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Effects of atmospheric turbulence on the scintillation and fade probability of flattened Gaussian beams

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Abstract. In an attempt to mitigate the effects of the atmosphere on the coherence of an optical (laser) beam, interest has recently been shown in changing the beam shape to determine if a different power distribution at the transmitter will reduce the effects of random fluctuations in the refractive index. Here, a model is developed for the field of a flattened Gaussian beam as it propagates through atmospheric turbulence, and the resulting effects on the scintillation of the beam and beam wander are determined. A comparison of these results is made with the like effects on a standard TEM₀₀ Gaussian beam. The theoretical results are verified by comparison with a computer simulation model for the flattened Gaussian beam (FGB). Further, a determination of the probability of fade and of mean fade time under weak fluctuation conditions is determined using the widely accepted lognormal model. Although this model has been shown to be somewhat optimistic when compared to results obtained in field tests, it has value here in allowing us to compare the effects of atmospheric conditions on the fade statistics of the FGB with those of the lowest order Gaussian beam. © 2008 Society of Photo-Optical Instrumentation Engineers. [DOI: 10.1117/1.2844715]

Subject terms: flattened Gaussian beam; scintillation; fade probability; atmospheric propagation; beam wander; spot size.

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1 Introduction

In recent years, there have been a number of studies directed at the propagation of various beam profiles other than that of the lowest-order Gaussian beam.^{1–18} Most of these studies have concentrated on free-space propagation characteristics, but some have involved propagation paths through optical turbulence.^{3,12,14–18} Motivation for these various studies is not always the same. In some cases, higher-order Gaussian modes are studied because these are more representative of the actual transmitted laser beam. Also, there are some applications, such as laser processing and manufacturing, where laser beams with a flat-topped intensity profile are encountered or desirable. In the presence of atmospheric turbulence, some of these studies were motivated by the question of whether a particular shape beam intensity profile might be more desirable in the field of free-space optical (FSO) communications. Preliminary work in some cases is promising,^{12,16–18} but a more definitive answer has yet to be determined.

One of the models for describing flat-topped profiles is the super-Gaussian function, which has been defined in prior works as^{8,10}

$$SG(x, w_{SG}; \tau) = A \exp \left[- \left(\frac{|x|}{w_{SG}} \right)^\tau \right], \quad (1)$$

where A is the maximum value of the function, x is radial distance, w_{SG} is a scale factor related to spot size, and τ is the super-Gaussian index. The value of τ affects the rapidity with which the function goes from its maximum value to zero. The function reduces to the standard Gaussian for $\tau=2$, while for higher values of τ the profile tends to be more and more flat. As τ approaches infinity, the cross section of the function tends to the rectangle function $rect(x/2w_{SG})$, which has a value of unity in the interval $-w_{SG} \leq x \leq w_{SG}$ and is equal to zero otherwise.

Although the super-Gaussian beam (SGB) is described by a relatively simple expression, in the propagation of flat-topped beams it has, to date, been dealt with only through numerical methods. Consequently, alternative models have been proposed whose analytical treatments are more tractable. The flattened Gaussian beam (FGB) was first introduced by Gori⁴ in 1994, and has been intensely studied (primarily in free space) from several perspectives using a variety of models. Gori's model, which uses the series form of the exponential function to flatten the beam, then truncates the sum at a finite value N , is

$$U_N(r,0) = A \exp\left(-\frac{Nr^2}{W_0^2}\right) \sum_{m=0}^N \frac{1}{m!} \left(\frac{Nr^2}{W_0^2}\right)^m, \quad N = 1, 2, 3, \dots, \quad (2)$$

where W_0 represents the spot size of the beam, r represents radial distance from the optical axis, and N is the order of the beam. This equation was adjusted by Bagini et al.⁵ to allow for the inclusion of $N=0$ leading to the standard Gaussian beam, namely

$$U_{N0}(r,0) = A_0 \exp\left[-\frac{(N+1)r^2}{W_0^2}\right] \sum_{m=0}^N \frac{1}{m!} \left(\frac{\sqrt{N+1}r}{W_0}\right)^{2m}, \quad N = 0, 1, 2, \dots \quad (3)$$

Gori's model has sparked a number of studies of the FGB in the past decade. In particular, Santarsiero et al.⁶ have studied the focusing characteristics of the beam, and Lu et al.^{9,13} have analyzed its intensity distribution after propagation and its pointing stability.

In comparing Gori's FGB to the SGB, Santarsiero and Borghi⁸ noted that, although the two were identical only in the limiting cases, the differences between them in all other cases were so small as to be indistinguishable in experimental situations. Lu, Luo, and Ji¹⁰ also noted that, given the same propagation conditions, the SGB and the FGB models would demonstrate nearly identical characteristics as they propagated through a paraxial ABCD system. However, the FGB model is preferable to the SGB model, because the former has the advantage of permitting tractable analytic solutions in propagation scenarios.

