Estimation and the Stress-Strength Model

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ESTIMATION IN THE STRESS-STRENGTH MODEL

by

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Abstract

The paper considers statistical inference for $R = P(X < Y)$ in the case when both $X$ and $Y$ have generalized gamma distributions. The maximum likelihood estimators for $R$ are developed in the case when either all three parameters of the generalized gamma distributions are unknown or when the shape parameters are known. In addition, objective Bayes estimators based on noninformative priors are constructed when the shape parameters are known. Finally, the uniform minimum variance unbiased estimators (UMVUE) are derived in the case when only the scale parameters are unknown.
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1 Introduction

Since the mid-twentieth century, statisticians, mathematicians, and scientists have been studying the stress-strength problem. At its heart is a special quantity called reliability, denoted by $R$. If $X$ and $Y$ are random variables, then $R = P(X < Y)$. Historically, $X$ is referred to as "stress", $Y$ as "strength", and $R$ represents the probability that the stress of a component is less than its strength, i.e., the probability that the component does not fail. For example, if $X$ represents the strength of a bridge, and $Y$ represents the stress it withstands, then $Y < X$ is the event that the bridge collapses, and $R$ is the probability that the bridge remains intact. Similarly, if $X$ represents the operating pressure of a rocket motor - the stress it undergoes - and $Y$ represents its strength, then $R$ represents the reliability of the rocket motor.

Statistical inference about the value of $R$ is of great interest to statisticians and practitioners in various fields of science (see Kotz et al. (2003) and references therein). Psychologists study the effects of psychological stress on the body and probabilities of overcoming it. Medical researchers compare reactions to drugs in clinical trials, and engineers compare materials, such as different types of steel or carbon fibers; both draw conclusions about the superiority of one over the other using reliability data. The stress-strength model even applies to the military. In 1983, M.A. Johnstone of the US Military Academy utilized Bayesian theory and analyzed reliability, which he called, "the probability that a given round will penetrate its target" in a battle against the Soviet Union.

While the problem of estimating $R$ has been studied in many different contexts, the most popular and well-researched cases are that of $X$ and $Y$ being normally or exponentially distributed (see Beg (1980), Church and Harris (1970), Downton (1973), Enis and Geisser (1971), Gupta and Gupta (1990), Guttman et al. (1988), Kelley et al. (1976), Nandi and Aich (1994a), Reiser and Guttman (1986), Tong (1974) and Weerahandi and Johnson (1992) among others). The other version of the problem which attracted a lot of effort is estimation of $R$ when both $X$ and
Y have independent gamma distributions or independent Weibull distributions (see, for example, Constantine et al. (1986) and Johnson (1988)). The importance of estimation of $P(X < Y)$ in all these cases is due to the fact that normal distribution is instrumental in modeling stress and strength while exponential, gamma and Weibull distributions are used to model lifetimes in order to compare longevity of various items.

The purpose of this thesis is to conduct estimation and testing in the stress-strength model. If $X$ and $Y$ are random variables, the quantity of interest is reliability, defined as $R = P(X < Y)$.

We shall study point estimation of $R = P(X < Y)$ in the case when $X$ and $Y$ have generalized gamma distributions. The generalized gamma distribution family is important to study because it contains as its special cases practically every distribution used in reliability and survival analysis. The exponential, Weibull, gamma, one-sided normal, and other distributions, are special cases of the generalized gamma distribution.

In the cases when all three parameters of the generalized gamma distributions are unknown or when the shape parameters are known, we shall derive the maximum likelihood estimators (MLE) of $R = P(X < Y)$ and objective Bayes estimators based on noninformative priors. We also derive uniform minimum variance unbiased estimators (UMVUE) in the case when only the scale parameters are unknown. We are planning to compare those estimators to each other via Monte Carlo simulations in future.

The importance of the theory which we are going to develop is that it enables one to estimate $R$ when the probability density functions of $X$ and $Y$ belong to the same, as well as to the different, distribution families, e.g., $X$ has a gamma and $Y$ has a Weibull distribution. The theory provides point estimators for the combinations of distributions which have never been considered before. Also, procedures enable one to obtain as special cases the existing point estimators for the combination of gamma
distributions or of Weibull distributions.
2 Generalized Gamma Distribution

The probability density function (pdf) of the generalized gamma GG3(a, b, λ) distribution is of the form

\[
f(x|a, b, \lambda) = \frac{\lambda x^{b-1} \exp \left\{ - \left( \frac{x}{a} \right)^{\lambda} \right\} \Gamma(b)}{a^{b\lambda}} I(x > 0),
\]

(2.1)

where \( I(\cdot) \) is the indicator function.

