Finite Element Analysis Of Left-handed Waveguides

Balasubramaniam, Vellakkinar

University of Central Florida

Find similar works at: https://stars.library.ucf.edu/etd

This Masters Thesis (Open Access) is brought to you for free and open access by STARS. It has been accepted for inclusion in Electronic Theses and Dissertations, 2004-2019 by an authorized administrator of STARS. For more information, please contact STARS@ucf.edu.

STARS Citation
https://stars.library.ucf.edu/etd/256
FINITE ELEMENT ANALYSIS OF LEFT-HANDED WAVEGUIDES

by

SATISH VELLAKKINAR BALASUBRAMANIAM
B.E. Kumaraguru College of Technology 2000

A thesis submitted in partial fulfillment of the requirements
for the degree of Master of Science
in the Department of Electrical and Computer Engineering
in the College of Engineering and Computer Science
at the University of Central Florida
Orlando, Florida

Fall Term
2004
ABSTRACT

In this work, waveguides with simultaneous negative dielectric permittivity and magnetic permeability, otherwise known as left-handed waveguides, are investigated. An approach of formulating and solving an eigenvalue problem with finite element method resulting in the dispersion relation of the waveguides is adopted in the analysis. Detailed methodology of one-dimensional scalar and two-dimensional vector finite element formulation for the analysis of grounded slab and arbitrary shaped waveguides is presented. Based on the analysis, for waveguides with conventional media, excellent agreement of results is observed between the finite element approach and the traditional approach. The method is then applied to analyze left-handed waveguides and anomalous dispersion of modes is found. The discontinuity structure of a left-handed waveguide sandwiched between two conventional dielectric slab waveguides is analyzed using mode matching technique and the results are discussed based on the inherent nature of the materials. The scattering characteristics of a parallel plate waveguide partially filled with left-handed and conventional media are also analyzed using finite element method with eigenfunction expansion technique.
ACKNOWLEDGEMENTS

First and foremost, I wish to express my deepest gratitude to my advisor, Dr. Thomas X. Wu, for his guidance and valuable advice for the completion of this work. His unabated enthusiasm, encouragement and support have been continually felt and are greatly appreciated.

I would like to thank Dr. Parveen F. Wahid and Dr. Kalpathy B. Sundaram for serving on my committee and for sharing with me their expertise and insight.

Special thanks to my parents for their love, wisdom and patience.

I would also like to thank my friends and colleagues who, one time or another, offered me their much appreciated help. These include Hao Dong, Liping Zheng, and all other friends in the Applied Electromagnetics Group at University of Central Florida.
# TABLE OF CONTENTS

**LIST OF FIGURES** .......................................................................................................................... vi

**CHAPTER 1 : INTRODUCTION** ............................................................................................................. 1

1.1 Overview ........................................................................................................................................ 1

1.2 Left Handed Media: A Review ........................................................................................................... 2

1.3 Computational Methods ....................................................................................................................... 6

1.3.1 Finite Element Method .................................................................................................................... 7

1.3.2 Mode Matching Technique ............................................................................................................. 8

1.4 Thesis Organization ............................................................................................................................ 10

**CHAPTER 2 : ANALYSIS OF GROUNDED LH SLAB WAVEGUIDE** ............................................. 12

2.1 Overview ........................................................................................................................................ 12

2.2 Grounded LH Slab Waveguide ......................................................................................................... 12

2.3 Finite Element Analysis .................................................................................................................... 14

2.4 Results and Discussion ...................................................................................................................... 19

2.4.1 TM modes in a conventional grounded slab waveguide ............................................................... 19

2.4.2 TE modes in a conventional grounded slab waveguide ............................................................... 20

2.4.3 TM modes in a grounded LH slab waveguide ............................................................................ 21

2.4.4 TE modes in a grounded LH slab waveguide ............................................................................ 23

**CHAPTER 3 : ANALYSIS OF DISCONTINUITIES IN LH SLAB WAVEGUIDE** ............................ 27

3.1 Overview ........................................................................................................................................ 27

3.2 Scattering Characteristics of Discontinuities in LH Slab Waveguide ............................................... 27

3.3 Results and Discussion ...................................................................................................................... 34
LIST OF FIGURES

Figure 1.1: Wave propagation characteristics in (a) DPS medium and (b) DNG medium ........... 3

Figure 2.1: Geometry of grounded LH slab waveguide ......................................................... 13

Figure 2.2: Grounded LH slab with an electrical wall ............................................................. 14

Figure 2.3: Dispersion diagram of TM modes in a conventional grounded slab waveguide with \( \varepsilon_r = 2.55 \) and \( \mu_r = 1 \) ................................................................................ 19

Figure 2.4: Dispersion diagram of TE modes in a conventional grounded slab waveguide with \( \varepsilon_r = 2.55 \) and \( \mu_r = 1 \) ................................................................................ 20

Figure 2.5: Dispersion diagram of TM modes in a grounded LH slab waveguide
with \( \varepsilon_r = -2.55 \) and \( \mu_r = -1 \) ................................................................................ 22

Figure 2.6: Dispersion diagram of TE modes in a grounded LH slab waveguide
with \( \varepsilon_r = -2.55 \) and \( \mu_r = -1 \) ................................................................................ 24

Figure 2.7: Dispersion curve of TE mode in a grounded LH slab waveguide
with \( \varepsilon_r = -2.55 \) and \( \mu_r = -1 \) ................................................................................ 25

Figure 3.1: Grounded LH slab waveguide sandwiched between two conventional grounded slab waveguides ........................................................................................................... 28

Figure 3.2: Discontinuity structure due to symmetry ................................................................. 29

Figure 3.3: Equivalent transmission line model of the symmetrical discontinuity structure..... 29

Figure 3.4: Percentage of reflected power of TM\(_0\) mode for \( d = 0.3 \lambda_0 \) waveguide thickness ................................................................. 34

Figure 3.5: Percentage of transmitted power of TM\(_0\) mode for \( d = 0.3 \lambda_0 \) waveguide thickness ................................................................. 35

Figure 3.6: Percentage of reflected power of TM\(_0\) mode for \( d = 0.31938 \lambda_0 \) waveguide thickness ................................................................. 36

Figure 3.7: Percentage of transmitted power of TM\(_0\) mode for \( d = 0.31938 \lambda_0 \) waveguide thickness ................................................................. 37
Figure 3.8: Percentage of reflected power of TM₀ mode for d = 0.35 \(\lambda₀\) waveguide thickness .......................................................... 38

Figure 3.9: Percentage of transmitted power of TM₀ mode for d = 0.35 \(\lambda₀\) waveguide thickness .......................................................... 38

Figure 3.10: Percentage of reflected power of TM₀ mode for d = 0.6 \(\lambda₀\) waveguide thickness .......................................................... 39

Figure 3.11: Percentage of transmitted power of TM₀ mode for d = 0.6 \(\lambda₀\) waveguide thickness .......................................................... 40

Figure 3.12: Percentage of reflected power of TM₁ mode for d = 0.6 \(\lambda₀\) waveguide thickness .......................................................... 40

Figure 3.13: Percentage of transmitted power of TM₁ mode for d = 0.6 \(\lambda₀\) waveguide thickness .......................................................... 41

Figure 4.1: Schematic of a parallel plate waveguide partially filled with LH media .................. 43

Figure 4.2: Parallel plate waveguide partially filled with LH media and with fictitious boundaries ........................................................................................................... 44

Figure 4.3: Transmitted power of a parallel plate waveguide partially filled with conventional media ........................................................................................................... 50

Figure 4.4: Reflected power of a parallel plate waveguide partially filled with conventional media ........................................................................................................... 50

Figure 4.5: Transmitted power of a parallel plate waveguide partially filled with LH media .... 51

Figure 4.6: Reflected power of a parallel plate waveguide partially filled with LH media ....... 52

Figure 5.1: Arbitrary shaped LH waveguide with electrical wall ........................................... 54

Figure 5.2: Configuration of (a) tangential edge elements (b) node elements ......................... 56

Figure 5.3: Schematic of a LH waveguide with circular cross section ..................................... 60

Figure 5.4: Circular LH waveguide with electrical wall ........................................................ 60

Figure 5.5: Dispersion diagram of conventional waveguide with circular cross section .......... 62

Figure 5.6: Dispersion diagram of LH waveguide with circular cross section ....................... 63
Figure 5.7: Dispersion curve of $\text{HE}_1$ mode in a LH waveguide with circular cross section .... 64

Figure 5.8: Dispersion curve of $\text{TE}_{01}$ and $\text{TE}_{01}$ mode in a LH waveguide with circular cross section............................................................................................................. 65

Figure 5.9: Dispersion curve of $\text{TM}_{01}$ and $\text{TM}_{01}$ mode in a LH waveguide with circular cross section............................................................................................................. 67
CHAPTER 1 : INTRODUCTION

1.1 Overview

In recent years, the study of electromagnetic properties of complex materials with simultaneous negative real permittivity and permeability has attracted a lot of attention in research. These media, typically referred as left-handed (LH) media, possess interesting features that may lead to unconventional phenomena in guidance, radiation, and scattering of electromagnetic waves. Many research groups are now exploring various aspects of this class of complex media and several potential future applications have been speculated. Recent research advancements have proved the feasibility of realizing these artificial materials, which were considered hypothetical. The novel and interesting features of these materials and their possible applications to future design of new devices are the primary motivation of this research work.

In this thesis, a comprehensive finite element analysis of left-handed waveguides is provided. It is also the first study of arbitrary shaped LH waveguide. Instead of the traditional approach of solving the waveguide characteristic equations, a different approach of using finite element method is followed to get the dispersion characteristics of the waveguides. This approach takes advantage of the inherent ability of the finite element method to solve an eigenvalue problem for all possible eigenvalues. In this approach an eigenvalue problem is formulated for the structure to be analyzed and then depending on the geometry of the structure, a one-dimensional or two-dimensional finite element program is developed to solve the problem. Based on this methodology, the grounded slab waveguide with conventional media is analyzed using one-dimensional scalar finite element formulation and the results are compared with those
obtained from the traditional method and an excellent agreement is found. Then the grounded LH slab waveguide is analyzed and the anomalous dispersion characteristics are obtained. A discontinuity structure, with a LH slab waveguide sandwiched between two dielectric slab waveguides, is investigated using the mode matching technique and the physical insights of the wave interaction between the two media are presented. The scattering characteristics of a parallel plate waveguide partially filled with LH and conventional media are analyzed using finite element method with eigenfunction expansion technique. A robust, two-dimensional vector finite element formulation for the analysis of arbitrary shaped waveguide structures is presented in detail. The method is then applied to analyze conventional and LH waveguides with a circular cross section. From the analysis, there is excellent agreement of the results for the conventional waveguide and anomalous dispersion is noted in LH waveguides. The anomalous dispersion characteristics of LH waveguides are compared in association with the conventional waveguides and the physical insights are discussed based on the inherent nature of the materials.

1.2 Left Handed Media: A Review

Electromagnetic properties of materials can be characterized by dielectric permittivity ($\varepsilon$) and magnetic permeability ($\mu$). Propagation properties of electromagnetic waves in a material are determined by $\varepsilon$ and $\mu$; which regulate the relationship between the electric field and magnetic field. Wave propagation in a hypothetical material with simultaneous negative dielectric permittivity and magnetic permeability was first envisioned in the 1960’s [1] by the Russian physicist Veselago. He predicted that electromagnetic wave propagation in these media should give rise to several peculiar characteristics. According to Maxwell’s equations, the direction of energy flow of a plane wave is given by the direction of the Poynting vector ($S$), which is the
cross product of electric field \( \mathbf{E} \) and magnetic field \( \mathbf{H} \). For plane waves propagating in isotropic regular media having simultaneously positive \( \varepsilon \) and \( \mu \) (double positive (DPS) medium), the cross product of electric field \( \mathbf{E} \) and magnetic field \( \mathbf{H} \) gives both the direction of energy flow (the Poynting vector) and the wave itself (that is, the phase velocity, or wave vector), and \( \mathbf{E} \), \( \mathbf{H} \) and wave vector \( \mathbf{k} \) form a right-handed triplet of vectors. But Veselago predicted that in a medium having simultaneously negative \( \varepsilon \) and \( \mu \) (DNG medium), while \( \mathbf{E} \times \mathbf{H} \) for a plane wave still gives the direction of energy flow, the phase velocity (wave vector) shall be in the opposite direction of energy flow, and \( \mathbf{E} \), \( \mathbf{H} \) and wave vector \( \mathbf{k} \) shall form a left-handed triplet of vectors.

Figure 1.1 (a) and (b) shows the “right-handed” wave propagation behavior in a DPS medium and the “left-handed” wave propagation behavior in a DNG medium, respectively.

Figure 1.1: Wave propagation characteristics in (a) DPS medium and (b) DNG medium

Due to this left-handed characteristic, Veselago termed such type of materials as left-handed medium (LHM), and all regular materials were correspondingly termed right-handed
medium (RHM). In addition to this left-handed characteristic, LHM materials have several other dramatically different electrodynamic properties compared with regular materials, stemming from a simultaneous change of the signs of $\varepsilon$ and $\mu$, including anomalous refraction, reversal of both the Doppler shift and the Cherenkov radiation, and reversal of radiation pressure to radiation tension [1]. Although these counterintuitive properties follow directly from Maxwell’s equations, due to the absence of naturally occurring materials having simultaneously negative $\varepsilon$ and $\mu$, Veselago’s prediction was considered as a theoretical concept and there has been little effort to better understand the electromagnetic behaviors of these materials for almost three decades. For LH materials, several other names and terminologies have also been suggested, such as media with negative refractive index, backward wave media (BW media), double negative (DNG) media, and negative index media (NIM), to name a few.

