The Effect Of Impurities on the Superconductivity of BSCCO-2212

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THE EFFECT OF IMPURITIES ON THE SUPERCONDUCTIVITY OF BI$_2$SR$_2$CACU$_2$O$_{8+\delta}$

by

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A thesis submitted in partial fulfillment of the requirements for the Honors in the Major Program in Physics in the College of Sciences and in the Burnett Honors College at the University of Central Florida

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ABSTRACT

$\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$ is a high-temperature cuprate superconductor whose microscopic behavior is currently poorly understood. In particular, it is unclear whether its order parameter is consistent with s-wave or d-wave symmetry. It has been suggested that its order parameter might take one of several forms that are consistent with d-wave behavior. We present some calculations using the many-body theory approach to superconductivity that suggest that such order parameters would lead to a suppression of the critical temperature in the presence of impurities. Because some experiments have suggested the critical temperature of $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$ is relatively independent of the concentration of impurities, this lends support to the hypothesis that its order parameter has s-wave symmetry.
Dedicated to Future
for constantly motivating me to continue
wherever I’m at
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CHAPTER 1: INTRODUCTION

The word *superconductivity* refers to a remarkable phenomenon observed to occur in a variety of elements and materials. The most immediately striking feature of superconductivity is that materials that exhibit it have *zero* electrical resistance below a characteristic temperature (hence the name). The next most obvious feature is the *complete expulsion* of magnetic field lines below that temperature; this is known as the *Meissner effect*, after Walter Meissner, one of the physicists that discovered it.

Elements and materials that exhibit superconductivity under some conditions are called *superconductors*. Interestingly, it is currently unclear what can and what cannot be a superconductor; metals like lead can go *superconducting* (exhibit superconductivity in some circumstance), but so can organic compounds like fullerenes [10] and nonmetallic compounds like hydrogen sulfide [3]. In fact, hydrogen sulfide—at extremely high pressures and below a certain temperature—was recently found to exhibit superconductivity at temperatures higher than anything ever examined before!

The *mechanism* for superconductivity—the microscopic properties of a superconductor that cause it to behave the way it does—is only well understood in some materials. It is thought to be most understood in so-called *conventional* superconductors, which are well-described by the *BCS theory of superconductivity*. The BCS theory—named for John Bardeen, Leon Cooper, and John Schrieffer—holds that the formation of *Cooper pairs* is responsible for superconductivity. A Cooper pair is a pair of electrons bound together by some attractive interaction. In most cases, it is thought that this interaction arises from the interaction between free electrons and the mate-
On the other hand, there exist *unconventional* superconductors which are for one reason or another not well-described by BCS theory (or some extension of it). Superconductivity in the *cuprates* (materials that contain copper anions), for example, tends to involve an amount of anisotropy that is uncharacteristic of BCS superconductors. For these superconductors, it is unclear if the mechanism is the same as in BCS theory, or if it is seriously different.

In the case of unconventional superconductors, the apparatus of *many-body theory* has proven to be a fruitful means of attacking the problem of understanding their microscopic behavior [2]. Many-body theory involves reducing the completely intractable problem of accounting for the detailed interactions between huge numbers of particles to the much more manageable (but still often very difficult!) problem of understanding the system using some reasonable average properties. It is well-suited to attacking many different kinds of condensed matter problems, and has the calculational advantage that everything can be done perturbatively.

In this thesis, the plan is to use the many-body formalism for superconductivity—in particular, as outlined by Abrikosov, Gorkov, and Dzyaloshinski in [2]—to examine some properties of $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$, a layered cuprate superconductor which has proven somewhat controversial [6]. The properties of interest, as well as the root of the controversy, will be explained briefly in the next few sections.
Layered Superconductors and Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$

Superconductivity was first discovered by Kamerlingh Onnes in 1911 when a mercury wire was cooled to below 4.2 K while immersed in liquid helium [1]. At that temperature, mercury goes superconducting, and has the strange and wonderful properties described previously. In general, we refer to the temperature at which a material goes superconducting as its critical temperature. For metals like mercury—the only superconductors known for many decades after the discovery of superconductivity—critical temperatures are less than 30 K.

However, it turns out that materials with higher critical temperatures can be manufactured by putting superconductors and other compounds together in special ways. Superconductors made by layering materials are, quite predictably, called layered superconductors, and can have critical temperatures as high as 134 K [7]; that is a significant jump! Creating layered superconductors with higher and higher critical temperatures was once thought to be the only real way to obtain a room temperature superconductor (one with a critical temperature roughly equal to room temperature); however, the discovery of superconductivity in hydrogen sulfide may have open up a new route.

One fairly typical family of layered superconductors is Bismuth strontium calcium copper oxide, or BSCCO. These superconductors are created by stacking CuO$_2$ layers (which are conducting) with BaO, SrO, and Ca layers in different ways. BSCCO and Yttrium barium copper oxide (YBCO) are two of the most studied unconventional superconductors, and are the most studied of the cuprates.

BSCCO is in general not a stoichiometric material [8] [4]; its stoichiometry reads Bi$_2$Sr$_2$Ca$_{n-1}$Cu$_n$O$_{2n+4+x}$. 
where \( n \) is an integer, and \( x \) is a small number. Its layers are highly incommensurate, which is responsible for the tunneling between layers being completely incoherent! In other words, an electron will scatter thousands of times when it moves from one layer to another, causing the direction of its motion in the new layer to be completely random.

The most studied variant of BSCCO corresponds to the choice \( n = 2 \) in the stoichiometry above, which means that we are considering \( \text{Bi}_2\text{Sr}_2\text{Ca}_1\text{Cu}_2\text{O}_{8+x} \). Its lattice structure is pictured in (1). It is commonly abbreviated as BSCCO-2212, and stoichiometries that correspond to other values of \( n \) are abbreviated similarly.

![The composition and lattice structure of BSCCO.](image)

The critical temperatures of BSCCO with \( n = 1 \), \( n = 2 \), \( n = 3 \), and \( n = 4 \) are 2 K, 95 K, 108 K, and 104 K respectively \([7]\). These temperatures are clearly much higher than the critical temperatures of normal metals. The lack of applicability of the BCS theory suggests that it is possible that the mechanism for superconductivity in these materials might be a little—or perhaps radically—different. This key question is the source of the controversy.
In condensed matter, the *symmetry* of a material says a great deal about its properties. The particular symmetries we are interested in are the rotations and flips that leave a unit cell of the material invariant. Mathematically, we are interested in elements of the dihedral group [11].

As we will see, the *superconducting order parameter* is a temperature-dependent quantity that characterizes superconductivity. Its physical interpretation is that its magnitude is the difference in energy between the ground state of the superconductor and the lowest possible energy of a quasiparticle excitation. Its magnitude is also known as the *superconducting gap*, and is directly related to (and can be used to calculate) the critical temperature [2].

It turns out that the order parameter must possess the same symmetry as the superconductor, so that only certain order parameters are allowed. In the case of BSCCO-2212, this includes $s$, $d_{x^2-y^2}$, $d_{xy}$, and $g_{xy(x^2-y^2)}$ [5].

Of the order parameters that are allowed, they are considered *compatible* if they agree on the symmetry operations that leave BSCCO invariant. If they are compatible, we can create a new valid order parameter by adding the two together. For example, we can add $s$ and $d_{x^2-y^2}$ order parameters together to obtain an order parameter $\Delta'(p) = \Delta_1 + \Delta_2 \cos(2\phi)$, where $\Delta_1$ and $\Delta_2$ are constants that are not necessarily the same. If two order parameters are *not* compatible, we can put them together by adding an $i$ to one of them. For example, we can create a valid order parameter by adding $d_{x^2-y^2}$ and $d_{xy}$, which reads $\Delta''(p) = \Delta_1 \cos(2\phi) + i\Delta_2 \sin(2\phi)$. If an order parameter is valid, it can only definitively be ruled out by experiment.
If the order parameter contains two components with different magnitudes $|\Delta_1|$ and $|\Delta_2|$, the presence of the second component introduces another critical temperature in addition to the original.

If $|\Delta_1| > |\Delta_2|$, then there is a temperature $T_{c1}$ at which both $\Delta_1 = \Delta_2 = 0$, and another temperature $T_{c2}$ at which only $\Delta_2 = 0$. Between these temperatures, only $\Delta_1$ is nonzero; below these temperatures, both are nonzero. If the two components have the same magnitude, as before, there is only one critical temperature.

Experiments do not seem to indicate the presence of two distinct critical temperatures, casting doubt on the idea that there are two components to the order parameter. If there are two components, it must be that either the two critical temperatures are immeasurably close to one another, or that the lower one is extremely low.
CHAPTER 2: Our Model

The Green Function Approach

Systems of interest to condensed matter are generally very low-energy; this means that relativistic effects can be safely ignored, and the dynamics of the system can in principle be completely described by nonrelativistic quantum mechanics. However, the quantum mechanics of more than two interacting bodies very quickly becomes computationally intractable, and suggests that a good approximation—rather than trying to solve the Schrödinger equation exactly for each particle in the system—is the way forward.

For this reason, and for the reason that it is necessary to have a model that takes the creation and annihilation of particles into account (in quantum mechanics, the number of particles is essentially fixed), the quantum field-theoretic methods of many-body theory are a standard approach to the microscopic theory of superconductivity.

In The Green function approach, the interesting properties (thermodynamic and otherwise) of a system are encoded in its Green functions; in fact, in the mean field approximation, the system is completely described by them. The Green functions of interest are solved exactly for the bare Hamiltonian, and perturbatively via Dyson’s equation for the interaction Hamiltonian in terms of the bare Green functions. Once the Green functions for the full Hamiltonian are calculated, they can be used to calculate any physical quantity (specific heat, entropy, the superconducting gap, the critical temperature, and so on) of interest.
It is worth noting that the Green functions of interest in many-body theory are not the same as the Green functions encountered in the study of linear differential equations or in ordinary quantum mechanics. They are referred to as Green functions because they are a generalization of the ones encountered in ordinary quantum mechanics, and not because they have the same properties.

