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I’m Being Framed: Phase Retrieval and Frame Dilation in Finite-Dimensional Real Hilbert Spaces

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I'M BEING FRAMED: PHASE RETRIEVAL AND FRAME DILATION IN FINITE-DIMENSIONAL REAL HILBERT SPACES

by

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ABSTRACT

Research has shown that a frame for an n-dimensional real Hilbert space offers phase retrieval if and only if it has the complement property. There is a geometric characterization of general frames, the Han-Larson-Naimark Dilation Theorem, which gives us the necessary and sufficient conditions required to dilate a frame for an n-dimensional Hilbert space to a frame for a Hilbert space of higher dimension k. However, a frame having the complement property in an n-dimensional real Hilbert space does not ensure that its dilation will offer phase retrieval. In this thesis, we will explore and provide what necessary and sufficient conditions must be satisfied to dilate a phase retrieval frame for an n-dimensional real Hilbert space to a phase retrieval frame for a k-dimensional real Hilbert.

DEDICATIONS

I dedicate this work to my friend, Mike Jefferson, for the incessant badgering that helped push me to re-enroll, to my wife, Allison, for showing me true strength and also keeping me in check, to my sister, Sarah, for keeping me humble and grounded, to my mum, Pamela, for showing me how to be resilient enough to get through anything, and to my dad, Garry, the original Greuling mathematician who's footsteps I will forever follow in, for teaching me patience, dedication, and integrity. I would not be where I am today without all of you.

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CHAPTER 1: INTRODUCTION

Reconstructing a signal without its phases presented a challenge for many years as it had long been thought that the use of a signal's phase was necessary for that signal's reconstruction [6]. The process involves taking a signal from its domain space and projecting its image through some linear transformation onto a space of coefficients before eventually projecting coefficients through some linear transformation back onto the domain space in order to synthesize the signal [2]. By passing a signal through a linear operator, it is often the case that only the intensity measurements are retained, resulting in the loss of the signal's phase in the process [6]. The loss of phase presented a daunting problem, known as the phase retrieval problem, in many engineering applications [5]. This led to the study of recovering signals without phase and its importance is evident from its appearance in the applications of many areas such as optics, X-ray crystallography, electron microscopy, coherence theory, diffractive imaging, astronomical imagining, X-ray tomography, speech recognition technology, and quantum state tomography [1][2][3][4][6][9].

Whenever a signal gets transmitted and received, there can be data loss and this complicates the signal's reconstruction [4]. This makes defining a linear operator to perform signal reconstruction from an orthonormal basis impractical despite the fact that, by the Parseval formula, an orthonormal basis provides perfect reconstruction [8]. Naturally, we utilize frames in the study of phase retrieval since frames can have a redundancy greater than 1 and are spanning sets of vectors for a finite-dimensional Hilbert space $\mathbb{H}$ such that any signal in $\mathbb{H}$ can be synthesized from its frame coefficients [2]. Because we desire recovering a signal from its intensity measurements rather than using its phase, we want frames that have the same intensity measurements for any two vectors in $\mathbb{H}$ differing by a unimodular constant [6]. We call these phase retrieval frames.

Let $\mathcal{F} = \{\mathbf{f}_1, ..., \mathbf{f}_l\}$ is any frame that offers phase retrieval frame for n-dimensional real Hilbert space $\mathbb{H}$. Suppose we want utilize $\mathcal{F}$ to recover signals in a real Hilbert space $\mathbb{K}$ of higher dimension k. There exists a very useful geometric characterization for redundant frames referred to as the Han-Larson-Naimark Dilation Theorem [7]: given a collection of vectors $\mathcal{F} = \{\mathbf{f}_1, ..., \mathbf{f}_l\}$ in an n-dimensional Hilbert space $\mathbb{H}$, $\mathcal{F}$ is a frame for $\mathbb{H}$ if and only if it is the orthogonal compression of a basis for a Hilbert space $\mathbb{L}$ of higher dimension l. Hence, we know $\mathcal{F}$ can be dilated to a basis for $\mathbb{L}$. However, a frame requires at least $2l - 1$ terms to offer phase retrieval for an l-dimensional real Hilbert space [6]. Thus, the dilation of $\mathcal{F}$ won't offer phase retrieval for $\mathbb{L}$. However, the dilation of $\mathcal{F}$ can be compressed to a frame for $\mathbb{K}$ [8]. So the natural question is, does this compression offer phase retrieval for $\mathbb{K}$?

CHAPTER 2: FRAMES

Before discussing frame dilation and phase retrieval frames, we need to recall the formal definition of a frame and some relevant terms and properties (c.f. [8]). The reconstruction of a signal in a Hilbert space from a set of coefficients is our primary motivating factor. Essentially this means passing a vector from some space of coefficients through a linear operator in order to synthesize our signal.

Definition 2.1. A **linear operator** between vector spaces $\mathbb{X}$ and $\mathbb{Y}$ over field $\mathbb{F}$ is any mapping $F : \mathbb{Y} \to \mathbb{Y}$ such that given any $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{X}$ and scalars a and b in $\mathbb{F}$, we have

$$F(a\mathbf{x} + b\mathbf{y}) = aF(\mathbf{x}) + bF(\mathbf{y}).$$

Definition 2.2. The vector space $\mathbb{H}$ is a **Hilbert space** provided that it is a complete normed inner product space.

Definition 2.3. We call any collection of n pairwise orthogonal unit vectors $\{\mathbf{u}_1, ..., \mathbf{u}_n\}$ in an n-dimensional Hilbert space $\mathbb{H}$ an **orthonormal basis for** $\mathbb{H}$.

Theorem 2.1. The Reconstruction Formula: Let $\{\mathbf{u}_1, ..., \mathbf{u}_n\}$ be an orthonormal basis for an n-dimensional Hilbert space $\mathbb{H}$. Then, given any $\mathbf{x}$ in $\mathbb{H}$,

$$\mathbf{x} = \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{u}_i \rangle \mathbf{u}_i.$$

Theorem 2.2. The Parseval Formula: Let $\{\mathbf{u}_1, ..., \mathbf{u}_n\}$ be an orthonormal basis for an n-dimensional Hilbert space $\mathbb{H}$. Then, given any $\mathbf{x}$ in $\mathbb{H}$,

$$||\mathbf{x}||^2 = \sum_{i=1}^{n} |\langle \mathbf{x}, \mathbf{u}_i \rangle|^2.$$

Hence, we see that an orthonormal basis can be utilized to calculate the norm of any signal $\mathbf{x}$ and reconstruct $\mathbf{x}$ in a finite-dimensional Hilbert space from a set of coefficients. We call these coefficients **linear measurement coefficients** and refer to their absolute value as **intensity measurements** [6]. Due to the fact that we are using orthonormal basis vectors for $\mathbb{H}$, the linear measurement coefficients of a signal $\mathbf{x}$ are exactly the terms within $\mathbf{x}$ that define it. As such, an orthonormal basis provides the ideal reconstruction of any signal in $\mathbb{H}$ from a set of coefficients. This of course assumes that we have all of the necessary coefficients available. But often in the transmission of information, data is lost, and so we may not have all of the necessary coefficients required to synthesize our desired signal [4]. Thus, we need a redundant sequence of vectors with similar properties to that of orthonormal bases that can be utilized to perform signal reconstruction.

