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An Investigation of Students' Modes of Thinking Concerning Linearity in Linear Algebra

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AN INVESTIGATION OF STUDENTS' MODES OF THINKING CONCERNING LINEARITY IN LINEAR ALGEBRA

by

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ABSTRACT

The intent of this thesis is to investigate student approaches to linearity within a linear algebra context, focusing on definitional, computational, and theoretical skills. Linear algebra's abstract nature constitutes a major challenge for a significant sector of STEM students, with the course often serving as undergraduates' first encounter with mathematical proofs and extrapolations. The current student struggle is reflected through the prominent gap in knowledge derived from a lack of a concrete understanding of rudimentary concepts (like linearity), pivotal to student success. As such, this investigation aimed to bridge this gap by considering students' modes of thinking regarding the elementary notion of linearity to improve the current course delivery and curriculum. Students were given three assessment questions targeting different skills integral to the mastery of linearity. Their responses were categorized using Action, Process, Object, Schema (APOS) and analyzed through Sierpinska's (2000) proposed modes of thinking. About 26% of the participants responded correctly to question 1, 77% to question 2, and 59% to question 3. The analytic mode proved pivotal, specifically when considering definition application and computational abilities. The synthetic-geometric mode, however, was integral to the practical application of the concept. Further discussion and suggestions regarding the results and their implications on the current structure of linear algebra instruction are provided.

To my parents, Yuval-Shira, Danielle, Lia, and Ethan

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CHAPTER ONE: INTRODUCTION

Linear algebra holds a pivotal role within the realms of scientific study and innovation. Through its efficacious nature in a multitude of domains, linear algebra is of significance even to non-mathematics students, serving as a prerequisite in several academic paths, including undergraduate engineering, physics, computer science, chemistry, biology, statistics, and the like (Pearlmutter & Šmigoc, 2018). However, while linear algebra constitutes an important milestone within a plethora of fields, the subject itself is challenging, posing an obstacle for most students. Literature investigating the learning and teaching of linear algebra structures offers several theories to explain the student struggle and elaborates upon the gap in student understanding. Research highlighting the student experience with the course identified that the abstract and theoretical nature of the material was a prime factor hindering student progress (Çelik, 2015).

Dorier et al. (2000) coined the term *formalism obstacle* to describe this phenomenon. The formalism obstacle poses an obstruction to one's understanding derived from a grappling attempt to grasp a multitude of new concepts, theorems, and notations simultaneously (Dorier et al., 2000; Çelik, 2015). Additionally, students limited (if any) prior knowledge and experience with linear algebra concepts (vector spaces, basis, linearity, span, dimension, etc.) feeds into the obstacle of formality and setback student comprehension (Dorier et al., 2000; Çelik, 2015). Notably, the lack of experience and knowledge in writing proofs, along with inadequate and incomplete ideas of mathematical logic are key factors impacting student learning in linear algebra (Britton & Henderson, 2009; Dorier et al., 2000; Çelik, 2015). Linear algebra often serves as students' first encounter with rigorous mathematical procedures and proofs, requiring students to deliberate concepts not only in specific, calculation-based encounters (e.g., computing the product of two matrices) but also in a general manner (e.g., considering if a set of vectors V is a vector space). Consequently, students are frequently required to extrapolate and rationalize advanced abstractions while still struggling to master rudimentary concepts (Dorier et al., 2000; Çelik, 2015). Dubinsky (1997) notes that when navigating higher-level mathematics education, an epistemological and thorough analysis of the fundamental structures that undergraduate students deem conceptually challenging, is of merit; particularly, defining the methods by which students reason and explain linear algebra concepts (Dubinsky, 1997; Çelik, 2015). Therefore, the pedagogical structures implemented should foster a learning environment that encourages the conceptualization of the prime concepts of linear algebra. Hence, to resolve the current gap in knowledge, an emphasis on the elemental, introductory notions must be placed prior to the integration of more advanced materials.

One such concept is that of linearity and linear transformations, the backbone of linear algebra. Linear transformations are unique functions that foster an additive and homogeneous structure. The useful qualities of a linear map transcend the realms of theoretical mathematics and posit real-world applications; most notably, they are used in machine learning algorithms and are implemented in the study of metric and kernel learning (Jain et al., 2012). An investigation of the student approach to the concept of linearity within a linear algebra context is therefore proposed.

The purpose of this research—keeping the objective of correcting the gap of knowledge in mind—is to understand what impact students' thinking modes and approach have on their comprehension and retention of linearity within a linear algebra context. The established paths towards answering this question are detailed in this investigation. The structures implemented for the purposes of this research are paved with an Action, Process, Object, Schema (APOS) outlook; an analysis of students' operations and thought processes within this context is pivotal to improving upon the knowledge gap and fostering an environment dedicated to student success.

CHAPTER TWO: THEORETICAL FRAMEWORK

The theoretical framework for this research is manifested through the works of Harel and Dubinsky (1991), Roa-Fuentes and Oktaç (2010) and Sierpinska (2000). An adaptation of the groundwork brought forth by Dubinsky's APOS theory (Dubinsky, 1984; Arnon et al., 2014) provides guidance throughout this research. APOS outlines differences in mathematical thinking processes, focusing on how one performs *Action* applications to mental *Objects*. A repetition of and reflection upon an Action transforms it into a *Process* through an interiorization mechanism; since APOS is cyclic, Processes convert into Objects so that Actions can be further applied to them. As such, these connections are considered as a collective, interwoven *Schema* (Dubinsky, 1984; Arnon et al., 2014; Oktaç et al., 2019).

APOS breaks down mental structures into their basic components to be studied, detailing the processes and mechanisms required for a specific subject to be learned. This is known as a concept's *genetic decomposition* (Arnon et al., 2014). Given new mathematical concepts often arise as transformations of existing concepts, a genetic decomposition frequently consists of an account of the Actions that a student needs to perform on current mental Objects, along with an explanation of how said Actions are interiorized into Processes (Arnon et al., 2014).

Harel and Dubinsky (1991) detail a genetic decomposition and approach to the study of functions. They suggest that the construction of a function (as a concept to be learned) begins with Actions taken on a set. These Actions range in complexity and involve the performance of an operation (explicit rule) on an element in one set and the assignment of a unique element– from the second set– to it (Arnon et al., 2014; Harel & Dubinsky, 1991). As these Actions are performed on various sets, students reflect on them and consider them as dynamic transformations. Therefore, a mental structure that performs the same transformations as Action has now been constructed in the minds of the students (Arnon et al., 2014; Harel & Dubinsky, 1991). Students who demonstrate an understanding of the Process procedure as it relates to a function will often conceptualize a function in terms of accepted inputs being altered through some rule and resulting in an output. This mental connection will occur without the explicit need for the students to operate or perform calculations. Students at a Process conceptual understanding point could elaborate on the concept of inputs and outputs drawn previously. For example, the ability to determine if a function has an inverse would demonstrate a successful Process (Arnon et al., 2014; Harel & Dubinsky, 1991).

Applications of Actions or additional Processes applied to the initial function Process pave the path to its mental capture as a cognitive Object. The procedure of encapsulation allows for a jump in conceptualization; the idea of a function is now transformed from a dynamic alteration of inputs to a static element that itself can be examined and operated on. Students who demonstrate the ability to form and recognize functional relationships between two entities and can manage a variety of Processes to determine the domain and range of a function may indicate the construction of a function Schema (Arnon et al., 2014; Harel & Dubinsky, 1991).

Since linear transformations are special cases of functions, a similar genetic decomposition can be built using the map suggested by Harel and Dubinsky. This decomposition serves as an extension of the *type 2* genetic decomposition brought forth by Roa-Fuentes and Oktaç (2010). The type 2 decomposition begins with the establishment of the notion of a (general) transformation between two vector spaces. Since the ability to recognize and apply the perseverance of addition and scalar multiplication is an Action directed to the transformation itself, the notion of a transformation must first be captured as a mental Object (Arnon et al., 2014; Roa-Fuentes & Oktaç, 2010).

As such, it follows that an encapsulation process of a transformation would draw a parallel between a transformation and a function, denoting a transformation, T, as a function defined between two vector spaces–the domain and codomain of the transformation (Arnon et al., 2014; Roa-Fuentes & Oktaç, 2010). As students become capable of viewing transformations as an extension of functions—recognizing the domain and range as the vector spaces of domain and codomain—the parallel between the concepts will be more evident.

Applications of Actions to the transformation Process would allow students to consider these transformations as singular entities to be studied. Indicators of the encapsulation process might include students' ability to form sets of or perform arithmetic operations on transformations. The idea of linearity, therefore, is introduced as a unique case of a transformation. Roa-Fuentes and Oktaç (2010) note that the de-encapsulation of the transformation Object is necessary for the construction of the properties of linearity as well-rounded Processes. Since the Process corresponding to the transformation provides students with the opportunity to consider the images of the domain vectors under the transformation, students are able to perform the following: (1) create a sum of any two vectors in the domain and apply the transformation operation to that sum and (2) determine the images of any two vectors in the domain and sum them together (Arnon et al., 2014; Roa-Fuentes & Oktaç, 2010). This, in turn, initiates the path for the capture of linear transformations as satisfying the conditions of perseverance of vector addition and scalar multiplication.

In a follow-up study, Roa-Fuentes and Oktaç (2012) concluded that function structures (as Schemas) and vector spaces (as Objects) are indispensable for the construction of the linear transformation concept (Roa-Fuentes & Oktaç, 2012). They further observed that the two suggested genetic decompositions brought forth in their preliminary 2010 paper point to a potential disconnect between the instructional treatment and the textbook structure and the content that the students received (Arnon et al., 2014; Roa-Fuentes & Oktaç, 2012). This barrier was referenced by Alves-Dias and Artigue (1995). They noted that students are not offered the *flexibility* necessary to develop an understanding of linear algebra; in particular, the tasks and exercises given to students are narrow in their scope (Dorier & Sierpinska, 2001; Alves-Dias & Artigue, 1995).

The delivery and accessibility of the content and instruction are prime factors in the conversation surrounding the retention and understanding of material, especially in a linear algebra course. As such, *cognitive flexibility* is of merit within this context. Sierpinska (2000), further elaborates upon the characteristics of thinking that are essential for the development of linear algebra. She notes that linear algebra requires a shift from, what she coined, *practical thinking* to *theoretical thinking*. Practical thinking (PT) is characterized as an auxiliary activity that directs other activities. Theoretical thinking (TT), on the other hand, is a specialized mental activity.

PT is manifested through direct action while TT is brought to light through written words or texts (Sierpinska, 2000; Dorier & Sierpinska, 2001). Sierpinska (2000) further identifies three prime modes of thinking governing students' approach to linear algebra: synthetic-geometric, analytic-arithmetic, and analytic-structural (Sierpinska, 2000; Çelik, 2015). Students operating under the synthetic mode demonstrate fundamental differences in their approach compared to their analytic mode counterparts. Synthetic thinkers aim to describe given mathematical objects without defining them while analytical thinkers attempt to comprehend the objects using their definition and mathematical properties (Sierpinska, 2000; Çelik, 2015). In general, it is recognized that the synthetic mode favors geometric representations, and the analytic mode favors algebraic and numerical representations. Therefore, the synthetic mode corresponds to the practical way of thinking while the analytical mode corresponds to the theoretical way of thinking (Sierpinska, 2000; Çelik, 2015).

As mentioned above, the analytic mode is split into two sub-categories: analytic-arithmetic and analytic-structural. Sierpinska (2000) draws a clear distinction between the two modes of analytical thinking; while the analytic-arithmetic mode focuses on computations and simplifying calculations, the structural mode aims to elaborate upon one's knowledge of concepts. Through an analytic-arithmetic lens, an object is best defined by a formula (set of rules) that allows one to manipulate and perform calculations on it; through a structural lens, an object is best defined through a set of mathematical properties (Sierpinska, 2000; Çelik, 2015). In this study, the instructional material and test problems are constructed following the guide of the genetic decomposition and representations proposed by Harel and Dubinsky (1991) and Roa-Fuentes and Oktaç (2010). An analysis of students' approach to linearity is conducted following the framework laid forth by Sierpinska (2000).

