On Saturation Numbers of Ramsey-minimal Graphs

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ON SATURATION NUMBERS OF RAMSEY-MINIMAL GRAPHS

by

HUNTER DAVENPORT

A thesis submitted in partial fulfilment of the requirements for the degree of Honors in the Major Program in Mathematics in the College of Sciences in the College of The Burnett Honors College at the University of Central Florida

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Thesis Chair: Dr. Zi-Xia Song
Dating back to the 1930’s, Ramsey theory still intrigues many who study combinatorics. Roughly put, it makes the profound assertion that complete disorder is impossible. One view of this problem is in edge-colorings of complete graphs. For forbidden graphs $H_1, \ldots, H_k$ and a graph $G$, we write $G \rightarrow (H_1, \ldots, H_k)$ if every $k$-edge-coloring of $G$ contains a monochromatic copy of $H_i$ in color $i$ for some $i \in \{1, \ldots, k\}$. If $\pi$ is a $\{\text{red, blue}\}$-edge-coloring of $G$, we say $\pi$ is a bad coloring if $G$ contains no red $K_3$ or blue $K_{1,t}$ under $\pi$. A graph $G$ is $(H_1, \ldots, H_k)$-Ramsey-minimal if $G \rightarrow (H_1, \ldots, H_k)$ but no proper subgraph of $G$ has this property. Given a family $\mathcal{F}$ of graphs, we say that a graph $G$ is $\mathcal{F}$-saturated if no member of $\mathcal{F}$ is a subgraph of $G$, but for any edge $xy \notin E(G)$, $G + xy$ contains a member of $\mathcal{F}$ as a subgraph. Letting $\mathcal{R}_{\min}(K_3, K_{1,t})$ be the family of $(K_3, K_{1,t})$-Ramsey minimal graphs, we study the saturation number, denoted $\text{sat}(n, \mathcal{R}_{\min}(K_3, K_{1,t}))$, which is the minimum number of edges among all $\mathcal{R}_{\min}(K_3, K_{1,t})$-saturated graphs on $n$ vertices. We believe the methods and constructions developed in this thesis will be useful in studying the saturation numbers of $(K_4, K_{1,t})$-saturated graphs.
I would like to thank my advisor, Dr. Zi-Xia Song, for her inspiring love for the subject of graph theory and support throughout the most challenging undertaking of my undergraduate career. Her course in combinatorics changed the way that I think about mathematics entirely. I must also thank Dr. Michael Reid, for my first introduction to graph theory, for his guidance, and for the frequent reality checks whenever I think I know all there is to know. I have been lucky to have such valuable mentors and to have the love and support of my family and friends whenever academic life was particularly difficult. Finally, I would like to thank the Department of Mathematics and the Burnett Honors College as a whole for the numerous opportunities to enrich my undergraduate experience and prepare for further education.
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Table 2.1: Various values of $r_2(K_n, K_m)$.

Table 3.1: Values of $|B|$ and $|C|$ for graph $J$. 
We begin this thesis with an overview of basic concepts and definitions in graph theory. Let \( \mathbb{N} \) be the set of natural numbers. For any \( n \in \mathbb{N} \), define \( [n] := \{1, 2, \ldots, n\} \). For a finite set \( V \), we use \( |V| \) to denote the number of elements of \( V \). For any pair of sets \( A, B \), we use \( A \setminus B \) to denote the elements of \( A \) which do not also belong to \( B \). We use \( \binom{V}{2} \) to denote the set of all unordered pairs of elements of \( V \). A graph \( G \) is composed of a set \( V(G) \) of vertices and a set \( E(G) \) of edges, where \( E(G) \subseteq \binom{V(G)}{2} \). We say that \( |V(G)| \) is the order of \( G \), and frequently abbreviate this notation to \( |G| \). For notational convenience, instead of writing \( \{v_1, v_2\} \) to represent an edge with endpoints \( v_1 \) and \( v_2 \), we write \( v_1v_2 \). Two vertices \( x, y \in V(G) \) are adjacent if \( xy \in E(G) \). The complement of a graph \( G \), denoted \( \overline{G} \), is a graph with vertex set \( V(G) \) and edge-set \( E(\overline{G}) = \left( \binom{V(G)}{2} \right) \setminus E(G) \). A graph \( H \) is a subgraph of \( G \) if \( V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \). Furthermore, we say a subgraph \( H \) of \( G \) is a proper subgraph of \( G \) if \( H \neq G \). If \( G \) does not contain \( H \) as a subgraph, then we say \( G \) is \( H \)-free. For any \( S \subseteq V(G) \), the subgraph of \( G \) induced by \( S \), denoted \( G[S] \), is the graph with vertex set \( S \) and edge set \( \{xy \in E(G) : x, y \in S\} \). A graph \( H \) is an induced subgraph of a graph \( G \) if \( H = G[S] \) for some \( S \subseteq V(G) \). The neighborhood \( N(x) \) of a vertex \( x \in V(G) \) is the set of all vertices \( y \in V(G) \) such that \( xy \in E(G) \). The degree of a vertex \( v \in V(G) \), denoted \( d(v) \), is the number of neighbors, namely \( |N(v)| \), of \( v \) in \( G \). The minimum degree of \( G \), denoted \( \delta(G) \), is the smallest \( d(v) \) among all \( v \in V(G) \). Similarly, the maximum degree of \( G \), denoted \( \Delta(G) \), is the largest \( d(v) \) among all \( v \in V(G) \). Given a subgraph \( H \subseteq G \) and a vertex \( v \in V(H) \), let \( N_H(v) \) denote the neighborhood of \( v \) in \( H \), and similarly let \( d_H(v) := |N_H(v)| \). A complete graph \( K_n \) is a graph with \( n \) vertices where \( xy \in E(K_n) \) for all \( x, y \in V(K_n) \) with \( x \neq y \). A triangle is a \( K_3 \). For all \( t \geq 1 \), a star \( K_{1,t} \) is a graph with one center vertex \( x \) and vertices \( v_1, \ldots, v_t \) with edge set \( E(K_{1,t}) = \{xv_i \mid i \in [t]\} \). A \( k \)-edge-coloring of a graph \( G \) is
a function $\pi : E(G) \rightarrow [k]$ that assigns a number to each edge in $G$. For the purpose of visualization, when $k = 2$ we often view colorings as a function $\pi : E(G) \rightarrow \{\text{red, blue}\}$. $G$ is monochromatic with respect to $\pi$ if $\pi(E(G)) = i \in [k]$ for some fixed $k \in \mathbb{N}$. Given a graph $G$, for any disjoint sets $A, B \subseteq V(G)$, $A$ is complete to $B$ if for all $a \in A$, $b \in B$, we have $ab \in E(G)$, and $A$ is anticomplete to $B$ if for all $a \in A$, $b \in B$, we have $ab \notin E(G)$. We define $e(G) := |E(G)|$ to be the number of edges in $G$. A subset $S \subset V(G)$ is said to be an independent set if for all pairs $u, v \in S$, $uv \notin E(G)$. The independence number of $G$, denoted $\alpha(G)$, is the maximum size of an independent set in $G$. The join $G + H$ (resp. union $G \cup H$) of two vertex disjoint graphs $G$ and $H$ is the graph having vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$ (resp. $E(G) \cup E(H)$). Given two isomorphic graphs $G$ and $H$, we may (with a slight but common abuse of notation) write $G = H$.

