Asymptotic Formulas For Large Arguments Of Hypergeometric-type Functio

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ASYMPTOTIC FORMULAS FOR LARGE ARGUMENTS OF HYPERGEOMETRIC-TYPE FUNCTIONS USING THE BARNES INTEGRAL

By

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ABSTRACT

Hypergeometric type functions have a long list of applications in the field of sciences. A brief history is given of Hypergeometric functions including some of their applications. A development of a new method for finding asymptotic formulas for large arguments is given. This new method is applied to Bessel functions. Results are compared with previously known methods.

ACKNOWLEDGEMENTS

To my Lord and Savior Jesus Christ who has never failed to be by my side. I owe you everything. To my parents, Tom and Kathy Heck, who have been more and done more for me than I could ever ask. To Dr. Andrews for his continual support throughout this process. I suspect that this probably took longer than it should have and his patience, wisdom, and kindness are amazing. To all of my students who have been understanding and supportive over the past two years. I hope that each of you continue to strive to reach the dreams that you envision.

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1 INTRODUCTION

There are many different types of functions that are used to model physical situations. Each of these functions or family of functions (i.e. Trigonometric functions) has their own unique set of properties. The discovery of Hypergeometric functions allowed mathematicians to further generalize previously unrelated functions. For instance the following is a partial list of functions that have hypergeometric representations: [2]

$$e^x = {}_0F_0\left(-;-;x\right)$$

$$\cos x = {}_0F_1\left(-;\tfrac{1}{2};-\tfrac{x^2}{4}\right)$$

$$\left(1-x\right)^{-a} = {}_1F_0\left(a;-;x\right)$$

$$P_n\left(x\right) = {}_2F_1\left(n+1,-n;1;\tfrac{(1-x)}{2}\right)$$

$$\frac{\sin^{-1} x}{x} = {}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};x^2\right)$$

By representing each of these functions as a hypergeometric type function, we can identify properties that apply to all functions of this type. In this paper we'll be looking at large argument asymptotic formulas for these hypergeometric functions. As examples, we'll look at various Bessel functions, which have hypergeometric representations. Finally, we will take a look at an application from optics.

2 GAMMA FUNCTIONS

The gamma function

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!n^x}{x(x+1)(x+2)\cdots(x+n)}$$

whose properties were developed by Euler is a highly useful special function. As we will see in this paper it is used quite extensively with Bessel functions and hypergeometric functions. More useful to us in this paper is the integral representation

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \quad x > 0 \tag{2.1}$$

discovered by Euler. Some of the important properties of the gamma function are the well known recurrence relation

$$\Gamma(x) = \frac{\Gamma(x+1)}{x} \tag{2.2}$$

and Legendre's duplication formula

$$2^{2x-1}\Gamma(x)\Gamma\left(x+\tfrac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2x). \tag{2.3}$$

A useful special value for $\Gamma(x)$ is

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}\ . \tag{2.4}$$

Finally, we'll have need of the identity

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}. \tag{2.5}$$

3 HYPERGEOMETRIC FUNCTIONS

Much of what is done in this paper deals with hypergeometric functions. Before introducing the hypergeometric function we introduce the Pochhammer symbol

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{3.1}$$

If we make the substitution $n \to k-n$ in (3.1), we are left with

$$(a)_{k-n} = \frac{\Gamma(a+k-n)}{\Gamma(a)}. \tag{3.2}$$

We then multiply the top and bottom of the right hand side of (3.2) by $\Gamma(a+k)$ and perform a bit of simple algebra.

$$(a)_{k-n} = \frac{\Gamma(a+k-n)}{\Gamma(a)} \cdot \frac{\Gamma(a+k)}{\Gamma(a+k)}$$

$$= \frac{\Gamma(a+k)}{\Gamma(a)} \cdot \frac{\Gamma(a+k-n)}{\Gamma(a+k)}$$

$$= \frac{(a)_k}{\dfrac{\Gamma(a+k)}{\Gamma(a+k-n)}}$$

$$= \frac{(a)_k}{(a+k-1)(a+k-2)\cdots(a+k-n)}$$

$$(a)_{k-n} = \frac{(-1)^n (a)_k}{(1-a-k)_n} \tag{3.3}$$

In a future section of this paper we will need a representation for $(a)_{-n}$. If we set $k=0$, we see that

$$(a)_{-n} = \frac{(-1)^{-n}}{(1-a)_n}. \tag{3.4}$$

The general hypergeometric function is given by

$$ {}_pF_q\left(a_1 \cdots a_p; b_1 \cdots b_q; x\right) = \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{p} (a_k)_n}{\prod_{k=1}^{q} (b_k)_n} \frac{x^n}{n!}. \tag{3.5}$$

Some of what we deal with in this paper will concern (3.5), but for the most part we will deal with specific cases of the general hypergeometric function. For instance if we let $p = 2$ and $q = 1$, we have the function

$$F(a,b;c;x) = {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \tag{3.6}$$

where semicolons separate the numerator parameters from those of the denominator. This function is called the Hypergeometric function. It can be easily shown, by using the ratio test, that this series converges for $|x| < 1$. The integral representation of (3.6) is

$${}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-xt)^{-a}\, dt. \tag{3.7}$$

Another special case is that of the confluent hypergeometric function

$$M(a;c;x) = {}_1F_1(a;c;x) \tag{3.8}$$

with integral representation

$${}_1F_1(a;c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{xt} t^{a-1} (1-t)^{c-a-1}\, dt. \tag{3.9}$$

An interesting result is obtained if we make the substitution $t = 1 - s$.

$$ {}_1F_1(a;c;x) = -\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_1^0 e^{x(1-s)} (1-s)^{a-1} (s)^{c-a-1}\, ds $$

$$ = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} e^x \int_0^1 e^{-xs} (s)^{c-a-1} (1-s)^{a-1}\, ds $$

The integral in the last line is, of course, ${}_1F_1(a;c;x)$. Therefore, we have established the identity

$$ {}_1F_1(a;c;x) = e^x\, {}_1F_1(c-a;c;-x). \qquad (3.10) $$

We will also be working with the confluent hypergeometric function of the second kind, which written in terms of the confluent hypergeometric function is

$$ U(a;c;x) = \frac{\Gamma(1-c)}{\Gamma(1+a-c)}\, {}_1F_1(a;c;x) + \frac{\Gamma(c-1)}{\Gamma(a)} x^{1-c}\, {}_1F_1(1+a-c;2-c;x) \qquad (3.11) $$

with associated integral representation,

$$ U(a;c;x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-xt} t^{a-1} (1+t)^{c-a-1}\, dt. \qquad (3.12) $$

4 THE MELLIN TRANSFORM

The famous Fourier Integral Theorem, named after the French Physicist Jean Baptiste Joseph Fourier, is our basis for developing the Mellin-Barnes integral.

Theorem 1: (Fourier Integral Theorem) If f and f' are piecewise continuous functions on every finite interval, and if

$$\int_{-\infty}^{\infty} \left| f(x) \right| dx < \infty ,$$

then

$$f(x) = \tfrac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t) \cos\left[s(t-x) \right] dtds$$

at points, x, where f is continuous. If x is a point of discontinuity of f, the above integral converges to the average value $\tfrac{1}{2}\left[f(x^{+}) + f(x^{-}) \right]$ of the right-hand and left-hand limits.

