w \),

\[
\begin{align*}
\dot{r}(t) &= r - \frac{\rho a}{E} \left( r_f + 2r_f t - l_0 \right), \\
\dot{\theta}(t) &= 2\pi t, \\
\dot{z}(t) &= r_f + 2r_f t + \frac{\rho a}{2E} \left( (r_f + 2r_f t - l_0) - l_0 \right)^2 - l_o^2 + \sigma r_c^2).
\end{align*}
\]
Now, by following the same line of reasoning which led to equations (2.17) thru (2.19), it is clear that the length of the optical fiber wrapped along the lateral surface of the deformed cylinder, $L_{\text{deformed}}$, becomes

$$
L_{\text{deformed}} = \int_{t=0}^{t_o} \left\{ \left( r' \right)^2 + \left( r' \theta' \right)^2 + \left( z' \right)^2 \right\}^{1/2} \, dt,
$$

(2.23)

where it is understood that all terms in the integrand are functions of the parameter $t$ and that a dot above a particular term denotes differentiation with respect to $t$, e.g.,

$$
r' = \frac{d}{dt} (r'(t)).
$$

If the expressions for the primed coordinates, (2.22), are now inserted into the integrand of (2.23) it will lead to an integral of the following general form:

$$
L_{\text{deformed}} = \int_{t=0}^{t_o} \left\{ A t^2 + B t + C \right\}^{1/2} \, dt.
$$

(2.24)

When Equation (2.24) is evaluated in closed form, it produces inverse hyperbolic functions. Although it was pleasing to have an exact expression for the deformed fiber length, its form was not deemed practical for use by design engineers. Therefore, an alternative tech-
nique was developed for approximating the integral defined in (2.24). Under suitable conditions, it was found that the integrand of (2.24) could be expanded in a binomial series prior to integration. Those wishing to follow the step-by-step maneuvers used in both the exact and the approximation technique are referred to Appendix B which contains the lengthy details. The approximation technique yields the useful form

\[ L_{\text{deformed}} = L_{\text{undeformed}} + \frac{\pi \rho \sigma L^2}{c^0 \frac{a}{2E_f}}. \]  

(2.25)

which shows that the change in length, \( \Delta L \), of the optical fiber due to acceleration is

\[ \Delta L = \frac{\pi \rho \sigma L^2}{c^0 \frac{a}{2E_f}}. \]  

(2.26)

It is this expression which will be useful in a subsequent development of the performance model for the proposed fiber optic laser accelerometer.
CHAPTER III

ANALYSIS OF THE PROPOSED SENSOR

Scale Factor

The phase of an optical beam after traversing the undeformed single-mode optical fiber is given by

\[ \phi = \beta L, \]  

(3.1)

where \( \beta \) is the propagation constant of the mode and \( L \) the undeformed fiber length. Straining the fiber in the axial direction alters the phase by an amount

\[ \Delta \phi = \beta \Delta L + L \Delta \beta. \]  

(3.2)

The first term in (3.2) represents a phase shift due to a change in the physical length of the fiber. Here, we will make use of the results of Chapter II where the change in length of the fiber due to cylinder deformation was found to be

\[ \Delta L = \frac{2 \pi \sigma r^2 a}{E r_f} \]  

(3.3)

The change in the propagation constant, \( \Delta \beta \), can come about from two effects: \(^2\) the strain optics effect whereby the strain changes the
refractive index of the fiber, and a waveguide mode dispersion effect due to a change in the fiber diameter. Hocker\(^3\) has shown that in the single-mode region of the dispersion curve that the effects of waveguide mode dispersion are negligible, and so that effect will be ignored. Therefore, the second term in (3.2) can be written as

\[ L \Delta \beta = L \frac{d\beta}{dn} \Delta n. \quad (3.4) \]

Now, with \( \beta = nk_0 \) it follows that

\[ \frac{d\beta}{dn} = k_0 = \frac{\beta}{n} \]

where \( k_0 \) is the free-space wavenumber of the optical wave.