By choosing specific values for the parameters, \( a, b \) and \( \lambda \) in (2.1) one can obtain a wide variety of classically studied pdfs (see Table 1). Since these distributions are useful in reliability studies, the generalized gamma distribution is beneficial to study.

<table>
<thead>
<tr>
<th>parameters</th>
<th>distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda = 1 )</td>
<td>Gamma</td>
</tr>
<tr>
<td>( b = 1 )</td>
<td>Weibull</td>
</tr>
<tr>
<td>( \lambda = 1, a = 2 )</td>
<td>Chi-squared</td>
</tr>
<tr>
<td>( \lambda = 2, a = \sqrt{2b}, b = 1 )</td>
<td>Rayleigh</td>
</tr>
<tr>
<td>( \lambda = 2, a = \sigma\sqrt{2}, b = \frac{1}{2} )</td>
<td>Half-normal</td>
</tr>
</tbody>
</table>

We shall find sufficient statistics. Let \( X = (X_1, X_2, \cdots, X_n) \) be an i.i.d. sample with the density function (2.1). Then

\[
f(X|a, b, \lambda) = \prod_{i=1}^{n} f(X_i|a, b, \lambda)
= \prod_{i=1}^{n} \frac{\lambda x_i^{b-1} e^{-x_i^{\lambda}}}{a^{b\lambda} \Gamma(b)}
= \left( \frac{\lambda}{a^{b\lambda} \Gamma(b)} \right)^n \left[ \prod_{i=1}^{n} x_i^{\lambda b-1} \right] e^{-\left( \frac{1}{a} \right)^{\lambda} \sum_{i=1}^{n} x_i^{\lambda}}
= \left( \frac{\lambda}{a^{b\lambda} \Gamma(b)} \right)^n \exp(-\lambda b - 1) \exp \left( -\frac{T}{a^{\lambda}} \right)
\]

(2.2)
where sufficient statistics $T$ and $t$ are of the forms

\[ T = \sum_{i=1}^{n_1} X_i^3 \]  

(2.3)

and

\[ t = \sum_{i=1}^{n_1} \ln X_i. \]  

(2.4)
3 Maximum likelihood estimation of $R = P(X < Y)$

To construct the MLE for $R$ we first calculate $R$ as a function of $\lambda_1$, $\lambda_2$, $b_1$, $b_2$, $a_1$ and $a_2$ and then substitute the MLEs for the unknown parameter values. Note that $R = \int \int f_1(x)f_2(y)I(x < y)\,dx\,dy$ which in the case of generalized gamma distributions takes the form

$$R = \int_0^\infty \int_0^\infty \frac{\lambda_1 x^{b_1-1} \exp\left\{-(\frac{x}{a_1})^{\lambda_1}\right\}}{\Gamma(b_1)} \frac{\lambda_2 y^{b_2-1} \exp\left\{-(\frac{y}{a_2})^{\lambda_2}\right\}}{\Gamma(b_2)} \,dx\,dy.$$

Changing variables $u = x/a_1$, $v = y/a_2$ we rewrite $R$ as

$$R = H(\lambda_1, \lambda_2, b_1, b_2, z)$$

where

$$H(\lambda_1, \lambda_2, b_1, b_2, z) = \int_0^\infty \int_0^u \frac{\lambda_1 u^{b_1-1} \exp\left\{-u^{\lambda_1}\right\} \lambda_2 v^{b_2-1} \exp\left\{-v^{\lambda_2}\right\}}{\Gamma(b_1) \Gamma(b_2)} \,du\,dv.$$