Green functions and the perturbative calculation method were borrowed from high-energy theory, where they were used to calculate the detailed properties of particles under the influence of various interactions. In most cases, systems of interest to high-energy are (appropriate) at high energies, but also at zero temperature. In other words, because the interesting properties of systems under consideration were due to their high energies, the effects of a finite (nonzero) temperature could be safely neglected.

In our case, taking the system’s finite temperature into account is absolutely crucial, which means that the Green functions and diagram technique must be suitably modified. This can be done using the method of Matsubara frequencies, named for physicist Takeo Matsubara, who first introduced it.

The program of the many-body theory approach, then, is as follows:

1. Write down a Hamiltonian that is thought to describe the system in its second-quantization representation. This will be separated into a bare part and an interaction part, where the bare part should be able to be easily solved exactly.

2. Use the Heisenberg representation equations of motion of \( \psi \) and \( \psi^+ \) to derive equations of motion for the bare Green functions (the Green functions that correspond to only the bare
Hamiltonian). Solve for them exactly using Dyson’s equation if an exact solution is possible.

3. Using Dyson’s equation, write the interaction Green functions (the Green functions that correspond to the \textit{entire} Hamiltonian) in terms of the original Green functions. Because the resulting equations are probably not exactly solvable, solve them to some first approximation.

4. If working at a \textit{finite} temperature $T$, replace integrals over $\omega$ (the Fourier-transformed time variable) in momentum space with \textit{sums} over Matsubara frequencies $i\omega_n$, where $\omega_n \equiv (2n + 1)\pi T$ and the sum runs over all (positive, negative, and zero) integers. That is, make the correspondence

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega) \, d\omega \rightarrow T \sum_{n=-\infty}^{\infty} f(i\omega_n).$$

(2.1)

Usually we will write the more compact $\sum_{\omega_n}$ instead of $\sum_{-\infty}^{\infty}$ to ease notation. Note that the Matsubara sum agrees with the integral in the $T \rightarrow 0$ limit.

In the next sections we will introduce our Hamiltonian and give a sketch of how the steps above will work. Step 3 will not be done in full until the next chapter, because different integrals must be calculated for different order parameters.

Our work exactly follows [2].
Choice of Hamiltonian

In BCS theory, we recall that the formation of Cooper pairs due a phonon-mediated effective attraction between electrons is the mechanism principally responsible for superconductivity. Hence, our interaction Hamiltonian will include an attractive potential. For convenience, we will choose the simplest possible interaction: one that is constant when two particles are ‘close’ enough to each other to interact. Because this involves the creation of pairs of particles, this term will involve four $\psi$ operators (in its second quantization representation) rather than the two required for a two-particle interaction.

The electrons near the Fermi energy of a material (the electrons that are free or close to it) are the ones that determine its properties, and are the ones that interact with one another. Hence, we will include a cutoff that says only electrons near the Fermi surface will interact via this attractive interaction.

Also, working within a given layer of BSCCO-2212 is an effectively two-dimensional problem, so we will work in two dimensions. In a more complicated (and more accurate) model, we would take tunneling between adjacent layers into account; however, we will neglect tunneling here for simplicity.

In the second quantization representation, the effective interaction Hamiltonian $H_e$ of one electron with another (neglecting impurities) is given by

$$H_e = \frac{\lambda}{2(2\pi)^2} \sum_{p_1+p_2=p_3+p_4} a_{p_1\sigma_1}^\dagger a_{p_2\sigma_2}^\dagger a_{p_3\sigma_2} a_{p_4\sigma_1} \theta_{p_1} \theta_{p_2} \theta_{p_3} \theta_{p_4},$$
where $\lambda$ dictates the strength of our interaction, $a_{p_1\sigma_1}^\dagger$ is a creation operator that creates an electron with momentum $p_1$ and spin $\sigma_1$, $a_{p_2\sigma_2}^\dagger$ is a creation operator that creates an electron with momentum $p_2$ and spin $\sigma_2$, $a_{p_3\sigma_2}$ is an annihilation operator that creates an electron with momentum $p_3$ and spin $\sigma_2$, $a_{p_4\sigma_1}$ is an annihilation operator that creates an electron with momentum $p_4$ and spin $\sigma_1$, and $\theta_{p_i}$ are cutoff factors that are zero outside of width $2\omega_D$ (where $\omega_D$ is the Debye frequency) away from the Fermi surface.

Figure 2.1: Two electrons interacting with one another via the phonon-created effective attraction.

In the coordinate representation, the effective interaction Hamiltonian $H_e$ can be written

$$H_e = \frac{\lambda}{2} \int \psi_\alpha^\dagger(x)\psi_\beta^\dagger(x)\psi_\beta(x)\psi_\alpha(x)d^2r,$$

where $x \equiv (\tau, \mathbf{r})$, and the cutoff factors are understood to be included despite not explicitly being written. Note that the usual fermionic equal time commutation relations

$$\{\psi_\alpha(r), \psi_\beta^\dagger(r')\} = \delta_{\alpha\beta}\delta(\mathbf{r} - \mathbf{r}')$$

$$\{\psi_\alpha(r), \psi_\beta(r')\} = 0$$

$$\{\psi_\alpha^\dagger(r), \psi_\beta^\dagger(r')\} = 0$$

are obeyed.
We will assume that spin-dependent interactions are negligible from here forward, and suppress all spin indices. Then $H_e$ becomes simply

$$H_e = \frac{\lambda}{2} \int \psi_\alpha^\dagger(x) \psi_\beta^\dagger(x) \psi_\beta(x) \psi_\alpha(x) d^2\mathbf{r}. \quad (2.2)$$

We must also include the effect of scattering by impurities. In a real material, a great variety of processes fall under the umbrella of ‘scattering by impurities’, and the result in general depends on geometry involved and the particular character of the impurities. In a mean field approximation, however, we can account for the average effect of scattering by an impurity at the position $\mathbf{r}_a$ using a scattering potential $u(\mathbf{r} - \mathbf{r}_a)$. In the coordinate representation, this part of the interaction takes the form

$$H_a = \int u(\mathbf{r} - \mathbf{r}_a) \psi^+(x) \psi(x) \ d^2\mathbf{r}. \quad (2.3)$$

To account for scattering by each impurity in the material, we simply sum over $a$, so that the full Hamiltonian is given by

$$H = H_0 + H_e + \sum_a H_a \quad (2.4)$$
$$= \int -\left( \frac{\nabla^2}{2m} \psi(x) \right) + \frac{\lambda}{2} \psi^\dagger(x) \psi^\dagger(x) \psi(x) \psi(x) + \sum_a u(\mathbf{r} - \mathbf{r}_a) \psi^+(x) \psi(x) \ d^2\mathbf{r}, \quad (2.5)$$

where $H_0$ is the Hamiltonian of a free particle.

Green Functions In A Pure Superconductor

We will take our bare Hamiltonian to be $H_0 + H_e$: everything but the impurity-dependent part, since it complicates the problem of finding the system’s Green functions immensely. In this sec-
tion, we will simply sketch the results for the Green functions in this case (i.e. in the case of a pure superconductor). These calculations can be found in more detail in [2].

Green Functions At $T = 0$

At zero temperature, the electron Green function is defined as

$$G(x, x') = -i \langle T(\psi(x), \psi^+(x')) \rangle,$$

where we now think of $\psi$ and $\psi^+$ in the Heisenberg picture (where operators carry the time-dependence), $T$ is the time-ordering operator, and we are averaging our the ground state of the system (in the $T \neq 0$ case, we use the Gibbs average, which involves $T$ and $\mu$). The Heisenberg equations of motion of $\psi$ and $\psi^+$ are given by

$$\left(\frac{i}{\partial t} + \frac{\nabla^2}{2m}\right) \psi(x) - \lambda(\psi^+(x)\psi(x))\psi(x) = 0 \quad (2.6)$$

$$\left(\frac{i}{\partial t} - \frac{\nabla^2}{2m}\right) \psi^+(x) + \lambda \psi^+(x)(\psi^+(x)\psi(x)) = 0. \quad (2.7)$$

From (2.6), we obtain

$$\left(\frac{i}{\partial t} + \frac{\nabla^2}{2m}\right) G(x, x') + i\lambda \langle T((\psi^+(x)\psi(x))\psi(x)) \rangle = \delta(x - x'), \quad (2.8)$$

the equation of motion for $G(x, x')$. Expanding the time-ordered product using Wick’s theorem and making some approximations, we find that we can write it more evocatively as

$$\left(\frac{i}{\partial t} + \frac{\nabla^2}{2m}\right) G(x - x') - i\lambda F(0+)F^+(x - x') = \delta(x - x'), \quad (2.9)$$
where we have defined

\[ F^+(x - x') \equiv e^{2i\mu t} \langle N | T(\psi(x)\psi(x')) | N + 2 \rangle \]  
\[ F(x - x') \equiv e^{-2i\mu t} \langle N + 2 | T(\psi(x)\psi^+(x')) | N \rangle \]  
\[ F(0+) \equiv \lim_{r \to r', t \to t'} F(x - x'), \]  

and where we have also noted that, by symmetry, \( G(x, x') \) should depend more specifically on the difference of \( x \) and \( x' \).

Just as \( G(x - x') \) is the electron Green function, the so-called anomalous Green functions (or Gor’kov Green functions) \( F(x - x') \) and \( F^+(x - x') \) represent Green functions for the system’s Cooper pairs. Because we expect the pairs to base like single particles (bosons, in fact, since the electron pair should be in the singlet spin state), having Green functions for them makes physical sense.