What remains is to calculate the change in index, \( \Delta n \), due to the longitudinal strain associated with the elongation of the fiber as expressed in (3.3). The so-called strain optics or photo-elastic effect manifests itself as a change in the optical indicatrix

\[ \Delta \left( \frac{1}{n} \right)_{ij} = \sum_s^{\Sigma} p_{ijkl} s_{ijkl} \quad (3.6) \]

where \( s \) and \( p \) are, respectively, components of the strain and photo-elastic tensors. Usually, texts on this subject prefer to convert the photoelastic tensor of rank 4 to a matrix by grouping the \( ij \) terms into \( m \) and the \( kl \) terms into \( n \), according to the correspondence
Therefore, in contracted form the change in the optical indicatrix can be expressed as

$$
\Delta \left( \frac{1}{z} \right)_m = \Sigma_{n=1}^{6} p_{mn} s_n.
$$

(3.7)

Using the above results the change in index is computed as

$$
(\Delta n)_m = -\frac{n}{2} \Delta \left( \frac{1}{z} \right)_m = -\frac{n}{2} \Sigma_{n=1}^{6} p_{mn} s_n.
$$

(3.8)

For our present purposes we will assume the fiber is strained longitudinally by an amount $\varepsilon$ in the x ($m = 1$) direction, and we will calculate the change in index an optical wave propagating in the x direction "sees." The associated strain vector for this scenario is easily shown by basic elasticity theory to be

$$
S = \begin{bmatrix}
\varepsilon \\
-\mu \varepsilon \\
-\mu \varepsilon \\
o \\
o \\
o
\end{bmatrix}.
$$
where μ is Poisson's ratio for the optical fiber. Sapriel has shown that the 6 x 6 matrix representation of the photoelastic tensor for an isotropic medium has only two numerical values, designated as \(p_{11}\) and \(p_{12}\), and their appropriate placement in said matrix is

\[
\begin{bmatrix}
  P_{11} & P_{12} & P_{12} & 0 & 0 & 0 \\
  P_{11} & P_{12} & 0 & 0 & 0 & 0 \\
  P_{12} & P_{12} & P_{11} & 0 & 0 & 0 \\
  0 & 0 & 0 & \frac{1}{2} (p_{11} - p_{12}) & 0 & 0 \\
  0 & 0 & 0 & 0 & \frac{1}{2} (p_{11} - p_{12}) & 0 \\
  0 & 0 & 0 & 0 & 0 & \frac{1}{2} (p_{11} - p_{12}) \\
\end{bmatrix}
\]

Therefore, carrying out the matrix multiplication as given by (3.8) yields

\[
(\Delta n)_2 = (\Delta n)_3 = -\frac{n}{2} \varepsilon \left( \mu p_{11} + (1-\mu) p_{12} \right). \tag{3.9}
\]

and, since \(\varepsilon = \frac{\Delta L}{L}\),

\[
\Delta n = -\frac{n}{2} \frac{\Delta L}{L} \left( \mu p_{11} + (1-\mu) p_{12} \right). \tag{3.10}
\]

Now, if we insert (3.10), (3.5), and (3.4) into (3.2), and after some algebraic manipulation, we arrive at

\[
\Delta \phi = \beta \Delta L \left( 1 - \frac{n}{2} \left\{ (1-\mu)p_{12} - \mu p_{11} \right\} \right). \tag{3.11}
\]
The \( \Delta L \) has previously been shown to be related to the applied acceleration, \( a \), as:

\[
\Delta L = \frac{\pi \rho r L_0^2}{2E_r f} a.
\]

From the above results, it is easily seen that the scale factor \( \frac{\Delta \phi}{a} \) for the sensor becomes

\[
\frac{\Delta \phi}{a} = \beta \left( \frac{\pi \rho r L_0^2}{2E_r f} \right) \left( 1 - \frac{n^2}{2} \left\{ (1-\mu)p_{12} - \mu p_{11} \right\} \right). \tag{3.12}
\]

**Mass Loading Effect**

Recently, Kersey et al.\(^5\) reported on a technique by which the scale factor of a fiber optic accelerometer composed of a solid right circular cylinder wrapped with optical fiber could be increased without altering the baseline design of the solid sensing cylinder or the optical fiber. The technique is commonly referred to as the "mass loading effect." Here, the solid sensing cylinder of mass \( M_c \) is pre-stressed with a loading mass \( M \) as shown in Figure 8.