CHAPTER THREE: METHODOLOGY

The methodology and application, along with the evaluative tools used in this investigation were approved by the University Institutional Review Board (IRB) (see Appendix A for approval form). A discussion of the participants and research settings for this study follows below.

Participants

This study was administered to 39 undergraduate matrix and linear algebra students from the University of Central Florida (UCF) during the Fall 2023 semester. Before this course, all participants took the prerequisite mathematics classes relevant to the linear algebra curriculum. As such, participants should have been familiar with and have had ample knowledge of vector arithmetic and functional relationships.

The research was conducted in a face-to-face, classroom setting; the linear algebra content the participants studied, particularly the introduction to the concept of transformations, was performed using the genetic decompositions suggested by Harel and Dubinsky (1991) and Roa- Fuentes and Oktaç (2010). A review of functions and their properties was provided before integrating transformations. The notion of a general transformation, defined from one vector space to another, was established through an example—the students were told to think of a machine that, like a function, takes in certain inputs and returns an output upon conducting some operation on the input. Linear transformations were introduced as special instances of transformations, with a formal definition of the conditions of linearity being provided to the participants. Classroom practices, along with homework exercises, concerning the definition of linearity and linear transformations were carried out; the goal of these assignments was to provide students with the opportunity to engage with the definition and understand its conditions. To assess students' comprehension of linearity, two data collection tools—a test and an interview—were used. A description of the tools is provided in the following sub-section.

Evaluative Instruments

Upon the integration of transformations and linearity into the curriculum, student evaluation could be conducted. In this study, students were tasked with completing three test questions (see Appendix B for list of test questions) concerning linearity and linear transformations. Since a well-rounded Schema involves the application of a multitude of Actions and Processes (ranging in complexity and type), each of the three questions focused on a different skill, unique to linearity. With definitional, computational, and practical applications corresponding to questions 1, 2, and 3, respectively.

An elaborate explanation and breakdown of the questions is provided in the results and analysis sections. These categories correspond to Sierpinska's (2000) identified modes of thinking, with the skill application for questions 1 and 2 aligning with analytical thinking skills and question 3 with a combination of synthetic-geometric and analytical thinking skills. The test was implemented during regularly scheduled class time, with students receiving the entire period to complete the set of questions. The tests were de-identified and then analyzed using APOS theory and Sierpinska's (2000) modes of thinking, as a guide. Each test was assessed individually, with similar responses being grouped under one general category. A total of eight categories (see Table 1) emerged.

Table 1: Explanation of categories classifying students' modes of thinking

CODE	*Category*	*Explanation of Category*
NR	*No Response*	*Participant did not attempt the question*
RWJ	*Response without Justification*	*Participant attempted the question but did not provide support/ justification for their response*
DWD	*Definition of Linearity without Deduction*	*Participant used/applied the definition of linearity in their response but did not deduce/ achieve a conclusive result*
DWC	*Definition of Linearity with Wrong Conclusion*	*Participant used/applied the definition of linearity in their response but achieved a wrong result*
ILI	*Incorrect logic; Incorrect Conclusion*	*Participant used/applied a linear algebra concept (other than linearity) incorrectly, resulting in an incorrect conclusion*
AIL	*Affinity instead of Linearity*	*Participant used/applied the concept of linear functions (first-degree polynomials) instead of the definition of linearity, resulting in an incorrect conclusion*

CODE	*Category*	*Explanation of Category*
ILC	*Incorrect Logic; Correct Conclusion*	*Participant used/applied a linear algebra concept (other than linearity) incorrectly, however, the conclusion achieved was correct*
DLC	*Definition of Linearity with Correct Conclusion*	*Participant used/applied the definition of linearity in their response and achieved a correct conclusion*

Upon the assessment's completion, and in accordance with the ethics guidelines outlined by the IRB, participants were offered the opportunity to take part in a voluntary follow-up interview. Students were given the IRB-approved consent form to review and sign prior to their participation. 13 out of 39 students consented to take part in the interview process. The post-test interview contained 7 questions (see Appendix C for list of interview questions) and took approximately 15 minutes to complete. The interviews were administered in a one-on-one setting at a quiet, reserved study room at the UCF Library.

During the interview, students were offered a blank piece of paper to draw, write, and jot down their thoughts and explanations as they saw fit. In the interest of consistency, students' names were collected at the beginning of the interview to ensure the test they were provided with was their own. However, when discussing student responses, both in the analyses of questions and in reference to the interview discussion, participants will be referred to using a code name—P*n*, where *n* corresponds to their assigned number for their de-identified test. The interviews were audio recorded—with the consent of the participants—and later transcribed.

CHAPTER FOUR: RESULTS

The findings of this research are discussed in this section. Each of the three assessment questions provided focused on a different application of linearity. While both question 1 and question 2 required the use of vector arithmetic and an understanding of the definition of linearity, question 1 put an emphasis on the conditions of linearity itself, tasking students with checking the linearity of the given transformation.

On the other hand, question 2, while utilizing linearity (the students are given that the transformation is linear) primarily focused on students' ability to correctly perform the necessary computations under said conditions. Question 3 focused on a practical application, asking students, given a word-problem scenario, to set up the implied transformation. While most students were able to think of linearity within a non-mathematical context, the disconnect occurred when applying the definition of linearity directly. The following tables present the total number of student responses corresponding to the emergent categories.

Table 2: Frequency of responses per category for question 1

CODE	*Category*	*frequency*	*Percentage*
NR	*No Response*	*4*	*10.26*
RWJ	*Response without Justification*	*2*	*5.13*
DWD	*Definition of Linearity without Deduction*	*1*	*2.56*
DWC	*Definition of Linearity with Wrong Conclusion*	*6*	*15.39*

CODE	*Category*	*frequency*	*Percentage*
ILI	*Incorrect Logic; Incorrect Conclusion*	*6*	*15.39*
AIL	*Affinity instead of Linearity*	*6*	*15.39*
ILC	*Incorrect Logic; Correct Conclusion*	*4*	*10.26*
DLC	*Definition of Linearity with Correct Conclusion*	*10*	*25.64*

Table 3: Frequency of responses per category for question 2

CODE	*Category*	*frequency*	*Percentage*
NR	*No Response*	*0*	*0.00*
RWJ	*Response without Justification*	*0*	*0.00*
DWD	*Definition of Linearity without Deduction*	*1*	*2.56*
DWC	*Definition of Linearity with Wrong Conclusion*	*5*	*12.82*
ILI	*Incorrect Logic; Incorrect Conclusion*	*3*	*7.69*
AIL	*Affinity instead of Linearity*	*0*	*0.00*
ILC	*Incorrect Logic; Correct Conclusion*	*0*	*0.00*

CODE	*Category*	*frequency*	*Percentage*
DLC	*Definition of Linearity with Correct Conclusion*	*30*	*76.92*

Table 4: Frequency of responses per category for question 3

CODE	*Category*	*frequency*	*Percentage*
NR	*No Response*	*7*	*17.95*
RWJ	*Response without Justification*	*1*	*2.56*
DWD	*Definition of Linearity without Deduction*	*0*	*0.00*
DWC	*Definition of Linearity with Wrong Conclusion*	*3*	*7.69*
ILI	*Incorrect Logic; Incorrect Conclusion*	*5*	*12.82*
AIL	*Affinity instead of Linearity*	*0*	*0.00*
ILC	*Incorrect Logic; Correct Conclusion*	*0*	*0.00*
DLC	*Definition of Linearity with Correct Conclusion*	*23*	*58.97*

Across all three questions, DLC is the most common category, corresponding to approximately 37% of responses. This result is significant, indicating that over a third of the class correctly applied and utilized the definition of linearity across multiple domains. Responses which fell under DWC, ILI, and AIL (accounting for 29% of student replies) featured mistakes, misconceptions, and leaps in logic that students performed to achieve their conclusions.

The AIL category demonstrates students' use of prior knowledge of linear functions when attempting to use the new definition of linearity in the context of transformations. This mistake appeared frequently throughout question 1, that it felt necessary to include it as its own category to be analyzed. Responses belonging to this section indicate that students think of linearity as a linear function rather than considering the definition of linearity or the conditions defined in the transformation.

When considering the DWC and DWD categories, it is apparent that students, while able to recognize the definition of linearity (and partially apply it), had gaps in their ideas and application and as such were left with a wrong conclusion or no conclusion at all. Replies in the range of ILI, ILC, and RWJ demonstrated a prominent lack of understanding of linearity; with the concept being redefined and altered by the students. RWJ responses, particularly, did not explain the linear (or nonlinear) relationships given in the sample questions, rather providing a conclusion without an adequate basis.

ILC responses, however, while reaching a correct conclusion, did not follow the course's definition of linearity. Students, to justify their thoughts, elaborated on concepts previously established in the course (such as matrices, rank, and vector arithmetic) and applied them to the questions. However, the application or definitions were utilized incorrectly within the context of the questions, hence providing a false foundation for the students' conclusions.

Among all three questions, question 2 had the highest correct number of responses (about 77%) followed by question 3 (about 59%). Comparatively, question 1 featured the lowest number of correct responses (about 26%). This is a remarkable result. The prominent difference in students' ability to apply linearity in questions 1 and 2 reveals the nature of students' modes of thinking regarding linear transformations. Notably, said difference points to a strong, analytic-arithmetic-influenced thought process (Sierpinska, 2000). Given the frequency of student replies in each classified category, an analysis of each sample question was performed. What follows is a thorough, question-by-question investigation of the distinct categories, their significance, and the emergent modes of thinking.

CHAPTER FIVE: ANALYSIS

The analysis of participant responses reveals important insights into students' modes of thinking and approach to linearity within a linear algebra context. An evaluation of these responses is conducted using the modes of thinking identified by Sierpinska (2000). Keeping the research objective in mind, the three test questions were designed to test the completion of the transformation Schema (focusing specifically on linearity).

Since this investigation’s goal is to bridge the knowledge gap through the mastery of elementary concepts (like linearity), each question represents a unique component within the decomposition of the linear transformation map: (1) definition application, (2) computational skills, and (3) practical application. First, let us consider the responses for question 1.

<u>Question 1</u>

The purpose of question 1, within this investigation, was to determine if the students’ foundation of linearity, as it pertains to definition application, is sound. This question required the participants to apply the definition of linearity to a given transformation and deduce its behavior (linear or nonlinear). About 49% of students were unable to correctly reason about the nature of the transformation’s linearity.

An additional 15% of students left the question blank or provided no justification for their response. Only about 26% of participants were able to both correctly apply and deduce the linearity of the given transformation, with the remaining students correctly concluding about the transformation’s linearity through incorrect logical application of concepts. Let us analyze and compare the results of a select group of students, each corresponding to a unique code within the list of categories. Consider the responses provided by P3 (AIL), P4 (DWC), P8 (ILI), and P15

(ILC). P3's response (see Figure 1) consisted of a deconstruction of the transformation $T(x)$ into two separate components to be studied: $x + 1$ and $3x$.

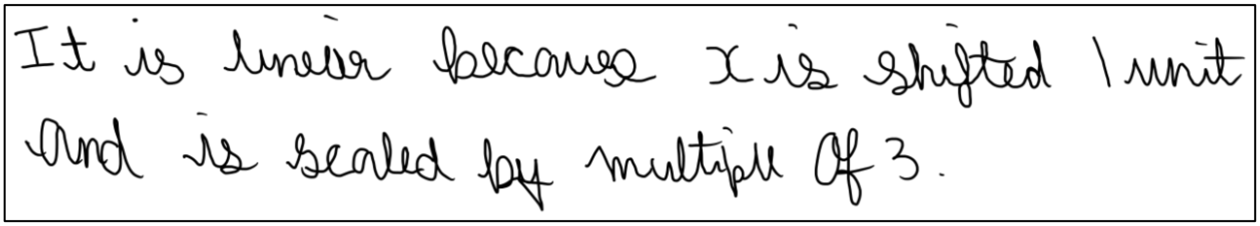

It is linear because x is shifted 1 unit
And is scaled by multiple Of 3.