Expanding $\exp\{-u^{\lambda_1}\}$ into a power series in $u^{\lambda_1}$ and integrating with respect to $u$ we obtain

$$H(\lambda_1, \lambda_2, b_1, b_2, z) = \frac{\lambda_1 \lambda_2}{\Gamma(b_1) \Gamma(b_2)} \sum_{k=0}^{\infty} \frac{(-1)^k z^{\lambda_1(b_1+k)}}{k! \lambda_1(b_1+k)} \int_0^\infty v^{\lambda_1(b_1+b_2+k)-1} e^{-v^{\lambda_2}} \,dv.$$

Using the change of variables $y = v_2^{1/z}$ we rewrite the last expression as

$$H(\lambda_1, \lambda_2, b_1, b_2, z) = \frac{\lambda_1}{\Gamma(b_1) \Gamma(b_2)} \sum_{k=0}^{\infty} \frac{(-1)^k z^{b_1+k+1}}{k! (b_1+k)} \Gamma\left(\frac{\lambda_1}{\lambda_2}(b_1+b_2+k)\right).$$

However, representation (3.2) is valid only if the series in the right-hand side of (3.4) is convergent. To check absolute convergence of the series in (3.4) we use the limit ratio test (see e.g. 0.222 of Gradshteyn and Ryzhik (1980)). Let $a_k$ be the $k-th$ term of the series and let $\varrho = \lim_{k \to \infty} |a_{k+1}/a_k|$. Then the series is absolutely convergent if $\varrho < 1$ and divergent if $\varrho > 1$ while the case of $\varrho = 1$ requires additional investigation. For the series in (3.4)

$$\frac{|a_{k+1}|}{a_k} = \frac{z^{\lambda_1} k(b_1 + \lambda_1) \Gamma\left((k + 1)(b_1 + b_2 + \frac{\lambda_1}{\lambda_2})\right)}{(b_1 + \lambda_1)(k + 1)^2 \Gamma\left(k(b_1 + b_2 + \frac{\lambda_1}{\lambda_2})\right)}.$$
Taking the limit in (3.5) and applying formula 8.328 of Gradshtein and Ryzhik (1980) we obtain

\[ \varrho = \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \left( \frac{a_1}{a_2} \right)^{\lambda_1} \lim_{k \to \infty} \frac{(k(b_1 + b_2 + \frac{\lambda_1}{\lambda_2}))^{\lambda_1/\lambda_2}}{k + 1}. \]  

(3.6)

Dividing the top and the bottom of the right-hand side of (3.6) by \( k \) we arrive at

\[ \varrho = \left( \frac{a_2}{a_1} \right)^{\lambda_1} \lim_{k \to \infty} \left( (b_1 + b_2 + \frac{\lambda_1}{\lambda_2})^{\lambda_1/\lambda_2} \right), \]

so that \( \varrho = 0 \) whenever \( \lambda_2 > \lambda_1 \) and \( \varrho = \infty \) for \( \lambda_2 < \lambda_1 \).

Thus, presentation (3.4) is valid only for \( \lambda_2 > \lambda_1 \).

If \( \lambda_2 < \lambda_1 \) we can use the relationship \( P(X < Y) = 1 - P(Y < X) \) to derive

\[ H(\lambda_1, \lambda_2, b_1, b_2, z) = 1 - \frac{\lambda_2}{\Gamma(b_2) \Gamma(b_1)} \sum_{k=0}^{\infty} \frac{(-1)^k z^{b_2+k\lambda_2}}{k!(b_2+k\lambda_2)} \Gamma \left( \frac{\lambda_2}{\lambda_1} (b_1 + b_2 + k) \right), \lambda_2 < \lambda_1. \]

(3.7)

Now, we need to calculate \( R \) in the case of \( \lambda_2 = \lambda_1 \). This problem can be solved with the minimal effort if one notes that \( \xi = X_1^\lambda \) and \( \eta = Y_2^\lambda \) have gamma distributions

\[ \xi = X_1^\lambda \sim \text{Gamma}(b_1, a_1^{\lambda_1}), \quad \eta = Y_2^\lambda \sim \text{Gamma}(b_2, a_2^{\lambda_2}) \]

(3.8)

and, in the case of \( \lambda_1 = \lambda_2 \), the probability \( R \) can be expressed as \( P(X < Y) = P(\xi < \eta) \). Therefore, applying the result of Constantine et al. (1986), we obtain that

\[ R = I_{[(a_1/a_2)^{\gamma_1}]}(b_1, b_2). \]

