Nonlinear Adaptive Estimation And Its Application To Synchronization Of Lorenz System

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NONLINEAR ADAPTIVE ESTIMATION AND ITS APPLICATION TO SYNCHRONIZATION OF LORENZ SYSTEM

by

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Summer Term
2004

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ABSTRACT

Synchronization and estimation of unknown constant parameters for Lorenz-type transmitter are studied under the assumption that one of the three state variables is not transmitted and that transmitter parameters are not known apriori. An adaptive algorithm is proposed to estimate both the state and system parameters. Since Lorenz system shows the property of sensitivity to initial conditions and evolves in different mode with parameter variation, an equivalent system is introduced. The adaptive observer is designed based on this equivalent system without any requirement on initial conditions of the observer. It is shown by Lyapunov arguments and persistent excitation analysis that exponential stability of state and parameter estimation is guaranteed. Simulation results are included to demonstrate properties of the algorithm. In a practical communication system, the received signals presented at the receiver part differ from those which were transmitted due to the effects of noise. The proposed synchronization scheme is robust with regard to external bounded disturbance. When an additive white gaussian noise (AWGN) channel model is considered, estimates of state and parameter converge except for small errors. The results show promise in either coherent detection or the message decoding in telecommunication systems.
ACKNOWLEDGMENTS

I would like to express my appreciation to my advisor Dr. Zhihua Qu for his many suggestions, research guidance and constant support during my research. Dr. Qu taught me not only how to be an effective researcher, but also taught me a sound philosophy of life. I also wish to express my gratitude to my committee members, Dr. S.Roy Choudhury, Dr. Roger Johnson, Dr. Michael G. Haralambous, and Dr. Lei Wei. Special Thanks go to Dr. Michael G. Haralambous for his kind help on improving my English writing.

I am thankful to David Huibregtse, Reiner Fenske and the Microtronic people and also thankful of their support and funding during part of my Ph.D program.

A special acknowledgement goes to Apiwat Saengdeejing, Guangyu Chai and Nattorn Pongratananukul for their support and help during many years of research development and projects. I also appreciate the help of my friends at the University of Central Florida - in particular, Hong Mao, Yangyang Wen, Songquan Deng, and Yujing Yang for their consistent encouragement during the past years.

Finally, I wish to thank and express my gratitude to my parents and my brother and sister for their patience, love and support.

Yufang Jin
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CHAPTER 1

INTRODUCTION

1.1 Application of Chaotic Signals in Communication Systems

Synchronization of chaotic systems has grown to be one of the richest research areas during the past decade. It is motivated from the pioneering research of Pecora and Carrol in [57, 58]. In [57], they proposed a cascade scheme in which two identical chaotic systems can be synchronized by a scalar driving signal with different initial conditions. The underlying assumptions require that the chaotic system be decomposed into two subsystems: driving subsystem and response subsystem which has negative Lyapunov exponents. Moreover, they assume all parameters of the system are known. They point out the possible application of chaotic signal to the communication area at the end of [57].

Application of chaotic signals in secure telecommunication is based on several interesting properties. One of these properties is the wide band spectrum of a chaotic signal, which leads to difficulty in extracting the message by spectral methods. In addition, a chaotic signal is sensitive to initial conditions (SIC). Given slightly different initial conditions, the resulting states of a chaotic system will diverge exponentially from each other. A final concern is related to parameter variation. When system parameters change, a chaotic
system displays different evolution behavior.

The early stage of this research is devoted to the application schemes of chaotic signaling in telecommunication systems. Researchers have considered several modulation schemes, such as masking, parameter modulation and chaotic switching [12, 31, 33, 59, 78]. Masking consists of adding the key, or message signal, on one state of the driving subsystem, namely the transmitter, and then recovering the message at the receiver. Parameter modulation means modulating the message to the parameters of the transmitter [15, 16, 17]. This communication scheme is based on the Pecora-Carroll cascaded synchronization scheme. In [15] the authors specify the circuit implementation of Lorenz system for transmitting speech signals and later on they show that without adding the message signal, the scheme is globally asymptotically stable. In this scheme, the information signal is added to a chaotic carrier but not injected into the dynamic system constituting the transmitter. Thus, the receiver is forced by a summation of the chaotic signal and the information signal, while this is not the case for the transmitter. Because of this slight difference, the receiver cannot be driven by the chaotic signal generated by the transmitter exactly and the message can only be recovered with an error. This inevitable error is investigated further in [17]. In a chaotic switching scheme, a binary message is transmitted by switching between two chaotic attractors generated by two different parameter sets. Other kinds of communication schemes have also been proposed, for example, state variable modulation, which uses a signal \( s(t) = h(x) \) or \( \dot{s} = h(x, s) \) to drive the transmitter. The restriction of this method is that \( s(t) \) must be chosen in a way that the transmitter and the receiver remain chaotic. Modulation schemes have been studied extensively in the available literature, while demodulation is done stochas-
tically or the message signal is detected roughly in some region. Thus far, little effort has been devoted to accurate parameter estimation for demodulation.

The essence of all schemes described above leads to the problem of synchronization or state reconstruction. With this consideration, researchers in different areas have provided various approaches, such as a master-slave subsystem [15, 16, 17, 57, 33], nonlinear dynamic forecasting (NLD) [1, 68, 69, 70] and observer-based synchronization [9, 21, 26, 41, 47, 48, 54]. In most existing results, when synchronization or reconstruction of chaotic states is considered in a master-slave scheme, one basic assumption is that all the parameters in chaotic systems are known. The nonlinear dynamic forecasting method is really a short term forecasting of system behavior. The model is based on higher dimension reconstruction of the state-space using either time delay or derivative coordinates and autocorrelation techniques. It focuses mainly on the transmitter section. Recently, an observer-based approach has been investigated to deal with synchronization and parameter estimation [71]. However, few results have been proposed to establish the link between synchronization and parameter estimation. In [71], a full state observer is proposed to estimate parameters of Lorenz system, but whether the estimates converge to their true values or not is not considered. In a real system, there are always measurement errors of initial conditions and parameter mismatches. Thus, parameter estimation of the transmitter with different initial conditions and partial state observation is an important unsolved problem. Though synchronization of chaotic systems with mismatched parameters has been discussed in [21, 73], full static-state error or dynamic-output error feedback is required.

The following problem will be addressed in this dissertation: an adaptive nonlinear
observer is proposed with only partial state measurement for reconstructing the unknown state and for estimating the unknown parameters of the transmitter. Related research areas are adaptive observer design and persistent excitation for parameter convergence.

1.2 Adaptive Observer Design

Adaptive observer design is a natural extension of observer design for linear time invariant (LTI) systems having known parameters. For LTI systems with known parameters, it is well established that estimates of the state variables can be generated from the inputs and outputs using a Luenberger observer. Moreover, the error dynamics between system states and their estimates are asymptotically stable. When the parameters of the system are unknown, theories developed for LTI system can no longer be applied. In such cases, an adaptive observer is designed to simultaneously estimate the parameters and state variables of the dynamical system. This was first accomplished in [11, 36, 38] in the early 1970’s. Traditionally the adaptive observer is obtained by replacing the unknown parameters with their on-line estimates updated by the adaptation laws. The idea behind this is that the adaptation laws will generate reasonably good parameter estimates, so that the observer can achieve the goals that the observer designed with known parameters does. Different representations have been discussed in detail for LTI systems [27, 49, 67]. Basic methods used to design adaptive laws are the Lyapunov-based design, gradient methods, and least squares methods. Research techniques have subsequently expanded to include adaptive observer design for nonlinear systems, linear time-varying systems and systems with disturbances [60].
Design procedures of adaptive observers and adaptation laws for the adjustment of parameters are based on stability analysis, so that all the signals in the system remain bounded based on Lyapunov arguments. The next step is to check whether the behavior of the overall system meets the desired performance requirements, such as convergence rate of parameter estimates.

### 1.3 Persistent Excitation

Convergence of parameter estimates to their true values as well as the rate of convergence are related to the persistent excitation (PE) property of certain signals in the system. Along with research in adaptive observer/control design, an equal amount of effort has been directed toward the study of the conditions for parameter convergence to their true values. Significant contributions were made in [2, 7, 49, 51, 53, 79], where necessary and sufficient condition for exponential stability of the overall adaptive system were determined based on PE property of certain signals in the system. In order to check the PE property of a signal, a detailed study of both linear and nonlinear, algebraic as well as dynamic transformations is necessary. The concept of PE is closely related to the convergence rate of the parameter estimates to their true values in adaptive systems. In essence, adaptive systems are nonlinear systems, however, available results mainly focus on algebraic transformation by a full rank matrix, dynamic transformation by a single input single output (SISO) for LTI systems. While the original definition of persistent excitation is given in the time domain, results for LTI systems are based on frequency domain analysis. Persistent excitation property for nonlinear system and multi-input multi-output (MIMO) system are limited. The latest research
results available are in [5, 7, 43, 52, 56]. In [43, 52], hierarchical algorithms are applied and parameter convergence is achieved globally under certain necessary and sufficient conditions on the system variables and the underlying nonlinear functions. In [56], a relaxed persistent excitation condition is obtained from a uniformly globally stability proof with restriction on the boundedness of system variables and the underlying nonlinear functions.

1.4 Main Results and Organization of The Dissertation

An adaptive observer is proposed in this dissertation to estimate the unknown state and parameters of Lorenz-type systems by partial observation. The research is based on the assumption that the structure of the system is known. The proposed observer is novel as follows: First, an equivalent system with specified initial conditions is proposed. The equivalence is proven using Lyapunov arguments. This equivalent system is proposed only for analysis and not for implementation. An adaptive nonlinear observer is designed based on this equivalent system and online estimates of unknown parameters. Since there is no requirement for setting specific initial conditions of the observer, SIC is not a issue. The second step is conducted to show that the error dynamics of the equivalent/original system and the observer are exponentially stable with certain persistent excitation signals in the system. The persistent excitation property of these signals is proven using invariant set theory and properties of the chaotic systems. Then, exponential convergence of parameter estimates to their true values can be concluded. The unknown state estimation error goes to zero, i.e. state reconstruction is achieved as well.

Existing results on persistent excitation are limited to single input single output
(SISO) systems in frequency domain analysis, even though the original definition of persistent excitation is given in time domain based on a stability proof. A lemma extending the existing results to vector and nonlinear signals is proven. Further research on the relationship between time varying signal and its frequency components is carried out in this dissertation. Specifically, condition of persistent excitation property of summation of positive semi-definite matrices is proposed based on checking the rank of its null space instead.

In a practical communication system, due to effects of noise, the received signals presented at the receiver differ from those which are transmitted. A noise-free observer is given in section 4 and the impact of noise is studied in section 5.

Chapter 2 presents necessary theoretical background, basic definitions and important theorems for further understanding. It includes fundamental stability theorems (Lyapunov method, input-output stability), and techniques for stability analysis of nonlinear systems. In addition to observer design techniques such as adaptive observer design, nonlinear adaptive estimation are introduced and some examples of designing adaptation laws are given. Since the proposed application is for chaotic signal, a brief introduction to chaotic systems, especially Lorenz-type system is presented. It contains some useful items for further stability and persistent excitation analysis.

In chapter 3 existing results of persistent excitation are first reviewed. Then further research on a special matrix, and general results to check persistent excitation property for signal in time domain are presented. Also, the link between time domain and frequency domain is studied. For the first time, the condition of persistent excitation property of summation of positive semi-definite matrices is established based on checking the rank of
their null spaces.

In chapter 4, observer designs with full and partial state measurement for Lorenz system are presented. All designs are based on noise-free transmission. Exponential stability proofs are conducted with Lyaponov arguments. The persistent excitation property of signals is analyzed in detail in this chapter. Two simulation examples for the full state and partial state measurement are given to show the effectiveness of the designs. Later on, in chapter 5, the robustness of the scheme with additive noises is discussed.

In chapter 5, a detailed proof of the robustness of the scheme is presented. Since frequency domain techniques are not applicable, an adaptive filter is applied to eliminate the effect of additive white gaussian channel noise (AWGN). Simulation results show that the system is bounded with bounded disturbance.

Conclusions and discussion of future research are included in chapter 6. Appendix A lists all m-file codes and Simulink simulation diagrams.
CHAPTER 2

THEORETICAL BACKGROUND

Mathematical description is fundamental in the quantitative analysis of any dynamic system. Typically, system dynamics are described through a set of vector differential equations,

\[ \dot{x}(t) = f(x, t), \quad f(0, t) = 0, \; \forall t \geq t_0, \]  

(2.0.1)

where \( x(t_0) = x_0 \) and \( f : \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^n \) is such that a solution \( x(t, x_0, t_0) \) exists for all \( t \geq t_0 \). States of the system, \( x \in \mathbb{R}^n \) can be obtained by integrating these differential equations. Stability analysis of systems consists of defining whether the system states diverge to infinity, converge to some fixed points, become periodic, or stay in some bounded region. The concept of stability of a system has been extensively investigated in the past and much literature exists \([35, 39, 40, 74, 81]\). Lyapunov stability theory and input-output stability theory based on functional analysis techniques are the most widely used stability analysis tools among controls groups especially for nonlinear systems, since explicit solutions to nonlinear differential equations are not easily obtained. Lyapunov stability considers stability as an internal property of the system. Input-output stability considers the effect of external inputs on the system. In the following sections we will review some important definitions and stability theorems, and give some background and techniques for nonlinear observer design.
2.1 Stability Theory

Lyapunov stability theory plays an important role in both system analysis and control design. It provides an effective means of analyzing the stability of nonlinear differential equations where the solutions to these equations are difficult to obtain. The fundamental approach of Lyapunov stability analysis consists of finding an energy-like scalar function, or so-called Lyapunov function $V(t, x(t, x_0))$, and examining its time derivative $\dot{V}(t, x(t, x_0))$ along the state trajectory of the system $x(t, x_0)$. If the system is dissipative, the energy of the system is always positive and its time derivative is non-positive. This means the energy of the system is always being dissipated with time when the state is in a certain region. Therefore, the system under investigation will become stable in that particular region. There are two main approaches in Lyapunov stability analysis: the first Lyapunov method, namely the Lyapunov indirect method, can be applied to determine local stability properties of the linearized system around the equilibrium point and its neighborhood. Although the first method yields only local results, it provides a basic design tool for linearized systems achieving stability around an operating point. The second Lyapunov method, namely Lyapunov’s direct method, is based on concept of energy dissipation. The stability problem is solved by using the form of $f(x(x_0, t_0), t)$ instead of the explicit solution of the system.

2.1.1 Lyapunov Stability Theory

Though most of the Lyapunov stability results have been covered in existing literature, some of the main definitions and stability results are summarized in the following sections [25, 29, 32, 61].
**Definition 2.1.1.** The autonomous nonlinear system described by

\[ \dot{x}(t) = f(t, x(t, x_0)), \]

has an equilibrium point \( x = x_e \) if, once the state arrives at \( x(t) = x_e \) it remains at this state for all time. Equivalently, the derivative of the system state is equal to zero at the equilibrium state, \( x = x_e \), thus \( f(x_e) = 0 \).

**Definition 2.1.2.** The equilibrium point \( x = 0 \) is said to be Lyapunov stable (LS) if, for each \( \varepsilon > 0 \) at time \( t_0 \), there exists a constant \( \delta(t_0, \varepsilon) > 0 \) such that

\[ \| x(t_0) \| < \delta(t_0, \varepsilon) \implies \| x(t) \| \leq \varepsilon, \quad \forall t \geq t_0. \]

It is said to be uniformly Lyapunov stable (ULS) if, for each \( \varepsilon > 0 \), the constant \( \delta(t_0, \varepsilon) = \delta(\varepsilon) > 0 \) is independent of initial time \( t_0 \).

**Definition 2.1.3.** The equilibrium point \( x = 0 \) is said to be attractive at time \( t_0 \) if, for some \( \delta > 0 \) and each \( \varepsilon > 0 \), there exists a finite time interval \( T(t_0, \delta, \varepsilon) \) such that

\[ \| x(t_0) \| < \delta \implies \| x(t) \| \leq \varepsilon, \quad \forall t \geq t_0 + T(t_0, \delta, \varepsilon). \]

It is said to be uniformly attractive (UA) if, for all \( \varepsilon \) satisfying \( 0 < \varepsilon < \delta \), the finite time interval \( T(t_0, \delta, \varepsilon) = T(\delta, \varepsilon) \) is independent of initial time \( t_0 \).

Stability and attractiveness are different concepts. The definition of Lyapunov stability indicates that if the initial state lies within a ball \( x(t) < \delta \) then the state will stay within the ball \( x(t) < \varepsilon \) for all \( t > t_0 \); also, \( \delta \) is dependent on \( \varepsilon \). Attractive means that if the system is initialized at any point in \( \| x(t_0) \| < \delta \) then in a finite time interval the trajectory will stay...
inside a ball \(\|x\| < \epsilon\), and as the size of \(\delta \to 0\), with \(\epsilon < \delta\) the trajectory will converge to 
\(x(t) = 0\); also, \(\delta\) and \(\epsilon\) are independent of each other.

**Definition 2.1.4.** The equilibrium point \(x = 0\) is asymptotically stable (AS) at time \(t_0\) if it is Lyapunov stable at time \(t_0\) and if it is attractive, or equivalently there exists \(\delta > 0\) such that

\[\|x(t_0)\| < \delta \implies \|x(t)\| \to 0 \text{ as } t \to \infty.\]

It is uniformly asymptotically stable (UAS) if it is uniformly Lyapunov stable and if \(x = 0\) is uniformly attractive.

Asymptotic stability means that if the initial states stay within some region \(\|x(t_0)\| < \delta\) then the state will converge to zero as time \(t \to \infty\). If this region \(\delta\) can be extended to the entire space, the system is globally asymptotically stable (GAS), otherwise it is locally asymptotically stable (LAS).

**Definition 2.1.5.** A trajectory \(x(t), x(t_0) = x_0\), is said to be uniformly bounded (UB) if, for some \(\delta > 0\), there is a positive constant \(d(\delta) < \infty\), possibly dependent on \(\delta\) but not on \(t_0\), such that, for all \(t \geq t_0\)

\[\|x(t_0)\| < \delta \implies \|x(t)\| \leq d(\delta).\]

The uniformly bounded (UB) condition is less restrictive than the Lyapunov stability condition. If system is Lyapunov stable it is uniformly bounded but the converse is not true. In Lyapunov stability \(\delta(\epsilon, t_0) \leq \epsilon\) as \(\epsilon \to 0 \implies \delta(0, t_0) = 0\) but it is not necessary for \(d(\delta = 0) \geq 0\) (UB).
Definition 2.1.6. A continuous function \( \gamma: \mathbb{R}^+ \to \mathbb{R}^+ \) is a class \( \mathcal{K} \) function if \( \gamma(0) = 0 \) and if it is strictly increasing. It is said to belong to class \( \mathcal{K}_\infty \) if \( \gamma(x) \to \infty \) as \( x \to \infty \).

Definition 2.1.7. A function \( V: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^+ \) is called locally positive definite if there exists a class \( \mathcal{K} \) function \( \gamma_1: \mathbb{R}^+ \to \mathbb{R}^+ \) such that,

\[
\gamma_1(\|x(t)\|) \leq V(x(t), t),
\]

for some neighborhood of the origin \( \Omega \subset \mathbb{R}^n \). Function \( V \) is said to be locally decrescent if there exists a class \( \mathcal{K} \) function \( \gamma_2: \mathbb{R}^+ \to \mathbb{R}^+ \) such that, for some neighborhood of the origin \( \Omega \subset \mathbb{R}^n \)

\[
V(x(t), t) \leq \gamma_2(\|x(t)\|).
\]

The following theorems deal with stability and uniform boundedness cases. Detailed proofs can be found in [61].

Theorem 2.1.1. Consider the system \( \dot{x} = f(x, t) \) where \( \gamma_1, \gamma_2: \mathbb{R}^+ \to \mathbb{R}^+ \) are class \( \mathcal{K}_\infty \) function, and \( \gamma_3: \mathbb{R}^+ \to \mathbb{R}^+ \) is a class \( \mathcal{K} \) function. If there exists a continuously differentiable Lyapunov function \( V: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^+ \), that satisfies

\[
\gamma_1(\|x(t)\|) \leq V(x(t)) \leq \gamma_2(\|x(t)\|),
\]

and the time derivative of \( V \) along the trajectory \( x(t) \) is

\[
\dot{V}(t) \leq -\gamma_3(\|x(t)\|),
\]

where \( \gamma_3(0) = 0 \), then the system has the following stability results:

1) if \( \gamma_3(\|x\|) \) is positive semi-definite then the system is uniformly stable.
2) if $\gamma_3(\|x\|)$ is positive definite then the system is uniformly asymptotically stable.

3) if $\gamma_3(\|x\|) \geq \beta V(x, t)$, $\exists \beta > 0$ then the system is exponentially stable.

4) if $\gamma_3(\|x\|) \geq \beta V^p(x, t)$, $\exists \beta > 0$, and $0 < p < 1$ then the system is exponentially stable with finite convergence time.