Figure 1: P3 response to question 1

The student concluded that $T(x)$ was linear as both functions comprising the transformation are known linear functions. P3 reasoned that since a vertical shift (by 1 unit for $x + 1$) and a stretch (by a factor of 3 for $3x$) are acceptable transitions for "linear functions" (first-degree polynomials), that the transformation at hand is linear.

Two interesting observations are to be noted: (1) the student disregarded, or was unable to, utilize the definition of linearity, and (2) the student attempted to apply previous knowledge of functional relationships (established within the creation of the function Schema) demonstrating a clear case of the obstacle of formalism phenomena discussed by Dorier et al. (2000). This form of rationale aligns with Sierpinska's (2000) synthetic mode of thinking. P3's approach to the question suggests a preference for a pragmatic thought process. Their analysis of the components of the transformation as "linear functions" is indicative of the PT approach. In fact, among all students within the AIL category, signs of synthetic thinking were manifested; particularly with students' attempts to graph and plot the transformation (see Figures 2 and 3), pointing to a gravitation toward geometrically based representations.

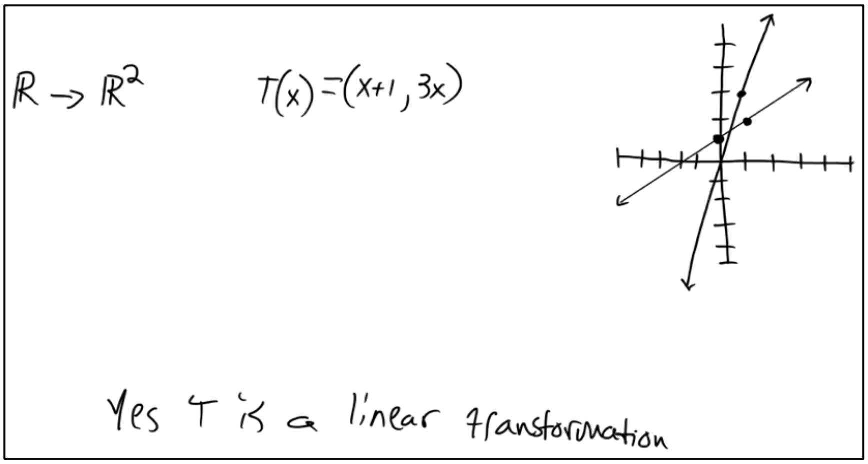


Figure 2: P21 response to question 1

P21's response falls under the AIL category; demonstrates an attempt to graph the transformation by plotting $x + 1$ and $3x$ on the plane.

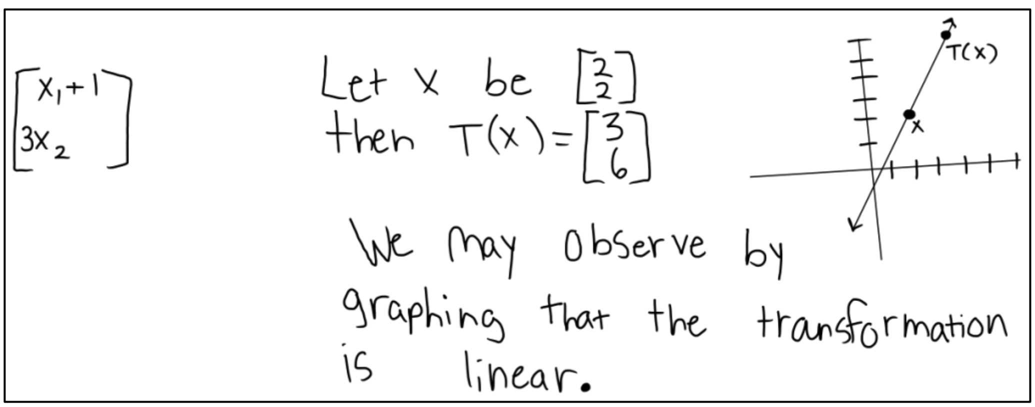


Figure 3:P10 response to question 1

P10's response falls under the AIL category; demonstrates an attempt to graph the transformation by considering a sample vector $x = \binom{2}{2}$ and comparing it to its image under T, $T(x) = \binom{3}{6}$. The participant used these two points to create a plot of the transformation.

It appears that students in the AIL category put greater emphasis on their understanding of functions, specifically first-degree polynomials, rather than the definition of linearity itself. It can be concluded that the AIL category reflects an initiation of the transformation Schema, however, the definition of linearity (within the function Schema context) took precedent.

Next, observe the response provided by P4 (see Figure 4). Unlike P3, this student attempted to deduce the linearity of the transformation using the definition of linearity. However, their portrayal of the definition and its application in their response were incomplete.

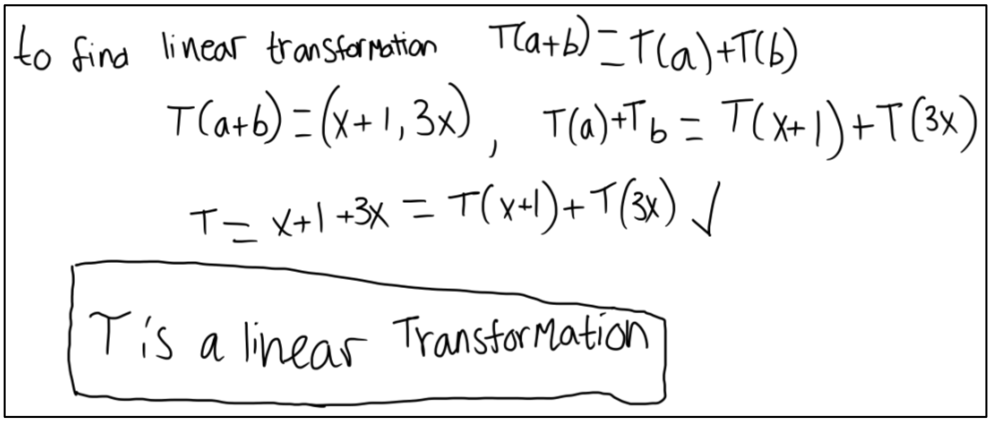


Figure 4: P4 response to question 1

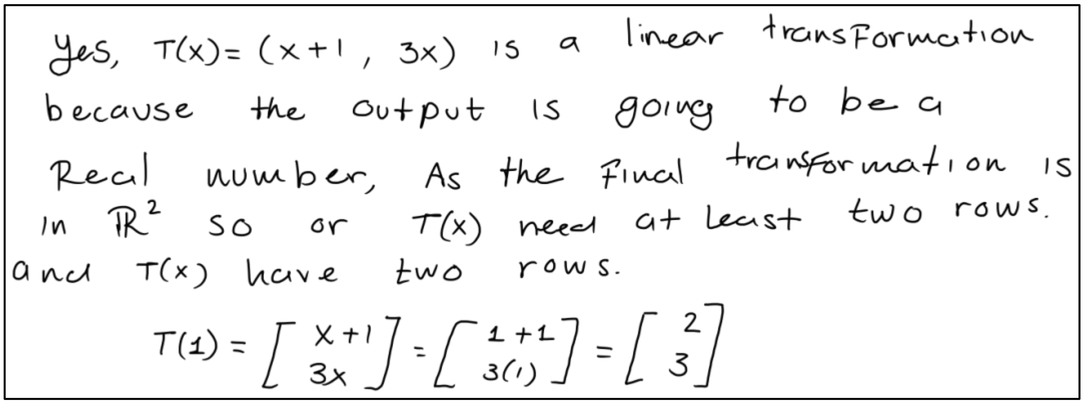
The student only listed one of the conditions for linearity, focusing solely on the additive property of linear transformations. Furthermore, their application of their definition was incorrect. While the student correctly noted condition (1) of the definition ($T(a+b) = T(a) + T(b)$), the student then applied the following step: $T(a) + T(b) = T(x+1) + T(3x)$, suggesting an insufficient understanding or familiarity with the notation.

P4 concluded that the relationship is linear since their initial condition for linearity was satisfied. The rationale used by P4 best aligns with Sierpinska's (2000) analytic thinking mode. The student favored the use and application of a definition when approaching the question at hand. However, their definition along with their ability to successfully articulate and work with the additive condition was lacking. As such, P4 leans more towards structural-arithmetic thinking, emphasizing the mathematical properties of the definition itself compared to manipulation and application of the definition's formula.

Moreover, when considering the student's application of their definition to the transformation, it is apparent that not only was the student unknowledgeable about or unfamiliar with the notation of the definition but also, they demonstrated a lack of understanding when attempting to apply it. This is seen through the student's inability to successfully substitute inputs into $T(x)$ to perform the test for linearity along with the attribution of the components of $T(x)$ as the inputs themselves. These properties are indicative of a deeper issue rooted within their function Schema. This student could not properly manipulate equations and their inputs or adequately apply the rule of the given function.

In fact, when considering the responses of other participants within the DWC category, a similar pattern can be observed within a major sector of the responses. The remaining responses in the category demonstrated an arithmetic-analytic approach. These students, while their definition of linearity is not always complete, demonstrate an ability to work with the conditions of linearity and perform the (partial) correct arithmetic corresponding to the given property.

P8's (see Figure 5) method, like P4, focused on a linear algebra approach to the question. However, unlike P4, this student did not focus on the definition of linearity within a linear algebra context. Instead, P8 tried to use concepts previously established in the course to justify their process.



Yes, $T(x) = (x+1, 3x)$ is a linear transformation because the output is going to be a Real number, As the final transformation is in $\mathbb{R}^2$ so or $T(x)$ need at least two rows. and $T(x)$ have two rows.

$$T(1) = \begin{bmatrix} x+1 \\ 3x \end{bmatrix} = \begin{bmatrix} 1+1 \\ 3(1) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Figure 5: P8 response to question 1

This student concluded that the transformation's linearity was directly correlated to their domain and codomain. Specifically, since the transformation's inputs are in $\mathbb{R}$ and the outputs are in $\mathbb{R}^2$, they reasoned that the output is going to be a real number; and, as the outputs are in $\mathbb{R}^2$, the transformation will require two rows for the output, which according to them, $T(x)$ possesses. Therefore, $T(x)$ is linear. P8 did not define or elaborate upon what constitutes or qualifies linearity within the context of transformations. Instead, they attempted to determine the transformation's behavior through the transformation's makeup (domain and codomain). However, P8's conceptual application is flawed, suggesting their understanding of the mathematical properties governing linear transformations is poor. The student made a false connection between the number of rows in a matrix and the transformation's linearity. Using this student's logic, any transformation T, defined $T: \mathbb{R} \rightarrow \mathbb{R}^2$, will be linear. This, of course, is not the case.

P8's applications of concepts throughout question 1 would suggest an analytic thought process aligning with the structural-analytic mode (Sierpinska, 2000). This is evident through P8's attempt to elaborate upon concepts (while not necessarily providing clear definitions of objects) to justify their rationale, and restructuring $T(x)$ as a 2×1 matrix to indicate that when inputs from the vector space $\mathbb{R}$ are plugged into $T(x)$, the outputs will be 2×1 matrices whose elements are real numbers. The student demonstrates the computation of $T(1)$ as an example.

A similar pattern can be observed among the other ILI responses. All the responses in this category made use of matrix representation of a transformation, referencing the number of rows/columns of the matrix (like P8) or the number of pivot positions.