In 2001, Tovar¹¹ introduced an alternate model for producing a flattened profile. By writing the profile as a sum of finite-width Gaussian functions side by side, his multi-Gaussian model produces a class of beams which, like Gori's model, has the shape of the SGB while having propagation characteristics that can be analytically described. Each of the Gaussian beams in the sum has an identical spot size w , but is offset by integer multiples of w from the beam axis. Thus, the normalized beam field at the transmitter becomes

$$G(r) = \frac{\sum_{m=-N}^N \exp\left[-\left(\frac{r-mw}{w}\right)^2\right]}{\sum_{m=-N}^N \exp(-m^2)}. \quad (4)$$

Tovar's analysis of the propagation characteristics of this model included both ABCD optical systems and free space. In free-space propagation, the multi-Gaussian beam approaches a standard Gaussian profile in the far field as the Gaussian beams overlap increasingly while the phase curvature of the individual beams becomes flat. This model compares favorably to the SGB profile in its numerical values, while retaining the benefit of analytic solutions in beam propagation situations.

Baykal and Eyyuboglu¹⁶ and Eyyuboglu, Arpali, and Baykal¹⁷ model the FGB as a superposition of Gaussian beams, where the field equation is

$$U(r,0) = \sum_{n=1}^N \frac{(-1)^{n-1}}{N} \binom{N}{n} \exp\left(-\frac{nr^2}{W_0^2}\right). \quad (5)$$

In Eq. (5), the amplitude of the field at the transmitter is taken to be unity, and the phase factor of the field is set to zero. The result is a beam which, as it flattens with increasing order N , also decreases in amplitude. This model has recently been the focus of study of the atmospheric effects on propagated intensity and on scintillation.¹⁶⁻¹⁸ Cai¹⁸ has developed an analytic formula for the average intensity of the beam in a turbulent atmosphere. Baykal and Eyyuboglu, and Eyyuboglu, Arpali, and Baykal¹⁷ have deduced the scintillation index, long-term beam size, and kurtosis parameter for the propagated beam modeled by Baykal.

In our study, we utilize Gori's model, as adjusted by Bagini, for the FGB. We first review the free-space propagation characteristics of the FGB, and then introduce atmospheric effects based on the Rytov approximation. By use of the derived spectral representation for the complex phase perturbation arising from the Rytov method, we are then able to find analytic solutions for the on-axis scintillation index, from which we eventually deduce various fade characteristics for a receiver with aperture smaller than the correlation width of irradiance.

2 Spectral Representation

2.1 Free-Space Analysis

Beginning with the field for a FGB at the transmitter,

$$U_{N0}(r,0) = A_0 \exp\left[-\frac{(N+1)r^2}{W_0^2}\right] \sum_{n=0}^N \frac{1}{n!} \left(\frac{\sqrt{N+1}r}{W_0}\right)^{2n}, \quad (6)$$

and making the substitution $W_{N0}^2 = W_0^2/N+1$, we have

$$U_{N0}(r,0) = A_0 \exp\left[-\frac{r^2}{W_{N0}^2}\right] \sum_{n=0}^N \frac{1}{n!} \left(\frac{r^2}{W_{N0}^2}\right)^n. \quad (7)$$

The free-space field of the optical wave in Eq. (7) at a receiver located distance L from the source can be obtained by using the Huygens-Fresnel integral:

$$U_{N0}(r,L) = -A_0 \frac{ik}{L} \exp\left(ikL + \frac{ik}{2L}r^2\right) \times \sum_{n=0}^N \frac{1}{n! W_{N0}^{2n}} \int_{-\infty}^{\infty} s^{2n+1} \exp\left[\frac{ik}{2L}\left(1 - \frac{2L}{ikW_{N0}^2}\right)s^2\right] \times J_0\left(\frac{kr}{L}s\right) ds, \quad (8)$$

where k is optical wave number and $J_0(x)$ refers to the Bessel function of the first kind.

Selecting, for convenience, $A_0=1$ and setting $\alpha_N = 2/kW_{N0}^2$, this now becomes

$$U_{N0}(r, L) = -\frac{ik}{L} \exp\left(ikL + \frac{ik}{2L}r^2\right) \sum_{n=0}^N \frac{1}{n! W_{N0}^{2n}} \times \int_{-\infty}^{\infty} s^{2n+1} \exp\left[\frac{ik}{2L}(1 + i\alpha_N L)s^2\right] J_0\left(\frac{kr}{L}s\right) ds. \quad (9)$$

Last, by solving the integral in Eq. (9) and making use of Kummer's transformation,¹⁹ we are led to the result

$$U_{N0}(r, L) = \frac{1}{(1 + i\alpha_N L)} \exp\left[ikL - \frac{k\alpha_N r^2}{2(1 + i\alpha_N L)}\right] \times \sum_{n=0}^N \frac{1}{W_{N0}^{2n}} \left[\frac{2iL}{k(1 + i\alpha_N L)}\right]^n \times {}_1F_1\left[-n; 1; \frac{ikr^2}{2L(1 + i\alpha_N L)}\right], \quad (10)$$

in which ${}_1F_1(a; c; x)$ is the confluent hypergeometric function.¹⁹

2.2 Optical Turbulence

Equation (10) provides us with an expression for the free-space field at distance L from the transmitter. In the presence of atmospheric turbulence, we use the Rytov approximation

$$U_N(\mathbf{r}, L) = U_{N0}(\mathbf{r}, L) \exp[\psi_1(\mathbf{r}, L) + \psi_2(\mathbf{r}, L)], \quad (11)$$

where $\psi_1(\mathbf{r}, L)$ and $\psi_2(\mathbf{r}, L)$ denote first-order and second-order complex phase perturbations. To calculate statistical moments of the optical field $U_N(\mathbf{r}, L)$, we develop spectral representations for the complex phase perturbations.

