On Saturation Numbers of Ramsey-minimal Graphs

2018

Hunter M. Davenport
University of Central Florida

ON SATURATION NUMBERS OF RAMSEY-MINIMAL GRAPHS

by

HUNTER DAVENPORT

A thesis submitted in partial fulfilment of the requirements
for the degree of Honors in the Major Program in Mathematics
in the College of Sciences
in the College of The Burnett Honors College
at the University of Central Florida

Spring 2018 Term

Thesis Chair: Dr. Zi-Xia Song

ABSTRACT

Dating back to the 1930's, Ramsey theory still intrigues many who study combinatorics. Roughly put, it makes the profound assertion that *complete disorder is impossible.* One view of this problem is in edge-colorings of complete graphs. For forbidden graphs $H_1, \ldots, H_k$ and a graph G, we write $G \longrightarrow (H_1, \ldots, H_k)$ if every k-edge-coloring of G contains a monochromatic copy of H_i in color i for some $i \in \{1, \ldots, k\}$. If π is a $\{red, blue\}$-edge-coloring of G, we say π is a *bad coloring* if G contains no red K_3 or blue $K_{1,t}$ under π. A graph G is $(H_1, \ldots, H_k)$-Ramsey-minimal if $G \longrightarrow (H_1, \ldots, H_k)$ but no proper subgraph of G has this property. Given a family $\mathcal{F}$ of graphs, we say that a graph G is $\mathcal{F}$-saturated if no member of $\mathcal{F}$ is a subgraph of G, but for any edge $xy \notin E(G)$, $G + xy$ contains a member of $\mathcal{F}$ as a subgraph. Letting $\mathcal{R}_{\min}(K_3, K_{1,t})$ be the family of $(K_3, K_{1,t})$-Ramsey minimal graphs, we study the *saturation number*, denoted $\mathrm{sat}(n, \mathcal{R}_{\min}(K_3, K_{1,t}))$, which is the minimum number of edges among all $\mathcal{R}_{\min}(K_3, K_{1,t})$-saturated graphs on n vertices. We believe the methods and constructions developed in this thesis will be useful in studying the saturation numbers of $(K_4, K_{1,t})$-saturated graphs.

ACKNOWLEDGMENTS

I would like to thank my advisor, Dr. Zi-Xia Song, for her inspiring love for the subject of graph theory and support throughout the most challenging undertaking of my undergraduate career. Her course in combinatorics changed the way that I think about mathematics entirely. I must also thank Dr. Michael Reid, for my first introduction to graph theory, for his guidance, and for the frequent reality checks whenever I think I know all there is to know. I have been lucky to have such valuable mentors and to have the love and support of my family and friends whenever academic life was particularly difficult. Finally, I would like to thank the Department of Mathematics and the Burnett Honors College as a whole for the numerous opportunities to enrich my undergraduate experience and prepare for further education.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER 1: INTRODUCTION

We begin this thesis with an overview of basic concepts and definitions in graph theory. Let $\mathbb{N}$ be the set of natural numbers. For any $n \in \mathbb{N}$, define $[n] := \{1, 2, ..., n\}$. For a finite set V, we use $|V|$ to denote the number of elements of V. For any pair of sets A, B, we use $A \setminus B$ to denote the elements of A which do not also belong to B. We use $\binom{V}{2}$ to denote the set of all unordered pairs of elements of V. A *graph* G is composed of a set $V(G)$ of vertices and a set $E(G)$ of edges, where $E(G) \subseteq \binom{V(G)}{2}$. We say that $|V(G)|$ is the *order* of G, and frequently abbreviate this notation to $|G|$. For notational convenience, instead of writing $\{v_1, v_2\}$ to represent an edge with endpoints v_1 and v_2, we write v_1v_2. Two vertices $x, y \in V(G)$ are *adjacent* if $xy \in E(G)$. The *complement* of a graph G, denoted $\overline{G}$, is a graph with vertex set $V(G)$ and edge-set $E(\overline{G}) = \binom{V(G)}{2} \setminus E(G)$. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Furthermore, we say a subgraph H of G is a *proper subgraph* of G if $H \neq G$. If G does not contain H as a subgraph, then we say G is *H-free*. For any $S \subseteq V(G)$, *the subgraph of G induced by S*, denoted $G[S]$, is the graph with vertex set S and edge set $\{xy \in E(G) : x, y \in S\}$. A graph H is an *induced subgraph* of a graph G if $H = G[S]$ for some $S \subseteq V(G)$. The *neighborhood* $N(x)$ of a vertex $x \in V(G)$ is the set of all vertices $y \in V(G)$ such that $xy \in E(G)$. The *degree* of a vertex $v \in V(G)$, denoted $d(v)$, is the number of neighbors, namely $|N(v)|$, of v in G. The *minimum degree* of G, denoted $\delta(G)$, is the smallest $d(v)$ among all $v \in V(G)$. Similarly, the *maximum degree* of G, denoted $\Delta(G)$, is the largest $d(v)$ among all $v \in V(G)$. Given a subgraph $H \subseteq G$ and a vertex $v \in V(H)$, let $N_H(v)$ denote the neighborhood of v in H, and similarly let $d_H(v) := |N_H(v)|$. A *complete graph* K_n is a graph with n vertices where $xy \in E(K_n)$ for all $x, y \in V(K_n)$ with $x \neq y$. A *triangle* is a K_3. For all $t \geq 1$, a *star* $K_{1,t}$ is a graph with one center vertex x and vertices $v_1, \ldots, v_t$ with edge set $E(K_{1,t}) = \{xv_i \mid i \in [t]\}$. A *$k$-edge-coloring* of a graph G is a function $\pi : E(G) \longrightarrow [k]$ that assigns a number to each edge in G. For the purpose of visualization, when $k = 2$ we often view colorings as a function $\pi : E(G) \longrightarrow \{\text{red, blue}\}$. G is *monochromatic* with respect to π if $\pi(E(G)) = i \in [k]$ for some fixed $k \in \mathbb{N}$. Given a graph G, for any disjoint sets $A, B \subseteq V(G)$, A is *complete* to B if for all $a \in A$, $b \in B$, we have $ab \in E(G)$, and A is *anticomplete* to B if for all $a \in A$, $b \in B$, we have $ab \notin E(G)$. We define $e(G) := |E(G)|$ to be the number of edges in G. A subset $S \subset V(G)$ is said to be an *independent set* if for all pairs $u, v \in S$, $uv \notin E(G)$. The *independence number* of G, denoted $\alpha(G)$, is the maximum size of an independent set in G. The *join* $G+H$ (resp. *union* $G \cup H$) of two vertex disjoint graphs G and H is the graph having vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$ (resp. $E(G) \cup E(H)$). Given two isomorphic graphs G and H, we may (with a slight but common abuse of notation) write $G = H$.

Ramsey theory originated in the work of Frank Ramsey [20]. We introduce the following notation: For "forbidden" graphs $H_1, \ldots, H_k$ and a graph G, we write $G \longrightarrow (H_1, \ldots, H_k)$ if every k-edge-coloring of G contains a monochromatic H_i in color i for some $i \in [k]$. The *Ramsey number* $r_k(H_1, \ldots, H_k)$ is the minimum integer n such that $K_n \longrightarrow (H_1, \ldots, H_k)$. If $H_1 = \cdots = H_k = H$, then we simply write $r_k(H)$ instead of $r_k(H, \ldots, H)$.

Theorem 1.1 (Ramsey's Theorem [20])**.** *For any positive integer k and any collection of graphs $H_1, \ldots, H_k$, the Ramsey number $r_k(H_1, \ldots, H_k)$ exists.*

Ramsey theory is a notoriously difficult branch of combinatorics. Many questions in the field have remained wide open for years. For example, we have exact values for only a few of the *classical Ramsey numbers* $r_k(K_{t_1}, \ldots, K_{t_k})$ with $t_i \in \mathbb{N}$ despite active research for almost a century. We introduce the study of classical Ramsey theory in Chapter 2, demonstrating some well-known results of the field and stating special results pertaining to certain classes of graphs.