Ramsey theory originated in the work of Frank Ramsey [20]. We introduce the following notation: For “forbidden” graphs $H_1, \ldots, H_k$ and a graph $G$, we write $G \rightarrow (H_1, \ldots, H_k)$ if every $k$-edge-coloring of $G$ contains a monochromatic $H_i$ in color $i$ for some $i \in [k]$. The Ramsey number $r_k(H_1, \ldots, H_k)$ is the minimum integer $n$ such that $K_n \rightarrow (H_1, \ldots, H_k)$. If $H_1 = \cdots = H_k = H$, then we simply write $r_k(H)$ instead of $r_k(H, \ldots, H)$.

**Theorem 1.1** (Ramsey’s Theorem [20]). For any positive integer $k$ and any collection of graphs $H_1, \ldots, H_k$, the Ramsey number $r_k(H_1, \ldots, H_k)$ exists.

Ramsey theory is a notoriously difficult branch of combinatorics. Many questions in the field have remained wide open for years. For example, we have exact values for only a few of the classical Ramsey numbers $r_k(K_{t_1}, \ldots, K_{t_k})$ with $t_i \in \mathbb{N}$ despite active research for almost a century. We introduce the study of classical Ramsey theory in Chapter 2, demonstrating some well-known results of the field and stating special results pertaining to certain classes.
of graphs.

In Chapter 2, we present some more background and history on Ramsey theory. We explore the notions of Ramsey-minimality and saturation numbers, most importantly defining the class of $\mathcal{R}_{\min}(K_3, K_{1,t})$-saturated graphs.

In Chapter 3, we present some of the most recent work on Ramsey-minimal graphs. We include here several known results, placing particular emphasis on the results which will be used in this thesis.

In Chapter 4, we begin presenting our original research, in which we determine the exact value of the saturation number of $\mathcal{R}_{\min}(K_3, K_{1,3})$-saturated graphs on $n \geq 13$ vertices.

In chapter 5, we present an asymptotic bound on the saturation number of $\mathcal{R}_{\min}(K_3, K_{1,t})$-saturated graphs for $t \geq 4$ and $n$ sufficiently large. We also discuss potential future work.
CHAPTER 2: RAMSEY-MINIMAL GRAPHS AND BACKGROUND

Background

Given graphs $H_1, \ldots, H_k$, to compute $r_k(H_1, \ldots, H_k)$ we show that $K_{n-1} \not\rightarrow (H_1, \ldots, H_k)$, that is, there exists a $k$-edge-coloring of $K_{n-1}$ that does not contain a monochromatic $H_i$ in color $i$ for any $i \in [k]$. This shows that $r_k(H_1, \ldots, H_k) \geq n$. Then, we show that $K_n \rightarrow (H_1, \ldots, H_k)$, that is, every $k$-edge-coloring of $K_n$ contains a monochromatic $H_i$ in color $i$ for some $i \in [k]$. This shows that $r_k(H_1, \ldots, H_k) \leq n$, and thus $r_k(H_1, \ldots, H_k) = n$.

We first observe a classical example of this technique in practice.

Example. $r_2(K_3) = 6$.

Proof. Figure 2.1 illustrates that $K_5 \not\rightarrow (K_3, K_3)$. Thus, $r_2(K_3) \geq 6$. Next, let $G := K_6$.

Figure 2.1: A monochromatic triangle free coloring of $K_5$. 
Let $\pi : E(G) \rightarrow \{\text{red}, \text{blue}\}$ be any 2-edge-coloring of $G$. We claim $G$ contains a monochromatic $K_3$. Consider a fixed vertex $x \in V(G)$ and let $v_1, v_2, \ldots, v_5$ be the five neighbors of $x$. By the Pigeonhole Principle, we may assume that $xv_1, xv_2, xv_3$ are colored blue under $\pi$. If $v_1v_2, v_1v_3, v_2v_3$ are all colored red, then $G[\{v_1, v_2, v_3\}]$ is a red $K_3$ under $\pi$. So, we may assume that $v_1v_2$ is colored blue. Then, $G[\{x, v_1, v_2\}]$ is a blue $K_3$ under $\pi$, as desired. 

Although the above proof is relatively simple, determining the exact values of Ramsey numbers for larger graphs quickly becomes incredibly difficult. Table 2.1, compiled by Radziszowski [19], aggregates some of the known Ramsey numbers of complete graphs when two colors are used.

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<td>282 - 1870</td>
<td>329 - 3583</td>
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<td>183 - 780</td>
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Table 2.1: Various values of $r_2(K_n, K_m)$.

Ramsey-minimal and Saturated Graphs

Although classical Ramsey theory typically examines colorings of complete graphs, given graphs $H_1, \ldots, H_t$, we can also look at graphs $G$ which are, roughly put, minimal with respect to the property that $G \rightarrow (H_1, \ldots, H_t)$.

A graph $G$ is $(H_1, \ldots, H_t)$-Ramsey-minimal if $G \rightarrow (H_1, \ldots, H_t)$, but for any proper
subgraph $G'$ of $G$, $G' \rightarrow (H_1, \ldots, H_t)$. We define $\mathcal{R}_{\text{min}}(H_1, \ldots, H_t)$ to be the family of $(H_1, \ldots, H_t)$-Ramsey-minimal graphs. It is straightforward to prove by induction that a graph $G$ satisfies $G \rightarrow (H_1, \ldots, H_t)$ if and only if there exists a subgraph $G'$ of $G$ such that $G'$ is $(H_1, \ldots, H_t)$-Ramsey-minimal. Ramsey’s theorem [20] implies that $\mathcal{R}_{\text{min}}(H_1, \ldots, H_t) \neq \emptyset$ for all integers $t$ and all finite graphs $H_1, \ldots, H_t$. As pointed out in a recent paper of Fox, Grinshpun, Liebenau, Person, and Szabó [13], “it is still widely open to classify the graphs in $\mathcal{R}_{\text{min}}(H_1, \ldots, H_t)$, or even to prove that these graphs have certain properties”. Some properties of $\mathcal{R}_{\text{min}}(H_1, \ldots, H_t)$ have been studied, such as the minimum degree $s(H_1, \ldots, H_t) := \min \{\delta(G) : G \in \mathcal{R}_{\text{min}}(H_1, \ldots, H_t)\}$, which was first introduced by Burr, Erdős, and Lovász [4]. Recent results on $s(H_1, \ldots, H_t)$ can be found in [12, 13]. For more information on Ramsey-related topics, the readers are referred to a very recent informative survey due to Conlon, Fox, and Sudakov [6].