The recent resurgence of interest in this medium began when Pendry suggested its first application [2]; he proposed the possibility that a “perfect” lens can be made by using LH medium which with a negative index of refraction might overcome known problems with common lenses. Ziolkowski investigated the propagation of electromagnetic waves in LH media from both analytical and numerical points of view. His analytical solution for a matched LH slab demonstrated that the Pendry “perfect lens” effect could be realized only in the presence of a nondispersive, lossless LH medium [3]. The lens effect was shown not to exist for any realistic dispersive, lossy LH medium. Inspired by the work of Pendry, Smith, Schultz and Shelby constructed such a composite medium by arranging arrays of small metallic wires and split ring resonators [4, 5]. This discovery aroused great interest in the unusual electrodynamic properties of LHM. Ziolkowski considered LH materials comprised of a substrate with embedded
capacitively loaded strips and split ring resonators and demonstrated that LH material can be
designed, fabricated, and tested with microwave engineering tools [6].

Engheta proposed the LH waveguide and analyzed the dispersion diagram of LH slab
waveguide, and found that the portion of the guided mode inside the slab has the Poynting vector
anti-parallel to the direction of phase flow of the mode and the portion of guided mode outside
the slab has the Poynting vector parallel with phase flow [7]. Alu and Engheta investigated the
mode coupling between a conventional slab waveguide placed next to a LH slab waveguide and
demonstrated the anti-directional coupling nature exhibited between the layers [8]. They also
analyzed various properties of the guided modes in parallel plate waveguide filled with pairs of
layers made of any two of the lossless epsilon-negative, mu-negative, DPS, and DNG materials
[9]. In the analysis, they used the characteristic equation of the waveguide to get the dispersion
relation of different material combinations [9]. Based on the analysis, they suggested some
potential applications in the design of novel devices and components, such as ultra-thin
waveguides, single-mode thick fibers with less restriction and more flexibility on the fiber
thickness, and very thin cavity resonators.

The analysis of guided modes in a LH grounded slab and its possible application as novel
substrates for microstrip antennas and arrays is presented in [10]. Ziolkowski and Kipple
investigated an electrically small dipole antenna with a shell of LH material and found that
properly designed dipole-LH shell combination increases the real power radiated by more than
an order of magnitude over the corresponding free space case [11]. Based on the waveguide
characteristics equation, Nefedov and Tretyakov discussed the influence of layer thicknesses and
material parameters over the waveguide propagation characteristics of various modes [12]. From
the recent research work it looks very promising that these LH materials will have a strong
impact on the technological world. Much research exploring the exotic properties of these materials is underway and many future applications have been speculated. However, the dispersion characteristics of arbitrary shaped LH waveguides have not been analyzed to date.

1.3 Computational Methods

Electromagnetic analysis in many engineering and scientific discipline is based on the electromagnetic principles governed by the well known set of equations formulated by James Clerk Maxwell in 1873. Electromagnetic field problems arise in many areas such as electrical machines, communication systems and electronics, the complex geometry and high accuracy requirements of these problems make it inevitable to seek numerical solution. Many highly sophisticated numerical methods have been developed over the years and new techniques are always being introduced. A computational electromagnetic method may be viewed as a computational algorithm capable of solving Maxwell’s equations subject to appropriate boundary constraints in a general configuration and in the presence of materials with different properties.

Currently available computational methods include the Finite Element Method (FEM) [13, 14], Finite Difference Time Domain method (FDTD), Mode Matching Technique (MMT), Transmission Line Method (TLM) and Method of Moments (MoM), Monte Carlo Method (MCM), Method of Lines (MOL) [14]. These methods are used to solve problems that are represented by means of differential equations and other mathematical forms. The innumerable research publications and the number of software packages verify the importance of these techniques in solving variety of problems.
1.3.1 Finite Element Method

The origin of Finite Element Method (FEM) dates back to 1943 when Courant published his work manuscript of the address he delivered to the American Mathematical Society [15]. In its primitive stage of development, FEM was used as a structural analysis tool to help aerospace engineers in designing better aircraft structures. Later on, aided by the rapid increase of computer power, the method has been continually developed until it became a very sophisticated generic tool for accomplishing a wide array of engineering tasks. Its development and success is not paralleled by any other numerical analysis technique. The technique is based on the premise that an approximate solution to any complex engineering problem can be reached by subdividing the problem into smaller more manageable (finite) elements. Using finite elements, solving complex partial differential equations that describe the behavior of certain structures can be reduced to a set of linear equations that can easily be solved using the standard techniques of matrix algebra.

Although the earlier mathematical treatment of the FEM was provided in 1943, the method was not applied to electromagnetic problems until 1968. Since then the method has gained importance and the systematic generality of the method makes it possible to construct general purpose computer programs for solving a wide range of problems in diverse areas such as waveguide problems, electric machines, microstrips and semiconductor devices.

In FEM, according to the geometry of the problem, the entire structure domain is divided into several subdomains, called elements. Then the unknown functions in each element are expanded by the nodal values (which are values of the functions in some particular points (nodes or edges) of the element), and the corresponding shape functions (interpolation functions) [13]. Thus the original problem with an infinite number of degrees of freedom is converted into a
problem with finite number of degrees of freedom. Then using Rayleigh-Ritz or Galerkin procedure [13, 16], a system of algebraic equations is formed from which a submatrix of each element can be obtained. Each time when a submatrix of a new element is obtained it is assembled with the existing system matrix. Finally, when all the elements are called in one gets a system matrix equation from which the nodal or edge values are solved.

Since its first application to classical electromagnetic problem of guided propagation [17] many improvements have enhanced the accuracy, geometrical modeling capability and efficiency of FEM. Also when combined with other techniques (like Green’s function, eigenfunction expansion method, absorbing boundary condition) FEM can, in general, tackle arbitrary geometries with inhomogeneous, anisotropic, and/or lossy medium. Consequently, computer programs developed for a particular discipline have been applied successfully to solve problems in a different field with little or no modification. With the sparsity of the coefficient matrices, FEM exhibits the rather pleasing characteristic of computational economy in numerical modeling. The success of FEM in electromagnetics can be largely attributed to their great versatility and flexibility, which allow the treatment of geometrically complex structures with inhomogeneous anisotropic or even nonlinear materials.

1.3.2 Mode Matching Technique

The concept of matching modes in electromagnetic analysis was first proposed by Wexler in the 1967 [18]. The usefulness and accuracy of his method assured it a great deal of attention from researchers since that time [19, 20]. Due to the limited computer power available at that time it was not possible to do more than simple numerical computations. The computational
emphasis was on reducing the number of modes to the minimum so that a numerical solution could be obtained. It was the arrival of powerful computers that enabled the concept to be applied to the analysis of complicated structures. Mode Matching Technique (MMT) is a powerful method for the analysis and design of many electromagnetic components and devices. The essence of the method is to divide the complex structure to be analyzed into small sections and to match the total mode fields at each junction between sections.

In analyzing a structure using MMT, the field in the structure is expanded in a complete set of vector wave functions, these functions are usually denoted as *modes*. The modes which can propagate in each section, including in some cases evanescent modes, are set up and the amplitudes of the modes are matched across the boundary of the section to the next section with appropriate treatment of the boundary conditions. The amplitudes of modes at the output of a junction can be deduced in terms of the amplitudes of the mode spectrum at the input to the junction. The strength of MMT stems from the fact that the amplitudes of the modes can be expressed as the components of a scattering matrix. Each junction along the structure has its own scattering matrix. The matrices for all junctions can be cascaded and results in an overall scattering matrix for the structure. The process of computing the overall scattering matrix can be decoupled to obtain the scattering matrix of particular section. Due to its numerical efficiency and robustness, mode matching analysis has been widely employed for designing microwave components.
1.4 Thesis Organization

In this thesis, we report the modeling and analysis of LH waveguides using FEM and MMT. In order to emphasize the unusual and exotic properties of LH media, we compare them with the conventional media as well. The technical analyses are contained in Chapter 2 through Chapter 5 and their associated appendices.

In chapter 2, we illustrate the methodology to frame an eigenvalue problem for the grounded LH slab waveguide and the one-dimensional scalar (node based) finite element formulation in detail. We report the dispersion characteristics of the modes for the conventional grounded slab in order to validate the accuracy of the finite element method. Then the anomalous dispersion characteristics of the modes in a grounded LH slab waveguide are reported. A detailed discussion of the dispersion characteristics of both the waveguides is provided and the unusual behavior of LH waveguides is elaborated.

In chapter 3, we provide the mode matching analysis of the discontinuity structure of a LH waveguide sandwiched between two dielectric slab waveguides. Considering the symmetrical properties of the structure, the scattering characteristics of discontinuities in the longitudinal direction are investigated using equivalent transmission line model. Based on this analysis, four cases with different waveguide heights are discussed and relevant physical insights are presented.

In chapter 4, the eigenfunction expansion technique in FEM is explained and the methodology to solve unbounded field problems is illustrated. Based on this technique, the scattering characteristics of a parallel plate waveguide partially filled with LH and conventional media are analyzed.
In chapter 5, we provide the in-depth details of the two dimensional vector finite element formulation for the analysis of arbitrary shaped waveguide structures. The method is then applied to analyze conventional and LH waveguides with a circular cross section. From the analysis, the anomalous dispersion characteristics of LH waveguides are compared in association with the conventional waveguides and the physical insights are discussed based on the inherent nature of the materials.

A summary of conclusions of this research work is given in chapter 6.
CHAPTER 2: ANALYSIS OF GROUNDED LH SLAB WAVEGUIDE

2.1 Overview

The main concern of this research is to analyze the dispersion characteristics of LH waveguides, otherwise known as DNG waveguides. This chapter provides an outline on grounded LH slab waveguide and the one-dimensional scalar finite element approach to obtain its dispersion characteristics. Based on finite element analysis, the dispersion characteristics for conventional grounded slab waveguide and grounded LH slab waveguide are compared. The anomalous dispersion in LH waveguide is discussed in detail.

2.2 Grounded LH Slab Waveguide

Dielectric slabs and rods, with or without any associated metal, are used to contain the energy associated with a wave within a given space and guide it in particular direction. Typically these are referred to as dielectric waveguide, and the field modes that they can support are known as surface wave modes [21]. Surface waves are represented by a field that decays exponentially away from the dielectric surface, with most of the field contained in or near the dielectric. At higher frequencies the field generally becomes more tightly bound to the dielectric, making such waveguides practical [22].

A grounded LH slab waveguide is a dielectric type of waveguide that has a ground plane covered with a dielectric slab of height “d”, as shown in Figure 2.1, and negative relative permittivity $\varepsilon_r$ and relative permeability $\mu_r$. The objective of the waveguide to contain the energy within the structure and direct it toward a given direction is accomplished by having the wave
bounce back and forth between its upper and lower interfaces at an incidence angle greater than
the critical angle. When this is accomplished, the refracted fields outside the dielectric form
evanescent (decaying) waves and all the real energy is reflected and contained within the
waveguide. The characteristics of this waveguide can be analyzed by treating the structure as a
boundary-value problem whose modal solution is obtained by solving the wave equation and
enforcing the boundary conditions.

![Diagram of grounded LH slab waveguide](Image)

**Figure 2.1: Geometry of grounded LH slab waveguide**

For a grounded LH slab there exist two sets of distinct surface modes. One set of modes
has no magnetic field component in the propagation direction; these modes are referred as
transverse magnetic (TM) modes and the other set of modes has no electric field component in
the propagation direction; these modes are referred to as transverse electric (TE) modes.
2.3 Finite Element Analysis

For a grounded LH slab, in addition to the guided wave modes, the radiation and evanescent waves comprise a continuous spectrum. We place an electrical wall at a height “$h$” above the structure to discretize the continuous spectrum. For the grounded LH slab waveguide, as shown in Figure 2.2, the TM and TE modes can be considered separately.

![Figure 2.2: Grounded LH slab with an electrical wall](image)

From source free Maxwell’s curl equation, we obtain

$$\nabla \times \mathbf{E} = -j \omega \mu_0 \mu_r \mathbf{H} \tag{2.1a}$$

$$\nabla \times \mathbf{H} = j \omega \varepsilon_0 \varepsilon_r \mathbf{E} \tag{2.1b}$$

where $\varepsilon_r$ and $\mu_r$ are the relative permittivity and permeability.
For the TM modes, we have

\[
\frac{\partial H_y}{\partial z} = -j \omega \varepsilon_r \varepsilon_j E_x \tag{2.2a}
\]

\[
\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} = j \omega \mu_r \mu_j H_y \tag{2.2b}
\]

\[
\frac{\partial H_y}{\partial x} = j \omega \varepsilon_r E_z \tag{2.2c}
\]

From the above equations (2.2a-2.2c) we can get the Helmholtz wave equation for \(H_y\),

\[
\frac{\partial}{\partial x} \left( \frac{1}{\varepsilon_r} \frac{\partial H_y}{\partial x} \right) + \left( k_0^2 \mu_r - \frac{k_z^2}{\varepsilon_r} \right) H_y = 0 \tag{2.3}
\]

where \(k_0\) is the free space wavenumber.

Similarly for the TE modes, we have

\[
\frac{\partial E_x}{\partial z} = j \omega \mu_r \mu_j H_z \tag{2.4a}
\]

\[
\frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z} = -j \omega \varepsilon_r \varepsilon_j E_y \tag{2.4b}
\]

\[
\frac{\partial E_y}{\partial x} = -j \omega \mu_r \mu_j H_z \tag{2.4c}
\]

From the above equations (2.4a-2.4c), the Helmholtz wave equation for \(E_y\) can be deduced as,

\[
\frac{\partial}{\partial x} \left( \frac{1}{\mu_r} \frac{\partial E_y}{\partial x} \right) + \left( k_0^2 \varepsilon_r - \frac{k_z^2}{\mu_r} \right) E_y = 0 \tag{2.5}
\]

Equations (2.3) and (2.5) have the form of the generalized eigenvalue problem. The dispersion characteristics of the grounded LH slab waveguide can be obtained by solving these eigenvalue problems. This class of problems can be dealt with using the finite element method, and the resultant system of equations has the form of the generalized eigenvalue equation.
\[
[A][\phi] - \lambda[B][\phi] = \{0\} \quad (2.6)
\]

where \([A]\) and \([B]\) are known matrices and \(\lambda\) and \(\{\phi\}\) are unknowns. In the finite element method we will solve for the eigenvalue \(\lambda\), which makes the system singular, or in other words, which makes the determinant of \([A-\lambda B]\) vanish. As a result, there will be a corresponding nontrivial solution for \(\{\phi\}\) which is called eigenvector.