(3.9)

Here \( I_z(a, b) \) is the incomplete gamma function of the form (see 8.392 of Gradshtein and Ryzhik (1980))

\[ I_z(a, b) = [B(a, b)]^{-1} \int_0^z t^{a-1}(1 - t)^{b-1} \, dt \]

(3.10)

where \( B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b) \) is the beta function. Therefore,

\[ H(\lambda_1, \lambda_2, b_1, b_2, z) = I_{\frac{z}{1+z}}(b_1, b_2), \quad \lambda_1 = \lambda_2 = \lambda. \]

(3.11)

To construct the MLE of \( R \) we now need to substitute unknown parameters in (3.4) or (3.9) by their MLEs. Again using the relation between the generalized gamma and
the gamma distributions (3.8), we derive that the MLEs \((\hat{b}_j, \hat{a}_j)\) for \((b_j, a_j), j = 1, 2\), are the solutions of the following equations (see e.g. Johnson and Kotz (1970))

\[
\lambda_j n_j^{-1} T_j = \hat{b}_j \hat{a}_j^{\lambda_j},
\]

\[
n_j^{-1} t_j = \lambda_j \ln(\hat{a}_j) + \Psi(\hat{b}_j)
\]

(3.12)

where, following (2.3) and (2.4) we denoted

\[
T_1 = \sum_{i=1}^{n_1} X_i^{\lambda_1}, \quad T_2 = \sum_{i=1}^{n_2} Y_i^{\lambda_2},
\]

\[
t_1 = \sum_{i=1}^{n_1} \ln X_i, \quad t_2 = \sum_{i=1}^{n_2} \ln Y_i
\]

and \(\Psi(z) = d \ln \Gamma(z)/dz\) is the derivative of the logarithm of the gamma function (see Section 8.36 of Gradshtein and Ryzhik (1980)). Note that when \(a_j\) is known, we replace \(\hat{a}_j\) by its exact value in the second equation in (3.12) while \(\hat{a}_j = (T_j/b_j)^{1/\lambda_j}\) whenever \(b_j\) is available.

Finally, the maximum likelihood estimator of \(R\) is of the form

\[
\hat{R} = H(\lambda_1, \lambda_2, \hat{b}_1, \hat{b}_2, \hat{a}_2/\hat{a}_1),
\]

(3.13)

where \(H(\lambda_1, \lambda_2, b_1, b_2, z)\) is given by (3.4), (3.7) and (3.11), whenever \(\lambda_1 < \lambda_2, \lambda_1 > \lambda_2\) or \(\lambda_1 = \lambda_2\), respectively.

When all parameters are unknown, according to Hager and Bain (1970), the maximum likelihood estimators of the parameters are the solutions of the following equations for \(j = 1, 2\):

\[-n \hat{b}_j + \sum_i^n \left( \frac{X_i}{\hat{a}_j} \right)^{\lambda_j} = 0\]

\[
\frac{n}{\lambda_j} + \hat{b}_j \sum_i^n \ln \left( \frac{X_i}{\hat{a}_j} \right) - \sum_i^n \left( \frac{X_i}{\hat{a}_j} \right)^{\lambda_j} \ln \left( \frac{X_i}{\hat{a}_j} \right) = 0
\]
\[-n \Phi(\hat{b}_j) + \hat{\lambda}_j \sum_i^n \ln \left( \frac{X_i}{\hat{a}_j} \right) = 0.\]

In this case, the maximum likelihood estimator of $R$ is given by

\[\hat{R} = H(\hat{\lambda}_1, \hat{\lambda}_2, \hat{b}_1, \hat{b}_2, \hat{a}_2/\hat{a}_1),\]

where $H(\lambda_1, \lambda_2, b_1, b_2, z)$ is given by (3.4), (3.7) and (3.11), whenever $\lambda_1 < \lambda_2$, $\lambda_1 > \lambda_2$ or $\lambda_1 = \lambda_2$, respectively.