The notion of saturated graphs was first defined by Zykov [22] in 1949, though the study of saturated graphs in their own right remained relatively unexplored until Erdős, Hajnal, and Moon [9] began to study saturation numbers in 1964. Given a family $\mathcal{F}$ of forbidden graphs, we say a graph $G$ is $\mathcal{F}$-saturated if $G$ contains no member of $\mathcal{F}$ as a subgraph, but for any edge $e \notin E(G)$, $G + e$ contains a member of $\mathcal{F}$ as a subgraph. We simply say that $G$ is $H$-saturated if $\mathcal{F} = \{H\}$. Until recently, typically only $K_t$-saturated graphs were studied. The saturation number of a family $\mathcal{F}$ of forbidden graphs is defined to be

$$\text{sat}(n, \mathcal{F}) = \min \{e(G) : G \text{ is } \mathcal{F}\text{-saturated on } n \text{ vertices}\}.$$ 

In [9], it was determined that for all $n > k$, $\text{sat}(n, K_k) = n(k - 2) - \binom{k-1}{2}$, and that the unique graph attaining this minimum is $K_{k-2} + \overline{K}_{n-k+2}$.

Combining the ideas of Ramsey-minimality and saturation, we consider $\mathcal{R}_{\text{min}}(H_1, \ldots, H_t)$-
saturated graphs for some forbidden graphs $H_1,\ldots,H_t$. This notion was first studied by Nešetřil [17] in 1986 when he asked whether there are infinitely many $R_{\text{min}}(H_1,\ldots,H_t)$-saturated graphs. At the time, such graphs were called $(H_1,\ldots,H_t)$-co-critical graphs. This was answered in the positive by Galluccio, Siminovits, and Simonyi [14] in 1992, when they studied $R_{\text{min}}(K_3,K_3)$-saturated graphs. Galluccio, Siminovits, and Simonyi also ask whether the stronger proposition holds: are there infinitely many $R_{\text{min}}(H_1,\ldots,H_t)$-saturated graphs, none of which contain another as a subgraph? This question remains open. The first to study $\text{sat}(n,R_{\text{min}}(H_1,\ldots,H_t))$ were Hanson and Toft [16] in 1987, where they took $H_1,\ldots,H_t$ to be complete graphs. They proposed the following conjecture.

Conjecture 2.1 ([16]). Let $r = r(K_{k_1},\ldots,K_{k_t})$ be the classical Ramsey number for complete graphs. Then

$$\text{sat}(n,R_{\text{min}}(K_{k_1},\ldots,K_{k_t})) = \begin{cases} \binom{n}{2} & n < r \\ (r-2)(n-r+2)+\binom{r-2}{2} & n \geq r \end{cases}$$
CHAPTER 3: KNOWN RESULTS

In 2011, Chen, Ferrara, Gould, Magnant, and Schmitt [5] proved that \( sat(n, R_{\text{min}}(K_3, K_3)) = 4n - 10 \) for \( n \geq 56 \). This settles the first non-trivial case of Conjecture 2.1 for sufficiently large \( n \), and is so far the only settled case. In 2014, Ferrara, Kim, and Yeager [11] proved that \( sat(n, R_{\text{min}}(m_1K_2, \ldots, m_tK_2)) = 3(m_1 + \cdots + m_t - t) \) for \( m_1, \ldots, m_t \geq 1 \) and \( n > 3(m_1 + \cdots + m_t - t) \). The problem of finding \( sat(n, R_{\text{min}}(K_3, T_k)) \) when \( k = 3 \) was also explored in [5], where \( T_k \) denotes the family of all trees on \( k \) vertices. This parameter remained undetermined until 2017 when Rolek and Song [21] proved the following.

**Theorem 3.1** ([21]). For all \( n \geq 18 \), \( sat(n, R_{\text{min}}(K_3, T_4)) = \lfloor \frac{5n}{2} \rfloor \). Furthermore, for any integers \( k \geq 5 \) and \( n \geq 2k + ([k/2] + 1)[k/2] - 2 \), there exist constants \( c = \left( \frac{1}{2} \left\lceil \frac{k}{2} \right\rceil + \frac{3}{2} \right) k - 2 \) and \( C = 2k^2 - 6k + \frac{3}{2} - \left\lceil \frac{k}{2} \right\rceil (k - \frac{1}{2} \left\lceil \frac{k}{2} \right\rceil - 1) \) such that

\[
\left( \frac{3}{2} + \frac{1}{2} \left\lceil \frac{k}{2} \right\rceil \right)n - c \leq sat(n, R_{\text{min}}(K_3, T_k)) \leq \left( \frac{3}{2} + \frac{1}{2} \left\lceil \frac{k}{2} \right\rceil \right)n + C.
\]

This is the result which inspires our present work. Rather than forbidding any tree on 4 vertices, we will investigate the parameter \( sat(n, R_{\text{min}}(K_3, K_{1,3})) \). When discussing a particular coloring \( \pi \) of a \( R_{\text{min}}(K_3, K_{1,3}) \)-saturated graph \( G \), we will call \( \pi \) a “bad” coloring if \( G \) does not contain a red \( K_3 \) or a blue \( K_{1,3} \) under \( \pi \).

There are several other known results which we will use throughout. One is due to Duffus and Hanson in 1986, when they studied \( K_3 \)-saturated and color-critical graphs with prescribed minimum-degree.

**Lemma 3.2** ([8]). If \( G \) is a \( K_3 \)-saturated graph on \( n \geq 10 \) vertices and \( \delta(G) = 3 \), then \( e(G) \geq 3n - 15 \).
The work of Duffus and Hanson led Bollobás [15] to conjecture that \( \text{sat}_p(n, K_t) = pn - O(1) \), where \( \text{sat}_p(n, K_t) \) is the minimum number of edges among \( K_t \)-saturated graphs on \( n \) vertices with minimum degree at least \( p \). This conjecture has remained open since 1965, with the first nontrivial case where \( t = 3 \) proven in 2017 by Day.

**Theorem 3.3 ([7]).** For any integers \( p \geq 1 \) and \( t \geq 3 \), there exists a constant \( c = c(p) \) such that if \( G \) is a \( K_t \)-saturated graph on \( n \) vertices with \( \delta(G) \geq p \), then \( e(G) \geq pn - c \).

