1973

# A Study of Finite Element Grid Optimizations

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# A STUDY OF FINITE ELEMENT GRID OPTIMIZATIONS

BY

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RESEARCH REPORT

Submitted in partial fulfillment of the requirements
for the degree of Master of Engineering
in the Graduate Studies Program of
Florida Technological University

Orlando, Florida
1973

ABSTRACT

Any structural analysis which gives stresses and displacements for some predefined structure is governed by some physical domain of loading, geometry and boundary conditions. Let this domain be called the structures "problem space."

In applying finite element analysis, the solution to any one problem space may be one of many admissible solutions all of which satisfy some given set of boundary conditions. Admissibility is determined by the stated problem with its boundary conditions along with computer storage capacity considerations. Obtaining the most exact approximate solutions is of major concern to insure adequate results. This problem has been approached from a number of viewpoints [4-9] all of which employ some version of minimum potential energy [5, 10]. This report is a study of current approaches to this problem and their effect on finite element grid optimizations.

Selected optimizations [4-9] are shown to be effective in producing better solutions but it is noted that the implementation of these optimizations may be difficult. To survey the situation two fixed problem spaces of a tapered beam and a cantilever beam are chosen for investigation.

Conclusions based on this study display that optimizations methods applied to a finite element model give an optimum space arrangement that is a function of the selected element geometry and displacement function. When changes in the element geometry are

introduced a new optimum results. Comparing test problem results leads to some speculation employing uniform strain energy as a better guide to "first guess" grid arrangement and a recommendation for further investigation in this direction.

ACKNOWLEDGMENTS

I would like to gratefully acknowledge my indebtedness to Professors D. R. Jenkins, A. H. Hagedoorn and W. E. Carroll of Florida Technological University for their valuable suggestions and criticisms without which this report would not have been possible. Special thanks is extended to Professor Carroll for all the time and effort he patiently contributed all through my research.

James G. Ladesic

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# 1. INTRODUCTION

For any structure which is to be analyzed to obtain stresses and displacements there must exist some domain of definition for the structure. This domain of definition describes the geometry and loading conditions that the structure experiences. Let this domain of definition be called the structure's "problem space."

The finite element method in structural analysis allows a wide variety of element configurations for any one problem space. In order to provide a reasonable approximation the element divisions or grid should be relatively fine [1, 2]. The size of the problem's finite element model can then be characterized by the number of elements in the model. The maximum size of any problem is limited to computer capacity. For problem spaces, grid refinements obtained by increasing the size of the model may make the model too large for the available computer. When this happens a typical practice is to increase the number of elements in areas of high strain gradients while leaving low gradient areas coarse [1-4]. This, too, may prove impractical and the task reduces to one of finding the most efficient solution for a fixed number of elements or, plainly, grid optimization.

In the past the selection of refined grids was left to the finite element user who relied heavily on his past experience and intuitive "feel" for the problem [4]. More recently the problem of grid optimization through theoretical and analytical methods has been explored by a number of different authors [4-9].

A variational approach developed by Turcke and McNeice [7] and the similar variational method of Carroll and Barker [5] use the minimization of potential energy with respect to a change in element length to determine an optimum grid. Oliveira [6] introduced the concept of isoenergetic lines as a criterion for optimum grid arrangement. In each, however, as the optimization formulation was solved a more distressing problem arose making implementation of the theory a sizable task. Turcke and McNeice [7] found their variational method "intractable" for problems of two and three dimensional nodal variation. Oliveira [6] states "It is fair to remark that the requirements of disposing of elements along the isoenergetic lines is not always easy to follow" and goes on to point out that "...isoenergetic lines are not known a priori." The residue convergence method employed by Carroll and Barker [5] appears to be the best defined and the most applicable to computer usage of those investigated but, as indicated by them, fine meshes may preclude the justification of the residue convergence technique due to economic considerations.

With all the above in mind, a study to compare any of these possible methods, indeed, seems in order. The net result of such a study shall provide a tabulation of sample problems which will cross check the techniques used. More germane to tractable applications, this study is intended to shed some needed light on the subject of finite element grid optimization.

## 2. DESCRIPTION OF SELECTED PROBLEM SPACES

The criterion for selection of the problem space to use for this study is determined by the relative ease of application each space displays with respect to the optimization methods. Two such problem spaces are selected. A tapered bar with axial load and a cantilever beam with end load. Each display favorable traits to one or both optimization approaches.

The tapered bar [Figure (1)] is easily modeled in six degree of freedom (6 DOF), constant strain, triangular elements. Analysis may then proceed with the aid of a linear displacement formulated finite element program [1]. Isoenergetics for the tapered bar are particularly simple.

Oliveira [6] demonstrates that a better approximate solution is obtained when the grid is arranged so the element nodes fall on lines of constant strain energy density (isoenergetics [6]). For any but the simplest of problems, these isoenergetic lines are generally non-linear and require sophisticated element descriptions, such as iso-parametric elements, to provide this desirable alignment. Interpreting isoenergetic lines to imply equipotential lines of constant strain energy density furnishes a link between the known exact solution and the major element divisions of the approximate solution. If the number of equipotential lines is set equal to the number of major element divisions desired in the model the alignment of nodes and isoenergetics is simplified.

The tapered bar with an applied axial load possesses straight isoenergetic lines normal to the axis of symmetry. This is one of the simple cases referenced above. By keeping the number of major element divisions small the tapered bar is also easily optimized by trial and error calculations on the approximate solution by making use of an existing plane stress finite element program [1].

In adapting Oliveira's [6] isoenergetic concept to shells of revolution Sen [9] indicates that the element strain energy as opposed to strain energy density should be monitored to isolate inflexibilities in the finite element grid. When the strain energy change from element to element is observed to be relatively uniform the grid arrangement is considered adequate. The tapered bar is a simple problem space for calculating strain energy by elements to allow this observation.