Based on previous analysis, the representation for $\psi_1(\mathbf{r}, L)$ is given by²⁰

$$\psi_1(\mathbf{r}, L) = \frac{k^2}{2\pi} \int_0^L dz \int_{-\infty}^{\infty} d^2s \exp\left[ik(L-z) + \frac{ik}{2(L-z)}|\mathbf{s} - \mathbf{r}|^2\right] \times \frac{U_{N0}(\mathbf{s}, z) n_1(\mathbf{s}, z)}{U_{N0}(\mathbf{r}, L) (L-z)}, \quad (12)$$

where $n_1(\mathbf{s}, z)$ is the refractive index at arbitrary position z along the propagation path. The substitution of the free-space field in Eq. (10) into Eq. (12) leads to

$$\psi_1(\mathbf{r}, L) = \frac{k^2}{2\pi} \int_0^L dz \int_{-\infty}^{\infty} d^2s \exp\left[ik(L-z) + \frac{ik|\mathbf{s} - \mathbf{r}|^2}{2(L-z)}\right] \frac{n_1(\mathbf{s}, z)}{L-z} \times \frac{1}{1 + i\alpha_N z} \exp\left[ikz - \frac{k\alpha_N s^2}{2(1 + i\alpha_N z)}\right] \sum_{n=0}^N \left\{ \left[\frac{2iz}{k(1 + i\alpha_N z)}\right]^n \frac{1}{W_{N0}^{2n}} {}_1F_1\left[-n; 1; \frac{iks^2}{2z(1 + i\alpha_N z)}\right] \right\} \times \frac{1}{1 + i\alpha_N L} \exp\left[ikL - \frac{k\alpha_N r^2}{2(1 + i\alpha_N L)}\right] \sum_{n=0}^N \left\{ \left[\frac{2iL}{k(1 + i\alpha_N L)}\right]^n \frac{1}{W_{N0}^{2n}} {}_1F_1\left[-n; 1; \frac{ikr^2}{2L(1 + i\alpha_N L)}\right] \right\}, \quad (13)$$

which can be reduced to

$$\psi_1(\mathbf{r}, L) = \frac{k^2}{2\pi} \sum_{n=0}^N \left\{ \left[\frac{2iL}{k(1 + i\alpha_N L)}\right]^n \frac{1}{W_{N0}^{2n}} {}_1F_1\left[-n; 1; \frac{ikr^2}{2L(1 + i\alpha_N L)}\right] \right\} \int_0^L dz \sum_{m=0}^N \left\{ \left[\frac{2iz}{k(1 + i\alpha_N z)}\right]^m \frac{1}{W_{N0}^{2m}} \right\} \times \int_{-\infty}^{\infty} d^2s {}_1F_1\left[-m; 1; \frac{iks^2}{2z(1 + i\alpha_N z)}\right] \exp\left[\frac{ik|\mathbf{s} - \mathbf{r}|^2}{2(L-z)} - \frac{k\alpha_N s^2}{2(1 + i\alpha_N z)} + \frac{k\alpha_N r^2}{2(1 + i\alpha_N L)}\right] \left(\frac{1 + i\alpha_N L}{1 + i\alpha_N z}\right) \frac{n_1(\mathbf{s}, z)}{L-z}. \quad (14)$$

To express this result in the form of a spectral representation, we replace the refractive index term by its Riemann-Stieltjes integral²⁰

$$n_1(\mathbf{s}, z) = \int_{-\infty}^{\infty} \exp(i\mathbf{K} \cdot \mathbf{s}) d\nu(\mathbf{K}, z), \quad (15)$$

where $d\nu(\mathbf{K}, z)$ is a random amplitude. Subsequently, after simplification, we obtain

$$\psi_1(\mathbf{r}, L) = \frac{ik}{A_n(r)} \int_0^L dz \int_{-\infty}^{\infty} d\nu(\mathbf{K}, z) \times \exp\left[\frac{\gamma\kappa^2(L-z)}{2ik} + i\gamma\mathbf{K} \cdot \mathbf{r}\right] \times \sum_{m=0}^N \left[\frac{2iL}{k(1 + i\alpha_N L)}\right]^m \times \frac{1}{W_{N0}^{2m}} L_m \left\{ \frac{i[\mathbf{K}(L-z) - k\mathbf{r}]^2}{2kL(1 + i\alpha_N L)} \right\}, \quad (16)$$

in which $L_m(x)$ refers to the Laguerre polynomial. For no-

Fig. 1 The scintillation index of FGBs of order $N=5$ and $N=10$ is compared to that of a standard Gaussian beam in the case where $W_0=1$ cm.

tational simplicity in the spectral representation in Eq. (16), we have introduced the parameters

$$\gamma = \frac{1 + i\alpha_N z}{1 + i\alpha_N L}, \quad (17)$$

and

$$A_n(r) = \sum_{n=0}^N \left\{ \left[\frac{2iL}{k(1 + i\alpha_N L)} \right]^n \frac{1}{W_{N0}^{2n}} L_n \left[\frac{ikr^2}{2L(1 + i\alpha_N L)} \right] \right\}. \quad (18)$$