We shall need a structural result on \( K_3 \)-saturated graphs with minimum degree at most 2. This structural lemma due to Rolek and Song [21] is fundamental to our proof of the lower bound on \( \text{sat}(n, R_{\min}(K_3, K_{1,3})) \). The graph \( J \) depicted in Figure 3.1 is a \( K_3 \)-saturated graph with minimum degree 2, where \( A \neq \emptyset \) and either \( B = C = \emptyset \) or \( B \neq \emptyset \) and \( C \neq \emptyset \); \( A \), \( B \) and \( C \) are independent sets in \( J \) and pairwise disjoint; \( A \) is anti-complete to \( B \cup C \) and \( B \) is complete to \( C \); \( N_J(y) = A \cup B \) and \( N_J(z) = A \cup C \); and \( |A| + |B| + |C| = |J| - 2 \). It is straightforward to check that \( e(J) = 2(|J| - 2) + |B||C| - |B| - |C| \geq 2|J| - 5 \). Moreover,
\( e(J) = 2|J| - 5 \) when \(|B| = 1 \) or \(|C| = 1 \). That is, \( e(J) = 2|J| - 5 \) when \( J \) is obtained from \( C_5 \) by repeatedly duplicating vertices of degree 2.

**Lemma 3.4** ([21]). Let \( G \) be a \( K_3 \)-saturated graph on \( n \geq 5 \) vertices with \( \delta(G) = 2 \). Then \( G = J \). Furthermore, if \( e(G) = 2n - k \) for some \( k \in \{0, 1, 2, 3, 4, 5\} \), then \(|B||C| - |B| - |C| = 4 - k\), where \( A, B, C, \) and \( J \) are as depicted in Figure 3.1 and the values of \(|B|\) and \(|C|\) are summarized as in Table 3.1.

| \( k \) | \( e(J) \) | values of \(|B|\) and \(|C|\) with \(|B| \leq |C|\) |
|---|---|---|
| 5 | \( 2n - 5 \) | \(|B| = 1 \) and \(|C| \geq 1\) |
| 4 | \( 2n - 4 \) | \(|B| = |C| = 2 \) or \(|B| = |C| = 0\) |
| 3 | \( 2n - 3 \) | \(|B| = 2 \) and \(|C| = 3\) |
| 2 | \( 2n - 2 \) | \(|B| = 2 \) and \(|C| = 4\) |
| 1 | \( 2n - 1 \) | \(|B| = 2 \) and \(|C| = 5 \) or \(|B| = |C| = 3\) |
| 0 | \( 2n \) | \(|B| = 2 \) and \(|C| = 6\) |

Table 3.1: Some explicit values for \( e(J), |B|, \) and \( |C| \).
CHAPTER 4: DETERMINING $\text{sat}(n, R_{\min}(K_3, K_{1,3}))$

In this section, we determine the exact value of $\text{sat}(n, R_{\min}(K_3, K_{1,3}))$ for every integer $n$ at least 13. For notational convenience, given a coloring $\pi : E(G) \rightarrow \{\text{red, blue}\}$, we let $E_r$ and $E_b$ denote the color classes of $\pi$. We use $G_r$ and $G_b$ to denote the spanning subgraphs of $G$ with edge sets $E_r$ and $E_b$ respectively, and define $d_r(v) := d_{G_r}(v)$ and $d_b(v) := d_{G_b}(v)$.

We will utilize the following Lemma 4.1(i) to force a unique bad 2-coloring of certain graphs in order to establish an upper bound for $\text{sat}(n, R_{\min}(K_3, K_{1,3}))$. Lemma 4.1(ii) and Lemma 4.1(iii) will be applied to establish a lower bound for $\text{sat}(n, R_{\min}(K_3, K_{1,t-1}))$. The proof of Lemma 4.1 is similar to the proof of Lemma 1.6 in [21]. For completeness, we include a proof here.

**Lemma 4.1.** For every integer $t \geq 3$, let $\pi : E(G) \rightarrow \{\text{red, blue}\}$ be a bad 2-coloring of a graph $G$ on $n \geq t + 3$ vertices.

(i) If $e \in E(G)$ belongs to at least $2t - 3$ triangles in $G$, then $e \in E_b$.

(ii) If $G$ is $R_{\min}(K_3, K_{1,t})$-saturated and $v_1, \ldots, v_p \in V(G)$ are all the vertices in $G_b$ satisfying $d_b(v_i) \leq t - 2$ for all $i \in \{1, \ldots, p\}$, then for any pair of distinct vertices $v_i, v_j$ with $v_i v_j \notin E_b, v_i v_j \in E_r$. Moreover, $\alpha(G_b[\{v_1, \ldots, v_p\}]) \leq 2$.

(iii) If $G$ is $R_{\min}(K_3, K_{1,t})$-saturated, and among all bad 2-colorings of $G$, $\pi$ is chosen so that $|E_r|$ is maximum, then $\Delta(G_r) \leq n - 3$ and $G_r$ is 2-connected.

**Proof.** To prove (i), suppose that there exists an edge $e = uv \in E_r$ such that $e$ belongs to at least $2t - 3$ triangles in $G$. Since $G_r$ is $K_3$-free, we see that either $d_b(u) \geq t - 1$ or $d_b(v) \geq t - 1$. In either case, $G_b$ contains $K_{1,t-1}$ as a subgraph, a contradiction.
To prove (ii), let $v_1, \ldots, v_p \in V(G)$ be given as in the statement (ii). Since $G$ is $\mathcal{R}_{\text{min}}(K_3, K_{1,t})$-saturated, we see that, for any edge $e$ in $\overline{G}$, $G + e$ admits no bad 2-coloring. We next show that, for any $i, j \in \{1, \ldots, p\}$ with $i \neq j$, if $v_i v_j \notin E_b$, then $v_i v_j \in E_r$. Suppose there exist $v_i, v_j$ such that $v_i v_j \notin E_b$ and $v_i v_j \notin E_r$. But then $v_i v_j \notin E(G)$ and so we obtain a bad 2-coloring of $G + u_i u_j$ from $\pi$ by coloring the edge $u_i u_j$ blue, a contradiction. Thus $v_i v_j \in E_r$ for any $i, j \in \{1, \ldots, p\}$ with $i \neq j$ and $v_i v_j \notin E_b$. Since $G_r$ is $K_3$-saturated, it follows that $\alpha(G_b[\{v_1, \ldots, v_p\}]) \leq 2$.