Through the use of Euler's formula, $\cos x = \tfrac{1}{2}\left(e^{ix} + e^{-ix} \right)$, we can derive the exponential form of this theorem

$$f(x) = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} \int_{-\infty}^{\infty} e^{ist} f(t) dtds . \tag{4.1}$$

For convenience let's change (4.1) to

$$g(p) = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} e^{-imp} \int_{-\infty}^{\infty} e^{imt} g(t) dtdm .$$

where we have made the simple substitutions, $x = p$, $s = m$, and $f = g$. In order to develop the Mellin transform we make the substitutions $x = e^{t}$ $y = e^{p}$, and $s = c + im$. Upon making the substitution and changing the order of integration we obtain

$$g(\ln y) y^{-c} = \tfrac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^{-s} \int_{0}^{\infty} g(\ln x) x^{-c} x^{s-1} dxds .$$

If we define the function $f(x) = g(\ln x)x^{-c}$, we obtain the Mellin transform formulas

$$F(s) = \int_0^\infty x^{s-1} f(x)\,dx \tag{4.2}$$

$$f(x) = \tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s} F(s)\,ds \tag{4.3}$$

where (4.2) is called the Mellin transform, $M\{f(x);s\}$, and (4.3) is the inverse Mellin transform, $M^{-1}\{F(s);x\}$.

It can be shown quite trivially that $M\{e^{-x};s\} = \Gamma(s)$ by looking at the definition. It is a bit more challenging to show that $M^{-1}\{\Gamma(s);x\} = e^{-x}$. Let's start with the inverse Mellin transform,

$$M^{-1}\{\Gamma(s);x\} = \tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\Gamma(s)\,ds\,. \tag{4.4}$$

Observe that all of the poles occur at values of the gamma function where $s = -n \ (n = 0,1,2,...)$. This implies that in the right half plane the gamma function is analytic. We can evaluate the integral by considering the contour in Figure 1.

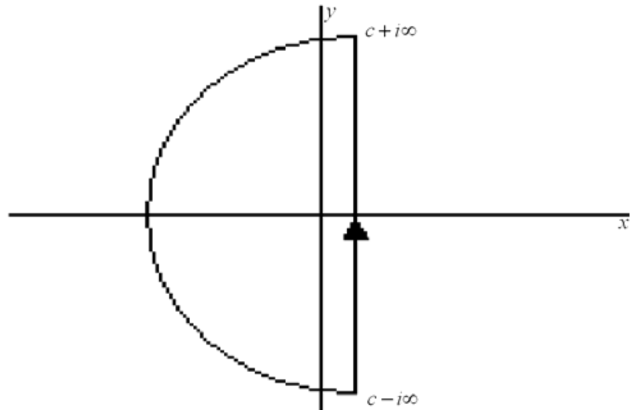

Figure 1: Inverse Mellin Transform Contour

Based on residue theory, we can conclude that

$$M^{-1}\{\Gamma(s);x\}=\sum_{n=0}^{\infty}\operatorname{Res}\{x^{-s}\Gamma(s);-n\}. \tag{4.5}$$

In order to show that $M^{-1}\{\Gamma(s);x\}=e^{-x}$ we recall equation (2.5) and rewrite it as

$$\Gamma(s)=\frac{\pi}{\Gamma(1-s)\sin(\pi s)}. \tag{4.6}$$

By substituting (4.6) into (4.5) we obtain

$$M^{-1}\{\Gamma(s);x\}=\sum_{n=0}^{\infty}\operatorname{Res}\left\{x^{-s}\frac{\pi}{\Gamma(1-s)\sin \pi s};-n\right\}. \tag{4.7}$$

Because the poles in (4.7) are simple poles (ie., the zeros of $\sin \pi s$), the following theorem will be useful.

Theorem 2 - Let $f(z)=\dfrac{P(z)}{Q(z)}$, where the functions $P(z)$ and $Q(z)$ are both analytic at z_0, and Q has a simple zero at z_0, and $P(z_0)\neq 0$, then

$$\operatorname{Res}(f;z_0)=\frac{P(z_0)}{Q'(z_0)}. \tag{4.8}$$

Applying Theorem 2 to (4.7), we find the following result.

$$\operatorname{Res}\left\{x^{-s}\Gamma(s);-n\right\}=\left.\frac{x^{-s}}{\Gamma(1-s)\cos(\pi s)}\right|_{s=-n}=\sum_{n=0}^{\infty}\frac{x^{n}}{\Gamma(1+n)\cos(-n\pi)}$$

$$=\sum_{n=0}^{\infty}\frac{x^{n}}{n!(-1)^{n}}=\sum_{n=0}^{\infty}\frac{(-x)^{n}}{n!}=e^{-x}$$

Thus we have shown that

$$M^{-1}\left\{\Gamma(s);x\right\}=\sum_{n=0}^{\infty}\operatorname{Res}\left\{x^{-s}\Gamma(s);-n\right\}=\sum_{n=0}^{\infty}\frac{(-1)^{n}x^{n}}{n!}=e^{-x} \tag{4.9}$$

and consequently,

$$e^{-x}=\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}x^{-s}\Gamma(s)\,ds\,. \tag{4.10}$$

As stated previously (see Figure 1), the poles of $\Gamma(s)$ occur in the left hand plane. Because the number of poles is infinite, this will produce an ascending infinite series as seen in (4.9). If our function has poles in the right hand plane, this will produce a descending series.

5 THE MELLIN-BARNES INTEGRAL

Recall the Mellin-Barnes integral from the previous section

$$e^{-z} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} z^s \Gamma(-s)\, ds\,. \tag{5.1}$$

Notice that we have made the substitution $s \to -s$ and replaced the real variable x with the complex variable z. This allows us to use a contour in the right-hand plane for evaluating (5.1) using the poles $s = n, \quad n = 0, 1, 2, 3, \ldots$. Because we made the change of variable $s \to -s$, we will now produce an ascending series by enclosing the poles in the right hand plane. A descending series will be found by enclosing the poles in the left hand plane. We wish to generalize (5.1) to other functions defined by

$$f\left(a_1, \ldots, a_p; c_1, \ldots, c_q; z\right) = \frac{1}{2\pi i}\int_L \chi(s)\, z^s ds \tag{5.2}$$

where, for non-negative integers p and q,

$$\chi(s) = \frac{\prod_{j=1}^{p}\Gamma\left(a_j + s\right)}{\prod_{j=1}^{q}\Gamma\left(c_j + s\right)}\Gamma(-s)\,. \tag{5.3}$$

For this new function, semicolons are used to separate the numerator parameters, $a_1, \ldots, a_p$, from those in the denominator, $c_1, \ldots, c_q$. The contour L is the contour starting at $i\infty$ and running to $-i\infty$ curving around the poles of $\Gamma\left(a_j + s\right)$ so that they are to the left of the contour, while keeping the poles of $\Gamma(s)$ on the opposite side (see Figure 2).

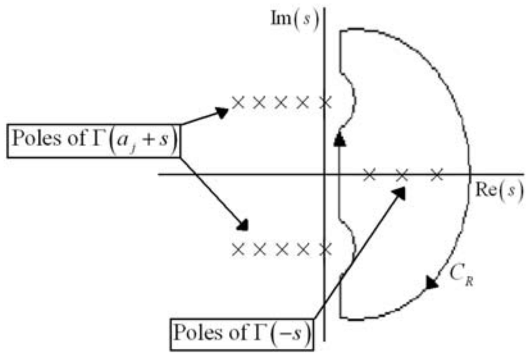

Figure 2: Sample Barnes Integral Contour enclosing the poles in the right hand plane

We make the assumption that a_j is a non-integer and $a_j - c_k \neq 0,1,2,...$ for all j,k . Recall that as $R \to \infty$, C_R will enclose all of the poles of $\Gamma(-s)$, and $\int_{C_R} \chi(s) z^s ds \to 0$. The integral converges absolutely if

$$\left|\arg(z)\right| < (1 + p - q)\tfrac{\pi}{2}. \tag{5.4}$$

This last condition will be of great importance as we near the end of the section.