2.3 Statistical Moments of the Rytov Approximations

It has previously been shown that all statistical quantities of interest can be determined from the three second-order statistical moments defined by²⁰

$$E_1(\mathbf{r}, \mathbf{r}) \equiv E_1(0, 0) = \langle \psi_2(\mathbf{r}, L) \rangle + \frac{1}{2} \langle \psi_1^2(\mathbf{r}, L) \rangle$$

$$E_2(\mathbf{r}_1, \mathbf{r}_2) = \langle \psi_1(\mathbf{r}_1, L) \psi_1^*(\mathbf{r}_2, L) \rangle$$

$$E_3(\mathbf{r}_1, \mathbf{r}_2) = \langle \psi_1(\mathbf{r}_1, L) \psi_1(\mathbf{r}_2, L) \rangle, \quad (19)$$

where $\langle \rangle$ represents the ensemble average and $*$ refers to the complex conjugate. In particular, based on the spectral representation in Eq. (16), we have that

$$E_2(\mathbf{r}_1, \mathbf{r}_2) = \frac{k^2}{A_n(r_1)A_n^*(r_2)} \sum_{m=0}^N B_m \sum_{p=0}^N B_p^* \int_0^L dz \int_0^L dz' \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle d\nu(\mathbf{K}, z) d\nu^*(\mathbf{K}', z') \rangle \exp \left[\frac{\gamma \kappa^2 (L - z)}{2ik} - \frac{\gamma^* \kappa'^2 (L - z')}{2ik} \right] \\ \times \exp(i\gamma \mathbf{K} \cdot \mathbf{r}_1 - i\gamma^* \mathbf{K}' \cdot \mathbf{r}_2) L_m \left[\frac{i|\mathbf{K}(L - z) - k\mathbf{r}_1|^2}{2kL(1 + i\alpha_N L)} \right] L_p \left[\frac{-i|\mathbf{K}'(L - z') - k\mathbf{r}_2|^2}{2kL(1 - i\alpha_N L)} \right], \quad (20)$$

where

$$B_m = \left[\frac{2iL}{k(1 + i\alpha_N L)} \right]^m \frac{1}{W_{N0}^{2m}}. \quad (21)$$

As in Chap. 5 of Ref. 20, we use the relationship

$$\langle d\nu(\mathbf{K}, z) d\nu^*(\mathbf{K}', z') \rangle = F_n(\mathbf{K}, |z - z'|) \delta(\mathbf{K} - \mathbf{K}') d^2 \kappa d^2 \kappa', \quad (22)$$

where δ represents the Kronecker delta function, and

Fig. 2 The scintillation index of FGBs of order $N=5$ and $N=10$ is compared to that of a standard Gaussian beam in the case where $W_0=3$ cm.

$$F_n(\mathbf{K}, |z - z'|) \equiv F_n(\kappa_x, \kappa_y, 0, |z - z'|)$$

is a 2-D spectral density of the index of refraction, which has an appreciable value only when $|z - z'|$ is close to zero.

The 3-D power spectrum obeys the relation

$$F_n(\kappa_x, \kappa_y, 0, |\mu|) = \int_{-\infty}^{\infty} \Phi_n(\kappa_x, \kappa_y, \kappa_z) \cos(\mu \kappa_z) d\kappa_z. \quad (23)$$

Performing an inverse Fourier transform on Eq. (23), we have

$$\Phi_n(\kappa_x, \kappa_y, \kappa_z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_n(\kappa_x, \kappa_y, 0, |\mu|) \cos(\mu \kappa_z) d\mu, \quad (24)$$

which, for $\kappa_z=0$, gives

$$\int_{-\infty}^{\infty} F_n(\mathbf{K}, |\mu|) d\mu = 2\pi \Phi_n(\kappa_x, \kappa_y, 0) = 2\pi \Phi_n(\mathbf{K}). \quad (25)$$

Using the substitution $\eta = 1/2(z + z')$ and the fact that $z \approx z' \approx \eta$, we can perform three of the integrations in Eq. (20) to arrive at

Fig. 3 The scintillation index of FGBs of order $N=5$ and $N=10$ is compared to that of a standard Gaussian beam in the case where $W_0=5$ cm.

Fig. 4 The scintillation index of FGBs of order $N=5$ and $N=10$ is compared to that of a standard Gaussian beam in the case where $W_0=10$ cm.

$$\begin{aligned}
 E_2(\mathbf{r}_1, \mathbf{r}_2) &= \frac{2\pi k^2}{A_n(r_1)A_n^*(r_2)} \sum_{m=0}^N B_m \sum_{p=0}^N B_p^* \\
 &\times \int_0^L d\eta \int_{-\infty}^{\infty} \Phi_n(\mathbf{K}) d^2\kappa \exp\left[\frac{\kappa^2(L-\eta)}{2ik}(\gamma - \gamma^*)\right] \\
 &\times \exp[i\mathbf{K} \cdot (\gamma\mathbf{r}_1 - \gamma^*\mathbf{r}_2)] L_m \left[\frac{i|\mathbf{K}(L-\eta) - k\mathbf{r}_1|^2}{2kL(1+i\alpha_N L)} \right] \\
 &\times L_p \left[\frac{-i|\mathbf{K}(L-\eta) - k\mathbf{r}_2|^2}{2kL(1-i\alpha_N L)} \right]. \quad (26)
 \end{aligned}$$

Further simplification of Eq. (26) will depend on the choice of the power spectrum $\Phi_n(\mathbf{K})$.