It remains to prove (iii). By the choice of $\pi$, $G_r$ is $K_3$-free but $G_r + e$ contains a $K_3$ for any $e \in E(\overline{G}_r)$, and $G_b$ is $K_{1,t}$-free. Since $G$ is $\mathcal{R}_{\text{min}}(K_3, K_{1,t})$-saturated, we see that, for any edge $e$ in $\overline{G}$, $G + e$ admits no bad 2-coloring. Suppose that $\Delta(G_r) \geq n - 2$. Let $x \in V(G)$ with $d_r(x) = \Delta(G_r)$ and let $v$ be the unique non-neighbor of $x$ in $G_r$ if $d_r(x) = n - 2$. Since $G_r$ is $K_3$-free, we see that $N_r(x)$ is an independent set in $G_r$. By the choice of $\pi$, $v$ must be complete to $N_r(x)$ in $G_r$, else there exists a vertex, say $v' \in N_r(x)$, such that $vv'$ is colored blue or $vv' \notin E(G)$. In the former case, recoloring $uv'$ red yields a bad 2-coloring of $G$ with $|E_r| + 1$ red edges, contradicting our choice of $\pi$. In the latter case, we obtain a bad 2-coloring of $G + vv'$ from $\pi$ by coloring the edge $vv'$ blue, a contradiction. Since $n \geq t + 3$, we have $|N_r(x)| \geq t + 1 \geq 4$. Let $u \in N_r(x)$. Since $d_b(u) \leq t - 1$, there exists a vertex $w \in N_r(x)$ such that $uw \notin E_b$. Clearly, $uw \notin E(G)$. But then we obtain a bad 2-coloring of $G + uw$ from $\pi$ by coloring the edge $uw$ red, and then recoloring all edges incident with $u$ in $G_r$ blue and all edges incident with $u$ in $G_b$ red, a contradiction. This proves that $\Delta(G_r) \leq n - 3$.

Finally, we show that $G_r$ is 2-connected. Suppose that $G_r$ is not 2-connected. Since $G_r$ is $K_3$-free but $G_r + e$ contains a $K_3$ for any $e \in E(\overline{G}_r)$, we see that $G_r$ is connected and must have a cut vertex, say $u$. Since $\Delta(G_r) \leq n - 3$, $u$ has a non-neighbor, say $v$, in $G_r$. Let $G_1$ and $G_2$ be two components of $G_r \setminus \{u\}$ with $v \in V(G_2)$. Let $w \in V(G_1)$. By the choice of
\(\pi, wv \notin E_b\), otherwise we obtain a bad 2-coloring of \(G\) from \(\pi\) by recoloring the blue edge \(wv\) red, contrary to the maximality of \(|E_r|\). Thus \(wv \notin E(G)\) and then we obtain a bad 2-coloring of \(G + wv\) from \(\pi\) by coloring the edge \(wv\) red, a contradiction. This proves that \(G_r\) is 2-connected.

This completes the proof of Lemma 4.1. \(\square\)

We next prove the main result of this chapter.

**Theorem 4.2.** For all \(n \geq 13\), \(sat(n, R_{\min}(K_3, K_{1,3})) = 3n - 4\).

**Proof.** To obtain the desired upper bound for \(sat(n, R_{\min}(K_3, K_{1,3}))\), let \(G\) be the graph depicted in Figure 4.1, where \(G[A] = K_3 \cup C_{n-10}\), \(z\) is complete to \(V(G) \setminus \{y, z\}\), \(y\) is complete to \(V(G) \setminus \{y, z_2\}\), \(\{z_1, z_2\}\) is complete to \(\{y_1, y_2, y_3\}\), and \(z_1z_2, y_1y_4, y_2y_4, y_2y_5, y_3y_5 \in E(G)\) and \(C\) is an independent set in \(G\). Note that \(e(G) = 3n - 4\).

We next show that \(G\) is \(R_{\min}(K_3, K_{1,3})\)-saturated by showing that the \{red, blue\}-coloring of \(G\) as depicted in Figure 4.1 is the unique bad 2-coloring of \(G\). Let \(\pi : E(G) \rightarrow \{\text{red, blue}\}\) be a bad coloring of \(G\). Then \(G_r\) is \(K_3\)-free and \(G_b\) is \(K_{1,3}\)-free. Thus \(d_b(v) \leq 2\) for all \(v \in V(G)\). Since \(z_1z_2\) belongs to five triangles in \(G\), by Lemma 4.1(i), \(z_1z_2\) must be colored blue under \(\pi\). Since \(d_b(z) \leq 2\), we may assume that \(zy_\ell\) is colored red under \(\pi\), where \(y_\ell \in C\).

Suppose \(yz_1\) is colored blue under \(\pi\). Then \(z_1\) must be red-complete to \(C \cup \{z\}\) under \(\pi\). But then \(G[\{z, z_1, y_\ell\}]\) is a red \(K_3\) under \(\pi\), contrary to \(\pi\) being bad. Thus \(yz_1\), and similarly, \(yz_2\), is colored red under \(\pi\). Next, suppose \(zz_1\) is colored red under \(\pi\). Then \(z_1y_\ell\) must be colored blue under \(\pi\), else \(G[\{z, z_1, y_\ell\}]\) is a red \(K_3\). Then \(z_1\) must be red-complete to \(C \setminus \{y_\ell\}\) and so \(z\) is blue-complete to \(C \setminus \{y_\ell\}\) under \(\pi\). Then \(zz_2\) is colored red and thus \(z_2y_\ell\) is colored blue. Let \(y'_\ell \in A\) such that \(y_\ell y'_\ell \in E(G)\). Then, \(y_\ell y'_\ell\) must be colored red under \(\pi\). But then \(G[\{z, y_\ell, y'_\ell\}]\) is a red \(K_3\), a contradiction. Thus \(zz_1\), and similarly, \(zz_2\) must be colored
blue under $\pi$. It follows that $B$ must be red-complete to $C$ and $z$ must be red-complete to $C \cup A$. Thus all edges in $G[A]$ and $\{yy_1, yy_2, y_1y_4, y_2y_4, y_2y_5, y_3y_5\}$ must be colored blue under $\pi$. Finally, $y$ must be red-complete to $A$ under $\pi$. This proves that $\pi$ is the unique bad 2-coloring of $G$ as depicted in Figure 4.1. It can be easily checked that for any $u, v \in V(G)$ with $uv \notin E(G)$, $G + uv$ has a red $K_3$ if $uv$ is colored red and a blue $K_{1,3}$ if $uv$ is colored blue. Thus $G$ is indeed $R_{\min}(K_3, K_{1,3})$-saturated. Hence, $\text{sat}(n, R_{\min}(K_3, K_{1,3})) \leq e(G) = 3n - 4$ for all $n \geq 13$.