Definition 2.4. Let $\mathbb{H}$ be an n-dimensional Hilbert space and $\mathcal{F} = \{\mathbf{f}_1, ..., \mathbf{f}_l\}$ be a collection of vectors in $\mathbb{H}$. We call $\mathcal{F}$ a **frame for** $\mathbb{H}$ provided that there exist positive real constants $a \leqslant b$, called **frame bounds**, such that for every $\mathbf{x}$ in $\mathbb{H}$,

$$a||\mathbf{x}||^2 \leqslant \sum_{i=1}^{l} |\langle \mathbf{x}, \mathbf{f}_i \rangle|^2 \leqslant b||\mathbf{x}||^2.$$

We call the inner products $\langle \mathbf{x}, \mathbf{f}_1 \rangle, ..., \langle \mathbf{x}, \mathbf{f}_l \rangle$ the **frame coefficients of x under** $\mathcal{F}$, the positive integer $l \geqslant n$ the **length of** $\mathcal{F}$, and define the **redundancy of** $\mathcal{F}$ as $\frac{l}{n}$. If $a = b$, we call $\mathcal{F}$ a **tight frame** and if $a = b = 1$, we call $\mathcal{F}$ a **Parseval frame**.

Remark 2.1. Similar to orthonormal bases, or any other basis for that matter, frames are also spanning sets for $\mathbb{H}$, which is an essential property we need in order to perform signal reconstruction.

From the definition, we can see that a frame allows us to estimate the norm of $\mathbf{x}$ between an interval of positive constants. We can narrow that range by utilizing a tight frame, allowing us to calculate the norm of $\mathbf{x}$ up to a single positive constant. Given a Parseval frame we can calculate the exact norm of $\mathbf{x}$. As useful as it is to find the norm of a signal we wish to reconstruct, ideally we desire to synthesize the signal itself. Like orthonormal bases, we can reconstruct $\mathbf{x}$ using the vectors that make up a frame $\mathcal{F}$. But first, we need to recall some useful linear operators naturally defined by the vectors in $\mathcal{F}$.

Definition 2.5. Let $\mathcal{F} = \{\mathbf{f}_1, ..., \mathbf{f}_l\}$ be a frame for an n-dimensional Hilbert space $\mathbb{H}$. The **analysis operator of** $\mathcal{F}$ is the linear operator $\Theta : \mathbb{H} \to \mathbb{R}^l$ defined as

$$\Theta\mathbf{x} = \begin{bmatrix} \langle \mathbf{x}, \mathbf{f}_1 \rangle \\ \vdots \\ \langle \mathbf{x}, \mathbf{f}_l \rangle \end{bmatrix} = \sum_{i=1}^{l} \langle \mathbf{x}, \mathbf{f}_i \rangle \mathbf{e}_i$$

for the standard orthonormal basis $\{\mathbf{e}_1, ..., \mathbf{e}_l\}$ in $\mathbb{R}^l$. The adjoint of the analysis operator, called the **synthesis operator of** $\mathcal{F}$, is the linear operator $\Theta^* : \mathbb{R}^l \to \mathbb{H}$ defined as

$$\Theta^*\mathbf{y} = \sum_{i=1}^{l} \langle \mathbf{y}, \mathbf{e}_i \rangle \mathbf{f}_i.$$

The **frame operator of** $\mathcal{F}$ is the positive self-adjoint linear operator $\Theta^*\Theta = \Phi : \mathbb{H} \to \mathbb{H}$ defined as

$$\Phi\mathbf{x} = \Theta^*\Theta\mathbf{x} = \sum_{i=1}^{l} \langle \mathbf{x}, \mathbf{f}_i \rangle \mathbf{f}_i.$$

By combining the analysis and synthesis operators to construct the frame operator, we allow for a very important counterpart of $\mathcal{F}$.

Definition 2.6. Let $\mathcal{F} = \{\mathbf{f}_1, ..., \mathbf{f}_l\}$ be a frame for an n-dimensional Hilbert space $\mathbb{H}$ with frame operator $\Phi : \mathbb{H} \to \mathbb{H}$. The **canonical dual frame of** $\mathcal{F}$, denoted as $\mathcal{D}_{\mathcal{F}} = \{\mathbf{d}_1, ..., \mathbf{d}_l\}$, is the frame given by

$$\mathbf{d}_i = \Phi^{-1}\mathbf{f}_i.$$

Remark 2.2. It can be shown that the frame operator is injective and, as such, invertible. Consequently,

$$\mathbf{x} = \Phi\Phi^{-1}\mathbf{x} = \sum_{i=1}^{l} \langle \Phi^{-1}\mathbf{f}_i, \mathbf{x} \rangle \mathbf{f}_i = \sum_{i=1}^{l} \langle \mathbf{x}, \Phi^{-1}\mathbf{f}_i \rangle \mathbf{f}_i = \sum_{i=1}^{l} \langle \mathbf{x}, \mathbf{d}_i \rangle \mathbf{f}_i$$

and

$$\mathbf{x} = \Phi^{-1}\Phi\mathbf{x} = \Phi^{-1} \sum_{i=1}^{l} \langle \mathbf{x}, \mathbf{f}_i \rangle \mathbf{f}_i = \sum_{i=1}^{l} \langle \mathbf{x}, \mathbf{f}_i \rangle \Phi^{-1}\mathbf{f}_i = \sum_{i=1}^{l} \langle \mathbf{x}, \mathbf{f}_i \rangle \mathbf{d}_i.$$

Hence, we have achieved the ability to reconstruct any signal $\mathbf{x}$ in $\mathbb{H}$ from a linear combination of the vectors in $\mathcal{F}$ and the frame coefficients for $\mathcal{D}_{\mathcal{F}}$ or vice versa. This idea is captured in the following theorem.