By proper non-dimensionalization of the tapered bar analysis on the approximate solution, use can be made of a one-dimensional linear displacement bar model optimized variationally by Turcke and McNeice [7]. A comparison of optimum gird arrangements with respect to the type of element selected may then be made.

The second problem space is selected to best utilize existing residual optimization techniques developed by Carroll and Barker [5]. A cantilever beam [Figure (2)] with an end load possesses simple characteristics which directly apply to the residue method and still has relatively simple isoenergetics. By using an eight degree of freedom (8 DOF) rectangular element to model the beam a suitable geometry is produced which is compatible with the residue method. A computer finite element program containing an iterative subroutine to optimize the rectangular element model by minimal residues is then useful. Carroll [5] has developed such a program which is used to make these optimizations. Since this same cantilever beam is of simple geometry it can also be easily modeled in 6 DOF triangular elements. A separate computer analysis on this triangular element configuration supplies an interesting comparison of the two models as shown in Figure (6).

As in the tapered bar, the cantilever beam lends itself to simple calculation of individual element strain energies. The concept of uniform strain energy for optimization purposes may then be applied with ease.

# 3. EXACT SOLUTIONS AND ISOENERGETICS

The strength of materials solutions to each of the two problem spaces are elementary and may be found in any basic strength of materials text such as Timoshenko [12] for solutions of displacements, stresses and strain energies. For the tapered bar the axial stress is given by

$$\sigma_x = P/A(x) \tag{3.1}$$

where $\sigma_x$ is the axial stress, P is the axially applied load and A(x) is the cross-sectional area of the bar as a function of x [Figure (1)]. The cantilever beam axial stress is given by

$$\sigma_x = (Fxy)/I \tag{3.2}$$

where $\sigma_x$ is the stress in the x direction (Figure (2)], F is the applied shear end load and I is the cross-sectional area moment of inertia.

The isoenergetics are calculated for each exact solution by employing an equipotential concept on the strain energy density function. For a linearly elastic, homogeneous, isotropic continuum in a plane stress formulation the strain energy density is given by [10]

$$\bar{u} = \frac{1}{2E}[(\sigma_x + \sigma_y)^2 - 2(1-\nu)(\sigma_x\sigma_y - \sigma_{xy}{}^2)] \tag{3.3}$$

where $\bar{u}$ represents the strain energy density per unit volume, $\nu$ is Poisson's ratio and E is Young's modulus. If the geometry of each problem is selected such that the length of the beam or bar is large compared to the height and thickness of the cross-section the contribution of $\sigma_y$ and $\sigma_{xy}$ to $\bar{u}$ in (3.3) is insignificant [12, 15]. It can be shown that (3.3) reduces further to

$$\bar{u} = \frac{1}{2E}(\sigma_x)^2 \tag{3.4}$$

with only a small error.

Orienting $\sigma_x$ with respect to the coordinates shown in Figure (1) and Figure (2) gives

$$\sigma_x = \sigma_x(x,y) \tag{3.5}$$

Oliveira [6] has shown it desirable to find the locus of points in the continuum for which $\bar{u}$ is equal to a constant. Referring to Figure (3) this may be stated in equation form as

$$\bar{u}_1 - \bar{u}_0 = \bar{u}_2 - \bar{u}_1 = \bar{u}_3 - \bar{u}_2 = \bar{u}_4 - \bar{u}_3 = C \tag{3.6}$$

Equation (3.6) may be generalized to

$$\bar{u}_n - \bar{u}_{n-1} = C;\ n = 1,2,\ldots,n \tag{3.7}$$

where C is a constant. From (3.4) and (3.5) then, an equation for $\sigma_x(x,y)$ in terms of the strain energy density can be written as

$$[\sigma_x(x,y)]^2 = 2E\bar{u} \tag{3.8}$$

For the tapered bar shown in Figure (1) equation (3.8) is seen to be essentially invarient in y and for the cantilever beam in Figure (2) equation (3.8) holds as stated. By now using equations (3.1) and (3.2) for these two problem spaces [12, 15] and the geometric descriptions given in Figures (1) and Figure (2) it can be shown that a parametric form of (3.8) for the tapered beam is given by

$$\Phi_T = [(b-a)\frac{x}{L} + a]^{-2} \qquad (3.9)$$

where $\Phi_T = \frac{8\bar{u}Et^2}{P^2}$. For the cantilever beam, a corollary of (3.9) is

seen to be

$$\Phi_c = x^2y^2 \qquad (3.10)$$

where $\Phi_c = \frac{2\bar{u}EI^2}{F^2}$. By recalling (3.7) the equipotential concept for the

tapered beam gives

$$\Phi_T(1)-\Phi_T(0) = \Phi_T(2)-\Phi_T(1) = \ldots = \Phi_T(n)-\Phi_T(n-1) \qquad (3.11)$$

for n representing the number of major element divisions desired in the approximate solution. Similarly, by preselecting an arbitrary value for $\Phi_c(n)$ at $x = L$, $y = \frac{h}{2}$ [Figure 2] the corollary of (6.10) for the cantilever beam is

$$\Phi_c(1)-\Phi_c(0) = \Phi_c(2)-\Phi_c(1) = \ldots = \Phi_c(n)-\Phi_c(n-1) \qquad (3.12)$$

and n has the same definition given above.

Evaluating (3.9) for $x_0 = 0$ and $x_n = L$ gives

$$\Phi_T(0) = a^{-2},\ \Phi_T(n) = b^{-2} \qquad (3.13)$$

Equations (3.11) then represents a system of (n-1) equations in (n-1) unknowns which will yield x-position values directly. Equations (3.12), on the other hand, represents a family of n hyperbolas which may be plotted by ordered pairs in the problem space once n is assigned a value from the approximate solution.

In the approximate solutions values of n = 2 is used for the tapered bar and $n = 3$ for the cantilever beam. Solving equation (3.11) then provides a possible location for the major element division to compare with the optimum approximate solution as shown in Figure (1). Equation (3.12) is plotted as a family of 3 hyperbolas on the cantilever beam in Figure (2) to observe possible similarities with its approximate.