In Chapter 2, we present some more background and history on Ramsey theory. We explore the notions of Ramsey-minimality and saturation numbers, most importantly defining the class of $\mathcal{R}_{\min}(K_3, K_{1,t})$-saturated graphs.

In Chapter 3, we present some of the most recent work on Ramsey-minimal graphs. We include here several known results, placing particular emphasis on the results which will be used in this thesis.

In Chapter 4, we begin presenting our original research, in which we determine the exact value of the saturation number of $\mathcal{R}_{\min}(K_3, K_{1,3})$-saturated graphs on $n \geq 13$ vertices.

In chapter 5, we present an asymptotic bound on the saturation number of $\mathcal{R}_{\min}(K_3, K_{1,t})$-saturated graphs for $t \geq 4$ and n sufficiently large. We also discuss potential future work.

CHAPTER 2: RAMSEY-MINIMAL GRAPHS AND BACKGROUND

Background

Given graphs $H_1, \ldots, H_k$, to compute $r_k(H_1, \ldots, H_k)$ we show that $K_{n-1} \not\longrightarrow (H_1, \ldots, H_k)$, that is, there exists a k-edge-coloring of K_{n-1} that does not contain a monochromatic H_i in color i for any $i \in [k]$. This shows that $r_k(H_1, \ldots, H_k) \geq n$. Then, we show that $K_n \longrightarrow (H_1, \ldots, H_k)$, that is, every k-edge-coloring of K_n contains a monochromatic H_i in color i for some $i \in [k]$. This shows that $r_k(H_1, \ldots, H_k) \leq n$, and thus $r_k(H_1, \ldots, H_k) = n$. We first observe a classical example of this technique in practice.

Example. $r_2(K_3) = 6$.

Proof. Figure 2.1 illustrates that $K_5 \not\longrightarrow (K_3, K_3)$. Thus, $r_2(K_3) \geq 6$. Next, let $G := K_6$.

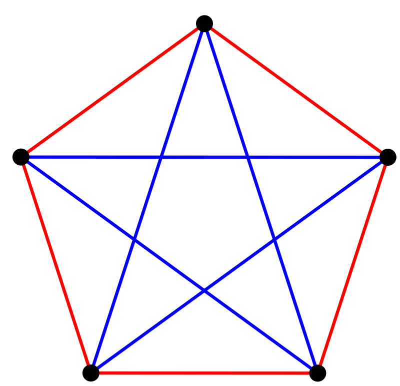

Figure 2.1: A monochromatic triangle free coloring of K_5.

	3	4	5	6	7	8	9
3	6	9	14	18	23	28	36
4	9	18	25	36 - 41	49 - 61	59 - 84	73 - 115
5	14	25	43-48	58 - 87	80 - 143	101 - 216	133 - 316
6	18	36 - 41	58 - 87	102 - 165	115 - 298	134 - 495	183 - 780
7	23	49 - 61	80 - 143	115 - 298	205 - 540	217 - 1031	252 - 1713
8	28	59 - 84	101 - 216	134 - 495	217 - 1031	282 - 1870	329 - 3583
9	36	73 - 115	133 - 316	183 - 780	252 - 1713	329 - 3583	565 - 6588

Table 2.1: Various values of $r_2(K_n, K_m)$.

Let $\pi : E(G) \longrightarrow \{\text{red}, \text{blue}\}$ be any 2-edge-coloring of G. We claim G contains a monochromatic K_3. Consider a fixed vertex $x \in V(G)$ and let $v_1, v_2, \ldots, v_5$ be the five neighbors of x. By the Pigeonhole Principle, we may assume that xv_1, xv_2, xv_3 are colored blue under π. If v_1v_2, v_1v_3, v_2v_3 are all colored red, then $G[\{v_1, v_2, v_3\}]$ is a red K_3 under π. So, we may assume that v_1v_2 is colored blue. Then, $G[\{x, v_1, v_2\}]$ is a blue K_3 under π, as desired. $\square$

Although the above proof is relatively simple, determining the exact values of Ramsey numbers for larger graphs quickly becomes incredibly difficult. Table 2.1, compiled by Radziszowski [19], aggregates some of the known Ramsey numbers of complete graphs when two colors are used.

Ramsey-minimal and Saturated Graphs

Although classical Ramsey theory typically examines colorings of complete graphs, given graphs $H_1, \ldots, H_t$, we can also look at graphs G which are, roughly put, minimal with respect to the property that $G \longrightarrow (H_1, \ldots, H_t)$.

A graph G is $(H_1, \ldots, H_t)$-*Ramsey-minimal* if $G \longrightarrow (H_1, \ldots, H_t)$, but for any proper subgraph G' of G, $G' \not\longrightarrow (H_1, \ldots, H_t)$. We define $\mathcal{R}_{\min}(H_1, \ldots, H_t)$ to be the family of $(H_1, \ldots, H_t)$-Ramsey-minimal graphs. It is straightforward to prove by induction that a graph G satisfies $G \longrightarrow (H_1, \ldots, H_t)$ if and only if there exists a subgraph G' of G such that G' is $(H_1, \ldots, H_t)$-Ramsey-minimal. Ramsey's theorem [20] implies that $\mathcal{R}_{\min}(H_1, \ldots, H_t) \neq \emptyset$ for all integers t and all finite graphs $H_1, \ldots, H_t$. As pointed out in a recent paper of Fox, Grinshpun, Liebenau, Person, and Szabó [13], "it is still widely open to classify the graphs in $\mathcal{R}_{\min}(H_1, \ldots, H_t)$, or even to prove that these graphs have certain properties". Some properties of $\mathcal{R}_{\min}(H_1, \ldots, H_t)$ have been studied, such as the minimum degree $s(H_1, \ldots, H_t) := \min\{\delta(G) : G \in \mathcal{R}_{\min}(H_1, \ldots, H_t)\}$, which was first introduced by Burr, Erdős, and Lovász [4]. Recent results on $s(H_1, \ldots, H_t)$ can be found in [12, 13]. For more information on Ramsey-related topics, the readers are referred to a very recent informative survey due to Conlon, Fox, and Sudakov [6].

The notion of saturated graphs was first defined by Zykov [22] in 1949, though the study of saturated graphs in their own right remained relatively unexplored until Erdős, Hajnal, and Moon [9] began to study saturation numbers in 1964. Given a family $\mathcal{F}$ of forbidden graphs, we say a graph G is *$\mathcal{F}$-saturated* if G contains no member of $\mathcal{F}$ as a subgraph, but for any edge $e \notin E(G)$, $G + e$ contains a member of $\mathcal{F}$ as a subgraph. We simply say that G is H-saturated if $\mathcal{F} = \{H\}$. Until recently, typically only K_t-saturated graphs were studied. The *saturation number* of a family $\mathcal{F}$ of forbidden graphs is defined to be

$$\mathrm{sat}(n, \mathcal{F}) = \min\{e(G) : G \text{ is } \mathcal{F}\text{-saturated on } n \text{ vertices}\}.$$

In [9], it was determined that for all $n > k$, $\mathrm{sat}(n, K_k) = n(k-2) - \binom{k-1}{2}$, and that the unique graph attaining this minimum is $K_{k-2} + \overline{K}_{n-k+2}$.