Note, however, that, while the MLEs certainly exist when the sample size is large, they may not be well-behaved or exist at all when the $n$ is small.
4 Unbiased Estimation of $R$

If $\lambda_1$, $\lambda_2$, $b_1$ and $b_2$ are known and $a_1$ and $a_2$ are the only unknown parameters, we can derive the UMVUE of $R$ which can be written as

$$
\tilde{R} = \int_0^\infty \int_0^\infty \tilde{f}_1(x)\tilde{f}_2(y)I(x < y)dx\,dy \quad (4.1)
$$

where $\tilde{f}_1(x)$ and $\tilde{f}_2(y)$ are unbiased estimators of the pdfs $f_1(x)$ and $f_2(y)$ based on observations $(X_1, \cdots, X_{n_1})$ and $(Y_1, \cdots, Y_{n_2})$, respectively. To derive expressions for $\tilde{f}_1(x)$ and $\tilde{f}_2(y)$ we use the relation (3.8) between the generalized and the regular gamma distribution. Recall that the unbiased estimator of the gamma pdf with the known shape parameter $a$ and unknown scale parameter $\sigma$ based on the the sum $S$ of $n$ observations is represented by

$$
\tilde{g}(x) = \frac{\Gamma(na)x^{a-1}(S-x)^{(n-1)a-1}}{\Gamma(a)\Gamma(a(n-1))S^{na-1}}I(0 < x < S)
$$

(see Voinov and Nikulin (1993), page 351). Then $\tilde{f}_j(x)$, $j = 1, 2$, can be obtained from $\tilde{g}(x)$ by replacing $n$ by $n_j$, $a$ by $b_j$, $x$ by $x^{\lambda_j}$, $S$ by $S_j^{\lambda_j}$ and multiplying the result by the Jacobian of the transformation $\lambda_jx^{\lambda_j-1}$. Here,

$$
S_1 = \left[ \sum_{i=1}^{n_1} X_i^{\lambda_1} \right]^{1/\lambda_1} = \left[ \frac{n_1T_1}{\lambda_1} \right]^{1/\lambda_1}, \quad S_2 = \left[ \sum_{i=1}^{n_2} Y_i^{\lambda_2} \right]^{1/\lambda_2} = \left[ \frac{n_2T_2}{\lambda_2} \right]^{1/\lambda_2}. \quad (4.2)
$$

Hence,

$$
\tilde{f}_j(x) = \frac{\lambda_j\Gamma(n_jb_j)x^{b_j\lambda_j-1}(S_j^{\lambda_j} - x^{\lambda_j})^{(n_j-1)b_j-1}}{\Gamma(b_j)\Gamma(b_j(n_j-1))S_j^{n_jb_j\lambda_j-\lambda_j}}I(0 < x < S_j), \quad j = 1, 2. \quad (4.3)
$$

Substituting estimators (4.3) into (4.1) and changing variables $u = x/S_1$, $v = y/S_2$ we obtain

$$
\tilde{R} = C_{12} \lambda_1\lambda_2 \int_0^1 \int_0^1 (1-u^{\lambda_1})^{(n_1-1)b_1-1}(1-v^{\lambda_2})^{(n_2-1)b_2-1}u^{b_1\lambda_1-1}v^{b_2\lambda_2-1}I \left( u < \frac{S_2}{S_1}v \right) du\,dv \quad (4.4)
$$

where

$$
C_{12} = \left[ B(b_1, b_1(n_1-1)) B(b_2, (n_2-1)b_2) \right]^{-1}. \quad (4.5)
$$
Let us consider the case when \( S_2 < S_1 \). Integrating over \( 0 < u < S_2v/S_1 \) in (4.4) and using representation of an incomplete beta function via hypergeometric series (see 8.391 of Gradshteyn and Ryzhik (1980)), we rewrite \( \tilde{R} \) as

\[
\tilde{R} = \frac{C_{12}}{b_2} \left( \frac{S_2}{S_1} \right)^{b_1 \lambda_1} \int_0^1 (1 - u^{b_2})(n_2-1)b_2-1 \nu^{\lambda_1 b_1 + \lambda_2 b_2 - 1} Q(v) dv.
\]  

(4.6)

where (see 9.100 of Gradshteyn and Ryzhik (1980))

\[
Q(v) = _2F_1 \left( b_1, 1 - b_1(n_1 - 1); b_1 + 1; \left( \frac{S_2 v}{S_1} \right)^{\lambda_1} \right) = \sum_{k=0}^{\infty} \frac{\left( \frac{S_2 v}{S_1} \right)^{k+1}}{k! (b_1 + k)} \prod_{j=1}^{k} (j - (n_1 - 1)b_1).
\]