We next show that every $R_{\min}(K_3, K_{1,3})$-saturated graph on $n \geq 13$ vertices has at least $3n - 4$ edges. Suppose this is not true. Let $G$ be an $R_{\min}(K_3, K_{1,3})$-saturated graph on $n \geq 13$ vertices with $e(G) \leq 3n - 5$. Since $G$ is $R_{\min}(K_3, K_{1,3})$-saturated, it must have a bad 2-coloring. Among all bad 2-colorings of $G$, let $\pi : E(G) \rightarrow \{\text{red, blue}\}$ be a bad 2-coloring

![Figure 4.1: $R_{\min}(K_3, K_{1,3})$-saturated graph with a unique bad $\{\text{red, blue}\}$-coloring.](image-url)
of $G$ with $|E_r|$ maximum. By the choice of $\pi$, $G_r$ is $K_3$-saturated. By Lemma 4.1(iii), $G_r$ is 2-connected and $\Delta(G_r) \leq n - 3$. By Lemma 3.4, $e(G_r) \geq 2n - 5$. As $\Delta(G_b) \leq 2$, $e(G_b) \leq n$, with equality holding when $\delta(G_b) = 2$. When $\delta(G_b) \leq 1$, let $v_1, \ldots, v_\ell \in V(G)$ be all the vertices in $G_b$ satisfying $d_b(v_i) \leq 1$ for all $i \in [\ell]$. We may further assume that $0 = d_b(v_1) = \cdots = d_b(v_s) < 1 = d_b(v_{s+1}) \leq \cdots \leq d_b(v_\ell)$, where $0 \leq s \leq \ell$. By Lemma 4.1(ii), $\alpha(G_b[\{v_1, \ldots, v_\ell\}]) \leq 2$. Thus either $\ell = s = 2$, or $\ell \leq 4$ and $s = 0$, or $\ell \leq 3$ and $s = 1$. It follows that

Claim 1. $n - 2 \leq e(G_b) \leq n$, and $e(G_r) \leq 2n - 3$.

We next show that

Claim 2. $\delta(G_r) = 2$, and thus $G_r = J$ with $A \neq \emptyset, B \neq \emptyset$, and $C \neq \emptyset$ as pictured in Figure 3.1.

Proof. Since $e(G_r) \leq 2n - 3$, we see that $\delta(G_r) \leq 3$. By Lemma 4.1(iii), $\delta(G_r) \geq 2$. Thus $2 \leq \delta(G_r) \leq 3$. Suppose $\delta(G_r) = 3$. By Theorem 3.2, $e(G_r) \geq 3n - 15$. By Claim 1, $e(G_b) \geq n - 2$. But then $e(G) = e(G_r) + e(G_b) \geq 4n - 17 > 3n - 5$ because $n \geq 13$, a contradiction. Thus $\delta(G_r) = 2$. By Lemma 3.4, $G_r = J$ with $A \neq \emptyset, B \neq \emptyset$, and $C \neq \emptyset$ as pictured in Figure 3.1.

By Claim 2, $G_r = J$. We may assume that $|B| \leq |C|$. Since $e(G_r) \leq 2n - 3$, by Lemma 3.4, $|B| \leq 2$. We claim that


Proof. Suppose $|B| \leq 1$. If $|B| = 0$, then by Lemma 3.4, $|B| = |C| = 0$. But then $d_r(y) = d_r(z) = n - 2$, contrary to Lemma 4.1(iii). Thus $|B| = 1$. Let $B = \{u\}$. By symmetry of $A$ and $C$, we may assume that $|A| \geq |C|$. Then $|A| \geq 5$ because $n \geq 13$. 

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Since $d_b(u) \leq 2$, there must exist a vertex $v \in A$ such that $uv \notin E(G)$. But then we obtain a bad 2-coloring of $G + uv$ from $\pi$ by first coloring the edge $uv$ red, then recoloring $yu$ blue, and recoloring all edges between $y$ and $C$, and all edges incident with $u$ in $G_b$ red, a contradiction.

By Claim 3, $|C| \geq |B| \geq 2$. Since $e(G_r) \leq 2n - 3$, by Lemma 3.4, $|B| = 2$, $2 \leq |C| \leq 3$, and $e(G_r) \leq 2n - 4$. Thus,

**Claim 4.** $n - 2 \leq e(G_b) \leq n - 1$.

Let $B = \{z_1, z_2\}$ and let $C = \{y_1, \ldots, y_{|C|}\}$.

**Claim 5.** $|C| = 3$.

**Proof.** Suppose $|C| = 2$. Then $C = \{y_1, y_2\}$. Since $d_b(z_1) \leq 2$ and $|A| \geq 7$, let $w \in A$ be such that $wz_1 \notin E_b$. We first consider the case when $yz \in E_b$. Since $d_b(y) \leq 2$ and $d_b(z) \leq 2$, we may assume that $yy_1, zz_1 \notin E_b$. We claim that $yy_2, zz_2 \in E_b$. Suppose, say $yy_2 \notin E_b$. Then we obtain a bad 2-coloring of $G + wz_1$ from $\pi$ by first coloring the edge $wz_1$ red, and then recoloring $yz_1$ blue and all edges $y_1w^*$ red for any $w^* \in N_b(z_1) \cap A$, a contradiction. Thus $yy_2, zz_2 \in E_b$, as claimed. But then we obtain a bad 2-coloring of $G + wz_1$ by first coloring the edge $wz_1$ red, and then recoloring edges $yy_2, zz_2$ and edges between $\{z_1, y_1\}$ and $A$ in $G_b$ red, and finally recoloring edges $y_1z_1, y_2z_2, yz_1$, and $zy_1$ blue as in Figure 4.2, a contradiction.

It remains to consider the case when $yz \notin E_b$. Then $yz \notin E(G)$. By Lemma 4.1(ii), either $d_b(y) = 2$ or $d_b(z) = 2$. Since $|B| = |C| = 2$, we may assume that $d_b(y) = 2$. Then $yy_1, yy_2 \in E_b$. Furthermore, $d_b(z) = 2$, otherwise let $d_b(z) = 1$. We may assume $zz_1 \notin E_b$.

But then $G + wz_1$ has a bad 2-coloring obtained from $\pi$ by first coloring the edge $wz_1$ red,
and then recoloring edges $yz_1$, $yz_2$, $zy_1$, and $zy_2$ blue, and finally recoloring $yy_1$, $yy_2$, $zz_2$, and all edges between $z_1$ and $A$ in $G_b$ red, a contradiction. If we have $d_b(z) = 0$, then we obtain a bad 2-coloring of $G + wz_1$ from $\pi$ by first coloring $wz_1$ red, and then recoloring $yz_1$ and $yz_2$ blue, and finally recoloring $yy_1$, $yy_2$, and all edges between $A$ and $B$ in $G_b$ red. We also have $z_1z_2 \in E_b$, else $z_1z_2 \notin E(G)$. Then we obtain a bad 2-coloring of $G + wz_1$ from $\pi$ by first coloring the edge $wz_1$ red, and then recoloring edges $yy_1$, $yy_2$, and any edges between $A$ and $B$ in $G_b$ red, and finally recoloring edges $yz_1$ and $yz_2$ blue. By a symmetric argument, $y_1y_2 \in E_b$. But then $G + yz$ has a bad 2-coloring obtained from $\pi$ by first coloring recoloring $yz$ blue, and then recoloring edges $z_1y$, $z_2y_1$, and $z_2y_2$ blue, and finally recoloring $yy_1$, $yy_2$, $z_2z_1$, $z_2z$ red, as depicted in Figure 4.3.