Combining the ideas of Ramsey-minimality and saturation, we consider $\mathcal{R}_{\min}(H_1, \ldots, H_t)$-

saturated graphs for some forbidden graphs $H_1, \ldots, H_t$. This notion was first studied by Nešetřil [17] in 1986 when he asked whether there are infinitely many $\mathcal{R}_{\min}(H_1, \ldots, H_t)$-saturated graphs. At the time, such graphs were called $(H_1, \ldots, H_t)$-co-critical graphs. This was answered in the positive by Galluccio, Siminovits, and Simonyi [14] in 1992, when they studied $\mathcal{R}_{\min}(K_3, K_3)$-saturated graphs. Galluccio, Siminovits, and Simonyi also ask whether the stronger proposition holds: are there infinitely many $\mathcal{R}_{\min}(H_1, \ldots, H_t)$-saturated graphs, none of which contain another as a subgraph? This question remains open. The first to study $\mathrm{sat}(n, \mathcal{R}_{\min}(H_1, \ldots, H_t))$ were Hanson and Toft [16] in 1987, where they took $H_1, \ldots, H_t$ to be complete graphs. They proposed the following conjecture.

Conjecture 2.1 ([16]). *Let $r = r(K_{k_1}, \ldots, K_{k_t})$ be the classical Ramsey number for complete graphs. Then*

$$sat(n, \mathcal{R}_{\min}(K_{k_1}, \ldots, K_{k_t})) = \begin{cases} \binom{n}{2} & n < r \\ (r-2)(n-r+2) + \binom{r-2}{2} & n \geq r \end{cases}$$

CHAPTER 3: KNOWN RESULTS

In 2011, Chen, Ferrara, Gould, Magnant, and Schmitt [5] proved that $sat(n, \mathcal{R}_{\min}(K_3, K_3)) = 4n - 10$ for $n \geq 56$. This settles the first non-trivial case of Conjecture 2.1 for sufficiently large n, and is so far the only settled case. In 2014, Ferrara, Kim, and Yeager [11] proved that $sat(n, \mathcal{R}_{\min}(m_1K_2, \ldots, m_tK_2)) = 3(m_1 + \cdots + m_t - t)$ for $m_1, \ldots, m_t \geq 1$ and $n > 3(m_1 + \cdots + m_t - t)$. The problem of finding $sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_k))$ when $k = 3$ was also explored in [5], where $\mathcal{T}_k$ denotes the family of all trees on k vertices. This parameter remained undetermined until 2017 when Rolek and Song [21] proved the following.

Theorem 3.1 ([21]). *For all $n \geq 18$, $sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_4)) = \left\lfloor \frac{5n}{2} \right\rfloor$. Furthermore, for any integers $k \geq 5$ and $n \geq 2k + (\lceil k/2 \rceil + 1)\lceil k/2 \rceil - 2$, there exist constants $c = \left(\frac{1}{2}\left\lceil \frac{k}{2} \right\rceil + \frac{3}{2}\right)k - 2$ and $C = 2k^2 - 6k + \frac{3}{2} - \left\lceil \frac{k}{2} \right\rceil \left(k - \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil - 1\right)$ such that*

$$\left(\frac{3}{2} + \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil\right) n - c \leq sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_k)) \leq \left(\frac{3}{2} + \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil\right) n + C.$$

This is the result which inspires our present work. Rather than forbidding any tree on 4 vertices, we will investigate the parameter $sat(n, \mathcal{R}_{\min}(K_3, K_{1,3}))$. When discussing a particular coloring π of a $\mathcal{R}_{\min}(K_3, K_{1,3})$-saturated graph G, we will call π a "bad" coloring if G does not contain a red K_3 or a blue $K_{1,3}$ under π.

There are several other known results which we will use throughout. One is due to Duffus and Hanson in 1986, when they studied K_3-saturated and color-critical graphs with prescribed minimum-degree.

Lemma 3.2 ([8]). *If G is a K_3-saturated graph on $n \geq 10$ vertices and $\delta(G) = 3$, then $e(G) \geq 3n - 15$.*

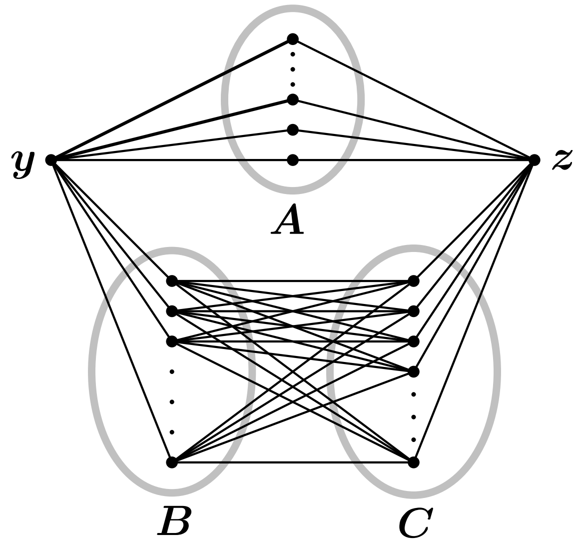

Figure 3.1: Graph J.

The work of Duffus and Hanson led Bollobás [15] to conjecture that $\mathrm{sat}_p(n, K_t) = pn - O(1)$, where $\mathrm{sat}_p(n, K_t)$ is the minimum number of edges among K_t-saturated graphs on n vertices with minimum degree at least p. This conjecture has remained open since 1965, with the first nontrivial case where $t = 3$ proven in 2017 by Day.

Theorem 3.3 ([7]). *For any integers $p \geq 1$ and $t \geq 3$, there exists a constant $c = c(p)$ such that if G is a K_t-saturated graph on n vertices with $\delta(G) \geq p$, then $e(G) \geq pn - c$.*

We shall need a structural result on K_3-saturated graphs with minimum degree at most 2. This structural lemma due to Rolek and Song [21] is fundamental to our proof of the lower bound on $\mathrm{sat}(n, \mathcal{R}_{\min}(K_3, K_{1,3}))$. The graph J depicted in Figure 3.1 is a K_3-saturated graph with minimum degree 2, where $A \neq \emptyset$ and either $B = C = \emptyset$ or $B \neq \emptyset$ and $C \neq \emptyset$; A, B and C are independent sets in J and pairwise disjoint; A is anti-complete to $B \cup C$ and B is complete to C; $N_J(y) = A \cup B$ and $N_J(z) = A \cup C$; and $|A| + |B| + |C| = |J| - 2$. It is straightforward to check that $e(J) = 2(|J| - 2) + |B||C| - |B| - |C| \geq 2|J| - 5$. Moreover,

$e(J) = 2|J| - 5$ when $|B| = 1$ or $|C| = 1$. That is, $e(J) = 2|J| - 5$ when J is obtained from C_5 by repeatedly duplicating vertices of degree 2.

Lemma 3.4 ([21]). *Let G be a K_3-saturated graph on $n \geq 5$ vertices with $\delta(G) = 2$. Then $G = J$. Furthermore, if $e(G) = 2n-k$ for some $k \in \{0,1,2,3,4,5\}$, then $|B||C|-|B|-|C| = 4-k$, where A, B, C, and J are as depicted in Figure 3.1 and the values of $|B|$ and $|C|$ are summarized as in Table 3.1.*

k	$e(J)$	values of $\lvert B\rvert$ and $\lvert C\rvert$ with $\lvert B\rvert \leq \lvert C\rvert$
5	$2n-5$	$\lvert B\rvert = 1$ and $\lvert C\rvert \geq 1$
4	$2n-4$	$\lvert B\rvert = \lvert C\rvert = 2$ or $\lvert B\rvert = \lvert C\rvert = 0$
3	$2n-3$	$\lvert B\rvert = 2$ and $\lvert C\rvert = 3$
2	$2n-2$	$\lvert B\rvert = 2$ and $\lvert C\rvert = 4$
1	$2n-1$	$\lvert B\rvert = 2$ and $\lvert C\rvert = 5$ or $\lvert B\rvert = \lvert C\rvert = 3$
0	$2n$	$\lvert B\rvert = 2$ and $\lvert C\rvert = 6$

Table 3.1: Some explicit values for $e(J)$, $|B|$, and $|C|$.

