Macmahon's Master Theorem And Infinite Dimensional Matrix Inversion

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MACMAHON'S MASTER THEOREM AND INFINITE DIMENSIONAL MATRIX INVERSION

by

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ABSTRACT

MacMahon's Master Theorem is an important result in the theory of algebraic combinatorics. It gives a precise connection between coefficients of certain power series defined by linear relations. We give a complete proof of MacMahon's Master Theorem based on MacMahon's original 1960 proof. We also study a specific infinite dimensional matrix inverse due to C. Krattenthaler.

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1 INTRODUCTION

In his ground breaking book, *Combinatory Analysis* [6], MacMahon proved, among other things, his famous Master Theorem. This states that given matrices $X = \mathrm{diag}(x_1, \ldots, x_n)$ and $A = (a_{ij})$, the coefficient of $x_1^{\nu_1} \cdots x_n^{\nu_n}$ in the power series expansion of $\det(I_n - AX)^{-1}$ is identical to the coefficient of $x_1^{\nu_1} \cdots x_n^{\nu_n}$ in the product expansion of $\prod_{i=1}^{n}(a_{i1}x_1 + \cdots + a_{in}x_n)^{\nu_i}$. MacMahon uses this to immediately prove several important results in algebraic combinatorics, especially in the theory of displacements, which is referred to today as the theory of derangements, as well as other types of permutations. However, we found MacMahon's proof of his Master Theorem difficult to read, primarily due to the lack of definitions in his notation, and in this thesis, we try to provide a more detailed proof.

In Chapter 5, we investigate the principal minors and determinants of finite dimensional matrix inverses in the form $(I_n - AX)^{-1}$, which leads us to the proof of MacMahon's Master Theorem. The lemmas prior to the proof do not concentrate heavily on matrix inversion, however, the result of this theorem yields a specific inverse relation.

Another important modern day technique in the study of combinatorics is the use of hypergeometric series. General q-analogues of well-known combinatorial theorems have been developed using these techniques. For example, the formula for binomial coefficients, $\binom{n}{k}$, which yields a positive integer, has the q-analogue, $\begin{bmatrix} n \\ k \end{bmatrix}_q$, which yields a polynomial. Also, the famous Pascal relation

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

has a q-analogue of the form

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q q^k + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$$

Though the definition and study of these techniques is much beyond the scope of this paper, it is worth noting that there is a q-analog of MacMahon's Master Theorem. The q-analog of this theorem was first provided by Krattenthaler and Schlosser in 1991 [4]. Their proof was due in part to a r-dimensional matrix inversion theorem which is an extension of an infinite dimensional matrix inversion theorem originally proved in 1980 by Krattenthaler [3]. This thesis will study and compute the infinite dimensional matrix.

In Chapters 2 and 3, we will prove important results in matrix theory which will lead us to the study of a specific infinite dimensional matrix inverse, whose entries are

$$f_{nk} = \frac{\prod_{j=k}^{n-1}(a_j + c_k b_j)}{\prod_{k+1}^{n}(c_j - c_k)}$$

such that $(a_i)_{i\in\mathbb{Z}}$, $(b_i)_{i\in\mathbb{Z}}$, and $(c_i)_{i\in\mathbb{Z}}$ are arbitrary sequences. The proof and computation of its inverse will be provided in Chapter 4. We will follow exactly the approach given by Krattenthaler [3]. In his paper, Krattenthaler briefly examines a method used for solving Lagrange inversion problems, properties of bilinear forms, and specific linear operators to arrive at the inverse relation. This thesis will provide the missing details of Krattenthaler's proof.

2 DEFINITIONS IN MATRIX THEORY

In this chapter we mention specific definitions from elementary linear algebra with respect to formal Laurent series. In particular, we define linear operators, bilinear forms, and adjoints in connection with formal Laurent series. We leave to the reader the general definitions and properties from elementary linear alegebra.

Definition 2.1. A Laurent series is an infinite power series which allows terms of negative degree. A formal Laurent series is a Laurent series centered at zero where all but finitely many of the negative degree coefficients are zero. Let fLs denote the infinite dimensional vector space of formal Laurent series with standard power series operations.

Definition 2.2. The bilinear form $\langle\ ,\rangle$ for formal Laurent series $a(z)$ and $b(z)$ is given by

$$\langle a(z), b(z)\rangle = \langle z^0\rangle a(z)\cdot b(z),$$

where $\langle z^0\rangle c(z)$ denotes the coefficient of z^0 in $c(z)$.

Theorem 2.3. *Let $a(z)$, $b(z)$, and $c(z)$ be formal Laurent series with real coefficients and α a scalar in $\mathbb{R}$. Then the following are true*

(i) $\langle a(z), b(z)\rangle = \langle b(z), a(z)\rangle$

(ii) $\langle \alpha a(z), b(z)\rangle = \alpha\langle a(z), b(z)\rangle$ *and* $\langle a(z), \alpha b(z)\rangle = \alpha\langle a(z), b(z)\rangle$

(iii) $\langle a(z)+c(z), b(z)\rangle = \langle a(z), b(z)\rangle + \langle c(z), b(z)\rangle$

(iv) $\langle a(z), b(z)+c(z)\rangle = \langle a(z), b(z)\rangle + \langle a(z), c(z)\rangle$

Proof. (i) We can obtain the result from the definition of bilinear forms since formal Laurent series are commutative, thus

$$\langle a(z), b(z)\rangle = \langle z^0\rangle(a(z)b(z)) = \langle z^0\rangle(b(z)a(z)) = \langle b(z), a(z)\rangle$$

(ii) α is a scalar, thus it multiplies all terms of $a(z)b(z)$, hence we can multiply the z^0 term by α before or after the expansion, giving us

$$\langle \alpha a(z), b(z)\rangle = \langle z^0\rangle(\alpha a(z)b(z)) = \alpha \cdot \langle z^0\rangle(a(z)b(z)) = \alpha\langle a(z), b(z)\rangle \text{ and}$$

$$\langle a(z), \alpha b(z\rangle = \langle z^0\rangle(a(z)\alpha b(z)) = \alpha \cdot \langle z^0\rangle(a(z)b(z)) = \alpha\langle a(z), b(z)\rangle$$

(iii) By distributing $b(z)$, we can see that the z^0 term of $((a(z) + c(z))b(z))$ is the same as the z^0 term of $(a(z)b(z) + c(z)b(z))$. But this is simply the sum of the z^0 terms of the series $a(z)b(z)$ and $c(z)b(z)$. Thus we have,

$$\begin{aligned}\langle a(z) + c(z), b(z)\rangle &= \langle z^0\rangle((a(z) + c(z))b(z)) \\ &= \langle z^0\rangle(a(z)b(z)) + \langle z^0\rangle(c(z)b(z)) \\ &= \langle a(z), b(z)\rangle + \langle c(z), b(z)\rangle\end{aligned}$$

(iv) Similar to (iii), the z^0 term of $a(z)(b(z)+c(z))$ is the sum of the z^0 terms of the series $a(z)b(z)$ and $a(z)c(z)$. Thus we have,

$$\begin{aligned}\langle a(z), b(z) + c(z)\rangle &= \langle z^0\rangle(a(z)(b(z) + c(z))) \\ &= \langle z^0\rangle(a(z)b(z)) + \langle z^0\rangle(a(z)c(z)) \\ &= \langle a(z), b(z)\rangle + \langle a(z), c(z)\rangle\end{aligned}$$

□

Definition 2.4 (Adjoints). Given any linear operator L acting on a formal Laurent series, then L^* is defined as the adjoint of L with respect to $\langle\ ,\rangle$ if for all formal Laurent series $a(z)$ and $b(z)$, $\langle La(z), b(z)\rangle = \langle a(z), L^*b(z)\rangle$.

Lemma 2.5. *Given linear operators A and B such that A^* and B^* exist and α a real scalar, then*

(i) $(AB)^* = B^*A^*$

(ii) $(A+B)^* = A^* + B^*$

(iii) $(\alpha A)^* = \alpha A^*$

(iv) $(A^*)^* = A$

Proof. (i) Let $a(z)$ and $b(z)$ to be formal Laurent series, and note $(AB)a(z) = A(Ba(z))$. Computing the bilinear form and using the definition of adjoints yields

$$\langle A(Ba(z)), b(z)\rangle = \langle Ba(z), A^*b(z)\rangle = \langle a(z), B^*(A^*b(z))\rangle = \langle a(z), B^*A^*b(z))\rangle$$

(ii) Let $a(z)$ and $b(z)$ to be formal Laurent series, and note $(A+B)a(z) = Aa(z) + Ba(z)$. By using Theorem 2.3 to compute the bilinear form, we have

$$\langle Aa(z) + Ba(z), b(z)\rangle = \langle Aa(z), b(z)\rangle + \langle Ba(z), b(z)\rangle = \langle a(z), A^*b(z)\rangle + \langle a(z), B^*b(z)\rangle$$

Applying Theorem 2.3 once more will obtain our desired result:

$$\langle a(z), A^*b(z)\rangle + \langle a(z), B^*b(z) = \langle a(z), A^*b(z) + B^*b(z)\rangle = \langle a(z), (A^* + B^*)b(z)\rangle$$

(iii) It is clear that $\langle(\alpha A)a(z), b(z)\rangle = \langle \alpha Aa(z), b(z)\rangle$. From Theorem 2.3, we can factor α out to obtain $\alpha\langle Aa(z), b(z)\rangle$. Using the definition of adjoints and Theorem 2.3 on this quantity gives us

$$\alpha\langle Aa(z), b(z)\rangle = \alpha\langle a(z), A^*b(z)\rangle = \langle a(z), \alpha A^*b(z)\rangle$$

(iv) By using the commutativity of bilinear forms from Theorem 2.3, we know that

$$\langle A^* a(z), b(z)\rangle = \langle b(z), A^* a(z)\rangle$$

But A^* is the adjoint of A, so we can rewrite the quantity as $\langle Ab(z), a(z)\rangle$. Again by commutativity, this is equivalent to $\langle a(z), Ab(z)\rangle$, thus $(A^*)^* = A$. $\square$

3 PRELIMINARIES FOR MATRIX INVERSION

This section contains preliminary lemmas necessary in our approach to proving Krattenthaler's Theorem. For convenience, in addition to denoting formal Laurent series in the form $\sum_{n=i}^{\infty} \alpha_n z^n$, we will also denote them in the form $\sum_{n=i}^{-\infty} \alpha_n z^{-n}$ for some $i \in \mathbb{Z}$. The latter series will be computed in the same manner as if it were written in the more conventional form $\sum_{n=-\infty}^{i} \alpha_n z^{-n}$. In addition to series expansions, we will also consider product expansions for our proof. Products in the form $\prod_{j=k}^{k-1} \alpha_j$ will be defined as 1, while products in the form $\prod_{j=k}^{k-m} \alpha_j$ where $m > 1$ will be defined as 0.

Theorem 3.1. *Let $F = (f_{nk})_{n,k\in\mathbb{Z}}$ be an infinite lower triangular matrix with $f_{kk} \neq 0 \; \forall k \in \mathbb{Z}$. For $k \in \mathbb{Z}$, the formal Laurent series $f_k(z)$ and $\tilde{f}_k(z)$ are defined by $f_k(z) = \sum_{n=k}^{\infty} f_{nk} z^n$ and $\tilde{f}_k(z) = \sum_{l=k}^{-\infty} \tilde{f}_{kl} z^{-l}$ where $(\tilde{f}_{kl})_{k,l\in\mathbb{Z}}$ is the uniquely determined inverse matrix of F. Suppose that for $k \in \mathbb{Z}$ holds:*

$$U f_k(z) = c_k \cdot V f_k(z),$$

where U, V are linear operators acting on formal Laurent series, U being bijective and $(c_k)_{k\in\mathbb{Z}}$ is a sequence of constants with $c_k \neq 0 \; \forall k \in \mathbb{Z}$. If the formal Laurent series $h_k(z)$ is a solution of

$$U^* h_k(z) = c_k \cdot V^* h_k(z),$$

with $h_k(z) \not\equiv 0 \; \forall k \in \mathbb{Z}$, then

$$\tilde{f}_k(z) = \frac{1}{\langle f_k(z), V^* h_k(z) \rangle} V^* h_k(z).$$

This remainder of this section is dedicated to proof of this theorem. The result obtained will provide us with a solid foundation for our proof of Krattenthaler's Theorem. Because of the complexity of this theorem, we have chosen to decompose the proof into several lemmas and corollaries. A formal proof to Theorem 3.1 will be provided at the end of this section.