Lastly, consider the response given by P15 (see Figure 6). This student, like P8, used a faulty matrix representation of the transformation to determine its linearity. However, while their logical basis was incorrect, their conclusion was. P15 determined that since the number of rows was altered, a new matrix was created and therefore, the transformation was not linear.

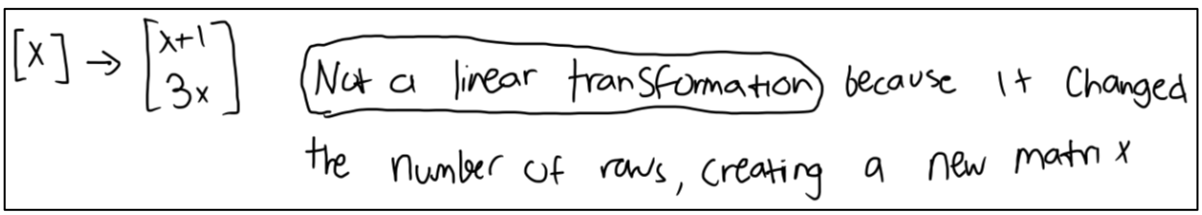


Figure 6:P15 response to question 1

While the transformation was, indeed, nonlinear, this leap in logic would suggest that any transformation where the domain and codomain have different dimensions is non-linear as well. However, this is a false, easily refutable, conclusion. It appears that both P8 and P15 used a similar approach to justify their results. In fact, all the students under the ILC category appeared to be making similar leaps in logic to P8 and P15 to deduce the transformation's linearity. P15's approach, unsurprisingly, like P8's and the rest of the ILI sector, is indicative of Sierpinska's (2000) structural-analytic mode.

The analysis of question 1 yielded significant results. In the quest to improve upon the gap in student knowledge, specifically, as it relates to the concept of linearity, this question served as a test of student's understanding of the core definition of linearity within linear algebra. This question focused on definition application and required understanding of the notation and arithmetic necessary to successfully complete it. As such, it appears that students who are analytical thinkers had a slight upper hand when attempting to understand and answer this question. The structural-analytical mode garnered an additional advantage, as this mode is heavily oriented around the application of mathematical properties, a quality necessary to answer question 1.

It is important to note, however, that students who exemplified the structural-analytic mode did not always complete the question successfully, as evident by students like P8 and P15 of the ILI and ILC categories. These students demonstrated the ability to focus on the mathematical properties of objects and draw conclusions based on them, yet, their knowledge of how to use or apply said properties was lacking. This suggests students in these categories possess the potential to comprehend and correctly apply the conditions of linearity. However, further work with the definition itself is necessary to ingrain the principles of linearity for said groups.

Question 2

The purpose of question 2, within this investigation, was to test students' ability to perform computations given that the conditions of linearity hold. This question saw the highest success rate among the three questions (about 77%). The skills necessary to successfully complete this question seemed to have been acquired by a major sector of the class compared to question 1's necessary skill set. This fact is unsurprising given that while both questions required knowledge of proper arithmetic, question 2 utilized simple computation techniques compared to question 1's focus on the definition of linearity, requiring a deeper understanding of notation, which, in its nature, is more abstract a concept. Nevertheless, let us consider two sample responses. Observe the answers from P4 (DWD), and P14 (ILI).

P4's response (see Figure 7) is of particular interest as out of all the responses to question 2, their response was the only one classified under the DWD category.

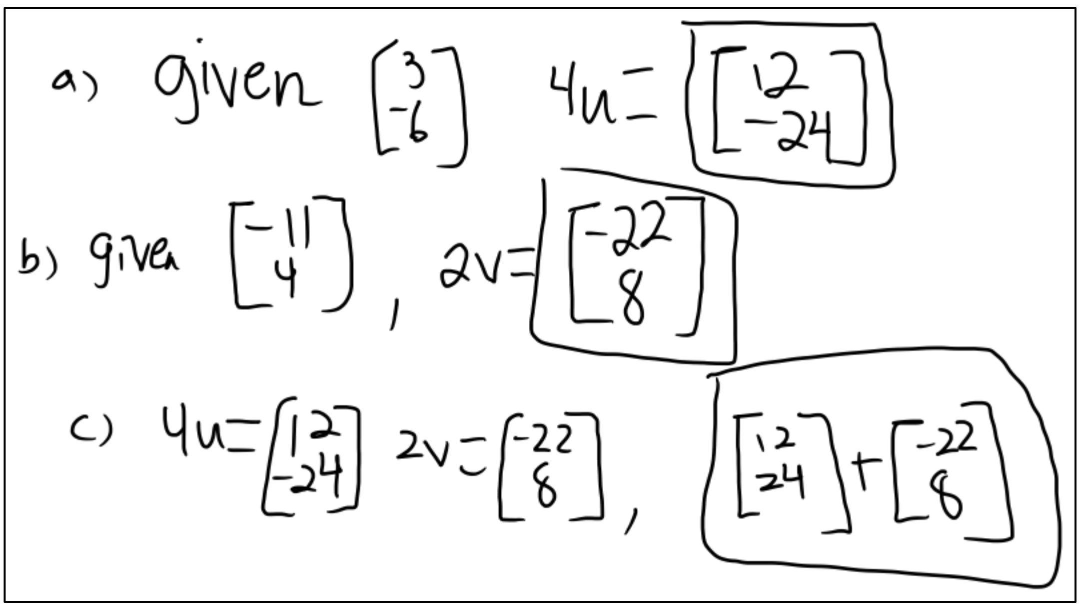


Figure 7: P4 response to question 2

Their understanding of the notation, in a similar fashion to their previous response, was incomplete. This is evident through their classification of the images under the transformation as scalar multiples of the vectors u and v. P4, while utilizing incorrect notation, did manage to perform the correct computation of the images under T of $4u$ and $2v$, however, when considering the image of $4u + 2v$ their response was incomplete. The student completed the setup for the addition but did not conduct the process itself. This could indicate that either: (1) the student ran out of time and could not complete the computation, or (2) the student could not perform the necessary vector arithmetic. Given P4's previous demonstration of their vector arithmetic in question 1, option (2) seems plausible. This student may need more practice with the properties of vectors; P4 demonstrates signs of initiation of their transformation Schema evidenced through partial application and knowledge of the definition of linearity and notation, however, the Schema is yet to be fully integrated. P4, once again, displays an analytically based thought process (Sierpinska, 2000). Their ability to conduct the correct arithmetic for the scalar multiplication section of the question suggests that their application aligns with the analytic-arithmetic mode.

It is important to note that the shift from P4's structural- analytic to arithmetic-analytic approach indicates that thinking modes are dynamic; one's thought process can be fluid and shift based on the question at hand. When analyzing the three-test questions, it was apparent that synthetic thinkers tended to stick to the synthetic approach while the analytical thinkers showed more flexibility in their thinking, fluctuating between arithmetic and structural processes.

Next, observe the response of P14 (see Figure 8). Unlike P4, who demonstrated a base level of the necessary arithmetic skills, P14's application of vector properties was poor. Their understanding of the notation was lacking or non-existent.

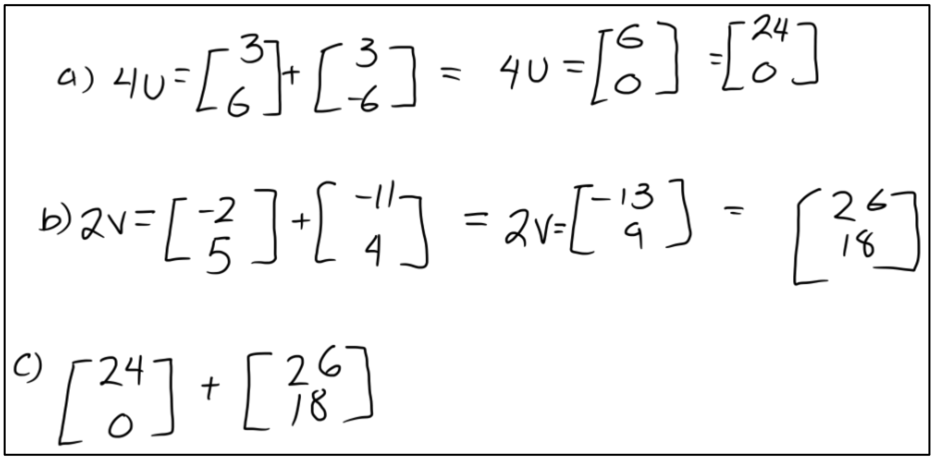


Figure 8: P14 response to question 2

For instance, P14 attempted to define the transformation as taking a vector x and returning $x + T(x)$ as the new vector. However, not only is this portrayal unrepresentative of the transformation at hand but the definition of a function $T(x)$ cannot depend on $T(x)$ itself. This suggests a disconnect within P14's function Schema. It is possible that the student may have attempted to define $T(x)$ as a recursive function, however, even if that were the case, it would indicate a poor understanding of recursive functions and their properties; a skill obtained within a function Schema (distinguishing types of functions and their domain and range).

Using their definition of $T(x)$, P14 then manipulated the given vectors. Assuming their definition was correct, their scalar multiplication and addition skills themselves can be considered "correct" as well. The severe lack of proper notation, along with a lacking understanding of the conditions of linearity, point to a struggle with analytical thinking and could suggest P14 is more compatible with the synthetic-geometric approach (Sierpinska, 2000).

The analysis of question 2 responses produced noteworthy observations. As previously mentioned, about 77% of the participants were able to correctly perform the vector addition and multiplication required in the problem. However, even among the DLC responses, a prominent lack of proper, correct notation is present. In fact, 18 out of the 30 DLC responses (60%) demonstrated issues with notation, specifically, writing $4u, 2v,$ and,$4u + 2v$ when referring to $T(4u), T(2v),$ and, $T(4u + 2v)$.

While these students were capable of correctly computing the images under the transformation, the improper notation indicates a slight gap in the conceptual perception of the definition of linearity. This could also explain the jump in responses in the DLC category from question 1 to question 2. When considering the significance of correct notation to the understanding of linearity conditions, additional work and practice are needed. It seems the student struggle with question 2 was primarily derived from gaps in the definition of linearity and its proper notation.

This is consistent with the established literature (Dorier et al., 2000, Britton & Henderson, 2009) investigating student struggle in linear algebra; students tend to attribute difficulty to theoretical and abstract concepts rather than to practical, calculation-based notions. Question 2 represents an arithmetic-focused, computational problem which, often, will not pose a major obstacle to student understanding.

Question 3

The purpose of question 3, within this investigation, was to test students' ability to apply the conditions of linearity outside of a mathematical framework. This exercise required students to think about a given scenario which described a color machine taking color inputs (of any positive, negative, or zero amount) and returning, as an output, the input added with two units of red color. The students, being familiar with the transformation set up and notation, should have been able to recognize and dissect this scenario into mathematical objects: the color machine constitutes the transformation (or function) itself and the domain and codomain of the transformation is the set of all colors. This question produced the most creative results, with students navigating the challenge of translating the given scenario into its mathematical components. Question 3 also saw the greatest number of no responses, with seven students leaving the question completely blank.

Let us analyze the results of P32 (ILI) and P35 (DLC). The response provided by P32 (see Figure 9) utilized a matrix representation to approach the scenario. This student dissected the scenario into the following transformation: $T = (2r + 2c)$.