The first step and most important step in the finite element analysis is the discretization of the domain. In this step the solution domain \(\Omega\), that is \((0, h)\), is subdivided into number of small domains usually referred to elements “\(e\)”. Due to one-dimensional nature of the domain, the elements are short line segments of length “\(l\)” that are interconnected to form the original line. The problem is formulated in terms the unknown function \(\{\phi\}\), that is \(E_y\) or \(H_y\), at two nodes associated with the linear line element. The second step is the selection of an interpolation function that provides an approximation of the unknown solution within each element. For the linear element, a linear interpolation function is selected which are nonzero within the element and vanish outside the element. Once the interpolation function is selected, we can derive an expression for the unknown solution in an element, say element \(e\), in the following form,

\[
\phi^e = \sum_{j=1}^{n} N_j^e \phi_j^e \quad (2.7)
\]

where \(n\) is the number of nodes in the element, \(\phi_j^e\) is the value of \(\phi\) at node \(j\) of the element, and \(N_j^e\) is the interpolation function for node \(j\).

With the expansion of \(\phi\) given in equation (2.7) we can formulate the system of equations using Ritz variational method, in which an equivalent variational problem is
formulated. A detailed procedure on the formulation of the variational problem is illustrated in Appendix A. The corresponding functional for the equations (2.3) and (2.5) is given by

\[ F(\phi) = -\frac{1}{2} \int_0^h \left[ \frac{1}{\varepsilon_r} \left( \frac{\partial \phi}{\partial x} \right)^2 - \left( k_0^2 \mu_r - \frac{k_z^2}{\varepsilon_r} \right) \phi^2 \right] dx \quad \text{for TM mode} \quad (2.8a) \]

\[ F(\phi) = -\frac{1}{2} \int_0^h \left[ \frac{1}{\mu_r} \left( \frac{\partial \phi}{\partial x} \right)^2 - \left( k_0^2 \varepsilon_r - \frac{k_z^2}{\mu_r} \right) \phi^2 \right] dx \quad \text{for TE mode} \quad (2.8b) \]

Without loss of generality the finite element method is explained with TM mode. The functional can be written as,

\[ F(\phi) = \sum_{e=1}^M F^e(\phi^e) \quad (2.9) \]

where \( M \) denotes the total number of elements and \( F^e \) is the subfunctional for the \( e \)th element given by

\[ F^e(\phi^e) = -\frac{1}{2} \int_{\ell_e} \left[ \frac{1}{\varepsilon_r} \left( \frac{\partial \phi^e}{\partial x} \right)^2 - \left( k_0^2 \mu_r - \frac{k_z^2}{\varepsilon_r} \right) \phi^e \right] dx \quad (2.10) \]

where \( \ell_e \) is the length of the \( e \)th element. Introducing expression (2.7) for \( \phi^e \) and differentiating \( F^e \) with respect to \( \phi_i^e \) yields

\[ \frac{\partial F^e}{\partial \phi_i} = \sum_{j=1}^2 \phi_j^e \int_{\ell_e} \left[ -\frac{1}{\varepsilon_r} \left( \frac{\partial N_i^e}{\partial x} \frac{\partial N_j^e}{\partial x} \right) + k_0^2 \mu_r \left( N_i^e N_j^e \right) - \frac{k_z^2}{\varepsilon_r} \left( N_{i}^e N_{j}^e \right) \right] dx \quad i = 1, 2 \quad (2.11) \]

In matrix form, this can be written as

\[ \begin{bmatrix} \frac{\partial F^e}{\partial \phi^1} \\ \frac{\partial F^e}{\partial \phi^2} \end{bmatrix} = \begin{bmatrix} K^e \end{bmatrix} \{ \phi^e \} \quad (2.12) \]

where \( \begin{bmatrix} \frac{\partial F^e}{\partial \phi^1} & \frac{\partial F^e}{\partial \phi^2} \end{bmatrix}^T \) and \( \{ \phi^e \} = \{ \phi_1^e \phi_2^e \}^T \)
Since $k_z^2$ is the unknown that has to be solved, we split $K^e$ into two parts,

$$\begin{align*}
\left\{ \frac{\partial F^e}{\partial \phi^e} \right\} &= [A^e]\{\phi^e\} - k_z^2[B^e]\{\phi^e\} \\
\text{(2.13)}
\end{align*}$$

The elements of the matrices $[A^e]$ and $[B^e]$ are given by

$$\begin{align*}
A_{ij}^e &= \int_f \left[ -\frac{1}{\varepsilon_r} \left( \frac{\partial N_i^e}{\partial x} \frac{\partial N_j^e}{\partial x} \right) + k_0^2 \mu_r (N_i^e N_j^e) \right] dx \\
\text{(2.14a)}
B_{ij}^e &= \int_f \left[ \frac{1}{\varepsilon_r} (N_i^e N_j^e) \right] dx \\
\text{(2.14b)}
\end{align*}$$

The elemental equations (2.13) are assembled for all $M$ elements and the stationary requirement on $F$ is imposed to find the system of equations

$$\begin{align*}
\left\{ \frac{\partial F}{\partial \phi} \right\} &= \sum_{c=1}^M \left\{ \frac{\partial F_c^e}{\partial \phi^e} \right\} = \sum_{c=1}^M \left( [A^e]\{\phi^e\} - k_z^2[B^e]\{\phi^e\} \right) = \{0\} \\
\text{(2.15)}
\end{align*}$$

The above system of equations can be written compactly as,

$$\left[ A \right]\{\phi\} = k_z^2\left[ B \right]\{\phi\}$$

(2.16)
this is recognized as the generalized eigenvalue problem defined in (2.6). Solving the system of equations with appropriate boundary conditions yields the eigenvalues or the longitudinal propagation constants $k_z$. The effective relative dielectric constant $\varepsilon_{re}$ of the LH slab is obtained from the relation

$$\varepsilon_{re} = \left( \frac{k_z}{k_0} \right)^2$$

(2.17)

Similar procedure is followed for TE modes and the dispersion characteristics of grounded LH slab waveguide are obtained.
2.4 Results and Discussion

Based on the finite element formulation described above the dispersion diagrams of TM modes and TE modes in a conventional grounded slab waveguide with $\varepsilon_r = 2.55$ and $\mu_r = 1$ are obtained.

2.4.1 TM modes in a conventional grounded slab waveguide

Figure 2.3: Dispersion diagram of TM modes in a conventional grounded slab waveguide with $\varepsilon_r = 2.55$ and $\mu_r = 1$
From the dispersion diagram, shown in Figure 2.3, it can be seen that for any nonzero thickness slab, with a permittivity greater than unity, there is at least one propagating TM mode. This TM$_0$ mode is the dominant mode of the dielectric slab waveguide and has a zero cutoff frequency. It can be seen that the next TM mode, the TM$_1$ mode, will not propagate until the height of the slab becomes greater $0.4\lambda_0$. Also, all the dispersion curves increase monotonically.

2.4.2 TE modes in a conventional grounded slab waveguide

![Figure 2.4: Dispersion diagram of TE modes in a conventional grounded slab waveguide with $\varepsilon_r = 2.55$ and $\mu_r = 1$](image)

Figure 2.4: Dispersion diagram of TE modes in a conventional grounded slab waveguide with $\varepsilon_r = 2.55$ and $\mu_r = 1$
For TE modes, from the dispersion diagram it can be seen that the first mode does not start to propagate until the height of the slab becomes greater $0.2\lambda_0$. This TE$_1$ mode is the dominant mode of dielectric slab waveguide. All the TE modes in the conventional dielectric slab have cutoff frequencies. For the TE modes the curves of effective dielectric constant $\varepsilon_{re}$ versus normalized thickness $d/\lambda_0$ increase monotonically.

2.4.3 TM modes in a grounded LH slab waveguide

The relative permittivity $\varepsilon_r$ and relative permeability $\mu_r$ are assumed negative values, $\varepsilon_r = -2.55$ and $\mu_r = -1$, and the finite element analysis is carried for grounded LH slab. The dispersion characteristics of TM modes in a grounded LH slab are shown in Figure 2.5. In order to understand the physics of LH media, we compare the dispersion diagram of the conventional grounded slab waveguide and the grounded LH slab waveguide. In the conventional grounded slab waveguide, the TM$_0$ mode has no cutoff frequency and the dispersion curve increase monotonically. On contrast, for the grounded LH slab waveguide all the modes have cutoff frequencies. Moreover, in the dispersion diagram the curve no longer increases monotonically, but is bent in a special region (shaded in Figure 2.5). When the height $(d/\lambda_0)$ of the slab is increased from 0, there is no propagating mode in the grounded LH slab waveguide. When the value of $(d/\lambda_0)$ reaches 0.31938 (point $A_0$ in Figure 2.5), the first TM mode appears. For explanatory purpose, this point where the first mode appears is defined as the critical point $A_0$. At the critical point, the portion of the guided mode inside the grounded LH slab waveguide shows the Poynting vector to be anti-parallel to the direction of the mode’s phase flow and the
portion of this mode outside the slab shows the Poynting vector to be parallel with the phase flow. But the net power is 0 for this case.

Figure 2.5: Dispersion diagram of TM modes in a grounded LH slab waveguide with $\varepsilon_r = -2.55$ and $\mu_r = -1$

With the increase of $(d/\lambda_0)$ beyond the critical point, there exist two TM$_0$ modes in the special region. In this region, the dispersion curve gets bifurcated in two different directions, that is, curve I is decomposed into curve $A_0A_1$ and curve $A_0A_2$. The reason for calling it a special region is as follows. For the curve $A_0A_2$, the portion of the guided mode inside the grounded LH
slab waveguide shows the Poynting vector to be anti-parallel to the direction of the mode’s phase flow and the portion of guided mode outside the slab shows the Poynting vector to be parallel with the phase flow. However, in this case the total power flow of the part of the mode inside the slab is greater than that of the total power flow of the part of the mode outside. In the other words, the net total power flow of the guided mode is antiparallel with the direction of the phase flow. On the other hand, for the curve A₀A₁, the portion of the guided mode inside the grounded LH slab waveguide shows the Poynting vector to be antiparallel to the direction of the mode’s phase flow and the portion of guided mode outside the slab shows the Poynting vector to be parallel with the phase flow. But here the total power flow of the part of the mode inside the slab is smaller than that of the total power flow of the part of the mode outside. In this case, the net total power flow of the guided mode is parallel with the direction of the phase flow, which is in contrary to that of curve A₀A₂ [23].

When the slab height \(d/\lambda_0\) reaches 0.40092, the curve A₀A₁ in the special region vanishes and the dispersion curve I is out of the special region. Curve A₂A₃ has the same characteristics as that of curve A₀A₂. In curve I, the A₀A₃ portion that has net negative power flow is defined as TM₀ mode and the A₀A₁ portion that has net positive power flow is defined as TM₁ mode. A similar analysis can be done for other modes as well.

2.4.4 TE modes in a grounded LH slab waveguide

The dispersion characteristics of TE modes in a grounded LH slab are shown in Figure 2.6. In a conventional grounded slab waveguide all the TE modes have cutoff frequencies and
the curves increase monotonically. On contrast, the first mode, TE₁ mode, in a grounded LH slab waveguide has no cutoff frequency. Moreover, in the dispersion diagram the curves no longer increase monotonically, but are bent in a special region (shaded in Figure 2.6).

![Dispersion diagram of TE modes in a grounded LH slab waveguide with \( \varepsilon_r = -2.55 \) and \( \mu_r = -1 \)](image)

**Figure 2.6: Dispersion diagram of TE modes in a grounded LH slab waveguide with \( \varepsilon_r = -2.55 \) and \( \mu_r = -1 \)**

From Figure 2.7, it can be seen that when the height \( (d/\lambda_0) \) of the slab is increased from 0 the TE₁ mode starts to appear. Also the dispersion curve decreases monotonically. For this mode, the portion of the guided mode inside the grounded LH slab waveguide shows the Poynting
vector to be antiparallel to the direction of the mode’s phase flow and the portion of guided mode outside the slab shows the Poynting vector to be parallel with the phase flow. The total power flow of the part of the mode inside the slab is smaller than that of the total power flow of the part of the mode outside. Hence in this case, the net total power flow of the guided mode is parallel with the direction of the phase flow. Since this mode has net positive power flow the mode is defined as $\text{TE}_1^-$ mode, in association with our TM mode analysis. For this TE mode, there is no critical point and also there is no net negative power flow region in the dispersion curve. The $\text{TE}_1^-$ mode reaches cutoff when the value of $(d/\lambda_0)$ reaches 0.19819.