Integrating (4.6) term by term, we arrive at

\[
\tilde{R} = C_{12} \sum_{k=0}^{\infty} \frac{B \left( \frac{\lambda_1 b_1 + \lambda_2 b_2 + \lambda_1 k}{\lambda_2 b_2 + \lambda_1 k}, (n_2 - 1)b_2 \right)}{k! (b_1 + k)} \left( \frac{S_2}{S_1} \right)^{(b_1+1)\lambda_1 k} \prod_{j=1}^{k} (j - (n_1 - 1)b_1).
\]

(4.7)

Using the ratio test similarly to Section 2 we derive that the limit of the ratio of the consecutive terms in (4.7) is equal to \( S_2 / S_1 \), so representation (4.7) is valid if \( S_2 < S_1 \). If \( S_2 > S_1 \), we again use the fact that \( P(X < Y) = 1 - P(Y < X) \) and calculate \( \tilde{R} \) as

\[
\tilde{R} = 1 - C_{12} \sum_{k=0}^{\infty} \frac{B \left( \frac{\lambda_1 b_1 + \lambda_2 b_2 + \lambda_1 k}{\lambda_2 b_2 + \lambda_1 k}, (n_1 - 1)b_1 \right)}{k! (b_2 + k)} \left( \frac{S_1}{S_2} \right)^{(b_2+1)\lambda_2 k} \prod_{j=1}^{k} (j - (n_2 - 1)b_2).
\]

(4.8)
5 Bayesian Estimation

The Jeffreys prior is one of the most widely used types of noninformative priors due to simplicity of its construction and its invariance under transformations of parameters of the model. If \( \theta = (\theta_1, \cdots, \theta_m) \) is the vector of parameters of the pdf \( f(x|\theta) \), then Jeffreys prior is equal to the positive square root of the determinant of the information matrix \( I \)

\[
\pi(\theta) = |\det I(\theta)|^{1/2}
\]

(5.9)

where \( I \) is the Fisher information matrix

\[
I_{ij}(\theta) = -E \left( \frac{\partial^2}{\partial \theta_i \partial \theta_j} \log f(X|\theta) \right), \quad i,j = 1, \cdots, m.
\]

(5.10)

Since Jeffreys is usually not a proper prior (i.e. does not have finite integral) and Fisher information matrix for the sample \( X \) is just a multiple of (5.10) and \( n \), one can use (5.9) and (5.10) for construction of Jeffreys prior for parameters \( \theta = (a, b, \lambda) \).

To obtain Jeffreys prior (5.9), define the polygamma function

\[
\Psi(m,y) = \frac{d^{m+1}}{dy^{m+1}} \log \Gamma(y).
\]

(5.11)

By direct calculations we derive that the entries of matrix \( I \) are of the forms

\[
I_{11} = \lambda^{-2} \left( 1 + b [\log a + \Psi(0, b + 1)]^2 + \Psi(1, b + 1) \right),
\]

\[
I_{12} = -(a\lambda)^{-1} b [\log a + \Psi(0, 1 + b)], \quad I_{13} = -\lambda^{-1} [\log a + \Psi(0, b)],
\]

\[
I_{22} = a^{-2} b, \quad I_{23} = a^{-1}, \quad I_{33} = \Psi(1, b).
\]

(5.12)

Calculation of the square root of the determinant of this matrix yields Jeffreys prior for parameters \( (a, b, \lambda) \)

\[
\pi_J(a, b, \lambda) = (a\lambda)^{-1} H(b) \quad \text{with} \quad H(b) = \sqrt{\Psi(1, b) \left( b^2\Psi(1, b) - 1 \right) - 1}.
\]

(5.13)

However, working with the model with all three parameters being unknown is rather hard. For this reason, in what follows, we consider a model where only parameters \( a \) and \( b \) are unknown.
Yang and Berger (1997) derived Jeffreys' prior for the two parameter gamma distribution with the pdf
\[
f(x|\alpha, \beta) = \frac{1}{\Gamma(\alpha) \beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}} I_{(0,\infty)}(x).
\]