Theorem 2.3. Let $\mathcal{F} = \{\mathbf{f}_1, ..., \mathbf{f}_l\}$ be a frame for an n-dimensional Hilbert space $\mathbb{H}$. Then there exists a frame $D_{\mathcal{F}} = \{\mathbf{d}_1, ..., \mathbf{d}_l\}$, called a **dual frame**, such that for any vector $\mathbf{x}$ in $\mathbb{H}$, we have

$$\mathbf{x} = \sum_{i=1}^{l} \langle \mathbf{x}, \mathbf{d}_i \rangle \mathbf{f}_i = \sum_{i=1}^{l} \langle \mathbf{x}, \mathbf{f}_i \rangle \mathbf{d}_i.$$

Suppose $\mathcal{F} = \{\mathbf{f}_1, ..., \mathbf{f}_l\}$ is a Parseval frame for an n-dimensional Hilbert space $\mathbb{H}$. Then both frame bounds for $\mathcal{F}$ are equal to 1 implying that $||\mathbf{x}||^2 = \sum_{i=1}^{l} |\langle \mathbf{x}, \mathbf{f}_i \rangle|^2$. If we let Θ be the analysis operator for $\mathcal{F}$, then it follows from the definition of Θ that $||\mathbf{x}||^2 = ||\Theta\mathbf{x}||^2$ for each $\mathbf{x}$ in $\mathbb{H}$. Consequently, Θ is an **isometry**, which means $\langle \Theta\mathbf{x}, \Theta\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$ for every $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{H}$, and from this it can be shown that $\mathbf{x} = \sum_{i=1}^{l} \langle \mathbf{x}, \mathbf{f}_i \rangle \mathbf{f}_i$. Therefore, by making $\mathcal{F}$ a Parseval frame, we are no longer required to find a dual frame in order to reconstruct a signal from a set of frame coefficients. More specifically, when $\mathcal{F}$ is a Parseval frame, its frame operator Φ is the identity operator $I : \mathbb{H} \to \mathbb{H}$ which means that $\Phi\mathbf{x} = I\mathbf{x} = \mathbf{x}$, making $\mathcal{F}$ its own canonical dual frame. Hence, a Parseval frame offers the same signal reconstruction capabilities as an orthonormal basis.

CHAPTER 3: FRAME DILATION

Given a frame $\mathcal{F}$ for $\mathbb{H}$, what if we want to utilize $\mathcal{F}$ to recover signals in a Hilbert space of higher dimension k? We know that $\mathcal{F}$ can be extended to form a frame for a higher dimensional space using orthogonal projections and orthogonal direct sums. We call this process **frame dilation** and it is the central topic in Dilation Theory. In this chapter, we recall a few key terms and results important to the dilation of frames and explore some of the intuition involved.

Definition 3.1. Let $\mathbb{H}$ be an n-dimensional Hilbert space with subspace $\mathbb{M}$. We call the linear operator $P : \mathbb{H} \to \mathbb{H}$ the **orthogonal projection from $\mathbb{H}$ onto $\mathbb{M}$** provided that

(i) $P = P^*$,

(ii) $P^2 = P$,

where P^* is the adjoint of P.

Let F be any given linear operator between Hilbert spaces $\mathbb{X}$ and $\mathbb{Y}$ and let F^* be its adjoint. Then we have that $\langle F\mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, F^*\mathbf{y}\rangle$ for every $\mathbf{x} \in \mathbb{X}$ and $\mathbf{y} \in \mathbb{Y}$. Hence, (i) implies that $\langle P\mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, P\mathbf{y}\rangle$ for every $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{H}$. In addition, for any $\mathbf{x} \in \mathbb{M}$, there exists some $\mathbf{y} \in \mathbb{H}$ such that $P\mathbf{y} = \mathbf{x}$. But $\mathbb{M}$ is a subspace contained within $\mathbb{H}$, which means that $\mathbf{x}$ is also in $\mathbb{H}$. Consequently, (ii) implies that $P\mathbf{x} = PP\mathbf{y} = P\mathbf{y} = \mathbf{x}$. Thus, $P\mathbf{x} = \mathbf{x}$ for every $\mathbf{x}$ in the subspace $\mathbb{M}$.

Definition 3.2. Let $\mathbb{H}$ be an n-dimensional Hilbert space with subspace $\mathbb{M}$ and P be the orthogonal projection from $\mathbb{H}$ onto $\mathbb{M}$. Given frame $\mathcal{F} = \{\mathbf{f}_1, ..., \mathbf{f}_l\}$ for $\mathbb{H}$, we call the collection of vectors $\mathcal{F}_P = \{P\mathbf{f}_1, ..., P\mathbf{f}_l\}$ in $\mathbb{M}$ the **orthogonal compression of $\mathcal{F}$ under P** and refer to $\mathcal{F}$ as **orthogonal dilation of $\mathcal{F}_P$ under** P.

One important result following from this definition is:

Theorem 3.1. Let $\mathbb{H}$ be an n-dimensional Hilbert space with subspace $\mathbb{M}$ and P be the orthogonal projection from $\mathbb{H}$ onto $\mathbb{M}$. Given frame $\mathcal{F} = \{\mathbf{f}_1, ..., \mathbf{f}_l\}$ for $\mathbb{H}$, the collection $\mathcal{F}_P = \{P\mathbf{f}_1, ..., P\mathbf{f}_l\}$ is a frame for $\mathbb{M}$ with the same frame bounds as $\mathcal{F}$.

Proof. Select any $\mathbf{x} \in \mathbb{M}$. Let positive constants a and b be the frame bounds for $\mathcal{F}$. As such,

$$a||\mathbf{x}||^2 \leqslant \sum_{i=1}^{l} |\langle \mathbf{x}, \mathbf{f}_i \rangle|^2 \leqslant b||\mathbf{x}||^2.$$

From the definition of orthogonal projection, $P\mathbf{x} = \mathbf{x}$ and $\langle \mathbf{x}, P\mathbf{f}_i \rangle = \langle P\mathbf{x}, \mathbf{f}_i \rangle$ for every $\mathbf{f}_i$ in $\mathbb{H}$. Thus, we can say that $\langle \mathbf{x}, \mathbf{f}_i \rangle = \langle \mathbf{x}, P\mathbf{f}_i \rangle$. Hence, we have

$$a||\mathbf{x}||^2 \leqslant \sum_{i=1}^{l} |\langle \mathbf{x}, P\mathbf{f}_i \rangle|^2 \leqslant b||\mathbf{x}||^2$$

for each $\mathbf{x} \in \mathbb{M}$. This implies that $\mathcal{F}_P = \{P\mathbf{f}_1, ..., P\mathbf{f}_l\}$ is a frame for $\mathbb{M}$ with the same frame bounds a $\mathcal{F}$.

QED.

It follows that if $\mathcal{F}$ is a tight frame, then $\mathcal{F}_P$ is a tight frame and, similarly, if $\mathcal{F}$ is a Parseval frame, then $\mathcal{F}_P$ is a Parseval frame.