In order to use uniform strain energy as a criterion for optimum grid arrangement it becomes necessary to determine a set of $x_i$'s which satisfy the equation

$$\int_{V_o} \bar{u}dV = \int_{V_1} \bar{u}dV = \ldots = \int_{V_n} \bar{u}dV. \tag{3.14}$$

For the tapered bar (3.14) can be shown to reduce to

$$\int_0^{x_1} [\Phi_T]^{\frac{1}{2}} dx = \int_{x_1}^{x_2} [\Phi_T]^{\frac{1}{2}} dx = \ldots = \int_{x_{n-1}}^{L} [\Phi_T]^{\frac{1}{2}} dx \tag{3.15}$$

where $\Phi_T$ is given by (3.9) and suitable geometric relations from Figure (2) are used to change the volume integrals into the definite integrals shown. For $n = 2$, (3.15) reduces to

$$\int_0^{x_1} [\Phi_T]^{\frac{1}{2}} dx = \int_{x_1}^{L} [\Phi_T]^{\frac{1}{2}} dx \tag{3.16}$$

which is directly solvable for $x_1$. It can also be shown that for the cantilever beam with $n = 3$, (3.14) reduces to

$$B\int_0^{x_1} xdx + \int_0^{x_1} x^2dx = B\int_{x_1}^{x_2} xdx + \int_{x_1}^{x_2} x^2dx = B\int_{x_2}^{L} xdx + \int_{x_2}^{L} x^2dx \tag{3.17}$$

where $B = 12(1 + \nu)h^2/5$ [Figure (2)]. Equation (3.17) represents two equations in two unknowns and can be solved directly for $x_1$ and $x_2$. These values, in turn, may then be compared to the approximate optimum grid arrangement. All of the comparisons mentioned above are made in a latter section of this report.

## 4. FINITE ELEMENT APPROXIMATE SOLUTIONS

The geometry of the tapered beam suggests use of the 6 DOF triangular element in the approximate solution as noted earlier. In order to contain the problem within reasonable bounds, two major element divisions are chosen [Figure (4)]. The line connecting nodes (2), (5) and (7) is varied laterally in position from $x = a$ to $x = b$ [Figure (5)] and a computer analysis is made for each. The potential energy for the system is calculated by multiplying the tip deflection obtained from this analysis by minus one-half the applied load P. From the exact solution the exact potential energy is also obtained. A ratio of approximate to exact potential energy is then tabulated for each new position of x. The result of this tabulation is given in Figure (5). An optimum configuration is selected from this plot by the interpolated value of the potential energy ratio most near unity. Figure (5) shows this maximum ratio value to occur at $(L-x_1)/L = .68$ along the ordinate. This same problem was investigated by Turcke and McNeice [7] as a one dimensional system using a 2 DOF linear displacement model. Their study produced an optimum division location at $(L-x_1)/L = .73$ along the ordinate. By solving equation (3.11) for $n = 2$ the isoenergetic prediction places the major division at $(L-x_1)/L = .9209$ along the ordinate. From equation (3.16) a value of $(L-x_1)/L = .71$ is obtained. All of these values are listed in Figures (5) and (7).

The cantilever beam is modeled using rectangular elements as stated previously. The results of the computer calculated optimization are given in Figure (6) verses the exact solution in terms of potential energies ratios. The $x_i$ values obtained from the solution of (3.17) are listed in Figure (7) to compare with the approximate solutions mentioned above. Modeling the same beam in triangular elements eliminates the possibility of using the residual subroutine which is designed to calculate optimums for rectangular element configurations only. Because of the selection of three major element divisions, the trial and error procedure used on the tapered beam is also impractical. Because of this attempts are not made in this formulation to actually optimize the beam but to ascertain if the optimum does occur at the same location as in the rectangular formulation. Problem space arrangements are selected around the rectangular element optimum and analyzed in the triangular element field by a method similar to Ward's [14]. A compilation of these calculations is shown in Figure (6). The value of $\Pi$(APPROX.)/$\Pi$(EXACT) for the triangular element formulation is calculated using the $x_i$'s found for the optimum rectangular formulation. Since a value of the energy ratio can be found which is larger than this value when a different set of $x_i$'s are chosen it is concluded that the optimum grid configurations do not have the same arrangement for the two formulations.

The large reduction in the energy ratios observed by changing from the rectangular to the triangular elements is attributed to the models themselves. The 6 DOF linear strain triangular element gives a less flexible system than the 8 DOF rectangular element. To equilibrate these ratios it becomes necessary to increase the number of element divisions in the triangular formulation which again destroys the desired similarities.

## 5. CONCLUSIONS AND RECOMMENDATIONS

By comparison of Figure (5) with the work done by Turcke and McNeice [3] on a one dimensional tapered beam it is seen that the optimums of the two formulations are at different major element divisions. Similarly the comparison of the rectangular optimum with the computations made on the triangular formulation of the cantilever beam are inconsistent. The results are compiled in tabular form in Figure (6) as energy ratios and non-dimensionalized measurements.

Attempting to predict the optimum location of the major element divisions by isoenergetic lines on the exact solution is seen to fall short of the anticipated results. The nodal locations determined by isoenergetics in every instance are far different from those obtained from approximate solution optimizations [Figures (5), (6)]. Reviewing Figure (7) shows the values predetermined on the exact solution of the cantilever beam by uniform strain energy are in close agreement with the major element divisions found in the approximate solution. In addition, when the uniform strain energy derived $x_i/L$ values are used as input, opposed to equidistant spacing, for the computer analysis of the approximate solution an iteration time saving is realized. For the cantilever beam described earlier up to thirty-five iterations are made in order to find the optimum. Upon inputing the uniform strain energy calculated $x_i$ values this time is reduced to five iterations. Enough similarity and usefulness is seen here to warrant comment and to recommend that further study possibly be directed toward investigation of this phenomenon.