Lemma 3.2. *Let $A = (a_{ij})_{i,k\in\mathbb{Z}}$ and $B = (b_{ij})_{i,k\in\mathbb{Z}}$ be infinite lower triangular matrices. Given the formal Laurent series $a_i(z) = \sum_{l=i}^{-\infty} a_{il} z^{-l}$ and $b_j(z) = \sum_{n=j}^{\infty} b_{nj} z^n$, the coefficient of z^m of their product is given by $\sum_{k=j}^{m+i} a_{i,k-m} b_{kj}$.*

Proof. Since A and B are infinite lower triangular matrices, then $a_{ik} = 0$ for $k > i$ and $b_{kj} = 0$ for $k < j$. Thus we can write

$$\sum_{l=i}^{-\infty} a_{il} z^{-l} = \sum_{l=-\infty}^{\infty} a_{il} z^{-l} = \sum_{l=-\infty}^{\infty} a_{i,-l} z^{l} \qquad \text{and} \qquad \sum_{n=j}^{\infty} b_{nj} z^n = \sum_{n=-\infty}^{\infty} b_{nj} z^n$$

Using these results, the product of $a_i(z)$ and $b_j(z)$ becomes

$$\sum_{l=-\infty}^{\infty} a_{i,-l} z^{l} \sum_{n=-\infty}^{\infty} b_{nj} z^n = \sum_{m=-\infty}^{\infty} c_m z^m$$

where c_m is defined by the Cauchy product such that

$$c_m z^m = \sum_{k=-\infty}^{\infty} a_{i,k-m} b_{kj} z^{m-k+k} = \sum_{k=-\infty}^{\infty} a_{i,k-m} b_{kj} z^m$$

Again we use the fact that A and B are lower triangular and note that we have $a_{i,k-m} = 0$ for $k - m > i$ or $k > m + i$. Similarly, $b_{kj} = 0$ for $k < j$. Incorporating this into the summation, we can compute the coefficient of z^m to be

$$\begin{aligned} c_m &= \sum_{k=-\infty}^{j-1} a_{i,k-m}b_{kj} + \sum_{k=j}^{m+i} a_{i,k-m}b_{kj} + \sum_{k=m+i+1}^{\infty} a_{i,k-m}b_{kj} \\ &= \sum_{k=j}^{m+i} a_{i,k-m}b_{kj} \end{aligned}$$

$\square$

Lemma 3.3. *Let $A = (a_{ij})_{i,k\in\mathbb{Z}}$ and $B = (b_{ij})_{i,k\in\mathbb{Z}}$ be infinite lower triangular matrices such that $a_{ii} \neq 0$ and $b_{ii} \neq 0\ \forall i \in \mathbb{Z}$. Then their product matrix $C = AB$ has entries*

$$c_{ij} = \sum_{k=j}^{i} a_{ik}b_{kj}$$

Proof. By definition of matrix multiplication $c_{ij} = \sum_{k=-\infty}^{\infty} a_{ik}b_{kj}$. However, since A is a lower triangular matrix, $a_{ik} = 0$ for $k > i$. Similarly, $b_{kj} = 0$ for $k < j$. Incorporating this into the summation, we can compute the entries of the product matrix to be

$$\begin{aligned} c_{ij} &= \sum_{k=-\infty}^{j-1} a_{ik}b_{kj} + \sum_{k=j}^{i} a_{ik}b_{kj} + \sum_{i+1}^{\infty} a_{ik}b_{kj} \\ &= \sum_{k=-\infty}^{j-1} a_{ik} \cdot 0 + \sum_{k=j}^{i} a_{ik}b_{kj} + \sum_{i+1}^{\infty} 0 \cdot b_{kj} \\ &= \sum_{k=j}^{i} a_{ik}b_{kj} \end{aligned}$$

$\square$

Lemma 3.4. *Let $A = (a_{ij})_{i,k\in\mathbb{Z}}$ and $B = (b_{ij})_{i,k\in\mathbb{Z}}$ be infinite lower triangular matrices such that $a_{ii} \neq 0$ and $b_{ii} \neq 0\ \forall i \in \mathbb{Z}$. Then their product matrix $C = AB$ has entries $c_{ij} =$*

$\langle a_i(z), b_j(z)\rangle$ *where* $a_i(z)$ *and* $b_j(z)$ *are formal Laurent series defined by* $a_i(z) = \sum_{l=i}^{-\infty} a_{il} z^{-l}$ *and* $b_j(z) = \sum_{n=j}^{\infty} b_{nj} z^n$.

Proof. First we compute the bilinear form of the Laurent series, and then we will compare the result to the entries of the product matrix. Computing the bilinear form gives us

$$\langle a_i(z), b_j(z)\rangle = \langle z^0\rangle \left(\sum_{l=i}^{-\infty} a_{il}^{-1} z^{-l} \sum_{n=j}^{\infty} b_{nj} z^n \right)$$

Using the result of Lemma 3.2 with $m = 0$, we compute the z^0 coefficient to be

$$\sum_{k=j}^{m+i} a_{i,k-m} b_{kj} = \sum_{k=j}^{i} a_{ik} b_{kj}$$

This is equal to the result obtained in Lemma 3.3, thus the bilinear form of $a_i(z)$ and $b_j(z)$ must be identical to the matrix entry c_{ij}. $\square$

Corollary 3.5. *Let* $A = (a_{ij})_{i,k\in\mathbb{Z}}$ *and* $B = (b_{ij})_{i,k\in\mathbb{Z}}$ *be infinite lower triangular matrices such that* $a_{ii} \neq 0$ *and* $b_{ii} \neq 0$ $\forall i \in \mathbb{Z}$. *Define* $a_i(z)$ *and* $b_j(z)$ *to be formal Laurent series defined by* $a_i(z) = \sum_{l=i}^{-\infty} a_{il} z^{-l}$ *and* $b_j(z) = \sum_{n=j}^{\infty} b_{nj} z^n$. *If* $\langle a_i(z), b_j(z)\rangle = \delta_{ij}$ *where* δ_{ij} *is the usual Kronecker delta, then* $A^{-1} = B$.

Lemma 3.6. *Let* fLs *be the space of formal Laurent series and define* U *to be a linear operator given by the function* $U : fLs \longrightarrow fLs$. *If* U *is surjective, then* U^* *is injective.*

Proof. Since U is surjective, we know that for every $d(z)$ in fLs we can find a $c(z)$ such that $Uc(z) = d(z)$. Suppose that $U^*a(z) = U^*b(z)$. To show the injectivity of U^*, we must show that $a(z) = b(z)$. To do so, we begin by computing the bilinear form, noting $\langle U^*a(z), c(z)\rangle = \langle U^*b(z), c(z)\rangle$. By applying the definition of adjoints we have

$$\langle a(z), Uc(z)\rangle = \langle b(z), Uc(z)\rangle$$

Subtracting both sides by $\langle b(z), Uc(z)\rangle$ and applying There om 2.3 gives us the following:

$$\langle a(z), Uc(z)\rangle - \langle b(z), Uc(z)\rangle = 0$$

$$\langle a(z) - b(z), Uc(z)\rangle = 0$$

$$\langle a(z) - b(z), d(z)\rangle = 0$$

Since U is surjective, we know this must hold true for every $d(z)$. If $a(z) - b(z) \neq 0$, we can always find a formal Laurent series $d(z)$ such that $\langle z^0\rangle((a(z) - b(z))d(z)) \neq 0$. Thus in order for the bilinear form to be 0, $a(z) - b(z)$ must be 0, hence $a(z) = b(z)$. $\square$

Lemma 3.7. *Let $F = (f_{nk})_{n,k\in\mathbb{Z}}$ be an infinite lower triangular matrix with $f_{kk} \neq 0\ \forall k \in \mathbb{Z}$. For $k \in \mathbb{Z}$, the formal Laurent series $f_k(z)$ is defined by $f_k(z) = \sum_{n=k}^{\infty} f_{nk}z^n$. Given a formal Laurent series $h(z)$,*

$$\langle h(z), f_k(z)\rangle = 0 \forall k \in \mathbb{Z} \qquad \textit{if and only if} \qquad h(z) = 0$$

Proof. $\Rightarrow$:) Let $\langle h(z), f_k(z)\rangle = 0\ \forall k \in \mathbb{Z}$ and suppose $h(z) \neq 0$. Since $h(z) \neq 0$, there must be some m such that if $h(z) = \sum_{n=-\infty}^{\infty} h_n z^n$ then $h_m \neq 0$. Without loss of generality, assume that m is the first index of $h(z)$ at which this occurs, thus we can write $h(z)$ as $\sum_{n=m}^{\infty} h_n z^n$. Consider the bilinear form computed at $k = -m$

$$\langle h(z), f_{-m}(z)\rangle = \langle z^0\rangle h(z) f_{-m}(z) = \langle z^0\rangle \left(\sum_{n=m}^{\infty} h_n z^n \sum_{n=-m}^{\infty} f_{n,-m} z^n\right)$$

It is clear that upon expansion of this product, the only terms that will contribute to the z^0 coefficient is the z^m term in $h(z)$ and the z^{-m} term in $f_{-m}(z)$. This leaves us with

$$\langle z^0 \rangle \left(\sum_{n=m}^{\infty} h_n z^n \sum_{n=-m}^{\infty} f_{n,-m} z^n \right) = h_m f_{-m,-m}$$

In order for $h_m f_{-m,-m}$ to be zero, either $h_m = 0$ or $f_{-m,-m} = 0$. By definition, $f_{-m,-m} \neq 0$, so h_m must be zero, but this is a contradiction. Thus $h(z) = 0$.

$\Leftarrow$:) Let $h(z) = 0$. Then $\langle h(z), f_k(z) \rangle = \langle z^0 \rangle (h(z) \cdot f_k(z)) = 0$ □

Lemma 3.8. *Let $F = (f_{nk})_{n,k\in\mathbb{Z}}$ be an infinite lower triangular matrix with $f_{kk} \neq 0 \,\forall k \in \mathbb{Z}$. For $k \in \mathbb{Z}$, the formal Laurent series $f_k(z)$ is defined by $f_k(z) = \sum_{n=k}^{\infty} f_{nk} z^n$. Suppose that for $k \in \mathbb{Z}$ holds:*

$$U f_k(z) = c_k \cdot V f_k(z),$$

where U, V are linear operators acting on formal Laurent series, U being bijective and $(c_k)_{k\in\mathbb{Z}}$ is a sequence of constants with $c_k \neq 0 \,\forall k \in \mathbb{Z}$. Furthermore, suppose the formal Laurent series $h_k(z)$ is a solution of

$$U^* h_k(z) = c_k \cdot V^* h_k(z),$$

with $h_k(z) \not\equiv 0 \,\forall k \in \mathbb{Z}$. If

$$\langle h_i(z), U f_j(z) \rangle = \langle h_i(z), c_j V f_j(z) \rangle$$

then $i = j$ or $\langle h_i(z), U f_j(z) \rangle = 0$.

Proof. Given $\langle h_i(z), U f_j(z) \rangle = \langle h_i(z), c_j V f_j(z) \rangle$, we can use Theorem 2.3 and the definition of adjoints to obtain

$$\langle h_i(z), Uf_j(z)\rangle = \langle h_i(z), c_j V f_j(z)\rangle = c_j \langle V^* h_i(z), f_j(z)\rangle$$

We are given that $Uf_i(z) = c_i \cdot Vf_i(z)$. Since $c_i \neq 0$ we divide both sides by c_i and substitute the quantity into the bilinear form and simplifying

$$c_j\langle V^* h_i(z), f_j(z)\rangle = c_j \left\langle \frac{U^* h_i(z)}{c_i}, f_j(z) \right\rangle = \frac{c_j}{c_i}\langle U^* h_i(z), f_j(z)\rangle$$

We now consider two cases of the resulting equation

$$\langle h_i(z), Uf_j(z)\rangle = \frac{c_j}{c_i}\langle h_i(z), Uf_j(z)\rangle$$

For $i \neq j$ we are given $c_j \neq c_i$, thus $\langle h_i(z), Uf_j(z)\rangle$ must be zero. Assume the same result for the case when $i = j$ and rewrite the bilinear form using U^*

$$\langle h_i(z), Uf_j(z)\rangle = \langle U^* h_i(z), f_j(z)\rangle = 0$$

This implies that $\langle U^* h_i(z), Uf_j(z)\rangle = 0$ for all i. Using Lemma 3.7, we construe that $U^* h_i(z) = 0$. Since U is bijective, it is clearly surjective. Thus by Lemma 3.6 we know that U^* is injective. From the injectivity of U^* we can now conclude that $h_i(z) = 0$. But this is a contradiction since we have defined $h_i(z) \not\equiv 0 \ \forall i \in \mathbb{Z}$, thus for $i = j$, $\langle h_i(z), Uf_i(z)\rangle \neq 0$ □