![Dispersion curve of $\text{TE}_1^-$ mode in a grounded LH slab waveguide with $\varepsilon_r = -2.55$ and $\mu_r = -1$](image)

Figure 2.7: Dispersion curve of $\text{TE}_1^-$ mode in a grounded LH slab waveguide with $\varepsilon_r = -2.55$ and $\mu_r = -1$
In the dispersion diagram shown in Figure 2.6, it can be noted that after the cutoff of $\text{TE}_1$ mode there is no mode propagation till $(d/\lambda_0)$ reaches 0.58868. When the $(d/\lambda_0)$ reaches 0.58868 the next higher order mode, $\text{TE}_2$ mode, starts to appear. At this critical point $A_0$, the portion of the guided mode inside the grounded LH slab waveguide shows the Poynting vector to be anti-parallel to the direction of the mode’s phase flow and the portion of this mode outside the slab shows the Poynting vector to be parallel with the phase flow. But the net power is 0 for this case. After the critical point, the curve II bifurcates in two different directions in the special region. The net total power flow of the guided mode in $A_0A_2$ portion of the curve is antiparallel with the direction of the phase flow, that is, the net power flow is negative. Similarly the net total power flow of the guided mode in $A_0A_1$ portion of the curve is parallel with the direction of the phase flow, that is, the net power flow is positive. When the slab height $(d/\lambda_0)$ reaches 0.60054, the curve $A_0A_1$ in the special region vanishes and the dispersion curve II is out of the special region. The $A_0A_1$ portion of curve II is defined as $\text{TE}_2^-$ mode and $A_0A_1$ portion is defined as $\text{TE}_2$ mode. A similar behavior is noted in $\text{TE}_3$ mode as well.
CHAPTER 3: ANALYSIS OF DISCONTINUITIES IN LH SLAB WAVEGUIDE

3.1 Overview

This chapter provides the finite element and mode matching analysis of the discontinuity structure of a LH waveguide sandwiched between two dielectric slab waveguides. The symmetrical properties of the structure are considered in the analysis and the scattering characteristics of discontinuities in the longitudinal direction are investigated using equivalent transmission line model. Based on the analysis, four cases with different waveguide heights are discussed and relevant physical insights are presented.

3.2 Scattering Characteristics of Discontinuities in LH Slab Waveguide

Discontinuities in dielectric waveguides, which arise due to differences in dimensions and/or material properties of the medium, have become indispensable part of many microwave component design and efforts have been made in the understanding the reflection and transmission phenomena at a discontinuity interface. The discontinuity structure of a grounded LH slab waveguide is sandwiched between two conventional grounded slab waveguides is shown in Figure 3.1. The effect of discontinuity in the structure shown in Figure 3.1 is an extremely complicated one and this complexity arises due to the left-handed characteristics of the media.
The continuous spectrum of radiation and evanescent waves, in addition to guided wave modes, is discretized by placing an electrical wall at a height “h” above the structure. For mode matching analysis of the discontinuity structure, we need to find the eigenvalues and eigenfunctions of the guided modes and the discretized radiation and evanescent modes in both LH and dielectric waveguide. Due to the symmetrical nature of the discontinuity structure, as shown in Figure 3.2, the scattering to the guided modes can be analyzed in terms of the symmetrical and antisymmetrical excitations for which we have the equivalent transmission line open circuit and short circuit respectively, as indicated in Figure 3.3. Without loss generality, the discontinuity structure is analyzed with TM mode.
Figure 3.2: Discontinuity structure due to symmetry

Figure 3.3: Equivalent transmission line model of the symmetrical discontinuity structure
Maxwell’s curl equation can be expressed as

\[ \nabla \times \mathbf{E} = -j \omega \mu_0 \mu_r \mathbf{H} \]  
(3.1a)

\[ \nabla \times \mathbf{H} = j \omega \varepsilon_0 \varepsilon_r \mathbf{E} \]  
(3.1b)

For the TM modes, we have

\[ \frac{\partial H_y}{\partial z} = -j \omega \varepsilon_0 \varepsilon_r E_x \]  
(3.2a)

\[ \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} = j \omega \mu_0 \mu_r H_y \]  
(3.2b)

\[ \frac{\partial H_y}{\partial x} = j \omega \varepsilon_0 \varepsilon_r E_z \]  
(3.2c)

From the above equations, we can also obtain

\[ E_x = e_x(x) U(z) \]  
(3.3a)

\[ E_z = e_z(x) J(z) \]  
(3.3b)

\[ H_y = h_y(x) J(z) \]  
(3.3c)

where the field components are expressed as

\[ e_x(x) = \begin{cases} 
\frac{1}{\varepsilon_r} A \cos(\bar{k}_x x), & 0 \leq x \leq d \\
\frac{1}{\varepsilon_r} A \cos[k_x (h - x)], & d \leq x \leq h
\end{cases} \]  
(3.4a)

\[ e_z(x) = \begin{cases} 
-\frac{\bar{k}_x}{j \omega \varepsilon} A \sin(\bar{k}_x x), & 0 \leq x \leq d \\
\frac{k_x}{j \omega \varepsilon} A \sin[k_x (h - x)], & d \leq x \leq h
\end{cases} \]  
(3.4b)

\[ h_y(x) = \begin{cases} 
A \cos(\bar{k}_x x), & 0 \leq x \leq d \\
A \cos[k_x (h - x)], & d \leq x \leq h
\end{cases} \]  
(3.4c)
In the equations (3.3a-3.3c), \( U(z) \) and \( J(z) \) satisfy the transmission line equations

\[
\frac{dU(z)}{dz} = -jk_z Z_c J(z) \tag{3.5a}
\]

\[
\frac{dJ(z)}{dz} = -jk_z Y_c U(z) \tag{3.5b}
\]

where the characteristic impedance \( Z_c \) is given by

\[
Z_c = \frac{1}{Y_c} = \frac{k_z}{\omega \epsilon_0} \tag{3.6}
\]

Based on the finite element approach discussed in chapter 2, the eigenvalues or the propagation constants and the corresponding eigenfunctions or the field components of the conventional slab and LH slab waveguides are determined. Before the mode matching treatment in the longitudinal direction, the eigenfunctions of each mode should be normalized by \( \sqrt{S} \), where

\[
S = \int_0^k e_x h_j dx \tag{3.7}
\]

The general field solution in each uniform region may be expressed in terms of the superposition of a complete set of mode functions. Hence the tangential field components in the \( z < z_l \) region are expressed as

\[
E_x(x, z) = \sum_{n=1}^{\infty} e_{xn}(x) U_n(z) \tag{3.8a}
\]

\[
H_y(x, z) = \sum_{n=1}^{\infty} h_{yn}(x) J_n(z) \tag{3.8b}
\]

and the tangential field components in the \( z > z_l \) region are given by

\[
E_x(x, z) = \sum_{n=1}^{\infty} \bar{e}_{xn}(x) \bar{U}_n(z) \tag{3.9a}
\]
\[ \overline{H}_y(x, z) = \sum_{n=1}^{\infty} \overline{h}_n(x) \overline{J}_n(z) \] (3.9b)

At the step discontinuity at \( z = z_l \), the tangential field components must be continuous satisfying the conditions

\[ \sum_{n=1}^{\infty} e_{xn}(x) U_n(z) = \sum_{n=1}^{\infty} \overline{e}_{xn}(x) \overline{U}_n(z) \] (3.10a)

\[ \sum_{n=1}^{\infty} h_{yn}(x) J_n(z) = \sum_{n=1}^{\infty} \overline{h}_{yn}(x) \overline{J}_n(z) \] (3.10b)

The field components satisfy the following orthogonality relation,

\[ \int_{0}^{h} e_{xn} h_{ym} \, dx = \delta_{nm} \] (3.11)

this yields

\[ \mathbf{U} = \mathbf{Q} \mathbf{U} \] (3.12a)

\[ \mathbf{J} = \mathbf{P} \mathbf{J} \] (3.12b)

where \( \mathbf{Q} \) and \( \mathbf{P} \) are the coupling matrices given by

\[ (Q_{il})_{mn} = \int_{0}^{h} \overline{e}_{xn} h_{ym} \, dx \] (3.13a)

\[ (P_{il})_{mn} = \int_{0}^{h} e_{xn} \overline{h}_{ym} \, dx \] (3.13b)

Making use of the orthogonality relation, equation (3.11), we can also obtain

\[ \mathbf{P}_l^T \mathbf{U} = \mathbf{U} \] (3.14a)

\[ \mathbf{Q}_l^T \mathbf{J} = \mathbf{J} \] (3.14b)
From equations (3.12) and (3.14), we can derive

\[ \mathbf{P}_i^T \mathbf{Q}_i = \mathbf{Q}_i^T \mathbf{P}_i = 1 \]  

(3.15)

where \( \mathbf{1} \) is the unit matrix. For the transmission line model, shown in Figure 3.3, the input impedance matrix at \( z = z_i^- \) plane, looking to the right of the structure, satisfies

\[ \mathbf{Z}(z_i^-) = \mathbf{Q}_i \mathbf{Z}(z_i^+) \mathbf{Q}_i^T \]  

(3.16)

and the reflection coefficient matrix at the \( z = z_i^- \) plane, looking to the right, is expressed as

\[ \mathbf{\Gamma}(z_i^-) = [\mathbf{Z}(z_i^-) + \mathbf{Z}_c]^{-1} [\mathbf{Z}(z_i^-) - \mathbf{Z}_c] \]  

(3.17)

With the bisectons in the longitudinal direction due to the symmetrical nature of the discontinuity structure, there are two different combinations of boundary condition. For an incident guided mode from the input waveguide, conventional slab waveguide, two separate substructures are analyzed with their respective boundary conditions. In each case, energy is reflected and the reflection coefficient matrices are denoted by \( \mathbf{R}_o \) and \( \mathbf{R}_s \) for the equivalent transmission line open circuit and short circuit, respectively. With the reflection matrices \( \mathbf{R}_o \) and \( \mathbf{R}_s \) of the substructure, the reflection coefficient matrix \( \mathbf{R} \) and transmission coefficient matrix \( \mathbf{T} \) of the entire structure are determined by [24, 25]

\[ \mathbf{R} = \frac{(\mathbf{R}_o + \mathbf{R}_s)}{2} \]  

(3.18a)

\[ \mathbf{T} = \frac{(\mathbf{R}_o - \mathbf{R}_s)}{2} \]  

(3.18b)
3.3 Results and Discussion

Based on the mode matching approach the discontinuity structure of a LH waveguide sandwiched between two dielectric slab waveguides is analyzed for different waveguide heights. The TM\textsubscript{0} mode with incident power of 1 is assumed to be incident from the slab waveguide and the percentage power of the reflected and transmitted mode is calculated for four different cases.

*Case 1: \( d = 0.3 \lambda_0 \)*

In this case, there is only the dominant TM\textsubscript{0} mode in the conventional grounded slab waveguide and all the modes are cutoff in the LH slab waveguide. From Figure 3.4, it is noted that when the normalized length \( (L/\lambda_0) \) of the LH slab waveguide is less than two, there is coupling between the two conventional slab waveguides.

![Figure 3.4: Percentage of reflected power of TM\textsubscript{0} mode for \( d = 0.3 \lambda_0 \) waveguide thickness](image)

Figure 3.4: Percentage of reflected power of TM\textsubscript{0} mode for \( d = 0.3 \lambda_0 \) waveguide thickness
But when the normalized $(L/\lambda_0)$ increases above two, the coupling effect vanishes and the percentage power of the reflected and transmitted TM$_0$ mode become constant. Because all the modes are cutoff in the grounded LH slab waveguide, from Figure 3.5, the percentage power of the transmitted TM$_0$ mode is 0. This is the case when the conventional dielectric slab waveguide is connected with an infinitely long LH slab waveguide.

![Graph](image)

Figure 3.5: Percentage of transmitted power of TM$_0$ mode for $d = 0.3 \lambda_0$ waveguide thickness

**Case 2: $d = 0.31938 \lambda_0$**

In this case, the height of the waveguide corresponds to the critical point on the dispersion curve of the LH slab waveguide, shown in Figure 2.5. For this height only one TM
mode with zero net total power appears in the LH slab waveguide and only the dominant TM$_0$ mode exists in the conventional grounded slab waveguide.

From Figure 3.5 and Figure 3.6, it can be inferred that 50% power of the TM$_0$ mode is reflected and there is small transmission. Also, for specific ($L/\lambda_0$) values of 22, 44, 66 and 88, strong resonance occurs. These normalized lengths correspond to the resonant frequencies of the LH resonator which has the same structure as shown in Figure 3.1. We can find that at these resonant points, the TM$_0$ mode has no reflection and most of the power is transmitted in the longitudinal direction.

Figure 3.6: Percentage of reflected power of TM$_0$ mode

for $d = 0.31938 \lambda_0$ waveguide thickness
Case 3: $d = 0.35 \lambda_0$

For this case, there is only TM$_0$ mode in the conventional grounded slab waveguide. For the grounded LH slab waveguide, this height is in the special region as shown in Figure 2.5. Hence there are two TM modes, the TM$_0$ mode with net negative total power flow and TM$'_0$ mode with net positive total power flow. These results are shown in Figure 3.8 and Figure 3.9. From the curves, we find that there are oscillations in the reflected and transmitted power which is due to the coupling of TM$_0$ and TM$'_0$ modes. Compared with case 2, the transmission of TM$_0$ mode takes the majority and the reflection is much smaller.
Figure 3.8: Percentage of reflected power of TM$_0$ mode for $d = 0.35\ \lambda_0$ waveguide thickness

Figure 3.9: Percentage of transmitted power of TM$_0$ mode for $d = 0.35\ \lambda_0$ waveguide thickness
Case 4: $d = 0.6 \lambda_0$

In this case, the conventional grounded slab waveguide has TM$_0$ and TM$_1$ modes. In the LH slab waveguide, this point is out of the special region in the dispersion curve, only the TM$_0$ mode exists. The plots of the reflected TM$_0$ and TM$_1$ mode and transmitted TM$_0$ and TM$_1$ mode are shown below.