CHAPTER 4: DETERMINING sat$(n, \mathcal{R}_{\min}(K_3, K_{1,3}))$

In this section, we determine the exact value of $\mathrm{sat}(n, \mathcal{R}_{\min}(K_3, K_{1,3}))$ for every integer n at least 13. For notational convenience, given a coloring $\pi : E(G) \longrightarrow \{\text{red, blue}\}$, we let E_r and E_b denote the color classes of π. We use G_r and G_b to denote the spanning subgraphs of G with edge sets E_r and E_b respectively, and define $d_r(v) := d_{G_r}(v)$ and $d_b(v) := d_{G_b}(v)$.

We will utilize the following Lemma 4.1(i) to force a unique bad 2-coloring of certain graphs in order to establish an upper bound for $sat(n, \mathcal{R}_{\min}(K_3, K_{1,3}))$. Lemma 4.1(ii) and Lemma 4.1(iii) will be applied to establish a lower bound for $sat(n, \mathcal{R}_{\min}(K_3, K_{1,t-1}))$. The proof of Lemma 4.1 is similar to the proof of Lemma 1.6 in [21]. For completeness, we include a proof here.

Lemma 4.1. *For every integer $t \geq 3$, let $\pi : E(G) \longrightarrow \{$red, blue$\}$ be a bad 2-coloring of a graph G on $n \geq t + 3$ vertices.*

(i) If $e \in E(G)$ belongs to at least $2t - 3$ triangles in G, then $e \in E_b$.

(ii) If G is $\mathcal{R}_{\min}(K_3, K_{1,t})$-saturated and $v_1, \dots, v_p \in V(G)$ are all the vertices in G_b satisfying $d_b(v_i) \leq t - 2$ for all $i \in \{1, \dots, p\}$, then for any pair of distinct vertices v_i, v_j with $v_iv_j \notin E_b$, $v_iv_j \in E_r$. Moreover, $\alpha(G_b[\{v_1, \dots, v_p\}]) \leq 2$.

(iii) If G is $\mathcal{R}_{\min}(K_3, K_{1,t})$-saturated, and among all bad 2-colorings of G, π is chosen so that $|E_r|$ is maximum, then $\Delta(G_r) \leq n - 3$ and G_r is 2-connected.

Proof. To prove (i), suppose that there exists an edge $e = uv \in E_r$ such that e belongs to at least $2t - 3$ triangles in G. Since G_r is K_3-free, we see that either $d_b(u) \geq t - 1$ or $d_b(v) \geq t - 1$. In either case, G_b contains $K_{1,t-1}$ as a subgraph, a contradiction.

To prove (ii), let $v_1, \ldots, v_p \in V(G)$ be given as in the statement (ii). Since G is $\mathcal{R}_{\min}(K_3, K_{1,t})$-saturated, we see that, for any edge e in $\overline{G}$, $G+e$ admits no bad 2-coloring. We next show that, for any $i, j \in \{1, \ldots, p\}$ with $i \neq j$, if $v_iv_j \notin E_b$, then $v_iv_j \in E_r$. Suppose there exist v_i, v_j such that $v_iv_j \notin E_b$ and $v_iv_j \notin E_r$. But then $v_iv_j \notin E(G)$ and so we obtain a bad 2-coloring of $G+u_iu_j$ from π by coloring the edge u_iu_j blue, a contradiction. Thus $v_iv_j \in E_r$ for any $i, j \in \{1, \ldots, p\}$ with $i \neq j$ and $v_iv_j \notin E_b$. Since G_r is K_3-saturated, it follows that $\alpha(G_b[\{v_1, \ldots, v_p\}]) \leq 2$.

It remains to prove (iii). By the choice of π, G_r is K_3-free but $G_r + e$ contains a K_3 for any $e \in E(\overline{G_r})$, and G_b is $K_{1,t}$-free. Since G is $\mathcal{R}_{\min}(K_3, K_{1,t})$-saturated, we see that, for any edge e in $\overline{G}$, $G+e$ admits no bad 2-coloring. Suppose that $\Delta(G_r) \geq n-2$. Let $x \in V(G)$ with $d_r(x) = \Delta(G_r)$ and let v be the unique non-neighbor of x in G_r if $d_r(x) = n-2$. Since G_r is K_3-free, we see that $N_r(x)$ is an independent set in G_r. By the choice of π, v must be complete to $N_r(x)$ in G_r, else there exists a vertex, say $v' \in N_r(x)$, such that vv' is colored blue or $vv' \notin E(G)$. In the former case, recoloring uv' red yields a bad 2-coloring of G with $|E_r|+1$ red edges, contradicting our choice of π. In the latter case, we obtain a bad 2-coloring of $G + vv'$ from π by coloring the edge vv' blue, a contradiction. Since $n \geq t+3$, we have $|N_r(x)| \geq t+1 \geq 4$. Let $u \in N_r(x)$. Since $d_b(u) \leq t-1$, there exists a vertex $w \in N_r(x)$ such that $uw \notin E_b$. Clearly, $uw \notin E(G)$. But then we obtain a bad 2-coloring of $G + uw$ from π by coloring the edge uw red, and then recoloring all edges incident with u in G_r blue and all edges incident with u in G_b red, a contradiction. This proves that $\Delta(G_r) \leq n-3$.

Finally, we show that G_r is 2-connected. Suppose that G_r is not 2-connected. Since G_r is K_3-free but $G_r + e$ contains a K_3 for any $e \in E(\overline{G_r})$, we see that G_r is connected and must have a cut vertex, say u. Since $\Delta(G_r) \leq n-3$, u has a non-neighbor, say v, in G_r. Let G_1 and G_2 be two components of $G_r \setminus \{u\}$ with $v \in V(G_2)$. Let $w \in V(G_1)$. By the choice of

π, $wv \notin E_b$, otherwise we obtain a bad 2-coloring of G from π by recoloring the blue edge wv red, contrary to the maximality of $|E_r|$. Thus $wv \notin E(G)$ and then we obtain a bad 2-coloring of $G + wv$ from π by coloring the edge wv red, a contradiction. This proves that G_r is 2-connected.

This completes the proof of Lemma 4.1. $\square$

We next prove the main result of this chapter.

Theorem 4.2. *For all $n \geq 13$, $sat(n, \mathcal{R}_{min}(K_3, K_{1,3})) = 3n - 4$.*

Proof. To obtain the desired upper bound for $sat(n, \mathcal{R}_{\min}(K_3, K_{1,3}))$, let G be the graph depicted in Figure 4.1, where $G[A] = \overline{K_3} \cup C_{n-10}$, z is complete to $V(G) \setminus \{y, z\}$, y is complete to $V(G) \setminus \{y, z, y_2\}$, $\{z_1, z_2\}$ is complete to $\{y_1, y_2, y_3\}$, and $z_1z_2, y_1y_4, y_2y_4, y_2y_5, y_3y_5 \in E(G)$ and C is an independent set in G. Note that $e(G) = 3n - 4$.

We next show that G is $\mathcal{R}_{\min}(K_3, K_{1,3})$-saturated by showing that the $\{red, blue\}$-coloring of G as depicted in Figure 4.1 is the unique bad 2-coloring of G. Let $\pi : E(G) \longrightarrow$ {red, blue} be a bad coloring of G. Then G_r is K_3-free and G_b is $K_{1,3}$-free. Thus $d_b(v) \leq 2$ for all $v \in V(G)$. Since z_1z_2 belongs to five triangles in G, by Lemma 4.1(i), z_1z_2 must be colored blue under π. Since $d_b(z) \leq 2$, we may assume that zy_ℓ is colored red under π, where $y_\ell \in C$. Suppose yz_1 is colored blue under π. Then z_1 must be red-complete to $C \cup \{z\}$ under π. But then $G[\{z, z_1, y_\ell\}]$ is a red K_3 under π, contrary to π being bad. Thus yz_1, and similarly, yz_2, is colored red under π. Next, suppose zz_1 is colored red under π. Then z_1y_ℓ must be colored blue under π, else $G[\{z, z_1, y_\ell\}]$ is a red K_3. Then z_1 must be red-complete to $C \setminus \{y_\ell\}$ and so z is blue-complete to $C \setminus \{y_\ell\}$ under π. Then zz_2 is colored red and thus z_2y_ℓ is colored blue. Let $y'_\ell \in A$ such that $y_\ell y'_\ell \in E(G)$. Then, $y_\ell y'_\ell$ must be colored red under π. But then $G[\{z, y_\ell, y'_\ell\}]$ is a red K_3, a contradiction. Thus zz_1, and similarly, zz_2 must be colored