From (2.7), we can find a similar equation of motion for \( F^+(x - x') \):

\[ \left( i \frac{\partial}{\partial t} - \frac{\nabla^2}{2m} - 2\mu \right) F^+(x - x') + i\lambda F^+(0+)G(x - x') = 0. \]  

In momentum space (where \((t, r) \to (\omega, p)\)), the equations of motion for \( G(x - x') \) and \( F^+(x - x') \) read

\[ \left( \omega - \frac{p^2}{2m} \right) G(p) - i\lambda F^+(0+) F^+(p) = 1 \]  
\[ \left( \omega + \frac{p^2}{2m} - 2\mu \right) F^+(p) + i\lambda F^+(0+) G(p) = 0. \]
To make these equations more symmetric, introduce the variable $\xi$, defined by

$$\xi \equiv \frac{p^2}{2m} - \mu,$$

and introduce the modified frequency $\omega'$, defined by

$$\omega' \equiv \omega - \mu.$$

In terms of $\omega'$ and $\xi$ instead of $\omega$, $p$, and $\mu$, these equations read

$$\begin{align*}
(\omega' - \xi) G(p) - i\lambda F(0+) F^{+}(p) &= 1 \\
(\omega' + \xi) F^{+}(p) + i\lambda F^{+}(0+) G(p) &= 0.
\end{align*}$$

According to our definition, $\xi$ ranges from $-\mu$ to $\infty$; however, because of the largeness of $\mu$ for a macroscopic system like a superconductor, we will take $\xi$ to range from $-\infty$ to $\infty$ in our subsequent integrations.

In what follows, for notational simplicity, we will drop the prime and write $\omega$.

Solving (2.18) and (2.19) for $G(x - x')$ and $F^{+}(x - x')$, we obtain

$$\begin{align*}
G(p) &= \frac{\omega + \xi}{\omega^2 - \xi^2 - |\Delta_0|^2} \\
F^{+}(p) &= \frac{\Delta_0}{\omega^2 - \xi^2 - |\Delta_0|^2},
\end{align*}$$

where we have defined $\Delta_0 \equiv i\lambda F^{+}(0+)$. The choice of notation is deliberate; it turns out $|\Delta_0|$ is the magnitude of our superconducting gap.
Green Functions At $T > 0$

At finite temperatures, instead of a time $t$, we work with a so-called imaginary time $\tau \equiv it$, which ranges from 0 to $i/T$. When we make the appropriate change of variables and take $T \to 0$, this reduces back to a variable $t$ that ranges from 0 to $\infty$.

Analogous to the S-matrix from the usual quantum field theory, we can define a matrix $S(\tau)$. In the interaction representation, with $H$ the full Hamiltonian and $H_0$ the bare Hamiltonian, it is defined by

\begin{align}
e^{-\left(H-\mu N\right)\tau} &= e^{-\left(H_0-\mu N\right)\tau}S(\tau) \\
e^{\left(H-\mu N\right)\tau} &= S^{-1}(\tau)e^{\left(H_0-\mu N\right)\tau},
\end{align}

where $\mu$ is the chemical potential and $N$ is the number operator.

At finite temperatures, we have an analog of averaging over the ground state of the system: the so-called Gibbs average, which we define by

$$\langle \ldots \rangle \equiv \text{Tr}(e^{\left(\Omega_0+\mu N-H_0\right)/T} \ldots),$$

where $\Omega_0$ is the thermodynamic potential in the absence of any interaction.
In this case, the electron Green function is defined as
\[ G(x, x') = -\frac{T_\tau(\psi(x), \psi^+(x'))S(1/T)}{\langle S(1/T) \rangle}, \]
where \( x \equiv (\tau, r) \). The anomalous Green functions are given by
\[ F(x, x') = \frac{\langle T_\tau(\psi(x), \psi(x'))S(1/T) \rangle}{\langle S(1/T) \rangle}, \tag{2.25} \]
\[ F^+(x, x') = \frac{\langle T_\tau(\psi^+(x), \psi^+(x'))S(1/T) \rangle}{\langle S(1/T) \rangle}, \tag{2.26} \]
and, similarly to before, we eventually find that the equations of motion are
\[ \left( -\frac{\partial}{\partial \tau} + \frac{\nabla^2}{2m} + \mu \right) G(x - x') + \Delta F^+(x - x') = \delta(x - x') \tag{2.27} \]
\[ \left( \frac{\partial}{\partial \tau} + \frac{\nabla^2}{2m} + \mu \right) F^+(x - x') - \Delta^* G(x - x') = 0. \tag{2.28} \]
This time, \( \Delta \equiv |\lambda|F(0+) \) and \( \Delta^* \equiv |\lambda|F^+(0+) \).

In momentum space (where \((\tau, r) \rightarrow (\omega, p)\)), after doing the same change of variables that we did in the zero temperature case, the equations of motion for \( G(x - x') \) and \( F^+(x - x') \) read
\[ (i\omega - \xi) G_\omega(p) + \Delta F^+_\omega(p) = 1 \tag{2.29} \]
\[ (i\omega + \xi) F^+_\omega(p) + \Delta^* G_\omega(p) = 0, \tag{2.30} \]
where we have placed an \( \omega \) subscript on our Green functions to emphasize that we will eventually sum over \( \omega \) instead of integrate over it. Also, we will again treat \( \xi \) as ranging from \(-\infty\) to \( \infty \) due to the largeness of \( \mu \).
Solving (2.29) and (2.30) for $G_\omega(p)$ and $F_\omega^+(p)$, we obtain

$$G_\omega(p) = -\frac{i\omega + \xi}{\omega^2 + \xi^2 + |\Delta|^2}$$

(2.31)

$$F_\omega^+(p) = \frac{\Delta^*}{\omega^2 + \xi^2 + |\Delta|^2}$$

(2.32)

As expected, this exactly corresponds to the solutions (REFS) from before, but with the substitution $\omega \to i\omega$.

Green Functions In A Superconducting Alloy

Now we will treat superconducting alloys; that is, we will treat superconductors that contain impurities. Specifically, we are interested in nonmagnetic impurities: impurities that do not affect the magnetic behavior of the superconductor. Because a superconductor need not contain a metal (hydrogen sulfide being a big example), calling superconductors with impurities ‘alloys’ is probably a misnomer. However, we will refer to them as such in alignment with [2].

After doing diagrams with $\sum_a H_a$ as our interaction Hamiltonian and picking the ones that contribute most (the so-called uncrossed impurity diagrams), we end up with a system of equations that mirrors (2.29) and (2.30):

$$\quad (i\omega - \xi - \overline{G_\omega})G(p) + (\Delta(p) + \overline{F_\omega})F^+(p) = 1$$

(2.33)

$$\quad (i\omega + \xi + \overline{G_\omega})F^+(p) + (\Delta^*(p) + \overline{F_\omega^+})G(p) = 0,$$

(2.34)
where
\[
\overline{G}_\omega = \frac{n}{(2\pi)^2} \int |u(p - p')|^2 G(p') \, d^2p' \tag{2.35}
\]
\[
\overline{F}^+_\omega = \frac{n}{(2\pi)^2} \int |u(p - p')|^2 F^+(p') \, d^2p' \tag{2.36}
\]
and we have used \( p \equiv (\omega, p) \) as shorthand. Note that the functions \( G(p) \) and \( F^+(p) \) are not the same as before even though our notation is very similar!

The general solution to this system of equations (in terms of itself: each function contains integrals of itself!) is
\[
G(p) = -\frac{i\omega - \overline{G}_\omega + \xi}{-(i\omega - \overline{G}_\omega)^2 + \xi^2 + |\Delta^* + \overline{F}^+_\omega|^2} \tag{2.37}
\]
\[
F^+(p) = \frac{\Delta^* + \overline{F}^+_\omega}{-(i\omega - \overline{G}_\omega)^2 + \xi^2 + |\Delta^* + \overline{F}^+_\omega|^2}. \tag{2.38}
\]

It is clear that \( G(p) \) and \( F^+(p) \) reduce to the previous result in the absence of scattering; taking \( \overline{G}_\omega \to 0 \) and \( \overline{F}^+_\omega \to 0 \), we get exactly what we did before. However, this correspondence can be made deeper. Suppose that we define \( \eta_{\omega_1} \) by
\[
i\omega - \overline{G}_\omega = i\eta_{\omega_1}\omega, \tag{2.39}
\]
and \( \eta_{\omega_2} \) by
\[
\Delta^* + \overline{F}^+_\omega = \eta_{\omega_2}\Delta^*. \tag{2.40}
\]

Then it is clear in some sense that the effect of scattering is to modify \( \Delta \) and \( \omega \) to new values \( \eta_{\omega_2}\Delta \) and \( \eta_{\omega_1}\omega \); defining \( \tilde{\Delta} \equiv \eta_{\omega_2}\Delta \) and \( \tilde{\omega} \equiv \eta_{\omega_1}\omega \) allows us to right our Green functions exactly like
before, but with modified quantities:

\[
G(p) = -\frac{i\tilde{\omega} + \xi}{\tilde{\omega}^2 + \xi^2 + |\Delta|^2} \\
F^+(p) = \frac{\tilde{\Delta}^*}{\tilde{\omega}^2 + \xi^2 + |\Delta|^2}.
\]

These equations are highly nonlinear and difficult (or perhaps impossible) to solve exactly for a reasonably general scattering potential. To first approximation, we can use the bare Green functions (which account for only the attractive electron-electron interaction) in the calculation of \(G\) and \(F^+\), so that

\[
G_{\omega} \approx -\frac{n}{(2\pi)^2} \int |u(p - p')|^2 \frac{i\omega + \xi}{\omega^2 + \xi^2 + |\Delta|^2} d^2p' \\
F^+_{\omega} \approx \frac{n}{(2\pi)^2} \int |u(p - p')|^2 \frac{\Delta^*}{\omega^2 + \xi^2 + |\Delta|^2} d^2p'.
\]

Once these quantities are calculated, we can modify \(\Delta\) to \(\tilde{\Delta}\) and \(\omega\) to \(\tilde{\omega}\), and then use the BCS self-consistency condition (discussed in the next section) to analyze what happens to the superconducting gap and critical temperatures.

In the next section, we will do exactly that for several choices of order parameters.

The BCS Self-Consistency Condition

Consider a superconductor in \(n\) space dimensions with a pairing interaction \(V(p, p')\), order parameter \(\Delta(p, T)\), and anomalous Green function \(F^+(p)\) (which one we use depends on whether there
are impurities are not). Then the BCS self-consistency condition \[1\] reads

\[
\Delta^*(p) = \frac{T}{(2\pi)^n} \sum_{\omega_n} \int V(p, p') F'^*(p') \, d^n p',
\]

where \(T\) is temperature. This condition is important because calculating the integral and sums on the right-hand side yield an expression for \(|\Delta(T)|\) (the \(p\)-independent magnitude of the order parameter) in terms of temperature. From this expression, the dependence of the gap on the critical temperature \(T_c\), as well as \(T_c\) itself, can be calculated.