![Mass loaded cylinder](image-url)

**Figure 8.** Mass loaded cylinder.
Kersey claims (and it is easily shown from basic elasticity theory) that the acceleration induced change in length of the optical fiber due to loading mass $M$ is given by

$$\delta L = \left( \frac{\sigma L M a}{E_r f C_o} \right).$$

(3.13)

It is instructive to compute the ratio of $\delta L$ with that already derived for the change in length of the optical fiber due to the deformation of the "unloaded" cylinder as given by (3.3),

$$\Delta L = \left( \frac{\pi r_c L^2 a}{2E_r f} \right).$$

We get

$$\frac{\delta L}{\Delta L} = \frac{2M}{2} = \frac{2M}{M_c},$$

(3.14)

and, by following the same line of reasoning which led to (3.12), the scale factor for the mass loaded configuration is by superposition,

$$\frac{\Delta \phi}{a} = \beta (\delta L + \Delta L) \left( 1 - \frac{n}{2} \left( (1 - \sigma) p_{11} - \sigma p_{12} \right) \right),$$

which, using (3.3) and (3.14), becomes

$$\frac{\Delta \phi}{a} = \beta \left( 1 + \frac{2M}{M_c} \right) \frac{\pi r_c L^2}{2E_r f} \left( 1 - \frac{n}{2} \left( (1 - \sigma) p_{11} - \sigma p_{12} \right) \right).$$

(3.15)
Chapter IV

EXPERIMENTAL RESULTS

Frequency Response and Scale Factor Linearity

In order to investigate the validity of the theory developed in the previous chapters, a working prototype was fabricated and tested. Two solid sensing cylinders were machined from a metal alloy of Columbium, Hafnium, and Titanium. The cylinders were tightly wrapped with separate, but identical, single-mode optical fibers and mounted in the housing structure as shown in Figure 9.

Figure 9. Fiber-wrapped cylinders in their housing structure.
As shown, the cylinders are affixed on opposite ends to their housing structure. In this configuration the cylinders react "oppositely" to an axial acceleration. When one cylinder experiences an acceleration induced radial reduction by an amount, say $\varepsilon$, the other cylinder undergoes a radial expansion by the same amount. This tandem configuration is attractive for at least two fundamental reasons. First, due to the "equal in magnitude but opposite in polarity" nature of the cylinder deformations, the relative phase shift between the two optical waves traversing the strained fibers is $2\Delta\phi$, where $\Delta\phi$ the magnitude of the acceleration induced phase shift in either fiber. In other words, the scale factor ($\frac{\Delta\phi}{a}$) is enhanced by a factor of 2 when the phase information for the tandem configuration is differentially processed. And secondly, when implemented differentially, the so-called common mode effects cancel. For example, when the tandem configuration experiences a lateral (perpendicular to the input axis) acceleration, the cylinders deform in an identical manner and the relative phase shift is zero.

One way of achieving this differential processing of the induced optical phase information is to insert the fiber from one cylinder into the reference leg and the fiber from the other cylinder into the signal leg of a Mach-Zehnder type interferometer. In this configuration a wide dynamic range is possible via optical phase lock techniques.

Figure 10 illustrates the system used to measure the performance of the dual cylinder transducer.
Figure 10. Synchronous optical phase detection system.