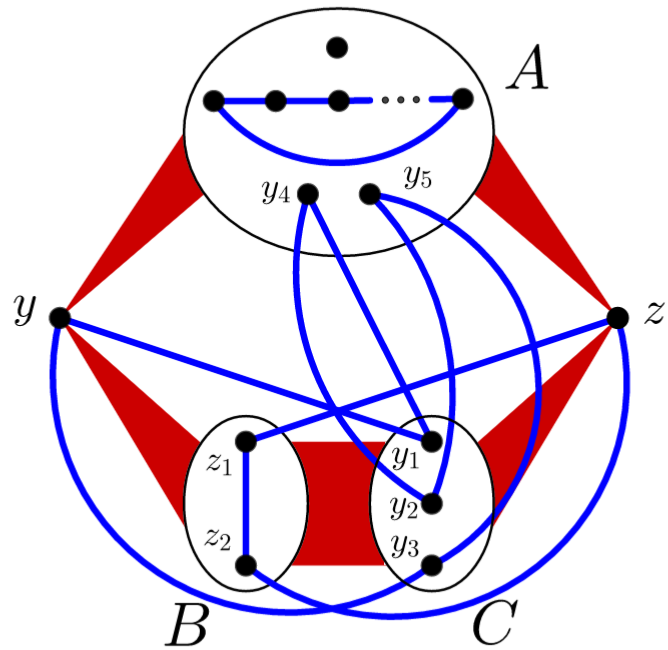

Figure 4.1: $\mathcal{R}_{\min}(K_3, K_{1,3})$-saturated graph with a unique bad $\{red, blue\}$-coloring.

blue under π. It follows that B must be red-complete to C and z must be red-complete to $C \cup A$. Thus all edges in $G[A]$ and $\{yy_1, yy_2, y_1y_4, y_2y_4, y_2y_5, y_3y_5\}$ must be colored blue under π. Finally, y must be red-complete to A under π. This proves that π is the unique bad 2-coloring of G as depicted in Figure 4.1. It can be easily checked that for any $u, v \in V(G)$ with $uv \notin E(G)$, $G+uv$ has a red K_3 if uv is colored red and a blue $K_{1,3}$ if uv is colored blue. Thus G is indeed $\mathcal{R}_{\min}(K_3, K_{1,3})$-saturated. Hence, $sat(n, \mathcal{R}_{\min}(K_3, K_{1,3})) \leq e(G) = 3n - 4$ for all $n \geq 13$.

We next show that every $\mathcal{R}_{\min}(K_3, K_{1,3})$-saturated graph on $n \geq 13$ vertices has at least $3n - 4$ edges. Suppose this is not true. Let G be an $\mathcal{R}_{\min}(K_3, K_{1,3})$-saturated graph on $n \geq 13$ vertices with $e(G) \leq 3n - 5$. Since G is $\mathcal{R}_{\min}(K_3, K_{1,3})$-saturated, it must have a bad 2-coloring. Among all bad 2-colorings of G, let $\pi : E(G) \longrightarrow \{red,\ blue\}$ be a bad 2-coloring of G with $|E_r|$ maximum. By the choice of π, G_r is K_3-saturated. By Lemma 4.1(iii), G_r is 2-connected and $\Delta(G_r) \le n-3$. By Lemma 3.4, $e(G_r) \ge 2n-5$. As $\Delta(G_b) \le 2$, $e(G_b) \le n$, with equality holding when $\delta(G_b) = 2$. When $\delta(G_b) \le 1$, let $v_1, \ldots, v_\ell \in V(G)$ be all the vertices in G_b satisfying $d_b(v_i) \le 1$ for all $i \in [\ell]$. We may further assume that $0 = d_b(v_1) = \cdots = d_b(v_s) < 1 = d_b(v_{s+1}) \le \cdots \le d_b(v_\ell)$, where $0 \le s \le \ell$. By Lemma 4.1(ii), $\alpha(G_b[\{v_1, \ldots, v_\ell\}]) \le 2$. Thus either $\ell = s = 2$, or $\ell \le 4$ and $s = 0$, or $\ell \le 3$ and $s = 1$. It follows that

Claim 1. $n-2 \le e(G_b) \le n$, and $e(G_r) \le 2n-3$.

We next show that

Claim 2. $\delta(G_r) = 2$, and thus $G_r = J$ with $A \ne \emptyset, B \ne \emptyset$, and $C \ne \emptyset$ as pictured in Figure 3.1.

Proof. Since $e(G_r) \le 2n-3$, we see that $\delta(G_r) \le 3$. By Lemma 4.1(iii), $\delta(G_r) \ge 2$. Thus $2 \le \delta(G_r) \le 3$. Suppose $\delta(G_r) = 3$. By Theorem 3.2, $e(G_r) \ge 3n-15$. By Claim 1, $e(G_b) \ge n-2$. But then $e(G) = e(G_r) + e(G_b) \ge 4n-17 > 3n-5$ because $n \ge 13$, a contradiction. Thus $\delta(G_r) = 2$. By Lemma 3.4, $G_r = J$ with $A \ne \emptyset, B \ne \emptyset$, and $C \ne \emptyset$ as pictured in Figure 3.1. $\square$

By Claim 2, $G_r = J$. We may assume that $|B| \le |C|$. Since $e(G_r) \le 2n-3$, by Lemma 3.4, $|B| \le 2$. We claim that

Claim 3. $|B| = 2$.

Proof. Suppose $|B| \le 1$. If $|B| = 0$, then by Lemma 3.4, $|B| = |C| = 0$. But then $d_r(y) = d_r(z) = n-2$, contrary to Lemma 4.1(iii). Thus $|B| = 1$. Let $B = \{u\}$. By symmetry of A and C, we may assume that $|A| \ge |C|$. Then $|A| \ge 5$ because $n \ge 13$.

Since $d_b(u) \leq 2$, there must exist a vertex $v \in A$ such that $uv \notin E(G)$. But then we obtain a bad 2-coloring of $G + uv$ from π by first coloring the edge uv red, then recoloring yu blue, and recoloring all edges between y and C, and all edges incident with u in G_b red, a contradiction. $\square$

By Claim 3, $|C| \geq |B| \geq 2$. Since $e(G_r) \leq 2n - 3$, by Lemma 3.4, $|B| = 2$, $2 \leq |C| \leq 3$, and $e(G_r) \leq 2n - 4$. Thus,

Claim 4. $n - 2 \leq e(G_b) \leq n - 1$.

Let $B = \{z_1, z_2\}$ and let $C = \{y_1, \ldots, y_{|C|}\}$.

Claim 5. $|C| = 3$.

Proof. Suppose $|C| = 2$. Then $C = \{y_1, y_2\}$. Since $d_b(z_1) \leq 2$ and $|A| \geq 7$, let $w \in A$ be such that $wz_1 \notin E_b$. We first consider the case when $yz \in E_b$. Since $d_b(y) \leq 2$ and $d_b(z) \leq 2$, we may assume that $yy_1, zz_1 \notin E_b$. We claim that $yy_2, zz_2 \in E_b$. Suppose, say $yy_2 \notin E_b$. Then we obtain a bad 2-coloring of $G + wz_1$ from π by first coloring the edge wz_1 red, and then recoloring yz_1 blue and all edges y_1w^* red for any $w^* \in N_b(z_1) \cap A$, a contradiction. Thus $yy_2, zz_2 \in E_b$, as claimed. But then we obtain a bad 2-coloring of $G + wz_1$ by first coloring the edge wz_1 red, and then recoloring edges yy_2, zz_2 and edges between $\{z_1, y_1\}$ and A in G_b red, and finally recoloring edges y_1z_1, y_2z_2, yz_1, and zy_1 blue as in Figure 4.2, a contradiction.