6 DIRECT RELATION BETWEEN $f\left(a_1,...,a_p;c_1,...,c_q;z\right)$ AND ${}_pF_q\left(a_1,...,a_p;c_1,...,c_q;-z\right)$

There is a direct relation between the functions $f\left(a_1,...,a_p;c_1,...,c_q;z\right)$ and the generalized hypergeometric functions denoted by ${}_pF_q\left(a_1,...,a_p;c_1,...,c_q;-z\right)$. For instance, if we set $p=q=0$, then we have the following

$$f\left(-;-;z\right)=e^{-z}={}_0F_0\left(-;-;-z\right).$$

Let's begin our discussion by looking at the case $f\left(a;-;z\right)$ where $p=1$ and $q=0$. Thus, we have the equation

$$f\left(a;-;z\right)=\tfrac{1}{2\pi i}\int_L\Gamma\left(a+s\right)\Gamma\left(-s\right)z^s ds,\quad a\neq 0,-1,-2,\dots. \tag{6.1}$$

For the purposes of this paper, we will assume that a is a real variable, although it can be complex as well. Using residue calculus and the contour shown in figure 2 to evaluate the integral in (6.1), we have the series

$$f\left(a;-;z\right)=-\sum_{n=0}^{\infty}\operatorname{Res}\left\{\Gamma\left(a+s\right)\Gamma\left(-s\right)z^s;n\right\}, \tag{6.2}$$

where the negative sign appears due to clockwise orientation of the contour. Using (2.5) as a substitution, we get

$$f\left(a;-;z\right)=\sum_{n=0}^{\infty}\operatorname{Res}\left\{\frac{\Gamma\left(a+s\right)z^{-s}\pi}{\Gamma\left(1+s\right)\sin\left(\pi s\right)};n\right\},$$

where we have also used the identity $\sin\left(-x\right)=-\sin x$. According to the theory of residues,

$$f\left(a;-;z\right)=\sum_{n=0}^{\infty}\frac{\Gamma\left(a+n\right)z^n}{\Gamma\left(n+1\right)\cos\left(n\pi\right)}=\sum_{n=0}^{\infty}\frac{\left(-1\right)^n\Gamma\left(a+n\right)z^n}{n!}.$$

Next we make use of the substitution $\Gamma(a+n)=(a)_n\,\Gamma(a)$ from (2.2).

$$f(a;-;z)=\Gamma(a)\sum_{n=0}^{\infty}(a)_n\frac{(-z)^n}{n!}=\Gamma(a)\,{}_1F_0(a;-;-z)$$

Thus we have shown that there is a direct relationship between $f(a;-;z)$ and ${}_1F_0(a;-;-z)$. By applying the ratio test, we can easily show that this series converges for $|z|<1$.

Let's continue this process for another example. This time we consider the function

$$f(a;c;z)=\tfrac{1}{2\pi i}\int_L\frac{\Gamma(a+s)\Gamma(-s)z^s}{\Gamma(c+s)}ds\,. \tag{6.3}$$

Notice that the contour does not change for this integral, as there are no poles for the additional $\Gamma(c+s)$ term. That is, $\dfrac{1}{\Gamma(s)}$ is an analytic function. Thus we can continue the process as done previously

$$f(a;c;z)=-\sum_{n=0}^{\infty}\mathrm{Res}\left\{\frac{\Gamma(a+s)\Gamma(-s)z^s}{\Gamma(c+s)};n\right\}$$

$$=\sum_{n=0}^{\infty}\mathrm{Res}\left\{\frac{\Gamma(a+s)\pi z^s}{\Gamma(c+s)\Gamma(1+s)\sin(\pi s)};n\right\}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(a+n)z^n}{\Gamma(c+n)\Gamma(1+n)}=\frac{\Gamma(a)}{\Gamma(c)}\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}\frac{(-z)^n}{n!}\,,$$

which yields

$$f(a;c;z)=\frac{\Gamma(a)}{\Gamma(c)}\,{}_1F_1(a;c;-z)\,. \tag{6.4}$$

It would then appear that there is a direct relationship between $f\left(a_1,\ldots,a_p;c_1,\ldots,c_q;z\right)$ and ${}_pF_q\left(a_1,\ldots,a_p;c_1,\ldots,c_q;-z\right)$. Indeed we can continue this process one more time

$$f\left(a_1,\ldots,a_p;c_1,\ldots,c_q;z\right)=\tfrac{1}{2\pi i}\int_L \frac{\Gamma\left(a_1+s\right)\cdots\Gamma\left(a_p+s\right)\Gamma\left(-s\right)z^s}{\Gamma\left(c_1+s\right)\cdots\Gamma\left(c_q+s\right)}ds$$

$$=-\sum_{n=0}^{\infty}\operatorname{Res}\left\{\frac{\Gamma\left(a_1+s\right)\cdots\Gamma\left(a_p+s\right)\Gamma\left(-s\right)z^s}{\Gamma\left(c_1+s\right)\cdots\Gamma\left(c_q+s\right)};n\right\}$$

$$=-\sum_{n=0}^{\infty}\operatorname{Res}\left\{\frac{\Gamma\left(a_1+s\right)\cdots\Gamma\left(a_p+s\right)\pi z^s}{\Gamma\left(c_1+s\right)\cdots\Gamma\left(c_q+s\right)\Gamma\left(1+s\right)\sin\pi s};n\right\}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma\left(a_1+n\right)\cdots\Gamma\left(a_p+n\right)z^n}{\Gamma\left(c_1+n\right)\cdots\Gamma\left(c_q+n\right)\Gamma\left(1+n\right)}$$

$$=\frac{\Gamma\left(a_1\right)\cdots\Gamma\left(a_p\right)}{\Gamma\left(c_1\right)\cdots\Gamma\left(c_q\right)}\sum_{n=0}^{\infty}\frac{\left(a_1\right)_n\cdots\left(a_p\right)_n}{\left(c_1\right)_n\cdots\left(c_q\right)_n}\frac{(-z)^n}{n!},$$

or,

$$f\left(a_1,\ldots,a_p;c_1,\ldots,c_q;z\right)=\frac{\Gamma\left(a_1\right)\cdots\Gamma\left(a_p\right)}{\Gamma\left(c_1\right)\cdots\Gamma\left(c_q\right)}\,{}_pF_q\left(a_1,\ldots,a_p;c_1,\ldots,c_q;-z\right). \qquad (6.5)$$

Therefore we have the following useful theorem.

Theorem 3 If $\operatorname{Re}(z)>0$ and if no a_m and c_j is zero or a negative integer,

then ${}_pF_q\left(a_1,\ldots,a_p;c_1,\ldots c_q;-z\right)=\dfrac{\prod_{m=0}^{p}\Gamma\left(c_n\right)}{\prod_{n=0}^{q}\Gamma\left(a_m\right)}\dfrac{1}{2\pi i}\displaystyle\int_L\frac{\Gamma\left(-s\right)z^{-s}\prod_{m=0}^{p}\Gamma\left(a_m+s\right)}{\prod_{n=0}^{q}\Gamma\left(c_n+s\right)},$

where L is the Barnes path as shown in figure 2.

Every ${}_pF_q$ function is defined by a series, which is at least valid, with $p \le q+1$, for all $|z|<1$. We can use this series to find asymptotic formulas for functions defined by hypergeometric series for small values of z. For this paper we seek a series, in reciprocal powers of z, that will be valid for $|z|>1$. We can find such a series by enclosing the poles in the left hand plane while excluding those in the right hand plane (see figure 3). As stated in chapter 5 (page 10), enclosing the poles in the left hand plane will produce the descending series that we seek. We can then use this series to find asymptotic formulas for large values of z.