**Theorem 2.1.2.** Consider the system $\dot{x} = f(x, t)$ where $\gamma_1, \gamma_2 : \mathbb{R}^+ \to \mathbb{R}^+$ are class $\mathcal{K}_\infty$ function, and $\gamma_3 : \mathbb{R}^+ \to \mathbb{R}^+$ is a class $\mathcal{K}$ function. If there exist a continuously differentiable Lyapunov function $V : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^+$, that satisfies

$$
\gamma_1(\|x(t)\|) \leq V(x(t)) \leq \gamma_2(\|x(t)\|),
$$

and positive constants $\eta_1$ and $\eta_2$ satisfying $\eta_2 > (\gamma_1^{-1} \circ \gamma_2)(\eta_1)$ such that, along the trajectory $x(t)$

$$
\dot{V}(x(t), t) < -\gamma_3(\|x\|), \forall x \in \{ x \in \mathbb{R}^n : \eta_1 < \|x(t)\| < \eta_2 \},
$$

then the trajectory of the system $x(t)$ with initial state $x(t_0) = x_0$ is locally uniformly bounded, $\|x(t_0)\| \leq s \implies \|x(t)\| \leq d(s)$, $\forall t \in [t_0, \infty)$ where

$$
d(s) = \begin{cases} 
(\gamma_1^{-1} \circ \gamma_2)(s) & \text{if } \eta_1 < s \leq (\gamma_2^{-1} \circ \gamma_1)(\eta_2) \\
(\gamma_1^{-1} \circ \gamma_2)(\eta_1) & \text{if } s \leq \eta_1
\end{cases}
$$

### 2.1.2 Input-Output Stability Theory

In the previous section, we focused on stability analysis of the system state. Assume the system dynamics take the general form

$$
\dot{x}(t) = f(t, x(t), u(t)),
$$

$$
y(t) = h(t, x(t), u(t)),
$$

(2.1.1)
where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^p \), and \( y \in \mathbb{R}^q \) are system states, control, and output respectively, \( f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n \) is continuously differentiable, and \( h : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^q \) is continuous. It is natural to relate the system output directly to input without any prior knowledge of the internal structure of the system. Input-output stability analysis is based on the application of functional analysis. Some stability results are reviewed. Detailed proofs of the theorem can be found in [29, 61].

**Theorem 2.1.3.** Consider equation (2.1.1) and assume that:

1) The origin \( x = 0 \) is an equilibrium point of (2.1.1) when \( u = 0 \); that is,

\[
f(t, 0, 0) = 0, \quad \forall t \geq 0.
\]

2) The origin \( x = 0 \) is a globally exponentially stable equilibrium point of the system,

\[
\dot{x} = f(t, x, 0).
\]

3) The Jacobian matrices \([\partial f / \partial x]\), evaluated at \( u = 0 \), and \([\partial f / \partial u]\) are globally bounded.

4) Function \( h(t, x, u) \) satisfies

\[
\|h(t, x, u)\| \leq k_1\|x\| + k_2\|u\| + k_3,
\]

(2.1.2)
globally for some nonnegative constants \( k_1 \), \( k_2 \) and \( k_3 \). For all \( \|x(0)\| \leq \eta \), there exist constants \( \gamma > 0 \) (independent of \( \eta \)) and \( \beta = \beta(\eta, k_3) \geq 0 \) such that

\[
\sup_{t \geq 0}\|y(t)\| \leq \gamma \sup_{t \geq 0}\|u(t)\| + \beta.
\]

Moreover, if \( x(0) = 0 \), and \( k_3 = 0 \), then \( \beta = 0 \).
From equation (2.1.2), the system is bounded input-bounded output stable and gives \( \gamma \) as an upper bound on the system gain. It also indicates one advantage of allowing a residual constant \( \beta \) in the definition of input-output stability. Without this constant \( \beta \), the initial state \( x(0) \) has to be restricted to zero.

Having reviewed general results for stability analysis, observer design techniques are presented next.

### 2.2 Observer and Estimator

Many nonlinear control design and adaptive system techniques assume state feedback; this implies that all the state variables are measured. In practice, this is not always true, either for economic or technical reasons, such as sensor failures. In most cases, system outputs are available instead of states. Intuitively, we want to use the outputs of the system and extend the state-dependent techniques to output-dependent techniques for system design. The idea is similar to what has been widely applied in LTI systems, i.e., build an observer that yields asymptotic estimates of the system state based on the output of the system, and then update the control/adaptation law using on-line estimation of the unmeasured states. This scheme is named output feedback [20]. Different observer design techniques, such as adaptive backstepping, tuning function design, modular designs, etc., have been investigated in [37, 61]. Among these methods, the high-gain observer and adaptive observer are widely used in nonlinear systems. Recently, advanced techniques have been established that realize not only asymptotic stability, but also that demonstrate recovery of sensor failures by using a high gain observer [62, 63].
2.2.1 High-Gain Observer

The application and use of high-gain observers guarantee that the output feedback scheme achieves the same performance as state feedback scheme with some uncertainties in the system. The basic idea can be illustrated by the following example. Let the nonlinear system be defined by

\[
\begin{align*}
\dot{x} &= A_c x + B_c \phi(x, z, u), \\
\dot{z} &= \psi(x, z, u), \\
y &= C x, \\
\zeta &= q(x, z), \\
u &= \varphi(v, \hat{x}, \zeta), \\
\dot{v} &= \Gamma(v, \hat{x}, \zeta),
\end{align*}
\]  
(2.2.1)

where \( x \in \mathbb{R}^n, \ z \in \mathbb{R}^m, \ u \in \mathbb{R} \) are the state vector and control input, respectively, \( y \) and \( \zeta \) are the measured output and \( \hat{x} \in \mathbb{R}^n \) is the state estimate of \( x \) obtained from a high gain observer. The matrix \( A_c \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times 1} \) are controllable normal form and \( C \in \mathbb{R} \times \mathbb{R}^{n} \) are given by

\[
A_c = \begin{bmatrix}
0 & 1 & \cdots & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 \\
0 & \cdots & \cdots & \cdots & 0
\end{bmatrix}, \quad B_c = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix}, \quad C = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0
\end{bmatrix}.
\]

The estimate \( \hat{x} \) satisfies

\[
\dot{\hat{x}} = A_c \hat{x} + B_c \phi_0(\hat{x}, z, u) + H(y - C \hat{x}),
\]  
(2.2.2)

for which the function \( \phi_0(\hat{x}, z, u) \) is required to be locally Lipschitz in its arguments over the domain of interest and globally bounded in \( x \). Moreover, \( \phi_0(0, 0, 0) = 0 \). The observer gain
$H \in \mathbb{R}^n \times 1$ is chosen as

$$H = \begin{bmatrix} \beta_1/\epsilon \\ \beta_2/\epsilon^2 \\ \vdots \\ \beta_n/\epsilon^n \end{bmatrix}_{n \times 1},$$

where $\epsilon > 0$ is a small positive constant, and the positive constants $\beta_i$ are chosen such that the roots of the characteristic polynomial

$$s^n + \beta_1 s^{n-1} + \beta_2 s^{n-2} + \cdots + \beta_{n-1} s + \beta_n = 0,$$

are in the open left half plane, for all $i = 1, 2, \ldots, n$. For error dynamics, the equivalent scaled estimated error will be considered and defined as

$$e_i = \frac{x_i - \hat{x}_i}{\epsilon^{n-i}},$$

for $i = 1, 2, \ldots, n$. Then the error dynamics become

$$\epsilon \dot{e} = A_o e + \epsilon B_c [\phi(x, z, \varphi(v, \hat{x}, \zeta)) - \phi_0(\hat{x}, \zeta, \varphi(v, \hat{x}, \zeta))],$$

where the matrix $A_o = \epsilon D^{-1}(\epsilon)(A_c - HC)D(\epsilon)$ is Hurwitz where $D = diag[e^{n-1}, \cdots, 1]$. Let $\chi = [x^T, z^T, v^T]^T$, it can be shown that for a compact set $\Omega = \{ (\chi, e) : \|\chi\| \leq c_1, \|e\| \leq c_2 \}$, there exists a constant $0 < \epsilon \leq \epsilon^*$ such that the system trajectory becomes bounded [4] [18] [19].

**Theorem 2.2.1.** For the system (2.2.1) with the functions $\phi$, $q$, $\psi$, $\Gamma$ and $\varphi$ satisfying the locally Lipschitz condition in their arguments over the domain of interest, $\Gamma$ and $\varphi$ are globally bounded functions of $x$, and the origin is uniformly globally asymptotically stable equilibrium point. Then there exists $\epsilon^*_1 > 0$ such that, for every $0 < \epsilon \leq \epsilon^*_1$ the trajectory $(\chi, e)$ starting in $\Omega$ become bounded for all $t \geq 0$.  

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Proof. Detailed proof can be found in \[4\].

One application of a high gain observer is the design of a derivative estimator. Assume \(y(t)\) be a measurable time-varying signal generated by a dynamic system. As the time derivative of \(y(t)\), i.e., \(\dot{y}(t)\) is not available, based on the measured \(y(t)\), an estimate of \(\dot{y}\) can be obtained as follows. Assuming \(\eta = \dot{\zeta} = \frac{1}{\varepsilon}(-\zeta + y)\), an auxiliary system is generated by

\[
\varepsilon \dot{\zeta} = -\zeta + y,
\]  

(2.2.3)

where \(\varepsilon\) is a small positive constant and the filter is always initiated in a small region decided by \(\zeta(0) - y(0) \leq k\varepsilon\) for some constant \(k > 0\). The transfer function from \(y\) to \(\eta\) is

\[
\eta = \frac{sy}{\varepsilon s + 1}.
\]

With \(\varepsilon\) a sufficiently small positive constant, \(\eta\) approaches \(sy\) for frequencies much smaller than \(1/\varepsilon\). The transfer function \(\frac{1}{\varepsilon s + 1}\) performs as a low pass filter with bandwidth \(\frac{1}{\varepsilon}\). The bandwidth expands for more signal components to pass through with smaller \(\varepsilon\) and the nominator is a pure differentiator.

This idea can be easily expanded from scalar systems to high-order systems. As a result of section \[2.2.1\] the high-gain observer has robust capability to estimate higher order differentiation, provided that the rate of change of the given signal is limited, in other words, their higher-order derivatives exist.

It can be seen that observer in equation \[2.2.2\] is nonlinear due to the term \(\phi_0(\dot{x}, z, u)\). Choosing the nominal function \(\phi_0(\dot{x}, z, u)\) as zero results in the linear observer

\[
\dot{x} = A_c \dot{x} + H(y - C \dot{x}).
\]  

(2.2.4)
As the result of [20], for any bounded $y^{(n-1)}$, it can be shown that the estimation error will be of order $O(\epsilon)$ after a small transient period, or its error will converge to zero as $\epsilon \to 0$. The differentiation power can also be seen by examining the transfer function of equation (2.2.4) from $y$ to $\hat{x}$ that given by

$$G(s) = (sI - A_c + HC)^{-1}H.$$ 

For example, let us consider a third order observer whose transfer function is given by

$$G(s) = \frac{1}{\Delta(\epsilon s)} \begin{bmatrix} \beta_1 \epsilon^2 s^2 + \beta_2 \epsilon s + \beta_3 \\ \beta_2 \epsilon s^2 + \beta_3 s \\ \beta_3 s^2 \end{bmatrix}$$

where $\Delta(\epsilon s) = \epsilon^3 s^3 + \beta_1 \epsilon^2 s^2 + \beta_2 \epsilon s + \beta_3$. It can be seen that the limit of $\Delta(\epsilon)$ as $\epsilon \to 0$ is $\beta_3$ and that the limit of $G(s)$ is

$$\lim_{\epsilon \to 0} G(s) = \begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix}.$$ 

As a result, $\hat{x}_2$ and $\hat{x}_3$ approach $\dot{y}$ and $\ddot{y}$, respectively.

From the above derivation, if the complete state vector is not available, the unmeasured states can be viewed as uncertainties generated by an exogenous system. The exogenous system can be completely or partially determined [65, 64] by applying a high-gain observer. This gives a promising way to deal with synchronization problem with partial state measurement.

### 2.2.2 Model Reference Adaptive Observer Design

The model reference approach for LTI system has been discussed in [27, 49, 67]. The most recent in-depth treatment and redesign can be found in [27]. In this subsection, we review the general model reference observer for a LTI system. One of the basic structures that has
been widely studied in adaptive observers is as follows. For any controllable and observable
\( n^{th} \) order continuous single-input-single-output (SISO) LTI system
\[
\begin{align*}
\dot{x} &= Ax + Bu, \\
y &= C^T x,
\end{align*}
\] (2.2.5)
where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R} \), and \( y \in \mathbb{R} \), matrices \( A \), \( B \), and \( C \) have proper dimensions, plant parameters are consist of \( n^2 + 2n \) elements in matrices \( (A, B, C) \). If equation (2.2.5) is in a normal canonical form, then \( n^2 \) elements of \( (A, B, C) \) have a value of either 0 or 1 and at most \( 2n \) parameters can be used to specify the unique properties of the input/output relationship of the plant. These \( 2n \) parameters are in fact the coefficients of the numerator and denominator of the transfer function
\[
\frac{Y(s)}{U(s)} = \frac{Z(s)}{R(s)}. 
\] (2.2.6)
Without loss of generality, we can assume that
\[ Z(s) = b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1s + b_0, \]
\[ R(s) = s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1s + a_0. \] (2.2.6)
For estimation and control problem, some parameterizations are much more convenient than others for analysis. Two commonly adopted parameterization models are presented.

A: Parameterization model 1

Plant equation (2.2.5) can be expressed as an \( n \)th-order differential equation as follows:
\[
y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0y = b_{n-1}u^{(n-1)} + b_{n-2}u^{(n-2)} + \cdots + b_0u. \] (2.2.7)
The above equation can also be expressed as a linear equation
\[
y^{(n)} = \theta^T Y, \] (2.2.8)
where \( y^{(n)} \) is \( n \)th-order derivative of \( y(t) \), \( \theta \) is a parameter vector given by

\[
\theta = [b_{n-1}, b_{n-2}, \ldots, b_0, a_{n-1}, a_{n-2}, \ldots, a_0],
\]

and \( Y \) is composed of the derivatives of the input and output signal vector, i.e.,

\[
Y = [u^{(n-1)}, u^{(n-2)}, \ldots, u, -y^{(n-1)}, -y^{(n-2)}, \ldots, -y]^T,
\]

(2.2.9)

with \( \alpha_i(s) \triangleq [s^i, s^{i-1}, \cdots, 1] \). In most situation, where the only available signals are input \( u \) and output \( y \). Differentiation of \( u \) and \( y \) should be avoided. One way to achieve this is to filter both sides of equation (2.2.8) with an \( n \)th-order stable \( \frac{1}{\wedge(s)} \) to obtain

\[
\begin{align*}
\hat{z} = \theta^T \phi, \\
\phi = \left[ \frac{\alpha_{n-1}^T(s) u - \alpha_{n-1}^T(s) y}{\wedge(s)} \right]^T,
\end{align*}
\]

(2.2.11)

where

\[
\begin{align*}
\hat{z} & \triangleq \frac{1}{\wedge(s)} y^{(n)} = \frac{s^n}{\wedge(s)} y, \\
\phi & \triangleq \left[ \frac{\alpha_{n-1}^T(s) u - \alpha_{n-1}^T(s) y}{\wedge(s)} \right]^T,
\end{align*}
\]

and \( \wedge(s) = s^n + \lambda_{n-1}s^{n-1} + \cdots + \lambda_0 \) is an arbitrary Hurwitz polynomial in \( s \).

If \( \wedge(s) = s^n + \lambda^T \alpha_{n-1}(s) \) with \( \lambda = [\lambda_{n-1}, \lambda_{n-2}, \ldots, \lambda_0]^T \), then

\[
\begin{align*}
\hat{z} &= \frac{s^n}{\wedge(s)} y = \frac{\wedge(s) - \lambda^T \alpha_{n-1}(s)}{\wedge(s)} y = y - \lambda^T \alpha_{n-1}(s) y.
\end{align*}
\]

This leads to

\[
y = \hat{z} + \lambda^T \frac{\alpha_{n-1}(s)}{\wedge(s)} y.
\]

Since \( \hat{z} = \theta^T \phi = \theta_1^T \phi_1 + \theta_2^T \phi_2 \), where

\[
\theta_1 = [b_{n-1}, b_{n-2}, \ldots, b_0]^T, \quad \text{and} \quad \theta_2 = [a_{n-1}, a_{n-2}, \ldots, a_0]^T
\]

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\[ \phi_1 = \frac{\alpha_{n-1}(s)}{\Lambda(s)} u, \quad \text{and} \quad \phi_2 = \frac{-\alpha_{n-1}(s)}{\Lambda(s)} y, \]

\( y \) can be rewritten as

\[ y = \theta^T_1 \phi_1 + \theta^T_2 \phi_2 - \lambda^T \phi_2. \]

Hence,

\[ y = \theta^T_\lambda \phi, \tag{2.2.12} \]

where \( \theta_\lambda = [\theta^T_1, \theta^T_2 - \lambda^T]^T \). A diagram of parameterization model 1 is shown in Figure 2.1 as follows to illustrate this relationship.

Figure 2.1: Diagram of parameterization model 1

Let a state-space generation of signals in equations (2.2.10) and (2.2.12) be obtained by using the following identity

\[ [\text{adj}(sI - \Lambda_c)] \hat{l} = \alpha_{n-1}(s), \]

where

\[ \Lambda_c = \begin{bmatrix} -\lambda_{n-1} & -\lambda_{n-2} & \cdots & -\lambda_0 \\ 1 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}, \quad \hat{l} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \]
and \( \det(sI - \wedge_c) = \wedge(s) \), it follows that

\[
(sI - \wedge_c)^{-1}l = \frac{\alpha_{n-1}(s)}{\wedge(s)}.
\]

Thus, the nominal parameters can be expressed as

\[
\hat{\phi}_1 = \wedge_c \phi_1 + lu,
\]

\[
\hat{\phi}_2 = \wedge_c \phi_2 - ly,
\]

\[
y = \theta^T \phi,
\]

\[
z = y + \chi^T \phi_2 = \theta^T \phi.
\]

Equation (2.2.13) is the input-output equivalent of the system described by equation (2.2.5) if all state initial conditions are equal to zero, i.e. \( x(0) = 0, \phi_1(0) = \phi_2(0) = 0 \). In a real system, this is not always the case. But a nonzero initial condition will decay exponentially since it will pass through a stable matrix \( \wedge_c \).

Corresponding observer and parameter estimator based on equation (2.2.13) can be designed based on this equivalent system as

\[
\hat{\phi}_1 = \wedge_c \hat{\phi}_1 + lu,
\]

\[
\hat{\phi}_2 = \wedge_c \hat{\phi}_2 - ly,
\]

\[
\hat{y} = \hat{\theta}^T \hat{\phi},
\]

\[
\hat{\theta} = \hat{\phi} (y - \hat{y}).
\]

**B: Parameterization 2**

Consider the representation (2.2.12) and the identity \( W_m(s)W_m^{-1}(s) = 1 \), where \( W_m(s) = \frac{Z_m(s)}{R_m(s)} \) is a transfer function with relative degree one, \( Z_m(s) \) and \( R_m(s) \) are Hurwitz polynomials, respectively. Output signal \( y \) can also be expressed as

\[
y = W_m(s)\theta^T \psi = W_m(s)\theta^T \psi.
\]
where
\[
\psi = \frac{1}{W_m(s)} \phi = \begin{bmatrix}
\alpha_{n-1}^T(s) \\
\frac{\alpha_{n-1}^T(s)}{W_m(s) \wedge (s) u} \\
\frac{-\alpha_{n-1}^T(s)}{W_m(s) \wedge (s) y}
\end{bmatrix}^T.
\]

Because \(\frac{\alpha_{n-1}^T(s)}{W_m(s) \wedge (s)}\) is stable transfer functions, \(\frac{\alpha_{n-1}^T(s)}{W_m(s) \wedge (s)} u\) and \(\frac{\alpha_{n-1}^T(s)}{W_m(s) \wedge (s)} y\) can be generated without differentiation of \(u\) and \(y\). Since the dimension of \(\psi\) depends on the order \(n\) of \(\wedge (s)\) and the order of \(Z_m(s)\), and \(Z_m(s)\) can be arbitrary, the dimension of \(\psi\) is also arbitrary.

Parameterization 2 is also called Model Reference Representation [45, 67] and is widely applied in adaptive systems to design parameter estimators with a strictly positive real transfer function \(W_m(s)\). Specifically, with \(W_m(s) = \frac{1}{s + \lambda_c}\), and \(s + \lambda_c\) a factor of \(\wedge (s)\), this leads to
\[
W_m(s) \wedge (s) = \wedge_q(s) = s^{n-1} + q_{n-2} s^{n-2} + \cdots + q_1 s + 1.
\]
Then, the biproper elements of \(\frac{\alpha_{n-1}^T(s)}{\wedge_q(s)}\) can be separated as follows. Choosing
\[
c \triangleq [c_{n-1}, c_{n-2}, \cdots, c_1, c_0]^T \in \mathbb{R}^n,
\]
we have
\[
\frac{c^T \alpha_{n-1}(s)}{\wedge_q(s)} = \frac{c_{n-1} s^{n-1}}{\wedge_q(s)} + \frac{\bar{c}^T \alpha_{n-2}(s)}{\wedge_q(s)} = c_{n-1} + \left(\bar{c} - c_{n-1} \bar{q}\right)^T \alpha_{n-2}(s),
\]
(2.2.15)
where \(\bar{c} \triangleq [c_{n-2}, \cdots, c_1, c_0]^T\), \(\alpha_{n-2}(s) \triangleq [s^{n-2}, \cdots, s, 1]^T\), and \(\bar{q} = [q_{n-2}, \cdots, q, 1]^T\).

