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Painlevé analysis and Lie group symmetries of the regularized long-wave equation

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The Painlevé formulation is given for the regularized long-wave equation, which has been used to describe shallow water waves and drift waves in a plasma. The Lie group symmetries of the equation is then studied and it is shown that the only symmetries are due to the three conserved quantities.

I. INTRODUCTION

Peregrine¹ originally derived the regularized long-wave equation as an alternative description to the Korteweg-de Vries equation for shallow water waves. Meiss and Horton² found it also was applicable to the study of drift waves in a plasma. Benjamin *et al.*³ has suggested that the regularized long-wave equation is more satisfactory than the Korteweg-de Vries equation as it does not have some of the difficulties than the latter. In contrast to the Korteweg-de Vries equation, only three conservation laws have been found (Olver⁴) and collisions between solitary wave solutions are inelastic (Bona *et al.*,⁵ Morrison *et al.*,⁶ and Gardner and Gardner⁷) for the regularized long-wave equation.

The Painlevé property can be shown directly for partial differential equations as proposed by Weiss *et al.*⁸ by checking if its solutions are single valued about a singular manifold. In this paper we give a Painlevé analysis for the regularized long-wave equation and then consider the Lie group symmetries of the equation. We find there to be only three symmetries.

II. THE PAINLEVÉ PROPERTY FOR THE REGULARIZED LONG-WAVE EQUATION

The nonlinear propagation of solitary waves in shallow water has been described by the regularized long-wave equation:

$$u_t + u_x - uu_x - u_{xx} = 0. \quad (1)$$

The local behavior of the solutions to Eq. (1) about a movable singular manifold,

$$\phi(x,t) = 0, \quad (2)$$

is determined by finding the dominant balance in the equation when

$$u(x,t) = u_0(x,t) [\phi(x,t)]^\alpha. \quad (3)$$

Substitution of (3) into (1) and balancing the highest derivative with the nonlinear term gives

$$\alpha = -2 \quad \text{and} \quad u_0(x,t) = -12\phi_x\phi_t. \quad (4)$$

The resonances are determined by setting

$$u = u_0\phi^{-2} + p\phi^{r-2}, \quad (5)$$

and balancing the most singular terms in Eq. (1), giving

$$p[-2u_0\phi_x + (r-2)u_0\phi_x + (r-2)(r-3)(r-4)\phi_x^2\phi_t] = 0. \quad (6)$$

Using (4), we find that the coefficient p is arbitrary when $r = -1, 4, 6$. The first resonance $r = -1$ corresponds to the arbitrariness of the function $\phi(x,t)$.

We then assume a generalized Laurent expansion of the form

$$u(x,t) = \phi^{-2} \sum_{j=0}^{\infty} u_j(x,t)\phi(x,t)^j, \quad (7)$$

where the $u_j(x,t)$ and $\phi(x,t)$ are analytic functions in the neighborhood of $\phi = 0$. Substitution of (7) into (1) will lead to recursion relations between the u_j 's. The analysis is simplified greatly by using the Kruskal ansatz for the form of the singularity manifold, namely,

$$\phi(x,t) = x - \psi(t), \quad (8)$$

which is permitted by the implicit function theorem, whenever the singular manifold is not a characteristic for the equation (Ward⁹). This form of ϕ allows one to write $u_j = u_j(t)$. The recursion relations are now coupled ordinary differential equations for the u_j 's.

Using (7) and (8) in (1), and collecting powers of ϕ through to $j = 6$ gives the following conditions:

$$\phi^{-5}: u_0^2 - 12u_0\psi_t = 0, \quad (9)$$

$$\phi^{-4}: u_1u_0 - 2u_1\psi_t - 2u_{0,t} = 0, \quad (10)$$

$$\phi^{-3}: 2u_0u_2 + u_1^2 - 2u_{1,t} - 2u_0(1 - \psi_t) = 0, \quad (11)$$

$$\phi^{-2}: u_0u_3 + u_1u_2 - (1 - \psi_t)u_1 + u_{0,t} = 0, \quad (12)$$

$$\phi^{-1}: 0 \cdot u_4 + u_{1,t} = 0, \quad (13)$$

$$\phi^0: u_0u_5 + 6u_5\psi_t + u_1u_4 + 2u_{4,t} + u_2u_3 - (1 - \psi_t)u_3 - u_{2,t} = 0 \quad (14)$$

$$\phi^1: 0 \cdot u_6 - 2u_1u_5 - 6u_{5,t} - 2u_2u_4 + 2(1 - \psi_t)u_4 + u_{3,t} - u_3^2 = 0. \quad (15)$$