Light from a single frequency helium neon laser was coupled into an acousto-optic frequency modulator (a Bragg cell) which produced two spatially separated optical beams of different frequency, each of which constituted one arm of an interferometer. The fiber from one cylinder functioned as the signal leg and the fiber from the other cylinder was fused into the reference leg of the system. Recombination of the beams produced a carrier frequency, $\omega_m$, with the acceleration (phase) information in the sidebands. The carrier signal was first processed with an automatic gain control (AGC) to maintain proper amplitude. The phase was then detected and synchronously tracked via feedback to a linear phase modulator in the reference leg. The phase modulator consisted of a length of fiber tightly wrapped around a thin-walled piezoelectric cylinder (PZT-4), which
had a flat frequency response to 40 KHZ. The complete details of this synchronous phase detection system have been documented elsewhere.6

The solid sensing cylinders were each 3.54 cm in length with a radius of 0.95 cm. The bulk properties of the cylinder material are as follows:

\[ E = 8.96 \times 10^{10} \text{ N}/\text{m}^2, \]

\[ \rho = 8.86 \times 10^3 \text{ Kg}/\text{m}^3, \]

\[ \sigma = 0.38. \]

Each sensing cylinder was tightly (at 20 KPSI) wrapped with AMFOX series 3020 optical fiber. The cylinders (in their housing structure) were mounted on a sinusoidal shaker table where the frequency response of the fiber optic transducer was measured by driving the shaker table with a "white" noise generator and monitoring the output of the Synchronous Phase Detector with a spectrum analyzer. The result of this experiment are shown in Figure 11.
As seen, the response is relatively flat except near 1.32 KHz where the fundamental resonance of the mass-spring arrangement comes into play. Also, the scale factor linearity was tested at 3 different frequencies over 3 orders of magnitude in acceleration. In this experiment a calibrated (traceable to NBS) mechanical accelerometer was mounted on the shaker table adjacent to the cylinders' housing structure to monitor the applied acceleration. The theoretical scale factor for the dual cylinder transducer is given by [see (3.15)]

$$\frac{\Delta \Phi}{a} = 2\beta \left( 1 + \frac{2M}{M_c} \right) \frac{\pi \rho \sigma \xi}{2E \ell_f} \left( 1 - \frac{n}{2} \left( (1-\mu) p_{12} - \mu p_{11} \right) \right), \quad (4.1)$$

where the factor of 2 in (4.1) has been added to account for the differential nature of the tandem configuration. Furthermore, it is
standard practice to express the scale factor in terms of induced optical radians of phase shift per "g" (Earth's gravitational acceleration). Using \( g = 9.8 \text{ m/s}^2 \), (4.1) becomes,

\[
\frac{\Delta \phi}{g} = (9.8) 2\beta \left( 1 + \frac{2M}{M_c} \right) \left( -\frac{\pi \sigma \tau g^2}{2 \text{Erf}_c} \right) \left( -\frac{n^2}{2} \left( (1 - \mu)p_{12} - \mu p_{11} \right) \right). \tag{4.2}
\]

The scale factor for the experimental set-up was calculated using (4.2) to be \( 1.705 \frac{\text{Radians}}{g} \). The following values (not previously referenced in this paper) were used in the computation:

\[
\beta = \frac{2\pi n}{\lambda_o}, \quad \text{and} \quad \lambda_o = 0.6328\mu\text{m (He-Ne Laser)},
\]

\[
n = 1.456,
\]

\[
P_{12} = 0.121,
\]

\[
P_{11} = 0.270,
\]

\[
\mu = 0.17,
\]

\[
\frac{M}{M_c} = 0.0460,
\]

\[
r_f = 100 \mu\text{m}.
\]

The results of the scale factor linearity test are shown in Figure 12.
As seen in Figure 12, the system closely tracks the theoretical prediction over the entire range of testing. A more quantitative measure of scale factor linearity involves a 2\textsuperscript{nd} order polynomial fit of the raw data in the form

$$y = a + bx + cx^2.$$ 

From the above coefficients, the percent scale factor linearity is calculated as

$$\% \text{ linearity} = \left( \frac{c}{b} \right)_{\text{max}} 100.$$
The $x_{\text{max}}$ is the maximum value at which data was taken. Using this technique, the percent scale factor linearity of the fiber optic accelerometer was calculated to be as follows:

<table>
<thead>
<tr>
<th>Frequency (Hz)</th>
<th>Scale Factor Linearity (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.18</td>
</tr>
<tr>
<td>300</td>
<td>0.06</td>
</tr>
<tr>
<td>500</td>
<td>0.62</td>
</tr>
</tbody>
</table>

In order to put the test results into perspective, a survey was made of the specifications for mechanical accelerometers currently used in commercial and military aircraft. In general, the best devices were quoted as having a scale factor linearity of not more than 1% at the maximum rated input acceleration. With this in mind, the results of testing the fiber optic prototype seem quite promising.
CHAPTER V

CONCLUDING REMARKS

This paper has been devoted to the initial investigation, both theoretically and experimentally, of a unique fiber optic laser accelerometer. Initial testing of the prototype has at least two significant results. First, the agreement between theoretical predictions and experimental results is excellent over the entire range of testing. And secondly, the scale factor linearity of the prototype appears competitive with even the best mechanical accelerometers currently available. Nonetheless, much additional work is still needed to determine the feasibility of such a device in a wide variety of applications. For example, the problems associated with thermal gradients and mechanical tolerances have not been addressed in this paper and the author hopes to investigate these, and others, in the very near future.
Appendix A

SOLUTION OF THE REDUCED SYSTEM EQUATIONS

Following the arguments outlined in the main body of this paper we arrived at the reduced system equations:

\[
\frac{\partial \tau_{rr}}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\partial \tau_{zz} - \tau_{\theta \theta}}{r} = 0,
\]

\[
\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \tau_{zz}}{\partial z} + \frac{\tau_{rz}}{r} = \rho a,
\]

\[
\frac{\partial u}{\partial r} = \frac{1}{E} \left( \tau_{rr} - \sigma (\tau_{\theta \theta} + \tau_{zz}) \right),
\]

\[
\frac{u}{r} = \frac{1}{E} \left( \tau_{\theta \theta} - \sigma (\tau_{\theta \theta} + \tau_{zz}) \right),
\]

\[
\frac{\partial w}{\partial z} = \frac{1}{E} \left( \tau_{zz} - \sigma (\tau_{\theta \theta} + \tau_{rr}) \right),
\]

\[
\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = \frac{2(1+\sigma)}{E} \tau_{rz}.
\]

A solution to these equations will now be formulated under the assumption that the stresses acting on each cross section of the cylinder are distributed uniformly. This will be satisfied if

\[
\tau_{rr} = \tau_{\theta \theta} = \tau_{rz} = 0.
\]
In addition, the boundary condition on the free surface requires that \( \tau_{zz} = 0 \) when \( z = l_0 \). Under these conditions the system equations become

\[
\frac{\partial \tau_{zz}}{\partial z} = \rho a. \tag{A.1}
\]
\[
\frac{\partial u}{\partial r} = -\frac{\sigma}{E} \tau_{zz}. \tag{A.2}
\]
\[
\frac{u}{r} = -\frac{\sigma}{E} \tau_{zz}. \tag{A.3}
\]
\[
\frac{\partial w}{\partial z} = \frac{1}{E} \tau_{zz}. \tag{A.4}
\]
\[
\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = 0. \tag{A.5}
\]

Now, integration of (A.1) gives

\[
\tau_{zz} = \rho a (z - z_0),
\]

where, the boundary condition that \( \tau_{zz} \) must vanish when \( z = l_0 \) requires that \( z_0 = l_0 \). Therefore,

\[
\tau_{zz} = \rho a (z - l_0). \tag{A.6}
\]

And substitution of (A.6) into (A.3) yields
\[ u = -\frac{\rho \sigma a}{E} (z - l_0). \quad (A.7) \]

With the results obtained thus far, \((A.4)\) and \((A.5)\) can be written
\[
\frac{\partial w}{\partial z} = \frac{\rho a}{E} (z - l_0), \quad \text{and} \quad \frac{\partial w}{\partial r} = \frac{\rho a}{E}.
\]

Integration gives, respectively
\[
w = \frac{\rho a}{2E} (z - l_0)^2 + w_1(r) + K_1, \quad \text{and} \quad w = \frac{\rho a}{2E} + w_2(z) + K_2.
\]