This leads to
\[
\theta_1^T \frac{\alpha_{n-1}(s)}{\wedge_q(s)} u = b_{n-1} u + \bar{\theta}_1^T \frac{\alpha_{n-2}(s)}{\wedge_q(s)} u,
\]
(2.2.16)
\[
(\lambda^T - \theta_2^T) \frac{\alpha_{n-1}(s)}{\wedge_q(s)} y = (\lambda_{n-1} - a_{n-1}) y - \theta_2^T \frac{\alpha_{n-2}(s)}{\wedge_q(s)} y,
\]
where \(\bar{\theta}_1 = \bar{b} - b_{n-1} \bar{q}\), \(\bar{\theta}_2 = \bar{a} - \lambda (a_{n-1} - \lambda_{n-1}) \bar{q}\) and \(\bar{a} = [a_{n-2}, \cdots, a_1, a_0]^T\), \(\bar{b} = [b_{n-2}, \cdots, b_1, b_0]^T\), \(\bar{\lambda} = [\lambda_{n-2}, \cdots, \lambda_1, \lambda_0]^T\). A diagram of parameterization 2 is shown in Figure 2.2 in which \(\wedge(s) = (s + \lambda_c) \wedge (s)\) and \(W_m(s) = \frac{1}{s + \lambda_c}\).
Based on the above representation scheme, an adaptive observer can be designed as follows. Equation (2.2.7) is the input-output equivalent of the LTI system described by the differential equations

\[
\begin{align*}
\dot{x}_1 &= -\lambda_c x_1 + \theta^T \psi, \\
\dot{\psi}_1 &= \lambda_c \psi_1 + l u, \\
\dot{\psi}_2 &= \lambda_c \psi_2 - l y, \\
y &= \bar{x}_1,
\end{align*}
\]

where the parameter vector \( \theta_\lambda \) is chosen as \([b_{n-1}, \bar{\theta}_1^T, \lambda_{n-1} - a_{n-1}, \bar{\theta}_2^T]^T, \psi = [u, \psi_1^T, y, \psi_2^T]^T, l = [1, 0, \cdots, 0]^T \) and \( \lambda_c \) is an \((n - 1) \times (n - 1)\) asymptotically stable matrix.

\[
\begin{pmatrix}
-q_{n-2} & -q_{n-3} & \cdots & -q_0 \\
1 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & 0
\end{pmatrix}.
\]

If nonzero initial conditions are considered, \( y = \bar{x} + \eta_0 \), where \( \eta_0 \) is the exponentially decaying term caused by the initial conditions.

Figure 2.2: Diagram of parameterization 2

The choice of the corresponding observer design to estimate parameters \( \theta \) and the
states $x$ is described as the following equations:

$$
\dot{x}_1 = -\lambda_c x_1 + \hat{\theta}^T \hat{\psi},
\dot{\psi}_1 = \lambda_c \dot{\psi}_1 + lu,
\dot{\psi}_2 = \lambda_c \dot{\psi}_2 - ly,
\hat{y} = \hat{x}_1,
$$

(2.2.19)

where $\hat{\theta}$ and $\hat{\psi}$ are defined as

$$
\hat{\theta} = \left[ b_{n-1}, \hat{\theta}_1^T, \lambda_{n-1} - \hat{a}_{n-1}, \hat{\theta}_2^T \right]^T, \text{ and } \psi = \left[ u, \psi_1^T, y, \psi_2^T \right]^T.
$$

Define $e_1 = \hat{y} - y$, $\bar{\theta} = \hat{\theta} - \theta$, and $\bar{\psi} = \hat{\psi} - \psi$, the error dynamics are

$$
\dot{e}_1 = -\lambda_c e_1 + \bar{\theta}^T \bar{\psi} + \bar{\theta}^T \bar{\psi},
$$

(2.2.20)

and

$$
\bar{\psi}_\lambda = \left[ \begin{array}{cc}
\lambda_c & 0 \\
0 & \lambda_c \\
\end{array} \right] \bar{\psi}_\lambda = \bar{\lambda} \bar{\psi}_\lambda,
$$

(2.2.21)

where $\bar{\psi}_\lambda = [\psi_1, \psi_2]$. Choose Lyapunov candidate as $V(e_1, \bar{\theta}, \bar{\psi}) = \frac{1}{2} (e_1^2 + \bar{\theta}^T \bar{\psi} + \beta \bar{\psi}^T P \bar{\psi})$

where $P$ is a symmetric positive-definite matrix such that $\bar{\lambda}^T P + P \bar{\lambda} = -Q < 0$ and $\beta$ is a positive constant. The adaptation law is designed as

$$
\dot{\bar{\theta}} = -(\hat{y} - y) \hat{\psi},
$$

then the derivative of Lyapunov function can be expressed as

$$
\dot{V} = -\lambda_c e_1^2 + e_1 \theta^T \bar{\psi} - \frac{\beta}{2} \bar{\psi}^T Q \bar{\psi} \leq 0.
$$

(2.2.22)

If $\beta \geq \frac{2\|\theta\|^2}{\lambda_c \lambda_Q}$ where $\lambda_Q$ is the minimum eigenvalue of $Q$, then the overall error system is uniformly stable at the origin. The conclusion of the stability proof that can be drawn for this adaptive scheme is that the parameters converge to some constants.
Example 2.2.2. Assume a SISO system transfer function

\[ W_p(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0}. \]

Use model reference representation to estimate the constant parameters \( a_0, a_1, b_1, b_0 \). Let the nominal values of the parameters be \( a_0 = 6, a_1 = 5, b_0 = 1, b_1 = 1 \). Three cases with different inputs \( u \) are considered for the two parameterization schemes, respectively.

Parameterization 2 For this parameterization model, \( b_0 = 1, b_1 = 1, a_0 = 6, a_1 = 5 \), with design parameter \( l = 1, \lambda_c = 2, \land_c = 1 \), the corresponding parameter estimate \( \theta = [1, 0, -2, 2] \), and adaptation laws are given as

\[
\begin{align*}
\dot{\theta}_3 &= -k_1 e_1 u, \\
\dot{\theta}_2 &= -k_2 e_1 y, \\
\dot{\theta}_1 &= -k_3 e_1 y, \\
\dot{\theta}_0 &= -k_4 e_1 \psi_2,
\end{align*}
\]

(2.2.23) with adaptation gains

\[ k_1 = 20, k_2 = 20, k_3 = 50, k_4 = 50. \]

Three cases are simulated here to show that the convergence of parameter estimates are intimately related to the signals in the whole system.

- case 1: Input is \( u = 5 \cos(t) \);
- case 2: Input is \( u = 5 \cos(t) + 10 \sin(3t) \);
- case 3: Input is \( u = 5 \cos(t) + 10 \sin(3t) + 6 \cos(3t) \);
Corresponding simulation results of output estimation error and parameter estimates are shown in Figure 2.3 to 2.8 at the end of this chapter, respectively. It can be seen that for case 1, the input is a sinusoid signal with a single frequency. Parameter estimates converge to certain constants, but not their true values. In case 2, an input with two distinct frequency components is feed into the system. The error dynamics are exponentially stable and parameter estimates converge to their true values. In case 3, the input has 3 frequency components and the convergence rate is faster than that in case 2.

![Graph showing estimation error of the output in case 1 for parameterization model 2](image)

**Figure 2.3**: Estimation error of the output in case 1 for parameterization model 2

**Parameterization 1** Consider the same plant, based on a different parameterization scheme, namely, \( \theta = [0, 1, -2, 0] \). Adaptation gains are chosen as

\[
  k_1 = 10, \quad k_2 = 10, \quad k_3 = 50, \quad k_4 = 50.
\]

The above three cases are simulated again to show the convergence of parameter estimates and property of the input signal. Corresponding simulation results are shown in Figure 2.9.
2.14 at the end of this chapter.

From the simulation results, it is observed that with different input signals, the parameter estimates converge to different values with different convergence speeds, though the estimation error of the system output always goes to zero asymptotically. Parameter convergence has nothing to do with the representation scheme. Convergence of parameter estimates to their true values is dependent on PE properties of the signal and will be considered in Chapter 3.

2.3 Review of Lorenz System

The Lorenz system is one of the most widely discussed systems for chaotic study. It was first recognized by E.N. Lorenz in 1963 [44] and is represented by the following set of equations,

\begin{align*}
\dot{x}_1 &= \sigma x_2 - \sigma x_1, \\
\dot{x}_2 &= \gamma x_1 - px_2 - x_1 x_3, \\
\dot{x}_3 &= x_1 x_2 - \beta x_3,
\end{align*}

(2.3.1)

where $x(t) = [x_1(t), x_2(t), x_3(t)]^T \in \mathbb{R}^3$ is the state, and $\sigma$, $\gamma$, $p$ and $\beta$ are positive constant parameters. Later, people recognized that many mechanical, electrical, biological and economic systems exhibit some properties described by chaotic equations. These phenomena had been considered as noises for a long time. Chaotic systems have become an attractive research area in the past decade. With more and more detailed studies going on, researchers realized that chaotic systems were deterministic systems and hold certain specific properties.
2.3.1 Properties of Chaotic systems

Three properties of chaos have been widely accepted \cite{22, 50, 72}: it is an *aperiodic* long-term behavior in a *nonlinear deterministic* system that exhibits *SIC* (sensitivity to initial conditions). The first property is related to the analysis of the system in time domain. The evolution of states, or the trajectories, in a chaotic system cannot settle down to fixed points, periodic orbits, or quasi-periodic orbits as time goes to infinity. When Lorenz first found this strange phenomenon in 1963 \cite{44}, he excluded all the mentioned possibilities and then concluded that something strange must be happening. Meanwhile, the system state must be bounded by a complicated set; this means all the states are bounded and cannot go to infinity as time goes to infinity. This attractor can also be explained by the volume point. The volume expands for a while (stretching) and then contracts and twists back on itself (folding) while dissipation comes in.

The second property is based on the deterministic nature of the system. The strange behavior is caused by its nonlinearities rather than by noisy driving forces or random parameters.

The third property is SIC. This means that two neighboring points can diverge exponentially from each other and we cannot predict where the state will be in the future, though the system is a deterministic system. The reason behind this SIC property is that chaotic system has some positive Lyapunov exponent $\lambda$. Strogatz explained this in a very straightforward way \cite{72}. Even if we measure the initial conditions very accurately; There is always some error $\delta_0$ between our measurement and the true value. After some time period
$t$, the difference goes to $\delta(t) = \delta_0 e^{\lambda t}$. If the state error within some tolerance threshold $a$ is acceptable, then we know after some time $t_{\text{horizon}} \sim \frac{1}{\lambda} \ln \frac{a}{\delta_0}$, our prediction will be intolerable since $\delta(t) \geq a$. We see this time limitation is always dependent on $\delta_0$. No matter how the initial state approaches the true value, we will always have some error, and we cannot predict the state beyond $t_{\text{horizon}}$.

Another interesting property of chaotic signals is that they are noise-like. Frequency spectrum of chaotic signals is wide band. This is the most interesting property for control and communication groups, since the wide band signal can be used as a carrier for communication systems, or from a control system point of view, this signal is sufficiently rich or persistent exciting for parameter estimation.

### 2.3.2 Properties of Lorenz Equation

Only two nonlinearities exist in the Lorenz system (2.3.1), the quadratic terms $x_1 x_3$ and $x_1 x_2$, where $\sigma$, $\gamma$, $p$ and $\beta$ are positive parameters. An important symmetry also exists. If we replace $(x_1, x_2)$ with $(-x_1, -x_2)$, the equation stays the same. Hence if $(x_1(t), x_2(t), x_3(t))$ is the solution, so is $(-x_1(t), -x_2(t), x_3(t))$.

There are two kinds of fixed points for the Lorenz equation. The origin $(0, 0, 0)$ is a fixed point for all values of the parameters. For $\gamma > 1$, there is also a symmetric pair of fixed points $x^* = y^* = \pm \sqrt{\beta(\gamma - p)}$, $z^* = \gamma - p$. We can call them $C^+$ and $C^-$. As $\gamma \to p$ from above, $C^+$ and $C^-$ coalesce with the origin in a pitchfork bifurcation.

By linearizing about the origin, we can show, for $\gamma > p$, the origin is a saddle point, and for $\gamma < p$ the origin is a stable node. This can be shown by Lyapunov arguments.
Choosing Lyapunov function as $V(x_1, x_2, x_3) = \frac{1}{\sigma} x_1^2 + x_2^2 + x_3^2$, the time derivative of this Lyapunov function can be derived as:

$$
\dot{V} = \frac{1}{\sigma} x_1 \dot{x}_1 + x_2 \dot{x}_2 + x_3 \dot{x}_3 \\
= x_2 x_1 - x_1^2 + \gamma x_1 x_2 - px_2^2 - x_1 x_2 x_3 + x_1 x_2 x_3 - \beta x_3^2 \\
= -x_1^2 + (1 + \gamma) x_1 x_2 - px_2^2 - \beta x_3^2 \\
= -\left(x_1 + \frac{1 + \gamma}{2} x_2\right)^2 - \left(p - \frac{(1 + \gamma)^2}{4}\right)x_2^2 - \beta x_3^2
$$

(2.3.3)

with $p \geq \frac{(1+\gamma)^2}{4}$, the origin is A.S. When $p = 1$, it can be proved that the origin is globally asymptotically stable for $\gamma < 1$. When $\gamma > 1$, from the above analysis $C^+$ and $C^-$ exist. They are linearly stable for $1 < \gamma < \gamma_H = \frac{\sigma(\sigma+\beta+3)}{\sigma-\beta-p}$ assuming $\sigma - \beta - p > 0$. A critical point $r_H$ exists, at $\gamma = \gamma_H = \frac{\sigma(\sigma+\beta+3)}{\sigma-\beta-p}$ a subcritical Hopf bifurcation occurs. An unstable limit cycle occurs when $\gamma < \gamma_H$. When $\gamma \to \gamma_H$ from below, the cycle shrinks down around the fixed point $C^+$ or $C^-$. At the Hopf bifurcation, the fixed point absorbs the saddle cycle and changes into a saddle point. For $\gamma > \gamma_H$ based on our analysis there are no attractors in the neighborhood. We also know the states are bounded, the trajectories cannot be driven to infinity, they must fly away to a distant attractor. This means they are confined to a bounded set and manage to move on this set forever without intersecting themselves. The boundedness of Lorenz system is demonstrated in the following subsection.

**A: Boundedness of the Lorenz Equation**

From the above review, when equation (2.3.2) cannot be satisfied, does this mean the states will go to infinity? The following Lemma shows an important property of the trajectories of the Lorenz system.
Lemma 2.3.1. Given the Lorenz system as in equation (2.3.1), all the state variables in chaotic evolution are uniformly bounded by outer bound $C_1$ and inner bound $C_2$, defined as
\begin{align*}
C_1 &\triangleq \left\{ x \in \mathbb{R}^3 : x_1^2 + x_2^2 + (x_3 - \gamma - \sigma)^2 = (\gamma + \sigma)^2 \max \left\{ 1, \frac{\beta}{2\sigma}, \frac{\beta}{2p} \right\} \right\}, \\
C_2 &\triangleq \left\{ x \in \mathbb{R}^3 : \sigma x_1^2 + px_2^2 + \beta(x_3 - \frac{\gamma + \sigma}{2})^2 = \beta(\gamma + \sigma)^2/4 \right\}.
\end{align*}

Proof: Consider a Lyapunov function $V_0 = x_1^2 + x_2^2 + (x_3 - \gamma - \sigma)^2$, it’s easy to see this Lyapunov function is globally positive definite and radially unbounded with respect to $\psi = [x_1, x_2, x_3 - \gamma - \sigma]^T$. Time derivative of $V_0$ can be obtained as
\begin{align*}
\dot{V}_0 &= 2x_1 \dot{x}_1 + 2x_2 \dot{x}_2 + 2(x_3 - \gamma - \sigma)\dot{x}_3 \\
&= 2x_1 \sigma (x_2 - x_1) + 2x_2 (\gamma x_1 - px_2 - x_1x_3) + 2(x_3 - \gamma - \sigma)(x_1x_2 - \beta x_3) \\
&= -2\sigma x_1^2 - 2px_2^2 - \beta x_3^2 - \beta(x_3 - \gamma - \sigma)^2 + \beta(\gamma + \sigma)^2 \\
&= -\beta x_3^2 - [2\sigma x_1^2 + 2px_2^2 + \beta(x_3 - \gamma - \sigma)^2 - \beta(\gamma + \sigma)^2].
\end{align*}
(2.3.5)

Thus, $\dot{V}_0 < 0$ will be satisfied when system states are outside a circle $C_1$ defined by
\begin{align*}
C_1 &\triangleq \left\{ x \in \mathbb{R}^3 : x_1^2 + x_2^2 + (x_3 - \gamma - \sigma)^2 = (\gamma + \sigma)^2 \max \left\{ 1, \frac{\beta}{2\sigma}, \frac{\beta}{2p} \right\} \right\},
\end{align*}
which means trajectories outside this circle will converge to a closure set. Thus all the states in Lorenz system are bounded. The boundary of state can be decided from the initial conditions of the system and the radius of the circle.

From equation (2.3.5), it can be shown that $\dot{V}_0 = 0$ holds if the chaotic state stays on an ellipse $C_2$ defined by
\begin{align*}
C_2 &\triangleq \left\{ x \in \mathbb{R}^3 : \sigma x_1^2 + px_2^2 + \beta(x_3 - \frac{\gamma + \sigma}{2})^2 = \beta(\gamma + \sigma)^2/4 \right\}.
\end{align*}

Thus, if the state goes within $C_2$, it will diverge. If the trajectory goes outside of $C_1$, it
will converge to a closure set within $C_1$. It means the chaotic attractor stays outside $C_2$ but within $C_1$.

B: Invariant Sets of the Lorenz Equation

From equation (2.3.1), if $x_1 = x_2 = 0$, then $\dot{x}_1 = \dot{x}_2 = 0$, $\dot{x}_3 = -\beta x_3$. This gives an invariant set of the Lorenz system: $x_1 = x_2 = 0, x_3 \in z-axis$. This means that any initial states starting within this invariant set will stay inside the set all the time. Also it is obvious that the system is not in chaotic evolution within the invariant set.

Another invariant set of the Lorenz system that can be obtained is

$$x_1 = x_2 = \pm \sqrt{\beta(\gamma - p)}, \quad x_3 = \frac{\gamma - p}{b},$$

where $\gamma - p \geq 0$ and $x_1 = x_2$ all the time. Obviously it is not in chaotic evolution either.

**Remark** 2.3.1. From Lemma 2.3.1 $C_1$ and $C_2$ has only one intersection at $x_1 = x_2 = x_3 = 0$, which stays in the invariant set of lorenz system: $x_1 = x_2 = 0, x_3 \in z-axis$. It is obvious the system is not in chaotic evolution within the invariant set. It also indicates that trajectory of $x_3$ will be positive if the initial condition of $x_3$ is positive. If $x_3(0) < 0$, it will go inside $C_1$ during the transient period and then keep to be positive. States $x_1$ and $x_2$ can be positive, zero or negative.

2.4 Summary

In this chapter, a brief review of some basic concepts in nonlinear system, important definitions, theorems and techniques such as Lyapunov’s direct method, input-output stability, high-gain observer, and adaptive observer design are reviewed. Persistent excitation properties relevant to parameter estimation have been introduced, though detailed information
will be presented in chapter 3. Also, fundamental properties of the Lorenz system are intro-
duced here. These concepts, definitions, and important properties are in preparation for the
following chapter.
Figure 2.4: Estimation of parameters in case 1 for parameterization model 2: \( \theta_3 \) - solid, \( \theta_2 \) - dashed, \( \theta_1 \) - dash dot, and \( \theta_0 \) - dotted

Figure 2.5: Estimation error of the output in case 2 for parameterization model 2
Figure 2.6: Estimation of parameters in case 2 for parameterization model 2: $\theta_3$ - solid, $\theta_2$ - dashed, $\theta_1$ - dash dot, and $\theta_0$ - dotted

Figure 2.7: Estimation error of the output in case 3 for parameterization model 2
Figure 2.8: Estimation of parameters in case 3 for parameterization model 2: $\theta_3$ - solid, $\theta_2$ - dashed, $\theta_1$ - dash dot, and $\theta_0$ - dotted

Figure 2.9: Estimation error of the output in case 1 for parameterization model 1
Figure 2.10: Estimation of parameters in case 1 for parameterization model 1: $\theta_3$ - solid, $\theta_2$ - dashed, $\theta_1$ - dash dot, and $\theta_0$ - dotted

Figure 2.11: Estimation error of the output in case 2 for parameterization model 1
Figure 2.12: Estimation of parameters in case 2 for parameterization model 1: $\theta_3$ - solid, $\theta_2$ - dashed, $\theta_1$ - dash dot, and $\theta_0$ - dotted

Figure 2.13: Estimation error of the output in case 3 for parameterization model 1
Figure 2.14: Estimation of parameters in case 1 for parameterization model 1: $\theta_3$ - solid, $\theta_2$ - dashed, $\theta_1$ - dash dot, and $\theta_0$ - dotted
CHAPTER 3

PERSISTENT EXCITATION AND PARAMETER ESTIMATION

Adaptation laws discussed so far involve the adjustment of the time derivative of the parameter estimates as functions of certain signals of the closed-loop system. Because these adjustment are considered as part of the system dynamics, the overall system should include the state variables of the system and the adjustable parameters together. Error dynamics should contain state error $\tilde{x}_i$ and the parameter (or parameter vector) estimation error $\tilde{\phi}$. The convergence of this estimation error depends on so called persistent excitation properties of signals inside the system. The persistent excitation (PE) plays an important role in adaptive systems. Detailed discussion of the PE property will be presented in this chapter.