Importantly, by using this condition both within the presence of impurities and not in the presence of impurities, we can calculate the suppression of \(T_c\) due to the presence of impurities. It is this purpose that we will mostly be interested in.
CHAPTER 3: Results

General Considerations

In the sections that follow, we will impose a pairing interaction and order parameter, calculate the adjustment to the Green functions due to scattering, and attempt to analyze the suppression of $T_c$. We will assume that the square of our scattering potential $u(p - p')$ can be written

$$|u(p - p')|^2 = u_0^2 + u_1^2 \cos(\phi_{pp'}) + u_2^2 \cos(2\phi_{pp'}),$$

(3.1)

where $u_0$, $u_1$, and $u_2$ are real constants, and we have defined $\phi_{pp'} \equiv \phi - \phi'$ for notational convenience. This has experimental precedent, and can be thought of in terms of the Born approximation’s expansion including an infinite series of $P_n(\cos(\phi_{pp'}))$ terms [9].

s-wave Order Parameter Calculations

Suppose that our order parameter $\Delta(p, T)$ takes the form

$$\Delta(p, T) = \Delta(T),$$

(3.2)

where $\Delta(T)$ is a complex-valued function of the temperature $T$. In what follows, it will be understood that the order parameter depends on $T$ even though the dependence will not be written explicitly.
Suppose that the pairing interaction is given by

\[ V(p, p') = \lambda, \lambda \in \mathbb{R}. \]  

(3.3)

Using the bare Green functions from before, to first order in \( \Delta(T) \),

\[
\overline{F}_\omega^+ = \frac{n}{(2\pi)^2} \int |u(p - p')|^2 F^+(p') d^2 p'
= \frac{nm}{(2\pi)^2} \int_{-\infty}^{\infty} \int_0^{2\pi} |u(p - p')|^2 \frac{\Delta^*}{\omega^2 + \xi^2 + |\Delta|^2} d\phi' d\xi.
\]

Since \( \sin(n\phi') \) and \( \cos(n\phi') \) both integrate to zero over \( 2\pi \) for any integer \( n \), the \( u_1^2 \) and \( u_2^2 \) terms do not contribute. Hence,

\[
\overline{F}_\omega^+ = \frac{nm}{(2\pi)^2} \int_{-\infty}^{\infty} \int_0^{2\pi} \frac{u_0^2 \Delta^*}{\omega^2 + \xi^2 + |\Delta|^2} d\phi' d\xi
= \frac{nmu_0^2 \Delta^*}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\omega^2 + \xi^2 + |\Delta|^2} d\xi
= \frac{nmu_0^2 \Delta^*}{2\pi} \left[ \frac{\tan \left( \xi/\sqrt{\omega^2 + |\Delta|^2} \right) }{\sqrt{\omega^2 + |\Delta|^2}} \right]_{-\infty}^{\infty}
= \frac{nmu_0^2 \Delta^*}{2\pi} \frac{\pi}{\sqrt{\omega^2 + |\Delta|^2}}
= \frac{nmu_0^2 \Delta^*}{2} \frac{1}{\sqrt{\omega^2 + |\Delta|^2}}.
\]

Almost exactly the same way (throwing away the \( \xi \) term from the numerator of \( G_\omega(p') \) since it...
integrates to zero),

\[
\overline{\mathcal{G}_\omega} = \frac{n}{(2\pi)^2} \int |u(p - p')|^2 \mathcal{G}_\omega(p') \, d^2 p'
\]

\[
= -\frac{nm}{(2\pi)^2} \int_{-\infty}^{\infty} \int_0^{2\pi} \frac{u_0^2 i\omega}{\omega^2 + \xi^2 + |\Delta|^2} \, d\phi' \, d\xi
\]

\[
= -\frac{nmu_0^2 i\omega}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\omega^2 + \xi^2 + |\Delta|^2} \, d\xi
\]

\[
= -\frac{nmu_0^2 i\omega}{2} \frac{1}{\sqrt{\omega^2 + |\Delta|^2}}.
\]

Define \( \frac{1}{\tau} \equiv nm u_0^2 \). Then we note that

\[
\Delta^* + \overline{F_\omega} = \eta_\omega \Delta^*
\]  

(3.4)

and

\[
i\omega - \overline{\mathcal{G}_\omega} = i\eta_\omega \omega,
\]  

(3.5)

where

\[
\eta_\omega \equiv 1 + \frac{1}{2\tau \sqrt{\omega^2 + |\Delta|^2}}.
\]  

(3.6)

Hence, the effect of scattering on the Green functions is simply to make the substitutions \( \omega \rightarrow \eta_\omega \omega \) and \( \Delta \rightarrow \eta_\omega \Delta \). For convenience, we will define \( \tilde{\omega} \equiv \eta_\omega \omega \) and \( \tilde{\Delta} \equiv \eta_\omega \Delta \).

Now we will calculate the effect of scattering on \( T_c \) using the BCS self-consistency condition.
Substituting, we have that

\[
\Delta^* = \frac{T}{(2\pi)^2} \sum_{\omega_n} \int V(p, p') \mathcal{F}_{\omega}^+(p') \, d^2p'
\]

\[
= \frac{\lambda T m}{(2\pi)^2} \sum_{\omega_n} \int_{0}^{2\pi} \int_{-\infty}^{\infty} \frac{\tilde{\Delta}^*}{\tilde{\omega}^2 + \xi^2 + |\Delta|^2} \, d\xi d\phi'
\]

\[
= \frac{\lambda T m}{2\pi} \sum_{\omega_n} \int_{-\infty}^{\infty} \frac{\tilde{\Delta}^*}{\tilde{\omega}^2 + \xi^2 + |\Delta|^2} \, d\xi
\]

\[
= \frac{\lambda T m}{2} \sum_{\omega_n} \frac{\tilde{\Delta}^*}{\sqrt{\tilde{\omega}^2 + |\Delta|^2}}
\]

\[
= \frac{\lambda T m}{2} \sum_{\omega_n} \frac{\eta_\omega \Delta^*}{\sqrt{\tilde{\omega}^2 + |\Delta|^2}}
\]

\[
= \frac{\lambda T m}{2} \sum_{\omega_n} \frac{\Delta^*}{\sqrt{\tilde{\omega}^2 + |\Delta|^2}}.
\]

Note that the factors of \(\eta_\omega\), which represented the effect of scattering on the Green functions, completely canceled. Then this condition is completely identical to the condition used to calculate \(T_c\) when there are no impurities; this indicates that there was no suppression, and that it remains exactly the same as before.

We can go ahead and calculate the sum. Dividing both sides by \(\Delta^*\), our self-consistency condition becomes

\[
1 = \frac{\lambda T m}{2} \sum_{\omega_n} \frac{1}{\sqrt{\tilde{\omega}^2 + |\Delta|^2}}. \quad (3.7)
\]

At zero gap (where \(T = T_c\), this reads

\[
1 = \frac{\lambda T_c m}{2} \sum_{\omega_n} \frac{1}{|\omega|}. \quad (3.8)
\]
This sum is divergent, and must be cut off at some finite (but large) frequency $\omega_D$ for us to obtain physical results. If we do this, then $(2n_{\text{max}} + 1)\pi T_c = \omega_D$, so that $n_{\text{max}} = \omega_D / 2\pi T_c - 1/2$. Writing our result in terms of the digamma function $\psi(x)$, which satisfies

$$\psi(x + N + 1) - \psi(x) = \sum_{n=0}^{\infty} \left( \frac{1}{n + x} - \frac{1}{n + x + N + 1} \right) = \sum_{n=0}^{N} \frac{1}{n + x},$$

we have

$$\sum_{\omega_n} \frac{1}{|\omega|} \approx \frac{\omega_D / 2\pi T_c - 1/2}{(2n + 1)\pi T_c} = \frac{2}{\pi T_c} \sum_{n=0}^{\infty} \left( \frac{1}{2n + 1} \right) = \frac{1}{\pi T_c} \sum_{n=0}^{\infty} \left( \frac{1}{n + 1/2} \right) \approx \frac{1}{\pi T_c} \left[ \psi \left( \frac{1}{2} + \frac{\omega_D}{2\pi T_c} \right) - \psi \left( \frac{1}{2} \right) \right]$$

due to the largeness of $\omega_D$. After some algebra, we get that

$$T_c = \frac{2\gamma}{\pi} \omega_D e^{-1/\lambda N(0)}$$

just as in [2], where we have defined the density of states $N(0) \equiv m / 2\pi$. 

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Suppose that our order parameter \( \Delta(p, T) \) takes the form

\[
\Delta(p, T) = \Delta(T)e^{2i\phi},
\]

(3.11)

and that the pairing interaction is given by

\[
V(p, p') = \lambda e^{-2i(\phi - \phi')} , \; \lambda \in \mathbb{R}.
\]

(3.12)

Using the bare Green functions, to first order in \( \Delta(T) \),

\[
\mathcal{F}_\omega^+ = \frac{n}{(2\pi)^2} \int \left| u(p - p') \right|^2 \mathcal{F}_\omega^+(p') \; d^2p'
\]

\[
= \frac{nm}{(2\pi)^2} \int_{-\infty}^\infty \int_0^{2\pi} \left| u(p - p') \right|^2 \frac{\Delta^* e^{-2i\phi'}}{\omega^2 + \xi^2 + |\Delta|^2} \; d\phi' d\xi
\]

\[
= \frac{nm}{(2\pi)^2} \int_{-\infty}^\infty \int_0^{2\pi} \left| u(p - p') \right|^2 \frac{\Delta^*[\cos(2\phi') - i \sin(2\phi')]}{\omega^2 + \xi^2 + |\Delta|^2} \; d\phi' d\xi.
\]