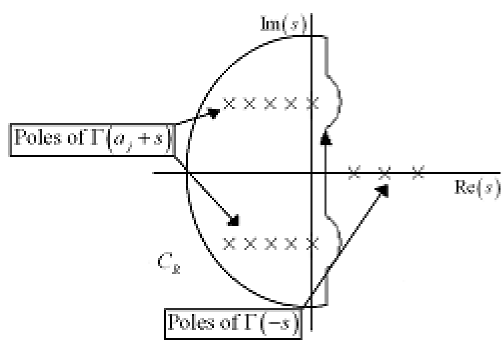

Figure 3: Sample Barnes Integral Contour enclosing the poles in the left hand plane

Let's re-examine the ${}_1F_0(a;-;-z)$ function. By Theorem 3,

$${}_1F_0(a;-;-z) = \frac{1}{2\pi i\Gamma(a)}\int_L \Gamma(-s)\Gamma(s+a)z^s ds.$$

This time we make a change of variable $s \to -(s+a)$. The Barnes integral will be very similar to the one used in Figure 2. Because of the change of variable, though, enclosing the poles in the right hand side will produce a descending series. The change of variable leads to

$$= \frac{z^{-a}}{2\pi i \Gamma(a)} \int_L \Gamma(s+a)\Gamma(-s) z^{-s} ds .$$

Observe that the integral has remained virtually the same. The only difference being that we have picked up an extra z^{-a} term outside the integral and that z^{-s} has replaced z^s. This will be a powerful change, as we shall see. We recognize the right hand side as being $z^{-a}\,{}_1F_0\left(a;-;-\frac{1}{z}\right)$. Thus we have discovered the identity

$${}_1F_0(a;-;-z) = z^{-a}\,{}_1F_0\left(a;-;-\tfrac{1}{z}\right).$$

Notice that the right hand side is valid for $|z| > 1$. Thus if we have a function that is represented by the hypergeometric function ${}_1F_0(a;-;-z)$, we can then find an asymptotic formula for this function for large values of z using the above identity.

Let's now look at another special case. We start with the hypergeometric function

$${}_1F_1(a;c;-z) = \frac{\Gamma(c)}{\Gamma(a)} \frac{1}{2\pi i} \int_L \frac{\Gamma(s+a)\Gamma(-s) z^s}{\Gamma(c+s)} ds . \tag{6.6}$$

Again we make the change of variable $s \to -(s+a)$, which produces

$${}_1F_1(a;c;-z) = \frac{\Gamma(c)}{\Gamma(a)} \frac{z^{-a}}{2\pi i} \int_L \frac{\Gamma(s+a)\Gamma(-s) z^{-s}}{\Gamma(c-a-s)} ds$$

By choosing the Barnes Integral in Figure 2, we enclose all of the poles of $\Gamma(-s)$

$$ {}_1F_1(a;c;-z) = -\frac{\Gamma(c)z^{-a}}{\Gamma(a)} \sum_{n=0}^{\infty} \operatorname{Res}\left\{ \frac{\Gamma(s+a)\Gamma(-s)z^{-s}}{\Gamma(c-a-s)}; n \right\}. \tag{6.7} $$

Again, we use (2.5), but we apply it to the $\Gamma(-s)$ term.

$$ {}_1F_1(a;c;-z) = \frac{\Gamma(c)}{\Gamma(a)} z^{-a} \sum_{n=0}^{\infty} \operatorname{Res}\left\{ \frac{\Gamma(a+s)\pi z^{-s}}{\Gamma(c-a-s)\Gamma(1+s)\sin\pi s}; n \right\} $$

$$ = \frac{\Gamma(c)}{\Gamma(a)} z^{-a} \sum_{n=0}^{\infty} \frac{\Gamma(a+n) z^{-n}}{\Gamma(c-a-n)\Gamma(1+n)\cos(\pi n)} $$

$$ = \frac{\Gamma(c)}{\Gamma(a)} z^{-a} \sum_{n=0}^{\infty} \frac{(a)_n \Gamma(a)(-1)^n}{\Gamma(c-a-n)} \frac{z^{-n}}{n!} \tag{6.8} $$

At this point it is wise to review our goal in this process. We are seeking to rewrite the hypergeometric function ${}_1F_1(a;c;-z)$ as another hypergeometric function that will converge for large values of z. We have almost achieved our goal except for the $\Gamma(c-a-n)$ term in (6.8). By using (2.2) and (3.4) we can rewrite $\Gamma(c-a-n)$ as

$$ \Gamma(c-a-n) = (c-a)_{-n}\Gamma(c-a) \tag{6.9} $$

$$ \Gamma(c-a-n) = \frac{(-1)^n \Gamma(c-a)}{(1-c+a)_n}. \tag{6.10} $$

After substituting (6.10) into (6.8), we obtain

$$ {}_1F_1(a;c;-z) = \frac{\Gamma(c)}{\Gamma(c-a)} z^{-a} \sum_{n=0}^{\infty} (a)_n (1-c+a)_n \frac{z^{-n}}{n!} $$

$$= \frac{\Gamma(c)}{\Gamma(c-a)} z^{-a} {}_2F_0\left(a, 1-c+a; -; \tfrac{1}{z}\right). \qquad (6.11)$$

Observe that the series on the right approaches zero for $a > 0$ and large values of z. This is the exact result that we wanted. We can then state that

$${}_1F_1(a; c; -z) \sim \frac{\Gamma(c)}{\Gamma(c-a)} z^{-a}, \quad |z| \to \infty. \qquad (6.12)$$

It will be useful for us to find a general result for the hypergeometric function

$${}_1F_k\left(a; c_1, \ldots, c_k; -z\right) = \frac{\prod_{n=1}^{k} \Gamma(c_n)}{\Gamma(a)} \frac{1}{2\pi i} \int_L \frac{\Gamma(a+s)\Gamma(-s)}{\prod_{n=1}^{k} \Gamma(c_n + s)} z^s ds$$

Appendix A offers a proof of the following result.

$${}_1F_k\left(a; c_1, \ldots, c_k; -z\right) = \frac{\prod_{j=1}^{k} \Gamma(c_j)}{\prod_{j=1}^{k} \Gamma(c_j - a)} z^{-a} {}_{k+1}F_0\left(a, 1+a-c_1, \ldots, 1+a-c_k; -; \frac{(-1)^{k+1}}{z}\right) \qquad (6.13)$$

7 ASYMPTOTICS FOR BESSEL FUNCTIONS

It is well known that for large arguments of x,

$$J_p(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left[x - \frac{(p+\frac{1}{2})\pi}{2}\right], \quad x \to \infty \tag{7.1}$$

$$K_p(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}, \quad x \to \infty \tag{7.2}$$

$$I_p(x) \sim \frac{e^x}{\sqrt{2\pi x}}, \quad x \to \infty. \tag{7.3}$$

In Appendix B, we verify these statements. We seek to achieve these same results using the Mellin-Barnes integral. In order to accomplish this, we first need to represent each of the above Bessel functions in terms of a hypergeometric function. Once this has been accomplished, we'll apply the asymptotic formulas that we've already developed. Although the results may also apply for complex arguments, we will use only the real variable x in the discussion below.