3.1 Definition of Persistent Excitation

The concept of persistent excitation signals was introduced along with the identification problem in 1970’s. An intuitive idea of PE is that the signal excites all modes of the system for perfect identification. It is closely linked to parameter convergence, and the convergence rate of the parameter estimates to their true values, as well as to the boundary of the magnitudes of the parameter errors. Two commonly adopted definitions of PE are given as
follows according to two general adaptive systems.

**Definition 3.1.1.** A bounded vector \( u : \mathbb{R}^+ \rightarrow \mathbb{R}^n \) is said to be a persistent excitation if constants \( t_0, T_0 \) and \( \varepsilon_0 \) exist such that

\[
\frac{1}{T_0} \int_{t_0}^{t_0 + T_0} |w^T u(\tau)| d\tau \geq \varepsilon_0, \quad \forall t \geq t_0,
\]

(3.1.1)

for all unit vectors \( w \in \mathbb{R}^n \).

This definition is commonly used in the literature and is a necessary and sufficient condition for the exponential stability of

\[
\dot{x}(t) = -u(t)u^T(t)x(t).
\]

**Definition 3.1.2.** A bounded vector \( u : \mathbb{R}^+ \rightarrow \mathbb{R}^n \) is said to be a persistent excitation if positive constants \( T_0, \delta_0 \) and \( \varepsilon_0 \) exist such that time \( t_2 \) exists with \( [t_2, t_2 + \delta_0] \subset [t, t + T_0] \) and

\[
\left| \frac{1}{T_0} \int_{t_2}^{t_2 + \delta_0} w^T u(\tau) d\tau \right| \geq \varepsilon_0, \quad \forall t \geq t_0,
\]

(3.1.2)

for all unit vectors \( w \in \mathbb{R}^n \).

This definition has been shown to be necessary and sufficient for the exponential stability of the differential equation

\[
\begin{align*}
\dot{x}_1 &= Ax_1(t) + bu^T(t)x_2(t), \\
\dot{x}_2 &= -u(t)b^T x_1(t),
\end{align*}
\]

(3.1.3)

where \( x_1 : \mathbb{R}^+ \rightarrow \mathbb{R}^m \), \( A \) is a \( \mathbb{R}^{m \times m} \) matrix and \( b \) is a \( \mathbb{R}^m \) vector, and \( u, x_2 : \mathbb{R}^+ \rightarrow \mathbb{R}^n \).
It is clear that the definition of persistent excitation used must depend on the context and the class of differential equations under investigation. It is also worth pointing out that a function \( u(\cdot) \) satisfying definition 2 also satisfies definition 1. The converse is not true, as shown by the following example:

\[
u(t) = \begin{cases} +1 & t \in \left[t_n, \ t_n + \frac{1}{2n+1}\right] \\ -1 & t \in \left[t_n + \frac{1}{2n+1}, \ t_{n+1}\right] \end{cases}, \quad n = 1, 2, \cdots \quad t_n = \sum_{i=1}^{n} \frac{1}{i}.
\]

(3.1.4)

When the class of signals is restricted to be piecewise continuous and piecewise differentiable functions, which is assumed in definition 3.1.3, the two definitions can be shown to be equivalent.

**Definition 3.1.3.** Let \( C_\delta \) be a set of points in \([0, \infty)\) for which there exists a \( \delta > 0 \) such that for all \( t_1, t_2 \in C_\delta, t_1 \neq t_2 \) implies \( |t_1 - t_2| \geq \delta \). Then \( \mathcal{P}_{(0, \infty)} \) is defined as the class of real-valued functions on \([0, \infty)\) such that for every \( u \in \mathcal{P}_{(0, \infty)} \) there corresponds some \( \delta \) and \( C_\delta \) such that

- \( u(t) \) and \( \dot{u}(t) \) are continuous and bounded on \([0, \infty))/C_\delta\), and
- for all \( t_1 \in C_\delta \), \( u(t) \) and \( \dot{u}(t) \) have finite limits as \( t \uparrow t_1 \) and \( t \downarrow t_1 \).

A vector \( u(\cdot) \) is said to belong to \( \mathcal{P}_{[0, \infty)} \) if every component of \( u(\cdot) \) belongs to \( \mathcal{P}_{[0, \infty)} \).

We shall also refer to the signal \( u(\cdot) \) as a persistent excitation over an interval \([t_0, t_0 + T]\) if it satisfies equations (3.1.1), and (3.1.2) over this interval.

### 3.2 Persistent Excitation Analysis in Time Domain

The definitions of persistent excitation are given as integration of time functions. If we look at this integration at some practical time instants, it means that if the signal \( u(t) \) is a
persistent excitation, there exist time instants $t_i$ such that the set containing all columns of $u(t_i)$ spans a whole space, i.e.

$$\text{rank } [u(t_1) \ u(t_2) \cdots u(t_i) \cdots] = n.$$ 

This interpretation will be applied in Chapter [4] for PE analysis. Based on the definition of persistent excitation, the following lemmas give preliminary results of PE property for time-varying signals.

**Lemma 3.2.1.** Given a diagonal matrix

$$\Omega(t) = \begin{bmatrix} \omega_1(t) & 0 & \cdots & 0 \\ 0 & \omega_2(t) & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \omega_n(t) \end{bmatrix},$$

where $\omega_i(t)$ are piecewise continuous signal matrices. Then $\Omega(t)$ is PE if and only if $\omega_i(t), \ i = 1, 2, \ldots n$ are PE.

**Proof.**

- When $n = 1$, the lemma is the case obviously from the definition of persistent excitation.

- When $n \geq 2$, since $\omega_i(t), \ i = 1, 2, \ldots n$ is PE, with $\omega_i(t) : \mathbb{R}^+ \to \mathbb{R}^{m_i}, \ \varepsilon_i > 0, \ t_0 > 0, \ T_i > 0$ it satisfies

$$\frac{1}{T_i} \int_{t_0}^{t_0 + T_i} \|\lambda_i^T \omega_i(\tau)\|d\tau \geq \varepsilon_i,$$

where $\lambda_i \in \mathbb{R}^{m_i}$ is a unit vector. Choose $T_0 = \max(T_1, T_2, \cdots, T_i), \ i = 1, 2, \ldots n$

$$\frac{1}{T_0} \int_{t_0}^{t_0 + T_0} \|\lambda_i^T \omega_i(\tau)\|d\tau \geq \frac{T_i}{T_0} \varepsilon_i \|\lambda_i\|. \quad (3.2.1)$$

Thus we get

$$\| (\lambda_1^T \lambda_2^T \cdots \lambda_n^T) \Omega(t) \| = \|\lambda_1^T \omega_1\| + \|\lambda_2^T \omega_2\| + \cdots + \|\lambda_n^T \omega_n\|.$$
\[
\frac{1}{T_0} \int_{t_0}^{t_0+T_0} \| (\lambda_1^T \lambda_2^T \ldots \lambda_n^T) \Omega(t) \| dt \geq \varepsilon_0 I, \tag{3.2.2}
\]
where \( \varepsilon_0 = \min \left( \frac{T_1}{T_0} \varepsilon_1, \frac{T_2}{T_0} \varepsilon_2, \ldots, \frac{T_i}{T_0} \varepsilon_i \right) \), \( i = 1, 2, \ldots, n \) and matrix \( \Omega(t) \) is PE by definition.

\textbf{Lemma 3.2.2.} Given a bounded triangular matrix

\[
\nu(t) = \begin{bmatrix} 
\nu_1(t) & \nu_{12}(t) \\
0 & \nu_2(t)
\end{bmatrix} \in \mathbb{R}^{(n_1+n_2) \times (m_1+m_2)},
\]

where \( \nu_i(t) \in \mathbb{R}^{n_i \times m_i} \) \( (i = 1, 2) \) and \( \nu_{12}(t) \) are piecewise continuous matrices. Then matrix \( \nu(t) \) is a persistent excitation if \( \nu_1(t) \) and \( \nu_2(t) \) are persistent excitation.

\textbf{Proof.} Let \( \eta \in \mathbb{R}^{m_1+m_2} \) be an arbitrary unit vector, \( \eta^T = \begin{bmatrix} \eta_1^T & \eta_2^T \end{bmatrix} \), \( \eta_1 \in \mathbb{R}^{m_1} \), and \( \eta_2 \in \mathbb{R}^{m_2} \). It follows that, for any \( t_0 \) and \( T > 0 \),

\[
\frac{1}{T} \int_{t_0}^{t_0+T} \| \nu(t) \eta \|^2 dt = \frac{1}{T} \int_{t_0}^{t_0+T} \| \nu_1(t) \eta_1 + \nu_{12}(t) \eta_2 \|^2 dt + \frac{1}{T} \int_{t_0}^{t_0+T} \| \nu_2(t) \eta_2 \|^2 dt. \tag{3.2.3}
\]

Both terms on the right hand of the above equation are positive semi-definite. It follows from \( \nu_2(t) \) being PE that, if \( \eta_2 \neq 0 \),

\[
\frac{1}{T} \int_{t_0}^{t_0+T} \| \nu_2(t) \eta_2 \|^2 dt > 0.
\]

On the other hand, \( \| \eta_1 \| = 1 \) if \( \eta_2 = 0 \). In this case,\n
\[
\frac{1}{T} \int_{t_0}^{t_0+T} \| \nu_1(t) \eta_1 + \nu_{12}(t) \eta_2 \|^2 dt = \frac{1}{T} \int_{t_0}^{t_0+T} \| \nu_1(t) \eta_1 \|^2 dt,
\]

which is also uniformly positive since \( \nu_1(t) \) is PE. This concludes the proof. \qed
**Remark 3.2.1.** From the above lemmas, it can be seen that the persistent excitation property of a vector or matrix is a stronger condition than linearly independent. Signal $\cos \omega t$ and $\cos \omega t + e^{-at}$ are linearly independent, where $a$ is any positive constant, while $u = [\cos \omega t \; \cos \omega t + e^{-at}]^T$ is not a PE vector. Also, PE property relies on both signal and structure of the vector or matrix, for example,

$$\begin{bmatrix} \cos \omega t & 0 \\ 0 & \cos \omega t + e^{-at} \end{bmatrix},$$

is a PE matrix.

**Lemma 3.2.3.** Assume $\Omega : \mathbb{R} \to \mathbb{R}^{n \times n}$ is a persistent excitation, and $\Omega, \dot{\Omega} \in \mathcal{L}_2$, then

1) any function belongs to $\mathcal{L}_1$ and $\mathcal{L}_2$ is not persistent excitation.

2) let $M \in \mathbb{R}^{r \times n}$, $r \leq n$, be a constant matrix, then $M\Omega$ is persistent excitation if and only if $M$ is full rank of $r$.

3) signal $\Omega + e$ is persistent excitation if $e \in \mathcal{L}_2$.

**Proof.** Proof of this lemma can be found in [27, 49, 67].

While the persistent excitation condition for signals and its application in adaptive control of linear systems is well understood, the derivation of persistent excitation conditions and their evaluation for general nonlinear systems are very difficult topics, they must be done on a case-by-case basis. Existing results are very limited. The following theorem relates stability analysis of a family of linear-like systems (commonly arising from adaptive control and/or estimation) to the persistent excitation condition of some matrix in the dynamics.
Theorem 3.2.4. Let \( A(t) \in \mathbb{R}^{n \times n}, B(t) \) and \( C(t) \in \mathbb{R}^{m \times n} \) be bounded, piecewise continuous matrix functions. Assume that \( \dot{B}(t) \) is uniformly bounded. Then, system

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
A & -B^T \\
B & 0
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix},
\]

(3.2.4)
is exponentially stable if and only if there exists a symmetric, positive definite matrix \( P(t) \) such that \( \dot{P}(t) + P(t)A(t) + A^T(t)P(t) \) is negative definite and if matrix \( B(t) \) is persistent excitation.

Proof. Proof of this lemma can be found in [2, 27, 30, 53, 67, 49, 80, 55]. \( \square \)

Corollary 3.2.5. Consider the system

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
A & -B^T \\
MB & 0
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix},
\]

(3.2.5)
where \( M \in \mathbb{R}^{n \times n} \) is a full rank constant matrix, \( A(t), B(t), x_1, x_2 \) are the same as in Theorem 3.2.4, the system is exponentially stable.

Proof. Define \( z = M^{-1/2}x_2 \), the equation (3.2.5) can be rewritten as

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{z}
\end{bmatrix} = \begin{bmatrix}
A & -(M^{1/2}B)^T \\
M^{1/2}B & 0
\end{bmatrix} \begin{bmatrix}
x_1 \\
z
\end{bmatrix}.
\]

(3.2.6)
By Lemma 3.2.3, \( M^{1/2}B \) is still a PE matrix of order \( n \), and by Theorem 3.2.4, the system is exponentially stable. \( \square \)

Corollary 3.2.5 is useful since it includes the effect of adaptive gains for real systems. It will be applied in stability proofs in Chapter 4.
3.3 Persistent Excitation Analysis in Frequency Domain

Since parameter convergence depends on the PE property of signals in the close-loop system, it is difficult to check this property beforehand. A simple condition can be applied is to check the sufficient richness of the signal. Most results presented here are existing results, except Lemma 3.3.4 which extends the current result to vector and nonlinear signals. This lemma was proven with the assistance of Dr. Zhihua Qu.

3.3.1 General Results for LTI System in Frequency Domain Analysis

Definition 3.3.1. A signal $u : \mathbb{R}^+ \rightarrow \mathbb{R}$ is called sufficiently rich of order $n$ if it consists of at least $\frac{n}{2}$ distinct frequencies.

Remark 3.3.1. One frequency has two spectral measure points, $\pm \omega$, except DC signal. If the signal has $\frac{n}{2}$ distinct nonzero frequencies, there $n$ spectral lines in its spectra.

Generally speaking, most signals in real system are stationary signals [8, 24, 67].

Auto-correlation of a stationary signal is intimately related with its persistent excitation property and is reviewed as follows.

Definition 3.3.2. A signal $u : \mathbb{R}^+ \rightarrow \mathbb{R}^n$ is said to be stationary if the following limit exists uniformly to $t_0$

$$R_u(t) = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} u(\tau)u^T(t + \tau)d\tau$$

(3.3.1)

The matrix $R_u(t) \in \mathbb{R}^{n \times n}$ is called the auto-correlation of $u$.
Definition 3.3.3. A stationary scalar signal $u: \mathbb{R}^+ \rightarrow \mathbb{R}$ is called sufficiently rich of order $n$ if the support of the spectral measure $S_u(\omega)$ of $u$ contains at least $n$ points.

The definition of sufficient richness really depends on the frequency components a signal has. The relationship between a time domain signal with sufficient richness of order $n$ can be shown as follows. $R_u(t)$ is a positive semi-definite matrix and its Fourier transform is given as

$$S_u(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega \tau} R_u(\tau) d\tau. \quad (3.3.2)$$

If $u$ has any frequency component at $\omega_0$, then $S_u(\omega)$ has a point mass (a $\delta$ function) at $\pm \omega_0$. $R_u(t)$ can be also calculated by inverse Fourier transform as

$$R_u(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} S_u(\omega) d\omega. \quad (3.3.3)$$

When $t = 0$, $e^{j\omega t} = 1$ and this leads to

$$R_u(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_u(\omega) d\omega. \quad (3.3.4)$$

Based on the definition of auto-correlation, the following lemma can be obtained.

Lemma 3.3.1. Let a function $u: \mathbb{R}^+ \rightarrow \mathbb{R}^{n \times m}$ be such that its auto-correlation function $R_u$ exists and is uniform with respect to $t_0$. Then $u$ is persistent exciting vector of dimension $n$ if and only if $R_u(0)$ is positive definite \cite{27, 49}.

Some useful lemmas have been proposed for linear transformation. Lemma 3.3.2 considers PE property for SISO LTI system, Lemma 3.3.3 discusses PE property of SIMO LTI system and Lemma 3.3.4 focus on PE property of MIMO nonlinear system.
Lemma 3.3.2. Given \( y(t) = H(s)u(t) \) where \( u, y : \mathbb{R}^+ \to \mathbb{R} \), if auto-correlation of \( u \) as defined in equation (3.3.1) is positive with respect to \( t_0 \), and \( H(s) \) is an asymptotically stable transfer function with zeros in the open left-half plane.

- If \( u \) is persistent exciting over period \( T_0 \) for all \( t \geq t_0 \), i.e. \( R_u(0) > 0 \), then \( y \) is persistent exciting over \( [t_1, T_1] \) where \( t_1 \geq t_0 \) and \( T_1 \geq T_0 \).

- If \( u \) is sufficient rich of order \( r \), \( r \leq n \), then, system output \( y \) is sufficient rich of order \( r \) \([27, 49, 67]\);

Proof. The proof of this lemma can be conducted in a very simple way. The necessary and sufficient condition for \( u \) to be persistent exciting is that \( R_u(0) > 0 \). Hence,

\[
R_u(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_u(\omega) d\omega
\]

(3.3.5)

where \( S_u(\omega) \) is spectral measure of \( u \). Since the transfer function is asymptotically stable, the initial states will converge to zero, the auto-correlation of the output signal \( y \) can be expressed as:

\[
R_y(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_y(\omega) d\omega
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega)S_u(\omega)H(j\omega)^* d\omega
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(j\omega)|^2 S_u(\omega) d\omega.
\]

(3.3.6)

Since \( H(j\omega) \) has no zeroes on the imaginary axis, from equation (3.3.5), \( R_y(0) > 0 \) and hence, the output signal is PE.

Moreover, since there is no frequency cancellation, any spectrum line appear in \( u \) is kept in the output \( y \). Thus \( y \) has the same order of sufficient richness as the input signal \( u \). \( \Box \)
Remark 3.3.2. When $H(s)$ has zeroes on the imaginary axis, some frequency cancellation may happen. For example: $H(s) = \frac{s^2 + a^2}{(s+b)^m}$ where $a > 0$, $b > 0$, $m \geq 3$. When $\omega = a$, $H(j\omega) = 0$ and $y$ is not PE if $u$ is only sufficient rich of order 1 at frequency $\omega = a$. However, when $u$ has any other frequency that $\omega \neq \pm a$, $y$ is PE. This simple example indicates that when a PE input passes through a filter, the PE property of the output depends on both the structure of the filter and also the frequency distribution of the input.

Lemma 3.3.3. Consider a linear system described by the equation

$$\dot{x} = Ax + bu, \quad x : \mathbb{R}^+ \to \mathbb{R}^n,$$

where $A$ is asymptotically stable, matrix pair $(A, b)$ is controllable.

- If $x$ is the state of the dynamic system, then the necessary and sufficient condition for the state $x$ to belong to a $n$-dimensional PE vector, is that the spectral measure of $u$ be sufficient rich of order $n$;

- Define $z = [x^T, u^T]$, the necessary and sufficient condition for $z$ to belong to a $n+1$ dimensional PE vector, is that $u$ is sufficient rich of order $n+1$.

Proof. The proof can be referred to [27, 49, 67], therefor is omitted here. \hfill \Box

Lemma 3.3.4. Consider the following model that describes linear filtering of nonlinear signals: given $y \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and real vector function $g(\cdot) : \mathbb{R}^m \to \mathbb{R}^l$,

$$y(t) = H(s)g(u),$$

where $H(s) \in \mathbb{R}^{n \times l}$ is a proper and stable transfer function matrix. Then, $y(t)$ is persistent excitation if the following conditions hold:
The spectral measure, \( S_{g(u)}(w) \), has non-zero mass (a matrix multiplied by a discrete delta function) at no less than \( p \) distinct points (say, \( w_i \) and \(-\omega_i \) for \( i = 1, \ldots, p \)) of \( w \), that is,

\[
S_{g(u)}(jw) = S'_{g(u)}(jw) + \sum_{i=1}^{p} D(jw_i)\delta(w - w_i),
\]

where \( S'_{g(u)}(jw_i) = 0 \) and \( D_i(w_i) \neq 0 \).

In \( \mathbb{C}^n \) and among the columns of matrices \( H^*(jw_i)D(jw_i)H^T(jw_i) \) for \( i = 1, \ldots, p \), there are \( n \) linearly independent vectors.

Proof. It follows from (3.3.8) that

\[
S_y(jw) = H^*(jw)S_{g(u)}(jw)H^T(jw),
\]

and that

\[
R_y(0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H^*(jw)S_{g(u)}(jw)H^T(jw)dw.
\]

Consequently, it yields that

\[
R_y(0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H^*(jw)S'_{g(u)}(jw)H^T(jw)dw
+ \frac{1}{2\pi} \sum_{i=1}^{p} H^*(jw_i)D(jw_i)H^T(jw_i).
\]

Clearly, matrices \( S'_{g(u)}(jw) \), \( D(jw_i) \), and \( H^*(jw_i)D(jw_i)H^T(jw_i) \) in (3.3.9) have the same properties as \( S_{g(u)}(jw) \), i.e., their real parts are symmetric, their imaginary parts are anti-symmetric, and their integrals are real, symmetric, and positive semi-definite. In particular, the property of their integrals being real and positive semi-definite means that

\[
\frac{1}{2\pi} \int_{-\infty}^{+\infty} H^*(jw)S'_{g(u)}(jw)H^T(jw)dw \geq 0, \text{ and }
\]

\[
\frac{1}{2\pi} \sum_{i=1}^{p} H^*(jw_i)D(jw_i)H^T(jw_i) \geq 0.
\]
Therefore, we have

\[ R_y(0) \geq \frac{1}{2\pi} \sum_{i=1}^{p} H^*(jw_i)D(jw_i)H^T(jw_i) \geq 0. \]  

(3.3.11)

The rest of the proof is done by contradiction. If \( y \) is not PE, there exists a non-zero vector \( \eta \in \mathbb{R}^n \) such that \( \eta^T R_y(0) \eta = 0 \). It follows from (3.3.11) that

\[ \sum_{i=1}^{p} \eta^T H^*(jw_i)D(jw_i)H^T(jw_i) \eta = 0, \]

which contradicts the hypothesis of matrices \( H^*(jw_i)D(jw_i)H^T(jw_i) \) being linearly independent. \( \square \)

**Remark 3.3.3.** If \( g(u) = u \) is linear and if \( u \in \mathbb{R} \) is one-dimensional, the above lemma reduces to the well-known lemma on linear filter and sufficient richness of scalar input [27, 67]. That is, the above lemma extends the existing results to vector and nonlinear signals.