Lemma 3.9. *Let $F = (f_{nk})_{n,k\in\mathbb{Z}}$ be an infinite lower triangular matrix with $f_{kk} \neq 0 \ \forall k \in \mathbb{Z}$. For $k \in \mathbb{Z}$, the formal Laurent series $f_k(z)$ is defined by $f_k(z) = \sum_{n=k}^{\infty} f_{nk} z^n$. Suppose that for $k \in \mathbb{Z}$ holds:*

$$Uf_k(z) = c_k \cdot Vf_k(z),$$

where U, V are linear operators acting on formal Laurent series, U being bijective and $(c_k)_{k\in\mathbb{Z}}$ is a sequence of constants with $c_k \neq 0\ \forall k \in \mathbb{Z}$. Furthermore, suppose the formal Laurent series $h_k(z)$ is a solution of

$$U^* h_k(z) = c_k \cdot V^* h_k(z),$$

with $h_k(z) \not\equiv 0\ \forall k \in \mathbb{Z}$. If

$$\tilde{f}_k(z) = \frac{1}{\langle f_k(z), V^* h_k(z)\rangle} V^* h_k(z)$$

then $\langle \tilde{f}_i(z), f_j(z)\rangle = \delta_{ij}$

Proof. Since a bilinear form produces a scalar value, note that we can use Theorem 2.3 to extrapolate the scalar value $\dfrac{1}{\langle f_i(z), V^* h_i(z)\rangle}$.

$$\langle \tilde{f}_i(z), f_j(z)\rangle = \left\langle \frac{1}{\langle f_i(z), V^* h_i(z)\rangle} V^* h_i(z), f_j(z) \right\rangle = \frac{1}{\langle f_i(z), V^* h_i(z)\rangle} \langle V^* h_i(z), f_j(z)\rangle$$

We are given $U f_i(z) = c_i \cdot V f_i(z)$ and $U^* h_i(z) = c_i \cdot V^* h_i(z)$. Since $c_i \neq 0$ we divide both sides of both equations by c_i. Use the definition of adjoints and substitute these quantities into the resulting bilinear form

$$\frac{1}{\langle V f_i(z), h_i(z)\rangle} \left\langle \frac{U^* h_i(z)}{c_i}, f_j(z) \right\rangle = \frac{1}{\left\langle \frac{U f_i(z)}{c_i}, h_i(z) \right\rangle} \frac{\langle h_i(z), U f_j(z)\rangle}{c_i}$$

Then by extrapolating scalar values we get

$$\frac{c_i}{\langle U f_i(z), h_i(z)\rangle} \frac{\langle h_i(z), U f_j(z)\rangle}{c_i} = \frac{\langle h_i(z), U f_j(z)\rangle}{\langle U f_i(z), h_i(z)\rangle}$$

But from Lemma 3.8 we know that $\langle h_i(z), U f_j(z)\rangle = 0$ only if $i \neq j$. Thus

$$\langle \tilde{f}_i(z), f_i(z)\rangle = \frac{\langle h_i(z), U f_i(z)\rangle}{\langle U f_i(z), h_i(z)\rangle} = 1$$

$$\langle \tilde{f}_i(z), f_j(z)\rangle = \frac{\langle h_i(z), U f_j(z)\rangle}{\langle U f_i(z), h_i(z)\rangle} = 0$$

This is just the definition of the Kronecker delta, therefore $\langle \tilde{f}_i(z), f_j(z)\rangle = \delta_{ij}$ $\square$

Proof of Theorem 3.1. From Lemma 3.9 we have concluded that given

$$Uf_k(z) = c_k \cdot Vf_k(z) \qquad \text{and} \qquad U^*h_k(z) = c_k \cdot V^*h_k(z)$$

for a particular $f_k(z)$ and $h_k(z)$ we will obtain the result that $\langle \tilde{f}_i(z), f_j(z)\rangle = \delta_{ij}$ provided

$$\tilde{f}_k(z) = \frac{1}{\langle f_k(z), V^*h_k(z)\rangle} V^*h_k(z)$$

From Corollary 3.5 we know that if $\langle \tilde{f}_i(z), f_j(z)\rangle = \delta_{ij}$ such that $F = (f_{ij})$ and $\tilde{F} = (\tilde{f}_{ij})$, then $F^{-1} = \tilde{F}$. Since we know the inverse to be unique, it must be the case that

$$\tilde{f}_k(z) = \frac{1}{\langle f_k(z), V^*h_k(z)\rangle} V^*h_k(z)$$

$\square$

4 INFINITE DIMENSIONAL MATRIX INVERSION THEOREM

Theorem 4.1 (Krattenthaler's Theorem). *Let* $(a_i)_{i\in\mathbb{Z}}$*,* $(b_i)_{i\in\mathbb{Z}}$*, and* $(c_i)_{i\in\mathbb{Z}}$ *be arbitrary sequences such that* $c_i \neq c_j$ *if* $i \neq j$ *and* $c_i \neq 0\ \forall i$*. Denote* $F = (f_{nk})$ *and* $F^{-1} = (\tilde{f}_{kl})$ *to be infinite lower triangular matrices. If*

$$f_{nk} = \frac{\prod_{j=k}^{n-1}(a_j + c_k b_j)}{\prod_{k+1}^{n}(c_j - c_k)}$$

then

$$\tilde{f}_{kl} = \frac{a_l + c_l b_l}{a_k + c_k b_k} \frac{\prod_{j=l+1}^{k}(a_j + c_k b_j)}{\prod_{j=l}^{k-1}(c_j - c_k)}$$

The proof of this theorem is obtained directly from the results of Theorem 3.1. However, Theorem 3.1 requires linear operators U and V with U being bijective that satisfy

$$Uf_k(z) = c_k \cdot Vf_k(z) \qquad \text{and} \qquad U^*h_k(z) = c_k \cdot V^*h_k(z)$$

for a given $f_k(z)$ and $h_k(z)$. In order to complete our proof of Krattenthaler's Theorem we must first define these operators. The remainder of this section is dedicated to this effort and is concluded by a formal proof of Krattenthaler's Theorem.

Lemma 4.2. *Let* $f_k(z) = \sum_{n=k}^{\infty} f_{nk} z^n$ *and let* $(a_i)_{i\in\mathbb{Z}}$*,* $(b_i)_{i\in\mathbb{Z}}$*, and* $(c_i)_{i\in\mathbb{Z}}$ *be arbitrary sequences such that* $c_i \neq c_j$ *if* $i = j$ *and* $c_i \neq 0\ \forall k \in \mathbb{Z}$*. If*

$$f_{nk} = \frac{\prod_{j=k}^{n-1}(a_j + c_k b_j)}{\prod_{j=k+1}^{n}(c_j - c_k)},$$

then for $n \geq k$

$$(c_n - c_k) f_{nk} = (a_{n-1} + c_k b_{n-1}) f_{n-1,k}.$$

Proof. By writing the $(n-1)$th term from the numerator and the nth term from the denominator outside of the product we have

$$f_{n,k} = \frac{\prod_{j=k}^{n-1} (a_j + c_k b_j)}{\prod_{k+1}^{n} (c_j - c_k)}$$

$$= \frac{(a_{n-1} + c_k b_{n-1})}{(c_n - c_k)} \frac{\prod_{j=k}^{n-2} (a_j + c_k b_j)}{\prod_{k+1}^{n-1} (c_j - c_k)}$$

However, we now recognize that $f_{n-1,k} = \dfrac{\prod_{j=k}^{n-2} (a_j + c_k b_j)}{\prod_{k+1}^{n-1} (c_j - c_k)}$ and substitute to obtain

$$f_{n,k} = \frac{(a_{n-1} + c_k b_{n-1})}{(c_n - c_k)} f_{n-1,k}$$

Multiplying both sides by $(c_n - c_k)$ will obtain the result

$$(c_n - c_k) f_{nk} = (a_{n-1} + c_k b_{n-1}) f_{n-1,k}.$$

$\square$

Lemma 4.3. *Let* $f_k(z) = \sum_{n=k}^{\infty} f_{nk} z^n$ *and let* $(a_i)_{i \in \mathbb{Z}}$, $(b_i)_{i \in \mathbb{Z}}$, *and* $(c_i)_{i \in \mathbb{Z}}$ *be arbitrary sequences such that* $c_i \neq c_j$ *if* $i = j$ *and* $c_i \neq 0$ $\forall k \in \mathbb{Z}$*. Define the linear operators* $\mathcal{A}$, $\mathcal{B}$, *and* $\mathcal{C}$, *by* $\mathcal{A} z^k = a_k z^k$, $\mathcal{B} z^k = b_k z^k$, $\mathcal{C} z^k = c_k z^k$*. If*

$$f_{nk} = \frac{\prod_{j=k}^{n-1} (a_j + c_k b_j)}{\prod_{j=k+1}^{n} (c_j - c_k)},$$

then

$$(\mathcal{C} - z\mathcal{A})f_k(z) = c_k(1 + z\mathcal{B})f_k(z)$$

Proof. Starting with the result from Lemma 4.2, we multiply by z^n and sum over the valid values for the equation, mainly for $n \geq k$

$$(c_n - c_k)f_{nk} = (a_{n-1} + c_k b_{n-1})f_{n-1,k}$$

$$(c_n - c_k)f_{nk}z^n = (a_{n-1} + c_k b_{n-1})f_{n-1,k}z^n$$

$$\sum_{n=k}^{\infty}(c_n - c_k)f_{nk}z^n = \sum_{n=k}^{\infty}(a_{n-1} + c_k b_{n-1})f_{n-1,k}z^n$$

Note that for $n = k$, $(a_{k-1} + c_k b_{k-1})f_{k-1,k}z^k = (c_k - c_k)f_{kk}z^k = 0$. Using this fact and applying the defined linear operators gives us

$$\sum_{n=k}^{\infty}(c_n - c_k)f_{nk}z^n = \sum_{n-1=k}^{\infty}(a_{n-1} + c_k b_{n-1})f_{n-1,k}z^n$$

$$\sum_{n=k}^{\infty} c_n z^n f_{nk} - c_k \sum_{n=k}^{\infty} f_{nk}z^n = z \sum_{n-1=k}^{\infty} a_{n-1}z^{n-1}f_{n-1,k} + c_k z \sum_{n-1=k}^{\infty} b_{n-1}z^{n-1}f_{n-1,k}$$

$$\sum_{n=k}^{\infty} \mathcal{C} z^n f_{nk} - c_k \sum_{n=k}^{\infty} f_{nk}z^n = z \sum_{n-1=k}^{\infty} \mathcal{A}z^{n-1}f_{n-1,k} + c_k z \sum_{n-1=k}^{\infty} \mathcal{B}z^{n-1}f_{n-1,k}$$

Recall from the definition of linear operators, that we can rewrite this as

$$\mathcal{C}\left(\sum_{n=k}^{\infty} f_{nk}z^n\right) - c_k \sum_{n=k}^{\infty} f_{nk}z^n = z\mathcal{A}\left(\sum_{n-1=k}^{\infty} f_{n-1,k}z^{n-1}\right) + c_k z\mathcal{B}\left(\sum_{n-1=k}^{\infty} f_{n-1,k}z^{n-1}\right)$$

Now we can replace the summations by the defined Laurent series

$$\mathcal{C}f_k(z) - c_k f_k(z) = z\mathcal{A}f_k(z) + c_k z\mathcal{B}f_k(z)$$

Through some rearranging and properties of linear operators, we have the desired result

$$\mathcal{C} f_k(z) - z\mathcal{A} f_k(z) = c_k f_k(z) + c_k z \mathcal{B} f_k(z)$$

$$(\mathcal{C} - z\mathcal{A}) f_k(z) = c_k (1 + z\mathcal{B}) f_k(z)$$

□

Lemma 4.4. *Let $(a_i)_{i\in\mathbb{Z}}$, $(b_i)_{i\in\mathbb{Z}}$, and $(c_i)_{i\in\mathbb{Z}}$ be arbitrary sequences such that $c_i \neq c_j$ if $i \neq j$ and $c_i \neq 0\ \forall i$. Define the linear operators $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, by $\mathcal{A}z^k = a_k z^k$, $\mathcal{B}z^k = b_k z^k$, $\mathcal{C}z^k = c_k z^k$. Then their respective adjoints are given by $\mathcal{A}^* z^{-k} = a_k z^{-k}$, $\mathcal{B}^* z^{-k} = b_k z^{-k}$, and $\mathcal{C} z^{-k} = c_k z^{-k}$.*

Proof. Using the given definitions, we compute bilinear forms for both operators and then compare.

$$\langle \mathcal{A}z^k, z^{-k}\rangle = \langle z^0\rangle (a_k z^k)(z^{-k}) = a_k$$

$$\langle z^k, \mathcal{A}^* z^{-k}\rangle = \langle z^0\rangle (z^{-k})(a_k z^k) = a_k$$