Figure 3.10: Percentage of reflected power of TM$_0$ mode
for $d = 0.6 \lambda_0$ waveguide thickness
Figure 3.11: Percentage of transmitted power of TM$_0$ mode
for $d = 0.6 \lambda_0$ waveguide thickness

Figure 3.12: Percentage of reflected power of TM$_1$ mode
for $d = 0.6 \lambda_0$ waveguide thickness
From the e TM0 mode is transmitted. The reflection of the TM0 and TM1 modes are all small. The reason for this to occur is as follows. In this case, the effective dielectric constants of the TM0 and TM1 modes in the conventional slab waveguide are 2.4032 and 1.3676 respectively.

For the LH slab waveguide the effective dielectric constant of the TM0 mode is 2.3408. The TM0 mode in the conventional slab waveguide is closely matched with the TM0 mode in the LH slab waveguide. Hence with the TM0 mode incident, the transmitted TM0 mode is the majority component and the reflected TM0 and TM1 modes and the transmitted TM1 mode are all small.

Figure 3.13: Percentage of transmitted power of TM1 mode

for d = 0.6 \lambda \text{ waveguide thickness}
CHAPTER 4: ANALYSIS OF PARALLEL PLATE WAVEGUIDE
PARTIALLY FILLED WITH LH MEDIA

4.1 Overview

In this chapter, the finite element method together with eigenfunction expansion technique for the analysis of a parallel plate waveguide partially filled with LH media is illustrated. It also provides the implementation of the method to analyze a parallel plate waveguide partially filled with conventional and LH media. The scattering characteristics of the parallel plate waveguide with partial filling are observed and the insights are discussed.

4.2 Parallel Plate Waveguide Partially Filled with LH Media

A parallel plate waveguide is the simplest type of guide consisting of two parallel perfectly conducting plates that trap propagating energy between them. The electromagnetic waves inside bounce back and forth between the plates as the wave propagate down the waveguide so as to satisfy the plate boundary conditions. A parallel plate waveguide can support TE, TM and TEM modes. Figure 4.1 shows a parallel plate waveguide partially filled with LH media. In the geometry of the parallel plate waveguide the metal plate width is assumed to be much greater than the separation, \( a \), so that fringing fields and any \( z \) variation can be ignored. Air is assumed to fill the region between the plates and a LH media of height \( d \) and width \( w \) is placed in between the plates.
4.3 Finite Element Analysis

The eigenfunction expansion technique with the finite element method is widely used in the treatment of unbounded field problems [13]. In this method the unbounded region is divided into an interior and an exterior region, and the finite element method is employed to formulate the field in the interior region. The exterior fields are represented by an expansion of eigenfunctions, which makes this method different from other methods. The eigenfunctions are a set of homogeneous solutions of a differential equation satisfying certain boundary conditions. Their expansions can be used to represent a solution of the corresponding inhomogeneous differential equation with any source function, subject to same boundary conditions. This technique allows multiple-mode propagation and, more important, allows the fictitious boundaries to be placed as close to the region of interest as possible. These are achieved by
including higher-order modes in the representation of the fields at the fictitious boundaries. This results in a relatively small finite element discretization and thereby decreases the computation time.

In the analysis of parallel plate waveguide partially filled with LH media we discretize the unbounded region by enclosing the structure with two fictitious boundaries at \( z = z_1 \) and \( z = z_2 \) respectively, shown in Figure 4.2. The fields in the interior region (II) are formulated using FEM and the fields in the exterior regions (I and III) are expressed by expansion of eigenfunctions.

![Parallel plate waveguide partially filled with LH media and with fictitious boundaries](image)

Figure 4.2: Parallel plate waveguide partially filled with LH media and with fictitious boundaries

For the structure shown in Figure 4.2, TE\(_1\) mode is assumed to be incident. The field at the fictitious plane at \( z = z_1 \) can be written as the superposition of the incident field and the field reflected from the LH media (discontinuity). Since this fictitious plane is placed close to the LH media (discontinuity), the reflected field cannot be represented by the dominant mode only;
rather, it is a superposition of the dominant and many higher-order modes excited by the discontinuity. Therefore, we have

$$E_y(x, z_1) = E_y^{inc}(x, z_1) + \sum_{m=1}^{\infty} a_m e_{jm}(x)e^{jk_m z_1}$$ \hspace{1cm} (4.1)$$

in which $a_m$ are the expansion coefficients and $e_{jm}(x)$ and $k_{zm}$ are given by,

$$e_{jm}(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{m \pi x}{a}\right)$$ \hspace{1cm} (4.2)$$

$$k_{zm} = \begin{cases} \sqrt{k_0^2 - \left(\frac{m \pi}{a}\right)^2}, & \text{if } \left(\frac{m \pi}{a}\right)^2 \leq k_0^2 \\ -j\sqrt{\left(\frac{m \pi}{a}\right)^2 - k_0^2}, & \text{if } \left(\frac{m \pi}{a}\right)^2 > k_0^2 \end{cases}$$ \hspace{1cm} (4.3)$$

where $a$ denotes the separation between the two plates. Since each mode given by $e_{jm}(x)e^{jk_m z}$ is a homogeneous solution of the Helmholtz equation subject to the boundary conditions applicable to $E_y$, their superposition (4.1) satisfies the Helmholtz equation and the required boundary condition as well. Further, since these modes form a complete set of eigenfunctions, they can represent any $E_y$ distribution within the waveguide. Using the orthogonality relation we obtain the expression for the expansion coefficients $a_m$ in terms of the field $E_y$ as,

$$a_m = e^{-jk_m z_1} \int_0^a [E_y(x, z_1) - E_y^{inc}(x, z_1)]e_{jm}(x)dx$$ \hspace{1cm} (4.4)$$

where $z_f$ denotes the position of the fictitious plane. Inserting this into (4.1), we obtain

$$E_y(x, z_1) = E_y^{inc}(x, z_1) + \sum_{m=1}^{\infty} e^{jk_m(z-z_1)} \cdot e_{jm}(x) \times \int_0^a [E_y(x', z_1) - E_y^{inc}(x', z_1)]e_{jm}(x')dx'$$ \hspace{1cm} (4.5)$$

Taking the partial derivative of this equation with respect to $z$ yields the boundary condition at $z = z_f$ which can be written in the generalized form as,
\[
\frac{\partial E_y(x,z_1)}{\partial n} + P[E_y(x,z_1)] = U^{\text{inc}}
\]  \hspace{1cm} (4.6)

where \( \mathbf{n} = -\mathbf{\hat{z}} \), \( P[E_y(x,z_1)] \) is the boundary operator given by

\[
P[E_y(x,z_1)] = -\sum_{m=1}^{\infty} jk_{zm} e_{ym}(x) \times \int_0^a E_y(x',z_1)e_{ym}(x')dx'
\]  \hspace{1cm} (4.7)

and \( U^{\text{inc}} \) is given by

\[
U^{\text{inc}} = \frac{\partial E_y^{\text{inc}}(x,z_1)}{\partial n} - \sum_{m=1}^{\infty} jk_{zm} e_{ym}(x) \times \int_0^a E_y^{\text{inc}}(x',z_1)e_{ym}(x')dx'
\]  \hspace{1cm} (4.8)

Similarly, the field at \( z = z_2 \) fictitious plane can be expressed as,

\[
E_y(x,z_2) = \sum_{m=1}^{\infty} b_m e_{ym}(x)e^{jk_{zm}z_2}
\]  \hspace{1cm} (4.9)

where \( e_{ym}(x) \) and \( k_{zm} \) are also given by (4.2) and (4.3) respectively. From (4.9), we can find

\[
b_m = e^{-jk_{zm}z_2} \int_0^a E_y(x,z_2)e_{ym}(x)dx
\]  \hspace{1cm} (4.10)

and further the boundary condition at \( z = z_2 \) is given by,

\[
\frac{\partial E_y(x,z_2)}{\partial z} + P[E_y(x,z_1)] = 0
\]  \hspace{1cm} (4.11)

where \( P[E_y(x,z_1)] \) is the boundary operator given by equation (4.7).

With boundary conditions (4.6) and (4.11), the boundary-value problem for the field inside the field inside the fictitious boundaries is uniquely defined. We can formulate the system of equations using Ritz variational method, in which an equivalent variational problem is formulated. A detailed procedure on the formulation of the variational problem is illustrated in Appendix A.
The functional for this problem is expressed as,
\[
F(E_y) = F_1(E_y) - F_2(E_y) - F_3(E_y)
\]  
(4.12)

where,
\[
F_1(E_y) = \frac{1}{2} \int_{\Omega} \left( \frac{\partial E_y}{\partial x} \right)^2 + \left( \frac{\partial E_y}{\partial z} \right)^2 - k_0 E_y^2 \, d\Omega
\]  
(4.13a)
\[
F_2(E_y) = \frac{1}{2} \int_{a_0}^{a} E_y P[E_y] \, dx \bigg|_{z=z_2}
\]  
(4.13b)
\[
F_3(E_y) = \frac{1}{2} \int_{a_0}^{a} \left( E_y - U^{inc} \right) P[E_y] \, dx \bigg|_{z=z_1}
\]  
(4.13c)

A detailed proof for the functional is given in Appendix B. Substituting the expressions for \( P[E_y] \) and \( U^{inc} \) into the above, we obtain a functional that is amenable to a finite element solution. With the suitable selection of interpolation function, we can derive an expression for the unknown solution in an element, say element \( e \), in the following form,
\[
\phi^e = \sum_{j=1}^{n} N_j^e \phi_j^e
\]  
(4.14)

where \( n \) is the number of nodes in the element, \( \phi_j^e \) is the value of \( \phi \) at node \( j \) of the element, and \( N_j^e \) is the interpolation function for node \( j \). Introducing expression (4.14) for \( \phi^e \) into equation (4.13a) and differentiating the subfunctional \( F_1^e \) with respect to \( \phi_i^e \) yields
\[
\frac{\partial F_1^e(\phi^e)}{\partial \phi_i^e} = \left[ A^e \right] \{ \phi^e \} - k_0^2 \left[ D^e \right] \{ \phi^e \}
\]  
(4.15)

where \( \left[ \frac{\partial F^e}{\partial \phi^e} \right] = \begin{bmatrix} \frac{\partial F^e}{\partial \phi_1^e} & \frac{\partial F^e}{\partial \phi_2^e} & \frac{\partial F^e}{\partial \phi_3^e} \end{bmatrix}^T \); \( \{ \phi^e \} = \begin{bmatrix} \phi_1^e & \phi_2^e & \phi_3^e \end{bmatrix}^T \) and the element matrices \( [A^e] \) and \( [D^e] \) are given by
\[ A_{ij}^e = \frac{1}{\alpha_e} \int_{\Omega^e} \left( \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial z} \frac{\partial N_j}{\partial z} \right) d\Omega^e \] (4.16)

\[ D_{ij}^e = \rho_e \int_{\Omega^e} (N_i N_j) d\Omega^e \] (4.17)

Following a similar approach for the other two subfunctionals we have,

\[ \frac{\partial F_2^e(\phi^e)}{\partial \phi^e_i} = [K_{st}^e] \{\phi^e\} \] (4.18)

\[ \frac{\partial F_3^e(\phi^e)}{\partial \phi^e_i} = [T_{st}^e] \{\phi^e\} + \{b_i^e\} \] (4.19)

where the element matrices \([K_{st}^e], [T_{st}^e]\) and the element vector \(\{b^e\}\) are given by

\[ K_{st}^e = \left( \frac{2}{a} \right)^2 \sum_{m=1}^{M} j_k^m \left[ \int_{e_{i-1}}^{W_{e_i}^{e-1}} \sin \left( \frac{m \pi}{a} x \right) dx + \int_{e_i}^{W_{e_i}^{e}} \sin \left( \frac{m \pi}{a} x \right) dx \right] \times \left[ \int_{e_{i-1}}^{W_{e_i}^{e-1}} \sin \left( \frac{m \pi}{a} x \right) dx + \int_{e_i}^{W_{e_i}^{e}} \sin \left( \frac{m \pi}{a} x \right) dx \right] \] (4.20)

\[ T_{st}^e = K_{st}^e \bigg|_{z=z_i} \] (4.21)

\[ b_i^e = 2 j_k^m e^{-j_{k_{x_{i-1}}}^m} \sqrt{\frac{2}{a}} \int_{e_i}^{W_{e_i}^{e}} \sin \left( \frac{\pi}{a} x \right) dx \] (4.22)

Since the subfunctional \(F_2\) and \(F_3\) represent the boundary condition at the two fictitious planes, \(W\) is assumed to be one-dimensional linear interpolation function. The elemental equations are assembled for all \(M\) elements and the stationarity requirement on \(F\) is imposed to find the system of equations,

\[ \left\{ \frac{\partial F}{\partial \phi} \right\} = \sum_{e=1}^{M} \left( \frac{\partial F^e}{\partial \phi^e} \right) = \sum_{e=1}^{M} \left( [A^e] \{\phi^e\} - k_e^z [D^e] \{\phi^e\} - [K_{st}^e] \{\phi^e\} - [T_{st}^e] \{\phi^e\} - \{b\} \right) = \{0\} \] (4.23)

48
The above system of equations can be written compactly as,

\[
[A] \{\phi\} - k_o^2 [D] \{\phi\} - [K] \{\phi\} - [T] \{\phi\} - \{b\} = 0
\]

(4.24)

where \( \phi \) represents the unknown field \( E_y \).

4.4 Results and Discussion

Using the finite element formulation with eigenfunction expansion technique, the scattering characteristics of a parallel plate waveguide partially filled with conventional media and LH media are investigated. The results are also compared with mode matching technique as well.