Since \( \sin(m\phi') \) and \( \cos(n\phi') \) are orthogonal on \([0, 2\pi]\) when \( m \) and \( n \) are distinct integers, the \( u_0^2 \), \( u_1^2 \), \( u_{12} \), and \( u_4 \) terms do not contribute. This leaves

\[
\mathcal{F}_\omega^+ = \frac{nm}{(2\pi)^2} \int_{-\infty}^\infty \int_0^{2\pi} \frac{u_2^2 \cos(2(\phi - \phi')) \Delta^* [\cos(2\phi) - i \sin(2\phi)]}{\omega^2 + \xi^2 + |\Delta|^2} \; d\phi' d\xi
\]

\[
= \frac{nm}{(2\pi)^2} \int_{-\infty}^\infty \int_0^{2\pi} \frac{u_2^2 \Delta^* [\cos(2\phi) \cos(2\phi') + \sin(2\phi) \sin(2\phi')] [\cos(2\phi) - i \sin(2\phi)]}{\omega^2 + \xi^2 + |\Delta|^2} \; d\phi' d\xi
\]

\[
= \frac{nm}{(2\pi)^2} \int_{-\infty}^\infty \int_0^{2\pi} \frac{u_2^2 \Delta^* [\cos(2\phi) \cos^2(2\phi') - i \sin(2\phi) \sin^2(2\phi')]}{\omega^2 + \xi^2 + |\Delta|^2} \; d\phi' d\xi
\]
since \( \sin(2\phi') \cos(2\phi') \) integrated on \([0, 2\pi] \) is zero. Recalling that the integral of both \( \sin^2(n\phi') \) and \( \cos^2(n\phi') \) from 0 to \(2\pi\) is \(\pi\) for any nonzero integer \(n\), we find

\[
\mathcal{F}_{\omega}^+ = \frac{nm}{(2\pi)^2} \int_{-\infty}^{\infty} u_2^2 \Delta^* \left[ \cos(2\phi) \frac{\pi}{\omega^2 + \xi^2 + |\Delta|^2} - i \sin(2\phi) \frac{\pi}{\omega^2 + \xi^2 + |\Delta|^2} \right] d\xi
\]

\[
= \frac{nm u_2^2 \Delta^* e^{-2i\phi}}{4\pi} \int_{-\infty}^{\infty} \frac{1}{\omega^2 + \xi^2 + |\Delta|^2} d\xi
\]

\[
= \frac{nm u_2^2 \Delta^* e^{-2i\phi}}{4\pi} \frac{\pi}{\sqrt{\omega^2 + |\Delta|^2}}
\]

\[
= \frac{nm u_2^2 \Delta^*(p)}{4} \frac{1}{\sqrt{\omega^2 + |\Delta|^2}}
\]

As in the s-wave case,

\[
\overline{G}_\omega = - \frac{nm u_0^2 i\omega}{2} \frac{1}{\sqrt{\omega^2 + |\Delta|^2}}.
\]  

Define \(\frac{1}{\tau_1} \equiv nm u_0^2\) and \(\frac{1}{\tau_2} \equiv \frac{nm u_2^2}{2}\). Then, as before, we note that

\[
i\omega - \overline{G}_\omega = i\eta_\omega \omega
\]  

and

\[
\Delta^*(p) + \mathcal{F}_{\omega}^+ = \eta_\omega \Delta^*(p),
\]  

where

\[
\eta_\omega \equiv 1 + \frac{1}{2\tau_1^2 \sqrt{\omega^2 + |\Delta|^2}}
\]  

\[
\eta_\omega \equiv 1 + \frac{1}{2\tau_2^2 \sqrt{\omega^2 + |\Delta|^2}}.
\]
Hence, the effect of scattering on the Green functions is simply to make the substitutions $\omega \rightarrow \eta_{\omega_1}\omega$ and $\Delta(p) \rightarrow \eta_{\omega_2}\Delta(p)$. For convenience, we will define $\tilde{\omega} \equiv \eta_{\omega_1}\omega$ and $\tilde{\Delta}(p) \equiv \eta_{\omega_2}\Delta(p)$.

Now we will calculate the effect of scattering on $T_c$ using the BCS self-consistency condition. Substituting, we have that

$$\Delta^*(p) = \frac{T}{(2\pi)^2} \sum_{\omega_n} \int V(p, p') \frac{F^+_{\omega}(p')}{\omega} d^2 p'$$

$$= \frac{\lambda T m}{(2\pi)^2} \sum_{\omega_n} \int_{0}^{2\pi} \int_{-\infty}^{\infty} \frac{e^{-2i(\phi-\phi')}\Delta^*}{\tilde{\omega}^2 + \xi^2 + |\Delta|^2} d\xi d\phi'$$

$$= \frac{\lambda T m \Delta^*(p)}{2\pi} \sum_{\omega_n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\eta_{\omega_2}}{\tilde{\omega}^2 + \xi^2 + |\Delta|^2} d\xi d\phi'$$

$$= \frac{\lambda T m \Delta^*(p)}{2\pi} \sum_{\omega_n} \int_{-\infty}^{\infty} \frac{\eta_{\omega_2}}{\sqrt{\tilde{\omega}^2 + |\Delta|^2}} d\xi$$

$$= \frac{\lambda T m \Delta^*(p)}{2} \sum_{\omega_n} \frac{\eta_{\omega_2}}{\sqrt{\eta_{\omega_1}^2 \omega^2 + \eta_{\omega_2}^2 |\Delta|^2}}.$$ 

Dividing both sides by $\Delta^*(p)$, our self-consistency condition becomes

$$1 = \frac{\lambda T m}{2} \sum_{\omega_n} \frac{\eta_{\omega_2}}{\sqrt{\eta_{\omega_1}^2 \omega^2 + \eta_{\omega_2}^2 |\Delta|^2}}.$$ 

(3.18)
At \( T_c \), where \( \Delta = 0 \), we have

\[
1 = \frac{\lambda T_c m}{2} \sum_{\omega_{1\omega}} \frac{\eta_{\omega 2}}{\eta_{\omega 1} |\omega|} \\
= \frac{\lambda T_c m}{2} \sum_{\omega_{1\omega}} \frac{1 + \frac{1}{2\tau_2 |\omega|}}{\left(1 + \frac{1}{2\tau_1 |\omega|}\right) |\omega|} \\
= \frac{\lambda T_c m}{2} \sum_{\omega_{1\omega}} \frac{|\omega| + \frac{1}{2\tau_2}}{|\omega| + \frac{1}{2\tau_1} |\omega|}.
\]

After some algebra analogous to the s-wave case, we get the result that

\[
\ln \left( \frac{T_c'}{T_c} \right) + \left(1 - \frac{\tau_1}{\tau_2}\right) \left[ \psi \left( \frac{1}{2} + \frac{1/2\tau_1}{2\pi T_c'} \right) - \psi \left( \frac{1}{2} \right) \right] = 0. \quad (3.19)
\]

This reduces to the standard pair-breaking formula when we take \( 1/\tau_2 \to 0 \) (when there is no d-wave scattering).

\textbf{\( d_{x^2-y^2} \) Order Parameter Calculations}

Suppose that our order parameter \( \Delta(p, T) \) takes the form

\[
\Delta(p, T) = \Delta(T) \cos(2\phi), \quad (3.20)
\]

and that the pairing interaction is given by

\[
V(p, p') = \lambda \cos[2(\phi - \phi')], \quad \lambda \in \mathbb{R}. \quad (3.21)
\]
Using the bare Green functions, to first order in $\Delta(T)$,

\[
\mathcal{F}_\omega^+ = \frac{n}{(2\pi)^2} \int |u(p - p')|^2 \mathcal{F}_\omega^+(p') \, d^2p' \\
= \frac{nm}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{0}^{2\pi} |u(p - p')|^2 \frac{\Delta^* \cos(2\phi')}{\omega^2 + \xi^2 + |\Delta|^2 \cos^2(2\phi')} \, d\phi' \, d\xi.
\]

Only the $\cos[2(\phi - \phi')]$ term from the scattering potential contributes. Then we have

\[
\mathcal{F}_\omega^+ = \frac{nm}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{0}^{2\pi} u_2^2 \cos[2(\phi - \phi')] \Delta^* \cos(2\phi') \frac{\Delta^* \cos(2\phi')}{\omega^2 + \xi^2 + |\Delta|^2 \cos^2(2\phi')} \, d\phi' \, d\xi \\
= \frac{\Delta^* \cos(2\phi)}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{0}^{2\pi} \left[ \cos(2\phi) \cos(2\phi') + \sin(2\phi) \sin(2\phi') \right] \cos(2\phi') \frac{\cos(2\phi')}{\omega^2 + \xi^2 + |\Delta|^2 \cos^2(2\phi')} \, d\phi' \, d\xi.
\]

The $\sin(2\phi')$ term integrates to zero. Now,

\[
\mathcal{F}_\omega^+ = \frac{\Delta^* \cos(2\phi) \cos(2\phi')}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{0}^{2\pi} \frac{\cos(2\phi) \cos(2\phi') \cos(2\phi')}{\omega^2 + \xi^2 + |\Delta|^2 \cos^2(2\phi')} \, d\phi' \, d\xi \\
= \frac{\Delta^* \cos(2\phi)}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{0}^{2\pi} \frac{\cos^2(2\phi')}{\omega^2 + \xi^2 + |\Delta|^2 \cos^2(2\phi')} \, d\phi' \, d\xi.
\]
Using the integral, this becomes

\[
\mathcal{F}_\omega^+ = \Delta^* \cos(2\phi) \frac{nmu^2_2}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{2\pi}{|\Delta|^2} \left[ 1 - \frac{\sqrt{\omega^2 + \xi^2}}{\sqrt{\omega^2 + \xi^2 + |\Delta|^2}} \right] d\xi
\]

\[
= \Delta^* \cos(2\phi) \frac{nmu^2_2}{2\pi} \int_{-\infty}^{\infty} \frac{1}{|\Delta|^2} \left[ 1 - \frac{\sqrt{\omega^2 + \xi^2}}{\sqrt{\omega^2 + \xi^2 + |\Delta|^2}} \right] d\xi
\]

\[
\approx \Delta^* \cos(2\phi) \frac{nmu^2_2}{2\pi} \int_{-\infty}^{\infty} \frac{1}{|\Delta|^2} \left[ 1 - \sqrt{\omega^2 + \xi^2} \left( \frac{1}{\sqrt{\omega^2 + \xi^2}} - \frac{|\Delta|^2}{2(\omega^2 + \xi^2)^{3/2}} \right) \right] d\xi
\]

\[
= \Delta^* \cos(2\phi) \frac{nmu^2_2}{4\pi} \int_{-\infty}^{\infty} \frac{1}{\omega^2 + \xi^2} d\xi
\]

\[
= \Delta^* \cos(2\phi) \frac{nmu^2_2}{4} \frac{\pi}{|\omega|}
\]

\[
= \Delta^* \cos(2\phi) \frac{nmu^2_2}{4} \frac{1}{|\omega|}
\]

Unlike the s-wave case, we must actually calculate \(\overline{G}_\omega\). It is

\[
\overline{G}_\omega = \frac{n}{(2\pi)^2} \int |u(p - p')|^2 \mathcal{F}_\omega^+(p') d^2p'
\]

\[
= -\frac{nm}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{0}^{2\pi} |u(p - p')|^2 \frac{i\omega + \xi}{\omega^2 + \xi^2 + |\Delta|^2 \cos^2(2\phi')} d\phi' d\xi
\]

\[
= -i\omega \frac{nm}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{0}^{2\pi} |u(p - p')|^2 \frac{1}{\omega^2 + \xi^2 + |\Delta|^2 \cos^2(2\phi')} d\phi' d\xi,
\]

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where we have thrown away the $\xi$ term in the numerator because we know it will integrate to zero.