Which imply that the general form of the solution is
\[ w = \frac{\rho a}{2E} (z - l_0)^2 + \frac{\rho a}{2E} + K_3, \]
or,
\[ w = \frac{\rho a}{2E} \left( (z - l_0)^2 + \sigma r^2 \right) + K. \]

Where the arbitrary constant \(K\) will be chosen to fix the point on the cylinder at the origin. This will be satisfied if
\[ w = \frac{\rho a}{2E} \left( (z - l_0)^2 + \sigma r^2 - l_0^2 \right). \quad (A.8) \]
Therefore, the complete solution to the deformation is

\[ \tau_{rr} = \tau_{\theta\theta} = \tau_{\theta r} = \tau_{\theta z} = \tau_{rz} = 0, \]

\[ \tau_{zz} = \rho a (z - l_0), \]

\[ u = -\frac{\sigma a}{E} (z - l_0), \]

\[ v = 0, \]

\[ w = \frac{\rho a}{2E} [(z - l_0)^2 + \sigma r^2 - l_0^2]. \]
APPENDIX B

THE DEFORMED FIBER LENGTH

Following the arguments outlined in the main body of this paper we arrived at the following expression for the length of the optical fiber when the cylinder is in the deformed state as

\[
L_{\text{deformed}} = \int_0^2 \left\{ \left( r' \right)^2 + \left( r' \theta' \right)^2 + \left( z' \right)^2 \right\}^{1/2} \, dt. \tag{B.1}
\]

Where,

\[
x'(t) = r_c - \frac{\rho a}{2} \left( r_f + 2r_f t - \ell_o \right),
\]

\[
y'(t) = 2\pi t,
\]

\[
z'(t) = r_f + 2r_f t + \frac{\rho a}{2E} \left( r_f + 2r_f t - \ell_o \right)^2 - \ell_o^2 + a r_c^2.
\]

Inserting the parametric equations shown above into the integrand of (B.1) and carrying out the prescribed operations leads to the following expression

\[
L_{\text{deformed}} = \int_0^2 \left( A t^2 + B t + C \right)^{1/2} \, dt. \tag{B.2}
\]
It can be shown that the coefficients in (B.2) are given by

\[ A = \left( \frac{2\sigma r_{av}}{E} \right)^2 \left( 1 + 4\pi^2 + 4 \left( \frac{r_f}{\sigma r_c} \right)^2 \right). \]

\[ B = 16 \left( \frac{\pi r_c^2 + \frac{2}{\sigma r_c}}{E} \right) \left( \frac{\sigma r_{av}}{E} \right) + 16 \left( 1 - \frac{L_0}{r_f} \right) \left( \frac{\sigma r_{av}}{E} \right) \left( \pi + \frac{r_f}{\sigma r_c} \right). \]

\[ C = \frac{(2\sigma r_{av})^2}{E} \left( 1 - \frac{L_0}{r_f} \right)^2 \left( \frac{\sigma r_{av}}{E} \right)^2 + r_f^2 + \frac{2\sigma r_{av}}{E} \left( \frac{1 - L_0}{r_f} \right) \left( (2r_f)^2 - \sigma (2\pi r_c)^2 \right) + (2\pi r_c)^2 + (2r_f)^2. \]

In order to simplify the notation associated with (B.2), we introduce an abbreviated form via letting

\[ R = At^2 + Bt + C. \]  \hspace{1cm} (B.3)

then,

\[ L_{\text{deformed}} = \int_{t=0}^{\frac{L_0}{2r_f}} (R)^{1/2} \, dt. \]  \hspace{1cm} (B.4)

the integral defined by (B.4) is well known and its evaluation in indefinite form is given by

\[ \int (R)^{1/2} \, dt = \frac{2At+B}{4A} (R)^{1/2} + \frac{4AC-B}{8A} \sinh^{-1} \left( \frac{2At+B}{(4AC-B)^{1/2}} \right). \]
As seen, the exact solution to the deformed fiber length yields inverse hyperbolic functions which are somewhat intractable, and this was the motivation for finding an approximation to the integral defined in (B.4). Fortunately, there are reasonable assumptions concerning the coefficients \((A, B, \text{ and } C)\) in the integrand of (B.4) which yield a more manageable solution. First, for reasonable values of acceleration (say, \(0 < a < 10\) g) it can be shown that