As evidenced by Figures (1-8) it is apparent that the ideal optimization for each element configuration is a function of the type of element selected for the model. The problem space is held constant and, realistically, will contain one and only one ootimization, the exact solution. In each approximate space, however, the only variable is the element models. Yet, in each situation, a better solution is obtained from the optimized element model than from the evenly space element model. In turn, each of these ideal solutions satisfied the well accepted concept of stationary potential energy [7-11] indicating a true optimum for the model but still failing to agree totally with the exact soltuion. Since this is observed to be so, the type of element must effect the approximate solution and therefore should be considered in any scheme for idealization.

Speculation in the area of isoenergetic lines seems to suggest field orientation as an important criterion for insuring the best idealization. Selecting an element that allows a general orientation of the mesh of more or less "flow" in the direction of the isoenergetics, as in the triangular formulation of the cantilever beam shown in Figure (8), produces a better approximate solution than if the mesh were arranged overlooking this consideration. The development of an iterative subroutine for the computer program used here which would optimize this triangular arrangement is recommended for future investigation.

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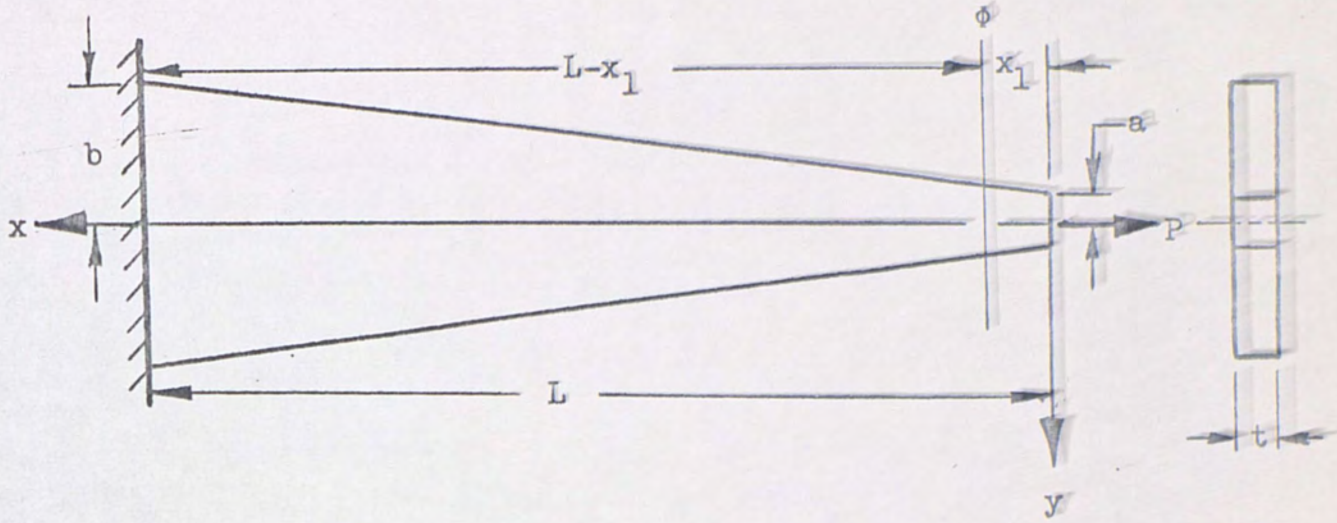



BEAM TAPER EQUATION: $y = \pm[(b-a)\frac{x}{L} + a]$ P = Applied Load

CROSS SECTION AREA: $A(x) = 2ty$

GEOMETRIC RATIOS: $\frac{b}{a} = \frac{6}{1}$, $\frac{a}{L} = \frac{1}{40}$, $\frac{a}{t} = \frac{5}{1}$

RATIOS OF MAJOR ELEMENT DIVISIONS:

Exact Isoenergetics Optimum ($\phi$), $(L-x_1)/L = .92$

Uniform Strain Energy, $(L-x_1)/L = .71$

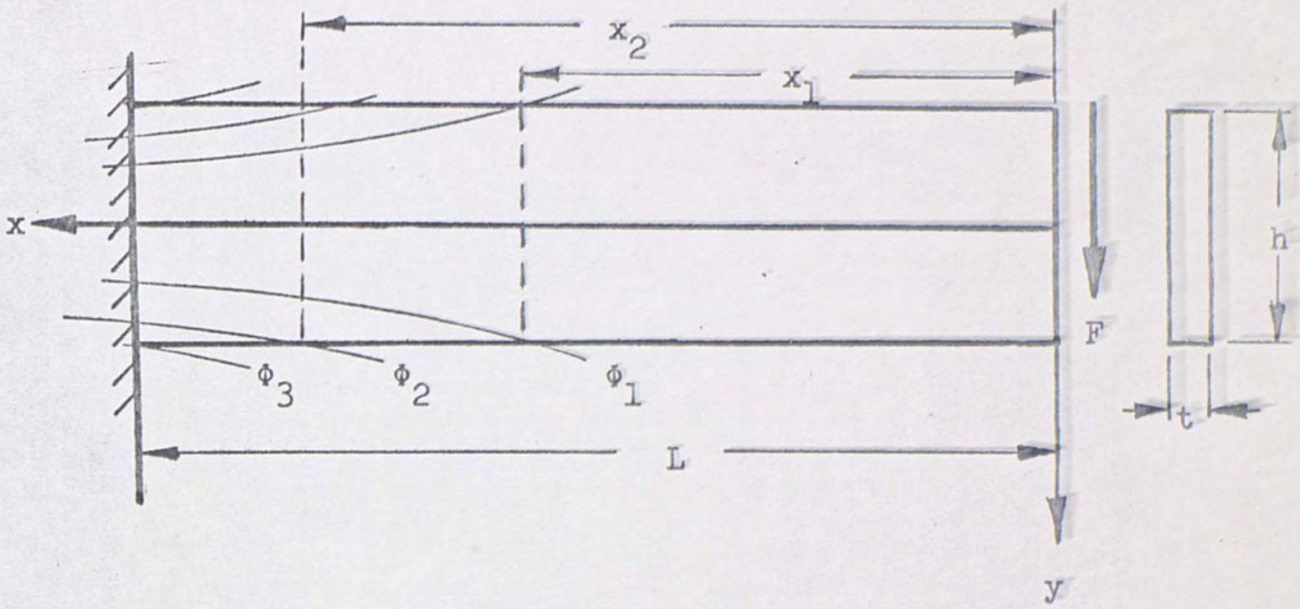
Approximate Optimum From Triangular Formulation,