7.1 The Modified Bessel Function of the First Kind

Let's start with the integral representation of $I_p(x)$, [2]

$$I_p(x) = \frac{\left(\frac{x}{2}\right)^p}{\sqrt{\pi}\,\Gamma\left(p+\frac{1}{2}\right)} \int_{-1}^{1} \left(1-t^2\right)^{p-\frac{1}{2}} e^{-xt} dt .$$

We make the substitution $t \to 1-2t$, to produce

$$I_p(x) = \frac{\left(\frac{x}{2}\right)^p}{\sqrt{\pi}\,\Gamma\left(p+\frac{1}{2}\right)} e^{-x} \int_0^1 2\left(4t-t^2\right)^{p-\frac{1}{2}} e^{2xt} dt$$

$$= \frac{\left(\frac{x}{2}\right)^p 2^{2p-1}}{\sqrt{\pi}\,\Gamma\left(p+\frac{1}{2}\right)} e^{-x} \int_0^1 (1-t)^{p-\frac{1}{2}} t^{p-\frac{1}{2}} e^{2xt} dt \,. \qquad (7.4)$$

Our goal is to represent the given Bessel function in terms of a hypergeometric function. This can by done by using (3.9), the integral definition of ${}_1F_1(a;c;x)$

$${}_1F_1(a;c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{xt} t^{a-1} (1-t)^{c-a-1} \, dt \,,$$

where, for equation (7.4), we recognize that $a = p+\frac{1}{2}$ and $c = 2p+1$. Applying the integral representation we obtain

$$I_p(x) = \frac{\left(\frac{x}{2}\right)^p 2^{2p-1} \Gamma\left(p+\frac{1}{2}\right)\Gamma\left(p+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(p+\frac{1}{2}\right)\Gamma(2p+1)} \, {}_1F_1\left(p+\tfrac{1}{2}; 2p+1; 2x\right)$$

$$I_p(x) = \frac{\left(\frac{x}{2}\right)^p}{\Gamma(p+1)} e^{-x} \, {}_1F_1\left(p+\tfrac{1}{2}; 2p+1; 2x\right). \qquad (7.5)$$

In the last step we used Legendre's duplication formula to obtain the result. At this point we are halfway to our goal of finding an asymptotic formula. Next we turn to (6.12), our asymptotic representation of ${}_1F_1(a;c;x)$.

$$I_p(x) \sim \frac{\left(\frac{x}{2}\right)^p}{\Gamma(p+1)} \frac{\Gamma(2p+1)}{\Gamma\left(p+\frac{1}{2}\right)} 2x^{-p-\frac{1}{2}} e^{-x}, \quad x \to \infty$$

$$\sim \frac{\Gamma(2p+1)}{\sqrt{2x}\, 2^{2p} \Gamma(p+1)\Gamma\left(p+\frac{1}{2}\right)} e^{-x}, \quad x \to \infty$$

or,

$$I_p(x) \sim \frac{1}{\sqrt{2\pi x}} e^{-x}, \quad x \to \infty. \tag{7.6}$$

7.2 The Modified Bessel Function of the Second Kind

The Modified Bessel function of the second kind, $K_p(x)$, presents a slightly different challenge. We start with the integral representation, [2]

$$K_p(x) = \frac{\sqrt{\pi}\left(\frac{x}{2}\right)^p}{\Gamma\left(p+\frac{1}{2}\right)} \int_1^\infty \left(t^2-1\right)^{p-\frac{1}{2}} e^{-xt} dt \tag{7.7}$$

and make the change of variable $t \to 2t+1$.

$$K_p(x) = \frac{2\sqrt{\pi}\left(\frac{x}{2}\right)^p}{\Gamma\left(p+\frac{1}{2}\right)} e^{-x} \int_0^\infty \left(4t^2+4t\right)^{p-\frac{1}{2}} e^{-2xt} dt$$

$$= \frac{\sqrt{\pi}\left(2x\right)^p}{\Gamma\left(p+\frac{1}{2}\right)} e^{-x} \int_0^\infty e^{-2xt} s^{p-\frac{1}{2}} \left(t+1\right)^{p-\frac{1}{2}} dt \tag{7.8}$$

The integral in (7.8) is an integral representation of (3.12), the Confluent Hypergeometric Function of the Second Kind

$$U(a;c;x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-xt} t^{a-1} \left(1+t\right)^{c-a-1} dt$$

with $a = p+\frac{1}{2}$ and $c = 2p+1$. Equation (7.8) then simplifies to

$$K_p(x) = \left(2x\right)^p \sqrt{\pi} e^{-x} U\left(p+\tfrac{1}{2}; 2p+1; 2x\right). \tag{7.9}$$

Thus far we have developed an asymptotic formula for ${}_1F_1(a;c;x)$, but now we are presented with the challenge for $U(a;c;x)$. The process is very similar although we now appeal to the Barnes integral representation for $U(a;c;x)$ [1]

$$U(a;c;z)=\frac{1}{\Gamma(a)\Gamma(1+a-c)}z^{-a}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(-s)\Gamma(a+s)\Gamma(1+a-c+s)z^{-s}ds \quad (7.10)$$

$$=-\frac{1}{\Gamma(a)\Gamma(1+a-c)}z^{-a}\sum_{n=0}^{\infty}\mathrm{Res}\left\{\Gamma(-s)\Gamma(a+s)\Gamma(1+a-c+s)z^{-s};n\right\}$$

$$=\frac{1}{\Gamma(a)\Gamma(1+a-c)}z^{-a}\sum_{n=0}^{\infty}\left\{\frac{\Gamma(a+s)\Gamma(1+a-c+s)}{\Gamma(1+s)\sin\pi s}\pi z^{-s};n\right\},$$

which simplifies to

$$U(a;c;z)=z^{-a}\sum_{n=0}^{\infty}(a)_n(1+a-c)_n\frac{(-z)^{-n}}{n!}.$$

The simple asymptotic relation is

$$U(a;c;z)\sim z^{-a},\ \ z\to\infty. \quad (7.11)$$

Applying (7.11) to (7.9) we find that

$$K_p(x)\sim(2x)^p\sqrt{\pi}e^{-x}(2x)^{-p-\frac{1}{2}},\ \ x\to\infty$$

or more simply,

$$K_p(x)\sim\sqrt{\frac{\pi}{2x}}e^{-x},\ \ x\to\infty$$

which is the result that we wanted.

7.3 The Bessel Function

We start with the integral representation of $J_p(x)$, [2]

$$J_p(x) = \frac{\left(\frac{x}{2}\right)^p}{\sqrt{\pi}\,\Gamma\left(p+\frac{1}{2}\right)} \int_{-1}^{1} \left(1-t^2\right)^{p-\frac{1}{2}} e^{ixt} dt \,. \tag{7.12}$$

We make the change of variable $t \to 2t-1$.

$$J_p(x) = \frac{\left(\frac{x}{2}\right)^p}{\sqrt{\pi}\,\Gamma\left(p+\frac{1}{2}\right)} \int_0^1 2\left(1-\left(2t-1\right)^2\right)^{p-\frac{1}{2}} e^{ix(2t-1)} dt$$

$$= \frac{\left(\frac{x}{2}\right)^p e^{-ix}}{\sqrt{\pi}\,\Gamma\left(p+\frac{1}{2}\right)} \int_0^1 2\left(4t-4t^2\right)^{p-\frac{1}{2}} e^{2ixt} dt$$

$$= \frac{\left(\frac{x}{2}\right)^p e^{-ix}}{\sqrt{\pi}\,\Gamma\left(p+\frac{1}{2}\right)} \int_0^1 2\left(4t-4t^2\right)^{p-\frac{1}{2}} e^{2ixt} dt$$

$$= \frac{\left(\frac{x}{2}\right)^p e^{-ix} 2^{2p}}{\sqrt{\pi}\,\Gamma\left(p+\frac{1}{2}\right)} \int_0^1 t^{p-\frac{1}{2}} \left(1-t\right)^{p-\frac{1}{2}} e^{2ixt} dt \tag{7.13}$$