First, the parallel plate waveguide is partially filled with a conventional media with permittivity \( \varepsilon_r = 2.55 \) and permeability \( \mu_r = 1 \). The width and height of the conventional media are assumed to be \( w = 20\text{mm} \) and \( d = 5.715\text{mm} \) respectively. For a parallel plate waveguide with \( a = 22.86\text{mm} \), the cutoff frequencies of TE\(_1\) and TE\(_2\) modes are 6.56 GHz and 13.12 GHz respectively. The TE\(_1\) mode with power of 1 is assumed to be incident from the left and the percentage of reflected and transmitted power is calculated. From Figure 4.3, it can be noted that from 8.5 GHz to 11.5 GHz there is complete power transmission. Also, there are some oscillations between 11.5 GHz and 13 GHz which is due to the next higher order mode in the parallel plate waveguide. The percentage of reflected power is shown in Figure 4.4. From the scattering plots it can be seen that there is an excellent agreement of results between the eigenfunction expansion technique and mode matching technique.
Figure 4.3: Transmitted power of a parallel plate waveguide partially filled with conventional media

Figure 4.4: Reflected power of a parallel plate waveguide partially filled with conventional media
The scattering plots of the parallel plate waveguide partially filled with LH media with permittivity $\varepsilon_r = -2.55$ and permeability $\mu_r = -1$ are shown in Figure 4.5 and Figure 4.6. The width and height of the LH media is kept the same as the conventional media case and the percentage of reflected and transmitted power are calculated. From Figure 4.5, it can be seen that there is no frequency with complete power transmission but the average power transmission over the entire frequency band is improved. It can also be noted that the higher frequency oscillations seen in conventional media case are not present in LH media case.

Figure 4.5: Transmitted power of a parallel plate waveguide partially filled with LH media
Figure 4.6: Reflected power of a parallel plate waveguide partially filled with LH media

From Figure 4.5 and Figure 4.6, it can be seen that there is an excellent agreement of results between the eigenfunction expansion technique and mode matching technique. The parallel plate waveguide partially filled with LH media has a wide transmission band without oscillations.
CHAPTER 5: ANALYSIS OF ARBITRARY SHAPED LH WAVEGUIDE

5.1 Overview

This chapter provides the two-dimensional vector finite element formulation for the analysis of arbitrary shaped LH waveguide structures. It also provides the implementation of the method to analyze conventional and LH waveguides with circular cross section. The anomalous dispersion characteristics of left-handed waveguides are compared in association with the conventional waveguides and the physical insights are discussed.

5.2 Vector Finite Element Formulation

The vector finite element method is widely used to compute the mode spectrum of an electromagnetic waveguide with arbitrary cross section [26]. The vector finite element method is most elegant and simple approach to eliminate the disadvantages of the scalar finite element approach. The choice of edge based elements makes this method immune against the undesired spurious modes or non-physical solutions and easy implementation of boundary conditions at material interfaces. In the analysis of any arbitrary shaped structure, we discretize the continuous spectrum by enclosing the structure with an electrical wall as shown in Figure 5.1.
The vector finite element formulation can be illustrated by using either the \( \mathbf{E} \) or \( \mathbf{H} \) field; here we explain the case for the \( \mathbf{E} \) field, which is the same for the \( \mathbf{H} \) field. The vector wave equation for the \( \mathbf{E} \) field is given by,

\[
\nabla \times \left( \frac{1}{\mu_r} \nabla \times \mathbf{E} \right) - k_0^2 \varepsilon_r \mathbf{E} = 0
\]

(5.1)

where \( \mu_r \) and \( \varepsilon_r \) are the permeability and permittivity, respectively, of the material in the waveguide. The transverse and longitudinal components are separated and are written as

\[
\nabla_r \times \left( \frac{1}{\mu_r} \nabla_r \times \mathbf{E}_t \right) + \frac{1}{\mu_r} \left( k_z^2 \nabla_z E_z + k_z^2 \mathbf{E}_t \right) = k_0^2 \varepsilon_r \mathbf{E}_t
\]

(5.2a)
\[-\frac{1}{\mu_r} \left[ \nabla_i \cdot \left( \nabla_i E_z + E_i \right) \right] = k_0^2 \varepsilon_r E_z \quad (5.2b)\]

To apply Galerkin’s method to the above equations, we multiply equation (5.2a) with the testing function \( T_t \) and equation (5.2b) with the testing function \( T_z \) and integrate both the equations over the cross section of the structure \( \Gamma \) [26]; that is,

\[
\int_T \left[ T_t \cdot \nabla_i \times \left( \frac{1}{\mu_r} \nabla_i \times E_i \right) + \frac{k_i^2}{\mu_r} (T_t \cdot \nabla_i E_z + T_t \cdot E_i) \right] ds = k_0^2 \varepsilon_r \int_T T_t \cdot E_i \ ds 
\]

(5.3a)

\[-\frac{1}{\mu_r} \int_T [T_z \cdot \nabla_i \cdot (\nabla_i E_z + E_i)] ds = k_0^2 \varepsilon_r \int_T T_z E_z \ ds \]

(5.3b)

A detailed explanation of Galerkin’s method is given in Appendix C. Using the vector identities in Appendix D; we can write the weak form of the above equations as

\[
\frac{1}{\mu_r} \int_T \left[ \left( \nabla_i \times T_t \right) \cdot \left( \nabla_i \times E_i \right) + \left( k_i^2 T_t \cdot \nabla_i E_z + k_i^2 T_t \cdot E_i \right) \right] ds 
= k_0^2 \varepsilon_r \int_T T_t \cdot E_i \ ds - \frac{1}{\mu_r} \int \nabla_i \cdot \left( \hat{n} \times \nabla \times E_i \right) ds 
\]

(5.4a)

\[-\frac{1}{\mu_r} \int_T \left( \nabla_i T_z \cdot \nabla_i E_z + \nabla_i T_z \cdot E_i \right) ds = k_0^2 \varepsilon_r \int_T T_z E_z \ ds + \frac{1}{\mu_r} \int \left( T_z \frac{\partial E_z}{\partial n} + T_z \hat{n} \cdot E_i \right) ds \]

(5.4b)

If the structure boundary \( d\Gamma \) in Figure 5.2 is assumed to be perfectly conducting, then \( T_t = 0 \) and \( T_z = 0 \) on \( d\Gamma \). Therefore the line integrals on the right hand side of equations (5.4a) and (5.4b) can be neglected. Multiplying equation (5.4b) with \( k_i^2 \) for the sake of symmetry and rearranging the equations we have

\[
\frac{1}{\mu_r} \int_T \left( \nabla_i \times T_t \right) \cdot \left( \nabla_i \times E_i \right) ds = \int_T T_t \cdot E_i \ ds \}
\]

(5.5a)

\[
\frac{k_i^2}{\mu_r} \int_T \nabla_i T_z \cdot \nabla_i E_z \ ds + \frac{k_i^2}{\mu_r} \int_T \nabla_i T_z \cdot E_i \ ds = k_0^2 k_i^2 \varepsilon_r \int_T T_z E_z \ ds
\]

(5.5b)
Since the vector Helmholtz equation is divided into two parts, equations (5.2a) and (5.2b), vector-based tangential edge elements, shown in Figure 5.2 (a), can be used to approximate the transverse fields, and nodal-based elements, shown in Figure 5.2 (b), can be used to approximate the longitudinal component.

![Figure 5.2: Configuration of (a) tangential edge elements (b) node elements](image)

For a single triangular element shown in the above figure the transverse electric field can be expressed as a superposition of edge elements. The edge elements permit a constant tangential component of the basis function along one triangular edge while simultaneously allowing a zero tangential component along the other two edges. Three such functions overlapping each triangular element provides the complete expansion, that is
\[ E_i = \sum_{m=1}^{3} e_m W_{tm} \]  

(5.6)

where \( m \) indicates the \( m \)th edge of the triangle and \( W_{tm} \) is the edge element for edge \( m \) given by

\[ W_{tm} = L_{tm} (\alpha_i \nabla \alpha_j - \alpha_j \nabla \alpha_i) \]  

(5.7)

\( L_{tm} \) is the length of edge \( m \) connecting nodes \( i \) and \( j \) and \( \alpha_i \) is the first-order shape function associated with nodes 1, 2 and 3 given by

\[ \alpha_i = \frac{1}{2A} (a_i + b_i x + c_i y) \]  

(5.8)

and \( a_i, b_i, \) and \( c_i \) are given by

\[ a_i = x_j y_k - x_k y_j \]  

(5.9a)

\[ b_i = y_j - y_k \]  

(5.9b)

\[ c_i = x_k - x_j \]  

(5.9c)

where \( i, j, \) and \( k \) are cyclical and \( A \) is given by

\[ A = \frac{1}{2} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} \]  

(5.10)

The longitudinal component is written as

\[ E_z = \sum_{i=1}^{3} e_z \alpha_i \]  

(5.11)

where \( i \) indicates the \( i \)th node and \( \alpha_i \) is given by equation (5.8). The testing functions \( T_t \) and \( T_z \) are chosen to be the same as the corresponding basis function in equations (5.5a) and (5.5b); that is \( T_t = W_{tm} \) and \( T_z = \alpha_i \)
Substituting equations (5.6) and (5.11) into equations (5.5a) and (5.5b), respectively, integrating over a single triangular element, and interchanging the integration and summation gives

\[
\frac{1}{\mu_r} \sum_{m=1}^{3} \int_{\Delta} (\nabla \times W_m) \cdot (\nabla \times W_m) e_m \, ds - k_0^2 \sum_{m=1}^{3} e_r \int_{\Delta} (W_m \cdot W_m) e_m \, ds
\]

\[
= -k_0^2 \left[ \frac{1}{\mu_r} \sum_{m=1}^{3} \int_{\Delta} (W_m \cdot \nabla \alpha_j) e_m \, ds + \frac{1}{\mu_r} \sum_{m=1}^{3} \int_{\Delta} (W_m \cdot W_m) e_m \, ds \right] \quad (5.12a)
\]

\[
\frac{k_0^2}{\mu_r} \sum_{i=1}^{3} \int_{\Delta} (\nabla \alpha_i \cdot \nabla \alpha_j) e_{zi} \, ds + \frac{k_0^2}{\mu_r} \sum_{i=1}^{3} \int_{\Delta} (\nabla \alpha_i \cdot W_m) e_m \, ds = k_0^2 \sum_{i=1}^{3} k_0^2 e_r \int_{\Delta} \alpha_i \alpha_j e_z \, ds \quad (5.12b)
\]

where the subscripts of \( \alpha \) and \( W_i \) indicate node number and edge numbers respectively.

Equations (5.12a) and (5.12b) can be written in matrix form as,

\[
\begin{bmatrix}
S_{e(t)} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
e_t \\
e_z
\end{bmatrix} = -k_0^2
\begin{bmatrix}
T_{e(t)} & T_{e(z)} \\
T_{e(z)} & T_{e(z)}
\end{bmatrix}
\begin{bmatrix}
e_t \\
e_z
\end{bmatrix} \quad (5.13)
\]

The element matrices are given by

\[
S_{e(t)} = \frac{1}{\mu_r} \int_{\Delta} (\nabla \times W_m) \cdot (\nabla \times W_m) \, ds - k_0^2 e_r \int_{\Delta} (W_m \cdot W_m) \, ds \quad (5.14a)
\]

\[
T_{e(t)} = \frac{1}{\mu_r} \int_{\Delta} (W_m \cdot W_m) \, ds \quad (5.14b)
\]

\[
T_{e(z)} = \frac{1}{\mu_r} \int_{\Delta} (W_m \cdot \nabla \alpha_j) \, ds \quad (5.14c)
\]

\[
T_{e(z)} = \frac{1}{\mu_r} \int_{\Delta} (\nabla \alpha_i \cdot W_m) \, ds \quad (5.14d)
\]

\[
T_{e(z)} = \frac{1}{\mu_r} \int_{\Delta} (\nabla \alpha_i \cdot \nabla \alpha_j) \, ds - k_0^2 e_r \int_{\Delta} \alpha_i \alpha_j \, ds \quad (5.14e)
\]
These element matrices are assembled over all the triangular elements in the cross section of the structure to obtain a global eigenvalue equation [26],

$$
\begin{bmatrix}
S_{zz} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
e_z
\end{bmatrix}
= -k_z^2
\begin{bmatrix}
T_{zz} & T_{zz}
\end{bmatrix}
\begin{bmatrix}
e_z
\end{bmatrix}
$$

(5.15)

Solving the above equation yields the eigenvalues or the longitudinal propagation constants $k_z$, from which the effective relative dielectric constant $\varepsilon_{re}$ is obtained using the relation

$$
\varepsilon_{re} = \left(\frac{k_z}{k_0}\right)^2
$$

(5.16)

Similar procedure can be followed with $\mathbf{H}$ field.

5.3 Finite Element Analysis of a LH Waveguide with Circular Cross Section

As an example of arbitrary shaped LH waveguide, a LH waveguide with circular cross section is analyzed. It is a cylindrical dielectric type of waveguide with negative relative permittivity ($\varepsilon_r$) and relative permeability ($\mu_r$), as shown in Figure 5.3. In general, these waveguides can support an infinite number of modes. But for a given set of permittivity, permeability, and radius “a” only a finite number of unattenuated waveguide modes exist with their fields localized in the central dielectric core. Generally a dielectric waveguide will have TE and/or TM modes but for the waveguide shown in Figure 5.3 pure TE and TM modes exist only when the field configurations are symmetrical and independent of $\phi$. Also there exist certain nonsymmetrical field configurations which possess angular $\phi$ variations and are combinations of TE and TM modes. These modes are usually referred as hybrid modes and are designated as $\text{HEM}_{mn}$. The hybrid mode with predominant TE modes is denoted as HE mode and when the TM modes are predominant they are denoted as EH mode.
In the analysis of LH waveguide with circular cross section we discretize the continuous spectrum by enclosing the structure with an electrical wall as shown in Figure 5.4.
5.4 Results and Discussion

Using the two dimensional vector finite element formulation described in the preceding section, the dispersion relation of conventional and LH waveguides with circular cross section is obtained.