$(L-x_1)/L = .66$

Approximate of Turcke and McNeice [3],

$(L-x_1)/L = .73$

Figure (1) Tapered Beam Description




GEOMETRIC RATIOS: $\frac{h}{L} = \frac{2}{10}$ , $\frac{t}{h} = \frac{1}{2}$

RATIOS OF MAJOR ELEMENT DIVISIONS:

Exact Isoenergetics Optimum ($\Phi_1$, $\Phi_2$),

$x_1/L = .5774$ , $x_2/L = .8165$

Uniform Strain Energy,

$x_1/L = .6209$ , $x_2/L = .8408$

Approximate Optimum From Rectangular Formulation,

$x_1/L = .6209$ , $x_2/L = .8408$

ISOENERGETIC MAPPING:

$\Phi_c = x^2y^2$

Figure (2) Cantilever Beam Description

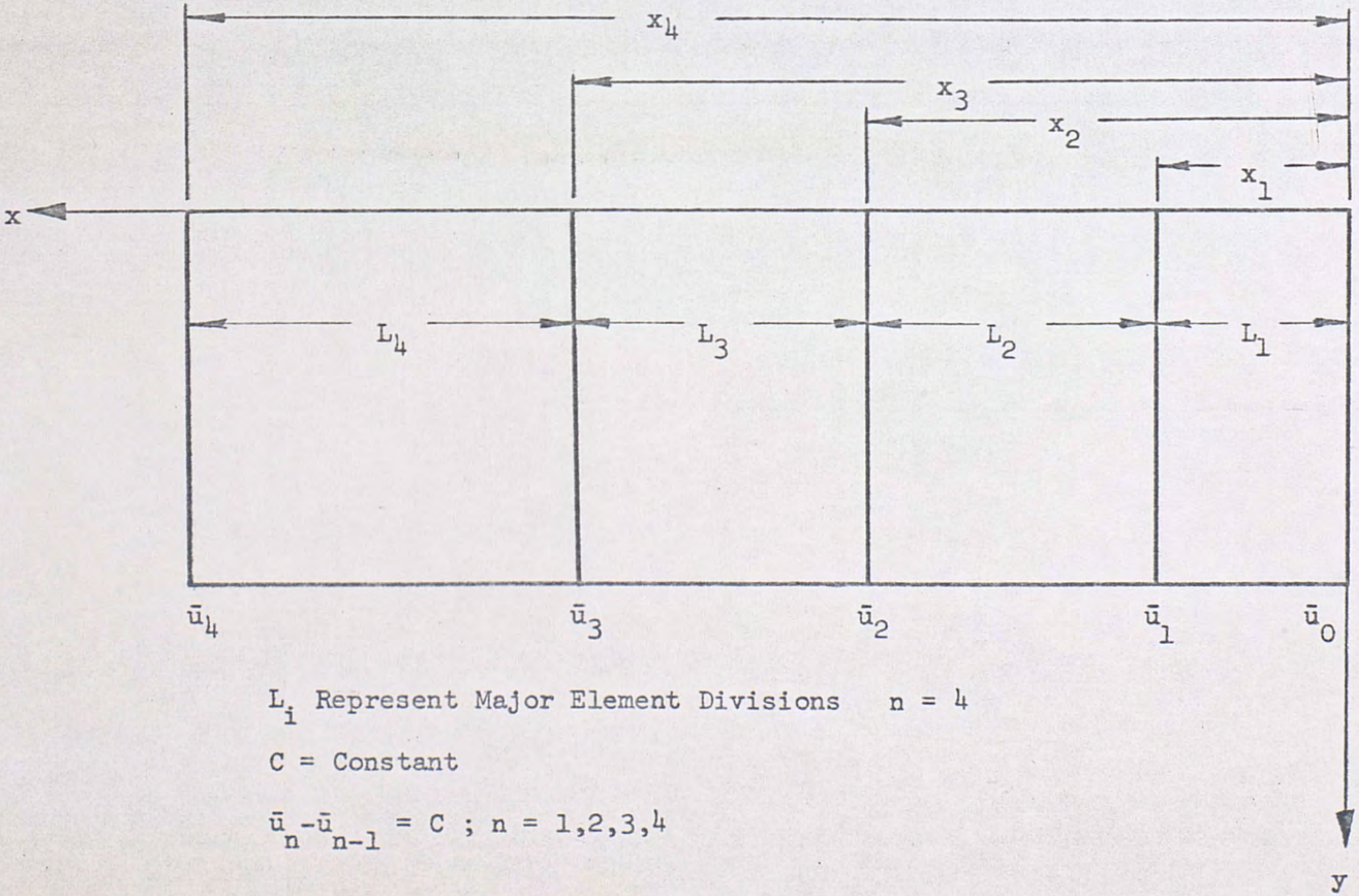
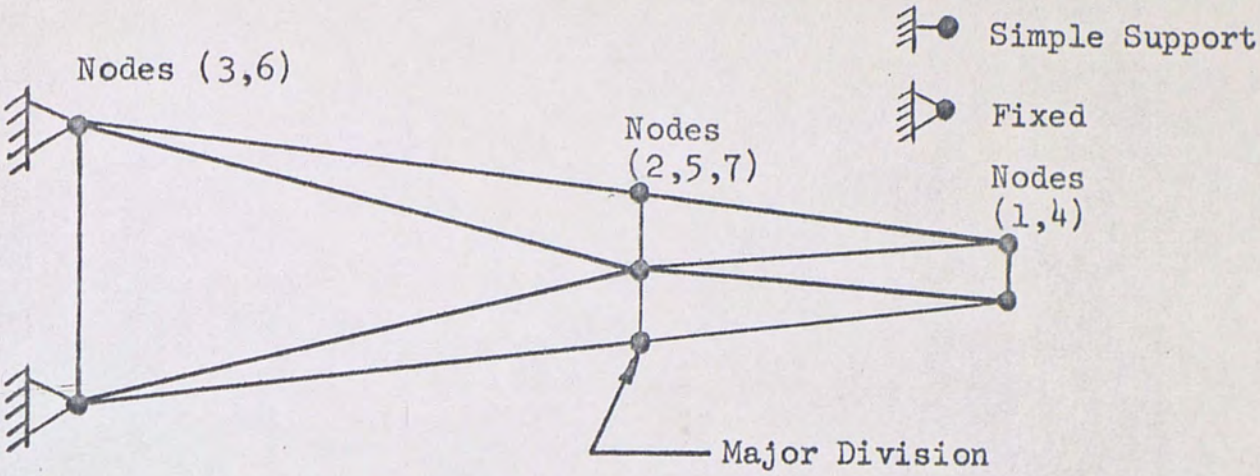
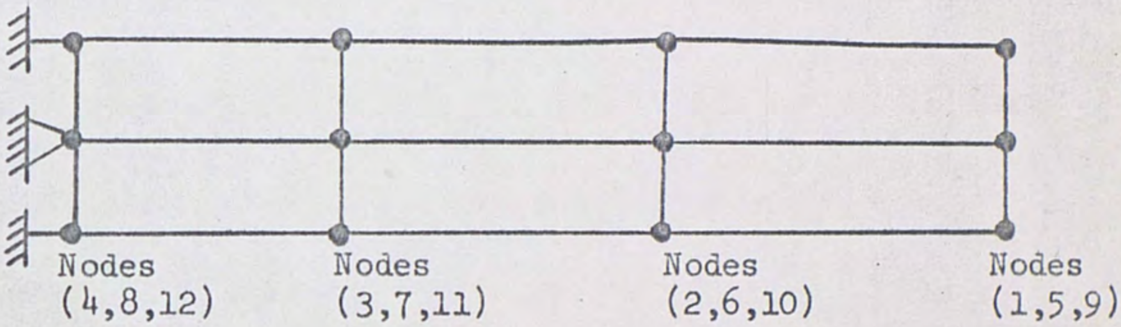