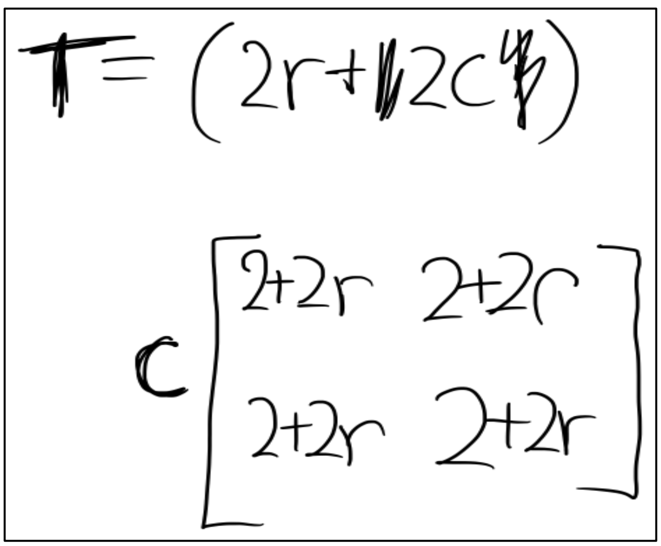


Figure 9: P32 response to question 3

One key observation to note here is that the student did not specify if T is a function of c or a function of r. Additionally, no mention of the domain or codomain of the transformation was included. P32 drew what appears to be a 2×2 matrix of the transformation with every entry in the matrix being $2 + 2r$. The matrix itself was multiplied by c. It is unclear if c is taken to be a scalar in this context or as a variable (as intended in the directions for the question).

P32 was unable to correctly break the scenario into mathematical objects. Moreover, their transformation setup suggests that the color input, c, is doubled then combined with two units of red color ($2r$), however, that is not the case. Their matrix representation is flawed, as well, with no supporting work to justify its elements. P32's thought process seems to best align with the synthetic mode (Sierpinska, 2000). Their struggle with the notation and baseless matrix representation suggests an insufficient understanding of the mathematical properties of transformations and matrices.

However, their ability to partially define the transformation (in terms of the constant addition of two units of red color) signifies a synthetically based approach; the objects introduced in the scenario are considered within the realms of practical thinking, treating the objects to be studied through direct action. P32 did not use any mathematical properties to define their transformation and neglected to include the domain and codomain. Their response points to a lack of understanding of the properties of a transformation, its notation, and the proper use of matrix representations, indicating that further work with the mathematical properties of transformations is needed for them to successfully derive the components of a transformation from a word problem scenario.

Comparatively, consider the response of P35 (see Figure 10). This student's response demonstrated a clear understanding of the necessary procedure to obtain the transformation from the given scenario. The student correctly identified the transformation to be $T(c) = c + 2r$.

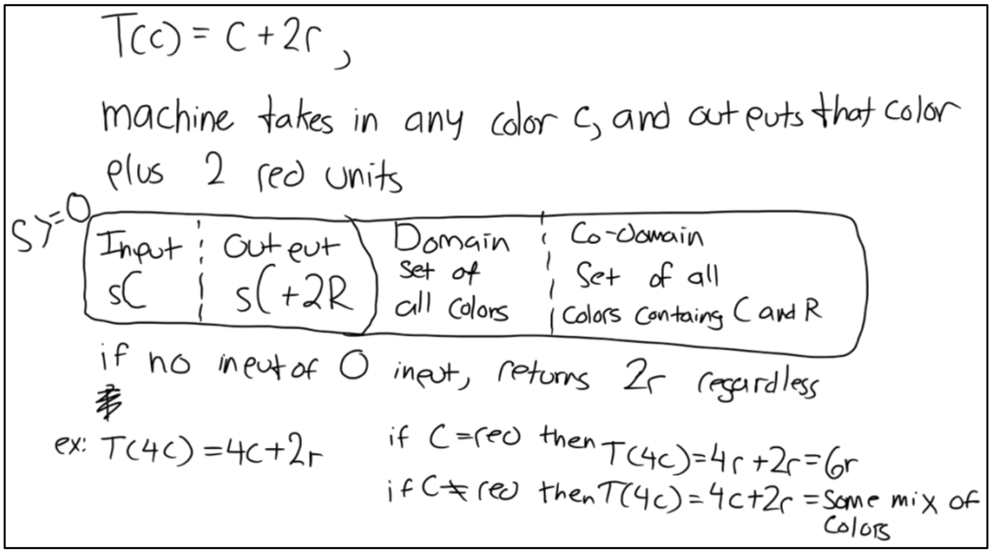


Figure 10: P35 response to question 3

P35 created a table listing the input, output, and the domain and codomain of their transformation. Their ability to correctly identify and construct the transformation implies a sound foundation of transformations and points to further development of their transformation Schema. The student also included miscellaneous calculations using their defined function to understand its behavior. They included an example of the transformation evaluated at $4c$. P35 indicated that if $c = r$, then the transformation would yield $6r$ as an output. Otherwise, it would yield $4c + 2r =$ *some mix of colors*. The student also mentioned that in the case that $c = 0$, the output of the transformation will be $2r$.

This process performed by P35 suggests a strong alignment with the structural-arithmetic mode (Sierpinska, 2000). The student also shows some synthetic-geometric thinking, with the student's ability to consider the objects of the scenario directly and break them successfully into mathematical objects to be studied. In fact, when considering students who ranked in the DLC category, it is apparent that the practical thinking skill associated with synthetic thinking was crucial when attempting to break the scenario into its mathematical objects and deriving the correct components for the transformation, including the domain and codomain.

The analysis of question 3 yielded significant results. In the quest to improve upon the knowledge gap, specifically as it pertains to linear algebra, this question served as a test of students' ability to draw upon a learned concept or definition (linearity and linear transformations) and use said knowledge to construct a solution. The composition of question 3 required students to both practically think and apply their idea of transformations to a non-mathematical scenario and then define the transformation's domain and codomain. This task requires some combination of the synthetic and arithmetic modes.

First, the student needs to break the scenario into mathematical objects to build their transformation (synthetic) then define said objects using the properties of transformations (arithmetic). When comparing the success rate among all three questions, an interesting pattern can be observed: questions targeted toward an analytic-arithmetic mode (question 2) saw a higher success rate compared to questions focused on an analytic-structural mode (question 1).

It is also apparent that the success rate of combined synthetic and analytic questions (question 3) was increased as well. Overall, students were able to correctly perform computations (based on the conditions of linearity) at a higher rate compared to using the conditions themselves to prove (or disprove) linearity. Hence, the analytic-arithmetic mode is more prominent among students compared to the analytic-structural mode.

Moreover, when asked to convert a word problem scenario into a transformation, practical thinking skills were essential, corresponding to the synthetic-geometric mode. Since question 3 only called for the transformation's setup, it can be concluded (based on the number of correct responses) that students are more comfortable identifying a function's components (variables, domain, range, etc.) compared to applying the definitions describing the behavior of said function.

Interviews

Upon the evaluative assessment's completion, the post-interview was conducted with the consenting participants. Every interviewee got the chance to demonstrate and elaborate upon their thought processes and explain their approach. As a majority (about 77%) of the original student group completed question 2 successfully, the interviews predominantly focused on questions 1 and 3. Each of the seven interview questions aimed to provoke thought and instigate a process of reflection among the students. The goal of the post-assessment interviews was to expand upon the student experience and garner a first-hand account of the students' method concerning the test questions. A sample of participant responses to a selection of the interview questions is presented in this section. First, let us consider the interview conducted with P3.

P3 Interview

The interview with P3 shed light on their understanding of and thought process surrounding linearity. Their responses to the interview questions suggested a faulty association between linear transformations and first-degree polynomials. For instance, P3's response (see Figure 11) to the fourth interview question demonstrated a disregard for or lack of understanding of the linear algebra context for linearity.

Researcher (Interview Q4): "Think about a general line of the form *y = mx + b, m, b ≠ 0*. Based on your knowledge of linearity, is a line of this form a linear transformation? Why/ why not? "

P3: "I think it is linear because lines are linear functions."

Researcher: "Can you elaborate on what you mean by linear functions?"

P3: "Yeah, so they are functions that are lines. Like in Q1, for example, I thought that since *x+1* and *3x* are linear, it (points to *x+1* on Q1 prompt) is shifted up by 1 and *3x* is scaled by 3. So it's linear."

Figure 11: An excerpt from interview with P3 (interview question 4)

Question 4 of the interview aimed to understand the existence of the student-made correlation between first-degree polynomials and linearity (as seen throughout question 1 of the assessment). P3 reasoned that a line of the form $y = mx + b; m, b \neq 0$ must be linear as it is a "linear function." They connected their rationale to their response for question 1 of the assessment; suggesting that there exists a relation between linear functions (like $x + 1$ and $3x$) and linear transformations. P3 expressed an AIL-based rationale throughout their interview. In their response to questions 3 and 5 (see Figure 12), for example, they noted that the transformation presented in the third question of the assessment must be linear as, it too, is a linear function. Specifically, P3 suggested that, again, since the translation portrayed in the question is a defined vertical shift for linear functions, the transformation described must, therefore, be linear as well.

Researcher (Interview Q3): "Could you explain your thought process for Q3?"

P3: "I tried to set up the transformation like it was described in the problem."

Researcher: "Could you elaborate upon your setup?"

P3: "Yeah so basically we're adding 2 units of red color each time so kind of like a vertical shift."

Researcher (Interview Q5): "So would you say the transformation described in Q3 is a linear transformation?"

P3: "Yeah."

Researcher: "Why?"

P3: "Because a vertical shift is linear. So like we're adding plus 2 of red so that's a shift of the transformation which makes it linear."

Figure 12: An excerpt from interview with P3 (interview questions 3 and 5)

It appears that P3, when approaching the concept of linearity, considered the possible alterations that can be acted upon standard linear functions (vertical/ horizontal shift, stretch/shrink, and reflection/rotation) and reasoned the behavior of the provided transformations through said actions. It is important to note, however, that these shifts are not exclusive to first-degree polynomials (which seemed to drive P3's rationale).

Furthermore, it is evident that this participant did not utilize or consider the linear algebra presentation of linearity provided in the course. In fact, throughout their interview, P3 made no mention of the conditions of linearity themselves, continuously referencing linearity and linear transformations within a fist-degree polynomial context. Similar lines of thought can be seen in P20's interview.

P20 Interview

The interview conducted with P20 demonstrated similar tones regarding the concept of linearity. Like P3, P20 reasoned that the transformation described in question 3 of the assessment is linear. However, this participant also indicated a level of confusion and uncertainty when approaching the question itself.

Researcher (Interview Q5): "Is the transformation described in Q3 a linear transformation? Why/ why not?"

P20: "I'm not entirely sure but I think it is (pauses to think). I'll be honest, this whole subject was confusing."

Researcher: "What do you think caused the confusion?"

P20: "I thought that linearity meant like linear (pauses) like a line. So like for Q3, I thought that it would be linear because we're adding 2r every time which is constant."

Figure 13: An excerpt from interview with P20 (interview question 5)

P20 described a sense of confusion with the terminology surrounding linearity derived from their previous knowledge and understanding of the concept. Linearity, as a mathematical term, is often introduced to students in the context of first-degree polynomials (described as “linear functions”). This previously established definition of linearity seems more institutive and far more ingrained within the students’ mathematical framework. As such, describing a function, whose graph is a line, as “linear” may seem more reasonable compared to considering “conditions for linearity.”

It is evident that this new idea of linearity was not regarded as its own concept to be learned and, therefore, the new terminology caused confusion among some of the participants. Linearity, within a linear algebra context, shakes the foundation of established linear functions;

suggesting that not all linear functions are indeed linear or linear transformations. P20 indicated difficulty with the concept as early as the first interview question (see Figure 14). When asked about their knowledge regarding a transformation's linearity, they revealed that they struggled grasping the definition of linearity, particularly, as it pertained to the notation or "terms" utilized when testing the conditions.

Researcher (Interview Q1): "What do you understand about a transformation being linear?"

P20: "I don't really know. Like I know there's the two things you test but I'm not sure."

Researcher: "Can you recall anything about those two conditions for linearity?"

P20: "Not really. (pauses to think) I think I got a little confused when doing this test, so like here (points to Q1 on their response paper) I didn't use the conditions."