Definition 3.3. Let $\mathbb{M}$ be an m-dimensional subspace of n-dimensional Hilbert space $\mathbb{H}$. The **orthogonal complement of** $\mathbb{M}$ is the subspace of $\mathbb{H}$ defined as

$$\mathbb{M}^{\perp} = \{\mathbf{x} \in \mathbb{H} : \langle \mathbf{x}, \mathbf{y} \rangle = 0, \forall \mathbf{y} \in \mathbb{M}\}.$$

We generally refer to any two subspaces $\mathbb{X}$ and $\mathbb{Y}$ of $\mathbb{H}$ as **orthogonal subspaces** provided that $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ for every $\mathbf{x} \in \mathbb{X}$ and $\mathbf{y} \in \mathbb{Y}$ and denote this by $\mathbb{X} \perp \mathbb{Y}$. Consequently, we can say that $\mathbb{M} \perp \mathbb{M}^{\perp}$. In [8], it is shown that for any $\mathbf{x} \in \mathbb{H}$, there is a unique decomposition of $\mathbf{x}$ such that $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ for some $\mathbf{x}_1 \in \mathbb{M}$ and $\mathbf{x}_2 \in \mathbb{M}^{\perp}$. Additionally, it is shown that, given orthonormal basis $\{\mathbf{u}_1, ..., \mathbf{u}_m\}$ for $\mathbb{M}$, the orthogonal projection P_M from $\mathbb{H}$ onto $\mathbb{M}$ defined as $P_M\mathbf{x} = \sum_{i=1}^{m} \langle \mathbf{x}, \mathbf{u}_i \rangle \mathbf{u}_i$ is the unique orthogonal projection such that $P_M\mathbb{H} = \mathbb{M}$ and $(I - P_M)\mathbb{H} = \mathbb{M}^{\perp}$ From this, we can conclude that $\mathbb{H}$ has an orthogonal decomposition $\mathbb{H} = \mathbb{M} + \mathbb{M}^{\perp}$. The use of orthogonal projections and the decomposition of the subspaces they give rise to leads us to our next definition.

Definition 3.4. Let $\mathbb{X}$ and $\mathbb{Y}$ be Hilbert spaces. Then their **orthogonal direct sum** is defined as

$$\mathbb{X} \oplus \mathbb{Y} = \{(\mathbf{x}, \mathbf{y}) : \mathbf{x} \in \mathbb{X}, \mathbf{y} \in \mathbb{Y}\}.$$

Remark 3.1. The direct orthogonal sum of finite-dimensional real Hilbert spaces forms a Hilbert space itself.

We can characterize orthogonal direct sums in terms of orthogonal subspaces of a Hilbert space with the following theorem.

Theorem 3.2. Let $\mathbb{X}$ and $\mathbb{Y}$ be subspaces of Hilbert space $\mathbb{H}$. Then $\mathbb{H} = \mathbb{X} \oplus \mathbb{Y}$ if and only if the following hold:

(i) $\mathbb{H} = \mathbb{X} + \mathbb{Y}$.

(ii) $\mathbb{X} \cap \mathbb{Y} = \mathbf{0}$.

(iii) $\mathbb{X} \perp \mathbb{Y}$.

By their definition, $\mathbb{M} \cap \mathbb{M}^{\perp} = \mathbf{0}$. In addition, we know that $\mathbb{M} \perp \mathbb{M}^{\perp}$ and $\mathbb{H} = \mathbb{M} + \mathbb{M}^{\perp}$. From this, we can say $\mathbb{H} = \mathbb{M} \oplus \mathbb{M}^{\perp}$. Let $\mathcal{V} = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ be a frame for $\mathbb{M}$. Then there a collection $\{\mathbf{x}_1, ..., \mathbf{x}_n\}$ in $\mathbb{H}$ such that each $\mathbf{x}_i$ has the unique decomposition $\mathbf{x}_i = \mathbf{v}_i \oplus \mathbf{w}_i$, where $\mathbf{w}_i \in \mathbb{M}^{\perp}$ for each $i = 1, ..., l$. It can be shown that the collection $\{\mathbf{x}_1, ..., \mathbf{x}_n\}$ is a basis for $\mathbb{H}$. This leads us to the Han-Larson-Naimark Dilation Theorem.

Theorem 3.3. [7] **Han-Larson-Naimark Dilation Theorem**: The collection $\mathcal{F} = \{\mathbf{f}_1, ..., \mathbf{f}_l\}$ is a frame for finite-dimensional Hilbert space $\mathbb{H}$ if and only if there exists an l-dimensional Hilbert space $\mathbb{L} \supset \mathbb{H}$ and a basis $\{\mathbf{b}_1, ..., \mathbf{b}_l\}$ for $\mathbb{L}$ such that, for every $i = 1, ..., l$,

$$\mathbf{f}_i = P\mathbf{b}_i,$$

where P is the orthogonal projection from $\mathbb{L}$ onto $\mathbb{H}$.

Let $\mathcal{F} = \{\mathbf{f}_1, ..., \mathbf{f}_l\}$ be a frame for an n-dimensional real Hilbert space $\mathbb{H}$. From Theorem 3.3, we are given an l-dimensional real Hilbert space $\mathbb{L}$, the orthogonal projection P_H from $\mathbb{L}$ onto $\mathbb{H}$, and a basis $\{\mathbf{b}_1, ..., \mathbf{b}_l\}$ for $\mathbb{L}$ such that $\mathbf{f}_i = P_H\mathbf{b}_i$ for every $i = 1, ..., l$. Since a basis is a frame, albeit with a redundancy of 1, by Theorem 3.1, $\{\mathbf{b}_1, ..., \mathbf{b}_l\}$ is the orthogonal dilation of $\mathcal{F}$ under P_H and, as such, has the same frame bounds as $\mathcal{F}$. Let P_K be the orthogonal projection from $\mathbb{L}$ onto a k-dimensional real Hilbert space $\mathbb{K} \supset \mathbb{H}$ such that $n < k < l$. Then, again by Theorem 3.1, the orthogonal compression of $\{\mathbf{b}_1, ..., \mathbf{b}_l\}$ under P_K is a frame for $\mathbb{K}$ with the same frame bounds as $\{\mathbf{b}_1, ..., \mathbf{b}_l\}$. Lastly, let Q be the orthogonal projection from $\mathbb{K}$ onto $\mathbb{H}$. It can be shown that the orthogonal projection of $\{\mathbf{b}_1, ..., \mathbf{b}_l\}$ under P_K is the orthogonal dilation of $\mathcal{F}$ under Q. Hence, if all we have to start with is $\mathcal{F}$, a frame for $\mathbb{H}$, not only can it be dilated up to a basis for $\mathbb{L}$, that dilation of $\mathcal{F}$ can be compressed to a frame for $\mathbb{K}$. Thus, we can essentially utilize $\mathcal{F}$ to recover signals in $\mathbb{H}$ and $\mathbb{K}$!

CHAPTER 4: PHASE RETRIEVAL

We've seen that a frame for a finite-dimensional Hilbert space offers some of the same qualities that an orthonormal basis possesses with regard to signal reconstruction; they span the space, permit the calculation of a signal's norm from the linear measurement coefficients, and allow us to represent the signal as a linear combination of vectors and those coefficients. We've also seen that frames for an n-dimensional Hilbert space can be dilated to a frame for a Hilbert space of higher dimension k. However, we still have noisy phase to contend with in order to recover a signal. And so we now turn our attention to phase retrieval frames.