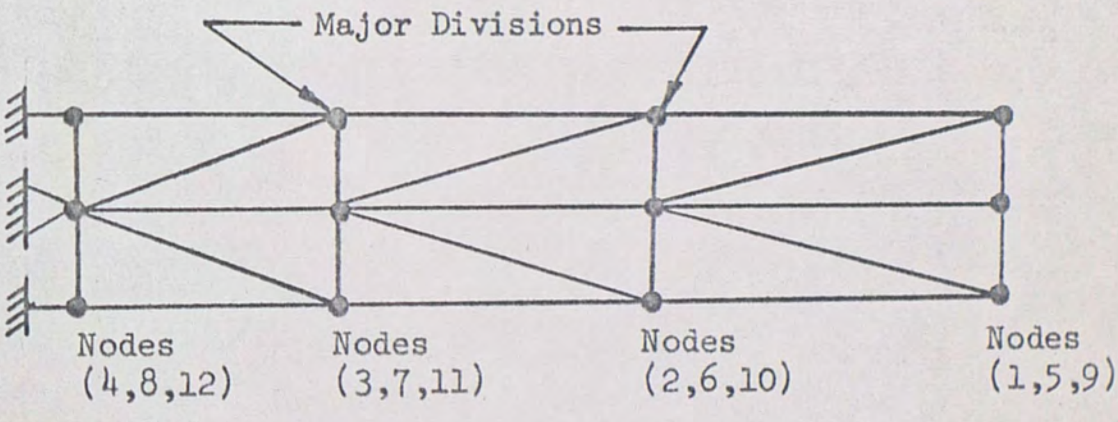

Figure (3) Major Element Divisions For Equipotential Strain Energy Density in a Simple Two Dimensional Problem Space.




Tapered Beam General Grid Arrangement For 6 DOF Triangular Element.