In a similar manner, we can use the fact that n_1 is a real function to equate $d\nu(\mathbf{K}', z')$ with $d\nu^*(-\mathbf{K}', z')$, and thus determine that

$$\langle d\nu(\mathbf{K}, z) d\nu(\mathbf{K}', z') \rangle = F_n(\mathbf{K}, |z - z'|) \delta(\mathbf{K} + \mathbf{K}') d^2\kappa d^2\kappa'. \quad (27)$$

Using the same η substitution demonstrated in the development of Eq. (26) allows the simplification of

$$\begin{aligned}
 E_3(\mathbf{r}_1, \mathbf{r}_2) &= \frac{k^2}{A_n(r_1)A_n(r_2)} \sum_{m=0}^N B_m \sum_{p=0}^N B_p \int_0^L dz \int_0^L dz' \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle d\nu(\mathbf{K}, z) d\nu(\mathbf{K}', z') \rangle \exp\left[\frac{\gamma\kappa^2(L-z)}{2ik} + \frac{\gamma\kappa'^2(L-z')}{2ik}\right] \\
 &\times \exp(i\gamma\mathbf{K} \cdot \mathbf{r}_1 + i\gamma\mathbf{K}' \cdot \mathbf{r}_2) L_m \left[\frac{i|\mathbf{K}(L-z) - k\mathbf{r}_1|^2}{2kL(1+i\alpha_N L)} \right] L_p \left[\frac{i|\mathbf{K}'(L-z') - k\mathbf{r}_2|^2}{2kL(1+i\alpha_N L)} \right], \quad (28)
 \end{aligned}$$

to

$$\begin{aligned}
 E_3(\mathbf{r}_1, \mathbf{r}_2) &= \frac{-2\pi k^2}{A_n(r_1)A_n(r_2)} \sum_{m=0}^N B_m \sum_{p=0}^N B_p \\
 &\times \int_0^L d\eta \int_{-\infty}^{\infty} d^2\kappa \Phi_n(\mathbf{K}) \exp\left[\frac{\gamma\kappa^2(L-\eta)}{ik}\right] \\
 &\times \exp[i\gamma\mathbf{K} \cdot (\mathbf{r}_1 - \mathbf{r}_2)] L_m \left[\frac{i|\mathbf{K}(L-\eta) - k\mathbf{r}_1|^2}{2kL(1+i\alpha_N L)} \right] \\
 &\times L_p \left[\frac{i|\mathbf{K}(L-\eta) + k\mathbf{r}_2|^2}{2kL(1+i\alpha_N L)} \right]. \quad (29)
 \end{aligned}$$

Last, although we have not shown a separate development of the spectral representation for the second-order complex phase perturbation $\psi_2(\mathbf{r}, L)$, we have found that the integral representation for the statistical moment $E_1(0, 0)$ is the same as previously derived for the lowest-order Gaussian-beam wave.²⁰

3 Scintillation Index

In the development of the scintillation index, we consider only the on-axis values where $\mathbf{r}_1 = \mathbf{r}_2 = 0$. In this case, the scintillation index $\sigma_I^2(L)$ is given as

Fig. 5 A comparison of theoretical and simulated data for the FGBs of order $N=5$ and $N=10$ with $W_0=3 \text{ cm}$, $\lambda=1.06 \mu\text{m}$, and $C_n^2=10^{-14} \text{ m}^{-2/3}$. The vertical line indicates the propagation distance at which the Rytov variance is unity.

$$\sigma_I^2(L) = 2 \operatorname{Re}[E_2(0,0) + E_3(0,0)], \quad (30)$$

where Re denotes the real part. The resulting integral expressions $E_2(0,0)$ and $E_3(0,0)$ deduced from Eqs. (26) and (29) reduce to

$$E_2(0,0) = \frac{2\pi k^2}{N} \sum_{n=0}^N \frac{B_n}{\sum_{q=0}^N B_q} \sum_{p=0}^N \frac{B_p^*}{\sum_{q=0}^N B_q} \int_0^L d\eta \int \int_{-\infty}^{\infty} d^2\kappa \Phi_n(\mathbf{K}) \times \exp\left[\frac{\kappa^2(L-\eta)}{2ik}(\gamma - \gamma^*)\right] \times L_m \left[\frac{i|\mathbf{K}(L-\eta)|^2}{2kL(1+i\alpha_N L)} \right] L_p \left[\frac{-i|\mathbf{K}(L-\eta)|^2}{2kL(1-i\alpha_N L)} \right], \quad (31)$$

and

$$E_3(0,0) = \frac{-2\pi k^2}{N} \sum_{n=0}^N \frac{B_n}{\sum_{q=0}^N B_q} \sum_{p=0}^N \frac{B_p}{\sum_{q=0}^N B_q} \int_0^L d\eta \int \int_{-\infty}^{\infty} d^2\kappa \Phi_n(\mathbf{K}) \times \exp\left[\frac{\gamma\kappa^2(L-\eta)}{ik}\right] L_m \left[\frac{i|\mathbf{K}(L-\eta)|^2}{2kL(1+i\alpha_N L)} \right] \times L_p \left[\frac{i|\mathbf{K}(L-\eta)|^2}{2kL(1+i\alpha_N L)} \right]. \quad (32)$$