Since we have $\langle \mathcal{A}z^k, z^{-k}\rangle = \langle z^k, \mathcal{A}^* z^{-k}\rangle$, then by the definition of adjoints we know that $\mathcal{A}^*$ must be the adjoint of $\mathcal{A}$. The proof is similar for $\mathcal{B}^*$ and $\mathcal{C}^*$. □

Lemma 4.5. *Let $f_k(z) = \sum_{n=k}^{\infty} f_{nk} z^n$ and let $(a_i)_{i\in\mathbb{Z}}$, $(b_i)_{i\in\mathbb{Z}}$, and $(c_i)_{i\in\mathbb{Z}}$ be arbitrary sequences such that $c_i \neq c_j$ if $i = j$ and $c_i \neq 0\ \forall k \in \mathbb{Z}$. Define the linear operators $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, by $\mathcal{A}z^k = a_k z^k$, $\mathcal{B}z^k = b_k z^k$, $\mathcal{C}z^k = c_k z^k$. Given $h_k(z) = \sum_{l=k}^{-\infty} h_{kl} z^{-l}$ and $h_{kl} = 0$ for $l > k$, then:*

(i) $(\mathcal{C} - z\mathcal{A})^* h_k(z) = (\mathcal{C}^* - \mathcal{A}^* z) h_k(z)$

(ii) $(1 + z\mathcal{B})^* h_k(z) = (1 + \mathcal{B}^* z) h_k(z)$

Proof. (i) Note that $f_{kk} = \dfrac{\prod_{j=k}^{k-1}(a_j + c_k b_j)}{\prod_{k+1}^{k}(c_j - c_k)} = 1$. Just as before, we will take the bilinear forms for both operators and compare.

$$\langle (\mathcal{C} - z\mathcal{A}) f_k(z), h_k(z)\rangle = \langle z^0 \rangle \left[\left(\sum_{n=k}^{\infty} c_n f_{nk} z^n - z \sum_{n=k}^{\infty} a_n f_{nk} z^n \right) \left(\sum_{l=k}^{-\infty} h_{kl} z^{-l} \right) \right]$$
$$\langle (f_k(z), \mathcal{C}^* - \mathcal{A}^* z h_k(z))\rangle = \langle z^0 \rangle \left[\left(\sum_{n=k}^{\infty} f_{nk} z^n \right) \left(\sum_{l=k}^{-\infty} c_l h_{kl} z^{-l} - \sum_{l=k}^{-\infty} a_l h_{kl} z^{-l+1} \right) \right]$$

In both cases, upon distribution of the product, note that only the terms that contribute to the z^0 coefficient are the z^k term in series involving coefficients from $f_k(z)$ and the z^{-k} term in the series involving coefficients from $h_k(z)$. Finding the coefficient gives us

$$\langle (\mathcal{C} - z\mathcal{A}) f_k(z), h_k(z)\rangle = (c_k f_{kk} z^k)(h_{kk} z^{-k}) = c_k h_{kk}$$
$$\langle (f_k(z), \mathcal{C}^* - \mathcal{A}^* z h_k(z))\rangle = (f_{kk} z^k)(c_k h_{kk} z^{-k}) = (c_k h_{kk})$$

Since we have $\langle (\mathcal{C} - z\mathcal{A}) f_k(z), h_k(z)\rangle = \langle (f_k(z), \mathcal{C}^* - \mathcal{A}^* z h_k(z))\rangle$, then by the definition of adjoints we know that $(\mathcal{C}^* - \mathcal{A}^* z)$ must be the adjoint of $(\mathcal{C} - z\mathcal{A})$.

(ii) Note that $f_{kk} = 1$. Just as before, we will take the bilinear forms for both operators and compare.

$$\langle (1 + z\mathcal{B}) f_k(z), h_k(z)\rangle = \langle z^0 \rangle \left[\left(\sum_{n=k}^{\infty} f_{nk} z^n + z \sum_{n=k}^{\infty} b_n f_{nk} z^n \right) \left(\sum_{l=k}^{-\infty} h_{kl} z^{-l} \right) \right]$$
$$\langle (f_k(z), 1 + \mathcal{B}^* z h_k(z))\rangle = \langle z^0 \rangle \left[\left(\sum_{n=k}^{\infty} f_{nk} z^n \right) \left(\sum_{l=k}^{-\infty} h_{kl} z^{-l} + \sum_{l=k}^{-\infty} b_l h_{kl} z^{-l+1} \right) \right]$$

In both cases, upon distribution of the product, note that only the terms that contribute to the z^0 coefficient are the z^k term in series involving coefficients from $f_k(z)$ and the z^{-k} term in the series involving coefficients from $h_k(z)$. Finding the coefficient gives us

$$\langle (1 + z\mathcal{B}) f_k(z), h_k(z) \rangle = (f_{kk} z^k)(h_{kk} z^{-k}) = h_{kk}$$

$$\langle (f_k(z), 1 + \mathcal{B}^* z h_k(z)) \rangle = (f_{kk} z^k)(h_{kk} z^{-k}) = h_{kk}$$

Since we have $\langle (1+z\mathcal{B}) f_k(z), h_k(z) \rangle = \langle (f_k(z), 1+\mathcal{B}^* z h_k(z)) \rangle$, then by the definition of adjoints we know that $(1 + \mathcal{B}^* z)$ must be the adjoint of $(1 + z\mathcal{B})$. $\square$

Lemma 4.6. *Let* $(a_i)_{i\in\mathbb{Z}}$*,* $(b_i)_{i\in\mathbb{Z}}$*, and* $(c_i)_{i\in\mathbb{Z}}$ *be arbitrary sequences such that* $c_i \neq c_j$ *if* $i \neq j$ *and* $c_i \neq 0\ \forall i$*. Also, let* $h_k(z) = \sum_{l=k}^{-\infty} h_{kl} z^{-l}$ *and* $h_{kl} = 0$ *for* $l > k$*. Define the linear operators* $\mathcal{A}$*,* $\mathcal{B}$*, and* $\mathcal{C}$*, by* $\mathcal{A} z^k = a_k z^k$*,* $\mathcal{B} z^k = b_k z^k$*,* $\mathcal{C} z^k = c_k z^k$*. If*

$$(\mathcal{C}^* - \mathcal{A}^* z) h_k(z) = c_k (1 + \mathcal{B}^* z) h_k(z)$$

then

$$(c_l - c_k) h_{kl} = (a_l + c_k b_l) h_{k,l+1}$$

Proof. First note that Lemma 4.4 provides us with the adjoints for the operators $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$. Using the properties of linear operators, we can write $c_k(1 + \mathcal{B}^* z) h_k(z)$ as $c_k + \mathcal{B}^* z h_k(z)$. Substituting this and the series expansion for $h_k(z)$ into the given equation will yield

$$(\mathcal{C}^* - \mathcal{A}^* z) h_k(z) = c_k h_k(z) + c_k \mathcal{B}^* z h_k(z)$$
$$(\mathcal{C}^* - \mathcal{A}^* z) \sum_{l=k}^{-\infty} h_{kl} z^{-l} = c_k \sum_{l=k}^{-\infty} h_{kl} z^{-l} + c_k \sum_{l=k}^{-\infty} \mathcal{B}^* z \cdot h_{kl} z^{-l}$$

Again, we see that we can regroup the equations using properties of linear operators

$$\mathcal{C}^* \left(\sum_{l=k}^{-\infty} h_{kl} z^{-l} \right) - \mathcal{A}^* \left(\sum_{l=k}^{-\infty} h_{kl} z^{-l+1} \right) = c_k \left(\sum_{l=k}^{-\infty} h_{kl} z^{-l} \right) + c_k \mathcal{B}^* \left(\sum_{l=k}^{-\infty} h_{kl} z^{-l+1} \right)$$

By changing the proper indices, we provide each term with a z^{-l} variable, we then apply the operators to obtain

$$\mathcal{C}^*\left(\sum_{l=k}^{-\infty} h_{kl}z^{-l}\right) - \mathcal{A}^*\left(\sum_{l+1=k}^{-\infty} h_{k,l+1}z^{-l}\right) = c_k\left(\sum_{l=k}^{-\infty} h_{kl}z^{-l}\right) + c_k\mathcal{B}^*\left(\sum_{l+1=k}^{-\infty} h_{k,l+1}z^{-l}\right)$$

$$\sum_{l=k}^{-\infty} c_l h_{kl}z^{-l} - \sum_{l+1=k}^{-\infty} a_l h_{k,l+1}z^{-l} = c_k\sum_{l=k}^{-\infty} h_{kl}z^{-l} + c_k\sum_{l+1=k}^{-\infty} b_l h_{k,l+1}z^{-l}$$

Because $h_{kl} = 0$ for $l > k$, we know that $a_k h_{k,k+1}z^{-k} = 0$ and $c_k b_k h_{k,k+1}z^{-k} = 0$. Let us add this into the equation and regroup the summations to obtain

$$\sum_{l=k}^{-\infty} c_l h_{kl}z^{-l} - \sum_{l+1=k}^{-\infty} a_l h_{k,l+1}z^{-l} - a_k h_{k,k+1}z^{-k} =$$

$$c_k\sum_{l=k}^{-\infty} h_{kl}z^{-l} + c_k\sum_{l+1=k}^{-\infty} b_l h_{k,l+1}z^{-l} + b_k h_{k,k+1}z^{-k}$$

$$\sum_{l=k}^{-\infty} c_l h_{kl}z^{-l} - \sum_{l=k}^{-\infty} a_l h_{k,l+1}z^{-l} = c_k\sum_{l=k}^{-\infty} h_{kl}z^{-l} + c_k\sum_{l=k}^{-\infty} b_l h_{k,l+1}z^{-l}$$

Now that all the indicies and powers of z coincide, we can compare like coefficients, which will give us the desired result

$$c_l h_{kl} - h_{k,l+1}a_l = c_k h_{kl} + c_k b_l h_{k,l+1}$$

$$c_l h_{kl} - c_k h_{kl} = h_{k,l+1}a_l + c_k b_l h_{k,l+1}$$

$$(c_l - c_k)h_{kl} = (a_l + c_k b_l)h_{k,l+1}$$

$\square$

Lemma 4.7. *Let* $(a_i)_{i\in\mathbb{Z}}$, $(b_i)_{i\in\mathbb{Z}}$, *and* $(c_i)_{i\in\mathbb{Z}}$ *be arbitrary sequences such that* $c_i \neq c_j$ *if* $i \neq j$ *and* $c_i \neq 0\ \forall i$. *Define the linear operators* $\mathcal{A}$, $\mathcal{B}$, *and* $\mathcal{C}$, *by* $\mathcal{A}z^k = a_k z^k$, $\mathcal{B}z^k = b_k z^k$, $\mathcal{C}z^k = c_k z^k$. *Then* $(\mathcal{C} - z\mathcal{A})$ *is surjective.*

Proof. Let $b(z) = \sum_{i=m}^{\infty} \beta_i z^i$ and $a(z) = \sum_{i=l}^{\infty} \alpha_i z^i$ for some $l, m \in \mathbb{Z}$. Define $\alpha_i = 0$ for $i < m$. To show surjectivity, we must show that given $(\mathcal{C} - z\mathcal{A})a(z) = b(z)$, we can find an $a(z)$ such that this holds true. First, using the properties of linear operators, simplify the expression and replace the formal Laurent series by their series expansion

$$\mathcal{C}(a(z)) - z\mathcal{A}(a(z)) = b(z)$$
$$\mathcal{C}\left(\sum_{i=l}^{\infty} \alpha_i z^i\right) - z\mathcal{A}\left(\sum_{i=l}^{\infty} \alpha_i z^i\right) = \sum_{i=m}^{\infty} \beta_i z^i$$

Next, we apply the linear operators and adjust indices so that each summation has a z^i as the coefficient

$$\sum_{i=l}^{\infty} c_i \alpha_i z^i - z \sum_{i=l}^{\infty} a_i \alpha_i z^i = \sum_{i=m}^{\infty} \beta_i z^i$$
$$\sum_{i=l}^{\infty} c_i \alpha_i z^i - \sum_{i=l}^{\infty} a_i \alpha_i z^{i+1} = \sum_{i=m}^{\infty} \beta_i z^i$$
$$\sum_{i=m}^{\infty} c_i \alpha_i z^i - \sum_{i=m+1}^{\infty} a_{i-1} \alpha_{i-1} z^i = \sum_{i=m}^{\infty} \beta_i z^i$$