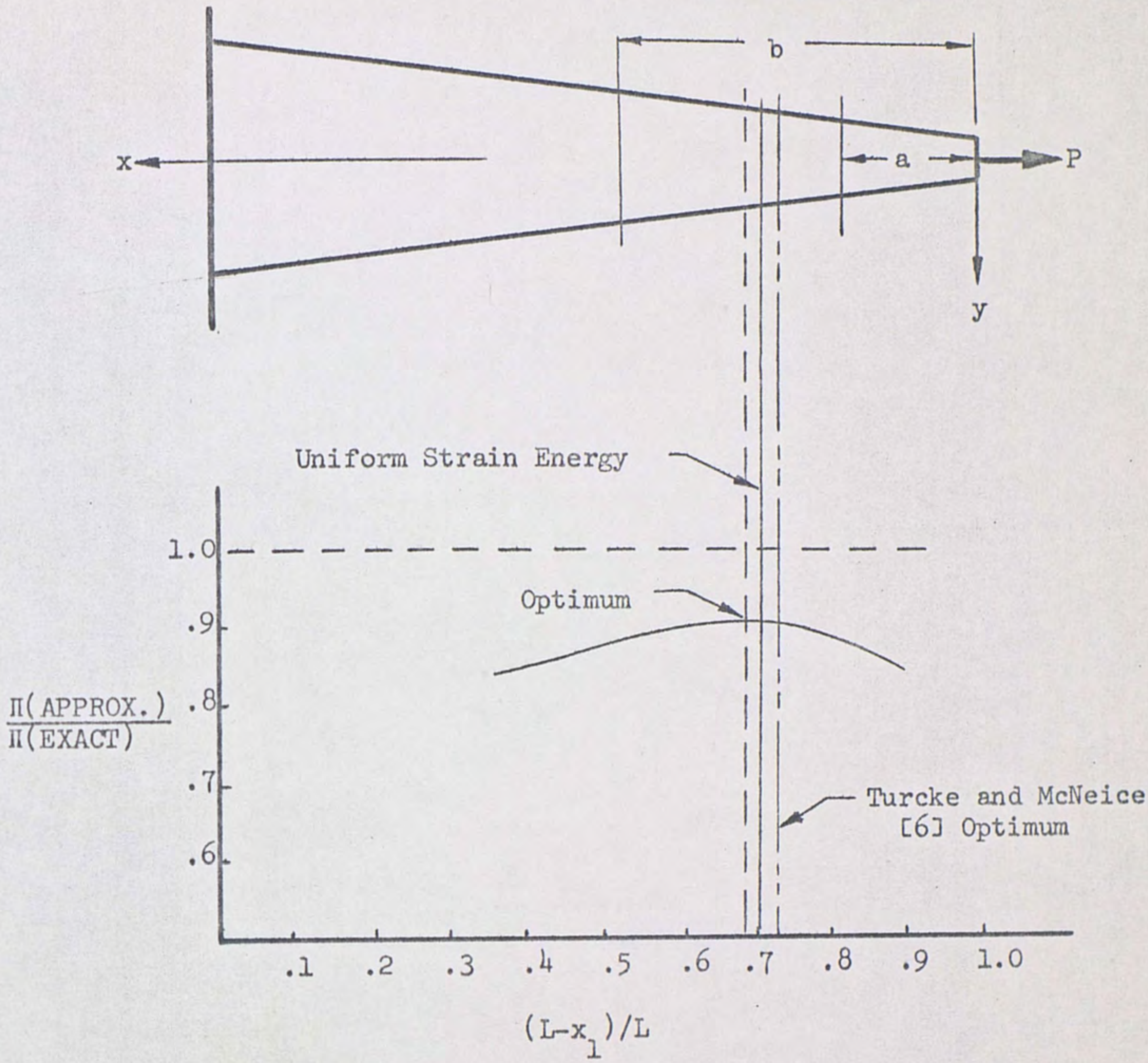



Cantilever Beam General Grid Arrangement For 8 DOF Rectangular Element.




Cantilever Beam General Grid Arrangement For 6 DOF Triangular Element.

Figure (4) Grid Arrangements For the Finite Element Analysis.

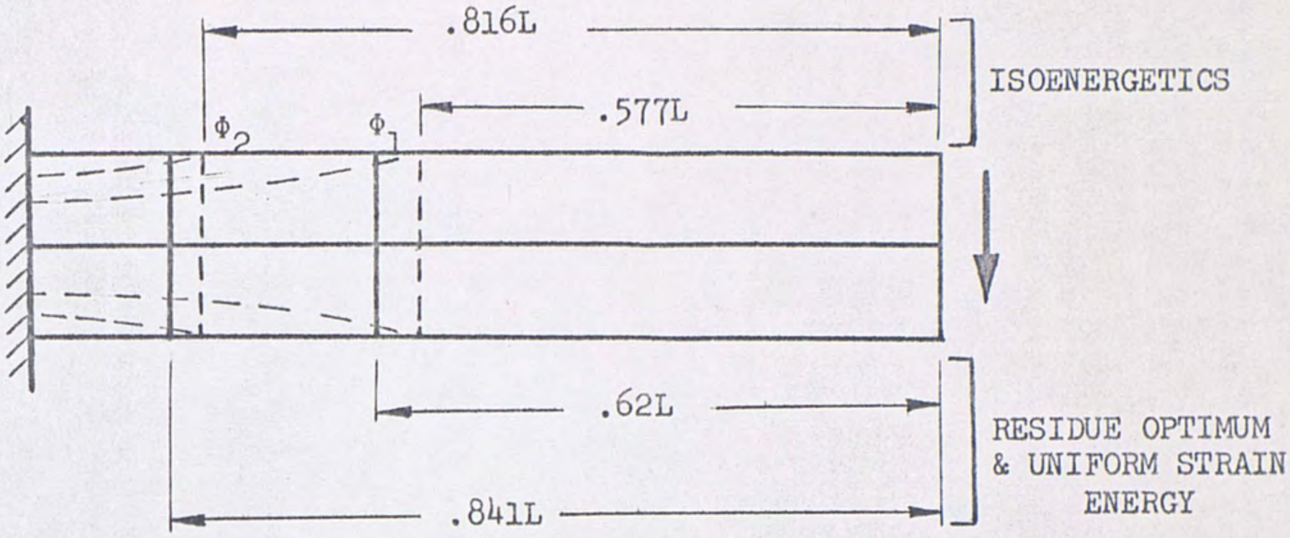



$$\Pi(\text{EXACT}) = -\frac{1}{2}\, Pu(\text{EXACT}) \;, \quad \Pi(\text{APPROX.}) = -\frac{1}{2}\, Pu(\text{APPROX.})$$

Desire Solution For

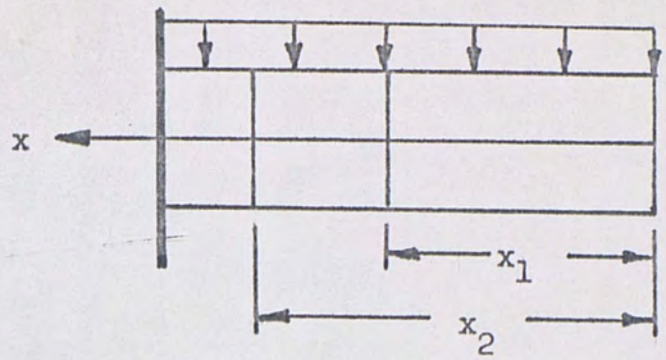
$$\frac{\Pi(\text{APPROX.})}{\Pi(\text{EXACT})} \implies \text{Maximum}$$

Figure (5) Tapered Beam, Triangular Element Optimization.




| | $x_1/L$ | $x_2/L$ | $\frac{\Pi(\text{APPROX.})}{\Pi(\text{EXACT})}$ |
|---|---|---|---|
| 8 DOF RECTANGULAR ELEMENT | .333 | .667 | .4680 |
| | .577 | .816 | .5530 |
| | .621 | .841 | .6173 |
| 6 DOF TRIANGULAR ELEMENT | .600 | .800 | .3596 |
| | .600 | .841 | .3690 |
| | .600 | .900 | .3423 |
| | .621 | .800 | .3606 |
| | .621 | .841 | .3728 |
| | .621 | .900 | .3484 |
| | .680 | .800 | .3514 |
| | .680 | .841 | .3750 |
| | .680 | .900 | .3624 |

Figure (6) $\Pi$(APPROX.)/$\Pi$(EXACT) For the Cantilever Beam.