### 3.3.2 Interpretation of Time-varying Signals With Frequency Viewpoint

In time domain, nonlinear terms appear in the form of \( u^2(t), u_1(t)u_2(t), u_1(t)e^{at} \) and even some inverse term like \( \frac{1}{u(t)} \). These nonlinear terms are signal multiplication in time domain. When this multiplication is viewed by auto-correlation, it related with frequency component, amplitude and phase difference of the signal. In fact it can be interpreted as summation of autocorrelation at distinct frequencies. Property of such nonlinear term will be considered and a lemma on PE property of summation of two positive semi-definite matrices will be proposed in this section.
Generally speaking, any periodic signal can be expressed in the form of

\[ u(t) = \sum_{i=0}^{n} a_i \cos(\omega_i t + \phi_i), \]

where DC signal can be considered as \( \omega_0 = 0 \). For an aperiodic signal, the frequency components can be obtained by wide sense Fourier transform, i.e., expand the system as a period function from \([0, T]\) where \( T \) is large enough to contain all the interested information. Given signal \( u_1(t) = \sum_{i=0}^{n} a_i \cos(\omega_i t + \phi_i) \), \( u_2(t) = \sum_{j=0}^{n} b_j \cos(\omega_j t + \varphi_j) \), \( u_3(t) = \sum_{k=0}^{n} c_k \cos(\omega_k t + \psi_k) \) if either of \( u_1 \) \( u_2 \), or \( u_3 \) doesn’t have the frequency \( \omega_i \), then the corresponding \( a_i \), \( b_i \) or \( c_i \) are set to zero. Multiplication of two signals can be expressed as

\[
u_1(t) \cdot u_2(t) = \sum_{i=0}^{n} a_i \cos(\omega_i t + \phi_i) \sum_{j=0}^{n} b_j \cos(\omega_j t + \varphi_j) \]

\[ = \sum_{i=0, j=0}^{n} a_i b_j \cos(\omega_i t + \phi_i) \cos(\omega_j t + \varphi_j) \]

\[ = \sum_{i=0, j=0}^{n} \frac{a_i b_j}{2} \{ \cos[(\omega_i - \omega_j)t + \phi_i - \varphi_j] + \cos[(\omega_i + \omega_j)t + \phi_i + \varphi_j] \} \]

\[ \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} u_1(t)u_2(t)dt = \begin{cases} \sum_{i=1}^{n} \frac{a_i b_i}{2} \cos(\phi_i - \varphi_i) & \text{if } i \neq j \\ 0 & \text{if } i = j, \end{cases} \]

since integration of the multiplication of two different frequencies is always a finite number, and when it’s divided by \( T \) and \( T \to \infty \), the limitation goes to zero.

Given vector \( u = [u_1(t) \ u_2(t) \ u_3(t)]^T \), where \( u_i \), \( i = 1, 2, 3 \) is persistent excitation respectively, its auto-correlation can be expresses as:

\[ R_u(0) = \begin{bmatrix} \sum_{i=0}^{n} \frac{a_i^2}{2} & \sum_{i=0}^{n} \frac{a_i b_i}{2} \cos(\phi_i - \varphi_i) & \sum_{i=0}^{n} \frac{a_i c_i}{2} \cos(\phi_i - \psi_i) \\ \sum_{i=0}^{n} \frac{a_i b_i}{2} \cos(\phi_i - \varphi_i) & \sum_{j=0}^{n} \frac{b_j^2}{2} & \sum_{i=0}^{n} \frac{b_j c_j}{2} \cos(\varphi_i - \psi_i) \\ \sum_{i=0}^{n} \frac{a_i c_i}{2} \cos(\phi_i - \psi_i) & \sum_{j=0}^{n} \frac{b_j c_j}{2} \cos(\varphi_i - \psi_i) & \sum_{k=0}^{n} \frac{c_k^2}{2} \end{bmatrix}. \]  

(3.3.14)
At each frequency $\omega_i$, define $\alpha_i = \phi_i - \varphi_i$, $\beta_i = \phi_i - \psi_i$, $\gamma_i = \varphi_i - \psi_i$, then $\alpha_i - \beta_i = \phi_i - \varphi_i - \phi_i + \psi_i = -\gamma_i$, auto-correlation of vector $u$ at frequency $\omega_i$ is

$$R_{\omega_i}(0) = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 \end{bmatrix} = \begin{bmatrix} \frac{a_i^2}{2} & \frac{a_i b_i}{2} \cos \alpha_i & \frac{a_i c_i}{2} \cos \beta_i \\
\frac{a_i b_i}{2} \cos \alpha_i & \frac{b_i^2}{2} & \frac{b_i c_i}{2} \cos \gamma_i \\
\frac{a_i c_i}{2} \cos \beta_i & \frac{b_i c_i}{2} \cos \gamma_i & \frac{c_i^2}{2} \end{bmatrix},$$  
(3.3.15)

and $R_u(0) = \sum_{i=1}^n R_{\omega_i}(0)$. If any component $u_i(t)$ doesn’t have frequency $\omega_i$, the corresponding magnitude will be zero.

- If $u_1$, $u_2$, and $u_3$ don’t share any frequency, $R_u(0)$ is a diagonal matrix. Since $u_1$, $u_2$, and $u_3$ are all PE signal, each of them contain at least one frequency component. Thus, $\sum_{i=1}^n a_i > 0$, $\sum_{i=1}^n b_i > 0$, $\sum_{i=1}^n c_i > 0$, the rank of $R_u(0)$ is 3.

- With $u_1$, $u_2$, and $u_3$ containing only one frequency $\omega_i$, $a_i$, $b_i$ and $c_i$ are not equal to zero. Determinant of $R_{\omega_i}(0)$ can be expressed as

$$\frac{8}{a_i^2 b_i^2 c_i^2} |R_{\omega_i}(0)| = 1 + 2 \cos \alpha_i \cos \beta_i \cos \gamma_i - \cos^2 \alpha_i - \cos^2 \beta_i - \cos^2 \gamma_i$$

$$= 1 + [\cos(\alpha_i + \beta_i) - \cos(\alpha_i - \beta_i)] \cos \gamma_i$$

$$= -1 - \frac{\cos 2\alpha_i + \cos 2\beta_i}{2} - \cos^2 \gamma_i$$

$$= [\cos(\alpha_i + \beta_i) - \cos(\alpha_i - \beta_i)] \cos \gamma_i$$

$$= \cos \gamma_i - \cos(\alpha_i + \beta_i) \cos(\alpha_i - \beta_i) - \cos^2 \gamma_i$$

$$= [\cos \gamma_i - \cos(\alpha_i + \beta_i)][\cos(\alpha_i - \beta_i) - \cos \gamma_i]$$

$$= 0,$$

since $\cos(\alpha_i - \beta_i) - \cos \gamma_i = 0$ always exists. Thus we know with only one frequency appearing in a three dimensional vector, it’s not going to be PE vector. Since $a_i^2 b_i^2 - a_i^2 b_i^2 \cos^2 \alpha_i \geq 0$ with equal sign satisfied if and only if $\cos \alpha_i = 0$, i.e. $u_1$, and $u_2$ have
the same frequency and phase. Thus, if any pair of \( u_1, u_2, \) and \( u_3 \) have different phase, they can contribute a 2-dimension PE vector.

- If all of them have the same phase at frequency \( \omega_i \), i.e. \( \alpha_i = \beta_i = \gamma_i = 0 \), they can only contribute to a 1-dimension PE signal though each of them may be PE.

Assume \( u_1, u_2 \) and \( u_3 \) share another frequency at \( \omega_j \), and

\[
R_{\omega_j}(0) = \begin{bmatrix} \eta_{j1} & \eta_{j2} & \eta_{j3} \end{bmatrix} = \begin{bmatrix} \frac{a_j^2}{2} & \frac{a_jb_j}{2} \cos \alpha_j & \frac{a_jc_j}{2} \cos \beta_j \\
\frac{a_jb_j}{2} \cos \alpha_j & \frac{b_j^2}{2} & \frac{b_jc_j}{2} \cos \gamma_j \\
\frac{a_jc_j}{2} \cos \beta_j & \frac{b_jc_j}{2} \cos \gamma_j & \frac{c_j^2}{2} \end{bmatrix}
\] (3.3.17)

where \( \eta_{j1} \) and \( \eta_{j2} \) are linearly independent also. From the above analysis, we know at each single frequency \( R_{\omega_j}(0) \) is positive semi-definite matrix. The following lemma gives proof on how to check summation of two positive semi-definite matrix to be positive matrix. By this theorem, we can check either a signal vector be persistent excitation or not based on its frequency analysis.

**Theorem 3.3.5.** Let \( A, B \in \mathbb{C}^{n \times n} \) be two positive semi-definite matrices, \( \mathcal{N}(\cdot) \) is null space of a matrix, and \( < \cdot, \cdot > \) denotes the inner product of two vectors, the following statements are equivalent.

1. Matrix \( A + B \) is positive definite;
2. Null space of \( A \), and \( B \) intersect only at \( \{0\} \), i.e. \( \mathcal{N}(A) \cap \mathcal{N}(B) = \{0\} \);
3. There exists \( 1 > \delta > 0 \) such that for all \( x \in \mathcal{N}(A), y \in \mathcal{N}(B) \),

\[
| < x, y > | \leq ||x|| \cdot ||y|| (1 - \delta).
\]
4. The set consisting the columns of \( A, \) and \( B \) spans a whole space.
Proof. Equivalence between (1) and (2): choose \( x, y, \zeta \in \mathbb{R}^n \),

\[ Ax = 0 \iff x \in \mathcal{N}(A), \quad By = 0 \iff y \in \mathcal{N}(B) \]

If \( A + B \) is positive definite, then

\[ (A + B)\zeta = 0 \iff \zeta = 0 \]

Since \( A\zeta = 0 \) and \( B\zeta = 0 \), \( \zeta \in \mathcal{N}(A) \cap \mathcal{N}(B) = 0 \).

Equivalence between (2) and (3): Inner product of any vectors with \( \{0\} \) is 0, then statement (3) holds all the time. For any nonzero vectors \( x \) and \( y \), \( < x, y > \) less than \( \|x\| \cdot \|y\| \) means \( x \) and \( y \) are linearly independent. Proof of equivalence of (2) and (3) can be done by contradicts the hypothesis of the statements.

- Necessary condition: Assume nonzero vectors \( x \in \mathcal{N}(A) \) and \( y \in \mathcal{N}(B) \) are linearly dependent, then \( x = ky \) with \( k \in \mathcal{C} \). Since \( Ax = kAy = 0 \), \( y \in \mathcal{N}(A) \), it means \( \mathcal{N}(A) \cap \mathcal{N}(B) = \{0\}, \{y\} \) or \( \{0\}, \{x\} \), which contradicts with statement (2).

- Sufficient condition: If \( \mathcal{N}(A) \cap \mathcal{N}(B) = \{0\}, \{y\} \), with \( y \neq 0 \), then \( Ay = 0, By = 0, < y, y > = \|y\| \cdot \|y\| \). It contradicts with statement (3). Thus the equivalence of (2) and (3) can be proved.

Equivalence between (1) and (4): Assuming \( A + B \) is not positive, there exists a non-zero vector \( \zeta \in \mathcal{C} \) such that \( \zeta^T(A + B)\zeta = 0 \), which contradicts with statement (4). Similarly, breaking the statement of (4) leads to contradiction of statement (2). Thus, the set consisting columns of \( A \) and \( B \) should contain \( n \) linearly independent vectors to span a whole space. \( \square \)
Remark 3.3.4. When the ranks of matrices are higher though not full rank, statement (3) reduces the number of vectors to check the positiveness. For example, if both $A$ and $B$ are $C_n^{n\times n}$ with rank $n - 1$. Then positive definiteness of $A + B$ can be checked by linearly independency of two vectors $x$ and $y$ where $x \in \mathcal{N}(A)$ and $y \in \mathcal{N}(B)$. While it’s much easier to find the null space of a matrix than to find the linearly independent basis of $n$ dimensional matrix.

3.4 Simulation Example

With the general results presented in the previous section, we are now ready to analyze the parameter convergence problem appeared in the nonlinear adaptive systems. Recall the example 2.2.2 we analyzed in Chapter 2. In parameterization model 1, by Theorem 3.2.4, the error dynamics is exponentially stable if the vector $\bar{\phi}$ is persistent excitation. Since

$$\bar{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} H_1(s) \\ -H_1(s)H(s) \end{bmatrix} u, \quad (3.4.1)$$

where $H_1(s) = \frac{\alpha_{n-1}(s)}{\lambda(s)}$ and $H(s) = \frac{Y(s)}{U(s)}$ are stable transfer functions. By Lemma 3.3.3 if $u$ is sufficient rich of order 4, then $\bar{\phi}$ is persistent excitation. From the simulation results, in case one, the control $u$ is only sufficient rich of order 2 since it contains only one frequency component. Thus, the parameter estimates cannot converge to their true values. While in case 2, $u$ is sufficient rich of order 4, all parameter estimates converge to their true values. In case 3, $u$ is sufficient rich of order 6, and since more modes of the system can be excited with this control, the convergence speed is faster than that in case 2.

Similar argument can be applied to parameterization model 2 and is omitted here.
3.5 Summary

In this chapter, general results of persistent excitation property of signal, vector, or matrix through dynamic transformation is discussed. New lemma extending the existing results to vector and nonlinear cases are proposed. Research on PE property of nonlinear signals is presented, and necessary and sufficient condition to check it is proposed. A simple example is also given to illustrate the technique for adaptive estimator with parameterization model discussed in Chapter 2.
CHAPTER 4

OBSERVER BASED SYNCHRONIZATION OF LORENZ SYSTEM

Synchronization of two chaotic systems is, in essence, observer design. Existing results can be found in [6, 9, 14, 23, 26, 41, 48, 54, 75]. Different nonlinear observer design techniques such as sliding differentiator and back stepping have been studied to approach synchronization of chaotic systems. The assumptions there are either the full state is measured or the higher order derivatives of the output are available. In real systems, these assumptions can not always be satisfied. In this chapter we try to illustrate for the Lorenz system that observer based synchronization can be achieved by partial state measurement and that the parameters can be estimated as well. An observer design with full state measurement will be presented first with exponential stability proof. Then, an adaptive observer with partial state measurement will be introduced. The persistent excitation condition will be checked and an exponential stability proof will be conducted based on Lyapunov arguments. This means that the reconstructed state tracks the un-transmitted state and that estimates of the unknown parameters converge to their true values. Though the nonlinear observer is proposed for the Lorenz system, this idea can be applied to the problem of synchronization and parameter estimation for other chaotic systems.
4.1 Full State Measurement Observer

With all the states of Lorenz system measurable, adaptation laws and observer design are given in this section. Once the two subsystems-transmitter and observer, can be synchronized, the parameters converge to their true values as well. Given the transmitter as Lorenz system,

\[
\begin{align*}
\dot{x}_1 &= \sigma(x_2 - x_1), \\
\dot{x}_2 &= \gamma x_1 - px_2 - x_1x_3, \\
\dot{x}_3 &= x_1x_2 - \beta x_3,
\end{align*}
\]

(4.1.1)

where \(\sigma, r, p, \beta\) are unknown positive constants, and the observer

\[
\begin{align*}
\dot{\hat{x}}_1 &= \hat{\sigma}(\hat{x}_2 - \hat{x}_1) + k_1\hat{x}_1, \\
\dot{\hat{x}}_2 &= \hat{\gamma}x_1 - \hat{p}\hat{x}_2 - x_1\hat{x}_3, \\
\dot{\hat{x}}_3 &= x_1\hat{x}_2 - \hat{\beta}\hat{x}_3 + k_2\hat{x}_3,
\end{align*}
\]

(4.1.2)

where \(\hat{\sigma}, \hat{\gamma}, \hat{p}, \hat{\beta}\) are the estimates of the unknown parameters, and design parameters \(k_1\) and \(k_2\) are arbitrary positive constants which satisfy \(k_2 \geq \frac{r_{\text{max}}}{p} - \beta\), and where \(r_{\text{max}}\) is the boundary of the states described by Lemma [2.3.1] Defining \(\hat{\sigma} = \sigma - \hat{\sigma}, \hat{\gamma} = \gamma - \hat{\gamma}, \hat{p} = p - \hat{p}, \hat{\beta} = \beta - \hat{\beta}\), and updating parameter estimates \(\hat{\sigma}, \hat{\gamma}, \hat{p}, \hat{\beta}\) by the following adaptation laws

\[
\begin{align*}
\dot{\hat{\sigma}} &= k_3\hat{x}_1(\hat{x}_2 - \hat{x}_1), \\
\dot{\hat{\gamma}} &= k_4\hat{x}_2x_1, \\
\dot{\hat{p}} &= -k_5\hat{x}_2\hat{x}_2, \\
\dot{\hat{\beta}} &= -k_6\hat{x}_3\hat{x}_3,
\end{align*}
\]

(4.1.3)
we have the stability theorem for the observer-based synchronization and estimation of unknown constant parameters.

4.1.1 Exponential Stability

In this section, theorem on exponential stability of the closed-loop system described by (4.1.1) with adaptation laws (4.1.3) is proposed and the proof based on Lyapunov argument is presented.

**Theorem 4.1.1.** An adaptive observer described by equation (4.1.2) for the plant (4.1.1) with adaptation laws designed as (4.1.3) guarantees that

- The synchronization error $\ddot{x}_i = x_i - \dot{x}_i$ converges to zero exponentially as $t \to \infty$.

- The parameter estimation error $\ddot{\theta} = [\ddot{\sigma} \; \ddot{\gamma} \; \ddot{\mu}]^T$ converges to zero exponentially, i.e, estimates converge to their true values exponentially.

**Proof.** The error dynamics of the system is

$$
\begin{align*}
\dot{\ddot{x}}_1 &= \sigma x_2 - \sigma x_1 - \sigma \ddot{x}_2 + \sigma \dot{x}_1 - k_1 \ddot{x}_1 \\
&= \sigma \dddot{x}_2 + \sigma \dot{x}_2 - \sigma \dddot{x}_1 - \sigma \dot{x}_1 - k_1 \ddot{x}_1 \\
&= -(k_1 + \sigma) \dddot{x}_1 + \sigma \dddot{x}_2 + (\dot{x}_2 - \dot{x}_1) \ddot{\sigma}, \\
\dot{\ddot{x}}_2 &= \gamma x_1 - px_2 - x_1 x_3 - \dot{\gamma} x_1 + \dot{p} \dddot{x}_2 + x_1 \dddot{x}_3 \\
&= -p \dddot{x}_2 - x_1 \dddot{x}_3 + x_1 \dddot{\gamma} - \dddot{x}_2 \ddot{p}, \\
\dot{\ddot{x}}_3 &= x_1 x_2 - \beta x_3 - x_1 \dot{x}_2 + \dot{\beta} \ddot{x}_3 - k_2 \ddot{x}_3 \\
&= x_1 \dddot{x}_2 - (\beta + k_2) \dddot{x}_3 - \dot{\beta} \ddot{x}_3.
\end{align*}
$$

(4.1.5)
Choose Lyapunov function as

\[ V = \frac{1}{2}[p \hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2 + \frac{p}{k_3 \sigma} \hat{\sigma}^2 + \frac{1}{k_4} \hat{\gamma}^2 + \frac{1}{k_5} \hat{p}^2 + \frac{1}{k_6} \hat{\beta}^2], \]  

(4.1.6)

and the time derivative of this Lyapunov function \( \dot{V} \) can be obtained as:

\[
\dot{V} = \frac{p}{\sigma} \hat{x}_1 \dot{\hat{x}}_1 + \frac{p}{\sigma} \hat{x}_2 \dot{\hat{x}}_2 + \frac{p}{\sigma} \hat{x}_3 \dot{\hat{x}}_3 + \frac{p}{\sigma} \hat{\sigma} \dot{\hat{\sigma}} + \frac{\hat{\gamma} \dot{\hat{\gamma}} + \hat{p} \dot{\hat{p}} + \hat{\beta} \dot{\hat{\beta}}}{k_3} \\
= \left[ -k_1 p \hat{x}_1^2 - p \hat{x}_1 \hat{x}_2 - p \hat{x}_2 - (\beta + k_2) \hat{x}_3^2 + \frac{p}{\sigma} \hat{x}_1 (\hat{x}_2 - x_1) \right] \\
+ \hat{\gamma} \hat{x}_2 \hat{x}_1 - \hat{\beta} \hat{x}_3 \hat{x}_3 \\
+ \frac{p}{k_3 \sigma} \hat{\sigma} + \frac{1}{k_4} \hat{\gamma} \hat{\gamma} + \frac{1}{k_5} \hat{p} \hat{p} + \frac{1}{k_6} \hat{\beta} \hat{\beta} \\
\leq \left[ -k_1 p \hat{x}_1^2 - p(\hat{x}_1 - \frac{1}{2} \bar{x}_2)^2 - \frac{3p}{4} \bar{x}_2^2 - (\beta + k_2) \hat{x}_3^2 + \hat{\sigma} \left[ \frac{p}{k_3 \sigma} \hat{\sigma} + \frac{p}{\sigma} \hat{x}_1 (\hat{x}_2 - x_1) \right] \right] \\
+ \hat{\gamma} \left( \frac{1}{k_4} \hat{\gamma} + \hat{x}_2 \hat{x}_1 \right) + \hat{\beta} \left( \frac{1}{k_6} \hat{\beta} - \bar{x}_3 \hat{x}_3 \right) \\
\leq 0, \quad (4.1.7)

where adaptation laws are defined as

\[
\hat{\sigma} = -k_3 \hat{x}_1 (\hat{x}_2 - \hat{x}_1), \\
\hat{\gamma} = -k_4 \hat{x}_2 \hat{x}_1, \\
\hat{p} = k_5 \hat{x}_2 \hat{x}_2, \\
\hat{\beta} = k_6 \hat{x}_3 \hat{x}_3. \quad (4.1.8)
\]