Since $\alpha_i = 0$ for $i < m$, we know that $a_{m-1}\alpha_{m-1}z^m = 0$. Incorporating this fact gives us

$$\sum_{i=m}^{\infty} c_i \alpha_i z^i - \sum_{i=m+1}^{\infty} a_{i-1} \alpha_{i-1} z^i + a_{m-1}\alpha_{m-1}z^m = \sum_{i=m}^{\infty} \beta_i z^i$$
$$\sum_{i=m}^{\infty} c_i \alpha_i z^i - \sum_{i=m}^{\infty} a_{i-1} \alpha_{i-1} z^i = \sum_{i=m}^{\infty} \beta_i z^i$$

We can now compare like coefficients such that

$$\alpha_i c_i - \alpha_{i-1} a_{i-1} = \beta_i$$
$$\alpha_i = \frac{\beta_i + \alpha_{i-1} a_{i-1}}{c_i}$$

This means that $a(z) = \sum_{i=m}^{\infty} \alpha_i z^i$ for some $m \in \mathbb{Z}$ where $\alpha_i = \dfrac{\beta_i + \alpha_{i-1}a_{i-1}}{c_i}$ and $\alpha_i = 0$ for $i < m$. Thus $(\mathcal{C} - z\mathcal{A})$ is surjective. $\square$

Lemma 4.8. *Let $(a_i)_{i\in\mathbb{Z}}$, $(b_i)_{i\in\mathbb{Z}}$, and $(c_i)_{i\in\mathbb{Z}}$ be arbitrary sequences such that $c_i \neq c_j$ if $i \neq j$ and $c_i \neq 0\ \forall i$. Define the linear operators $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, by $\mathcal{A}z^k = a_k z^k$, $\mathcal{B}z^k = b_k z^k$, $\mathcal{C}z^k = c_k z^k$. Then $(\mathcal{C} - z\mathcal{A})$ is injective.*

Proof. Let $b(z) = \sum_{i=m}^{\infty} \beta_i z^i$ and $a(z) = \sum_{i=l}^{\infty} \alpha_i z^i$ for some $l, m \in \mathbb{Z}$. Furthermore, define $\beta_i = 0$ for $i < m$ and $\alpha_i = 0$ for $i < l$. Without loss of generality, we assume that $m < l$. To show injectivity, we must show that given $(\mathcal{C} - z\mathcal{A})a(z) = (\mathcal{C} - z\mathcal{A})b(z)$, then it is the case that $a(z) = b(z)$. First, using the properties of linear operators, we simplify the expression and replace the formal Laurent series by their series expansion

$$\mathcal{C}(a(z)) - z\mathcal{A}(a(z)) = \mathcal{C}(b(z)) - z\mathcal{A}(b(z))$$
$$\mathcal{C}\left(\sum_{i=l}^{\infty} \alpha_i z^i\right) - z\mathcal{A}\left(\sum_{i=l}^{\infty} \alpha_i z^i\right) = \mathcal{C}\left(\sum_{i=m}^{\infty} \beta_i z^i\right) - z\mathcal{A}\left(\sum_{i=m}^{\infty} \beta_i z^i\right)$$

For values of $i < l < m$ notice that we will have $\alpha_i = 0$, thus we can start all summations at $i = m$

$$\mathcal{C}\left(\sum_{i=m}^{\infty} \alpha_i z^i\right) - z\mathcal{A}\left(\sum_{i=m}^{\infty} \alpha_i z^i\right) = \mathcal{C}\left(\sum_{i=m}^{\infty} \beta_i z^i\right) - z\mathcal{A}\left(\sum_{i=m}^{\infty} \beta_i z^i\right)$$

Next, we simplify again using properties of linear operators, then we adjust indices so that each summation has a z^i as the coefficient

$$\mathcal{C}\left(\sum_{i=m}^{\infty} \alpha_i z^i\right) - \mathcal{A}\left(\sum_{i=m}^{\infty} \alpha_i z^{i+1}\right) = \mathcal{C}\left(\sum_{i=m}^{\infty} \beta_i z^i\right) - \mathcal{A}\left(\sum_{i=m}^{\infty} \beta_i z^{i+1}\right)$$
$$\mathcal{C}\left(\sum_{i=m}^{\infty} \alpha_i z^i\right) - \mathcal{A}\left(\sum_{i=m+1}^{\infty} \alpha_{i-1} z^i\right) = \mathcal{C}\left(\sum_{i=m}^{\infty} \beta_i z^i\right) - \mathcal{A}\left(\sum_{i=m+1}^{\infty} \beta_{i-1} z^i\right)$$

Finally we apply the operators to obtain

$$\sum_{i=m}^{\infty} c_i\alpha_i z^i - \sum_{i=m+1}^{\infty} a_i\alpha_{i-1}z^i = \sum_{i=m}^{\infty} c_i\beta_i z^i - \sum_{i=m+1}^{\infty} a_i\beta_{i-1}z^i$$

By definition, since $i < m < l$ we know that $a_m\alpha_{m-1}z^m = 0$ and $a_m\beta_{m-1}z^m = 0$. Incorporating this into our summation yields

$$\sum_{i=m}^{\infty} c_i\alpha_i z^i - \sum_{i=m+1}^{\infty} a_i\alpha_{i-1}z^i - a_m\alpha_{m-1}z^m = \sum_{i=m}^{\infty} c_i\beta_i z^i - \sum_{i=m+1}^{\infty} a_i\beta_{i-1}z^i - a_m\beta_{m-1}z^m$$
$$\sum_{i=m}^{\infty} c_i\alpha_i z^i - \sum_{i=m}^{\infty} a_i\alpha_{i-1}z^i = \sum_{i=m}^{\infty} c_i\beta_i z^i - \sum_{i=m}^{\infty} a_i\beta_{i-1}z^i$$

We now show injectivity by comparing like coefficients and inducting on i. Recall that we know $\beta_{m-1} = 0$ and $\alpha_{m-1} = 0$. Thus for $i = m$ we have

$$c_m\alpha_m - a_m\alpha_{m-1} = c_m\beta_m - a_m\beta_{m-1}$$

$$c_m\alpha_m = c_m\beta_m$$

$$\alpha_m = \beta_m$$

Assume the inductive hypothesis: $\alpha_i = \beta_i$ for all $m \leq i \leq n$ for some $n \in \mathbb{Z}$. By comparing like coefficients, we have:

$$c_{n+1}\alpha_{n+1} - a_{n+1}\alpha_n = c_{n+1}\beta_{n+1} - a_{n+1}\beta_n$$

By the inductive hypothesis we have $c_{n+1}\alpha_{n+1} = c_{n+1}\beta_{n+1}$. Since $c_{n+1} \neq 0$ then $\alpha_{n+1} = \beta_{n+1}$. Thus by induction we have $\alpha_i = \beta_i \forall i \geq m$, so $a(z) = b(z)$. $\square$

Corollary 4.9. *Let $(a_i)_{i\in\mathbb{Z}}$, $(b_i)_{i\in\mathbb{Z}}$, and $(c_i)_{i\in\mathbb{Z}}$ be arbitrary sequences such that $c_i \neq c_j$ if $i \neq j$ and $c_i \neq 0\ \forall i$. Define the linear operators $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, by $\mathcal{A}z^k = a_k z^k$, $\mathcal{B}z^k = b_k z^k$, $\mathcal{C}z^k = c_k z^k$. Then $(\mathcal{C} - z\mathcal{A})$ is bijective.*

Proof of Krattenthaler's Theorem. To show that F is infinite lower triangular, it suffices to show that $f_{nk} = 0$ for $k = n+1$. From this it follows that $f_{nk} = 0$ for all $k > n$.

$$f_{n,n+1} = \frac{\prod_{j=n+1}^{n-1}(a_j + c_{n+1}b_j)}{\prod_{n+2}^{n}(c_j - c_{n+1})} = 0$$

Define the linear operators $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$. Then from Lemma 4.3 we have that

$$(\mathcal{C} - z\mathcal{A})f_k(z) = c_k(1 + z\mathcal{B})f_k(z)$$

where it has been shown by Corollary 4.9 that $(\mathcal{C} - z\mathcal{A})$ is bijective. Suppose we have

$$(\mathcal{C}^* - \mathcal{A}^* z)h_k(z) = c_k(1 + \mathcal{B}^* z)h_k(z)$$

then from Lemma 4.6 we know

$$(c_l - c_k)h_{kl} = (a_l + c_k b_l)h_{k,l+1}$$

and from Theorem 3.1 we know

$$\tilde{f}_k(z) = \frac{1}{\langle f_k(z), (1 + \mathcal{B}^* z)h_k(z)\rangle}(1 + \mathcal{B}^* z)h_k(z)$$

We now use the above results to solve for $\tilde{f}_k(z)$. Using $(c_l - c_k)h_{kl} = (a_l + c_k b_l)h_{k,l+1}$, let us first solve for h_{kl}. We will set $h_{kk} = 1$.

$$
\begin{aligned}
h_{kl} &= \frac{(a_l + c_k b_k)}{(c_l - c_k)} h_{k,l+1} \\
&= \frac{(a_l + c_k b_l)}{(c_l - c_k)} \cdot \frac{(a_{l+1} + c_k b_{l+1})}{(c_{l+1} - c_k)} h_{k,l+2} \\
&= \frac{\prod_{j=l}^{k-1} (a_j + c_k b_j)}{\prod_{j=l}^{k-1} (c_j - c_k)} h_{kk} \\
&= \frac{\prod_{j=l}^{k-1} (a_j + c_k b_j)}{\prod_{j=l}^{k-1} (c_j - c_k)}
\end{aligned}
$$

A result from Lemma 4.5(ii) tells us that $\langle f_k(z), (1 + z\mathcal{B}^* z) h_k(z) \rangle = h_{kk}$ which in our case is 1.

Using this, we now solve for $\tilde{f}_k(z)$

$$
\begin{aligned}
\tilde{f}_k(z) &= \frac{1}{\langle f_k(z), (1 + \mathcal{B}^* z) h_k(z) \rangle} \\
&= (1 + \mathcal{B}^* z) h_k(z) \\
&= (1 + \mathcal{B}^* z) \sum_{l=k}^{-\infty} h_{kl} z^{-l} \\
&= \sum_{l=k}^{-\infty} h_{kl} z^{-l} + \mathcal{B}^* \left(\sum_{l=k}^{-\infty} h_{kl} z^{-l+1} \right)
\end{aligned}
$$

Recall from Lemma 4.4 that $\mathcal{B}^* z^{-k} = b_k z^{-k}$. Applying this and adjusting the indices gives us

$$
\sum_{l=k}^{-\infty} h_{kl} z^{-l} + \sum_{l=k}^{-\infty} b_{l-1} h_{kl} z^{-l} = \sum_{l=k}^{-\infty} h_{kl} z^{-l} + \sum_{l-1=k}^{-\infty} b_l h_{k,l+1} z^{-l}
$$

Using the previously obtained definition of h_{kl}, we notice that $k_{k,k+1} = 0$. Incorporating this into the summation yields

$$\sum_{l=k}^{-\infty} h_{kl} z^{-l} + \sum_{l-1=k}^{-\infty} b_l h_{k,l+1} z^{-l} + b_k h_{k,k+1} z^{-k} = \sum_{l=k}^{-\infty} h_{kl} z^{-l} + \sum_{l=k}^{-\infty} b_l h_{k,l+1} z^{-l}$$

Recall from Theorem 3.1 that $\tilde{f}_k(z) = \sum_{l=k}^{\infty} \tilde{f}_{kl} z^{-l}$. Thus our equation reduces to

$$\sum_{l=k}^{\infty} \tilde{f}_{kl} z^{-l} = \sum_{l=k}^{-\infty} h_{kl} z^{-l} + \sum_{l=k}^{-\infty} b_l h_{k,l+1} z^{-l}$$

Finally, by comparing like coefficients and substituting our previous computation of h_{kl} we will obtain the desired result

$$\begin{aligned}
\tilde{f}_{kl} &= h_{kl} + b_l h_{k,l+1} \\
&= \frac{\prod_{j=l}^{k-1} (a_j + c_k b_j)}{\prod_{j=l}^{k-1} (c_j - c_k)} + b_l \cdot \frac{\prod_{j=l+1}^{k-1} (a_j + c_k b_j)}{\prod_{j=l+1}^{k-1} (c_j - c_k)} \\
&= \frac{(a_l + c_k b_l)}{\prod_{j=l}^{k-1} (c_j - c_k)} \cdot \frac{\prod_{j=l+1}^{k} (a_j + c_k b_j)}{(a_k + c_k b_k)} + b_l \cdot \frac{\prod_{j=l+1}^{k} (a_j + c_k b_j)}{\prod_{j=l}^{k-1} (c_j - c_k)} \cdot \frac{(c_l - c_k)}{(a_k + c_k b_k)} \\
&= \frac{(a_l + c_k b_l + b_l c_l - b_l c_k) \prod_{l+1}^{k} (a_j + c_k b_j)}{(a_k + c_k b_k) \prod_{j=l}^{k-1} (c_j - c_k)} \\
&= \frac{a_l + c_l b_l}{a_k + c_k b_k} \frac{\prod_{j=l+1}^{k} (a_j + c_k b_j)}{\prod_{j=l}^{k-1} (c_j - c_k)}
\end{aligned}$$

□

5 MACMAHON'S MASTER THEOREM

Definition 5.1. Let π be a permutation. The sign of π is given by $\operatorname{sgn}(\pi) = (-1)^k$ where k is the number of transpositions required to restore π to its natural order. A permutation is even if it has a sign of 1, otherwise it is odd.