It is of the form
\[
g(\alpha, \beta) = \frac{1}{\beta} \sqrt{\alpha \Psi(1, \alpha) - 1}
\]

where \(\Psi(1, x)\) is the PolyGamma function defined in (5.11). Now, to derive Jeffreys' prior for the generalized gamma distribution, use transformation of variables suggested in Brownstein and Pensky (2007). Application of \(b = \alpha\) and \(a = \beta^k\) yields Jeffreys' prior when parameter \(\lambda\) is known
\[
g_\lambda(a, b) = \frac{1}{a^\lambda} \sqrt{b \Psi(1, b) - 1} \cdot \lambda a^{\lambda-1}
= \frac{\lambda}{a} \sqrt{b \Psi(1, b) - 1}
\]

(5.14)

Let \(\mathbf{X}\) be a random with the pdf \(f(\mathbf{X} | \theta)\). In Bayes approach, vector parameter \(\theta\) is treated as a random vector, i.e. \(\mathbf{X} | \theta \sim f(\mathbf{X} | \theta)\). Let \(\varphi(\theta)\) be a function of \(\theta\). Then the Bayes estimator of \(\varphi(\theta)\) is given by
\[
\hat{\varphi} = E_{\theta | X} \varphi(\theta) = \frac{\int \varphi(\theta) f(\mathbf{X} | \theta) g(\theta) d\theta}{\int f(\mathbf{X} | \theta) g(\theta) d\theta}.
\]

(5.15)

5.1 Derivation of Bayes Estimator of \(R\)

Denote
\[
f(\mathbf{X}|a_1, b_1, \lambda_1) = \prod_{i=1}^{n_1} f(X_i | a_1, b_1, \lambda_1), \quad f(\mathbf{Y}|a_2, b_2, \lambda_2) = \prod_{i=1}^{n_2} f(Y_i | a_2, b_2, \lambda_2).
\]

Then, the Bayes estimator \(\hat{R}\) of \(R\) based on Jeffreys prior can be written as
\[
f f(x|a_1, b_1, \lambda_1) f(y|a_2, b_2, \lambda_2) f(\mathbf{X}|a_1, b_1, \lambda_1) g_\lambda_1(a_1, b_1) f(\mathbf{Y}|a_2, b_2, \lambda_2) g_\lambda_2(a_2, b_2) da_1 da_2 db_1 db_2 dx dy
f_0^\infty f_0^\infty \int f(\mathbf{X}|a_1, b_1, \lambda_1) g_\lambda_1(a_1, b_1) da_1 db_1 \int f_0^\infty f(\mathbf{Y}, a_2, b_2, \lambda_2) g_\lambda_2(a_2, b_2) da_2 db_2
\]

where the integral in the numerator is evaluated over the region
\[
\Omega = \{ a_1 > 0, a_2 > 0, b_1 > 0, b_2 > 0, 0 < x < y < \infty \}.
\]
Let
\[
F(x, X, \lambda_1) = \int_0^\infty \int_0^\infty f(x|a_1, b_1, \lambda_1) f(X|a_1, b_1, \lambda_1) g_{\lambda_1}(a_1, b_1) da_1 db_1,
\]
\[
G(X, \lambda_1) = \int_0^\infty \int_0^\infty f(X|a_1, b_1, \lambda_1) g_{\lambda_1}(a_1, b_1) da_1 db_1,
\]
and, similarly,
\[
F(y, Y, \lambda_2) = \int_0^\infty \int_0^\infty f(y|a_2, b_2, \lambda_2) f(Y|a_2, b_2, \lambda_2) g_{\lambda_2}(a_2, b_2) da_2 db_2,
\]
\[
G(Y, \lambda_2) = \int_0^\infty \int_0^\infty f(Y|a_2, b_2, \lambda_2) g_{\lambda_2}(a_2, b_2) da_2 db_2.
\]
Then \( \hat{R} \) can be re-written as
\[
\hat{R} = \int_0^\infty \int_0^\infty \frac{I(x < y) F(x, X, \lambda_1) F(x, Y, \lambda_2) dx dy}{G(X, \lambda_1) G(Y, \lambda_2)}.
\]
In order to calculate \( R \), we shall simplify expressions for \( F(x, X, \lambda_1), F(y, Y, \lambda_2), G(X, \lambda_1) \) and \( G(Y, \lambda_2) \).