It remains to consider the case when $yz \notin E_b$. Then $yz \notin E(G)$. By Lemma 4.1(ii), either $d_b(y) = 2$ or $d_b(z) = 2$. Since $|B| = |C| = 2$, we may assume that $d_b(y) = 2$. Then $yy_1, yy_2 \in E_b$. Furthermore, $d_b(z) = 2$, otherwise let $d_b(z) = 1$. We may assume $zz_1 \notin E_b$. But then $G + wz_1$ has a bad 2-coloring obtained from π by first coloring the edge wz_1 red,

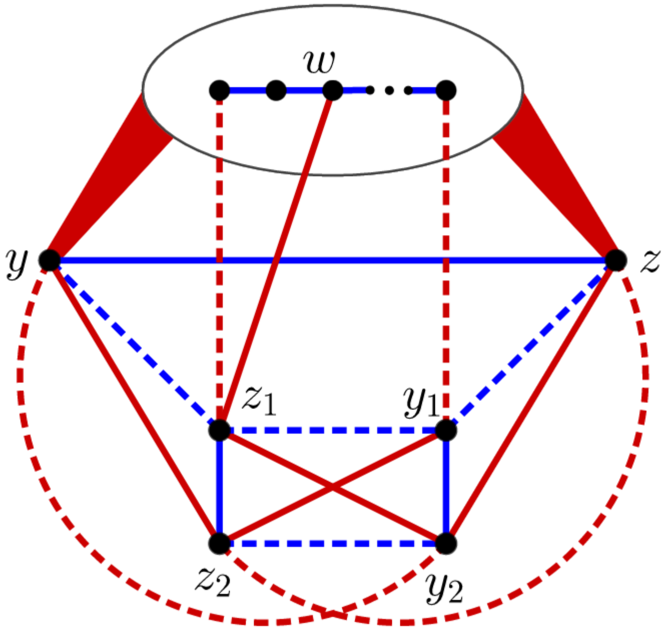

Figure 4.2: A bad 2-coloring of $G + z_1 w$ where dashed lines indicate edges which have been recolored.

and then recoloring edges yz_1, yz_2, zy_1, and zy_2 blue, and finally recoloring yy_1, yy_2, zz_2, and all edges between z_1 and A in G_b red, a contradiction. If we have $d_b(z) = 0$, then we obtain a bad 2-coloring of $G + wz_1$ from π by first coloring wz_1 red, and then recoloring yz_1 and yz_2 blue, and finally recoloring yy_1, yy_2, and all edges between A and B in G_b red. We also have $z_1z_2 \in E_b$, else $z_1z_2 \notin E(G)$. Then we obtain a bad 2-coloring of $G + wz_1$ from π by first coloring the edge wz_1 red, and then recoloring edges yy_1, yy_2, and any edges between A and B in G_b red, and finally recoloring edges yz_1 and yz_2 blue. By a symmetric argument, $y_1y_2 \in E_b$. But then $G + yz$ has a bad 2-coloring obtained from π by first coloring recoloring yz blue, and then recoloring edges z_1y, z_2y_1, and z_2y_2 blue, and finally recoloring yy_1, yy_2, z_2z_1, z_2z red, as depicted in Figure 4.3. $\square$

By Claim 5, $|C| = 3$. Then $C = \{y_1, y_2, y_3\}$. By Lemma 3.4, $e(G_r) = 2n - 3$. By Claim 1, $e(G_b) = n - 2$. Then,

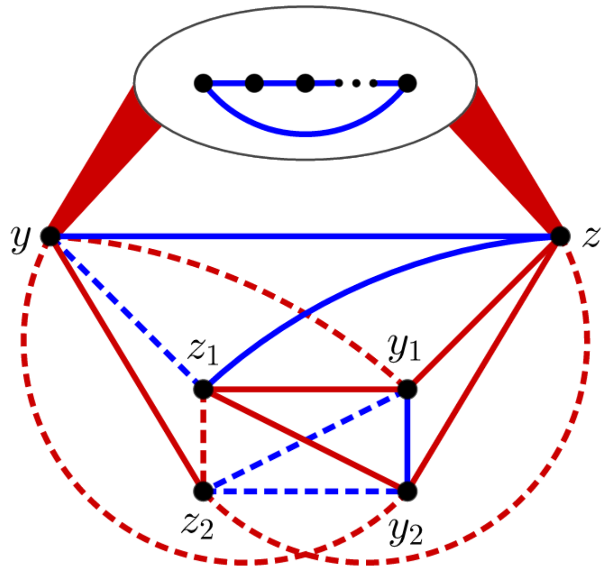

Figure 4.3: A bad 2-coloring of $G + yz$, where dashed lines indicate edges which have been recolored.

(∗) $\alpha(G_b[\{v_1, \ldots, v_\ell\}]) = 2$ and either $\ell = s = 2$, or $\ell = 4$ and $s = 0$, or $\ell = 3$ and $s = 1$. Furthermore, for any $i \in [\ell]$, there exists $j \in [\ell]$ with $i \neq j$ such that $v_i v_j \in E_r$.

We next show that $d_b(y) = d_b(z) = 2$. Suppose $d_b(y) \leq 1$. We may assume that $yy_2, yy_3 \notin E_b$. By Lemma 4.1(ii), $d_b(y_2) = d_b(y_3) = 2$. For any $v \in A$, if $d_b(v) \leq 1$, then either $vz_1 \notin E_b$ or $vz_2 \notin E_b$. We may assume that $vz_1 \notin E_b$. But then we obtain a bad 2-coloring of $G+vz_1$ from π by first coloring the edge vz_1 red and then recoloring the edge yv blue, a contradiction. Thus for any $v \in A$, $d_b(v) = 2$. Similarly, $d_b(z_i) = 2$ for all $i \in \{1, 2\}$, else, say $d_b(z_1) \leq 1$. Let $v \in A \setminus N_b(z_1)$. Then we obtain a bad 2-coloring of $G + z_1 v$ from π by first coloring the edge $z_1 v$ red and then recoloring the edge yz_1 blue, a contradiction. If $yy_1 \notin E_b$, then by Lemma 4.1(ii), $d_b(y_1) = 2$. But then, by (∗), $d_b(y) = d_b(z) = 0$ and so $G + yz$ has a bad 2-coloring obtained from π by coloring the edge yz blue. Thus $yy_1 \in E_b$. Then $yz \notin E(G)$ by (∗). We again obtain a bad 2-coloring of $G + yz$ from π by coloring the edge yz blue. This proves that $d_b(y) = 2$. Similarly, one can prove that $d_b(z) = 2$.