5.4.1 Dispersion diagram of conventional waveguide with circular cross section

From the dispersion diagram, shown in Figure 5.4, it can be seen that for any nonzero radius, with permittivity $\varepsilon_r = 2.55$ and permeability $\mu_r = 1$, there is at least one propagating mode. This mode, $HE_{11}$ (or $HEM_{11}$) mode, is a hybrid mode and is the dominant mode of the waveguide with zero cutoff frequency. It can be seen that the next higher order modes, the $TE_{01}$ and $TM_{01}$ mode, will not propagate until the diameter of the dielectric rod becomes greater than $0.615\lambda_0$. Also, all the dispersion curves increase monotonically.
5.4.2 Dispersion diagram of LH waveguide with circular cross section

The dispersion characteristics of LH waveguide with \( \varepsilon_r = -2.55 \) and \( \mu_r = -1 \) and with circular cross section is shown in Figure 5.5. From the dispersion diagram, it can be noted that curves no longer increase monotonically for LH waveguide which is in contrast with the conventional waveguide. Each mode is considered separately and the physical insights are explained based on their dispersion curves.
From Figure 5.6, it can be seen that when the diameter ($2a/\lambda_0$) of the LH rod is increased from 0 the HE$_{11}$ mode starts to appear. Also the dispersion curve decreases monotonically. For this mode, the portion of the guided mode inside the LH rod shows the Poynting vector to be antiparallel to the direction of the mode’s phase flow and the portion of guided mode outside the slab shows the Poynting vector to be parallel with the phase flow. The total power flow of the part of the mode inside the rod is smaller than that of the total power flow of the part of the mode outside. Hence in this case, the net total power flow of the guided mode is parallel with the
direction of the phase flow. Since this mode has net positive power flow the mode is defined as \( \text{HE}_{1}^{\prime} \) mode. For this mode, there is no critical point and also there is no net negative power flow region in the dispersion curve. The \( \text{HE}_{1}^{\prime} \) mode reaches cutoff when the value of \((2a/\lambda_0)\) reaches 0.612.

![Dispersion curve of \( \text{HE}_{1}^{\prime} \) mode in a LH waveguide with circular cross section](image)

**Figure 5.7:** Dispersion curve of \( \text{HE}_{1}^{\prime} \) mode in a LH waveguide with circular cross section

When the diameter \((2a/\lambda_0)\) of the waveguide reaches 0.60844 (point \(A_0\) in Figure 5.7), the \( \text{TE}_{01} \) mode appears. This point, \(A_0\), is defined as the critical point and the reason for this definition is as follows.
Figure 5.8: Dispersion curve of TE_{01}' and TE_{01} mode in a LH waveguide with circular cross section.
At $A_0$, the portion of the guided mode inside the LH rod shows the Poynting vector to be anti-parallel to the direction of the mode’s phase flow and the portion of this mode outside the slab shows the Poynting vector to be parallel with the phase flow. But the net power is 0 for this case. With increase in the diameter ($2a/\lambda_0$) beyond the critical point, there exist two TE$_{01}$ modes in the special region. In the special region, the dispersion curve gets bifurcated in two different directions, that is, curve I is decomposed into curve $A_0A_1$ and curve $A_0A_2$. For the curve $A_0A_1$, the portion of the guided mode inside the LH waveguide shows the Poynting vector to be antiparallel to the direction of the mode’s phase flow and the portion of guided mode outside the slab shows the Poynting vector to be parallel with the phase flow. But here the total power flow of the part of the mode inside the slab is smaller than that of the total power flow of the part of the mode outside. In this case, the net total power flow of the guided mode is parallel with the direction of the phase flow, in other words, the $A_0A_1$ portion has net positive power flow and is defined as TE$_{01}$ mode.

For the curve $A_0A_2$, the portion of the guided mode inside the LH waveguide shows the Poynting vector to be anti-parallel to the direction of the mode’s phase flow and the portion of guided mode outside the slab shows the Poynting vector to be parallel with the phase flow. However, in this case the total power flow of the part of the mode inside the slab is greater than that of the total power flow of the part of the mode outside. Hence the net total power flow of the guided mode is antiparallel with the direction of the phase flow. When the value of ($2a/\lambda_0$) reaches 0.614, the curve $A_0A_1$ in the special region vanishes and the dispersion curve I is out of the special region. Curve $A_2A_3$ has the same characteristics as that of curve $A_0A_2$; the $A_0A_3$ portion of curve I that has net negative power flow is defined as TE$_{01}$ mode.
Figure 5.9: Dispersion curve of $\text{TM}_{01}'$ and $\text{TM}_{01}$ mode in a LH waveguide with circular cross section
The dispersion curve of $\text{TM}_{01}$ mode in Figure 5.8 shows a similar pattern as that of $\text{TE}_{01}$ mode shown in Figure 5.7. Hence a similar analysis can be done for the $\text{TM}_{01}$ mode and the explanation for $\text{TE}_{01}$ mode holds for the dispersion behavior of $\text{TM}_{01}$ mode, as well.
CHAPTER 6 : CONCLUSION

This thesis is to investigate the dispersion characteristics of waveguides with simultaneous negative dielectric permittivity and magnetic permeability, otherwise known as left-handed waveguides, which may lead to unconventional phenomena in guidance, radiation, and scattering of electromagnetic waves. A different approach of formulating and solving an eigenvalue problem with finite element method resulting in the dispersion relation of the waveguides was illustrated.

Grounded slab waveguide with conventional media is analyzed using one-dimensional scalar finite element formulation. The results are compared with those obtained from traditional method and excellent agreement was found. Then the grounded slab waveguide is analyzed with LH media and the anomalous dispersion characteristics are found. In the case of TM modes, all the modes in the grounded LH slab waveguide have cutoff frequencies. Moreover, each dispersion curve is bent in a special region. In the special region, one guided mode has the net negative total power flow, which is antiparallel to the direction of the phase flow. However, another guided mode has the net positive total power flow, which is parallel to the direction of the phase flow. On the other hand, for TE modes, the dominant mode appears when the slab thickness is increased from zero and reaches cut off after a certain thickness. Also there exists a cut off region where there is no mode propagation in the LH slab. For higher order modes the dispersion curves are bent in the special region similar to the TM modes.

The discontinuity structure of a LH waveguide, sandwiched between two conventional dielectric slab waveguides, is analyzed using the mode matching technique. Four cases, each with a different height of the discontinuity structure, have been considered and the results have
been discussed. For case 1, when all the modes in the LH waveguide are cutoff, the percentage power of the transmitted TM$_0$ mode is 0. For case 2, when the guided mode in the LH waveguide is at the critical point, the reflection of TM$_0$ mode is the majority component except for the resonant points where the TM$_0$ mode has no reflection and most of the power is transmitted. For case 3, when the LH waveguide is in the special region, there are oscillations in the reflected and transmitted power because of the coupling of TM$_0$ and TM$'_0$ modes. The reflection of the TM$_0$ mode is much smaller and the transmission takes the majority component. For case 4, when only one TM mode exists in the LH waveguide, this mode is almost matched with the TM$_0$ mode in the dielectric waveguide. In this case, the reflected TM$_0$ and TM$_1$ modes and the transmitted TM$_1$ mode are very small. Therefore the transmitted TM$_0$ is the majority component.

The scattering characteristics of a parallel plate waveguide partially filled with LH and conventional media are analyzed using finite element method together with eigenfunction expansion technique. From the analysis, it is found that oscillations appear at higher frequencies in conventional media case but these oscillations disappear in the LH media case. The LH media case has a wide transmission band when compared with the conventional media case. Also, there is an excellent agreement of results between the eigenfunction expansion technique and mode matching technique.

A robust two-dimensional vector finite element formulation for the analysis of arbitrary shaped waveguide structures was presented in detail. With excellent agreement of results for the conventional waveguide the method is then applied to analyze LH waveguides with a circular cross section and anomalous dispersion was noted in LH waveguides. The dominant mode appears when the rod diameter is increased from zero and reaches cut off after a certain diameter.
For higher order modes, the dispersion curves are bent in the special region similar to the grounded LH slab waveguide.
APPENDIX A: RITZ VARIATIONAL PRINCIPLE
The Ritz method, also known as the Rayleigh-Ritz method, is a variational method in which the boundary-value problem is formulated in terms of a variational expression called functional [12]. The minimum of this corresponds to the governing differential equation under the given boundary conditions. The approximate solution is then obtained by minimizing the functional with respect to variables that define a certain approximation to the solution.

Variational method is one of the two methods often employed to formulate finite element solutions. The variational method starts from the variational formulation; therefore applicability of the method depends on the availability of such a variational formulation. In this method the variational formulation is stated first and then proved by taking the first variation of the functional, this is a widely adopted practice. When seeking a variational solution for problems in terms of differential equations a logical procedure is to derive the variational formulation rather than to prove it [12]. A detailed proof for the standard variational principle, used in chapter 2, and its limitations is given here.

Given a boundary value problem defined by the differential equation

\[ L\phi = f \]  \hspace{1cm} (a.1)

if the operator \( L \) is self–adjoint, that is,

\[ \langle L\phi, \psi \rangle = \langle \phi, L\psi \rangle \]  \hspace{1cm} (a.2)

and positive definite, that is,

\[ \langle L\phi, \psi \rangle \begin{cases} > 0 & \phi \neq 0 \\ = 0 & \phi = 0 \end{cases} \]  \hspace{1cm} (a.3)

then the solution to the original boundary value problem can be obtained by minimizing the functional by
\[ F(\phi) = \frac{1}{2} \langle L\phi, \phi \rangle - \frac{1}{2} \langle \phi, f \rangle - \frac{1}{2} \langle f, \phi \rangle \quad (a.4) \]

In equation (a.2) \( \psi \) denotes an arbitrary function satisfying the same boundary conditions as \( \phi \).

In the above equations the angular brackets denote the inner product defined by,

\[ \langle \phi, \psi \rangle = \int_{\Omega} \phi \psi^* \, d\Omega \quad (a.5) \]

where \( \Omega \) denote the domain of the problem, which could be one-, two-, or three dimensional, and the asterisk denotes the complex conjugate.

In order to prove this variational principle, we need to show that the differential equation (a.1) is the necessary consequence when the functional \( F \) is stationary, that is, \( \delta F = 0 \). Then, we need to show that the stationary point is at the minimum of the functional \( F \), which is equivalent to showing that \( \delta (\delta F) > 0 \).

Let us consider the first condition by taking the first variation of (a.4) to find

\[ \delta F = \frac{1}{2} \langle L\delta \phi, \phi \rangle + \frac{1}{2} \langle L\phi, \delta \phi \rangle - \frac{1}{2} \langle \delta \phi, f \rangle - \frac{1}{2} \langle f, \delta \phi \rangle. \quad (a.6) \]

Due to the self-adjoint property of \( L \) the first term on the right-hand side can be written as,

\[ \frac{1}{2} \langle L\delta \phi, \phi \rangle = \frac{1}{2} \langle \delta \phi, L\phi \rangle \quad (a.7) \]

resulting in

\[ \delta F = \frac{1}{2} \langle \delta \phi, L\phi - f \rangle + \frac{1}{2} \langle L\phi - f, \delta \phi \rangle. \quad (a.8) \]

From the definition of the inner product, we have

\[ \delta F = \frac{1}{2} \langle \delta \phi, L\phi - f \rangle + \frac{1}{2} \langle \delta \phi, L\phi - f \rangle^* = \text{Re}\langle \delta \phi, L\phi - f \rangle \quad (a.9) \]

where \( \text{Re}(\cdot) \) denotes the real part of (\( \cdot \)). Imposing the stationarity requirement \( \delta F = 0 \), we obtain
from which it can be concluded that $\phi$ must satisfy equation (a.1) since $\delta \phi$ is an arbitrary variation. From this we can find that the first condition is proved.

Now, let us consider the second condition by taking the first variation of $\delta F$ once again. This yields

$$\delta (\delta F) = \delta F (\phi + \delta \phi) - \delta F (\phi) = \text{Re} \langle \delta \phi, L \delta \phi \rangle$$ \hspace{1cm} (a.11)

Since $L$ is positive definite, from equation (a.3) we have $\delta (\delta F) > 0$ for nontrivial $\delta \phi$. Therefore the stationary point is indeed at the minimum of $F$.