Definition 4.1. [6] Let $\mathcal{F} = \{\mathbf{f}_1, ..., \mathbf{f}_l\}$ be a frame for an n-dimensional Hilbert space $\mathbb{H}$. We say that $\mathcal{F}$ offers phase retrieval and refer to it as a **phase retrieval frame** provided that any $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{H}$ with the same intensity measurements under $\mathcal{F}$ only differ by a unimodular constant. That is, if

$$|\langle \mathbf{x}, \mathbf{f}_i \rangle| = |\langle \mathbf{y}, \mathbf{f}_i \rangle|,$$

for $i = 1, ..., l$, then there is a constant c such that

$$\mathbf{x} = c\mathbf{y} \text{ and } |c| = 1.$$

There is an interesting property that phase retrieval frames exhibit which leads us to the minimum amount of redundancy a frame must possess to offer phase retrieval in a finite-dimensional real Hilbert space.

Definition 4.2. [2] **The Complement Property**: Let $\mathcal{F} = \{\mathbf{f}_1, ..., \mathbf{f}_l\}$ be a frame for an n-dimensional Hilbert space $\mathbb{H}$. We say that $\mathcal{F}$ has the **complement property** provided that for all subsets Λ of $\{1, ..., l\}$, either $\text{span}\{\mathbf{f}_i : i \in \Lambda\} = \mathbb{H}$ or $\text{span}\{\mathbf{f}_i : i \in \Lambda^c\} = \mathbb{H}$.

Theorem 4.1. [2]Let $\mathcal{F} = \{\mathbf{f}_1, ..., \mathbf{f}_l\}$ be a frame for an n-dimensional Hilbert space $\mathbb{H}$. Then $\mathcal{F}$ offers phase retrieval if and only if it has the complement property.

Proof. Suppose that $\mathcal{F}$ does not have the complement property. Then there is a subset Λ of $\{1, ..., l\}$ such that $\text{span}\{\mathbf{f}_i : i \in \Lambda\} \neq \mathbb{H}$ and $\text{span}\{\mathbf{f}_i : i \in \Lambda^c\} \neq \mathbb{H}$. Thus, it is possible to select nonzero $\mathbf{p}, \mathbf{q} \in \mathbb{H}$ such that for each $i \in \Lambda$,

$$\langle \mathbf{p}, \mathbf{f}_i \rangle = 0$$

and for each $i \in \Lambda^c$,

$$\langle \mathbf{q}, \mathbf{f}_i \rangle = 0.$$

Define $\mathbf{x}, \mathbf{y} \in \mathbb{H}$ as

$$\mathbf{x} = \mathbf{p} - \mathbf{q} \quad \text{and} \quad \mathbf{y} = \mathbf{p} + \mathbf{q}.$$

As such, $\mathbf{x}$ and $\mathbf{y}$ are not equal nor scalar multiples of each other. More specifically, $\mathbf{x}$ and $\mathbf{y}$ do not differ by a unimodular constant. Now observe that for each $i \in \Lambda$,

$$\begin{aligned}
|\langle \mathbf{x}, \mathbf{f}_i \rangle| &= |\langle \mathbf{p} - \mathbf{q}, \mathbf{f}_i \rangle| \\
&= |\langle \mathbf{p}, \mathbf{f}_i \rangle - \langle \mathbf{q}, \mathbf{f}_i \rangle| \\
&= |-\langle \mathbf{q}, \mathbf{f}_i \rangle| \\
&= |\langle \mathbf{q}, \mathbf{f}_i \rangle| \\
&= |\langle \mathbf{p}, \mathbf{f}_i \rangle + \langle \mathbf{q}, \mathbf{f}_i \rangle| \\
&= |\langle \mathbf{p} + \mathbf{q}, \mathbf{f}_i \rangle| \\
&= |\langle \mathbf{y}, \mathbf{f}_i \rangle|.
\end{aligned}$$

Similarly, for each $i \in \Lambda^c$,

$$\begin{aligned}
|\langle \mathbf{x}, \mathbf{f}_i \rangle| &= |\langle \mathbf{p} - \mathbf{q}, \mathbf{f}_i \rangle| \\
&= |\langle \mathbf{p}, \mathbf{f}_i \rangle - \langle \mathbf{q}, \mathbf{f}_i \rangle| \\
&= |\langle \mathbf{p}, \mathbf{f}_i \rangle| \\
&= |\langle \mathbf{p}, \mathbf{f}_i \rangle + \langle \mathbf{q}, \mathbf{f}_i \rangle| \\
&= |\langle \mathbf{p} + \mathbf{q}, \mathbf{f}_i \rangle| \\
&= |\langle \mathbf{y}, \mathbf{f}_i \rangle|.
\end{aligned}$$

Hence, $|\langle \mathbf{x}, \mathbf{f}_i \rangle| = |\langle \mathbf{y}, \mathbf{f}_i \rangle|$ for every $i = 1, ..., l$. That is, $\mathbf{x}$ and $\mathbf{y}$ have the same intensity measurements through $\mathcal{F}$. However, they do not differ by a unimodular constant. This implies that $\mathcal{F}$ is not a phase retrieval frame. Therefore, by the contrapositive, if $\mathcal{F}$ is a phase retrieval frame, then it has the complement property.

Conversely, suppose that $\mathcal{F}$ is not a phase retrieval frame. Then there are $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{H}$ not differing by a unimodular constant such that for every $i \in \{1, ..., l\}$,

$$|\langle \mathbf{x}, \mathbf{f}_i \rangle| = |\langle \mathbf{y}, \mathbf{f}_i \rangle|.$$

As such, for each $i = 1, ..., l$,

$$\langle \mathbf{x}, \mathbf{f}_i \rangle = \pm \langle \mathbf{y}, \mathbf{f}_i \rangle.$$

Define a subset $\Lambda \subseteq \{1, ..., l\}$ as

$$\Lambda = \{i : \langle \mathbf{x}, \mathbf{f}_i \rangle = \langle \mathbf{y}, \mathbf{f}_i \rangle\}.$$

Now observe that when $i \in \Lambda$,

$$0 = \langle \mathbf{x}, \mathbf{f}_i \rangle - \langle \mathbf{y}, \mathbf{f}_i \rangle = \langle \mathbf{x} - \mathbf{y}, \mathbf{f}_i \rangle.$$

Hence, $(\mathbf{x}-\mathbf{y}) \perp \mathbf{f}_i$ for each $i \in \Lambda$, meaning that $\{\mathbf{f}_i : i \in \Lambda\}$ does not span $\mathbb{H}$. Similarly, when $i \in \Lambda^c$,

$$0 = \langle \mathbf{x}, \mathbf{f}_i \rangle + \langle \mathbf{y}, \mathbf{f}_i \rangle = \langle \mathbf{x}+\mathbf{y}, \mathbf{f}_i \rangle.$$

Thus, $(\mathbf{x}+\mathbf{y}) \perp \mathbf{f}_i$ for each $i \in \Lambda^c$, so $\{\mathbf{f}_i : i \in \Lambda^c\}$ does not span $\mathbb{H}$ either. Consequently, this implies that $\mathcal{F}$ does not have the complement property. Therefore, by the contrapositive, if $\mathcal{F}$ has the complement property, then it is a phase retrieval frame.