If we make the assumption that the atmosphere is statistically homogeneous and isotropic in each transverse plane where $\eta=\text{constant}$, we can replace $\Phi_n(\mathbf{K})$ with $\Phi_n(\kappa)$. Also, by introducing the normalized distance variable $\xi = 1 - \eta/L$, we can write

$$\gamma = \frac{1 + i\alpha_N z}{1 + i\alpha_N L} = 1 - (\bar{\Theta}_N - i\Lambda_N)\xi, \quad (33)$$

$$\Lambda_{N0} = \frac{2L}{kW_{N0}^2}, \quad \bar{\Theta}_N = \frac{\Lambda_{N0}^2}{1 + \Lambda_{N0}^2}, \quad \Lambda_N = \frac{\Lambda_{N0}}{1 + \Lambda_{N0}^2}. \quad (34)$$

where

Consequently, by making these changes in notation and using the Kolmogorov spectrum, $\Phi_n(\kappa) = 0.033C_n^2\kappa^{-11/3}$, our integral expressions reduce to

$$E_2(0,0) = (0.033)C_n^2 \frac{4\pi^2 k^2}{N} \sum_{n=0}^N B_n \sum_{q=0}^N B_q^* \int_0^1 L d\xi \int_0^\infty \kappa^{-8/3} d\kappa \exp\left[\frac{-\kappa^2 L \xi}{k} (\Lambda_N \xi)\right] L_m \left[\frac{i|\kappa L \xi|^2}{2kL(1+i\alpha_N L)}\right] L_p \left[\frac{-i|\kappa L \xi|^2}{2kL(1-i\alpha_N L)}\right], \quad (35)$$

and

$$E_3(0,0) = \frac{-4\pi^2(0.033)k^2 C_n^2}{N} \sum_{n=0}^N B_n \sum_{q=0}^N B_q \int_0^1 L d\xi \int_0^\infty \kappa^{-8/3} d\kappa \exp\left[\frac{\kappa^2 L \xi [1 - (\bar{\Theta}_N + i\Lambda_N)\xi]}{ik}\right] L_m \left[\frac{i|\kappa^2 L \xi|^2}{2k(1+i\alpha_N L)}\right] L_p \left[\frac{i|\kappa^2 L \xi|^2}{2k(1+i\alpha_N L)}\right]. \quad (36)$$

On evaluation of the remaining integrals in Eqs. (35) and (36), the scintillation index in Eq. (30) takes the form

$$\begin{aligned} \sigma_I^2(L) = 0.397\sigma_R^2 \operatorname{Re} & \left[\frac{\Lambda_N^{5/6}}{N} \sum_{n=0}^N \sum_{q=0}^N B_m \frac{(-1)^b m! \left[\frac{i}{2\Lambda_N(1+i\Lambda_{N0})} \right]^b}{(b!)^2(m-b)!} \sum_{p=0}^N \sum_{j=0}^p B_p^* \frac{p! \left[\frac{i}{2\Lambda_N(1-i\Lambda_{N0})} \right]^j}{(j!)^2(p-j)!} \Gamma\left(-\frac{5}{6} + j + b\right) \right. \\ & - \frac{(i)^{5/6}}{N} \sum_{n=0}^N \sum_{q=0}^N B_m \frac{m! \left[\frac{-1}{2(1+i\Lambda_{N0})} \right]^b}{(b!)^2(m-b)!} \sum_{p=0}^N \sum_{j=0}^p B_p \frac{p! \left[\frac{-1}{2(1+i\Lambda_{N0})} \right]^j}{(j!)^2(p-j)!} \Gamma\left(-\frac{5}{6} + j + b\right) \frac{16}{6b+6j+11} \\ & \left. \times {}_2F_1\left(-\frac{5}{6} + j + b, \frac{11}{6} + j + b; \frac{17}{6} + j + b; \bar{\Theta}_N + i\Lambda_N\right) \right], \quad (37) \end{aligned}$$

in which ${}_2F_1(a, b; c; x)$ is the hypergeometric function and $\sigma_R^2 = 1.23C_n^2 k^{7/6} L^{11/6}$ is the Rytov variance. The case $N=0$ correctly reduces Eq. (37) to the on-axis scintillation index for the standard Gaussian beam:²⁰

$$\sigma_I^2(L) = 3.86\sigma_R^2 \times \operatorname{Re} \left[i^{5/6} {}_2F_1\left(-\frac{5}{6}, \frac{11}{6}, \frac{17}{6}, \bar{\Theta} + i\Lambda\right) - \frac{11}{16}\Lambda^{5/6} \right]. \quad (38)$$

3.1 Graphical Results

A comparison of the scintillation index in Eq. (37) with that of the lowest-order TEM₀₀ Gaussian beam in Eq. (38) is shown in Figs. 1–4 under conditions of weak irradiance

fluctuations. The wavelength of the optical wave in all cases is 1.06 μm , and C_n^2 is set at $10^{-14} \text{ m}^{-2/3}$ in Figs. 1 and 2 and $10^{-15} \text{ m}^{-2/3}$ in Figs. 3 and 4. The cases for the FGB corresponding to $N=5$ and $N=10$ are selected in our analysis as typical to illustrate the general trend for increasing values of N .