Since $d_b(y) = d_b(z) = 2$, by $(*)$, $d_b(v) = 2$ for any $v \in A$. We may further assume that $yy_1, zz_1 \in E_b$. By Lemma 4.1(ii), either $d_b(z_1) = 2$ or $d_b(z_2) = 2$ because $z_1z_2 \notin E_r$. Thus $|\{z_1, z_2\} \cap \{v_1, \ldots, v_\ell\}| \leq 1$. Suppose first that $yz \notin E_b$. Then $yz \notin E(G)$ and $zz_1, zz_2 \in E_b$. Then $|\{z_1, z_2\} \cap \{v_1, \ldots, v_\ell\}| \leq 1$ and $|\{z_1, z_2\} \cap \{v_1, \ldots, v_s\}| = 0$. We may assume that $yy_2 \in E_b$. Since C is an independent set in G_r, by Lemma 4.1(ii), $|C \cap \{v_1, \ldots, v_\ell\}| \leq 1$. Thus $\ell \leq 2$ and $s \leq 1$, contrary to $(*)$. This proves that $yz \in E_b$. Then $zz_2, yy_2, yy_3 \notin E(G)$ and $|\{z_1, z_2\} \cap \{v_1, \ldots, v_\ell\}| \leq 1$. Since C is an independent set in G_r, by Lemma 4.1(ii), $|C \cap \{v_1, \ldots, v_s\}| = 0$, else $\ell = 1$, contrary to $(*)$. Furthermore, $|C \cap \{v_{s+1}, \ldots, v_\ell\}| \leq 2$. By $(*)$, $d_b(z_2) = 0$, $y_2y_3 \in E_b$, and $d_b(y_2) = d_b(y_3) = 1$. Thus $z_1z_2 \notin E(G)$ and $d_b(z_1) = 2$. Let $w, w^* \in A$ be such that $z_1w \in E_b$ and $z_1w^* \notin E(G)$. But then we obtain a bad 2-coloring of $G + z_1w^*$ from π by first coloring the edge z_1w^* red, and then recoloring edges yz_1, z_2y_1 blue and edges yy_1, z_1w red as in Figure 4.4, a contradiction. □

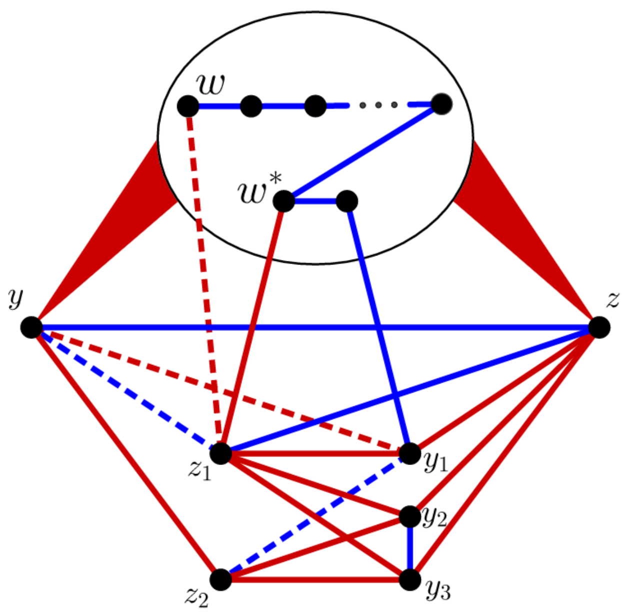

Figure 4.4: A bad recoloring of $G + z_1w^*$ where dashed lines indicate edges which have been recolored.

CHAPTER 5: BOUNDS FOR sat$(n, \mathcal{R}_{\min}(K_3, K_{1,t}))$ AND FUTURE WORK

Asymptotic bound for sat$(n, \mathcal{R}_{\min}(K_3, K_{1,t}))$

In this section, we generalize the result on sat$(n, \mathcal{R}_{\min}(K_3, K_{1,3}))$ to stars of any order to obtain an asymptotic bound for sat$(n, \mathcal{R}_{\min}(K_3, K_{1,t}))$ for $t \geq 4$ and n sufficiently large. Many of the methods used in the specific case continue to hold in general. For example, our upper bound construction for sat$(n, \mathcal{R}_{\min}(K_3, K_{1,t}))$ mimics the properties of the construction used to bound sat$(n, \mathcal{R}_{\min}(K_3, K_{1,3}))$.

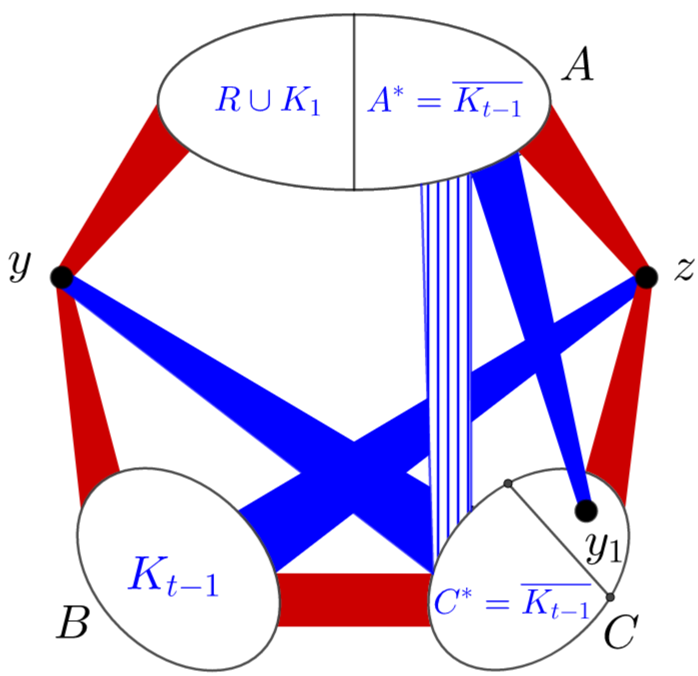

Figure 5.1: $\mathcal{R}_{\min}(K_3, K_{1,t})$-saturated graph with a unique bad $\{red, blue\}$-coloring.

Theorem 5.1. *For all integers $t \geq 4$ and $n \geq 4t + 2$, there exists $c = c(t)$ such that*

$sat(n, \mathcal{R}_{\min}(K_3, K_{1,t})) = \left(\frac{3}{2} + \frac{t}{2}\right) n + c$. *Furthermore,* $t - t\left\lfloor \frac{t}{2} \right\rfloor + \left\lfloor \frac{t}{2} \right\rfloor^2 - 6 \leq c \leq t^2 - 3t - 3$.

Proof. To obtain a desired upper bound for $sat(n, \mathcal{R}_{\min}(K_3, K_{1,t}))$, let G be the graph depicted in Figure 5.1, where $G[R]$ is a $(t-1)$-regular graph on $n - 3t - 1 - \varepsilon$ vertices where $\varepsilon = (n + t - 1) \bmod 2$, $G[A] = \overline{K}_t \cup R$, z is complete to $V(G) \setminus \{y, z\}$, y is complete to $V(G) \setminus \{y, z, y_1\}$, B is complete to C, y_1 is complete to A^*, and finally, $G[A^* \cup C^*]$ is a $(t-2)$-regular bipartite graph with $|A^*| = |C^*| = t - 1$, $G[B] = K_{t-1}$ and $G[C] = \overline{K}_t$. Note that $e(G) \leq \left(\frac{3}{2} + \frac{t}{2}\right) n + t^2 - 3t - 3$.

We show that the {red, blue}-coloring of G depicted in Figure 5.1 is the unique bad 2-coloring of G. Let $\pi : E(G) \longrightarrow$ {red, blue} be a bad coloring of G. Since π is bad, G_r is K_3-free and G_b is $K_{1,t}$-free. Thus $d_b(v) \leq t-1$ for all $v \in V(G)$. Let $B = \{z_1, \ldots, z_{t-1}\}$ and consider the vertex z_1. For all $i \in \{2, 3, \ldots, t-1\}$, z_1z_i belongs to $2t - 1$ triangles and must be colored blue under π by Lemma 4.1(i). Now, as $d_b(z) \leq t-1$, we may assume that zy_ℓ is colored red for some $y_\ell \in C$. Next, suppose that yz_1 is colored blue under π. Then z_1 is red-complete to $C \cup \{z\}$ under π. But then we have that $G[\{z, z_1, y_\ell\}]$ is a red K_3 under π, contradicting that π is a bad coloring. Thus, yz_1 is colored red under π, and by symmetry, yz_i is colored red under π for all $i \in [t-1]$. Now suppose instead that zz_1 is colored red under π. Then, z_1y_ℓ must be colored blue under π, or else $G\{[z, z_1, y_\ell]\}$ is a red K_3. Hence z_1 is red-complete to $C \setminus \{y_\ell\}$, and z is blue-complete to $C \setminus \{y_\ell\}$ under π. Furthermore, z is red-complete to A under π. Then, zz_i is colored red and z_iy_ℓ is colored blue under π for all $i \in \{2, 3, \ldots, t-1\}$. Now, let $y'_\ell \in A$ such that $y_\ell y'_\ell \in E(G)$, and hence $y_\ell y'_\ell \in E_r$. But then $G[\{z, y_\ell, y'_\ell\}]$ is a red K_3 under π, a contradiction. Thus zz_1 must be colored blue under π, and by symmetry, zz_i is colored blue under π for all $i \in [t-1]$. It then follows that B is red-complete to C, and that z is red-complete to $A \cup C$ under π. Thus, all edges in $G[A]$ and $G[\{y\} \cup C]$ must be colored blue under π. Finally, this implies that y is red-complete to A under π, proving that the coloring depicted in Figure 5.1 is indeed the unique bad coloring of G. It is easy to check that for any pair $u, v \in V(G)$ with $uv \notin E(G)$, $G + uv$ contains a red K_3 if uv is colored red and a blue $K_{1,t}$ if uv is colored blue. Thus, $\mathrm{sat}(n, \mathcal{R}_{\min}(K_3, K_{1,t})) \leq \left(\frac{3}{2} + \frac{t}{2}\right) n + t^2 - 3t - 3$.