QED.

Let $\mathbb{H}$ be an n-dimensional real Hilbert space and suppose $\mathcal{F}$ is a frame for $\mathbb{H}$ with the complement property. Consider that in order to span $\mathbb{H}$, a subset of $\mathcal{F}$ either has n linearly independent vectors or its complement does. From this we can conclude that $\mathcal{F}$ must have at least $2n-1$ vectors in it. Since any frame with the complement property offers phase retrieval, we see that a frame with less than $2n-1$ vectors cannot offer phase retrieval for a real Hilbert space with dimension n. Hence, we have found a minimum redundancy requirement for a frame to offer phase retrieval in an n-dimensional real Hilbert space.

CHAPTER 5: DILATION OF PHASE RETRIEVAL FRAMES

Given a phase retrieval frame $\mathcal{F}$ for an n-dimensional real Hilbert space $\mathbb{H}$, we know that any dilation of $\mathcal{F}$ to a k-dimensional real Hilbert space $\mathbb{K}$ requires the complement property in order to offer phase retrieval for $\mathbb{K}$. Consequently, any subset of this dilation of $\mathcal{F}$ either spans $\mathbb{K}$ or its complement does. It follows then that for any subset Λ of $\{1, ..., l\}$, either $|\Lambda| \geqslant k$ or $|\Lambda^c| \geqslant k$. This implies that $l \geqslant 2k-1$. Hence, $\mathcal{F}$ having no less than $2k-1$ vectors is a necessary condition in order for its dilation to offer phase retrieval. However, it is not a sufficient condition. Additionally, $\mathcal{F}$ having the complement property does not guarantee that its dilation does. We need a stronger condition, a stronger version of the complement property, if you will.

Definition 5.1. A Stronger Complement Property: Let n and k be positive integers such that $k > n$ and $\mathcal{F} = \{\mathbf{f}_1, ..., \mathbf{f}_l\}$ be a phase retrieval frame for an n-dimensional real Hilbert space $\mathbb{H}$. Then $\mathcal{F}$ has the **complement property with strength** k provided that for any subset $\Lambda \subseteq \{1, ..., l\}$, either

(i) $\{\mathbf{f}_i : i \in \Lambda\}$ spans $\mathbb{H}$ and $|\Lambda| \geqslant k$ or

(ii) $\{\mathbf{f}_i : i \in \Lambda^c\}$ spans $\mathbb{H}$ and $|\Lambda^c| \geqslant k$.

Let $\mathcal{F} = \{\mathbf{f}_1, ..., \mathbf{f}_l\}$ be a phase retrieval frame for an n-dimensional real Hilbert space $\mathbb{H}$ and P the orthogonal projection from k-dimensional real Hilbert space $\mathbb{K}$ onto $\mathbb{H}$. Suppose $\mathcal{F}$ has the complement property with strength k and consider the space

$$\mathbb{X} = \underbrace{\mathbb{H}^{\perp} \oplus \cdots \oplus \mathbb{H}^{\perp}}_{l\,\text{times}}$$

where $\mathbb{H}^{\perp}$ is the orthogonal complement to $\mathbb{H}$ under P. By definition, $\mathbb{X}$ is a km-dimensional real Hilbert space, where $m = k - n$. Define $\Omega(\Lambda)$ as

$$\Omega(\Lambda) = \begin{cases} \Lambda & \text{if } \{\mathbf{f}_i : i \in \Lambda\} \text{ spans } \mathbb{H} \text{ and } |\Lambda| \geqslant k \\ \\ \Lambda^c & \text{if } \{\mathbf{f}_i : i \in \Lambda^c\} \text{ spans } \mathbb{H} \text{ and } |\Lambda^c| \geqslant k \end{cases}$$

for each $\Lambda \subseteq \{1, ..., l\}$. The Ω operator essentially selects the subset of indices for which its associated indexed subset of vectors from $\mathcal{F}$ spans $\mathbb{H}$ and has no fewer than k vectors in it. We'll make use of this operator to define a subset of $\mathbb{X}$ wherein the necessary components can be found to construct a dilation of $\mathcal{F}$ that offers phase retrieval.

Proposition 5.1. Let $\mathcal{F} = \{\mathbf{f}_1, ..., \mathbf{f}_l\}$ be a phase retrieval frame for an n-dimensional real Hilbert space $\mathbb{H}$ that has the complement property with strength k and define $\Omega(\Lambda)$ as above. Let $\mathbb{K}$ be a k-dimensional real Hilbert space and define

$$D_{\Omega(\Lambda)} = \{(\mathbf{g}_1, ..., \mathbf{g}_l) \in \mathbb{X} : \text{span}\{\mathbf{f}_i \oplus \mathbf{g}_i\}_{i \in \Omega(\Lambda)} = \mathbb{K}\}.$$

Then $D_{\Omega(\Lambda)}$ is a nonempty open dense subset of $\mathbb{X}$.

Proof. We begin by first showing that $D_{\Omega(\Lambda)}$ is nonempty. By Theorem 4.1, we know $\text{span}\{\mathbf{f}_i : i \in \Omega(\Lambda)\} = \mathbb{H}$. By Theorems 3.1 and 3.3, there exists $\mathbf{g}_i \in \mathbb{H}^\perp$ for each $i \in \Omega(\Lambda)$ such that $\{\mathbf{f}_i \oplus \mathbf{g}_i : i \in \Omega(\Lambda)\}$ is a frame for $\mathbb{K}$. Let $\mathbf{g}_i = \mathbf{0}$ for every $i \notin \Omega(\Lambda)$. Then $(\mathbf{g}_1, ..., \mathbf{g}_l) \in D_{\Omega(\Lambda)}$. Hence, $D_{\Omega(\Lambda)}$ is nonempty.