In order to proceed we appeal to equation (3.9),

$${}_1F_1\left(a;c;x\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{xt} t^{a-1} \left(1-t\right)^{c-a-1} dt \,.$$

By comparing (3.9) with (7.13) we find that $a = p+\frac{1}{2}$ and $c = 2p+1$. This means that

$$J_p(x) = \frac{\left(\frac{x}{2}\right)^p e^{-ix} 2^{2p}}{\sqrt{\pi}\,\Gamma\left(p+\frac{1}{2}\right)} \frac{\Gamma\left(p+\frac{1}{2}\right)\Gamma\left(p+\frac{1}{2}\right)}{\Gamma\left(2p+1\right)} {}_1F_1\left(p+\tfrac{1}{2};2p+1;2ix\right)$$

$$= \frac{\left(\frac{x}{2}\right)^p e^{-ix} 2^{2p}\,\Gamma\left(p+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(2p+1\right)} {}_1F_1\left(p+\tfrac{1}{2};2p+1;2ix\right).$$

Finally we use (2.3) with $x = p+\frac{1}{2}$ to obtain

$$J_p(x) = \frac{\left(\frac{x}{2}\right)^p}{\Gamma(p+1)} e^{-ix} {}_1F_1\left(p+\tfrac{1}{2}; 2p+1; 2ix\right). \tag{7.14}$$

Again, we are at the halfway point. Notice that for the first time, we have pure imaginary arguments in the confluent hypergeometric function. This argument, with $p = q = 1$, does not meet condition (5.4), which says that the Barnes integral will converge absolutely if $\left|\arg(z)\right| < (1+p-q)\frac{\pi}{2}$. We need to devise another scheme for dealing with these arguments. We start with (3.12)

$$U(a;c;z) = \frac{\Gamma(1-c)}{\Gamma(1+a-c)} {}_1F_1(a;c;z) + \frac{\Gamma(c-1)}{\Gamma(a)} z^{1-c} {}_1F_1(1+a-c; 2-c; z).$$

By replacing a with $c-a$, and z with $-z$, equation (3.12) becomes

$$U(c-a;c;-z) = \frac{\Gamma(1-c)}{\Gamma(1-a)} {}_1F_1(c-a;c;-z) + \frac{\Gamma(c-1)}{\Gamma(c-a)} (-z)^{1-c} {}_1F_1(1-a; 2-c; -z)$$

$$= \frac{\Gamma(1-c)}{\Gamma(1-a)} {}_1F_1(c-a;c;-z) - \frac{\Gamma(c-1)}{\Gamma(c-a)} (z)^{1-c} e^{\pm c\pi i} {}_1F_1(1-a; 2-c; -z)$$

$$= e^{-z} \frac{\Gamma(1-c)}{\Gamma(1-a)} {}_1F_1(a;c;z) - \frac{\Gamma(c-1)}{\Gamma(c-a)} (z)^{1-c} e^{\pm c\pi i} e^{-z} {}_1F_1(1+a-c; 2-c; z),$$

where in the last step we used Kummer's transformation. Finally by multiplying by e^z we obtain

$$e^z U(c-a;c;-z) = \frac{\Gamma(1-c)}{\Gamma(1-a)} {}_1F_1(a;c;z) - \frac{\Gamma(c-1)}{\Gamma(c-a)} (z)^{1-c} e^{\pm c\pi i} {}_1F_1(1+a-c; 2-c; z). \tag{7.15}$$

Notice that in equations (7.15) and (3.12) we have the same ${}_1F_1$ functions. By eliminating the function ${}_1F_1(1+a-c; 2-c; z)$, we are left with

$$\frac{e^{z}}{\Gamma(a)}U(c-a;c;-z)+\frac{e^{\pm c\pi i}}{\Gamma(c-a)}U(a;c;z)=\frac{\Gamma(1-c)}{\Gamma(a)\Gamma(1-a)}\,{}_1F_1(a;c;z)+\frac{e^{\pm c\pi i}\Gamma(1-c)}{\Gamma(c-a)\Gamma(1+a-c)}\,{}_1F_1(a;c;z),$$

where we multiplied (3.12) by $\frac{e^{\pm c\pi i}}{\Gamma(c-a)}$ and (7.15) by $\frac{1}{\Gamma(a)}$ to eliminate ${}_1F_1(1+a-c;2-c;z)$.

Continuing with the simplification process we find

$$\left[\frac{\Gamma(1-c)}{\Gamma(a)\Gamma(1-a)}+\frac{e^{\pm c\pi i}\Gamma(1-c)}{\Gamma(c-a)\Gamma(1+a-c)}\right]{}_1F_1(a;c;z)=\frac{e^{z}}{\Gamma(a)}U(c-a;c;-z)+\frac{e^{\pm c\pi i}}{\Gamma(c-a)}U(a;c;z)\,. \tag{7.16}$$

We need to simplify the coefficient of ${}_1F_1(a;c;z)$. To do this we recall (2.5), and apply this to the coefficient of ${}_1F_1(a;c;z)$ to produce

$$\left[\frac{\Gamma(1-c)}{\Gamma(a)\Gamma(1-a)}+\frac{e^{\pm c\pi i}\Gamma(1-c)}{\Gamma(c-a)\Gamma(1+a-c)}\right]=\frac{1}{\Gamma(c)}\left(\frac{\sin\pi a+e^{\pm c\pi i}\sin\pi(c-a)}{\sin\pi c}\right)$$

$$=\frac{1}{\Gamma(c)}\left(\frac{e^{a\pi i}-e^{-a\pi i}+e^{\pm c\pi i}\left(e^{(c-a)\pi i}-e^{(a-c)\pi i}\right)}{e^{c\pi i}-e^{-c\pi i}}\right), \tag{7.17}$$

where we have written the sine functions in terms of complex exponentials. Upon simplification, (7.17) becomes

$$\left[\frac{\Gamma(1-c)}{\Gamma(a)\Gamma(1-a)}+\frac{e^{\pm c\pi i}\Gamma(1-c)}{\Gamma(c-a)\Gamma(1+a-c)}\right]=\frac{e^{\pm(c-a)\pi i}}{\Gamma(c)}\,. \tag{7.18}$$

Applying (7.18) to (7.16), we can represent ${}_1F_1(a;c;z)$ in terms of U functions.

$$\frac{e^{\pm(c-a)\pi i}}{\Gamma(c)}\,{}_1F_1(a;c;z)=\frac{e^{z}}{\Gamma(a)}U(c-a;c;-z)+\frac{e^{\pm c\pi i}}{\Gamma(c-a)}U(a;c;z)$$

$${}_1F_1(a;c;z)=\frac{\Gamma(c)}{\Gamma(a)}e^{\pm(a-c)\pi i}e^{z}U(c-a;c;-z)+\frac{\Gamma(c)}{\Gamma(c-a)}e^{\pm(a-c)\pi i}e^{\pm c\pi i}U(a;c;z)$$

$$ {}_1F_1(a;c;z) = \frac{\Gamma(c)}{\Gamma(c-a)} e^{\pm a\pi i} U(a;c;z) + \frac{\Gamma(c)}{\Gamma(a)} e^{\pm(a-c)\pi i} e^{z} U(c-a;c;-z) \qquad (7.19) $$