Researcher: "Did you find the definition itself confusing?"

P20: "Yeah, I just didn't understand how to use this definition I guess. (pauses) Like the terms didn't make sense to me."

Figure 14: An excerpt from interview with P20 (interview question 1)

The account provided by P20 is indicative of a struggle with the idea of linearity and linear transformations derived from two key components: (1) a faulty connection to previously established concepts, and (2) unfamiliarity with the notation and terminology. This phenomenon is further reflected in the interview conducted with P2.

P2 Interview

P2, like P3 and P20, also discussed difficulties with grasping the concept of linearity. However, their struggle was manifested through their application and connection to matrices and vectors. P2 revealed that they think of transformations as functions of vectors, with linear transformations consisting of "linear formulas" which are applied to transformations. P2 included a drawing (see Figure 16) to explain their thought process.

Researcher (Interview Q1): "What do you understand about a transformation being linear?"

P2: "I like to think of transformations as functions with linear transformations being an extension of them."

Researcher: "Could you elaborate on that?"

P2: "Yeah, so for example if you have a linear formula, so like let's say a vector v plus 2. So that function just adds by 2 so it will just take this one (points to the vector v they wrote on the paper) and add it. You could also multiply by 2, or any scalar, I guess, and just stretch the function. So linear transformations are just something you apply to a linear function, I guess."

Figure 15:An excerpt from interview with P2 (interview question 1)

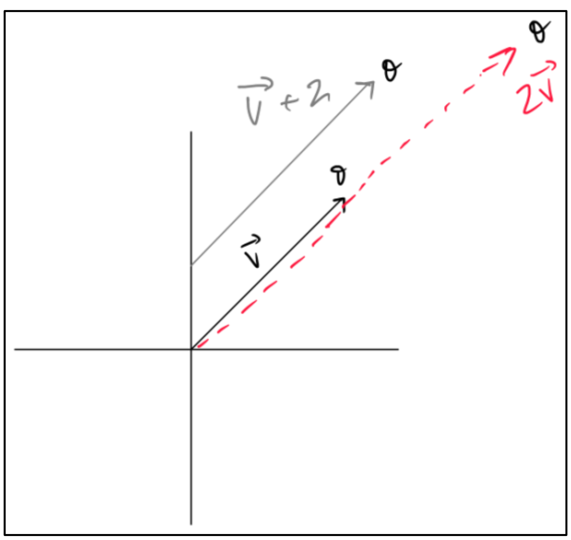


Figure 16: Drawing included by P2 in response to interview question 1

P2's explanation of linear functions. The graph contains three functions. A parent function v, $v + 2$ and $2v$. P2 discussed possible linear functions that, in their eyes, constitute an application of linear transformations.

When asked about their application for question 1 of the assessment, P2 admitted that they drew a blank when it came to the definition of linearity. Instead, they attempted to use their knowledge of vectors and matrices to complete the problem.

Researcher (Interview Q3): "Could you explain your thought process for Q1?"

P2: "I'll be honest, I blanked a little bit and completely forgot the definition and transformations. So I tried to play with the matrix representation."

Researcher: "I noticed you plugged in *e1* and *e2* into the transformation."

P2: "Yes, I was just trying to make sense of the transformation. I could tell from my result that the transformation was nonlinear, but I wasn't sure how to explain it."

Researcher:"What made you think it was nonlinear?"

P2:"When I think of something being linear, I think that when something is applied to one side it should also be applied to the other side. So here we had x+1 and 3x which are different from each other. So if it was linear it would be x+1 on both sides."

Figure 17: An excerpt from interview with P2 (interview question 3)

P2's responses demonstrated an overall lack of conceptual understanding of linearity within a linear algebra context. Both P2 and P20, when attempting to answer question 1, utilized some prior understanding of linearity to rationalize their result. However, while P2 demonstrated an ability to think about transformations as functions—which is pivotal for the formulation of the transformation Schema—they fell short in their understanding and use of linear functions within the scope of linear transformations.

Furthermore, their faulty logical application in question 1 suggests further work with the definition itself is necessary. Their idea of linearity appears to also be rooted in some form of symmetry, as they reference the need for balanced action to be taken on "both sides." P2 considered linear relationships as vectors where every component is identical, noting that the transformation described in question 1 was nonlinear due to its different functional elements. This, however, is a logical fallacy. Moreover, during the interview, P2 mentioned the term “weights” (see Figure 18) when discussing the setup for question 3.

They noted that the addition of a constant (in this instance, 2r) is the “weight” of the function. P2 claimed that the term helped them visualize the action of the transformation, referencing their computer science background in their explanation. This result is significant, indicating that students may draw on their existing knowledge not only from previous mathematics courses but also from various STEM-related fields (like computer science). This is consistent with the established importance of linear algebra within STEM itself (Pearlmutter & Šmigoc, 2018).

Researcher (Interview Q3): "Could you explain your thought process for Q3?"

P2: "I tried to draw the machine transformation but I thought that the output would just be 2r, which, re-reading the question now, I realize I got wrong."

Researcher: "What do you think caused you to think that?"

P2: "I understood the transformation to be c goes into the transformation but then you get 2r in return. So like no matter what input you plug in, you'll always get 2r."

Researcher: "I see. Do you think you could fix your setup right now?"

P2: "Yeah, (writes transformation on paper) so now you have c (with scalar = 1) as your input and 2r is your weight for the function."

Researcher: "Could you explain what you mean by weight?"

P2: "Oh, it's just how I think about constants being added, they're like a weight on the function. In computer science we talk about weighted functions like that so it makes sense to me."

Figure 18: An excerpt from interview with P2 (interview question 3 continued)

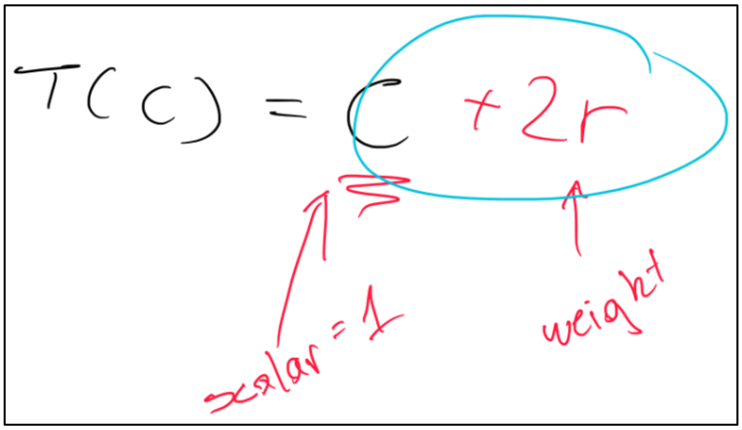


Figure 19: Work included by P2 in response to interview question 3

P2's new setup for question 3 of the assessment. They indicate that 2r is the "weight" of the transformation with c, the input, being scaled by a factor of 1.

The interview with P2 identified similar struggles to those present in P3's and P20's interviews. By identifying their mistake in question 3, P2 was able to correctly set up the transformation and apply their idea of weighted functions in their set up. This suggests that with additional exposure to the terminology of transformations and practice with the definition of linearity, P2 could improve upon their transformation Schema. A similar case can be observed in the interview conducted with P26.

P26 Interview

P26, like P3, P20, and P2, struggled with the definition application of linearity. Upon reintroducing the definition in the interview session, P26 was asked to think about what would happen if the given transformation in question 1 was altered from $T(x) = (x + 1, 3x)$ to $T(x) = (x, 3x)$. They concluded that based on the definition and conditions—as discussed and revisited in the interview—that the transformation would be linear. P26 then attempted to test the two conditions of linearity for the new $T(x)$ (see Figure 21).

Researcher (Interview Q7a): "What would happen if, for Q1, the given transformation was *T(x) = (x,3x)*?"

P26: "I think it would be linear."

Researcher: "Why?"

P26: "Based on what we've talked about so far, I feel like it would fit the definition." (proceeds to attempt to verify linearity conditions)

Researcher(Interview Q7b): "What does that tell you about the relationship between the addition of non-zero constants and linear transformations?"

P26: "They make it nonlinear."

Figure 20: An excerpt from interview with P26 (interview question 7)

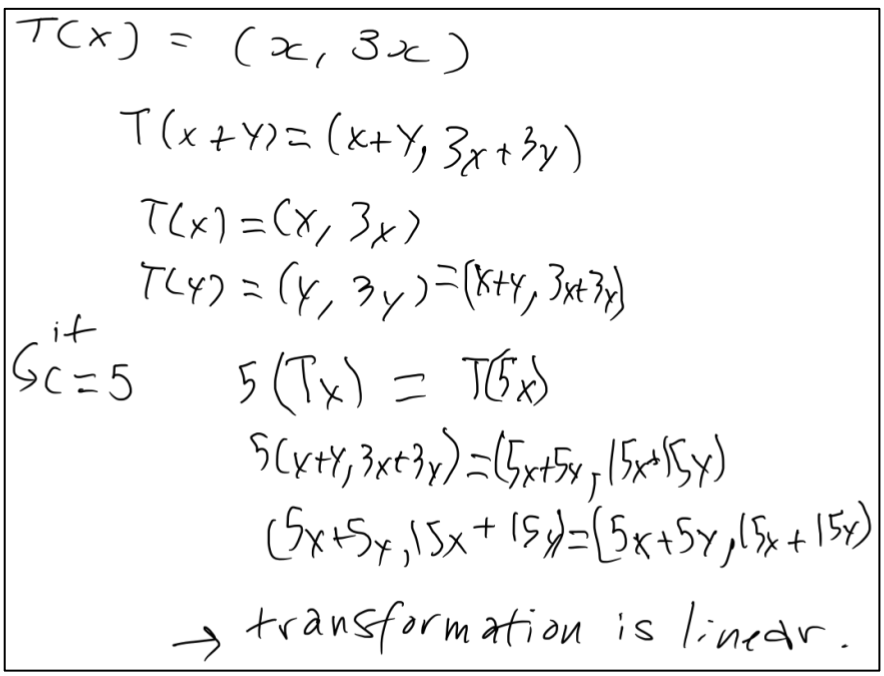


Figure 21: Work included by P26 in response to interview question 7

P26's test for the conditions of linearity for the new $T(x)$. P26 tested for additivity using arbitrary vectors x and y but used $c = 5$ for the scalar multiplication test.

During the interview, P26 noted that they are going to use $c = 5$ in their test to "see how the transformation behaves." This may suggest that P26, while comfortable using arbitrary vectors for vector addition, still benefits from the use of numbers when applying other vector arithmetic (like scalar multiplication). As such, while P26 demonstrated correct arithmetic and definition, their proof is incomplete; an application of the scalar multiplication test using an arbitrary c value needs to be performed to ensure that the condition holds true for all scalars, c.

P26, upon testing the two conditions, concluded that the transformation was linear. They also reasoned that adding non-zero constants constitute nonlinear behavior. This further exemplifies P26's new understanding of the material, suggesting they can form a distinction between linear functions and linear transformations. Although their proof, in its entirety, is insufficient, their application points to the initiation of a Process relevant to their establishment of their transformation Schema.

CHAPTER SIX: DISCUSSION

Both the analysis and interviews revealed significant information regarding students' modes of thinking in linear algebra. The analysis revealed that among all three test questions, an issue with notation and its proper usage was prevalent. The interviews further indicated that students faced difficulty understanding the definition of linearity within a linear algebra context, citing the terminology and misconceptions of the definition as prime factors in their confusion. These results are consistent with the literature on student struggle with mathematics, specifically as it pertains to the learning and understanding of mathematical notation.