Next, we show that $D_{\Omega(\Lambda)}$ is dense in $\mathbb{X}$. Consider some fixed $\mathbf{g} = (\mathbf{g}_1, ..., \mathbf{g}_l) \in D_{\Omega(\Lambda)}$ and select any $\mathbf{x} = (\mathbf{x}_1, ..., \mathbf{x}_l) \in \mathbb{X}$. Because $\{\mathbf{f}_i \oplus \mathbf{g}_i : i \in \Omega(\Lambda)\}$ spans $\mathbb{K}$, there are $i_1, ..., i_k \in \Omega(\Lambda)$ such that $\{\mathbf{f}_{i_1} \oplus \mathbf{g}_{i_1}, ..., \mathbf{f}_{i_k} \oplus \mathbf{g}_{i_k}\}$ form a linearly independent spanning set for $\mathbb{K}$. Next we define the function $d : [0,1] \to \mathbb{R}$ as

$$d(t) = \det[\mathbf{f}_{i_1} \oplus ((1-t)\mathbf{g}_{i_1} + t\mathbf{x}_{i_1}), ..., \mathbf{f}_{i_k} \oplus ((1-t)\mathbf{g}_{i_k} + t\mathbf{x}_{i_k})].$$

Note that

$$(1-t)\mathbf{g} + t\mathbf{x} \in D_{\Omega(\Lambda)} \text{ when } d(t) \neq 0$$

and

$$(1-t)\mathbf{g} + t\mathbf{x} \notin D_{\Omega(\Lambda)} \text{ when } d(t) = 0.$$

As $d(t)$ is a polynomial of a most degree k, the Fundamental Theorem of Algebra tells us $d(t) = 0$ can have at most k real solutions. Thus, $d(t) \neq 0$ for all but a finite number of points in $[0,1]$. Select any $\varepsilon > 0$ and observe that

$$(1-t)\mathbf{g} + t\mathbf{x} = \mathbf{x} \text{ when } t = 1.$$

It follows that for any value of $t < 1$, we can find a point $t_0 \in (t, 1)$ such that $d(t_0) \neq 0$ and

$$||(1-t_0)\mathbf{g} + t_0\mathbf{x} - \mathbf{x}|| < \varepsilon.$$

Hence, for any open ε-ball about $\mathbf{x}$, we can find a t_0 such that

$$(1-t_0)\mathbf{g} + t_0\mathbf{x} \in B_\varepsilon(\mathbf{x}).$$

Since $(1-t_0)\mathbf{g}+t_0\mathbf{x} \in D_{\Omega(\Lambda)}$, we see that $B_\varepsilon(\mathbf{x})$ has a nonempty intersection with $D_{\Omega(\Lambda)}$. The arbitrariness of ε means the result holds for any open ε-ball about $\mathbf{x}$ and the arbitrariness of $\mathbf{x}$ means the result holds for every $\mathbf{x}$. That is, any open ε-ball around any $\mathbf{x} \in \mathbb{X}$ intersects $D_{\Omega(\Lambda)}$ which shows $D_{\Omega(\Lambda)}$ is dense in $\mathbb{X}$.

To see that $D_{\Omega(\Lambda)}$ is open in $\mathbb{X}$, we use the following lemma:

If $\{\mathbf{v}_1, ..., \mathbf{v}_l\}$ is a frame for $\mathbb{K}$, then there exists $\delta > 0$ such that for any $\mathbf{w}_1, ..., \mathbf{w}_l \in \mathbb{K}$,

$$\sum_{i=1}^{l} ||\mathbf{w}_i - \mathbf{v}_i||^2 < \delta$$

implies $\{\mathbf{w}_1, ..., \mathbf{w}_l\}$ is a frame for $\mathbb{K}$.

Select any $\mathbf{g} = (\mathbf{g}_1, ..., \mathbf{g}_l) \in D_{\Omega(\Lambda)}$. Then $\{\mathbf{f}_1 \oplus \mathbf{g}_1, ..., \mathbf{f}_l \oplus \mathbf{g}_l\}$ is a frame for $\mathbb{K}$ and there is some $\delta > 0$ such that for any $\mathbf{w}_1, ..., \mathbf{w}_l \in \mathbb{K}$,

$$\sum_{i=1}^{l} ||\mathbf{w}_i - (\mathbf{f}_i \oplus \mathbf{g}_i)||^2 < \delta$$

implies $\{\mathbf{w}_1, ..., \mathbf{w}_l\}$ is a frame for $\mathbb{K}$. Set $\varepsilon = \sqrt{\delta}$ and choose any $\mathbf{x} = (\mathbf{x}_1, ..., \mathbf{x}_l) \in \mathbb{X}$ from the open ε-ball about $\mathbf{g}$. Since $||(\mathbf{f}_i \oplus \mathbf{x}_i) - (\mathbf{f}_i \oplus \mathbf{g}_i)|| = ||\mathbf{x}_i - \mathbf{g}_i||$ for each $1 \leqslant i \leqslant l$, we see that

$$\sum_{i=1}^{l} ||(\mathbf{f}_i \oplus \mathbf{x}_i) - (\mathbf{f}_i \oplus \mathbf{g}_i)||^2 = \sum_{i=1}^{l} ||\mathbf{x}_i - \mathbf{g}_i||^2 = ||\mathbf{x} - \mathbf{g}||^2 < \varepsilon^2 = \delta.$$

As such, $\{\mathbf{f}_1 \oplus \mathbf{x}_1, ..., \mathbf{f}_l \oplus \mathbf{x}_l\}$ is a frame for $\mathbb{K}$, placing $\mathbf{x}$ in $D_{\Omega(\Lambda)}$. Thus, $B_\varepsilon(\mathbf{g}) \subset D_{\Omega(\Lambda)}$. The arbitrariness of $\mathbf{g}$ shows that each $\mathbf{g} \in D_{\Omega(\Lambda)}$ has its own open ε-ball about it contained in $D_{\Omega(\Lambda)}$. Hence, $D_{\Omega(\Lambda)}$ is open in $\mathbb{X}$. Therefore, $D_{\Omega(\Lambda)}$ is a nonempty open dense subset of $\mathbb{X}$.

QED.

Having established that, for each subset $\Lambda \subseteq \{1, ..., l\}$, $D_{\Omega(\Lambda)}$ is an open dense subset of $\mathbb{X}$, we now have a collection $\{\mathbf{g}_1, ..., \mathbf{g}_l\}$ of vectors from $\mathbb{H}^\perp$ such that $\{\mathbf{f}_1 \oplus \mathbf{g}_1, ..., \mathbf{f}_l \oplus \mathbf{g}_l\}$ spans $\mathbb{K}$. However, each collection is dependent on the Λ associated with it. What we need is as collection such that $\{\mathbf{f}_1 \oplus \mathbf{g}_1, ..., \mathbf{f}_l \oplus \mathbf{g}_l\}$ spans $\mathbb{K}$ independent of Λ. Fortunately, we find some help in the Baire Catagory Theorem.

Theorem 5.2. The Baire Category Theorem: Let each D_i be a nonempty open dense subset of $\mathbb{R}^n$. Then $\bigcap\limits_{i=1}^{\infty} D_i$ is a nonempty open dense subset of $\mathbb{R}^n$.