We proceed to show that $\mathrm{sat}(n, \mathcal{R}_{\min}(K_3, K_{1,t})) \geq \left(\frac{3}{2} + \frac{t}{2}\right) n + t - t\left\lfloor\frac{t}{2}\right\rfloor + \left\lfloor\frac{t}{2}\right\rfloor^2 - 6$. Suppose that G is an $\mathcal{R}_{\min}(K_3, K_{1,t})$-saturated graph on $n \geq 4t + 2$ vertices, and among all bad 2-colorings of $E(G)$, choose $\pi : E(G) \longrightarrow \{\text{red, blue}\}$ to be a bad 2-coloring of $E(G)$ with $|E_r|$ maximum. By Lemma 4.1(iii), G_r is 2-connected and so $\delta(G_r) \geq 2$. By Lemma 3.2 and Lemma 3.4, $e(G_r) \geq 2n - 5$. Furthermore, Lemma 4.1(ii) implies that for any k such that $2 \leq k \leq t$, there can be at most $2k - 2$ vertices of degree at most $k - 2$. In particular, we have precisely $2k - 2$ vertices of degree $k - 2$ when $G[\{v_1, \ldots, v_p\}] = 2K_{k-1}$. Thus, $e(G_b) \geq \frac{1}{2}[(t-1)(n - 2k + 2) + (2k - 2)(k - 2)]$ for some fixed $2 \leq k \leq t$. Minimizing the right-hand-side of this inequality with respect to k, we obtain

$$e(G_b) \geq \frac{1}{2}\left[(t-1)\left(n - 2\left\lfloor\frac{t}{2}\right\rfloor + 2\right) + 2\left\lfloor\frac{t}{2}\right\rfloor\left(\left\lfloor\frac{t}{2}\right\rfloor - 1\right)\right].$$

Combining our lower bounds for $e(G_b)$ and $e(G_r)$, we have that

$$e(G) \geq \left(\frac{3}{2} + \frac{t}{2}\right) n + t - t\left\lfloor\frac{t}{2}\right\rfloor + \left\lfloor\frac{t}{2}\right\rfloor^2 - 6.$$

This completes our proof of the asymptotic bounds on $\mathrm{sat}(n, \mathcal{R}_{\min}(K_3, K_{1,t}))$. $\square$

Future Work

This research began as an extension of the study of $\mathrm{sat}(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_k))$ as in [21], where a precise value for $\mathrm{sat}(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_4))$ is computed. Note that $\mathcal{T}_4 = \{P_4, K_{1,3}\}$. We believe the techniques and insights developed in this thesis may be useful in studying the saturation number $\mathrm{sat}(n, \mathcal{R}_{\min}(K_3, P_4))$. We can also attempt to forbid larger complete graphs to study $\mathrm{sat}(n, \mathcal{R}_{\min}(K_4, K_{1,t}))$. Given a structural result regarding K_4-saturated graphs, the techniques regarding $K_{1,t}$-saturated graphs should still prove useful and may provide insight to the study of $\mathrm{sat}(n, \mathcal{R}_{\min}(K_4, \mathcal{T}_k))$.

LIST OF REFERENCES

[1] N. Alon, P. Erdős, R. Holzman and M. Krivelevich, On k-saturated graphs with restrictions on the degrees, J. Graph Theory 23 (1996) 1–20.

[2] B. Bollobás, On generalized graphs, Acta Math. Acad. Sci. Hungar. 16 (1965) 447–452.

[3] C. Bosse, Z-X. Song and J. Zhang, On $\mathcal{R}_{\min}(K_3, K_4)$-saturated graphs, in preparation.

[4] S. A. Burr, P. Erdős and L. Lovász, On graphs of Ramsey type, Ars Combin. 1 (1976) 167–190.

[5] G. Chen, M. Ferrara, R. J. Gould, C. Magnant and J. Schmitt, Saturation numbers for families of Ramsey-minimal graphs, J. Combinatorics 2 (2011) 435–455.

[6] D. Conlon, J. Fox and B. Sudakov, Recent developments in graph Ramsey theory, Surveys in Combinatorics 424 (2015) 49–118.

[7] A. N. Day, Saturated graphs of prescribed minimum degree, Combinatorics, Probability and Computing 26 (2017) 201–207.

[8] D. A. Duffus and D. Hanson, Minimal k-saturated and color critical graphs of prescribed minimum degree, J. Graph Theory 10 (1986) 55–67.

[9] P. Erdős, A. Hajnal and J. W. Moon, A problem in graph theory, Amer. Math. Monthly. 71 (1964) 1107–1110.

[10] J. R. Faudree, R. J. Faudree and J. R. Schmitt, A survey of minimum saturated graphs and hypergraphs, Electron. J. Combin. 18 (2011) DS19.

[11] M. Ferrara, J. Kim and E. Yeager, Ramsey-minimal saturation numbers for matchings, Discrete Math. 322 (2014) 26–30.

[12] J. Fox and K. Lin, The minimum degree of Ramsey-minimal graphs, J. Graph Theory 54 (2007) 167–177.

[13] J. Fox, A. Grinshpun, A. Liebenau, Y. Person and T. Szabó, On the minimum degree of minimal Ramsey graphs for multiple colors, J. Combin. Theory, Ser. B. 120 (2016) 64–82.

[14] A. Galluccio, M. Simonovits and G. Simonyi, On the structure of co-critical graphs, *Graph theory, combinatorics, and algorithms, Vol. 1,2 (Kalamazoo, MI, 1992)*, 1053–1071, Wiley-Intersci. Publ., Wiley, New York, 1995.

[15] R.L. Graham, M. Grötschel and L. Lovász, eds, *Handbook of Combinatorics, Vol. 2*, Elsevier, 1995.

[16] D. Hanson and B. Toft, Edge-colored saturated graphs, J. Graph Theory, 11 (1987) 191–196.

[17] J. Nešetřil, Problem, in Irregularities of Partitions, (eds G. Halász and V. T. Sós), Springer Verlag, Series Algorithms and Combinatorics, vol 8, (1989) P164. (Proc. Coll. held at Fertőd, Hungary 1986).

[18] O. Pikhurko, Results and open problems on minimum saturated graphs. Ars Combin. 72 (2004) 111–127.

[19] S.P. Radziszowski, Small Ramsey Numbers, Electronic Journal of Combinatorics, 15 (2017).

[20] F. P. Ramsey, On a problem of formal logic, Proc. Lond. Math. Soc. (3) 30 (1930) 264–286.

[21] M. Rolek and Z-X. Song, Saturation numbers for Ramsey-minimal graphs, submitted.

[22] A. A. Zykov, On some properties of linear complexes, Mat. Sb., N.S., 24 (66) (1949), 163-188 (in Russian).