The outstanding feature in these graphs is the presence of a “bump” in the scintillation index of the FGB, which exceeds the scintillation index of a TEM₀₀ Gaussian beam (solid curve). Moreover, the bump occurs with greater magnitude for successively larger values of N and at greater distances from the transmitter with increasing beam size at the transmitter. The reason for the bump is that the mean on-axis irradiance takes on a minimum value at the range where the bump occurs, leading to higher values of the scintillation index. For an initial beam radius of 1 cm or

Fig. 6 A comparison of theoretical and simulated data for the FGBs of order $N=5$ and $N=10$ with $W_0=5$ cm, $\lambda=1.06$ μm , and $C_n^2=10^{-15}$ $\text{m}^{-2/3}$.

less, the bump is quite small and there is little difference between the scintillation index of the FGB and that of the TEM_{00} Gaussian beam over the whole propagation path, as shown in Fig. 1. For spot radius at the transmitter of 10 cm or larger (see Fig. 4), the peak of the bump is of considerable magnitude and occurs far from the transmitter. In this latter case, the location of the bump is actually beyond the boundary between weak and strong fluctuations (approximately 6 km). At distances beyond the bump, the general trend of the FGB scintillation index in Figs. 1–3 is that it falls below that of the TEM_{00} Gaussian beam, but, through gradually increasing in magnitude with propagation distance, it will again rise above that of the TEM_{00} Gaussian beam. Similar results as shown here were previously reported in the numerical treatment of the scintillation index by Baykal and Eyyuboglu.¹⁶

3.2 Simulation Results

In this section we compare the results of the scintillation theory derived earlier with a numerical simulation of FGBs propagated through the atmosphere.²¹ The simulation results were obtained from a wave-optics code based on the split-step technique, in which the light waves are propagated in the transformed domain and the effect of atmospheric turbulence along the propagation path is simulated by a series of uncorrelated random phase screens.

The simulation results as a function of propagation distance are plotted in Figs. 5–7 along with theoretical results obtained from Eq. (37). Once again, the optical wavelength is 1.06 μm , and the refractive index structure constant C_n^2 is set at 10^{-14} $\text{m}^{-2/3}$ in Fig. 5 and 10^{-15} $\text{m}^{-2/3}$ in Figs. 6 and 7. These results demonstrate that the theoretical scintillation index in Eq. (37) tracks that obtained from the simulated

Fig. 7 A comparison of theoretical and simulated data for the FGBs of order $N=5$ and $N=10$ with $W_0=10 \text{ cm}$, $\lambda=1.06 \mu\text{m}$, and $C_n^2=10^{-15} \text{ m}^{-2/3}$.

data quite well. The best agreement between theory and simulation results was in the case of the 5-cm beam (Fig. 6). Excellent agreement was also achieved in the case of the 10-cm beam for the first 5 km (see Fig. 7), at which point the realm of strong irradiance fluctuations was reached when the scintillation index attained a value of unity.

3.3 Probability of Fade

Application of a FSO system is contingent on the ability of the receiver to properly detect a signal, which can be severely hampered by scintillation. In particular, we are interested in determining the probability that the power at the receiver will fall below a specified threshold, known as the fade threshold. We can determine this probability mathematically through the relation

$$\Pr(I \leq I_T) = \int_0^{I_T} p_I(I) dI, \quad (39)$$

where $p_I(I)$ is the probability density function (PDF) of the irradiance and I_T is the fade threshold.

Because this work is restricted to weak scintillation conditions, we use the traditionally accepted lognormal model of the PDF for our comparisons. For this model, the probability of fade becomes

$$\Pr(I \leq I_T) = \frac{1}{2} \left\{ 1 + \operatorname{erf} \left[\frac{1/2 \sigma_I^2(L) - 0.23 F_T}{\sqrt{2} \sigma_I(L)} \right] \right\}, \quad (40)$$

in which $\operatorname{erf}(x)$ refers to the error function and F_T is the fade threshold parameter defined by

Fig. 8 Probability of fade as a function of the fade threshold level F_T for a 1-cm beam.

$$F_T = 10 \log_{10} \left[\frac{\langle I(0, L) \rangle}{I_T} \right]. \quad (41)$$

The fade threshold parameter is measured in decibels and represents the decibel level below the on-axis mean irradiance at which the threshold I_T is set.