When investigating student difficulties with mathematical terminology, Mulwa (2015) found that students' inadequate grasp of the mathematical language yielded logical inconsistencies. She observed that this gap in student knowledge caused confusion, e.g., some terms were mistaken with others under the guise that both terms corresponded to the same mathematical operation, and some terms were given informal interpretations rather than a mathematical one (Mulwa, 2015). The observations described by Mulwa correlate with students' difficulty with the concept of linearity. For instance, students' confusion of linearity within a linear algebra context and linearity as an affine function (first-degree polynomial). The AIL category in question 1 points to a student-made connection between linear transformations and linear (first-degree polynomial) functions.

Furthermore, the difficulty with notation and retention of the conditions of linearity support the existence of the formalism obstacle suggested by Dorier et al. (2000). Students, to understand linearity as a new concept, attempted to draw upon previous knowledge (matrices, functions, rank, dimension, etc.) to form connections to the current material. The three thinking modes described by Sierpinska (2000) help guide a conversation to improve upon the gap in knowledge and the current linear algebra curriculum.

Consider the synthetic-geometric mode; students who operate under this mode of thinking tend to, primarily, favor geometric representations of mathematical concepts. Aspari et al. (2019) noted the benefits of said representations in their research; their study focused on the impact of geometric representations on student thinking in pre-algebra courses. They observed that the usage of geometric-based visuals enhanced students' ability to identify patterns and construct generalization within an algebraic context. A similar approach can be taken when considering linear transformations.

The difference between linear and non-linear transformations can be demonstrated through a graph (where each element of a transformation is considered as a separate entity). If all elements of a transformation are linear, then the transformation will also be linear. The figure below (Figure 22) displays such a representation using the transformation in question 1. The transformation $T(x)$ is broken into its components, with $f_1(x) = x + 1$ and $f_2(x) = 3x$. If the transformation is linear, then the additive condition will hold true for both $f_1(x)$ and $f_2(x)$; hence, every point $(x_0, f_1(x_0))$ and $(x_0, f_2(x_0))$ on the graph of T will have a one-to-one correspondence with $(x_0, f_1(x_1) + f_1(x_2) + \cdots + f_1(x_n))$ and $(x_0, f_2(x_1) + f_2(x_2) + \cdots + f_2(x_n))$, respectively, where $x_0 = x_1 + x_2 + \cdots + x_n, n \in \mathbb{N}$.

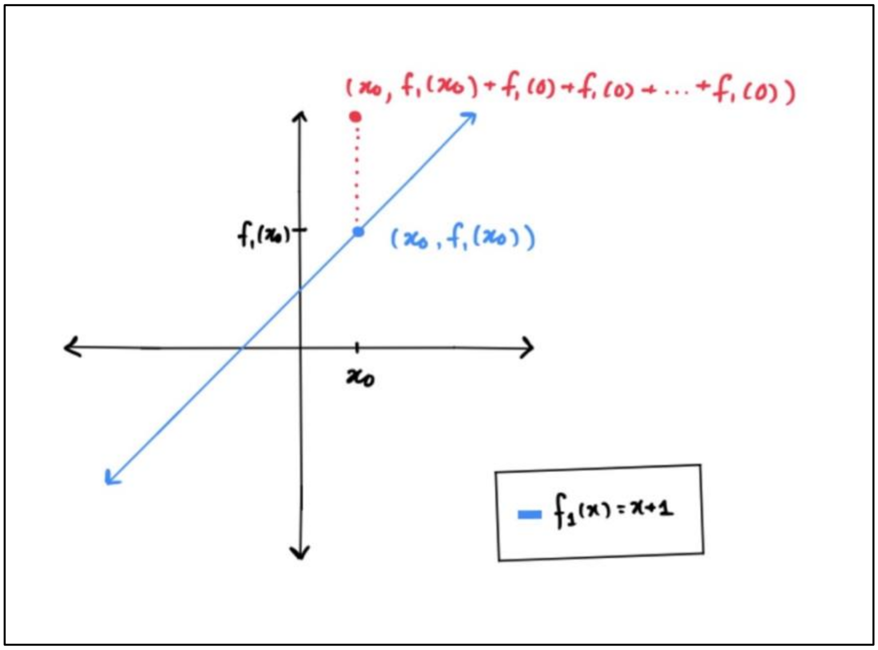


Figure 22:A graphical representation of the additive condition of linearity for the transformation component $x + 1$

An arbitrary x_0 is selected for the test and compared with $x_0 + 0 + 0 + \cdots + 0$. If $f_1(x)$ is linear, the position of $(x_0, f_1(x_0))$ on the graph will be equivalent to $(x_0, f_1(0) + f_1(0) + \cdots + f_1(0))$. Since the positions are not equal, $f_1(x)$ is nonlinear, and as such, $T(x)$ is nonlinear as well.

An interesting case to dissect is the additive condition of linearity where the input, $x \in \mathbb{R}$, is written as the trivial linear combination $x + 0 + 0 + \cdots + 0$. Synthetic thinkers may choose to plot these points to visualize the transformation. It can be observed that

$f_1(x_0) \neq f_1(x_0 + 0 + 0 + \cdots + 0)$. The point at $(x_0, f_1(x_0))$ has been shifted upward by $(n-1)f_1(0) = n - 1$ units (where n is equal to the total number of zeros present in the linear expansion of x_0). Therefore, $f_1(x)$ is nonlinear as it fails condition (1) for linearity. Since $f_1(x)$ is nonlinear, $T(x)$, as a whole, must be nonlinear as well.

Similarly, the scalar multiplication condition can be demonstrated in a like manner. Here, for any scalar c, if T is linear, every point $(x_0, cf_1(x_0))$ and $(x_0, cf_2(x_0))$ will have a one-to-one correspondence with $(x_0, f_1(cx_0))$ and $(x_0, f_2(cx_0))$, respectively. In the case of $f_1(x) = x + 1$, it can be shown that for a given x_0, $cf_1(x_0) \neq f_1(cx_0)$, unless $c = 1$. An interesting case to consider is the comparison at $x_0 = 0$, as this will constitute the y-intercept of the function. A graphical representation of the scalar multiplication condition for $f_1(x)$ is provided below (Figure 23).

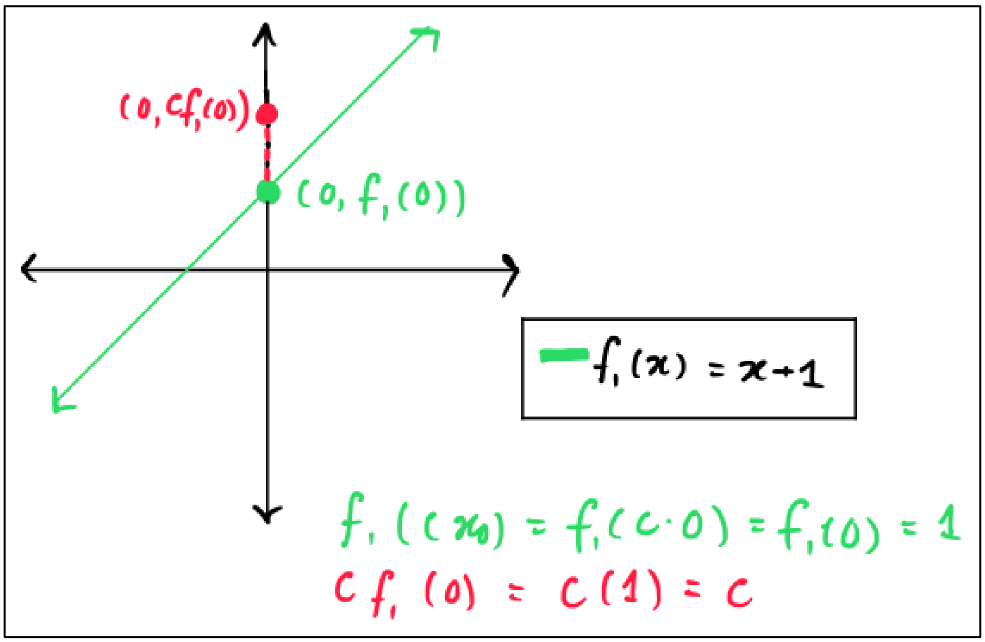


Figure 23: A graphical representation of the scalar multiplication condition of linearity for the transformation component $x + 1$

The value $x_0 = 0$ was selected for the test. If $f_1(x)$ is linear, then the position of $(0, cf_1(0))$ on the graph of T will be equivalent to $(0, f_1(0c))$. Since the positions are not equal, $f_1(x)$ is nonlinear, and as such, $T(x)$ is nonlinear as well.

It is important to note that throughout this presentation, the selection of specific values is done with the intention of providing students with a concrete example of the application of the conditions to further build their intuition towards linearity. When proving linear relationships, however, students should use arbitrary vectors and scalars in their solution. Since $f_1(x)$ failed condition (2) for linearity, it is proven, again, to be a nonlinear function. It should also be mentioned that once a component within a transformation's breakdown fails one of the two tests for linearity, it can be concluded that (1) the specific component is nonlinear (like in the case of $f_1(x)$) and (2) that the transformation, as a whole, is nonlinear as well.

Next, consider the analytic mode; for analytic thinkers, particularly arithmetic thinkers, algebraic representations, such as formulas and equations, are a primary source of reasoning and understanding of mathematical concepts. Otten and Wambua (2022) concluded that algebraic representations hold merit in building a strong foundation for the construction of proofs, with each new algebraic statement requiring some form of work which calls for the direct exercise of justification skills.

As such, comparing linear and nonlinear relationships, for analytical thinkers, may best be done through work with and manipulation of algebraic representations. Specifically, through the construction of a proof of the conditions of linearity. A direct proof demonstrating the distinction between first-degree polynomials and linear transformations is therefore proposed (see Figure 24).

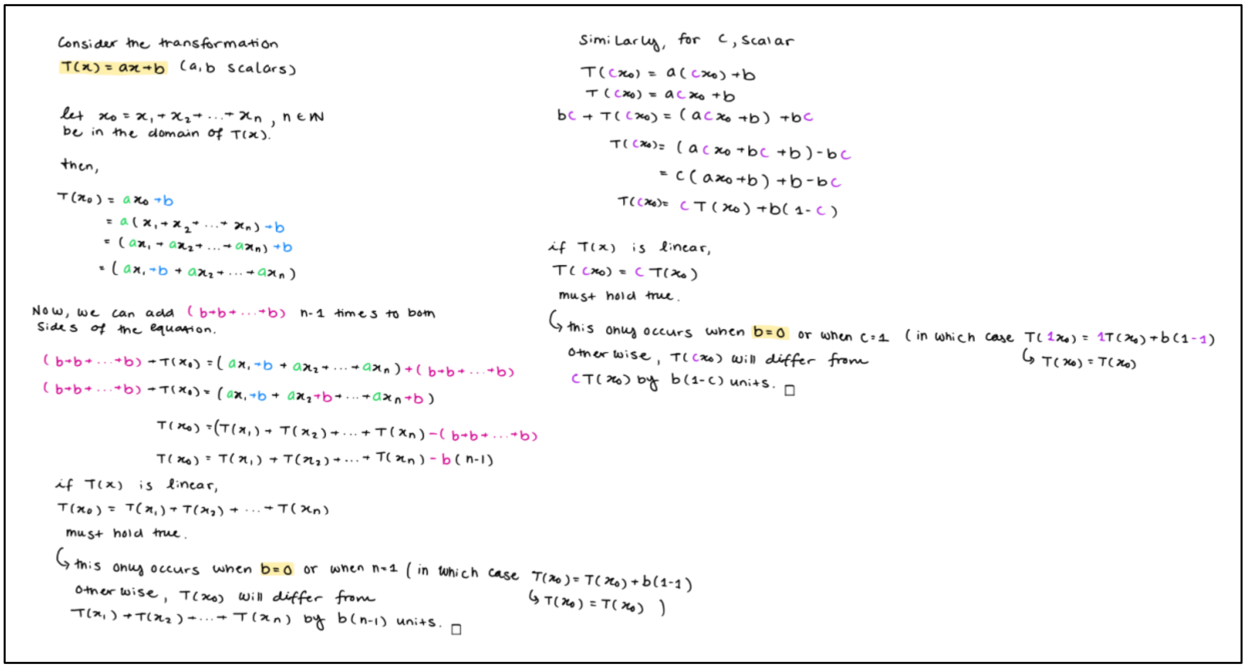


Figure 24:A direct proof of the conditions of linearity for the transformation T(x) $= ax + b; a, b$ *scalars*

The test shows that given the conditions of linearity hold true, either $b = 0$ or $n, c = 1$, respectively. This suggests that either no addition of non-zero constants is performed or, when $b \neq 0$, $T(x)$ is linear if and only if an input x_0 is not broken into a linear combination of more than one element (i.e. $x_0 = x_1$) and that it is not scaled by a factor other than one (i.e. $1T(x_0) = T(1x_0)$). As such, $T(x)$, as defined above, is linear, if and only if $T(x) = ax$.