Definition 5.2. Let $A = (a_{ij})$ be an $n \times n$ matrix. Define S_n be the set of all permutations of $\{1, \ldots, n\}$. The determinant of A is given by

$$\det A = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)}.$$

The following are elementary properties of the determinant that, though used, will not be proved in this paper

(i) $\det(AB) = \det A \det B$

(ii) $(\det A)^{-1} = \det A^{-1}$

(iii) If any row or column of A is multiplied by a scalar α, then the determinant of the resulting matrix is $\alpha \det A$.

(iv) If the rows or columns of A are linearly dependent, then $\det A = 0$.

(v) If A and B are diagonal matrices, then $AB = BA$

Definition 5.3. Let A be an $n \times n$ matrix and define the $k \times k$ principal submatrix $P_{i_1 \ldots i_k}$ to be the matrix formed by the $i_1, \ldots, i_k$ rows and columns of A. The determinant of such a matrix is a principal minor and can be written as $\det P_{i_1 \ldots i_k}$ or $|a_{i_1, i_1} \ldots a_{i_k, i_k}|$.

Remark 5.4. For an $n \times n$ matrix $A = (a_{ij})$, we can write the sum of all principal minors of A as $\displaystyle\sum_{(\xi_1,\ldots,\xi_n)=(0,\cdots,0)}^{(1,\ldots,1)} \left| a_{11}^{\xi_1} \cdots a_{nn}^{\xi_n} \right|$

Definition 5.5. Let $A = (a_{ij})$ be an $n \times n$ matrix and $x_1, \ldots, x_n$ be scalar indeterminants. Define the symbolic form

$$|(1 - a_{11}s_1x_1) \cdots (1 - a_{nn}s_nx_n)|$$

such that after multiplication, the product $a_{k_1,k_1} \cdots a_{k_m,k_m}$ in each term is to be written in determinant brackets.

Lemma 5.6. $\det(I - A) = \displaystyle\sum_{k=0}^{n} c_k$, *where* $c_k =$ *the sum of all* $k \times k$ *principal minors of* $-A$.

Lemma 5.7.

$$|(1 - a_{11}) \cdots (1 - a_{nn})| = \sum_{(\xi_1,\ldots,\xi_n)=(0,\cdots,0)}^{(1,\ldots,1)} (-1)^{\xi_1+\cdots+\xi_n} \left| a_{11}^{\xi_1} \cdots a_{nn}^{\xi_n} \right|$$

Proof. First consider the quantity $(1 - a_{11}) \cdots (1 - a_{nn})$. For $n = 1$, we only have the $(1 - a_{11}) = \displaystyle\sum_{\xi_1=0}^{1} (-1)^{\xi_1} a_{11}^{\xi_1}$. Assume the following inductive hypothesis for $n = 1, \ldots, k - 1$

$$(1 - a_{11}) \cdots (1 - a_{nn}) = \sum_{(\xi_1,\ldots,\xi_n)=(0,\cdots,0)}^{(1,\ldots,1)} (-1)^{\xi_1+\cdots+\xi_n} a_{11}^{\xi_1} \cdots a_{nn}^{\xi_n}$$

and consider $(1 - a_{11}) \cdots (1 - a_{kk})$. Note that this can be rewritten as

$$(1 - a_{11}) \cdots (1 - a_{k-1,k-1})(1 - a_{kk}) =$$
$$\sum_{(\xi_1,\ldots,\xi_{k-1})=(0,\cdots,0)}^{(1,\ldots,1)} (-1)^{\xi_1+\cdots+\xi_{k-1}} a_{11}^{\xi_1} \cdots a_{k-1,k-1}^{\xi_{k-1}} \cdot (1 - a_{kk})$$

This summation can further be simplified as

$$\sum_{(\xi_1,\ldots,\xi_{k-1})=(0,\cdots,0)}^{(1,\ldots,1)} (-1)^{\xi_1+\cdots+\xi_{k-1}} a_{11}^{\xi_1}\cdots a_{k-1,k-1}^{\xi_{k-1}}(-1)^0 a_{k,k}^0 +$$

$$\sum_{(\xi_1,\ldots,\xi_{k-1})=(0,\cdots,0)}^{(1,\ldots,1)} (-1)^{\xi_1+\cdots+\xi_{k-1}} a_{11}^{\xi_1}\cdots a_{k-1,k-1}^{\xi_{k-1}}(-1)^1 a_{k,k}^1 =$$

$$\sum_{(\xi_1,\ldots,\xi_k)=(0,\cdots,0)}^{(1,\ldots,1)} (-1)^{\xi_1+\cdots+\xi_k} a_{11}^{\xi_1}\cdots a_{kk}^{\xi_k}$$

Using this result, we can now compute $|(1-a_{11})\cdots(1-a_{nn})|$ to be

$$\left|\sum_{(\xi_1,\ldots,\xi_n)=(0,\cdots,0)}^{(1,\ldots,1)} (-1)^{\xi_1+\cdots+\xi_n} a_{11}^{\xi_1}\cdots a_{nn}^{\xi_n}\right|$$

From Definition 5.3, we now place the determinant brackets within each term, giving us

$$\sum_{(\xi_1,\ldots,\xi_n)=(0,\cdots,0)}^{(1,\ldots,1)} (-1)^{\xi_1+\cdots+\xi_n} \left|a_{11}^{\xi_1}\cdots a_{nn}^{\xi_n}\right|$$

The $(-1)^{\xi_1+\cdots+\xi_n}$ need not be in the determinant brackets since this only tells us whether to add or subtract the term. □

Corollary 5.8. *Given an $n\times n$ matrix $A=(a_{ij})$, $|(1-a_{11})\cdots(1-a_{nn})|$ = the sum of all principal minors of $-A$*

Proof. Recall from elementary properties of determinants, if any row or column of A is multiplied by a scalar α, then the resulting determinant of this matrix is equal to the determinant of A multiplied by α. Using the result of the Lemma 5.7, notice that we can rewrite

$$\sum_{(\xi_1,\ldots,\xi_n)=(0,\cdots,0)}^{(1,\ldots,1)} (-1)^{\xi_1+\cdots+\xi_n} \left|a_{11}^{\xi_1}\cdots a_{nn}^{\xi_n}\right| = \sum_{(\xi_1,\ldots,\xi_n)=(0,\cdots,0)}^{(1,\ldots,1)} \left|(-a_{11})^{\xi_1}\cdots(-a_{nn})^{\xi_n}\right|$$

Without loss of generality, assume that $-a_{kk}$ in the determinant brackets implies that the kth column of the corresponding principal submatrix is multiplied by -1. After this manipulation, it is clear that each term in the summation is a principal minor of $-A$, furthermore, note that $\left|(-a_{11})^{\xi_1}\cdots(-a_{nn})^{\xi_n}\right| = \det(-A)$ when $(\xi_1,\ldots,\xi_n) = (1,\cdots,1)$. Thus by previous remark, we have $|(1-a_{11})\cdots(1-a_{nn})|$ = the sum of all principal minors of $-A$ $\square$

Corollary 5.9. *Given an $n\times n$ matrix $A=(a_{ij})$, $\det(I-A) = |(1-a_{11})\cdots(1-a_{nn})|$.*

Proof. By Lemma 5.6, $\det(I-A)$ = the sum of all principal minors of $-A$. But we obtained from the last corollary that $|(1-a_{11})\cdots(1-a_{nn})|$ = the sum of all principal minors of $-A$, thus $\det(I-A) = |(1-a_{11})\cdots(1-a_{nn})|$ $\square$

Remark 5.10. Note from Lemmas 5.6 and 5.7 and Corollaries 5.8 and 5.9, we can conclude that the following are equivalent:

(i) $\det(I-A)$

(ii) $\sum_{k=0}^{n} c_k$, where c_k = the sum of all $k\times k$ principal minors of $-A$

(iii) $|(1-a_{11})\cdots(1-a_{nn})|$

Proposition 5.11. *Let $A=(a_{ij})$, $S=\operatorname{diag}(s_1,\ldots,s_n)$ and $X=\operatorname{diag}(x_1,\ldots,x_n)$. Then*

$$\det(I-ASX) = |(1-a_{11}s_1x_1)\cdots(1-a_{nn}s_nx_n)|$$

Proof. Let $B=ASX=(a_{ij}s_jx_j)$ such that

$$\det(I-B) = (-1)^{\xi_1+\cdots+\xi_n}\,|(1-b_{11})\cdots(1-b_{nn})| = \sum_{(\xi_1,\ldots,\xi_n)=(0,\cdots,0)}^{(1,\ldots,1)} (-1)^{\xi_1+\cdots+\xi_n}\left|b_{11}^{\xi_1}\cdots b_{nn}^{\xi_n}\right|$$

Note that the diagonal entries of B are of the form $a_{ii}s_ix_i$, thus we can rewrite the determinant bracket quantity as $\left|(a_{11}s_1x_1)^{\xi_1}\cdots(a_{nn}s_nx_n)^{\xi_n}\right|$. Now, reconsider the matrix ASX. Because S and X are diagonal matrices, the product ASX has entries $(ASX)_{ij} = (a_{ij}s_jx_j)$ with diagonal entries $a_{ii}s_ix_i$. Since each column of ASX is being multiplied by s_jx_j, we can write this quantity either inside or outside the determinant brackets giving us

$$\begin{aligned}\left|(a_{11}s_1x_1)^{\xi_1}\cdots(a_{nn}s_nx_n)^{\xi_n}\right| &= s_1^{\xi_1}\cdots s_n^{\xi_n}\left|(a_{11}x_1)^{\xi_1}\cdots(a_{nn}x_n)^{\xi_n}\right| \\ &= (s_1x_1)^{\xi_1}\cdots(s_nx_n)^{\xi_n}\left|a_{11}^{\xi_1}\cdots a_{nn}^{\xi_n}\right|\end{aligned}$$

This confirms that a term of the form $|b_{k_1k_1}\cdots b_{k_lk_l}|$, when $B = ASX = (a_{ij}s_jx_j)$is equivalent a term of the form $|(a_{k_1k_1}s_{k_1}x_{k_1})\cdots(a_{k_lk_l}s_{k_l}x_{k_l})|$, thus we can write

$$\det(I - ASX) = |(1 - a_{11}s_1x_1)\cdots(1 - a_{nn}s_nx_n)|$$

□

Lemma 5.12. *Let* $C = (c_{ij})$, $S = \mathrm{diag}(s_1, \ldots, s_n)$, $D = \mathrm{diag}(d_1, \ldots, d_n)$*, and* $\det D \neq 0$*. Then*

$$|(d_1 - c_{11}x_1)\cdots(d_n - c_{nn}x_n)| = \det(D - CS).$$

Proof. $\det D \neq 0$, thus none of the $d_1, \ldots, d_n$ can be zero since D is diagonal. Therefore we can factor such that

$$|(d_1 - c_{11}s_1)\cdots(d_n - c_{nn}s_n)| = \left|(1 - c_{11}s_1\frac{1}{d_1})d_1\cdots(1 - c_{nn}s_n\frac{1}{d_n})d_n\right|$$

Investigating the left hand side, we can deduce that the quantities $d_1, \ldots, d_n$ are multiplied by some row or column of the matrix $I - CSD^{-1}$. Without loss of generality, we can assume that row k of the matrix $I - CSD^{-1}$ is being multiplied by d_k, thus we can factor each d_k out of the determinant brackets, obtaining $\left|(1 - c_{11}s_1\frac{1}{d_1})\cdots(1 - c_{nn}s_n\frac{1}{d_n})\right| d_1\cdots d_n$. Note that because D is diagonal, $\det D = d_1\cdots d_n$. By replacing this quantity into the previous equation we obtain and using 5.11 we obtain:

$$\begin{aligned}\left|(1 - c_{11}s_1\frac{1}{d_1})\cdots(1 - c_{nn}s_n\frac{1}{d_n})\right| d_1\cdots d_n &= \left|(1 - c_{11}s_1\frac{1}{d_1})\cdots(1 - c_{nn}s_n\frac{1}{d_n})\right| \det D \\ &= \det(I - CSD^{-1})\det D \\ &= \det((I - CSD^{-1})D) \\ &= \det(D - CS)\end{aligned}$$