Equation (13) implies that u_4 will be arbitrary if the compatibility condition

$$u_{1,t} = 0 \quad (16)$$

is satisfied, which on using (9) and (10) can be written as

$$\psi_t\psi_{tt} - \psi_{tt}^2 = 0. \quad (17)$$

Using (14) and its time derivative in (15), and making use of the compatibility condition (16) implies

$$0 \cdot u_6 + 10u_1 u_{4,t} + 24u_{4,t} + (2u_{3,t} - u_3^2)u_0 + (u_{2,t} + (1 - \psi_t)u_3 - u_2 u_3)u_1 - 12(u_{2,t} - (\psi_t + u_{2,t})u_3 + (1 - u_2)u_{3,t}) = 0. \quad (18)$$

Since u_4 is arbitrary and the coefficient of $u_{4,t}$ never vanishes it is not possible for u_6 to be arbitrary and (15) cannot be satisfied. We conclude that the generalized Laurent series of (7) cannot be a valid local representation of the solution to (1) in the neighborhood of the singular manifold. Hence the regularized long-wave equation does not have the Painlevé property. Failing the Painlevé test does not imply nonintegrability but the evidence suggests that the regularized long-wave equation is not integrable.

III. THE LIE GROUP SYMMETRIES

To examine the Lie group symmetries of the regularized long-wave equation we first rewrite the equation in a different notation

$$u_2 + u_1 - uu_1 - u_{112} = 0 \quad (19)$$

to facilitate the calculation, where $x_1 = x$ and $x_2 = t$ and a subscript j will refer to differentiation with respect to x_j . Consider the invariance of (19) under the local Lie group of infinitesimal transformations ξ^j and η :

$$\bar{u} = u + \epsilon \eta(x_1, x_2, u) + O(\epsilon^2) \\ \bar{x}_j = x_j + \epsilon \xi^j(x_1, x_2, u) + O(\epsilon^2), \quad j = 1, 2. \quad (20)$$

The invariance condition on the generators and their extensions $\eta^{(k)}$, to the k th partial derivatives, is that

$$\eta_1^{(1)} + \eta_2^{(1)} - u_1 \eta - u \eta_1^{(1)} - \eta_{112}^{(3)} = 0. \quad (21)$$

The extensions are given by

$$\eta_j^{(1)} = \eta_j + \eta_u u_1 - \xi_j^k u_k \quad (22) \\ \eta_{112}^{(3)} = \eta_{112} + 2\eta_{12u} u_1 + \eta_{11u} u_2 + 2\eta_{1u} u_{12} \\ + \eta_{2u} u_{11} + \eta_u u_{112} - \xi_{112}^k u_k - \xi_{11}^k u_{2k} \\ - 2\xi_{12}^k u_{1k} - 2\xi_1^k u_{12k} - \xi_2^k u_{12k}, \quad (23)$$

where repeated indices are summed over $k = 1, 2$. Since the equation is quasilinear (see Bluman and Kumei¹⁰), the generators satisfy

$$\frac{\partial \xi^j}{\partial u} = 0, \quad j = 1, 2, \quad (24) \\ \frac{\partial^2 \eta}{\partial u^2} = 0 \Rightarrow \eta = f(x_1, x_2)u + g(x_1, x_2),$$

which will simplify the calculations. Substitution of (22)–(24) in (21) give 12 partial differential equations for the

generators ξ^j and η , five of which give the conditions

$$f_1 = 0, \quad \xi_2^1 = 0, \quad \xi_1^2 = 0. \quad (25)$$

These conditions simplify the remaining seven equations to

$$g_1 + g_2 - g_{112} = 0, \quad (26)$$

$$f_2 - g_1 = 0, \quad (27)$$

$$\xi_1^1 + \xi_2^2 - g = 0, \quad (28)$$

$$\xi_1^1 = 0, \quad (29)$$

$$f_2 = 0, \quad (30)$$

$$\xi_{11}^1 = 0, \quad (31)$$

$$\xi_1^1 + \xi_2^2 + f = 0. \quad (32)$$

Solving Eqs. (26)–(32) gives the generators for the infinitesimal transformations as

$$\xi^1 = \alpha, \\ \xi^2 = -\beta x_2 + \gamma, \\ \eta = \beta u - \beta. \quad (33)$$

This is a three parameter group (α, β, γ) with infinitesimal generators given by

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial x_2}, \\ X_3 = -x_2 \frac{\partial}{\partial x_2} + (u - 1) \frac{\partial}{\partial u}, \quad (34)$$

X_1 corresponds to the invariance under a spatial translation, X_2 to the invariance under a time translation and X_3 to the equation's invariance under the transformation:

$$\bar{x}_2 = x_2/k, \quad \bar{x}_1 = x_1, \quad \bar{u} = k(u - 1) + 1, \quad k = \text{const.} \quad (35)$$

These three invariances correspond to the conservation of the momentum, energy, and "mass," respectively. These symmetries and resulting invariances have been previously obtained by a different method.⁶ None of these transformations allow one to reduce the equation to an ordinary differential equation.

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