To demonstrate a specific example of the general proof, consider the established function $f_1(x) = x + 1$ from the transformation given in question 1 (see Figure 25). In this case, $a, b = 1$. Since the linear combination of the images under f_1 of $x_1, x_2, \cdots, x_n$ does not equal $f_1(x_0)$, $x + 1$ is a nonlinear function. Therefore, the transformation $T(x)$, described in question 1, is nonlinear.

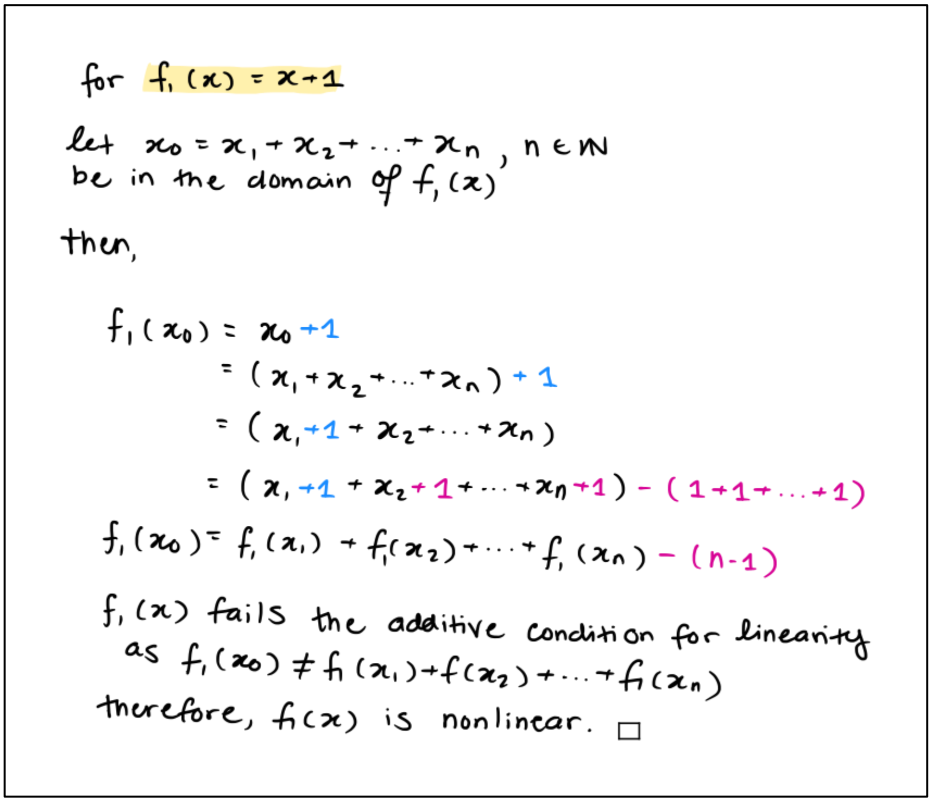


Figure 25: A sample case of the direct proof provided in Fig. 24

The proof demonstrates the test for the additive condition of linearity for the function $f_1(x) = x + 1$. Since $b = 1$ and $n > 1$, it follows that $f_1(x)$ fails the linearity test.

The proof's construction uses the mathematical properties of linear transformations and arithmetic and computational skills. Each line of work requires students to reason, based on the definition, the following logical step, which builds proof writing techniques, as suggested by Otten and Wambua (2022). Analytical and synthetic thinkers can both conceptualize linearity by presenting it through an algebraic or geometric lens, respectively. It can be observed that in both cases, the analytical and synthetic thinkers would reach the same conclusion regarding the transformation's linear behavior. The analytic thinkers would consider the transformation's mathematical properties and use algebra to deconstruct the given formula into the desired conditions of linearity.

Comparatively, the synthetic thinkers would consider the graphical representation of the transformation's components, concluding that the transformation is linear if and only if for all inputs, x_0, in the domain of T, there exists (1) a one-to-one correspondence between $(x_0, T(x_0))$ and $(x_0, T(x_1) + T(x_2) + \cdots + T(x_n))$ where $x_0 = x_1 + x_2 + \cdots + x_n$, $n \in \mathbb{N}$ and (2) a one-to-one correspondence between $(x_0, cT(x_0))$ and $(x_0, T(cx_0))$ for all scalars c and inputs x_0 in the domain of T.

Lastly, consider the gap in the students' conceptual component of their transformation Schema. While students could perform computational-based Actions on transformations (as seen in question 2) they could not make the necessary leap from computational Processes to structural Processes. Since a well- rounded Schema requires students to understand the mathematical properties of an Object—both in its definitional applications and computational applications—it appears that the establishment of the Schema was incomplete.

Piaget and García (1989) suggest that the development of a Schema requires students to recognize the Schema's components and their relations. To overcome the conceptual barrier in the acquisition of the Schema, therefore, a change from implicit application to consequent use of concepts is proposed (Piaget and García, 1989). For instance, providing students with additional assessments focusing on the definition of linearity directly (like in question 1), compared to assessments utilizing linearity implicitly (like in question 2), would benefit students' retention of the notion of linearity within a linear algebra context. Piaget and García (1989) coined the term *thematization* to describe this conceptualization process. The thematization of a Schema points to the mental establishment of a Schema with the objective of dissecting and assessing its parts, reassembling its components, and performing further Action on the Schema through the lens of an Object (Cooley et al., 2007; Piaget and García, 1989). As such, a thematization of the transformation Schema will focus on the Action application on the Object of transformations, with an emphasis on the definitional and conceptual Actions (as demonstrated in questions 1 and 3).

The student struggle surrounding work with and the understanding of the definition of linearity, along with the gap in the notational and terminological comprehension of linear transformations, could potentially be resolved through further work with linearity involving its thematization.

CHAPTER SEVEN: CONCLUSION

This investigation explored students' modes of thinking concerning linearity in linear algebra; focused on understanding how students' interactions with, and ideas of linearity impact their comprehension and success within a linear algebra context. Both evaluative measures (3-assessment questions, and a follow-up interview) utilized in this study revealed significant information pertaining to the root of the student struggle and experience with the concept of linearity and linear transformations. The following section discusses the derived conclusions from this study, along with suggestions for future work and replications of the research.

The results of this study indicated a noteworthy student-struggle with mathematical notation and terminology, which impacted student success with the acquisition of their transformation Schema. The genetic decomposition of functions and transformations (Harel & Dubinsky, 1991; Roa-Fuentes & Oktaç, 2010) implemented in the teaching of the course proved to be a useful indicator of student retention of the concept of linearity. The APOS breakdown of categories shed light on the various methods students integrated when attempting to solve a linearity problem.

Overall, students did not acquire a complete Schema for transformations, demonstrating difficulty with conceptual, definition-based Actions and Processes. While most students correctly applied the necessary vector arithmetic (given that the conditions of linearity hold true) correctly, they struggled using the conditions themselves to prove a transformation's behavior. The interviews further indicated the existence of a struggle with definition application derived from a lack of understanding of and misconceptions about the terminology surrounding linear relationships.

Three suggestions were made to address this disconnect between the student experience of linearity and its proper linear algebra applications. Two of the suggestions are manifested through Sierpinskia's (2000) modes of thinking and correlate to the established significance of both geometric and algebraic representations in mathematics (Aspari et al., 2019; Otten & Wambua, 2022). The last suggestion, concerning the conceptual component of a Schema, is seen through the lens of thematization discussed by Piaget and García (1989).

Future work with the ideas brought forth in this investigation should be focused on correcting the knowledge gap through improving students' understanding of notation and mathematical terminology. In particular, emphasizing the thematization of linearity within the transformation Schema to ensure the conceptual understanding of linearity is retained (Piaget and García, 1989). In addition, one of the limitations of this investigation was the student sample size. As such, future replications of this research should be conducted with a larger participant pool, to establish the findings within a wider sector of STEM related research.

APPENDIX A: IRB APPROVAL

UNIVERSITY OF CENTRAL FLORIDA

Institutional Review Board
FWA00000351
IRB00001138, IRB00012110
Office of Research
12201 Research Parkway
Orlando, FL 32826-3246

APPROVAL

October 10, 2023

Dear Katiuscia Teixeira:

On 10/10/2023, the IRB reviewed the following submission:

Type of Review:	Initial Study, Category
Title:	An Investigation of Students' Modes of thinking about Linearity in Linear Algebra
Investigator:	Katiuscia Teixeira
IRB ID:	STUDY00005916
Funding:	None, None
IND, IDE, or HDE:	None
Documents Reviewed:	• HRP-254-Interview Consent (2).pdf, Category: Consent Form; • HRP-254-Test Answers Consent(2).pdf, Category: Consent Form; • HRP-255(2).docx, Category: IRB Protocol; • Interview questions.pdf, Category: Interview / Focus Questions; • Invitation to Participate in Research.docx, Category: Recruitment Materials; • Test Questions.pdf, Category: Test Instruments;

The IRB approved the protocol on 10/10/2023.

In conducting this protocol, you are required to follow the requirements listed in the Investigator Manual (HRP-103), which can be found by navigating to the IRB Library within the IRB system. Guidance on submitting Modifications and a Continuing Review or Administrative Check-in is detailed in the manual. If continuing review is required and approval is not granted before the expiration date, approval of this protocol expires on that date.

APPENDIX B: LIST OF ASSESMENT QUESTIONS

1. (6 pts.) Consider the transformation given by
   $T: \mathbb{R} \rightarrow \mathbb{R}^2, T(x) = (x + 1, 3x)$. Is T a linear transformation?
   Justify your answer. Show your work.

2. (6 pts.) Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation that maps $u = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ into $\begin{bmatrix} 3 \\ -6 \end{bmatrix}$ and maps $v = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ into $\begin{bmatrix} -11 \\ 4 \end{bmatrix}$. Use the fact that T is linear to find the following images under T.

   a) $4u$
   b) $2v$
   c) $4u + 2v$

3. (5 pts.) Use the following information to set up the transformation:

   A color machine that takes an input of any color (of any positive, negative, or zero amount) and returns, as an output, that color added with two units of red color. Use c to represent the color input and r to represent the color red in your transformation setup.

APPENDIX C: LIST OF INTERVIEW QUESTIONS

1) What do you understand about a transformation being linear?

2) Think about the line $y = x + 2$. How does this line relate to the color transformation in question 3?

3) Explain your thought process for question 1/2/3….

4) Think about a general line of the form $y = mx + b,\ m, b \neq 0$. Based on your knowledge of linearity, is a line of this form a linear transformation? Why/ why not?

5) Is the transformation described in question 3 a linear transformation? Why/ why not?

6) Based on your knowledge of linear transformations could you come up with a NON-Linear COLOR example?

7) a) What would happen if, for question 1 in your exam, the given transformation was $T\colon \mathbb{R} \rightarrow \mathbb{R}^2,\ T(x) = (x, 3x)$?

   b) What does that tell you about the relationship between the addition of non-zero constants and linear transformations?

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