We have already discovered an asymptotic result for $U(a;c;z)$. Applying (7.11) to (7.19) we find that

$$ {}_1F_1(a;c;z) \sim \frac{\Gamma(c)}{\Gamma(c-a)} e^{\pm a\pi i} z^{-a} + \frac{\Gamma(c)}{\Gamma(a)} e^{\pm(a-c)\pi i} e^{z} (-z)^{-(c-a)} $$

$$ {}_1F_1(a;c;z) \sim \frac{\Gamma(c)}{\Gamma(c-a)} e^{\pm a\pi i} z^{-a} + \frac{\Gamma(c)}{\Gamma(a)} e^{z} (z)^{-(c-a)}. \qquad (7.20) $$

If we apply (7.20) to (7.14) we obtain

$$ J_p(x) \sim \frac{\left(\frac{x}{2}\right)^p}{\Gamma(p+1)} e^{-ix} \left(\frac{\Gamma(2p+1)}{\Gamma\left(p+\frac{1}{2}\right)} e^{\left(p+\frac{1}{2}\right)\pi i} (2ix)^{-\left(p+\frac{1}{2}\right)} + \frac{\Gamma(2p+1)}{\Gamma\left(p+\frac{1}{2}\right)} e^{2ix} (2ix)^{-\left(p+\frac{1}{2}\right)} \right) $$

$$ \sim \frac{\left(\frac{x}{2}\right)^p}{\Gamma(p+1)} \frac{\Gamma(2p+1)}{\Gamma\left(p+\frac{1}{2}\right)} e^{-ix} (2ix)^{-\left(p+\frac{1}{2}\right)} \left(e^{\left(p+\frac{1}{2}\right)\pi i} + e^{2ix} \right) $$

$$ \sim \frac{1}{\sqrt{2\pi x}} e^{-ix} (i)^{-\left(p+\frac{1}{2}\right)} \left(e^{\left(p+\frac{1}{2}\right)\pi i} + e^{2ix} \right). \qquad (7.21) $$

Finally, through a bit of complex algebra, we obtain our desired result

$$ J_p(x) \sim \frac{1}{\sqrt{2\pi x}} e^{-ix} (e)^{-\frac{\left(p+\frac{1}{2}\right)\pi}{2} i} \left(e^{\left(p+\frac{1}{2}\right)\pi i} + e^{2ix} \right) $$

$$ \sim \frac{1}{\sqrt{2\pi x}} \left(e^{i\left(\frac{p+\frac{1}{2}}{2}\pi - x\right)} + e^{i\left(x - \frac{p+\frac{1}{2}}{2}\pi\right)} \right) $$

or,

$$ J_p(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{p+\frac{1}{2}}{2}\pi\right),\ x \to \infty \qquad (7.22) $$

8 AN APPLICATION

Thus far we have spent time developing the above method for finding asymptotic formulas for hypergeometric type functions. In the paper by Andrews [4], a hypergeometric function arises in calculating the aperture averaging figure of various optical scintillations.

$$A = {}_2F_2\left(\tfrac{7}{6},\tfrac{3}{2};2,3;\tfrac{-\kappa_m^2 D^2}{4}\right), \tag{8.1}$$

where D is the diameter of the circular aperture, and $\kappa_m = \frac{5.92}{l_0}$, with l_0 being the turbulence. We would like to calculate the aperture averaging figure for large values of $\frac{-\kappa_m^2 D^2}{4}$. In his paper Dr. Anrdrews found that $A \sim 0.453\left(\frac{D}{l_0}\right)^{-\frac{7}{3}} - 0.215\left(\frac{D}{l_0}\right)^{-3}$ for large arguments of $\frac{D}{l_0}$. This time we are dealing with a hypergeometric type function of the form ${}_2F_2\left(a,b;c,d;x\right)$. Let's explore the asymptotics of such a function.

$${}_2F_2\left(a,b;c,d;z\right) = \frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)}\int_L \frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)\Gamma(d+s)}\Gamma(-s)z^s ds \tag{8.2}$$

Notice that we have two sets of poles in the left hand plane. By using the contour in Figure 3 and residue theory we can write (8.2) as

$$\begin{aligned}{}_2F_2\left(a,b;c,d;z\right) = &\frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\text{Res}\left\{\frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)\Gamma(d+s)}\Gamma(-s)z^s;-n-a\right\} \\ &+\frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\text{Res}\left\{\frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)\Gamma(d+s)}\Gamma(-s)z^s;-n-b\right\}.\end{aligned} \tag{8.3}$$

Next we'll need to apply (2.5) to $\Gamma(a+s)$ in the first summation and $\Gamma(b+s)$ in the second summation. By doing so, and applying Theorem 2, we can simplify (8.3) to

$$
= \frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(b-a-n)\Gamma(n+a) z^{-n-a}}{\Gamma(c-a-n)\Gamma(d-a-n)\Gamma(1+n)}
+ \frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(a-b-n)\Gamma(n+b) z^{-n-b}}{\Gamma(c-b-n)\Gamma(d-b-n)\Gamma(1+n)} . \tag{8.4}
$$

Finally we use (6.10) to find our desired asymptotic formula

$$
{}_2F_2(a,b;c,d;z) \sim \frac{\Gamma(c)\Gamma(d)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)\Gamma(d-a)} z^{-a} + \frac{\Gamma(c)\Gamma(d)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)\Gamma(d-b)} z^{-b}, \quad |z| \to \infty . \tag{8.5}
$$

If we apply (8.5) to (8.1) we find that

$$
A \sim \frac{\Gamma(2)\Gamma(3)\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{11}{6}\right)} \left(\frac{-\kappa_m^2 D^2}{4}\right)^{-\frac{7}{6}} + \frac{\Gamma(2)\Gamma(3)\Gamma\left(-\frac{1}{3}\right)}{\Gamma\left(\frac{7}{6}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)} \left(\frac{-\kappa_m^2 D^2}{4}\right)^{-\frac{3}{2}}
$$

$$
A \sim 0.453\left(\frac{D}{l_0}\right)^{-\frac{7}{3}} - 0.215\left(\frac{D}{l_0}\right)^{-3}, \tag{8.6}
$$

which agrees with the results achieved by Dr. Andrews.

9 CONCLUSION

We've presented an alternate method for developing large argument asymptotic formulas for Hypergeometric type functions. Specifically, we've looked at the Bessel functions. The method and its results have been verified by comparing it to previously obtained results.

The purpose of this paper has never been to show that this method is better than others, or that it is more efficient. The goal has simply been to show that another method does exist, which under other circumstances, may prove to be not only efficient, but also necessary. Continued research in this area would be beneficial to the development of asymptotic formulas and beyond.

APPENDIX A

PROOF OF THEOREM 3

By Theorem 1

$$ {}_1F_k\left(a;c_1,...,c_k;-z\right)=\frac{\prod_{j=1}^{k}\Gamma\left(c_j\right)}{\Gamma\left(a\right)}\frac{1}{2\pi i}\int_L \frac{\Gamma\left(a+s\right)\Gamma\left(-s\right)}{\prod_{j=1}^{k}\Gamma\left(c_j+s\right)}z^s ds $$

Make the change of variable $s\to-\left(s+a\right)$.

$$ =-\frac{\prod_{j=1}^{k}\Gamma\left(c_j\right)}{\Gamma\left(a\right)}z^{-a}\sum_{n=0}^{\infty}\operatorname{Res}\left\{\frac{\Gamma\left(a+s\right)\Gamma\left(-s\right)}{\prod_{j=1}^{k}\Gamma\left(c_j-a-s\right)}z^{-s};n\right\}. $$