In Fig. 8 there appears to be no reduction in the probability of fade for a FGB over that of a TEM_{00} Gaussian beam of this size at any range. This is not surprising, because for a small beam (beam radius 1 cm or less), the scintillation index for a FGB with $N=5$ or $N=10$ was nearly the same over the whole propagation path as that of the TEM_{00} Gaussian beam (see Fig. 1). However, in Fig. 9 we see a lowered probability of fade for a 3-cm FGB of orders 5 and 10 over that of a TEM_{00} Gaussian beam. This improvement shown in Fig. 9 for the fade probability of a

flattened beam over that of the standard Gaussian beam is significant and consistent with the scintillation results shown in Fig. 2. In Fig. 10 we compare the probability of fade for a 5-cm FGB at 5 km from the transmitter with that of the same size TEM_{00} Gaussian beam. As in the case of the 3-cm beam, there is significant improvement in the fade probability for the flattened Gaussian beam over that of the fundamental mode. Once again, this is consistent with the scintillation results shown in Fig. 3. For a 10-cm FGB, the initial jump in the scintillation index was comparatively large and occurred far enough from the transmitter that we did not see it drop below that of the Gaussian beam before the weak/strong fluctuation boundary was reached. Consequently, we expect no advantage in fade probability under weak fluctuation conditions for a FGB of this size.

Fig. 9 Probability of fade as a function of the fade threshold level F_T for a 3-cm beam.

Fig. 10 The probability of fade as a function of fade threshold level F_T for a 5-cm beam.

4 Spot Size and Beam Wander

It is known that beam wander for a TEM_{00} Gaussian beam depends almost exclusively on the spot size of the beam at the transmitter.²⁰ Consequently, to determine the effect of beam wander, we first examine the spot size of the beam. The spot size of the beam W_0 given in Figs. 1–10 is not the actual spot radius of a FGB at the transmitter, but rather the beam spot radius associated with a TEM_{00} Gaussian beam. The actual spot radius of the FGB is nonetheless closely related to this parameter.

It has been shown in several studies of the FGB intensity distribution profile that the initial flattened beam shape transforms into a circular ring as the FGB propagates in free space and also in turbulence, but gradually reverts to a circular Gaussian profile in the far field.^{5,9,18} To obtain an

estimate of spot size radius σ_s for the FGB, we use the definition suggested by Carter²² for higher-order Hermite-Gaussian beams, namely,

$$\sigma_s^2 = \frac{2 \int_0^\infty |U_N(r)|^2 r^3 dr}{\int_0^\infty |U_N(r)|^2 r dr} = \frac{W_0^2 \sum_{m=0}^N \frac{1}{m! 2^m} \sum_{p=0}^N \frac{1}{p! 2^p} \Gamma(m+p+2)}{N+1 \sum_{m=0}^N \frac{1}{m! 2^m} \sum_{p=0}^N \frac{1}{p! 2^p} \Gamma(m+p+1)}, \quad (42)$$

where W_0 is the spot size at the transmitter of the fundamental mode and $\Gamma(x)$ is the gamma function.

Fig. 11 A comparison of computer simulated beam wander statistics to those developed for theory.

The spot size deduced from Eq. (42) reduces to $\sigma_s = W_0$ for a beam of order $N=0$. However, it is simply a double finite sum for all other orders of FGB that will yield a multiple of the spot size W_0 of the TEM_{00} Gaussian beam at the transmitter. In particular, given beams of order 5 and 10, we have $\sigma_s = 0.9356W_0$ and $\sigma_s = 0.9427W_0$, respectively, just slightly smaller than W_0 . Similar results arise for other orders so that, for many purposes, it suffices to use the estimate $\sigma_s \approx W_0$ for all N .

As a result of the nearly identical spot sizes between the flattened and the standard Gaussian beams, we do not expect to see a significant difference in beam wander between them. In fact, a computer simulation of beam wander under the same conditions as the scintillation index revealed that, for indices $N=5$ and $N=10$, beam wander was virtually identical to that of the fundamental mode. These results for the centroid displacement are shown in Fig. 11 as a function of propagation distance for a 3-cm beam. Also shown is the theoretical result for a TEM_{00} beam obtained from^{23,24}

$$\langle \beta_c^2 \rangle = 1.36L^3 C_n^2 W_0^{-1/3}. \quad (43)$$

5 Concluding Remarks

The theoretical results presented for the scintillation index are based on weak scintillation statistics. Depending on beam size, order of the FGB, and propagation path length, the basic characteristics of the FGB scintillation index may show somewhat different behavior as compared with a TEM_{00} Gaussian beam. Consequently, we believe that the use of a FGB could have some benefit over the TEM_{00} Gaussian beam in a FSO communication system, for example, under certain restrictions on the beam radius and propagation path. Beams with spot radius of 1 cm or smaller and $N \leq 10$ do not show any improvement over the fundamental mode in either scintillation or probability of fade. We believe the same may be true for larger values of N . Also, beams with radii 10 cm or larger show scintillation values that are greater than that of the standard Gaussian beam under weak scintillation conditions. Where the beam radius is either 3 or 5 cm, the significant reduction in the scintillation index translates to a lower probability of fade for the flattened beam. It should be noted that the probability of fade determined in this work is based on the traditionally accepted lognormal model, which has been shown to be optimistic when compared to actual field results. However, the model is useful here for comparison purposes. Last, no difference is found between the FGB and the standard Gaussian beam in the case of beam wander when the two types of beam are of comparable size at the transmitter.

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