In this case, only the s-wave term will contribute, so we have

$$
\overline{G}_\omega = -i\omega \frac{nm}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{0}^{2\pi} \frac{u_0^2}{\omega^2 + \xi^2 + |\Delta|^2 \cos^2(2\phi')} \ d\phi' \ d\xi
$$

$$
= -i\omega \frac{nu_0^2}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{2\pi}{\sqrt{\omega^2 + \xi^2}} \ d\xi
$$

$$
= -i\omega \frac{nu_0^2}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\omega^2 + \xi^2}} \ d\xi.
$$

Expanding the second square root term in the integral in a Taylor series similarly to before, we have

$$
\overline{G}_\omega = -i\omega \frac{nu_0^2}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\omega^2 + \xi^2}} \left( \frac{1}{\sqrt{\omega^2 + \xi^2}} - \frac{|\Delta|^2}{2(\omega^2 + \xi^2)^{3/2}} \right) \ d\xi
$$

$$
= -i\omega \frac{nu_0^2}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\omega^2 + \xi^2} - \frac{|\Delta|^2}{2(\omega^2 + \xi^2)^2} \ d\xi
$$

$$
= -i\omega \frac{nu_0^2}{2\pi} \left( \frac{\pi}{|\omega|} - \frac{\pi |\Delta|^2}{4|\omega|^3} \right)
$$

$$
= -i\omega \frac{nu_0^2}{2} \left( \frac{1}{|\omega|} - \frac{|\Delta|^2}{4|\omega|^3} \right),
$$

where we have used the integrals from in the next to last step.

Define $\frac{1}{\tau_1} \equiv nm\nu_0^2$ and $\frac{1}{\tau_2} \equiv \frac{nu_0^2}{2}$. Note,

$$
i\omega - \overline{G}_\omega = i\eta_1 \omega
$$

(3.22)

and

$$
\Delta^*(\mathbf{p}) + \overline{F}_{\omega}^* = \eta_{\omega_2} \Delta^*(\mathbf{p}),
$$

(3.23)
where

\begin{equation}
\eta_{\omega 1} \equiv 1 + \frac{1}{2\tau_1} \left( \frac{1}{|\omega|} - \frac{|\Delta|^2}{4|\omega|^3} \right) \tag{3.24}
\end{equation}

\begin{equation}
\eta_{\omega 2} \equiv 1 + \frac{1}{2\tau_2 |\omega|}. \tag{3.25}
\end{equation}

Hence, the effect of scattering on the Green functions is simply to make the substitutions \( \omega \to \eta_{\omega 1} \omega \) and \( \Delta(p) \to \eta_{\omega 2} \Delta(p) \). For convenience, we will define \( \tilde{\omega} \equiv \eta_{\omega 1} \omega \) and \( \tilde{\Delta} \equiv \eta_{\omega 2} \Delta \).

Now we will calculate the effect of scattering on \( T_c \) using the BCS self-consistency condition. Substituting, we have that

\[
\Delta^*(p) = \frac{T}{(2\pi)^2} \sum_{\omega_n} \int V(p, p') F^+_{\omega}(p') \, d^2p'
\]

\[
= \frac{\lambda T m}{(2\pi)^2} \sum_{\omega_n} \int_{-\infty}^{\infty} \int_{0}^{2\pi} \frac{\cos[2(\phi - \phi')]|\tilde{\Delta} \cos(2\phi')|}{\tilde{\omega}^2 + \xi^2 + \tilde{\Delta}^2 \cos^2(2\phi')} \, d\phi' \, d\xi
\]

\[
= \frac{\lambda T m}{(2\pi)^2} \sum_{\omega_n} \int_{-\infty}^{\infty} \int_{0}^{2\pi} \frac{\eta_{\omega 2} \Delta \cos(2\phi) \cos(2\phi') + \sin(2\phi) \sin(2\phi') \cos(2\phi')}{\tilde{\omega}^2 + \xi^2 + \tilde{\Delta}^2 \cos^2(2\phi')} \, d\phi' \, d\xi
\]

\[
= \frac{\lambda T m \Delta^*(p)}{(2\pi)^2} \sum_{\omega_n} \int_{-\infty}^{\infty} \int_{0}^{2\pi} \eta_{\omega 2} \cos^2(2\phi') \, \frac{1 - \sqrt{\tilde{\omega}^2 + \xi^2 + \tilde{\Delta}^2}}{\tilde{\omega}^2 + \xi^2 + \tilde{\Delta}^2} \, d\phi' \, d\xi
\]

\[
= \frac{\lambda T m \Delta^*(p)}{2\pi} \sum_{\omega_n} \int_{-\infty}^{\infty} \eta_{\omega 2} \cos^2(2\phi') \, \frac{1 - \sqrt{\tilde{\omega}^2 + \xi^2 + \tilde{\Delta}^2}}{\tilde{\omega}^2 + \xi^2 + \tilde{\Delta}^2} \, d\xi,
\]

where we used the integral in the next to last step. Expanding the bottom square root in the right
part of the integrand in a Taylor series as before, we have approximately that

\[
\Delta^*(p) \approx \frac{\lambda T m \Delta^*(p)}{2\pi} \sum_{\omega_n} \int_{-\infty}^{\infty} \frac{\eta_{\omega^2}}{|\Delta|^2} \frac{1}{(\omega^2 + \xi^2)} d\xi
\]

\[
= \frac{\lambda T m \Delta^*(p)}{4\pi} \sum_{\omega_n} \int_{-\infty}^{\infty} \eta_{\omega^2} \frac{1}{\omega^2 + \xi^2} d\xi
\]

\[
= \frac{\lambda T m \Delta^*(p)}{4\pi} \sum_{\omega_n} \eta_{\omega^2} \frac{\pi}{|\omega|}
\]

\[
= \frac{\lambda T m \Delta^*(p)}{4} \sum_{\omega_n} \eta_{\omega^2} \frac{1}{|\omega|}.
\]

Dividing both sides by \(\Delta^*(p)\), our self-consistency condition becomes

\[
1 = \frac{\lambda T m}{4} \sum_{\omega_n} \eta_{\omega^2} \frac{1}{|\omega|}.
\]  

(3.26)

Keeping in mind the correspondence we established earlier, this self-consistency condition in the absence of impurities (where \(\eta_{\omega^2} \rightarrow 1\) and \(\tilde{\omega} \rightarrow \omega\)) is

\[
1 = \frac{\lambda T m}{4} \sum_{\omega_n} \frac{1}{|\omega|}.
\]  

(3.27)

Summing this gives the same result as in the previous case; in other words, the suppression of \(T_c\) has exactly the same behavior.

\textit{d}_{xy} \textit{Order Parameter Calculations}

Suppose that our order parameter \(\Delta(p, T)\) takes the form

\[
\Delta(p, T) = \Delta(T) \sin(2\phi),
\]  

(3.28)
and that the pairing interaction is (again) given by

\[ V(p, p') = \lambda \cos[2(\phi - \phi')] \, , \, \lambda \in \mathbb{R}. \]  

(3.29)

Using the bare Green functions, to first order in \( \Delta(T) \),

\[ \mathcal{F}_\omega^+ = \frac{n}{(2\pi)^2} \int |u(p - p')|^2 \mathcal{F}_\omega^+(p') \, d^2p' \]

\[ = \frac{nm}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{0}^{2\pi} \frac{|u(p - p')|^2 \Delta^* \sin(2\phi')}{\omega^2 + \xi^2 + |\Delta|^2 \sin^2(2\phi')} \, d\phi' d\xi. \]

Only the \( \cos[2(\phi - \phi')] \) term from the scattering potential contributes. Then we have

\[ \mathcal{F}_\omega^+ = \frac{nm}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{0}^{2\pi} \frac{u_2^2 \cos[2(\phi - \phi')] \Delta^* \sin(2\phi')}{\omega^2 + \xi^2 + |\Delta|^2 \sin^2(2\phi')} \, d\phi' d\xi \]

\[ = \Delta^* \frac{nm}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{0}^{2\pi} \frac{[\cos(2\phi) \cos(2\phi') + \sin(2\phi) \sin(2\phi')] \sin(2\phi')}{\omega^2 + \xi^2 + |\Delta|^2 \sin^2(2\phi')} \, d\phi' d\xi. \]

The \( \cos(2\phi') \) term integrates to zero. Now,

\[ \mathcal{F}_\omega^+ = \Delta^* \frac{nm\mu_2^2}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{0}^{2\pi} \frac{\sin(2\phi) \sin(2\phi') \sin(2\phi')}{\omega^2 + \xi^2 + |\Delta|^2 \sin^2(2\phi')} \, d\phi' d\xi \]

\[ = \Delta^* \sin(2\phi) \frac{nm\mu_2^2}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{0}^{2\pi} \frac{\sin^2(2\phi')}{\omega^2 + \xi^2 + |\Delta|^2 \sin^2(2\phi')} \, d\phi' d\xi. \]