From the Lyapunov arguments we see that synchronization errors will go to zero asymptotically, i.e. \( \bar{x}_1 \to 0, \bar{x}_2 \to 0, \bar{x}_3 \to 0 \) as \( t \to \infty \). In order for the estimates of parameters to converge to their real values, persistent excitation condition must apply. The error dynamics can be rewritten as the form of equation (3.2.5) with \( e = [\bar{x}_1, \bar{x}_2, \bar{x}_3, \hat{\sigma}, \hat{\gamma}, \hat{p}, \hat{\beta}] \) and

\[
\dot{e} = \begin{bmatrix} A & -B^T \\ MB & 0 \end{bmatrix} e, \quad A = \begin{bmatrix} -(k_1 + \sigma) & \sigma & 0 \\ 0 & -p & -x_1 \\ 0 & x_1 & -(\beta + k_2) \end{bmatrix},
\]

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Matrix $A$ is negative definite. Separate matrix $B$ into three parts,

$$B = \begin{bmatrix}
-(\hat{x}_2 - x_1) & 0 & 0 \\
0 & -x_1 & 0 \\
0 & \hat{x}_2 & 0 \\
0 & 0 & \hat{x}_3
\end{bmatrix}, \text{ and } M = \begin{bmatrix}
k_3 & 0 & 0 & 0 \\
k_4 & 0 & 0 & 0 \\
k_5 & 0 & 0 & 0 \\
k_6
\end{bmatrix}.$$  

From Lemma 3.2.1 and Lemma 3.3.3 the necessary and sufficient condition for matrix $B$ to be PE is that $\omega_1$, $\omega_2$ and $\omega_3$ is PE, respectively. Omitting $\tilde{x}_i$, $i = 1, 2, 3$, leads to the persistent excitation property of the matrix

$$\omega_1 = x_1 - \hat{x}_2, \quad \omega_2 = [-x_1 \ \hat{x}_2]^T, \quad \omega_3 = \hat{x}_3.$$  

Since $x_2$ cannot equal to $x_1$ all the time in chaotic mode, $\omega_1$ is persistent excitation. $x_3$ is a time varying signal in chaotic mode, thus, $\omega_3$ is persistent excitation. The vector $\omega_2$ is persistent excitation since $x_1 = \frac{1}{s + \sigma} x_2$, $x_2$ is a wide band signal and is sufficient rich of order 2. Thus, matrix $B$ is persistent matrix by lemma 3.2.1 From Corollary 3.2.5 the error dynamics will be exponentially stable thus the transmitter and receiver will synchronize and the estimates of the unknown parameters converge to their true values. \qed

### 4.1.2 Simulation Example

A simple example for $\sigma = 10$, $r = 30$, $p = 1$ and $\beta = 3$ is applied to show the exponentially stability of the whole system. Design parameters are chosen as $k_1 = 100$, $k_2 = 100$, $k_3 = 1$, $k_4 = 1$, $k_5 = 1$, $k_6 = 1$. Figure 4.1 shows that the synchronization errors of states go to zero exponentially fast. In Figure 4.2 we denote $\sigma$ as the solid line, $\gamma$ the dash dot line, $p$ is
the long dash line and $\beta$ is the dash line. The simulation results exhibit that the parameter estimates converge to their true values exponentially fast.

Figure 4.1: Synchronization error for full state observer

Figure 4.2: Estimates of parameters for full state observer

This example shows that unknown parameters in Lorenz system can be estimated while all states are available. However, this scheme is not a good application for real com-
munication systems since not all the states can be transmitted in a real system. If only partial state are available, the above adaptation laws are unapplicable. New scheme should be considered to obtain state reconstruction and parameter estimation with partial state transmission as well. However, this full state observer design really gives us some intuitive information: if we can reconstruct the three states we should estimate all the unknown parameters. Also we noticed that in the adaptation laws, what really matters is the error information between the un-transmitted state and their estimate to update the parameters.

A nonlinear adaptive observer with partial state measurement is proposed in Section 4.2.

4.2 Noise-Free Adaptive Observer Design with Partial State Measurement

In this section, estimation of unknown system parameters and reconstruction of the un-transmitted state variable $x_2$ are pursued without considering the effects of noises and parameter variations. To this end, the adaptive observer is proposed based on the following assumption.

Assumption 4.2.1. The structure of the system is fixed and the bounds on parameters are given by, $\tau_1 < \sigma < \tau_2$, $\tau_3 < \gamma < \tau_4$, $\tau_5 < p < \tau_6$, and $\tau_7 < \beta < \tau_8$, where $\tau_i > 0$, $i = 1, \ldots, 8$.

The observer is designed as

\[
\begin{align*}
\dot{x}_1 &= \hat{\sigma}\hat{\omega}_1 + [k_4(\hat{\sigma} + \hat{p}) + \hat{\sigma}(\hat{\gamma} - \hat{p}) - k_4^2]\hat{\omega}_2 + (k_4 - \hat{\sigma} - \hat{p})\hat{x}_1 + k_1(x_1 - \hat{x}_1), \\
\dot{x}_2 &= \frac{1}{\hat{\sigma}}[\hat{\sigma}\hat{\omega}_1 + [k_4(\hat{\sigma} + \hat{p}) + \hat{\sigma}(\hat{\gamma} - \hat{p}) - k_4^2]\hat{\omega}_2 + (k_4 - \hat{\sigma} - \hat{p})\hat{x}_1 + k_1(x_1 - \hat{x}_1)] + x_1, \\
\dot{x}_3 &= x_1\hat{x}_2 - \hat{\beta}\hat{x}_3 + k_2(x_3 - \hat{x}_3),
\end{align*}
\]
where \( \hat{w}_1 \) and \( \hat{w}_2 \) are auxiliary signals defined by

\[
\dot{\hat{w}}_1 = -k_4 \hat{w}_1 - x_1 \hat{x}_3, \quad \text{and} \quad \dot{\hat{w}}_2 = -k_4 \hat{w}_2 + \hat{x}_1,
\]

and initial conditions are set at

\[
[\hat{x}_1(t_0) \; \hat{x}_2(t_0) \; \hat{x}_3(t_0) \; \hat{w}_1(t_0) \; \hat{w}_2(t_0)] = [0 \; 0 \; 0 \; 0 \; 0].
\]

Estimates of unknown parameters are calculated online by the following adaptation laws:

\[
\begin{align*}
\dot{\sigma} &= \begin{cases} 0 & \text{if } \sigma = \tau_1 \text{ and if } \zeta(x_1, x_3, \hat{x}_1, \hat{x}_3, \hat{x}_2, \hat{w}_1) \leq 0, \\
k_5[(x_1 - \hat{x}_1)\hat{w}_1 + x_1(x_3 - \hat{x}_3)(\hat{w}_1 + x_1 - \hat{x}_2)], & \text{else}
\end{cases} \\
\dot{\hat{\theta}}_1 &= k_6[(x_1 - \hat{x}_1)\hat{w}_2 + (x_3 - \hat{x}_3)x_1\hat{w}_2], \\
\dot{\hat{\theta}}_2 &= k_7[(x_1 - \hat{x}_1)\hat{x}_1 + (x_3 - \hat{x}_3)x_1\hat{x}_1], \\
\dot{\hat{\beta}} &= -k_8(x_3 - \hat{x}_3)\hat{x}_3,
\end{align*}
\]

where \( \tau_1 \) is that in Assumption 4.2.1, \( \hat{\sigma}(t_0) = \tau_1, \; \hat{\theta}_1(t_0) = \hat{\theta}_2(t_0) = 0, \; \hat{p} = k_3 + k_4 - \dot{\sigma} - \dot{\hat{\theta}}_2, \; \gamma = \hat{p} + [\hat{\theta}_1 + k_4^2 - k_4(\dot{\sigma} + \hat{\beta})]/\sigma, \) and

\[
\zeta(x_1, x_3, \hat{x}_1, \hat{x}_3, \hat{x}_2, \hat{w}_1) \triangleq (x_1 - \hat{x}_1)\hat{w}_1 + x_1(x_3 - \hat{x}_3)(\hat{w}_1 + x_1 - \hat{x}_2).
\]

In (4.2.1) and (4.2.2), gains \( k_i \) \( (i = 1, 2, 3, 4) \) are chosen to be positive scalars satisfying the following inequalities:

\[
\begin{align*}
k_1 &\geq \max \left\{ \frac{\tau_2(\tau_4 - \tau_5)}{k_4}, \frac{\tau^2}{\tau_1^2 \tau_7}, \frac{4k_4^2(\tau_2 + \tau_5) - \tau_2(\tau_4 - \tau_5) - k_4^2 + 1}{k_4^2}, k_4 \right\}, \\
k_2 &\geq \frac{\|\hat{x}_1\|^2}{4\tau_1 k_4} \max \left\{ 9(\tau_2 - 1)^2, 16k_4(k_1 - k_4 + \tau_2 + \tau_6), [k_4(\tau_2 + \tau_6) + \tau_2(\tau_4 - \tau_5) - k_4^2]^2 \right\}.
\end{align*}
\]

Properties of the proposed adaptive scheme will be analyzed in terms of: input/output stability and convergence of parameter estimates.
4.2.1 Input-Output Asymptotic Stability

The following theorem shows that the observer is input-output asymptotically stable, namely, the estimates of $x_1$ and $x_3$ asymptotically converge to their true values, respectively. Convergence of the estimate of $x_2$ will be considered in section 4.2.2.

Theorem 4.2.2. Assume that system (4.1.1) satisfies Assumption 4.2.1. Then, the adaptive observer given by (4.2.1) and (4.2.2) guarantees that:

- All signals are uniformly bounded;

- $\hat{x}_1$ and $\hat{x}_3$ asymptotically converge to $x_1$ and $x_3$, respectively.

Proof. The proof proceeds in two parts. In the first part, it is shown that Lorenz system (4.1.1) with initial conditions $x_i(t_0)$ has an identical solution to that of the following system:

\[
\begin{align*}
\dot{\xi}_1 &= -k_3\xi_1 + \sigma w_1 + [k_4(\sigma + p) + \sigma(\gamma - p) - k_4^2]w_2 + (k_3 + k_4 - \sigma - p)\xi_1 + k_1(x_1 - \xi_1), \\
\xi_2 &= \frac{1}{\sigma}\{\sigma w_1 + [k_4(\sigma + p) + \sigma(\gamma - p) - k_4^2]w_2 + (k_4 - \sigma - p)\xi_1 + k_1(x_1 - \xi_1)\} + x_1, \\
\dot{w}_1 &= -k_4 w_1 - x_1 x_3, \\
\dot{w}_2 &= -k_4 w_2 + \xi_1, \\
\dot{\xi}_3 &= x_1 \xi_2 - \beta \xi_3, \\
\end{align*}
\]

where $\xi_1(t_0) = x_1(t_0)$, $\xi_3(t_0) = x_3(t_0)$, $w_1(t_0) = x_2(t_0)$ and $w_2(t_0) = -(k_4 - p)x_1(t_0)/[k_4(\sigma + p) + \sigma(\gamma - p) - k_4^2]$. Note that the equation for $\xi_2$ is algebraic and that system (4.2.5) is introduced only for the purpose of analysis, thus need not be implemented. To show systems...
\((4.1.1)\) and \((4.2.5)\) yield the same solution, let us define

\[
\tilde{\xi}_1 = x_1 - \xi_1, \quad \tilde{\xi}_2 = x_2 - \xi_2, \quad \tilde{\xi}_3 = x_3 - \xi_3,
\]

\[
\theta_1 = k_4(\sigma + p) + \sigma(\gamma - p) - k_4^2, \quad (4.2.6)
\]

\[
\theta_2 = k_3 + k_4 - \sigma - p.
\]

Thus, by direct differentiation of \((4.2.6)\), the error dynamics between system \((4.1.1)\) and \((4.2.5)\) are:

\[
\dot{\tilde{\xi}}_1 = \sigma \left[ x_2 - x_1 + \frac{k_3 - \theta_2}{\sigma} \xi_1 - w_1 - \frac{\theta_1}{\sigma} w_2 - \frac{k_4}{\sigma} \tilde{\xi}_1 \right] = \sigma \tilde{\xi}_2,
\]

\[
\dot{\tilde{\xi}}_2 = \dot{x}_2 - \dot{x}_1 + \frac{k_3 - \theta_2}{\sigma} \xi_1 - \dot{w}_1 - \frac{\theta_1}{\sigma} \dot{w}_2 - \frac{k_4}{\sigma} \dot{\tilde{\xi}}_1
\]

\[
= \gamma x_1 - p x_1 - \frac{p}{\sigma} \dot{x}_1 - x_1 x_3 - \dot{x}_1 + \frac{\sigma + p - k_4}{\sigma} \dot{\xi}_1 + k_4 w_1 + x_1 x_3 + \frac{\theta_1 k_4}{\sigma} w_2 - \frac{\theta_1}{\sigma} \xi_1 - \frac{k_4}{\sigma} \dot{\xi}_1
\]

\[
= (\gamma - p) x_1 - \frac{\sigma + p}{\sigma} \dot{x}_1 + \frac{\sigma + p}{\sigma} \xi_1 - \frac{k_4}{\sigma} \dot{\xi}_1 + k_4 w_1 + \frac{\theta_1 k_4}{\sigma} w_2 - \frac{\theta_1}{\sigma} \xi_1 - \frac{k_4}{\sigma} \dot{\xi}_1
\]

\[
= (\gamma - p) x_1 - \frac{\sigma + p}{\sigma} \dot{x}_1 + \frac{k_4 k_3}{\sigma} \xi_1 - k_4 w_1 - \frac{k_4 \theta_1}{\sigma} w_2 - \frac{k_4}{\sigma} \theta_2 \xi_1 - \frac{k_4 k_4}{\sigma} \dot{\xi}_1 + k_4 w_1
\]

\[
+ \frac{k_4 \theta_1}{\sigma} w_2 - \frac{\theta_1}{\sigma} \xi_1
\]

\[
= (\gamma - p) x_1 - \frac{k_4 k_4}{\sigma} \dot{\xi}_1 + \frac{k_4 (k_3 - \theta_2)}{\sigma} \xi_1 - (k_1 + \sigma + p) \tilde{\xi}_2
\]

\[
= - \left( \frac{k_1 k_4}{\sigma} + p - \gamma \right) \tilde{\xi}_1 - (k_1 + \sigma + p) \tilde{\xi}_2.
\]

\[
\dot{\tilde{\xi}}_3 = x_1 \tilde{\xi}_2 - \beta \tilde{\xi}_3.
\]

To show \(\tilde{\xi}_i(t) \equiv 0\) for all \(t\) and for \(i = 1, 2, 3\), a Lyapunov candidate is chosen as

\[
V_1(\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3) = \frac{1}{2} \left[ (k_1 k_4 + \sigma p - \sigma \gamma) \tilde{\xi}_1^2 + \sigma^2 \tilde{\xi}_2^2 + \tilde{\xi}_3^2 \right],
\]

which is guaranteed to be positive definite by \((4.2.3)\). Along the dynamics of \((4.2.7)\), the
time derivative of $V_1$ is

$$
\begin{align*}
\dot{V}_1 &= (k_1 k_4 + \sigma p - \sigma \gamma) \dot{\xi}_1 \dot{\xi}_1 + \sigma^2 \dot{\xi}_2 \dot{\xi}_2 + \dot{\xi}_3 \dot{\xi}_3 \\
&= (k_1 k_4 + \sigma p - \sigma \gamma) \sigma \dot{\xi}_1 \dot{\xi}_2 - (k_1 k_4 + \sigma p - \sigma \gamma) \sigma \ddot{\xi}_1 \ddot{\xi}_2 \\
&\quad - (k_1 + p + \sigma) \sigma^2 \ddot{\xi}_2 + x_1 \ddot{\xi}_3 - \beta \ddot{\xi}_3^2 \\
&\leq -\frac{1}{2} [(k_1 + p + \sigma) \sigma^2 \ddot{\xi}_2 + \beta \ddot{\xi}_3^2],
\end{align*}
$$

which is negative semi-definite as inequality (4.2.3) ensures $k_1 + p + \sigma \geq \frac{\|x_1\|^2}{\sigma^2 \beta}$. It follows from $V_1(t_0) = 0$, $V_1$ being positive definite and $\dot{V}_1(t) \leq 0$ that $V_1(t) \equiv 0$. Having shown $\ddot{\xi}_i(t) \equiv 0$, we can define the following state estimation errors: $\ddot{x}_i = \xi_i - \hat{x}_i = x_i - \hat{x}_i$, $\ddot{\sigma} = \sigma - \hat{\sigma}$, $\ddot{\theta}_j = \theta_j - \hat{\theta}_j$, $\ddot{w}_j = w_j - \hat{w}_j$, with $i = 1, 2, 3$, $j = 1, 2$. Considering $\sigma x_2 + \hat{\sigma} \hat{x}_2 = \sigma x_2 - \hat{\sigma} \hat{x}_2 + \sigma \hat{x}_2 - \hat{\sigma} \hat{x}_2 = \sigma \hat{x}_2 + \hat{\sigma} \hat{x}_2$ the error dynamics between the Lorenz system and the observer can be obtained from equation (4.2.1) and (4.2.5):

$$
\begin{align*}
\ddot{x}_1 &= - (k_1 + k_3 - \theta_2) \ddot{x}_1 + \sigma \ddot{w}_1 + \theta_1 \ddot{w}_2 + \ddot{\theta}_1 \ddot{w}_2 + \ddot{\theta}_2 \ddot{x}_1, \\
\sigma \ddot{x}_2 &= - (k_1 + k_3 - \theta_2) \ddot{x}_1 + \sigma \ddot{w}_1 + \theta_1 \ddot{w}_2 + \ddot{\theta}_1 \ddot{w}_2 + \ddot{\theta}_2 \ddot{x}_1, \\
\ddot{x}_3 &= x_1 \ddot{x}_2 - \beta \ddot{x}_3 - \beta \ddot{x}_3 - k_2 \ddot{x}_3, \\
\ddot{w}_1 &= - k_4 \ddot{w}_1 - x_1 \ddot{x}_3, \\
\ddot{w}_2 &= - k_4 \ddot{w}_2 + \ddot{x}_1.
\end{align*}
$$

Choosing Lyapunov function

$$
V_2(\ddot{x}_1, \ddot{x}_3, \ddot{w}_1, \ddot{w}_2) = \frac{1}{2} \left[ \ddot{x}_1^2 + \sigma \ddot{x}_3^2 + \ddot{w}_1^2 + \ddot{w}_2^2 + \frac{1}{k_5} \ddot{\sigma}^2 + \frac{1}{k_6} \ddot{\theta}_1^2 + \frac{1}{k_7} \ddot{\theta}_2^2 + \frac{\sigma}{k_8} \ddot{\beta}^2 \right],
$$