Cantilever Beam With
Evenly Distributed Load

| PROBLEM SPACE | PREDICTION FOR UNIFORM STRAIN ENERGY | | OPTIMUM FROM THE APPROXIMATE SOLUTION | |
|---|---|---|---|---|
| CANTILEVER BEAM WITH | $x_1/L$ | $x_2/L$ | $x_1/L$ | $x_2/L$ |
| SHEAR END LOAD | .621 | .841 | .621 | .841 |
| EVENLY DISTRIB. LOAD | .8034 | .9239 | .7893 | .9084 |

| PROBLEM SPACE | PREDICTION FOR UNIFORM STRAIN ENERGY | OPTIMUM FROM THE APPROXIMATE SOLUTION |
|---|---|---|
| TAPERED BEAM | $x_1/L$ | $x_1/L$ |
| AXIAL LOAD (ONE DIMENSIONAL) | .29 | .27 [3] |
| AXIAL LOAD (TWO DIMENSIONAL) | .29 | .34 |

Figure (7) Table of Coordinate Optimums Calculated By Uniform Strain Energy Verses Approximate Optimums.

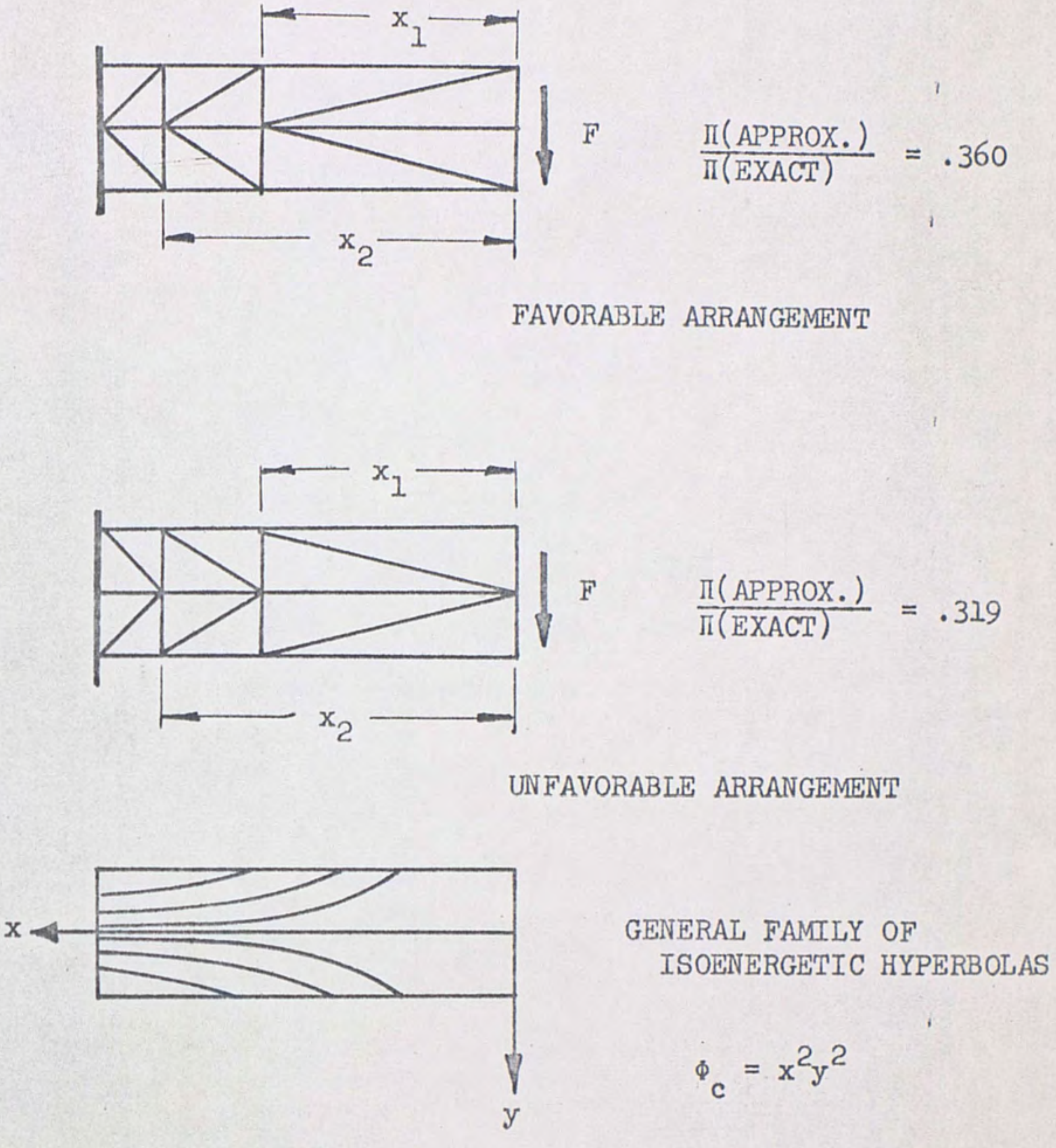



Figure (8) Comparison of Grid Orientation to Isoenergetics For the Cantilever Beam.