By Claim 5, $|C| = 3$. Then $C = \{y_1, y_2, y_3\}$. By Lemma 3.4, $e(G_r) = 2n - 3$. By Claim 1, $e(G_b) = n - 2$. Then,
Figure 4.3: A bad 2-coloring of $G + yz$, where dashed lines indicate edges which have been recolored.

\[
\begin{align*}
\alpha(G_b[\{v_1, \ldots, v_\ell\}]) & = 2 \text{ and either } \ell = s = 2, \text{ or } \ell = 4 \text{ and } s = 0, \text{ or } \ell = 3 \text{ and } s = 1. \\
\text{Furthermore, for any } i \in [\ell], \text{ there exists } j \in [\ell] \text{ with } i \neq j \text{ such that } v_iv_j \in E_r.
\end{align*}
\]

We next show that $d_b(y) = d_b(z) = 2$. Suppose $d_b(y) \leq 1$. We may assume that $yy_2, yy_3 \notin E_b$. By Lemma 4.1(ii), $d_b(y_2) = d_b(y_3) = 2$. For any $v \in A$, if $d_b(v) \leq 1$, then either $vz_1 \notin E_b$ or $vz_2 \notin E_b$. We may assume that $vz_1 \notin E_b$. But then we obtain a bad 2-coloring of $G + vz_1$ from $\pi$ by first coloring the edge $vz_1$ red and then recoloring the edge $yzv$ blue, a contradiction.

Thus for any $v \in A$, $d_b(v) = 2$. Similarly, $d_b(z_i) = 2$ for all $i \in \{1, 2\}$, else, say $d_b(z_1) \leq 1$. Let $v \in A \setminus N_b(z_1)$. Then we obtain a bad 2-coloring of $G + z_1v$ from $\pi$ by first coloring the edge $z_1v$ red and then recoloring the edge $yzv$ blue, a contradiction. If $yy_1 \notin E_b$, then by Lemma 4.1(ii), $d_b(y_1) = 2$. But then, by $(*), d_b(y) = d_b(z) = 0$ and so $G + yz$ has a bad 2-coloring obtained from $\pi$ by coloring the edge $yzv$ blue. Thus $yy_1 \in E_b$. Then $yz \notin E(G)$ by $(*).$ We again obtain a bad 2-coloring of $G + yz$ from $\pi$ by coloring the edge $yz$ blue.

This proves that $d_b(y) = 2$. Similarly, one can prove that $d_b(z) = 2$. 

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Since \(db(y) = db(z) = 2\), by \((\ast)\), \(db(v) = 2\) for any \(v \in A\). We may further assume that \(yy_1, zz_1 \in E_b\). By Lemma 4.1(ii), either \(db(z_1) = 2\) or \(db(z_2) = 2\) because \(z_1z_2 \notin E_r\). Thus \(|\{z_1, z_2\} \cap \{v_1, \ldots, v_\ell\}| \leq 1\). Suppose first that \(yz \notin E_b\). Then \(yz \notin E(G)\) and \(zz_1, zz_2 \in E_b\). Then \(|\{z_1, z_2\} \cap \{v_1, \ldots, v_\ell\}| \leq 1\) and \(|\{z_1, z_2\} \cap \{v_1, \ldots, v_s\}| = 0\). We may assume that \(yy_2 \in E_b\). Since \(C\) is an independent set in \(G_r\), by Lemma 4.1(ii), \(|C \cap \{v_1, \ldots, v_\ell\}| \leq 1\). Thus \(\ell \leq 2\) and \(s \leq 1\), contrary to \((\ast)\). This proves that \(yz \in E_b\). Then \(zz_2, yy_2, yy_3 \notin E(G)\) and \(|\{z_1, z_2\} \cap \{v_1, \ldots, v_\ell\}| \leq 1\). Since \(C\) is an independent set in \(G_r\), by Lemma 4.1(ii), \(|C \cap \{v_1, \ldots, v_s\}| = 0\), else \(\ell = 1\), contrary to \((\ast)\). Furthermore, \(|C \cap \{v_{s+1}, \ldots, v_\ell\}| \leq 2\). By \((\ast)\), \(db(z_2) = 0\), \(y_2y_3 \in E_b\), and \(db(y_2) = db(y_3) = 1\). Thus \(z_1z_2 \notin E(G)\) and \(db(z_1) = 2\). Let \(w, w^* \in A\) be such that \(z_1w \in E_b\) and \(z_1w^* \notin E(G)\). But then we obtain a bad 2-coloring of \(G + z_1w^*\) from \(\pi\) by first coloring the edge \(z_1w^*\) red, and then recoloring edges \(yz_1, zz_2y_1\) blue and edges \(yy_1, z_1w\) red as in Figure 4.4, a contradiction.

Figure 4.4: A bad recoloring of \(G + z_1w^*\) where dashed lines indicate edges which have been recolored.
CHAPTER 5: BOUNDS FOR \( \text{sat}(n, R_{\text{min}}(K_3, K_{1,t})) \) AND FUTURE WORK

Asymptotic bound for \( \text{sat}(n, R_{\text{min}}(K_3, K_{1,t})) \)

In this section, we generalize the result on \( \text{sat}(n, R_{\text{min}}(K_3, K_{1,3})) \) to stars of any order to obtain an asymptotic bound for \( \text{sat}(n, R_{\text{min}}(K_3, K_{1,t})) \) for \( t \geq 4 \) and \( n \) sufficiently large. Many of the methods used in the specific case continue to hold in general. For example, our upper bound construction for \( \text{sat}(n, R_{\text{min}}(K_3, K_{1,t})) \) mimics the properties of the construction used to bound \( \text{sat}(n, R_{\text{min}}(K_3, K_{1,3})) \).

\[ \text{Theorem 5.1.} \text{ For all integers } t \geq 4 \text{ and } n \geq 4t + 2, \text{ there exists } c = c(t) \text{ such that} \]

![Figure 5.1: \( R_{\text{min}}(K_3, K_{1,t}) \)-saturated graph with a unique bad \{red, blue\}-coloring.](image)
\[ \text{sat}(n, \mathcal{R}_{\text{min}}(K_3, K_{1,t})) = \left( \frac{3}{2} + \frac{t}{2} \right) n + c. \text{ Furthermore, } t - t \left\lfloor \frac{t}{2} \right\rfloor + \left\lfloor \frac{t}{2} \right\rfloor^2 - 6 \leq c \leq t^2 - 3t - 3. \]

**Proof.** To obtain a desired upper bound for \( \text{sat}(n, \mathcal{R}_{\text{min}}(K_3, K_{1,t})) \), let \( G \) be the graph depicted in Figure 5.1, where \( G[R] \) is a \((t - 1)\)-regular graph on \( n - 3t - 1 - \varepsilon \) vertices where \( \varepsilon = (n + t - 1) \mod 2 \), \( G[A] = K_t \cup R \), \( z \) is complete to \( V(G) \setminus \{y,z\} \), \( y \) is complete to \( V(G) \setminus \{y,z,y_1\} \), \( B \) is complete to \( C \), \( y_1 \) is complete to \( A^* \), and finally, \( G[A^* \cup C^*] \) is a \((t - 2)\)-regular bipartite graph with \( |A^*| = |C^*| = t - 1 \), \( G[B] = K_{t-1} \) and \( G[C] = K_t \). Note that \( e(G) \leq \left( \frac{3}{2} + \frac{t}{2} \right) n + t^2 - 3t - 3 \).