□

Theorem 5.13. *Define* $X_1, \ldots, X_n$ *to be linear functions such that* $X_k = a_{k1}x_1 + \cdots + a_{kn}x_n$. *Let* $S = \operatorname{diag}(s_1, \ldots, s_n)$, $X = \operatorname{diag}(x_1, \ldots, x_n)$, $Y = \operatorname{diag}(X_1, \ldots, X_n)$, *and* $A = (a_{ij})$. *Then*

$$\frac{|(1 - a_{11}s_1x_1)\cdots(1 - a_{nn}s_nx_n)|}{(1 - s_1X_1)\cdots(1 - s_nX_n)} = \sum_{(\xi_1,\ldots,\xi_n)=(0,\cdots,0)}^{(1,\ldots,1)} \frac{s_1^{\xi_1}\cdots s_n^{\xi_n}\left|(X_1 - a_{11}x_1)^{\xi_1}\cdots(X_n - a_{nn}x_n)^{\xi_n}\right|}{(1 - s_1X_1)^{\xi_1}\cdots(1 - s_nX_n)^{\xi_n}}$$

Proof. First note that we can rewrite the numerator of the left hand side as the following

$$|(1 - a_{11}s_1x_1)\cdots(1 - a_{nn}s_nx_n)| = |(1 - X_1s_1 + (X_1 - a_{11}x_1)s_1)\cdots(1 - X_ns_n + (X_n - a_{nn}x_n)s_n)|$$

Let $D = I - YS = \operatorname{diag}(1 - X_1s_1, \ldots, 1 - X_ns_n)$ and $C = -Y + AX = (-X_i + a_{ij}x_i)$ and rewrite the previous quantity as

$$|(1 - X_1 s_1 + (X_1 - a_{11}x_1)s_1) \cdots (1 - X_n s_n + (X_n - a_{nn}x_n)s_n)| =$$

$$|(d_1 - c_{11}s_1) \cdots (d_n - c_{nn}s_n)|,$$

which is, by Lemma 5.12, equivalent to $\det(D - CS) = \det((I - YS) + (Y - AX)S)$. Note that since both Y and S are diagonal matrices, $YS = SY$, thus $\det D = \det(I - YS) = \det(I - SY) = (1 - s_1 X_1) \cdots (1 - s_n X_n)$. Using these results we can simplify the left hand side to be:

$$\begin{aligned}
\frac{|(1 - a_{11}s_1x_1) \cdots (1 - a_{nn}s_nx_n)|}{(1 - s_1X_1) \cdots (1 - s_nX_n)} &= \frac{\det((I - YS) + (Y - AX)S)}{\det(I - YS)} \\
&= \det((I - YS) + (Y - AX)S)(\det(I - YS))^{-1} \\
&= \det((I - YS) + (Y - AX)S)\det(I - YS)^{-1} \\
&= \det(((I - YS) + (Y - AX)S)(I - YS)^{-1})) \\
&= \det(I + (Y - AX)S(I - YS)^{-1})) \\
&= \det(I + (Y - AX)S(I - SY)^{-1}))
\end{aligned}$$

This is just the sum of all the principal minors of the matrix $(Y - AX)S(I - SY)^{-1}$. Using Remark 5.4 and Proposition 5.11, we can compute the sum of all the principal minors to be

$$\sum_{(\xi_1,\ldots,\xi_n)=(0,\cdots,0)}^{(1,\ldots,1)} \left|(X_1 - a_{11}x_1)^{\xi_1} s_1^{\xi_1}(1 - s_1X_1)^{-\xi_1} \cdots (X_1 - a_{11}x_n)^{\xi_n} s_n^{\xi_n}(1 - s_nX_n)^{-\xi_n}\right| =$$

$$\sum_{(\xi_1,\ldots,\xi_n)=(0,\cdots,0)}^{(1,\ldots,1)} \left|(X_1 - a_{11}x_1)^{\xi_1} \cdots (X_1 - a_{11}x_n)^{\xi_n}\right| s_1^{\xi_1}(1 - s_1X_1)^{-\xi_1} \cdots s_n^{\xi_n}(1 - s_nX_n)^{-\xi_n} =$$

$$\sum_{(\xi_1,\ldots,\xi_n)=(0,\cdots,0)}^{(1,\ldots,1)} \frac{s_1^{\xi_1} \cdots s_n^{\xi_n} \left|(X_1 - a_{11}x_1)^{\xi_1} \cdots (X_n - a_{nn}x_n)^{\xi_n}\right|}{(1 - s_1X_1)^{\xi_1} \cdots (1 - s_nX_n)^{\xi_n}}$$

$\square$

Remark 5.14. Let $p(x_1, \ldots, x_n) \neq 0$ be a power series. If evaluating $p(x_1, \ldots, x_n)$ at some $x_i = 0$ yields 0, then each term of $p(x_1, \ldots, x_n)$ contains a factor of x_i. This is clear since in order for $p(x_1, \ldots, 0, \ldots, x_n) = 0$ to occur for $p(x_1, \ldots, x_n) \neq 0$, each term must be identically 0. Since only $x_i = 0$, then each term must be multiplied by x_i.

We can generalize this result to the following: Given a power series $p(x_1, \ldots, x_n) \neq 0$, if $p(x_1, \ldots, x_n) = 0$ for $x_{i_1} = \cdots = x_{i_n} = 0$, then each term of $p(x_1, \ldots, x_n)$ contains at least one of the quantities $x_{i_1}, \ldots, x_{i_n}$.

Lemma 5.15. *If* $p(x_1, \ldots, x_n, s_1, \ldots, s_n) = s_1^{\xi_1} \cdots s_n^{\xi_n} \left|(X_1 - a_{11}x_1)^{\xi_1} \cdots (X_n - a_{nn}x_n)^{\xi_n}\right|$ *where* $\xi_i = 0$ *or* $\xi_i = 1$*, then* $p(\xi_1 x_1, \ldots, \xi_n x_n, s_1, \ldots, s_n) = 0$ *except when* $\xi_1 = \cdots = \xi_n = 0$*.*

Proof. Notice if we define $Y = \operatorname{diag}(X_1, \ldots, X_n)$ and $Z = \operatorname{diag}(x_1, \ldots, x_n)$, then from a Lemma 5.12, we have

$$p(x_1, \ldots, x_n, s_1, \ldots, s_n) = s_1^{\xi_1} \cdots s_n^{\xi_n} \det(Y - AZ)$$

for $\xi_1 = \cdots = \xi_n = 1$. Without loss of generality, suppose

$$\xi_1, \ldots, \xi_t = 1 \qquad \text{and} \qquad \xi_{t+1}, \ldots, \xi_n = 0$$

for some $t < n$ and compute $p(\xi_1 x_1, \ldots, \xi_n x_n, s_1, \ldots, s_n)$. By plugging in the values of ξ, we can rewrite this as

$$p(x_1, \ldots, x_t, 0, \ldots, 0, s_1, \ldots, s_n) = s_1 \cdots s_t \left|(X_1 - a_{11}x_1) \cdots (X_t - a_{tt}x_t)\right|$$

Consider $|(X_1 - a_{11}x_1) \cdots (X_t - a_{tt}x_t)|$. This is a principal minor of $Y - AZ$, in particular, this is

$$\det \begin{vmatrix} X_1 - a_{11}x_1 & -a_{12}x_1 & \dots & -a_{1t}x_1 \\ -a_{21}x_2 & X_2 - a_{22}x_2 & \dots & -a_{2t}x_2 \\ \vdots & \vdots & \vdots & \vdots \\ -a_{t1}x_t & -a_{t2}x_t & \dots & X_t - a_{tt}x_t \end{vmatrix}$$

Each diagonal entry of the above matrix is in the form $X_k - a_{kk}x_k$. Before computing the determinant, let us first compute this quantity at $x_{t+1} = \cdots = x_n = 0$

$$\begin{aligned} X_k - a_{kk}x_k &= a_{k1}x_1 + \cdots + a_{kn}x_n - a_{kk}x_k \\ &= a_{k1}x_1 + \cdots + a_{kt}x_t - a_{kk}x_k \\ &= a_{k1}x_1 + \cdots + a_{k,k-1}x_{k-1} + a_{k,k+1}x_{k+1} + \cdots + a_{kt}x_t \end{aligned}$$

Thus upon substitution, the matrix becomes

$$\begin{vmatrix} a_{12}x_2 + \ldots + a_{1t}x_t & -a_{12}x_1 & \dots & -a_{1t}x_1 \\ -a_{21}x_2 & a_{21}x_1 + a_{23}x_3 + \ldots + a_{2t}x_t & \dots & -a_{2t}x_2 \\ \vdots & \vdots & \vdots & \vdots \\ -a_{t1}x_t & -a_{t2}x_t & \dots & a_{t1}x_1 + \ldots + a_{t,t-1}x_{t-1} \end{vmatrix}$$

Denote $C_1, \ldots, C_t$ to be the column vectors of this matrix and notice that $x_1C_1 + x_2C_2 + \cdots + x_tC_t = 0$, thus the columns of this matrix are linearly dependent, hence the determinant is 0 and $p(\xi_1 x_1, \ldots, \xi_n x_n) = 0$. It is clear that when $\xi_1 = \cdots = \xi_n = 0$ the expression becomes 1. $\square$

Lemma 5.16. *The expansion of* $s_1^{\xi_1} \cdots s_n^{\xi_n} \left|(X_1 - a_{11}x_1)^{\xi_1} \cdots (X_n - a_{nn}x_n)^{\xi_n}\right|$ *where* $\xi_i = 0$ *or* $\xi_i = 1$ *contains no terms which are functions of* $s_1^{\xi_1}x_1^{\xi_1}, \ldots, s_n^{\xi_n}x_n^{\xi_n}$ *only except when* $\xi_1 = \cdots = \xi_n = 0$.

Proof. From Lemma 5.15 we have

$$p(\xi_1 x_1, \dots, \xi_n x_n, s_1, \dots, s_n) = s_1^{\xi_1} \cdots s_n^{\xi_n} \left|(X_1 - a_{11}x_1)^{\xi_1} \cdots (X_n - a_{nn}x_n)^{\xi_n}\right| = 0$$

except when $\xi_1 = \cdots = \xi_n = 0$, thus from a Remark 5.14, each term of p must contain at least one of the quantities $x_{i_1}, \dots, x_{i_k}$ where $\xi_{i_1}, \dots, \xi_{i_k}$ are all zero and $k < n$. Clearly if one of these x_i's are in each quantity, the corresponding s_i is not as that particular s_i will be 1. Since all terms contain an x in which the corresponding ξ is 0, then there must be no terms in the expansion that only contain x's in which the corresponding ξ's are 1, hence there are no terms in the expansion that only contain an $s_i x_i$ for which $\xi_i = 1$.

It is clear that

$$s_1^{\xi_1} \cdots s_n^{\xi_n} \left|(X_1 - a_{11}x_1)^{\xi_1} \cdots (X_n - a_{nn}x_n)^{\xi_n}\right|$$

is a function of $s_1^{\xi_1} x_1^{\xi_1}, \dots, s_n^{\xi_n} x_n^{\xi_n}$ when $\xi_1 = \cdots = \xi_n = 0$. $\square$

Example 5.17. Let $p(x_1, x_2, x_3, s_1, s_2, s_3) = s_1 s_3 \left|(X_1 - a_{11}x_1)(X_3 - a_{33}x_3)\right|$. We can check Lemma 5.16 by simplifying p

$$s_1 s_3 \left|(X_1 - a_{11}x_1)(X_3 - a_{33}x_3)\right| = s_1 s_3 \det \begin{vmatrix} X_1 - a_{11}x_1 & -a_{13}x_1 \\ -a_{31}x_3 & X_3 - a_{33}x_3 \end{vmatrix}$$

Evaluating the determinant and simplifying further yields

$$
\begin{aligned}
p &= s_1 s_3 \left((X_1 - a_{11}x_1)(X_3 - a_{33}x_3) - a_{13}a_{31}x_1x_3\right) \\
&= s_1 s_3 \left((a_{11}x_1 + a_{12}x_2 + x_{13}x_3 - a_{11}x_1)(a_{31}x_1 + a_{32}x_2 + x_{33}x_3 - a_{33}x_3) - a_{13}a_{31}x_1x_3\right) \\
&= s_1 s_3 \left((a_{12}x_2 + x_{13}x_3)(a_{31}x_1 + a_{32}x_2) - a_{13}a_{31}x_1x_3\right) \\
&= s_1 s_3 \left(a_{12}a_{31}x_1x_2 + x_{13}a_{31}x_1x_3 + a_{12}a_{32}x_2^2 + x_{13}a_{32}x_2x_3 - a_{13}a_{31}x_1x_3\right) \\
&= s_1 s_3 \left(a_{12}a_{31}x_1x_2 + a_{12}a_{32}x_2^2 + x_{13}a_{32}x_2x_3\right) \\
&= a_{12}a_{31}s_1s_3x_1x_2 + a_{12}a_{32}s_1s_3x_2^2 + x_{13}a_{32}s_1s_3x_2x_3
\end{aligned}
$$

Notice, as expected from Lemma 5.16, p contains no terms which are functions of s_1x_1, s_3x_3 only. Equivalently, each term of p has a factor of x_2 in which s_2 is not present.