From the above proof, it is clear that to use the standard variation principle to construct the functional $F$ whose minimum corresponds to the original boundary-value problem; it must satisfy the self-adjoint and positive definite conditions stated in equations (a.2) and (a.3) [12]. Since the goal is to solve equation (a.1) the second (positive definite) condition is not necessary, although in many problems the solution indeed corresponds to the minimum. Therefore the only limitation attached to this variation principle is the condition of self-adjointness.
APPENDIX B: PROOF OF THE VARIATIONAL PRINCIPLE
A detailed proof for the variational principle used in the eigenfunction expansion technique in chapter 4 is given here. The functional for the partially filled waveguide problem is given as,

\[
F(E) = \frac{1}{2} \int_{s} \left[ \frac{1}{\mu_r} \left( \nabla \times E \right) \cdot \left( \nabla \times E \right) - k_0^2 \varepsilon_r E \cdot E \right] ds - \int_{z_1} \left[ \frac{1}{2} E \cdot P(E) - E \cdot U^{inc} \right] dl - \int_{z_2} \left[ \frac{1}{2} E \cdot P(E) \right] dl \quad \text{(b.1)}
\]

Taking the first variation of the above functional we have,

\[
\delta F(E) = \int_{s} \left[ \frac{1}{\mu_r} \left( \nabla \times E \right) \cdot \left( \nabla \times \delta E \right) \right] ds - \int_{s} \left[ k_0^2 \varepsilon_r E \cdot \delta E \right] ds - \int_{z_1} \left[ \frac{1}{2} E \cdot \delta P(E) + \frac{1}{2} \delta E \cdot P(E) - \delta E \cdot U^{inc} \right] dl - \int_{z_2} \left[ \frac{1}{2} E \cdot \delta P(E) + \frac{1}{2} \delta E \cdot P(E) \right] dl \quad \text{(b.2)}
\]

We know that,

\[
\nabla \cdot \left[ (\nabla \times E) \times \delta E \right] = \nabla \cdot \left[ (\nabla \times E) \times \delta E \right] + \nabla \cdot \left[ (\nabla \times E) \times \delta E \right] = -(\nabla \times E) \cdot (\nabla \times \delta E) + \delta E \cdot [(\nabla \times (\nabla \times E)] \quad \text{(b.3)}
\]

\[
(\nabla \times E) \cdot (\nabla \times \delta E) = -\nabla \cdot \left[ (\nabla \times E) \times \delta E \right] + \delta E \cdot [(\nabla \times (\nabla \times E)] \quad \text{(b.4)}
\]

Substituting equations (b.3) and (b.4) into equation (b.2) we get,

\[
\delta F(E) = \int_{s} \left\{ \left( -\frac{1}{\mu_r} \right) \left[ \nabla \times (\nabla \times E) \right] \right\} ds + \int_{s} \left\{ \left( \frac{1}{\mu_r} \right) \left[ \nabla \times (\nabla \times E) - k_0^2 \varepsilon_r E \right] \cdot \delta E \right\} ds - \int_{z_1} \left[ \frac{1}{2} E \cdot \delta P(E) + \frac{1}{2} \delta E \cdot P(E) - \delta E \cdot U^{inc} \right] dl - \int_{z_2} \left[ \frac{1}{2} E \cdot \delta P(E) + \frac{1}{2} \delta E \cdot P(E) \right] dl \quad \text{(b.5)}
\]
We know \( \int_{s} T(\nabla) \, ds = \int T(\hat{n}) \, dl \), \( \hat{n} \) is unit vector normal to the surface and the tangential vector on the surface is zero. Therefore we have,

\[
\delta F(E) = \int_{z_1} \left\{ \left( -\frac{1}{\mu_r} \right) (\hat{\mu}_r) \cdot \left[ (\nabla \times E) \times \delta E \right] + \left[ \frac{1}{2} \delta E \cdot \delta P(E) + \frac{1}{2} \delta E \cdot P(E) \right] \right\} \, dl
+ \int_{z_2} \left\{ \left( -\frac{1}{\mu_r} \right) (\hat{\mu}_r) \cdot \left[ (\nabla \times E) \times \delta E \right] - \frac{1}{2} \delta E \cdot \delta P(E) - \frac{1}{2} \delta E \cdot P(E) \right\} \, dl
- \int_{z_1} \frac{1}{2} \delta E \cdot \delta P(E) + \frac{1}{2} \delta E \cdot P(E) \right\} \, dl
- \int_{z_2} \left[ \frac{1}{2} \delta E \cdot \delta P(E) + \frac{1}{2} \delta E \cdot P(E) \right] \, dl
+ \int_{s} \left\{ \left( \frac{1}{\mu_r} \right) \nabla \times (\nabla \times E) - k_0^2 \varepsilon, E \right\} \cdot \delta E \right\} \, ds
\]

which can be written as

\[
\delta F(E) = \int_{z_1} \left\{ \left( -\frac{1}{\mu_r} \right) \delta E \cdot [-\hat{\mu}_r \times (\nabla \times E)] - \frac{1}{2} \delta E \cdot \delta P(E) - \frac{1}{2} \delta E \cdot P(E) + \delta E \cdot U^{inc} \right\} \, dl
+ \int_{z_2} \left\{ \left( -\frac{1}{\mu_r} \right) \delta E \cdot [-\hat{\mu}_r \times (\nabla \times E)] - \frac{1}{2} \delta E \cdot \delta P(E) - \frac{1}{2} \delta E \cdot P(E) \right\} \, dl
+ \int_{s} \left\{ \left( \frac{1}{\mu_r} \right) \nabla \times (\nabla \times E) - k_0^2 \varepsilon, E \right\} \cdot \delta E \right\} \, ds
\]  

(b.7)

Let us consider the term \( \int_{z_1} \frac{1}{2} \delta E \cdot \delta P(E) \) \, dl and simplify,

\[
P(E) = - \sum_{m} \gamma_m e_m(x) \int_{z_1} e_m(x') \cdot E(x') \, dx'
\]  

(b.8)

\[
\delta P(E) = - \sum_{m} \gamma_m e_m(x) \int_{z_1} e_m(x') \cdot \delta E(x') \, dx'
\]  

(b.9)

\[
E \cdot \delta P(E) = - \sum_{m} \gamma_m e_m(x) \cdot E(x) \int_{z_1} e_m(x') \cdot \delta E(x') \, dx'
\]  

(b.10)
\[ \frac{1}{2} \int_{z_1}^{z_2} \mathbf{E} \cdot \delta \mathbf{P}(\mathbf{E}) \, dx = -\frac{1}{2} \sum_{m} \int_{z_1}^{z_2} \gamma_m e_m(x) \cdot \mathbf{E}(x) \, dx \int_{z_1}^{z_2} e_m(x') \cdot \delta \mathbf{E}(x') \, dx' \quad \text{(b.11)} \]

Interchanging between \( x \leftrightarrow x' \) and \( y \leftrightarrow y' \) in the above expression we have

\[ \frac{1}{2} \int_{z_1}^{z_2} \mathbf{E} \cdot \delta \mathbf{P}(\mathbf{E}) \, dx = -\frac{1}{2} \sum_{m} \int_{z_1}^{z_2} \gamma_m e_m(x') \cdot \mathbf{E}(x') \, dx' \int_{z_1}^{z_2} e_m(x) \cdot \delta \mathbf{E}(x) \, dx \quad \text{(b.12)} \]

From the above equation we can find

\[ \int_{z_1}^{z_2} \left\{ \frac{1}{2} \mathbf{E} \cdot \delta \mathbf{P}(\mathbf{E}) \right\} \, dl = \int_{z_1}^{z_2} \left\{ \frac{1}{2} \delta \mathbf{E} \cdot \mathbf{P}(\mathbf{E}) \right\} \, dl \]

Using the above relation in equation (b.7) we get,

\[ \delta \mathbf{F}(\mathbf{E}) = \int_{z_1}^{z_2} \left\{ \left[ \left( -\frac{1}{\mu_r} \right) \left[ -\hat{\mathbf{z}} \times (\nabla \times \mathbf{E}) \right] - P(\mathbf{E}) + U^{\text{inc}} \right] \cdot \delta \mathbf{E} \right\} \, dl \]

\[ + \int_{z_1}^{z_2} \left\{ \left[ \left( -\frac{1}{\mu_r} \right) \left[ -\hat{\mathbf{z}} \times (\nabla \times \mathbf{E}) \right] - P(\mathbf{E}) \right] \cdot \delta \mathbf{E} \right\} \, dl \]

\[ + \oint_{s} \left\{ \left[ \frac{1}{\mu_r} \nabla \times (\nabla \times \mathbf{E}) - k_0^2 \varepsilon_r \mathbf{E} \right] \cdot \delta \mathbf{E} \right\} \, ds \quad \text{(b.13)} \]

Imposing the stationarity requirement \( \delta \mathbf{F}(\mathbf{E}) = 0 \), we obtain

\[ \int_{z_1}^{z_2} \left\{ \left[ \left( -\frac{1}{\mu_r} \right) \left[ -\hat{\mathbf{z}} \times (\nabla \times \mathbf{E}) \right] - P(\mathbf{E}) + U^{\text{inc}} \right] \cdot \delta \mathbf{E} \right\} \, dl \]

\[ + \int_{z_2}^{z_1} \left\{ \left[ \left( -\frac{1}{\mu_r} \right) \left[ -\hat{\mathbf{z}} \times (\nabla \times \mathbf{E}) \right] - P(\mathbf{E}) \right] \cdot \delta \mathbf{E} \right\} \, dl \]

\[ + \oint_{s} \left\{ \left[ \frac{1}{\mu_r} \nabla \times (\nabla \times \mathbf{E}) - k_0^2 \varepsilon_r \mathbf{E} \right] \cdot \delta \mathbf{E} \right\} \, ds = 0 \quad \text{(b.14)} \]

Since \( \delta \mathbf{E} \) is an arbitrary variation, the integral part of the above equation must vanish. Thus we have,

\[ \left( -\frac{1}{\mu_r} \right) \left[ -\hat{\mathbf{z}} \times (\nabla \times \mathbf{E}) \right] - P(\mathbf{E}) + U^{\text{inc}} = 0 \quad \text{(b.15)} \]
\[
\left( -\frac{1}{\mu_r} \right) \left[ \hat{z} \times (\nabla \times \mathbf{E}) \right] - P(\mathbf{E}) = 0
\]  
\hspace{2cm} \text{(b.16)}

\[
\left( \frac{1}{\mu_r} \right) \nabla \times (\nabla \times \mathbf{E}) - k_0^2 \varepsilon_r \mathbf{E} = 0
\]  
\hspace{2cm} \text{(b.17)}

In the above set of equations it can be noted that equations (b.15) and (b.16) represent the boundary conditions at the two fictitious planes at \( z_1 \) and \( z_2 \) respectively. Also, equation (b.17) refers to the equation of the partially filled waveguide. Thus the minimum of the functional corresponds to the governing differential equation of the problem under the given boundary conditions.
APPENDIX C: GALERKIN METHOD
Galerkin’s method is one of the weighted residual methods, which as the name indicates, seek the solution by weighting the residual of the differential equation. In this method, the weighting functions are selected to be the same as those used for expansion of the approximate solution. This usually leads to the most accurate solution and is hence a popular approach in developing finite element equations. In chapter 5, we used Galerkin’s method in the finite element formulation to analyze arbitrary shaped waveguide structures. A brief illustration of Galerkin’s method is given below.

Consider the one-dimensional Sturm-Liouville problem,

\[-d\left(p(x)\frac{dU}{dx}\right) + q(x)U(x) = f(x) \quad 0 < x < a\]  

(c.1)

where \(p(x)\), \(q(x)\), and \(f(x)\) are known functions and \(U(x)\) is the unknown field quantity. Methods of weighted residual are useful to obtain approximate solutions to the above differential equation. As a first step we introduce the residual,

\[R(x) = -d\left(p(x)\frac{dU}{dx}\right) + q(x)U(x) - f(x)\]  

(c.2)

which must of course be zero in accordance with the problem. However, it is impractical to enforce \(R(x) = 0\) at every point in the problem domain, that is from \(x = 0\) to \(x = a\). Since \(U(x)\) is not expected to show substantial variation over a small distance, say \(\Delta x\), we subdivide the domain into small segments and enforce the following condition over each segment.

\[\int_{\text{Domain of } W_m} W_m(x)R(x)dx = 0\]  

(c.3)

With \(W_m(x)\) being some weighting function the above condition enforces the differential equation on an average sense over the \(m\)th segment. A linear system of equations can be generated from equation (c.3), but this formulation requires the evaluation of the highest order of derivative term.
(second derivative) in the differential equation. The integral must have a non-zero finite value to yield a meaningful approximate solution to the differential equation, which means that second derivative should not vanish. This formulation is usually referred as strong formulation.

So to cast equation (c.3) in more suitable form we take advantage of the weighting function $W_m(x)$ to reduce the order of the derivative contained in $R(x)$. To do so we employ integration by parts, resulting in

$$
- \int_0^x W_m(x) \left( p(x) \frac{dU}{dx} \right) dx = - \left[ p(x) W_m(x) \frac{dU}{dx} \right]_{x=0}^{x=x_m} + \int_0^x p(x) \frac{dW(x)}{dx} \frac{dU}{dx} dx \quad (c.4)
$$

The first term on the right hand side of the above equation can be evaluated by enforcing the boundary conditions at the endpoints. Substituting equation (c.4) into (c.3) gives the weak form of the differential equation, which is

$$
\int_0^x \left[ p(x) \frac{dW(x)}{dx} \frac{dU(x)}{dx} + q(x) W_m(x) U(x) - W_m(x) f(x) \right] dx - \left[ p(x) W_m(x) \frac{dU(x)}{dx} \right]_{x=0}^{x=x_m} = 0 \quad (c.5)
$$

In the above equation it can be noted that the order of differentiation is reduced to one and because of the integral the weak form enforces the differential equation on an average (and therefore weaker) sense. The remarkable ability of the weak formulation is that it incorporates in a single mathematical statement the requirements imposed by the differential equation and the boundary conditions at the endpoints [27]. That is, upon substitution of the boundary conditions on $U(x)$ and $dU(x)/dx$, the weak form in not only an alternative statement of the differential equation (c.1), but also includes the information about the boundary conditions which are essential for the uniqueness of the solution.
APPENDIX D: VECTOR IDENTITIES
\[ \mathbf{A} \cdot (\nabla \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - \nabla \times (\mathbf{A} \times \mathbf{B}) \]  
(d.1)

\[
\int \int \nabla \cdot (\mathbf{A} \times \mathbf{B})\, ds = \int (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{n}\, dl = -\int \mathbf{A} \cdot (n \times \mathbf{B})\, dl 
\]  
(d.2)

\[ \nabla \cdot f \, \mathbf{A} = \mathbf{A} \cdot \nabla f + f \, \nabla \cdot \mathbf{A} \]  
(d.3)

\[
\int \int \nabla \cdot \mathbf{A}\, ds = \int \mathbf{A} \cdot \mathbf{n}\, dl 
\]  
(d.4)
LIST OF REFERENCES


9. A. Alu and N. Engheta, “Guided modes in a waveguide filled with a pair of single-negative (SNG), double-negative (DNG), and /or double-positive (DPS) layers,” IEEE Trans. on Microwave Theory and Techniques, vol.53, no.1, January 2004.