Applying (2.5) we find that,

$$ {}_1F_k=\frac{\prod_{j=1}^{k}\Gamma\left(c_j\right)}{\Gamma\left(a\right)}z^{-a}\sum_{n=0}^{\infty}\operatorname{Res}\left\{\frac{\pi}{\Gamma\left(1+s\right)\sin s\pi}\frac{\Gamma\left(a+s\right)}{\prod_{j=1}^{k}\Gamma\left(c_j-a-s\right)}z^{-s};n\right\} $$

$$ =\frac{\prod_{j=1}^{k}\Gamma\left(c_j\right)}{\Gamma\left(a\right)}z^{-a}\sum_{n=0}^{\infty}\operatorname{Res}\left\{\frac{\Gamma\left(a+s\right)}{\Gamma\left(1+s\right)\cos s\pi\prod_{j=1}^{k}\Gamma\left(c_j-a-s\right)}z^{-s};n\right\} $$

$$ =\frac{\prod_{j=1}^{k}\Gamma\left(c_j\right)}{\Gamma\left(a\right)}z^{-a}\sum_{n=0}^{\infty}\left(\frac{\Gamma\left(n+a\right)}{\Gamma\left(1+n\right)\cos n\pi\prod_{j=1}^{k}\Gamma\left(c_j-a-n\right)}z^{-n}\right) $$

By using (6.10), we come up with the desired result.

$$ {}_1F_k\left(a;c_1,\ldots,c_k;-z\right)=\frac{\prod_{j=1}^{k}\Gamma\left(c_j\right)}{\Gamma\left(a\right)}z^{-a}\sum_{n=0}^{\infty}\left(\frac{\left(-1\right)^n\left(a\right)_n\Gamma\left(a\right)\prod_{j=1}^{k}\left(-1\right)^n\left(1-c_j+a\right)_n}{\Gamma\left(1+n\right)\prod_{j=1}^{k}\Gamma\left(c_j-a\right)}z^{-n}\right) $$

$$= \frac{\prod_{j=1}^{k} \Gamma\left(c_j\right)}{\prod_{j=1}^{k} \Gamma\left(c_j - a\right)} z^{-a} \sum_{n=0}^{\infty} \left((a)_n \prod_{j=1}^{k} \left(1 - c_j + a\right)_n \frac{\left(-1\right)^n \left(-1\right)^{kn} z^{-n}}{n!} \right)$$

$$= \frac{\prod_{j=1}^{k} \Gamma\left(c_j\right)}{\prod_{j=1}^{k} \Gamma\left(c_j - a\right)} z^{-a} \sum_{n=0}^{\infty} \left((a)_n \prod_{j=1}^{k} \left(1 - c_j + a\right)_n \frac{\left(\left(-1\right)^{k+1} z\right)^{-n}}{n!} \right)$$

$$= \frac{\prod_{j=1}^{k} \Gamma\left(c_j\right)}{\prod_{j=1}^{k} \Gamma\left(c_j - a\right)} z^{-a} \ {}_{k+1}F_0 \left(a, 1 - c_1 + a, \ldots 1 - c_k + a; -; \frac{\left(-1\right)^{k+1}}{z} \right)$$

APPENDIX B

VERIFICATION OF ASYMPTOTIC FORMULAS

In order to derive asymptotic formulas for $K_p(x)$, $J_p(x)$, and $I_p(x)$ we start with the integral representation of $K_p(x)$,

$$K_p(x) = \frac{\sqrt{\pi}\left(\frac{x}{2}\right)^p}{\Gamma\left(p+\frac{1}{2}\right)} \int_1^\infty e^{-xt}\left(t^2-1\right)^{p-\frac{1}{2}}\,dt \quad p > -\tfrac{1}{2}, x > 0\,, \tag{8.1}$$

and immediately make the substitution $t = 1+\frac{u}{x}$. This leads to

$$K_p(x) = \frac{\sqrt{\pi}\left(\frac{x}{2}\right)^p}{\Gamma\left(p+\frac{1}{2}\right)x} \int_0^\infty e^{-x\left(1+\frac{u}{x}\right)}\left(\left(1+\tfrac{u}{x}\right)^2-1\right)^{p-\frac{1}{2}}\,du$$

$$= \frac{\sqrt{\pi}\left(\frac{x}{2}\right)^p e^{-x}}{\Gamma\left(p+\frac{1}{2}\right)x} \int_0^\infty e^{-u}\left(\tfrac{2u}{x}+\tfrac{u^2}{x^2}\right)^{p-\frac{1}{2}}\,du$$

$$= \frac{\sqrt{\pi}\left(\frac{x}{2}\right)^p e^{-x}}{\Gamma\left(p+\frac{1}{2}\right)x} \int_0^\infty e^{-u}\left(\tfrac{2u}{x}+\tfrac{u^2}{x^2}\right)^{p-\frac{1}{2}}\,du$$

$$K_p(x) = \frac{\sqrt{\pi}\left(\frac{x}{2}\right)^p}{\Gamma\left(p+\frac{1}{2}\right)}\left(\frac{2}{x}\right)^{p-\frac{1}{2}}\frac{e^{-x}}{x}\int_0^\infty e^{-u}u^{p-\frac{1}{2}}\left(1+\tfrac{u}{2x}\right)^{p-\frac{1}{2}}\,du\,.$$

For large values of x we use the relationship $\left(1+\frac{u}{2x}\right)^{p-\frac{1}{2}} \approx 1, \quad x \gg u$ to obtain

$$K_p(x) \approx \sqrt{\frac{\pi}{2x}}\,\frac{e^{-x}}{\Gamma\left(p+\frac{1}{2}\right)}\int_0^\infty e^{-u}u^{p-\frac{1}{2}}du \tag{8.2}$$

Notice that the integral in (8.2) is $\Gamma\left(p+\frac{1}{2}\right)$. This implies that

$$K_p(x) \approx \sqrt{\frac{\pi}{2x}}e^{-x}, \quad x \to \infty\,, \tag{8.3}$$

which is our desired asymptotic formula.

In order to develop asymptotic formulas for $J_p(x)$, $I_p(x)$ we need to refer to the relationship between $K_p(x)$ and the Hankel functions $H_p^{(1)}(x)$.

$$K_p(x) = \tfrac{1}{2}\pi i^{p+1} H_p^{(1)}(x) \tag{8.4}$$

After making the change of variable $x \to -ix$ and solving for $H_p^{(1)}(x)$, (8.4) becomes

$$H_p^{(1)}(x) = \frac{2}{\pi} i^{-(p+1)} K_p(-ix),$$

where, upon assuming that (8.4) is valid for complex arguments, we obtain

$$H_p^{(1)}(x) \sim \frac{2}{\pi} i^{-(p+1)} \sqrt{\frac{\pi}{-2ix}} e^{ix} \sim \sqrt{\frac{2}{\pi x}} i^{-\left(p+\frac{1}{2}\right)} e^{ix}.$$

Upon writing $i = e^{\frac{i\pi}{2}}$ we find

$$H_p^{(1)}(x) \sim \sqrt{\frac{2}{\pi x}} e^{i\left[x - \frac{\left(p+\frac{1}{2}\right)\pi}{2}\right]}$$

$$\sim \sqrt{\frac{2}{\pi x}} \left[\cos\left(x - \tfrac{\left(p+\frac{1}{2}\right)\pi}{2}\right) + i \sin\left(x - \tfrac{\left(p+\frac{1}{2}\right)\pi}{2}\right)\right], \quad x \to \infty .$$

Finally, we use the relationship $H_p^{(1)}(x) = J_p(x) + iY_p(x)$ and equat real and imaginary terms to obtain the desired result

$$J_p(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \tfrac{\left(p+\frac{1}{2}\right)\pi}{2}\right). \tag{8.5}$$

To reach an asymptotic result for $I_p(x)$ we use the relation $I_p(x) = i^{-p} J_p(ix)$ to find

$$I_p(x) \sim \frac{e^x}{\sqrt{2\pi x}}. \tag{8.6}$$

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