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its derivative is given by

\[ \dot{V}_2 = \ddot{x}_1 \dot{x}_1 + \sigma \dddot{x}_3 \dot{x}_3 + \ddot{w}_1 \dot{w}_1 + \ddot{w}_2 \dot{w}_2 + \frac{1}{k_5} \dddot{\sigma} + \frac{1}{k_6} \dddot{\theta}_1 \dot{\theta}_1 + \frac{1}{k_7} \dddot{\theta}_2 \dot{\theta}_2 + \frac{\sigma}{k_8} \ddot{\beta} \]

\[ = - (k_1 + k_3 - \theta_2)\dddot{x}_1^2 + \sigma \dddot{w}_1 \dot{x}_1 + \theta_1 \dddot{w}_2 \dot{x}_1 + \dddot{\sigma} \dot{x}_1 \dot{w}_1 + \dddot{\theta}_1 \dot{x}_1 \dot{w}_2 + \dddot{\theta}_2 \dot{x}_1 \dot{w}_1 \]

\[ + \sigma x_1 \dddot{x}_3 \dot{x}_3 - \sigma \beta \dddot{x}_3 \dot{x}_3 - \dddot{\sigma} \dddot{x}_3 \dot{x}_3 - k_4 \dddot{w}_1^2 - k_4 \dddot{w}_2^2 - \dddot{x}_1 \dddot{w}_1 \]}

\[ + \frac{1}{k_5} \dddot{\sigma} + \frac{1}{k_6} \dddot{\theta}_1 \dot{\theta}_1 + \frac{1}{k_7} \dddot{\theta}_2 \dot{\theta}_2 + \frac{\sigma}{k_8} \ddot{\beta} \]

\[ = - (k_1 + k_3 - \theta_3)\dddot{x}_1^2 - \sigma (\beta + k_2)\dddot{x}_3^2 - k_4 \dddot{w}_1^2 - k_4 \dddot{w}_2^2 + \sigma \dddot{w}_1 \dot{x}_1 \]

\[ + (\theta_1 + 1) \dddot{w}_2 \dot{x}_1 - x_1 \dddot{w}_3 \dot{w}_1 + x_1 \dddot{x}_3 \dot{w}_1 \]

\[ = - (k_1 + k_3 - \theta_2)\dddot{x}_1^2 - \sigma (\beta + k_2)\dddot{x}_3^2 - k_4 \dddot{w}_1^2 - k_4 \dddot{w}_2^2 + \sigma \dddot{w}_1 \dot{x}_1 + (\theta_1 + 1) \dddot{w}_2 \dot{x}_1 - x_1 \dddot{x}_3 \dot{w}_1 \]

\[ = - (k_1 + k_3 - \theta_2) \dddot{x}_1^2 - \sigma (\beta + k_2) \dddot{x}_3^2 - k_4 \dddot{w}_1^2 - k_4 \dddot{w}_2^2 + \sigma \dddot{w}_1 \dot{x}_1 + \dddot{\theta}_1 \dot{x}_1 \dot{w}_1 \]

\[ + \sigma (\dddot{w}_1 + x_1 - \dddot{x}_2) x_1 \dddot{x}_3 + \dddot{\theta}_1 \dot{x}_1 \dot{w}_2 + \dddot{\theta}_2 x_1 \dddot{x}_3 + \dddot{\sigma} x_1 x_3 + \dddot{\theta}_1 x_1 \dot{w}_2 + \dddot{\theta}_2 x_1 \dot{w}_1 - \sigma \beta \dddot{x}_3 \dot{x}_3 \]

\[ + \frac{1}{k_5} \dddot{\sigma} + \frac{1}{k_6} \dddot{\theta}_1 \dot{\theta}_1 + \frac{1}{k_7} \dddot{\theta}_2 \dot{\theta}_2 + \frac{\sigma}{k_8} \ddot{\beta} \]

\[ = - (k_1 + k_3 - \theta_2) \dddot{x}_1^2 - \sigma (\beta + k_2) \dddot{x}_3^2 - k_4 \dddot{w}_1^2 - k_4 \dddot{w}_2^2 + \sigma \dddot{w}_1 \dot{x}_1 + (\theta_1 + 1) \dddot{w}_2 \dot{x}_1 \]

\[ = - (k_1 + k_3 - \theta_2) \dddot{x}_1^2 - \sigma (\beta + k_2) \dddot{x}_3^2 - k_4 \dddot{w}_1^2 - k_4 \dddot{w}_2^2 + \sigma \dddot{w}_1 \dot{x}_1 + (\theta_1 + 1) \dddot{w}_2 \dot{x}_1 \]

\[ = - (k_1 + k_3 - \theta_2) \dddot{x}_1^2 - \sigma (\beta + k_2) \dddot{x}_3^2 - k_4 \dddot{w}_1^2 - k_4 \dddot{w}_2^2 + \sigma \dddot{w}_1 \dot{x}_1 + (\theta_1 + 1) \dddot{w}_2 \dot{x}_1 \]

\[ + \sigma (\dddot{w}_1 + x_1 - \dddot{x}_2) x_1 \dddot{x}_3 + \dddot{\theta}_1 x_1 \dot{w}_2 + \dddot{\theta}_2 x_1 \dddot{x}_3 + \dddot{\sigma} x_1 x_3 + \dddot{\theta}_1 x_1 \dot{w}_2 + \dddot{\theta}_2 x_1 \dot{w}_1 - \sigma \beta \dddot{x}_3 \dot{x}_3 \]

\[ + \frac{1}{k_5} \dddot{\sigma} + \frac{1}{k_6} \dddot{\theta}_1 \dot{\theta}_1 + \frac{1}{k_7} \dddot{\theta}_2 \dot{\theta}_2 + \frac{\sigma}{k_8} \ddot{\beta} \]

\[ = - (k_1 + k_3 - \theta_2) \dddot{x}_1^2 - \sigma (\beta + k_2) \dddot{x}_3^2 - k_4 \dddot{w}_1^2 - k_4 \dddot{w}_2^2 + \sigma \dddot{w}_1 \dot{x}_1 + (\theta_1 + 1) \dddot{w}_2 \dot{x}_1 \]

\[ + \sigma [\frac{1}{k_5} \dddot{\sigma} + (\dddot{w}_1 + x_1 - \dddot{x}_2) x_1 \dddot{x}_3 + \dddot{\theta}_1 \dot{w}_1] \]

\[ + \dddot{\theta}_1 \left[ \frac{1}{k_6} \dddot{\theta}_1 + \dddot{w}_2 x_1 \dddot{x}_3 + \dddot{\theta}_2 \dot{w}_1 \right] + \dddot{\theta}_2 \left[ \frac{1}{k_6} \dddot{\theta}_2 + \dddot{\theta}_2 x_1 \dddot{x}_3 + \dddot{\theta}_2 \dot{x}_1 \right] + \sigma \beta \left[ \frac{1}{k_8} \dddot{\beta} - \dddot{x}_3 \dot{x}_3 \right] \]

\[ (4.2.11) \]

\[ \leq - \frac{1}{4} (k_1 + k_3 - \theta_2) \dddot{x}_1^2 - \sigma \beta \dddot{x}_3^2 - \frac{1}{3} k_4 \dddot{w}_1^2 - \frac{1}{3} k_4 \dddot{w}_2^2 - \frac{1}{4} (k_1 + k_3 - \theta_2) \dddot{x}_1^2 + \sigma \dddot{w}_1 \dot{x}_1 \]

\[ - \frac{1}{4} k_4 \dddot{w}_1^2 - \frac{1}{4} (k_1 + k_3 - \theta_2) \dddot{x}_1^2 + (\theta_1 + 1) \dddot{w}_2 \dot{x}_1 - \frac{1}{4} k_4 \dddot{w}_2^2 - \frac{1}{4} (k_1 + k_3 - \theta_2) \dddot{x}_1^2 \]

\[ - (k_1 + k_3 - \theta_2) ||x_1|| \dddot{x}_3 x_3 - \frac{1}{4} \sigma k_2 \dddot{x}_3^2 - \frac{1}{3} \sigma k_2 \dddot{x}_3^2 + (\sigma - 1) x_1 \dddot{x}_3 \dddot{w}_1 - \frac{1}{3} k_4 \dddot{w}_1^2 \]

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\[-\frac{1}{3}\sigma k_2 \ddot{x}_3 + \dot{\theta}_1 x_1 \ddot{x}_3 \ddot{w} - \frac{1}{3} k_4 \ddot{w}_2 + \tilde{\sigma} \left[ \frac{1}{k_5} \dot{\tilde{\sigma}} + (\ddot{w}_1 + x_1 - \dddot{x}_2) x_1 \dddot{x}_3 + \dddot{x}_1 \dot{w}_1 \right] \]
\begin{align*}
+ \frac{1}{k_6} \dot{\theta}_1 + \dot{w}_2 x_1 \dddot{x}_3 + \dddot{x}_1 \dddot{w}_2 \right] + \frac{1}{k_7} \dot{\theta}_2 + \dddot{x}_1 x_1 \dddot{x}_3 + \dddot{x}_1 \dot{x}_1 \right] + \sigma \tilde{\beta} \left[ \frac{1}{k_8} \dot{\tilde{\beta}} - \dddot{x}_3 \dot{x}_3 \right]
\end{align*}
\leq - \frac{1}{4} (k_1 + k_3 - \theta_2) \dddot{x}_1^2 - \sigma \beta \dddot{x}_3^2 - \frac{1}{3} k_4 \dddot{w}_1^2 - \frac{1}{3} k_4 \dddot{w}_2^2
\]
\begin{align*}
+ \left\{ \begin{array}{ll}
\tilde{\sigma} \left[ (\dddot{w}_1 + x_1 - \dddot{x}_2) x_1 \dddot{x}_3 + \dddot{x}_1 \dot{w}_1 \right] & \text{if } \tilde{\sigma} = \tau_1 \text{ and } \zeta \leq 0 \\
0 & \text{if else}
\end{array} \right.
\end{align*}
\quad \text{(4.2.13)}

Thus, \( \dot{V}_2 \) is negative definite with design parameters

\[
k_1 \geq \max \left\{ \frac{4\sigma^2}{k_4}, \frac{4(\theta_1 + 1)^2}{k_4}, k_4 \right\}
\quad \text{(4.2.14)}
\]

\[
k_2 \geq \frac{\|x_1\|^2}{4\sigma k_4} \max \left\{ 9(\sigma - 1)^2, 9\theta_1^2, 16k_4(k_3 + k_1 - \theta_2) \right\},
\]

\[
\dot{\tilde{\sigma}} = \left\{ \begin{array}{ll}
0 & \text{if } \tilde{\sigma} = \tau_1 \text{ and } \zeta \leq 0 \\
-k_5 \dddot{x}_1 \dddot{w}_1 - k_3 x_1 \dddot{x}_1 \dddot{w}_1 + x_1 - \dddot{x}_2) & \text{if else}
\end{array} \right.
\]

\[
\dot{\theta}_1 = -k_6 (\dddot{x}_1 \dddot{w}_2 + \dddot{x}_3 x_1 \dddot{w}_2),
\]

\[
\dot{\theta}_2 = -k_7 (\dddot{x}_1 \dddot{x}_1 + \dddot{x}_3 x_1 \dddot{x}_1),
\]

\[
\dot{\beta} = k_8 \dddot{x}_3 \dddot{x}_3,
\]

where \( \tau_1 \) is chosen as Assumption \textbf{4.2.1} and \( \zeta, \theta_i \) are defined in \textbf{(4.2.2)} and \textbf{(4.2.6)}. This adaptation law ensures the lower boundary of \( \tilde{\sigma} \) will be \( \tau_1 \). Bounds on \( \gamma, \sigma, \theta_1 \) and \( \theta_2 \) can be obtained from Assumption \textbf{4.2.1} Lemma \textbf{2.3.1} in Chapter \textbf{2} and equation \textbf{(4.2.6)}, thus \( k_1, k_2 \) can be chosen from equation \textbf{(4.2.3)} as listed in Theorem \textbf{4.2.2} accordingly.

The conclusion can be drawn so far is that all the signals in the observer and adaptive estimator are bounded. Since \( \dddot{x}_1 \) and \( \dddot{x}_3 \) are bounded and belong to function 2-norm, from the error dynamics \textbf{(4.2.8)}, \( \dddot{x}_1, \dddot{x}_3, \dddot{w}_1 \) and \( \dddot{w}_2 \) are also bounded, thus \( \dddot{x}_1, \dddot{x}_3, \dddot{w}_1 \) and \( \dddot{w}_2 \) go to zero as \( t \to \infty \), which means the reconstructed signals \( \dddot{x}_1 \) and \( \dddot{x}_3 \) track the transmitted
signals $x_1$ and $x_3$ asymptotically, respectively. The estimates of the unknown parameters will converge to some constants as $t \to \infty$ since $\dot{\sigma} = 0$, $\dot{\theta}_i = 0$ and $\dot{\beta} = 0$.

In equation (4.2.8), replacing $\tilde{x}_2$ with the second line in that equation, $\dot{x}_3$ can be expressed as

$$
\dot{x}_3 = -\frac{1}{\sigma}(k_1 + k_3 - \theta_2)x_1\tilde{x}_1 + x_1\tilde{w}_1 + \frac{1}{\sigma}\theta_1 x_1\tilde{w}_2 \\
+ \frac{1}{\sigma}\tilde{\sigma}(\tilde{w}_1 + x_1 - \tilde{x}_2)x_1 + \frac{1}{\sigma}\tilde{\theta}_1 \tilde{w}_2 x_1 + \frac{1}{\sigma}\tilde{\theta}_2 x_1\tilde{x}_1 - (\beta + k_2)\tilde{x}_3 - \tilde{\beta}\tilde{x}_3
$$

(4.2.16)

Since $\tilde{w}_i$ go to zero as time goes to infinite, omitting the effect of $\tilde{w}_i$ equation (4.2.8), (4.2.15) lead to the following expression of the error dynamics with $\zeta > 0$, $e = [\tilde{x}_1 \quad \tilde{x}_3 \quad \tilde{\sigma} \quad \tilde{\theta}_1 \quad \tilde{\theta}_2 \quad \tilde{\beta}]$,

$$
\dot{e} = \begin{bmatrix} A & -B^T \\ C & 0 \end{bmatrix} e, \quad A = \begin{bmatrix} -(k_1 + k_3 - \theta_2) & 0 \\ -(k_1 + k_3 - \theta_2)x_1/\sigma & -(\beta + k_2) \end{bmatrix},
$$

(4.2.17)

$$
B = \begin{bmatrix} -\tilde{w}_1 & -x_1(\tilde{w}_1 + x_1 - \tilde{x}_2)/\sigma \\ -\tilde{w}_2 & -x_1\tilde{w}_2/\sigma \\ -\tilde{x}_1 & -x_1\tilde{x}_1/\sigma \\ 0 & \tilde{x}_3 \end{bmatrix}, \quad \text{and} \quad C = M_1 \begin{bmatrix} -\tilde{w}_1 & -x_1(\tilde{w}_1 + x_1 - \tilde{x}_2) \\ -\tilde{w}_2 & -x_1\tilde{w}_2 \\ -\tilde{x}_1 & -x_1\tilde{x}_1 \\ 0 & \sigma\tilde{x}_3 \end{bmatrix},
$$

$$
M_1 = \begin{bmatrix} k_5 & 0 & 0 & 0 \\ 0 & k_6 & 0 & 0 \\ 0 & 0 & k_7 & 0 \\ 0 & 0 & 0 & \frac{k_8}{\sigma} \end{bmatrix}
$$

4.2.2 Convergence of Parameter Estimates

To show that estimate $\hat{x}_2$ converges to state variable $x_2$ and to study the properties of synchronization and parameter estimation of the proposed adaptive scheme, we must show that the estimates of parameters converge. The limit of $\hat{x}_2$ as time approaches to infinity is given by Theorem 4.2.2 and from the second line in (4.2.8) that

$$
\lim_{t \to \infty} \hat{x}_2 = \frac{1}{\sigma} \left[ (\tilde{w}_1 + x_1 - \tilde{x}_2)\tilde{\sigma} + \tilde{w}_2\tilde{\theta}_1 + \tilde{x}_1\tilde{\theta}_2 \right].
$$
From Lemma 2.3.1, it is with $s$ that we consider convergence of parameter estimation depends on whether certain signals in the system are persistent excitation (PE).

The following lemma checks persistent excitation condition of matrix $B$ given by equation (4.2.17). This property will be used in the proof of Theorem 4.2.4.

**Lemma 4.2.3.** Let $B$ be the matrix in equation (4.2.17), where each element is generated from equation (4.1.1, 4.2.1, 4.2.5). $B$ is a persistent excitation.

**Proof.** The proof is executed by showing that there exist time instants $t_i$ such that

$$\text{rank}[B(t_1) \ B(t_2) \ \cdots \ B(t_i) \ \cdots] = 4.$$ 

Since $\tilde{x}_1$, $\tilde{w}_1$, $\tilde{w}_2$, $\tilde{x}_3$ go to zero as $t \to \infty$, $[B(t_1) \ B(t_2) \ \cdots]$ has the same rank as that of $[B'(t_1) \ B'(t_2) \ \cdots]$ as $t \to \infty$, where

$$B'(t) = \begin{bmatrix} 0 & \sigma x_3 \\ w_1 & -x_1(w_1 + x_1 - \tilde{x}_2) \\ w_2 & -x_1 w_2 \\ x_1 & -x_1 x_1 \end{bmatrix}.$$ 

$$[B'(t_1) \ B'(t_2)] = \begin{bmatrix} 0 & \sigma x_3(t_1) & 0 & \sigma x_3(t_2) \\ w_1(t_1) & -x_1(t_1)(w_1(t_1) + s(t_1)) & w_1(t_2) & -x_1(t_2)(w_1(t_2) + s(t_2)) \\ w_2(t_1) & -x_1(t_1) w_2(t_1) & w_2(t_2) & -x_1(t_2) w_2(t_2) \\ x_1(t_1) & -x_1(t_1)^2 & x_1(t_2) & -x_1(t_2)^2 \end{bmatrix},$$

with $s(t) = x_1(t) - \tilde{x}_2(t)$. Rank of $[B'(t_1) \ B'(t_2)]$ is the same as following matrix,

$$\begin{bmatrix} 0 & 0 & \sigma x_3(t_1) & \sigma x_3(t_2) \\ w_1(t_1) & w_1(t_2) & -x_1(t_1)s(t_1) & -x_1(t_2)s(t_2) \\ w_2(t_1) & w_2(t_2) & 0 & 0 \\ x_1(t_1) & x_1(t_2) & 0 & 0 \end{bmatrix}.$$ 

From Lemma 2.3.1, $x_1$ has zero crossing points and $x_3(t)$ is positive in the chaotic mode.

Signal $s(t)$ is a time varying signal and cannot be zero for all time. Thus, there exist $t_1$ and $t_2$ such that $x_1(t_1) = 0$, $x_1(t_2) \neq 0$ and $s(t_2) \neq 0$, rank $B_1 = 2$ where

$$B_1 = \begin{bmatrix} \sigma x_3(t_1) & \sigma x_3(t_2) \\ -x_1(t_1)s(t_1) & -x_1(t_2)s(t_2) \end{bmatrix}.$$ 

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If $w_2(t_1) \neq 0$ holds when $x_1(t_1) = 0$, resulting in rank $B_2 = 2$ where

$$B_2 = \begin{bmatrix} w_2(t_1) & w_2(t_2) \\ x_1(t_1) & x_1(t_2) \end{bmatrix}.$$  \hspace{1cm} (4.2.18)

This condition can be proven by contradiction. Assuming rank $B_2 < 2$, this means that for every zero crossing point of $x_1(t_i) = 0$, $w_2(t_i) = 0$. From equations (4.2.1, 4.2.5) \( \dot{w}_2 = -aw_2 + x_1 \), \( w_2(t_j) = e^{-at_j}x_1(t_{j-1}) + e^{-at_j} \int_{t_{j-1}}^{t_j} x_1(\tau)e^{a\tau} d\tau \). Letting $x_1(t_{j-1}) = x_1(t_j) = 0$, where $t_{j-1}$ and $t_j$ are neighboring zero crossing times of $x_1$, $w_2(t_j) = 0$, leads to $\int_{t_{j-1}}^{t_j} x_1(\tau)e^{a\tau} d\tau = 0$, i.e. $x_1(t) = 0$ within the period $[t_{j-1}, t_j]$. By Remark 2.3.1 if state $x_1$ being zero in a period means $\dot{x}_1 = 0$, then $x_1 = x_2 = 0$, and the system stays in its invariant set. This contradicts with the assumption that the transmitter evolves in the chaotic mode. Therefore, rank $B_2 = 2$, and rank of $[B'(t_1) \quad B'(t_2)]$ is 4, $B$ is a persistent excitation matrix. \hfill \Box

**Theorem 4.2.4.** Consider system (4.1.1), under observer (4.2.1) and adaptation law (4.2.2). Then

- all the parameters converge to their true values exponentially;
- state variable $x_2$ converges to its true value exponentially.

**Proof.** In the previous section, it is shown that the error dynamics (4.2.17) are in the form as considered in Corollary 3.2.5, that is,

$$\dot{e} = \begin{bmatrix} A & -B^T \\ C & 0 \end{bmatrix} e,$$  \hspace{1cm} (4.2.19)

where $e = [\ddot{x}_1 \quad \ddot{x}_3 \quad \ddot{\theta}_1 \quad \ddot{\theta}_2 \quad \ddot{\beta}]^T$. Let $p_1 = \frac{r_{\text{max}}(k_1 + k_3 - \theta_2)}{4\sigma(k_2 + \beta)}$, where $r_{\text{max}}$ is given in equation (2.3.4).

A positive definite constant matrix $P$ is defined as

$$P = \begin{bmatrix} p_1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad A^T P + PA = \begin{bmatrix} -(k_1 + k_3 - \theta_2)x_1/\sigma & -(k_1 + k_3 - \theta_2)x_1/\sigma \\ -(k_1 + k_3 - \theta_2)x_1/\sigma & -2(\beta + k_2) \end{bmatrix} < 0.$$
Thus matrix $A$ is a stable matrix which satisfies the requirement in Corollary 3.2.5. By Lemma 4.2.3, $B$ is a persistent excitation matrix. From Corollary 3.2.5, the error dynamics in equation (4.2.17) are exponentially stable, i.e., synchronization errors of $x_1$ and $x_3$, and the estimation errors of parameters $\sigma$, $\theta_1$, $\theta_2$ and $\beta$ will go to zero exponentially. Once $\theta_i$ can be obtained uniquely, the unknown constant parameters $\gamma$ and $p$ will converge to their true values. Thus all estimates of the unknown parameters converge to their true values. From theorem 4.2.2 and equation (4.2.8), $\tilde{x}_2$ goes to zero as time goes to infinity, which means that the estimate of state $x_2$ converges to its true value.