Using the integral, this becomes

\[ \mathcal{F}_\omega^+ = \Delta^* \sin(2\phi) \frac{nm\mu_2^2}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{0}^{2\pi} \frac{2\pi}{|\Delta|^2} \left[ 1 - \frac{\sqrt{\omega^2 + \xi^2}}{\omega^2 + \xi^2 + |\Delta|^2} \right] \, d\xi. \]
After this the integration is exactly the same, so we skip to the result, which is the same as before except for $\sin(2\phi)$ substituted for $\cos(2\phi)$:

$$\mathcal{F}_\omega^- = \Delta^* \sin(2\phi) \frac{nnu_2^2}{4} \frac{1}{|\omega|}. \quad (3.30)$$

The integration for $\mathcal{G}_\omega^-$ also works out the same as before (because the angular integral has the exact same properties), so we only note the result that

$$\mathcal{G}_\omega^- = -i\omega \frac{nnu_0^2}{2} \left( \frac{1}{|\omega|} - \frac{1}{4|\omega|} \right). \quad (3.31)$$

Define $\frac{1}{\tau_1} \equiv nnu_0^2$ and $\frac{1}{\tau_2} \equiv \frac{nnu_2^2}{2}$. Again,

$$i\omega - \mathcal{G}_\omega^- = i\eta_\omega \omega \quad (3.32)$$

and

$$\Delta^*(\mathbf{p}) + \mathcal{F}_\omega^- = \eta_\omega \Delta^*(\mathbf{p}), \quad (3.33)$$

where

$$\eta_\omega \equiv 1 + \frac{1}{2\tau_1} \left( \frac{1}{|\omega|} - \frac{1}{4|\omega|} \right) \quad (3.34)$$

$$\eta_\omega \equiv 1 + \frac{1}{2\tau_2 |\omega|}. \quad (3.35)$$

Hence, the effect of scattering on the Green functions is simply to make the substitutions $\omega \rightarrow \eta_\omega \omega$ and $\Delta(\mathbf{p}) \rightarrow \eta_\omega \Delta(\mathbf{p})$. For convenience, we will define $\tilde{\omega} \equiv \eta_\omega \omega$ and $\tilde{\Delta} \equiv \eta_\omega \Delta$.

Now we will calculate the effect of scattering on $T_c$ using the BCS self-consistency condition.
Substituting, we have that

\[
\Delta^*(p) = \frac{T}{(2\pi)^2} \sum_{\omega_n} \int V(p, p') F^+_{\omega} (p') \, d^2 p'
\]

\[
= \frac{\lambda T m}{(2\pi)^2} \sum_{\omega_n} \int_{-\infty}^{\infty} \int_{0}^{2\pi} \frac{\cos[2(\phi - \phi')]|\Delta| \sin(2\phi')}{\tilde{\omega}^2 + \xi^2 + \tilde{\Delta}^2 \sin^2(2\phi')} \, d\phi' \, d\xi
\]

\[
= \frac{\lambda T m}{(2\pi)^2} \sum_{\omega_n} \int_{-\infty}^{\infty} \int_{0}^{2\pi} \frac{\eta_{\omega} 2 \Delta [\cos(2\phi) \cos(2\phi') + \sin(2\phi) \sin(2\phi')] \sin(2\phi')}{\tilde{\omega}^2 + \xi^2 + \tilde{\Delta}^2 \sin^2(2\phi')} \, d\phi' \, d\xi
\]

\[
= \frac{2\pi}{(2\pi)^2} \sum_{\omega_n} \int_{-\infty}^{\infty} \int_{0}^{2\pi} \frac{\eta_{\omega} 2 \Delta [\cos(2\phi) \cos(2\phi') + \sin(2\phi) \sin(2\phi')] \sin(2\phi')}{\tilde{\omega}^2 + \xi^2 + \tilde{\Delta}^2 \sin^2(2\phi')} \, d\phi' \, d\xi
\]

where we used the integral in the next to last step. The rest of the analysis works out exactly the same as the \(d_{x^2-y^2}\) case since the angular part worked out the same.

\[\Delta_1 d_{x^2-y^2} + i \Delta_2 d_{xy}\] Order Parameter Calculations

Suppose that our order parameter \(\Delta(p, T)\) takes the form

\[
\Delta(p, T) = \Delta_1(T) \cos(2\phi) + i \Delta_2(T) \sin(2\phi),
\]

and that the pairing interaction is given by

\[
V(p, p') = \lambda \cos[2(\phi - \phi')], \ \lambda \in \mathbb{R}.
\]
Using the bare Green functions, to first order in $\Delta(T)$,

\[
\mathcal{F}_\omega^+ = \frac{n}{(2\pi)^2} \int |u(p - p')|^2 \mathcal{F}_\omega^+(p') \, dp'
\]

\[
= \frac{nm}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{0}^{2\pi} \frac{|u(p - p')|^2}{\omega^2 + \xi^2 + |\Delta_1|^2 \cos^2(2\phi') + |\Delta_2|^2 \sin^2(2\phi')} \, d\phi' \, d\xi.
\]

As we learned from examining $d_{x^2-y^2}$ and $d_{xy}$ separately, only the $\cos[2(\phi - \phi')]$ term from the scattering potential contributes. Then we have

\[
\mathcal{F}_\omega^+ = \frac{nmu_2^2}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{0}^{2\pi} \frac{\cos(2\phi) \cos(2\phi') + \sin(2\phi) \sin(2\phi')}{\omega^2 + \xi^2 + |\Delta_1|^2 \cos^2(2\phi') + |\Delta_2|^2 \sin^2(2\phi')} \, d\phi' \, d\xi.
\]

Only the $\cos^2(2\phi')$ and $\sin^2(2\phi')$ terms will be nonvanishing. Then

\[
\mathcal{F}_\omega^+ = \frac{nmu_2^2}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{0}^{2\pi} \frac{\Delta_1^* \cos(2\phi) \cos^2(2\phi') - i \Delta_2^* \sin(2\phi) \sin(2\phi')}{\omega^2 + \xi^2 + |\Delta_1|^2 \cos^2(2\phi') + |\Delta_2|^2 \sin^2(2\phi')} \, d\phi' \, d\xi
\]

\[
= \frac{nmu_2^2}{(2\pi)^2} \left[ \Delta_1^* \cos(2\phi) \int_{-\infty}^{\infty} \int_{0}^{2\pi} \frac{\cos^2(2\phi')}{\omega^2 + \xi^2 + |\Delta_1|^2 \cos^2(2\phi') + |\Delta_2|^2 \sin^2(2\phi')} \, d\phi' \, d\xi
\]

\[
- i \Delta_2^* \sin(2\phi) \int_{-\infty}^{\infty} \int_{0}^{2\pi} \frac{\sin^2(2\phi')}{\omega^2 + \xi^2 + |\Delta_1|^2 \cos^2(2\phi') + |\Delta_2|^2 \sin^2(2\phi')} \, d\phi' \, d\xi \right].
\]

The first integral is

\[
\int_{0}^{2\pi} \frac{\cos^2(2\phi')}{\omega^2 + \xi^2 + |\Delta_1|^2 \cos^2(2\phi') + |\Delta_2|^2 \sin^2(2\phi')} \, d\phi' = \frac{2\pi}{|\Delta_1|^2 - |\Delta_2|^2} \left[ 1 - \frac{\sqrt{\omega^2 + \xi^2 + |\Delta_2|^2}}{\sqrt{\omega^2 + \xi^2 + |\Delta_1|^2}} \right].
\]

We can expand the result in terms of $|\Delta_1|^2 - |\Delta_2|^2$, which is experimentally motivated by the fact that if there were two d-wave components, the difference in their critical temperatures must be
very small, since two critical temperatures have not been observed. Note,

\[ \frac{\sqrt{\omega^2 + \xi^2 + |\Delta_2|^2}}{\sqrt{\omega^2 + \xi^2 + |\Delta_1|^2}} \approx \frac{\sqrt{\omega^2 + \xi^2 + |\Delta_2|^2}}{\sqrt{\omega^2 + \xi^2 + |\Delta_2|^2 + (|\Delta_1|^2 - |\Delta_2|^2)}} \]

\[ \approx \sqrt{\omega^2 + \xi^2 + |\Delta_2|^2} \left[ \frac{1}{\sqrt{\omega^2 + \xi^2 + |\Delta_2|^2}} - \frac{1}{2} \frac{(|\Delta_1|^2 - |\Delta_2|^2)}{(\omega^2 + \xi^2 + |\Delta_2|^2)^{3/2}} \right] \]

\[ = 1 - \frac{1}{2} \frac{(|\Delta_1|^2 - |\Delta_2|^2)}{\omega^2 + \xi^2 + |\Delta_2|^2}. \]

The approximate result is then

\[ \int_0^{2\pi} \frac{\cos^2(2\phi')}{\omega^2 + \xi^2 + |\Delta_1|^2 \cos^2(2\phi') + |\Delta_2|^2 \sin^2(2\phi')} \, d\phi' \approx \frac{\pi}{\omega^2 + \xi^2 + |\Delta_2|^2}. \]

Using this approximation, we can do the next integration easily:

\[ \int_{-\infty}^{\infty} \int_0^{2\pi} \frac{\cos^2(2\phi')}{\omega^2 + \xi^2 + |\Delta_1|^2 \cos^2(2\phi') + |\Delta_2|^2 \sin^2(2\phi')} \, d\phi' \, d\xi \approx \frac{\pi}{\omega^2 + \xi^2 + |\Delta_2|^2} \int_{-\infty}^{\infty} \frac{\pi}{\omega^2 + \xi^2 + |\Delta_2|^2} \, d\xi \]

\[ \approx \frac{\pi^2}{\sqrt{\omega^2 + |\Delta_2|^2}}. \]