\[ Bt + C \gg At, \]

for all \(t\) in the interval \(0 < t < \frac{L_0}{2rf}\). Thus, there is little error associated with neglecting the quadratic term in the integrand of (B.4) and our simplified form becomes

\[
L_{\text{deformed}} = \int_{t = 0}^{2} (Bt + C)^{1/2} \, dt. \tag{B.5}
\]

Which can be expressed as

\[
L_{\text{deformed}} = \frac{L_0}{2rf} \left( C \right)^{1/2} \int_{t = 0}^{2} \left( 1 + \frac{Bt}{c} \right)^{1/2} \, dt. \tag{B.6}
\]

Now, it can also be shown that

\[
\frac{Bt}{c} \ll 1,
\]
for all \( t \) in the interval \( 0 < t < \frac{l_0}{2r_f} \). This result justifies expanding the integrand of (B.6) in a binomial series and neglecting terms of order 2 and higher in \( \frac{Bt}{C} \), i.e.

\[
\left(1 + \frac{Bt}{C}\right)^{1/2} = 1 + \frac{Bt}{2C},
\]

and (B.6) becomes

\[
L_{\text{deformed}} = (C)^{1/2} \int_{t=0}^{l_0} \left(1 + \frac{Bt}{2C}\right) \, dt,
\]

which gives

\[
L_{\text{deformed}} = (C)^{1/2} \frac{l_0}{2r_f} + (C)^{1/2} \frac{B}{4C} \left(\frac{l_0}{2r_f}\right)^2. \tag{B.7}
\]

The first term in (B.7) can be shown to approximate the undeformed fiber length by the following argument. For reasonable values of acceleration (as before, \( 0 \leq a \leq 10 \, \text{g} \)) and using the numerical values associated with the experimental set-up, the coefficient \( C \) is approximately

\[
C = \left(2\pi r_c\right)^2 + \left(2r_f\right)^2,
\]

therefore, (B.7) becomes

\[
L_{\text{deformed}} = \left(\left(2\pi r_c\right)^2 + \left(2r_f\right)^2\right)^{1/2} \frac{l_0}{2r_f} + \frac{B}{4} (C)^{1/2} \left(\frac{l_0}{2r_f}\right)^2. \tag{B.8}
\]
Comparing the first term in (B.8) with the results derived for the undeformed fiber length, (B.8) becomes

\[ L_{\text{deformed}} = L_{\text{undeformed}} + \frac{B}{4(\xi_0)} \left( \frac{\xi_0}{2rf} \right)^2. \]  

(B.9)

Therefore, the change in length of the optical fiber due to acceleration, \( \Delta L \), takes on the form

\[ \Delta L = \frac{B}{4(\xi_0)^{1/2}} \left( \frac{\xi_0}{2rf} \right)^2. \]  

(B.10)

And, following the same line of reasoning used before, for reasonable values of acceleration the coefficient \( B \) is approximately

\[ B = \frac{16 \pi r_c^2 \rho \sigma_f a}{E}, \]

therefore (B.10) becomes

\[ \Delta L = \frac{4 \pi r_c^2 \rho \sigma_f a}{(2\pi r_c^2 + (2rf)^2)^{1/2}} \left( \frac{\xi_0}{2rf} \right)^2. \]  

(B.11)

Or, since \( (2\pi r_c^2) \gg (2rf)^2 \)

\[ \Delta L \approx \frac{4 \pi r_c^2 \rho \sigma_f a}{2\pi r_c} \left( \frac{\xi_0}{2rf} \right)^2, \]

which gives

\[ \Delta L = \frac{\pi \rho \sigma_f a}{2Er_f} \left( \frac{\xi_0}{2rf} \right)^2. \]  

(B.12)
REFERENCES