We show that the \{red, blue\}-coloring of \( G \) depicted in Figure 5.1 is the unique bad 2-coloring of \( G \). Let \( \pi : E(G) \rightarrow \{ \text{red}, \text{blue} \} \) be a bad coloring of \( G \). Since \( \pi \) is bad, \( G_r \) is \( K_3 \)-free and \( G_b \) is \( K_{1,t} \)-free. Thus \( d_b(v) \leq t - 1 \) for all \( v \in V(G) \). Let \( B = \{ z_1, \ldots, z_{t-1} \} \) and consider the vertex \( z_1 \). For all \( i \in \{2, 3, \ldots, t - 1\} \), \( z_1z_i \) belongs to \( 2t - 1 \) triangles and must be colored blue under \( \pi \) by Lemma 4.1(i). Now, as \( d_b(z) \leq t - 1 \), we may assume that \( yz_1 \) is colored red for some \( y_1 \in C \). Next, suppose that \( yz_1 \) is colored blue under \( \pi \). Then \( z_1 \) is red-complete to \( C \cup \{z\} \) under \( \pi \). But then we have that \( G[\{z, z_1, y_1\}] \) is a red \( K_3 \) under \( \pi \), contradicting that \( \pi \) is a bad coloring. Thus, \( yz_1 \) is colored red under \( \pi \), and by symmetry, \( yz_i \) is colored red under \( \pi \) for all \( i \in [t - 1] \). Now suppose instead that \( zz_1 \) is colored red under \( \pi \). Then, \( z_1y_1 \) must be colored blue under \( \pi \), or else \( G[\{z, z_1, y_1\}] \) is a red \( K_3 \). Hence \( z_1 \) is red-complete to \( C \setminus \{y_1\} \), and \( z \) is blue-complete to \( C \setminus \{y_1\} \) under \( \pi \). Furthermore, \( z \) is red-complete to \( A \) under \( \pi \). Then, \( zz_i \) is colored red and \( z_iy_1 \) is colored blue under \( \pi \) for all \( i \in \{2, 3, \ldots, t - 1\} \). Now, let \( y'_1 \in A \) such that \( y_1y'_1 \in E(G) \), and hence \( y_1y'_1 \in E_r \). But then \( G[\{z, y_1, y'_1\}] \) is a red \( K_3 \) under \( \pi \), a contradiction. Thus \( zz_1 \) must be colored blue under \( \pi \), and by symmetry, \( zz_i \) is colored blue under \( \pi \) for all \( i \in [t - 1] \). It then follows that \( B \) is red-complete to \( C \), and that \( z \) is red-complete to \( A \cup C \) under \( \pi \). Thus, all edges in \( G[A] \) and \( G[\{y\} \cup C] \) must be colored blue under \( \pi \). Finally, this implies that \( y \) is red-complete to \( A \) under \( \pi \), proving that
the coloring depicted in Figure 5.1 is indeed the unique bad coloring of $G$. It is easy to check that for any pair $u, v \in V(G)$ with $uv \notin E(G)$, $G + uv$ contains a red $K_3$ if $uv$ is colored red and a blue $K_{1,t}$ if $uv$ is colored blue. Thus, $\text{sat}(n, R_{\min}(K_3, K_{1,t})) \leq \left(\frac{3}{2} + \frac{t}{2}\right) n + t^2 - 3t - 3$.

We proceed to show that $\text{sat}(n, R_{\min}(K_3, K_{1,t})) \geq \left(\frac{3}{2} + \frac{t}{2}\right) n + t - t \left\lfloor \frac{t}{2} \right\rfloor + \left\lfloor \frac{t}{2} \right\rfloor^2 - 6$. Suppose that $G$ is an $R_{\min}(K_3, K_{1,t})$-saturated graph on $n \geq 4t + 2$ vertices, and among all bad 2-colorings of $E(G)$, choose $\pi : E(G) \rightarrow \{\text{red}, \text{blue}\}$ to be a bad 2-coloring of $E(G)$ with $|E_r|$ maximum. By Lemma 4.1(iii), $G_r$ is 2-connected and so $\delta(G_r) \geq 2$. By Lemma 3.2 and Lemma 3.4, $e(G_r) \geq 2n - 5$. Furthermore, Lemma 4.1(ii) implies that for any $k$ such that $2 \leq k \leq t$, there can be at most $2k - 2$ vertices of degree at most $k - 2$. In particular, we have precisely $2k - 2$ vertices of degree $k - 2$ when $G[[v_1, \ldots, v_p]] = 2K_{k-1}$. Thus, $e(G_b) \geq \frac{1}{2}[(t - 1)(n - 2k + 2) + (2k - 2)(k - 2)]$ for some fixed $2 \leq k \leq t$. Minimizing the right-hand-side of this inequality with respect to $k$, we obtain

$$e(G_b) \geq \frac{1}{2} \left( (t - 1) \left( n - 2 \left\lceil \frac{t}{2} \right\rceil + 2 \right) + 2 \left\lceil \frac{t}{2} \right\rceil \left( \left\lceil \frac{t}{2} \right\rceil - 1 \right) \right).$$

Combining our lower bounds for $e(G_b)$ and $e(G_r)$, we have that

$$e(G) \geq \left(\frac{3}{2} + \frac{t}{2}\right) n + t - t \left\lfloor \frac{t}{2} \right\rfloor + \left\lfloor \frac{t}{2} \right\rfloor^2 - 6.$$

This completes our proof of the asymptotic bounds on $\text{sat}(n, R_{\min}(K_3, K_{1,t})). \quad \Box$

### Future Work

This research began as an extension of the study of $\text{sat}(n, R_{\min}(K_3, T_k))$ as in [21], where a precise value for $\text{sat}(n, R_{\min}(K_3, T_4))$ is computed. Note that $T_4 = \{P_4, K_{1,3}\}$. We believe
the techniques and insights developed in this thesis may be useful in studying the saturation number \( \text{sat}(n, \mathcal{R}_{\text{min}}(K_3, P_4)) \). We can also attempt to forbid larger complete graphs to study \( \text{sat}(n, \mathcal{R}_{\text{min}}(K_4, K_1,t)) \). Given a structural result regarding \( K_4 \)-saturated graphs, the techniques regarding \( K_{1,t} \)-saturated graphs should still prove useful and may provide insight to the study of \( \text{sat}(n, \mathcal{R}_{\text{min}}(K_4, \mathcal{T}_k)) \).
LIST OF REFERENCES