The proof of MacMahon's Master Theorem requires a great deal of reference to the coefficients of some power series $p(x_1, \ldots, x_n)$. For convenience, define the function $\kappa(p; x_1^{\nu_1} \cdots x_n^{\nu_n})$ to be the coefficient of the $x_1^{\nu_1} \cdots x_n^{\nu_n}$ term in $p(x_1, \ldots, x_n)$.

Theorem 5.18 (MacMahon's Master Theorem). *Define $X_1, \ldots, X_n$ to be linear functions such that $X_k = a_{k1}x_1 + \cdots + a_{kn}x_n$. Let $X = \operatorname{diag}(x_1, \ldots, x_n)$ and $A = (a_{ij})$. Then*

$$\kappa(\det(I_n - AX)^{-1}; x_1^{\nu_1} \cdots x_n^{\nu_n}) = \kappa(X_1^{\nu_1} \cdots X_n^{\nu_n}; x_1^{\nu_1} \cdots x_n^{\nu_n})$$

Proof. Let $S = \operatorname{diag}(s_1, \ldots, s_n)$. First let us notice that the fraction

$$\frac{1}{(1 - X_1) \cdots (1 - X_n)}$$

has a power series expansion in which the general term is $X_1^{\nu_1} \cdots X_n^{\nu_n}$. We also note that the only term in the expansion which contributes to the coefficient of the $x_1^{\nu_1} \cdots x_n^{\nu_n}$ term is $X_1^{\nu_1} \cdots X_n^{\nu_n}$.

Thus we have

$$\kappa\left(\frac{1}{(1-X_1)\cdots(1-X_n)}; x_1^{\nu_1}\cdots x_n^{\nu_n}\right) = \kappa(X_1^{\nu_1}\cdots X_n^{\nu_n}; x_1^{\nu_1}\cdots x_n^{\nu_n})$$

Furthermore, notice that the expansion will yield various exponents for each x since each X_k is a function of $x_1, \ldots, x_n$. This makes investigating the coefficient of the $x_1^{\nu_1}\cdots x_n^{\nu_n}$ term more difficult since every resulting term will be functions of $x_1, \ldots, x_n$.

Let us multiply each X_k by the auxiliary variable s_k, giving us

$$s_k X_k = a_{k1}s_k x_1 + \cdots + a_{kk}s_k x_k + \cdots + a_{kn}x_n$$

Replacing this into our original fraction yields

$$\frac{1}{(1-s_1X_1)\cdots(1-s_nX_n)}$$

which has a power series expansion in which the general term is in the form $s_1^{\nu_1}\cdots s_n^{\nu_n}X_1^{\nu_1}\cdots X_n^{\nu_n}$. We also note that the only term in the expansion which contributes to the coefficient of the $s_1^{\nu_1}\cdots s_n^{\nu_n}x_1^{\nu_1}\cdots x_n^{\nu_n}$ term is $s_1^{\nu_1}\cdots s_n^{\nu_n}X_1^{\nu_1}\cdots X_n^{\nu_n}$.

Furthermore, notice that even though each term in the expansion is still a function of $x_1, \ldots, x_n$, the term we are interested will be a function of $s_1x_1, \ldots, s_nx_n$ only. Note that our addition of $s_1, \ldots, s_n$ does not affect our results since we are merely concerned with the coefficient of a specific term, more specifically,

$$\kappa\left(\frac{1}{(1-X_1)\cdots(1-X_n)}; x_1^{\nu_1}\cdots x_n^{\nu_n}\right) = \kappa\left(\frac{1}{(1-s_1X_1)\cdots(1-s_nX_n)}; s_1^{\nu_1}\cdots s_n^{\nu_n}x_1^{\nu_1}\cdots x_n^{\nu_n}\right)$$

Let $f(x_1, \ldots, x_n, s_1, \ldots, s_n) = \dfrac{1}{(1 - s_1X_1)\cdots(1 - s_nX_n)}$.

Now consider our other power series. Because of the previous explanation, we now are interested in the coefficient of $s_1^{\nu_1}\cdots s_n^{\nu_n}x_1^{\nu_1}\cdots x_n^{\nu_n}$ However, the power series $\det(I_n - AX)^{-1}$ has no term of that form. Let

$SX = \text{diag}(s_1x_1, \ldots, s_nx_n)$. Since every x in this matrix is paired with a corresponding s it immediately becomes clear that

$$\kappa\left(\det(I_n - AX)^{-1}; x_1^{\nu_1}\cdots x_n^{\nu_n}\right) = \kappa\left(\det(I_n - ASX)^{-1}; s_1^{\nu_1}\cdots s_n^{\nu_n}x_1^{\nu_1}\cdots x_n^{\nu_n}\right)$$

This result can also be deduced by the result of Proposition 5.11. By the same Proposition, we can write

$$\det(I_n - ASX)^{-1} = \frac{1}{|(1 - a_{11}s_1x_1)\cdots(1 - a_{nn}s_nx_n)|}$$

Let $g(s_1x_1, \ldots, s_nx_n) = \det(I_n - ASX)^{-1}$.

Now, let us compute $\dfrac{f(x_1, \ldots, x_n, s_1, \ldots, s_n)}{g(s_1x_1, \ldots, s_nx_n)}$ and examine the terms in the result. From Theorem 5.13, we can write

$$\begin{aligned}
\frac{f(x_1, \ldots, x_n, s_1, \ldots, s_n)}{g(s_1x_1, \ldots, s_nx_n)} &= \frac{|(1 - a_{11}s_1x_1)\cdots(1 - a_{nn}s_nx_n)|}{(1 - s_1X_1)\cdots(1 - s_nX_n)} \\
&= \sum_{(\xi_1,\ldots,\xi_n)=(0,\cdots,0)}^{(1,\ldots,1)} \frac{s_1^{\xi_1}\cdots s_n^{\xi_n}\left|(X_1 - a_{11}x_1)^{\xi_1}\cdots(X_n - a_{nn}x_n)^{\xi_n}\right|}{(1 - s_1X_1)^{\xi_1}\cdots(1 - s_1X_n)^{\xi_n}}
\end{aligned}$$

To better examine the coefficient of $s_1^{\nu_1}\cdots s_n^{\nu_n}x_1^{\nu_1}\cdots x_n^{\nu_n}$, let us find all terms in the above summation which are functions of $s_1x_1, \ldots, s_nx_n$ only since they will contribute to the coefficient of interest. To do this, we must find the n-tuples of $\xi_1, \ldots, \xi_n$ that will obtain terms which are functions of $s_1x_1, \ldots, s_nx_n$ only. By Lemma 5.16, we know that there does not exist an n-tuple in which this will occur for $s_1^{\xi_1}\cdots s_n^{\xi_n}\left|(X_1-a_{11}x_1)^{\xi_1}\cdots(X_n-a_{nn}x_n)^{\xi_n}\right|$ unless $\xi_i=0$ for all i. Thus this will also not occur for $\dfrac{\left|(X_1-a_{11}x_1)^{\xi_1}\cdots(X_n-a_{nn}x_n)^{\xi_n}\right|}{(1-X_1)^{\xi_1}\cdots(1-X_n)^{\xi_n}}$ unless $\xi_i=0$ for all i.

Denote $h(x_1,\ldots,x_n,s_1,\ldots,s_n)$ to be the quantity in the quotient which are not functions of $s_1x_1,\ldots,s_nx_n$ only. We can simplify the quotient as

$$\frac{f(x_1,\ldots,x_n,s_1,\ldots,s_n)}{g(s_1x_1,\ldots,s_nx_n)}=1+h(x_1,\ldots,x_n,s_1,\ldots,s_n)$$

$$f(x_1,\ldots,x_n,s_1,\ldots,s_n)=g(s_1x_1,\ldots,s_nx_n)+g(s_1x_1,\ldots,s_nx_n)h(x_1,\ldots,x_n,s_1,\ldots,s_n)$$

Each of the terms in $g(x_1,\ldots,x_n,s_1,\ldots,s_n)$ are functions of $s_1x_1,\ldots,s_nx_n$, while each of the terms in $h(x_1,\ldots,x_n,s_1,\ldots,s_n)$ are not, thus the terms in their product will not be in terms of $s_1x_1,\ldots,s_nx_n$ only. If we rewrite f and substitute

$$f=f_1(s_1x_1,\ldots,s_nx_n)+f_2(x_1,\ldots,x_n,s_1,\ldots,s_n)$$

$$f_1(s_1x_1,\ldots,s_nx_n)+f_2(x_1,\ldots,x_n,s_1,\ldots,s_n)=$$
$$g(s_1x_1,\ldots,s_nx_n)+g(s_1x_1,\ldots,s_nx_n)h(x_1,\ldots,x_n,s_1,\ldots,s_n)$$

then we see that

$$f_1(s_1x_1,\ldots,s_nx_n)=g(s_1x_1,\ldots,s_nx_n)\qquad\text{and}$$
$$f_2(x_1,\ldots,x_n,s_1,\ldots,s_n)=g(s_1x_1,\ldots,s_nx_n)h(x_1,\ldots,x_n,s_1,\ldots,s_n)$$

Thus we have

$$\kappa(\det(I_n-AX)^{-1};x_1^{\nu_1}\cdots x_n^{\nu_n})=\kappa(X_1^{\nu_1}\cdots X_n^{\nu_n};x_1^{\nu_1}\cdots x_n^{\nu_n})$$

$\square$

6 FURTHER STUDY

q-Analogue of MacMahon's Master Theorem The infinite dimensional matrix inverse presented in Krattenthaler's theorem is used to derive summation formulas involving hypergeometric series [3]. Krattenthaler uses similar methods provided in his approach to matrix inversion to prove Lagrange inversion formulas [5]. By extending this into a multidimensional matrix inversion, Krattenthaler derives the following q-analogue of MacMahon's Master Theorem [4].

Theorem 6.1. *Let z_i, b_{ij}, for $i, j = 1, \ldots, r$ be indeterminate, and let $n_1, \ldots, n_r$ be arbitrary nonnegative integers. Then there holds*

$$\langle z^0 \rangle \prod_{i=1}^{r} \left(\sum_{j=1}^{r} b_{ij} z_j / z_i ; q_i \right)_{n_i} = \langle z^n \rangle \left(\det_{1 \le i,j \le r} ((1 - z_i)\delta_{ij} + z_i b_{ij} \mathcal{E}_i) \right)^{-1} \mathbf{1}$$

where $\mathcal{E}_i$ denotes the q-shift operator defined by $\mathcal{E}_i z_i = q_i z_i$, and $\mathbf{1}$ represents the constant polynomial 1.

Alternate proofs of MacMahon's Master Theorem Though we have chosen an algebraic approach, MacMahon's Master Theorem has been proved using other methods. In his paper pertaining to residue theory, Huang provides a proof to this theorem by method of residue maps [2]. We have also found a proof of MacMahon's Master Theorem using permutation digraphs. For this proof, see the lecture notes by Payne [7], which are based on the book by Brualdi and Ryser [1]. This book also contains several other references.

LIST OF REFERENCES

[1] Brualdi, Richard A. and Herbert J. Ryser, *Combinatorial matrix theory*, Cambridge University Press, Cambridge, 1991.

[2] Huang, I-Chuau, *Applications of residues to combinatorial identities*, Proceedings of the American Mathematical Society **125** (1997), 1011–1017.

[3] Krattenthaler, Christian, *A new matrix inverse*, Proceedings of the American Mathematical Society **124** (1996), 47–59.

[4] Krattenthaler, Christian, and Michael Schlosser, *A new multidimensional matrix inverse with applications to multiple q-series*, Discrete Mathematics **204** (1999), 249–279.

[5] Krattenthaler, Christian, *Operator methods and Lagrange inversion: A unified approach to Lagrange formulas,*, Transactions of the American Mathematical Society **305** (1988), 431–465.

[6] MacMahon, Percy A., *Combinatory analysis*, Chelsea Publishing Co., New York, 1960.

[7] Payne, S.E., *Applied Combinatorics - Math 6409*, `http://www-math.cudenver.edu/~spayne/classnotes/studentroot.ps`.