4.2.3 Simulation Example

The proposed observer design and adaptation law are applied to the Lorenz system as described in (4.1.1), with system parameters $\sigma = 10$, $\gamma = 28$, $p = 1$, $\beta = 3$, where the parameter range of the system in Assumption 4.2.1. The design parameter values are chosen as $\tau_1 = 7$, $\tau_2 = 15$, $\tau_3 = 15$, $\tau_4 = 45$, $\tau_5 = 0.5$, $\tau_6 = 4$, $\tau_7 = 2.5$, $\tau_8 = 5$, $k_1 = k_2 = 220$, $k_3 = 15$, $k_4 = 4$, $k_5 = 0.6$, $k_6 = 80$, $k_7 = 0.3$, $k_8 = 1$, $r_{\max} = 60$. Synchronization errors of states are shown in Figures (4.3) to (4.5). Estimates of unknown parameters and estimation errors are shown in Figures (4.6) to (4.10). Simulation results demonstrate the effectiveness of the observer design and the adaptation law. Errors between the reconstructed states and the transmitter states converge to zero exponentially. Also, parameter estimates converge to their true values exponentially.
4.3 Summary

In this chapter, the full state feedback observer to estimate the unknown parameters is presented with exponential stability proof of the error dynamics. Concerning power efficiency, a partial state measurement is more practical for communications transmission. An adaptive nonlinear observer is proposed to reconstruct the untransmitted state. Parameter estimates are updated by the reconstructed state. Exponential stability proof is conducted based on Lyapunov arguments. The persistent excitation property of signals in the Lorenz system is checked by arguing in terms of its invariant set and chaotic property. Simulation results for observer designs exhibit the effectiveness of the proposed schemes.
Figure 4.3: Synchronization error $\tilde{x}_1$

Figure 4.4: Synchronization error $\tilde{x}_2$
Figure 4.5: Synchronization error $\tilde{x}_3$

Figure 4.6: $\sigma$-dashed, $\gamma$-solid, $\hat{p}$-dash dot, $\hat{\beta}$-dotted
Figure 4.7: Estimation error of $\sigma$

Figure 4.8: Estimation error of $\gamma$
Figure 4.9: Estimation error of $p$

Figure 4.10: Estimation error of $\beta$
When signals are transmitted through a channel, signal distortion, transmission delay and noise are unavoidable. A good communication scheme shows robustness to such effects. When retrieving useful information from the embedded noisy signal, features are sought that distinguish the signal from noise. Conventional techniques such as linear filtering use differences between the spectra of the signal and noise to separate them. This technique performs well when signal and noise occupy distinct frequency bands. Since a chaotic signal is really noise like, it has a wide band spectra like the noise signal, these spectral methods are no longer applicable. However, it’s reasonable to consider noise reduction or cancelation instead of spectral separation for achieving a higher SNR.

Current research on the effect of additive noise in chaotic communication system has been focusing on new robust schemes to attenuate noise and improve the signal-to-noise ratio (SNR) by robust filtering [3, 13, 66]. The extreme situations are based on the knowledge of the signal generating systems:

- The chaotic system is completely known;
• The chaotic system is unknown but of low order, while the noise is generating by a higher order system.

For the first case, the chaotic system is known while the system generating noise is unknown and therefore, modeled by a stochastic process; available results are based on probabilistic approach [34, 46]. In the second case, the difference between the signal and the noise is argued based on the assumption that the noise is generated by a higher order system.

5.1 Boundedness with Additive Noise

From control systems point of view, consider the system

$$\dot{x} = f(t, x) + g(t, x), \quad (5.1.1)$$

where \( f : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n \), \( g : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) are piecewise continuous in \( t \), the systems can be viewed as a perturbation of the nominal system

$$\dot{x} = f(t, x), \quad (5.1.2)$$

with perturbation term \( g(t, x) \). Assume the nominal system has an equilibrium point at the origin, if \( g(t, x) = 0 \) as \( x = 0 \), then the origin is still an equilibrium point with the disturbance \( g(t, x) \). However, in most cases, the disturbance does not satisfy this assumption, i.e. \( g(t, 0) \neq 0 \). In this case, the origin is not an equilibrium point and no conclusion can be drawn about stability of the origin. The following Theorem shows that the best result we can expect is the uniform boundedness of the disturbed system when the origin is exponentially stable.
Theorem 5.1.1. Consider the system, observer and adaptation laws given by equations (4.1.1), (4.2.1), and (4.2.2), the system is uniformly bounded with disturbance \( \|g(t, x)\| \leq \delta \) where \( \delta \) is a positive constant. The upper bound \( b_u \) and lower bound \( b_l \) of the system is determined by

\[
    b_u = \frac{\alpha_4 \delta}{\alpha_3 \mu} \sqrt{\frac{\alpha_2}{\alpha_2}}, \quad \text{and} \quad b_l = \frac{\alpha_4 \delta}{\alpha_3 \mu},
\]

where \( 0 < \mu < 1 \), respectively.

Proof. From the proof of Theorem 4.2.2 and Theorem 4.2.4, the error dynamics of the whole system is exponentially stable at the origin. Thus, there exist a Lyapunov function \( V(\Phi) \), \( \Phi = [\tilde{x}_i, \tilde{\sigma}, \tilde{\theta}_i, \tilde{\beta}]^T \) satisfies the following condition:

\[
    \alpha_1 \|\Phi\|^2 \leq V(\Phi) \leq \alpha_2 \|\Phi\|^2,
\]

\[
    \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \Phi} f(t, \Phi) \leq -\alpha_3 \|\Phi\|^2,
\]

\[
    \frac{\partial V}{\partial \Phi} \leq \alpha_4 \|\Phi\|. \tag{5.1.3}
\]

With the additive noise disturbance \( g(t, \Phi) \) considered, the time derivative of \( V_2 \) is rewritten as

\[
    \dot{V} \leq -\alpha_3 \|\Phi\|^2 + \alpha_4 \|\Phi\| \cdot \|g(t, \Phi)\|
\]

\[
    \leq -\alpha_3 \|\Phi\|^2 + \alpha_4 \delta \|\Phi\|
\]

\[
    \leq -\alpha_3 (1 - \mu) \|\Phi\|^2 - \alpha_3 \mu \|\Phi\|^2 + \alpha_4 \delta \|\Phi\|
\]

\[
    \leq -\alpha_3 (1 - \mu) \|\Phi\|^2,
\]

for any \( \|\Phi\| \geq \frac{\delta \alpha_4}{\mu \alpha_3} \). This gives the lower bound as \( b_l = \frac{\delta \alpha_4}{\mu \alpha_3} \). From Theorem 2.1.2 the upper bound can be determined as \( b_u = \frac{\alpha_4 \delta}{\alpha_3 \mu} \sqrt{\frac{\alpha_2}{\alpha_2}} \).

From this theorem, it can be shown that with bounded noise, both the state estimation errors and parameter estimation errors are bounded. With disturbance bound \( \delta \to 0 \), the
estimation errors shrink to zero also.

## 5.2 Robustness of Adaptive Filter

From the above analysis, the best result we can expect is the uniform boundedness of the system with additive noise. While the convention technique to compensate for channel effects in communication system is the use of a training signal, which means that the transmitter sends a training signal to the receiver first, the receiver extracts property of the channel after receiving this training signal and then a filter is applied at the receiver part to minimize the channel effects. In this section an additive white gaussian noise (AWGN) channel model will be considered, i.e. the only impairment is the linear addition of wide band gaussian noise with a constant spectral density. An adaptive noise cancelation filter will be applied before the observer in the simulation. Coefficients of this adaptive filter are determined by implementing the sign-data variation of the Least Mean Square (LMS) algorithm [77]. The filtered signal of \( x_1 \) and \( x_3 \) will be used in the observer and adaptation law (4.2.1, 4.2.2) to reconstruct the state and estimates the unknown parameters. System setup is the same as that in the previous section. Power of AWGNs in channels transmitting \( x_1 \) and \( x_3 \) are modeled as -20dB, and repeated every 10 seconds continuously. Parameter estimation errors are shown in figures 5.1 to 5.4. From the simulations, estimates of parameters converge to a small region around their real values but not exactly the real values as it is in the ideal situation. Estimation errors get larger when larger noises are added.
5.3 Summary

In this chapter, effect of the additive noise on a nonlinear system is discussed in general first. It shows if the system is exponentially stable at the origin, with unvanishing noise, the best result can be obtained is that the system are bounded. In such a case, an adaptive filter is applied for noise reduction and the boundedness of the synchronization and parameter estimation error are guaranteed. Simulation results are presented to show the robustness of this scheme.
Figure 5.1: Estimation error of $\sigma$ with channel noise

Figure 5.2: Estimation error of $\gamma$ with channel noise
Figure 5.3: Estimation error of $p$ with channel noise

Figure 5.4: Estimation error of $\beta$ with channel noise
CHAPTER 6

CONCLUSIONS

6.1 Impact of Main Results

Observer based synchronization and parameter estimation for chaotic systems is an open problem. The main contribution of this dissertation is the novel nonlinear adaptive observer design on estimation based synchronization of chaotic system. None of existing results shows both state reconstruction and unknown parameter estimation with partial state measurement for Lorenz system. The design can be applied to solve parameter mismatch and initial condition differences for chaotic system since it does not require any accurate information of parameters and initial conditions. The only assumption is that the structure of the system is known and parameter ranges should make the system evolve in chaotic mode. Persistent excitation are used to check exponential stability of the error dynamics. Thus, state synchronization and parameter estimates convergence to their real values are obtained successfully. Proof the persistent excitation property of signals in the closed loop system is done by the invariant set and chaotic property of Lorenz system. Also, new results on PE property extending the existing ones to vector and nonlinear signals have been proposed which can be useful for the further research.
In chapter 2, mathematical and theoretical background of selected definitions and most important theorems in stability analysis are reviewed. Some useful technique and examples of observer design, such as Model Reference Algorithm, High-gain observer are briefly discussed for nonlinear adaptive system. Simple example on the design and implementation has been explained. Basic idea as has been applied in MRA based estimator that mainly used in chapter 4 is discussed. The procedure of estimating differentiation using robust estimator and high-gain observer is briefly explained. In addition to background in control systems, overview of property of chaotic, especially Lorenz system has been presented. Some property useful for PE analysis for chaotic system have been presented.

In chapter 3, the impact on persistent excitation property of signal, vector, or matrix through dynamic transformation is discussed. This discussion allows us to analyze the persistent excitation property both in time domain and frequency domain. Based on persistent excitation property, the error dynamics of a class of nonlinear system are exponentially stable. New results of PE property on vector and nonlinear signals are presented. A simple example is given to illustrate the technique for adaptive estimator.

In chapter 4, the full state feedback observer to estimate the unknown parameters is presented first. Exponential stability of the error dynamics at the origin is achieved by checking persistent excitation of signals in the closed-loop system. In the concern of power efficiency, a partial state measurement is more practical for communication transmission. An adaptive nonlinear observer is proposed to reconstruct the un-transmitted state. Parameter estimation are updated by the reconstructed state. Exponential stability proof is conducted based on Lyapunov arguments and persistent excitation signals in Lorenz system is checked.
by arguing its invariant set and chaotic property. Simulation results for both full state observer and partial state measurement nonlinear adaptive observer exhibit the effectiveness of the proposed schemes.

In chapter 5, effect of the additive noise on a nonlinear system is discussed in general first. It shows if the system is exponentially stable at the origin, with unvanishing disturbance, or noise, the best result can be obtained is that the system are bounded. In such a case, an adaptive filter is applied for noise reduction and the boundedness of the synchronization and parameter estimation error are guaranteed. Simulation results are presented to show the robustness of this scheme.

6.2 Future Research

Items listed below might become the future research in this area of nonlinear observer design.

- Investigate techniques to reduce the number of transmitted signal, for example only scalar signal $x_1$ is transmitted.

- Investigate the application of Extended Kalman Filter technique in chaotic communication for the concern of white noise.

- Extend the results to a class of chaotic systems.
APPENDIX:

M-FILES AND SIMULINK DIAGRAM

A.1 Code in Example 2.2.2: Adaptive Estimation for Model Reference Algorithm

Simulink diagrams are shown in Figure 7.1 and the related adaptive observer in .m file are listed separately for parameterization 1 and parameterization 2, respectively. The Simulink diagrams keep the same, only the observer and adaptation laws vary with different parameterization models.

% -------------------------------------------------------------
% Adaptive Observer Design for LTI System -- Parameterization
% Model 1
% %
% Author: Yufang Jin
% %
% Date: 3/6/2004
% -------------------------------------------------------------

function [sys, x0] = observer1(t,x,u,flag);

% x(1)=f (u)
% x(2)=g (u)
% x(3)=f (y)
% x(4)=g (y)
Figure 7.1: Simulink diagrams of parameter estimation

\[
\begin{align*}
\% u(1) &= y \\
\% u(2) &= u \\
\end{align*}
\]

\[
f = 1; \\
g = 2;
\]

\[
\text{switch flag}
\]

\[
\text{case 1,}
\]

\[
\% \text{ State Derivarives ( flag = 1 or -1 )}
\]

\[
\theta = [x(5) \ x(6) \ x(7) \ x(8)]; \\
w = [x(1) \ x(2) \ x(3) \ x(4)]';
\]

\[
e_1 = \theta w - u(1);
\]

\[
\mathbf{x}_{\dot{}} =
\begin{bmatrix}
-fx(1) + u(2) \\
-gx(2) + u(2) \\
-fx(3) + u(1) \\
-gx(3) + u(1)
\end{bmatrix}
\]

\[
\% \text{ adaptation laws}
\]

\[
-20e1x(1) \\
-20e1x(2) \\
-50e1x(3)
\]
-50*e1*x(4)];

sys = xdot;

case 3,

% Output Equations  
theta=[x(5) x(6) x(7) x(8)];  
w=[x(1) x(2) x(3) x(4)]';  
y=theta*w;

sys=[y; x((3:8),:) ];

case 0,

% sys=[state, 0, output, input, 0, 0]  
sys = [8,0,7,2,0,1];  
% Initial Conditions  
x0 = [0 0 0 0 0 0 0 0]';

end

% -------------------------------------------------------------
% Adaptive Observer Design for LTI System -- Parameterization  
% Model 2  
% % Author: Yufang Jin  
% % Date: 3/6/2004  
% -------------------------------------------------------------
function [sys, x0] = observer2(t,x,u,flag);

%x(1)=x1  
%x(2)=psi1  
%x(3)=psi2  
%u(1)=y  
%u(2)=u

lambda=2;
a=1;

switch flag
    case 1,  
    % State Derivarives ( flag = 1 or -1 )
theta=[x(4) x(5) x(6) x(7)];
w=[u(2) x(2) u(1) x(3)]’;
e1=x(1)-u(1);

xdot = [-lambda*x(1)+theta*w
-a*x(2)+u(2)
-a*x(3)+u(1)
% adaptation laws
-10*e1*u(2)
-10*e1*x(2)
-50*e1*u(1)
-50*e1*x(3)];

sys = xdot;

case 3,
% Output Equations
sys=[x];

case 0,
% sys=[state, 0, output, input, 0, 0]
sys = [7,0,7,2,0,1];

% Initial Conditions
x0 = [0 0 0 0 0 0]’;
% x0=initstate;
end

A.2 Code in Theorem 4.1.1: Full State Measurement Observer and Parameter Estimation

Simulink diagrams are shown in Figure 7.2. Initial condition of Lorenz system is chosen as

\[-1 9 5]’.

%-----------------------------------------------
% Lorenz System S-function for Full State Measurement Observer
% Design and Parameter Estimation
%
% Author: Yufang Jin

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Figure 7.2: Simulink diagrams of full state observer

function [sys, x0] = plant(t,x,u,flag);

% Date: 3/6/2004
% -----------------------------------------------------------

% parametersetting of Lorenz system
a=10; r=30; p=1; bm=3; if (abs(flag) == 1)
% State Derivarives ( flag = 1 or -1 )

% xstate=[x(1) x(2) x(3)]';
xdot = [-a*x(1)+a*x(2)
    r*x(1)-p*x(2)-x(1)*x(3)
    x(1)*x(2)-bm*x(3)];

sys = xdot;

elseif flag == 3

% Output Equations

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sys = [ x ];

elseif flag == 0
  \% sys=[state, 0, output, input, 0, 0]
sys = [3,0,3,0,0,0];

  \% Initial Conditions
x0 = [-1 9 5]';

end

% -------------------------------------------------------------
% Observer S-Function for Full State Observer and
% Parameter Estimation
% 
% Author: yufang Jin
%
% Date: 3/6/2004
% -------------------------------------------------------------

function [sys, x0] = observer(t,x,u,flag);

  \%x(1)=\hat x1
  \%x(2)=\hat x2
  \%x(3)=\hat x3
  \%x(4)=\hat \sigma
  \%x(5)=\hat \gamma
  \%x(6)=\hat p
  \%x(7)=\hat \beta
  \%u(1)=x1
  \%u(2)=x2
  \%u(3)=x3

  if (abs(flag) == 1)
    \% State Derivarives ( flag = 1 or -1 )

    \%xstate=[x(1) x(2) x(3)]';
    \%parameter state=[x(4) x(5) x(6) x(7)]';

    \%Error between the states measurement and state estimates
    e1=u(1)-x1; e2=u(2)-x2; e3=u(3)-x3;
    \%Error dynamics
    xdot = [x(4)*(x(2)-x(1))+100*e1 + 99]
A.3 Code in Theorem 4.2.2: Nonlinear Adaptive Observer Design with Partial State Measurement

Simulink diagrams are shown in Figure 7.3.
function [sys, x0] = mrdcmod(t,x,u,flag);

% x(1)=x1
% x(2)=x2
% x(3)=x3

sigma=10;
r=28;
p=1;
beta=3;

if (abs(flag) == 1)
  % State Derivarives ( flag = 1 or -1 )
  xdot = [sigma*(x(2)-x(1))
           r*x(1)-p*x(2)-x(1)*x(3)
           x(1)*x(2)-beta*x(3)];

end
sys = xdot;

elseif flag == 3

% Output Equations
x13=x(1)*x(3); sys=x((1:3),:); x13;

elseif flag == 0

% sys=[state, 0, output, input, 0, 0]
sys = [3,0,4,1,0,0];

% Initial Conditions
x10=15;
x20=12;
x30=30;
x0 = [x10 x20 x30]’;
end

% ______________________________________________________
% Observer and Adaptation Laws S-Function for Adaptive
% Nonlinear Observer Design with Partial State Measurement
% ______________________________________________________

function [sys, x0] = Observer(t,x,u,flag);

%\x(1)=$\hat{x_1}$
%\x(2)=filtered version of x1x3
%\x(3)=filtered version of x3

%Design parameter
k1=220;
k2=220;
lambda=15;
a=4;

switch flag
case 1,
% State Derivatives (flag = 1 or -1)
theta=[x(4) x(5) x(6)];
w=[x(2) x(3) x(1)]’;
tildex1=u(1)-x(1);
tildex3=u(2)-x(7);
hatx2=\(-\lambda x(1)+\theta w+k1 tildex1\)/x(4)+u(1);
slope=tildex1*x(2)+u(1)*tildex3*(x(2)+u(1)-hatx2);
if (x(4)>7)
slope=1*slope;
elseif x(4)==7 & (slope>0)
slope=slope*1;
elseif (slope>0)
slope=1*slope;
else
slope=0;
end

xdot = \([-\lambda x(1)+\theta w+k1 tildex1
-ax(2)-u(1)x(7)
-ax(3)+x(1)
% Adaptation laws
0.6*slope
80*(tildex1*x(3)+u(1)*tildex3*x(3))
0.3*(tildex1*x(1)+u(1)*x(1)*tildex3
% hat x3
u(1)*hatx2-x(8)*x(7)+k2*tildex3
% hat beta
-1*x(7)*tildex3
\]);

sys = xdot;

case 3,
% elseif flag == 3
% Output Equations
theta=[x(4) x(5) x(6)];
w=[x(2) x(3) x(1)]’;
% State estimation error
tildex1=u(1)-x(1);
tildex3=u(2)-x(7);
% Parameter estimates
hatp=a+\lambda x(4)-x(6);
\begin{verbatim}
hatx2=(-lambda*x(1)+theta*w+k1*tildex1)/x(4)+u(1);
hatr=hatp+(x(5)+a*x(6)-a*lambda)/x(4);

sys=[x;hatp;hatr;hatx2];

case 0,

\% sys=[state, 0, output, input, 0, 0]
sys = [8,0,11,2,0,1];

\% Initial Conditions
\% sigma unknown, with lower bound 8
x0 = [0 0 8 0 0 0 0]’;
end

\% ------------------------------------------------------------------------
\% Adaptive Filter For Adaptive Nonlinear Observer
\% Design with Partial State Measurement
\% \%
\% Author: Yufang Jin
\% \%
\% Date: 3/6/2004
\% ------------------------------------------------------------------------

function  [Y, nsignal]=noisecan(x0, N, tausamp)

    tend=(N-1)*tausamp; ts=[0:tausamp:tend];
    [T,Y] = ode45(@lorenz,ts,x0);
    signal1=Y(:,1); signal2=Y(:,3);

    \%Generate white gaussian noise signal
    noise1= 0.5*wgn(N,1,-20);

    \% Eleventh order lowpass filter
    nfilt1=fir1(201,0.9);

    \% Correlated noise data
    fnoise1=filter(nfilt1,1,noise1); d1=signal1+fnoise1;

    \% Set the filter initial conditions.
\end{verbatim}
coeffs = nfilt1.;

% Set the step size for algorithm updating.
mu = 0.0001;

warning off signal:dfilt:basefilter:warnifreset: PropWillBeReset

ha = adaptfilt.sd(202,mu);
set(ha,'Coefficients',coeffs);
set(ha,'ResetBeforeFiltering','off');

[y1,e1] = filter(ha,noise1,d1);

noise2= 0.5*wgn(N,1,-20);

% Eleventh order lowpass filter
nfilt2=fir1(201,0.9); fnoise2=filter(nfilt2,1,noise2);
d2=signal2+fnoise2; coeffs = nfilt2.'; ha = adaptfilt.sd(202,mu);
set(ha,'Coefficients',coeffs);
set(ha,'ResetBeforeFiltering','off');

[y2,e2] = filter(ha,noise2,d2);
nsignal=[ e1'; Y(:,2)'; e2'];
LIST OF REFERENCES


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