Similarly, the second integral is

\[ \int_0^{2\pi} \frac{\sin^2(2\phi')}{\omega^2 + \xi^2 + |\Delta_1|^2 \cos^2(2\phi') + |\Delta_2|^2 \sin^2(2\phi')} \, d\phi' = \frac{2\pi}{|\Delta_2|^2 - |\Delta_1|^2} \left[ 1 - \frac{\sqrt{\omega^2 + \xi^2 + |\Delta_1|^2}}{\sqrt{\omega^2 + \xi^2 + |\Delta_2|^2}} \right]. \]

Expanding again in terms of $|\Delta_1|^2 - |\Delta_2|^2$,

\[ \frac{\sqrt{\omega^2 + \xi^2 + |\Delta_1|^2}}{\sqrt{\omega^2 + \xi^2 + |\Delta_2|^2}} \approx \frac{\sqrt{\omega^2 + \xi^2 + |\Delta_2|^2 + (|\Delta_1|^2 - |\Delta_2|^2)}}{\sqrt{\omega^2 + \xi^2 + |\Delta_2|^2}} \]

\[ \approx \frac{1}{\sqrt{\omega^2 + \xi^2 + |\Delta_2|^2}} \left[ \sqrt{\omega^2 + \xi^2 + |\Delta_2|^2} + \frac{1}{2} \frac{(|\Delta_1|^2 - |\Delta_2|^2)}{\sqrt{\omega^2 + \xi^2 + |\Delta_2|^2}} \right] \]

\[ = 1 + \frac{1}{2} \frac{(|\Delta_1|^2 - |\Delta_2|^2)}{\omega^2 + \xi^2 + |\Delta_2|^2}. \]
The approximate result this time is

\[
\int_0^{2\pi} \frac{\sin^2(2\phi')}{\omega^2 + \xi^2 + |\Delta_1|^2 \cos^2(2\phi') + |\Delta_2|^2 \sin^2(2\phi')} \, d\phi' \approx \frac{\pi}{\omega^2 + \xi^2 + |\Delta_2|^2}.
\]

Integration yields that

\[
\int_{-\infty}^{\infty} \int_0^{2\pi} \frac{\sin^2(2\phi')}{\omega^2 + \xi^2 + |\Delta_1|^2 \cos^2(2\phi') + |\Delta_2|^2 \sin^2(2\phi')} \, d\phi' \, d\xi \approx \int_{-\infty}^{\infty} \frac{\pi}{\omega^2 + \xi^2 + |\Delta_2|^2} \, d\xi \approx \frac{\pi^2}{\sqrt{\omega^2 + |\Delta_2|^2}}.
\]

Putting these results together, we conclude that

\[
\mathcal{F}_\omega \approx \frac{nm \omega^2}{(2\pi)^2} \frac{1}{\sqrt{\omega^2 + |\Delta_2|^2}} \Delta^*(p).
\]

Now let us calculate \( \overline{\mathcal{G}}_\omega \). It is

\[
\overline{\mathcal{G}}_\omega = \frac{n}{(2\pi)^2} \int |u(p - p')|^2 \mathcal{F}_\omega^+(p') \, d^2p'
= -\frac{nm}{(2\pi)^2} \int_{-\infty}^{\infty} \int_0^{2\pi} |u(p - p')|^2 \frac{i\omega + \xi}{\omega^2 + \xi^2 + |\Delta_1|^2 \cos^2(2\phi') + |\Delta_2|^2 \sin^2(2\phi')} \, d\phi' \, d\xi
= -i\omega \frac{nm}{(2\pi)^2} \int_{-\infty}^{\infty} \int_0^{2\pi} |u(p - p')|^2 \frac{1}{\omega^2 + \xi^2 + |\Delta_1|^2 \cos^2(2\phi') + |\Delta_2|^2 \sin^2(2\phi')} \, d\phi' \, d\xi,
\]
where we have thrown away the $\xi$ term in the numerator (as usual) because we know it will integrate to zero. In this case, only the s-wave term will contribute, so we have

$$
\overline{G}_\omega = -i\omega \frac{nm}{(2\pi)^2} \int_{-\infty}^{\infty} \int_0^{2\pi} \frac{u_0^2}{\omega^2 + \xi^2 + |\Delta_1|^2 \cos^2(2\phi') + |\Delta_2|^2 \sin^2(2\phi')} d\phi' d\xi
$$

$$
= -i\omega \frac{nm u_0^2}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{2\pi}{\sqrt{\omega^2 + \xi^2 + |\Delta_2|^2} \sqrt{\omega^2 + \xi^2 + |\Delta_1|^2}} d\xi
$$

$$
= -i\omega \frac{nm u_0^2}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\omega^2 + \xi^2 + |\Delta_2|^2} \sqrt{\omega^2 + \xi^2 + |\Delta_1|^2}} d\xi.
$$

Expanding the second square root term in the integral in a Taylor series similarly to before, we have

$$
\overline{G}_\omega = -i\omega \frac{nm u_0^2}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\omega^2 + \xi^2 + |\Delta_2|^2}} \left( \frac{1}{\sqrt{\omega^2 + \xi^2 + |\Delta_2|^2}} - \frac{(|\Delta_1|^2 - |\Delta_2|^2)}{2(\omega^2 + \xi^2 + |\Delta_2|^2)^{3/2}} \right) d\xi
$$

$$
= -i\omega \frac{nm u_0^2}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\omega^2 + \xi^2 + |\Delta_2|^2} - \frac{(|\Delta_1|^2 - |\Delta_2|^2)}{2(\omega^2 + \xi^2 + |\Delta_2|^2)^2} d\xi
$$

$$
= -i\omega \frac{nm u_0^2}{2\pi} \left( \frac{\pi}{\sqrt{\omega^2 + |\Delta_2|^2}} - \frac{\pi(|\Delta_1|^2 - |\Delta_2|^2)}{4(\omega^2 + |\Delta_2|^2)^{3/2}} \right)
$$

$$
= -i\omega \frac{nm u_0^2}{2} \left( \frac{1}{\sqrt{\omega^2 + |\Delta_2|^2}} - \frac{(|\Delta_1|^2 - |\Delta_2|^2)}{4(\omega^2 + |\Delta_2|^2)^{3/2}} \right).
$$

Define $\frac{1}{\tau_1} \equiv nm u_0^2$ and $\frac{1}{\tau_2} \equiv \frac{nm u_0^2}{2}$. Note,

$$
i\omega - \overline{G}_\omega = i\eta_\omega \omega \quad (3.38)
$$

and

$$
\Delta^*(p) + \overline{F}_\omega^* = \eta_\omega \Delta^*(p), \quad (3.39)
$$

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where

\[
\eta_{\omega_1} \equiv 1 + \frac{1}{2\tau_1} \left( \frac{1}{\sqrt{\omega^2 + |\Delta_2|^2}} - \frac{(|\Delta_1|^2 - |\Delta_2|^2)}{4(\omega^2 + |\Delta_2|^2)^{3/2}} \right) \quad (3.40)
\]

\[
\eta_{\omega_2} \equiv 1 + \frac{1}{2\tau_2} \frac{1}{\sqrt{\omega^2 + |\Delta_2|^2}}. \quad (3.41)
\]

Hence, the effect of scattering on the Green functions is simply to make the substitutions $\omega \to \eta_{\omega_1}\omega$ and $\Delta(p) \to \eta_{\omega_2}\Delta(p)$. For convenience, we will define $\tilde{\omega} \equiv \eta_{\omega_1}\omega$ and $\tilde{\Delta} \equiv \eta_{\omega_2}\Delta$.

This time, because we have asymmetry in the order parameter, there will be two critical temperatures: $T_{c1}$, the temperature that corresponds to the $d_{x^2-y^2}$ part, and $T_{c2}$, the temperature that corresponds to the $d_{xy}$ part, with $T_{c1} > T_{c2}$.

At $T_{c1}$, $\Delta_1 = \Delta_2 = 0$. Between $T_{c1}$ and $T_{c2}$, the $d_{x^2-y^2}$ part becomes active, and $|\Delta_1| > 0$ while it is still true that $\Delta_2 = 0$. At $T_{c2}$, $\Delta_2$ becomes nonzero and $\Delta_1$ remains nonzero.

Now we will calculate the effect of scattering on $T_{c1}$ and $T_{c2}$ using the BCS self-consistency condition. Substituting, we have that

\[
\Delta^*(p) = \frac{T}{(2\pi)^2} \sum_{\omega_n} \int V(p, p') \mathcal{F}^+_{\tilde{\omega}}(p') \, d^3p' = \frac{\lambda T m}{(2\pi)^2} \sum_{\omega_n} \int_0^\infty \int \frac{\cos[2(\phi - \phi')][\tilde{\Delta}_1^* \cos(2\phi') - i\tilde{\Delta}_2^* \sin(2\phi')]}{\tilde{\omega}^2 + \xi^2 + |\Delta_1|^2 \cos^2(2\phi') + |\Delta_2|^2 \sin^2(2\phi')} \, d\phi' \, d\xi.
\]

This integral is the same as the one we did before, but with $\tilde{\Delta}_1, \tilde{\Delta}_2$, and $\tilde{\omega}$ instead of $\Delta_1, \Delta_2$, and $\omega$. 43
Reading off the result from before, all we have to do now is the sum over Matsubara frequencies:

$$\Delta^*(p) = \frac{\lambda T_m}{4} \sum_{\omega_n} \frac{\eta_{\omega}^2}{\sqrt{\omega^2 + |\tilde{\Delta}|^2}} \Delta^*(p).$$  \hfill (3.42)

Dividing both sides by $\Delta^*(p)$, our self-consistency condition in the presence of impurities becomes

$$1 = \frac{\lambda T_m}{4} \sum_{\omega_n} \frac{\eta_{\omega}^2}{\sqrt{\omega^2 + |\tilde{\Delta}|^2}}. $$  \hfill (3.43)

The analysis of this formula has yet to be done, so we will stop here.

Conclusion

We have used many-body theory to calculate the effects of impurity scattering on a 2D superconductor for a variety of order parameters. Interestingly, we have found that $d_{x^2-y^2} + id_{xy}$, $d_{x^2-y^2}$, and $d_{xy}$ lead to exactly the same pair-breaking formula giving the suppression of $T_c$. The analysis for the asymmetric case has yet to be completed.
LIST OF REFERENCES