Hence, the Baire Category Theorem implies that $D = \bigcap\limits_{\Lambda\subseteq\{1,...,l\}} D_{\Omega(\Lambda)}$ is a nonempty open dense subset of $\mathbb{X}$. This leads to the following proposition.

Proposition 5.3. For every

$$\mathbf{g} = (\mathbf{g}_1, ..., \mathbf{g}_l) \in D = \bigcap_{\Lambda\subseteq\{1,...,l\}} D_{\Omega(\Lambda)},$$

the collection $\{\mathbf{f}_1 \oplus \mathbf{g}_1, ..., \mathbf{f}_l \oplus \mathbf{g}_l\}$ in $\mathbb{K}$ has the complement property.

Proof. Select any $\mathbf{g} = (\mathbf{g}_1, ..., \mathbf{g}_l) \in D$. Then $\mathbf{g}$ is in every $D_{\Omega(\Lambda)}$. Hence, for each Λ, we have $|\Omega(\Lambda)| \geqslant k$ and by the definition of $\Omega(\Lambda)$, either

$$\text{span}\{\mathbf{f}_i \oplus \mathbf{g}_i : i \in \Lambda\} = \mathbb{K}$$

or

$$\text{span}\{\mathbf{f}_i \oplus \mathbf{g}_i : i \in \Lambda^c\} = \mathbb{K}.$$

Thus, $\{\mathbf{f}_1 \oplus \mathbf{g}_1, ..., \mathbf{f}_l \oplus \mathbf{g}_l\}$ has the complement property.

QED.

We now have all of the puzzle pieces assembled to state and prove the most important result sought after in this thesis.

Theorem 5.4. Let $\mathcal{F} = \{\mathbf{f}_1, ..., \mathbf{f}_l\}$ be a phase retrieval frame for an n-dimensional real Hilbert space $\mathbb{H}$. The dilation of $\mathcal{F}$ offers phase retrieval for a k-dimensional real Hilbert space $\mathbb{K}$ if and only if $\mathcal{F}$ has the complement property with strength k.

Proof. Suppose $\mathcal{F}$ has the complement property with strength k. Then for any subset $\Lambda \subseteq \{1, ..., l\}$, either $\{\mathbf{f}_i : i \in \Lambda\}$ spans $\mathbb{H}$ and $|\Lambda| \geqslant k$ or $\{\mathbf{f}_i : i \in \Lambda^c\}$ spans $\mathbb{H}$ and $|\Lambda^c| \geqslant k$. For each Λ, define $\Omega(\Lambda)$ as

$$\Omega(\Lambda) = \begin{cases} \Lambda & \text{if } \{\mathbf{f}_i : i \in \Lambda\} \text{ spans } \mathbb{H} \text{ and } |\Lambda| \geqslant k \\ \Lambda^c & \text{if } \{\mathbf{f}_i : i \in \Lambda^c\} \text{ spans } \mathbb{H} \text{ and } |\Lambda^c| \geqslant k \end{cases}$$

and consider the Hilbert space

$$\mathbb{X} = \underbrace{\mathbb{H}^\perp \oplus \cdots \oplus \mathbb{H}^\perp}_{l\,\text{times}},$$

where $\mathbb{H}^\perp$ is the orthogonal complement to $\mathbb{H}$ under P, the orthogonal projection from $\mathbb{K}$ onto $\mathbb{H}$. Set

$$D_{\Omega(\Lambda)} = \{(\mathbf{g}_1, ..., \mathbf{g}_l) \in \mathbb{X} : \text{span}\{\mathbf{f}_i \oplus \mathbf{g}_i\}_{i \in \Omega(\Lambda)} = \mathbb{K}\}.$$

By Proposition 5.1, $D_{\Omega(\Lambda)}$ is a nonempty open dense subset of $\mathbb{X}$. Let

$$D = \bigcap_{\Lambda \subseteq \{1,...,l\}} D_{\Omega(\Lambda)}.$$

From the Baire Category Theorem, we know that D is also a nonempty open dense subset of $\mathbb{X}$. Select any $\mathbf{g} = (\mathbf{g}_1, ..., \mathbf{g}_l) \in D$. By Proposition 5.3, we know that $\{\mathbf{f}_1 \oplus \mathbf{g}_1, ..., \mathbf{f}_l \oplus \mathbf{g}_l\}$, which is the dilation of $\mathcal{F}$ to a frame for $\mathbb{K}$, has the complement property. Hence, by Theorem 4.1, $\{\mathbf{f}_1 \oplus \mathbf{g}_1, ..., \mathbf{f}_l \oplus \mathbf{g}_l\}$ offers phase retrieval for $\mathbb{K}$.

Conversely, suppose $\mathcal{V} = \{\mathbf{v}_i, ..., \mathbf{v}_l\}$ is the dilation of $\mathcal{F}$ and that $\mathcal{V}$ offers phase retrieval for $\mathbb{K}$. By Theorem 4.1, we know that $\mathcal{V}$ has the complement property. As such, for any $\Lambda \subseteq \{1, .., s\}$, either

$$\{\mathbf{v}_i : i \in \Lambda\} \text{ spans } \mathbb{K} \text{ and } |\Lambda| \geqslant k$$

or

$$\{\mathbf{v}_i : i \in \Lambda^c\} \text{ spans } \mathbb{K} \text{ and } |\Lambda^c| \geqslant k.$$

Let P be the orthogonal projection from $\mathbb{K}$ onto $\mathbb{H}$. By Theorems 3.1 and 3.3, we have $P\mathbf{v}_i = \mathbf{f}_i$ for each $i = 1, ..., l$ It follows that either

$$\{P\mathbf{v}_i = \mathbf{f}_i : i \in \Lambda\} \text{ spans } \mathbb{H} \text{ and } |\Lambda| \geqslant k$$

or

$$\{P\mathbf{v}_i = \mathbf{f}_i : i \in \Lambda^c\} \text{ spans } \mathbb{H} \text{ and } |\Lambda^c| \geqslant k.$$

This implies that $\mathcal{F}$ has the complement property with strength k.

Therefore, we can see that the dilation of $\mathcal{F}$ offers phase retrieval $\mathbb{K}$ if and only if $\mathcal{F}$ has the complement property with strength k.

QED.

We've now seen that a phase retrieval frame for a finite-dimensional real Hilbert space can be dilated to a phase retrieval frame for a larger finite-dimensional Hilbert space. Considering that the scope of this thesis focuses solely on finite-dimensional real Hilbert spaces, there is still more subject matter to be explored on the topic of frame dilation and phase retrieval. For example, it remains to be seen how this result holds up in finite-dimensional complex Hilbert spaces. Furthermore, this thesis pays no consideration to infinite Hilbert spaces, both real and complex. Neither does it consider how to generate a phase retrieval frame dilation from a phase retrieval frame, only the proof of the frame's existence. Perhaps there is an algorithm or another method for generating this frame dilation. Regardless, such explorations will have to be undertaken in future bodies of research.

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