Note that by (2.1), we have
\[
f(X|a_1, b_1, \lambda_1) = \left( \frac{\lambda_1}{a_1^{\lambda_1} \Gamma(b_1)} \right)^{n_1} e^{\lambda_1 (b_1 - 1)} e^{-\frac{\lambda_1}{a_1} T_1},
\]
and
\[
f(Y|a_2, b_2, \lambda_2) = \left( \frac{\lambda_2}{a_2^{\lambda_2} \Gamma(b_2)} \right)^{n_2} e^{\lambda_2 (b_2 - 1)} e^{-\frac{\lambda_2}{a_2} T_2}.
\]
Then
\[
G(X, \lambda_1) = \int_0^\infty \int_0^\infty f(X|a_1, b_1, \lambda_1) g_{\lambda_1}(a_1, b_1) da_1 db_1
\]
\[
= \int_0^\infty \int_0^\infty \left( \frac{\lambda_1}{a_1^{b_1 \lambda_1} \Gamma(b_1)} \right)^{n_1} e^{\lambda_1 (b_1 - 1) \frac{\lambda_1}{a_1} \sqrt{\Psi'(b_1)(b_1 - 1)}} da_1 db_1
\]
\[
= \int_0^\infty \frac{\lambda_1^{n_1+1} e^{\lambda_1 (b_1 - 1)} \sqrt{\Psi'(b_1)(b_1 - 1)}}{[\Gamma(b_1)]^{n_1}} \left( \int_0^\infty \frac{1}{a_1^{b_1 \lambda_1+1}} e^{-\frac{\lambda_1}{a_1} T_1} da_1 \right) db_1.
\]
Transformation of variables \( z = T_1 a^{-\lambda_1} \) yields
\[
G(X, \lambda_1) = \int_0^\infty \frac{\lambda_1^{n_1+1} e^{\lambda_1 (b_1 - 1)}}{[\Gamma(b_1)]^{n_1}} \sqrt{\Psi'(b_1)(b_1 - 1)} \left( \frac{T_1^{-b_1 n_1}}{\lambda_1} \int z^{b_1 n_1-1} e^{-z} dz \right) db_1.
\]
Similarly

\[ G(Y, \lambda_2) = \int_0^\infty \left( \frac{\lambda_2}{\Gamma(b_2)} \right)^{n_2} \frac{e^{\lambda_2(b_2-1)}}{T_2^{b_2n_2}} \sqrt{\Psi'(b_2)(b_2-1)} \Gamma(b_2n_2) \, db_2. \]  

(5.17)

Now, let us derive an expression for \( F(x, X, \lambda_1) \):

\[
F(x, X, \lambda_1) = \int_0^\infty \int_0^\infty f(x|a_1, b_1, \lambda_1)f(X|a_1, b_1, \lambda_1)g_{\lambda_1}(a_1, b_1)da_1db_1
\]

\[
= \int_0^\infty \int_0^\infty \frac{\lambda_1}{a_1^{b_1}} \frac{1}{\Gamma(b_1)} \frac{e^{-\left(\frac{x}{a_1}\right)\lambda_1}}{\frac{T_1}{a_1^{b_1}\lambda_1}} e^{\frac{T_1}{a_1^{b_1}\lambda_1}} \frac{1}{\sqrt{\Psi'(b_1)(b_1-1)}} \, db_1 \, da_1
\]

\[
= \int_0^\infty \frac{\lambda_1^{n_1+2} x^{B_1 b_1-1} e^{\frac{T_1}{a_1^{b_1} b_1} b_1-1}}{\Gamma(b_1)^{n_1+1}} \sqrt{\Psi'(b_1)(b_1-1)} \left( \int a_1^{-\lambda_1 b_1(n_1+1)+1} e^{-\left(\frac{z}{a_1^{b_1} \lambda_1}\right)} \, da_1 \right) \, db_1.
\]

Again using transformation of variables

\[ z = \frac{T_1 + x^{\lambda_1}}{a_1^{b_1}}, \]

we may simplify:

\[
F(x, X, \lambda_1)
\]

\[
= \int \sqrt{\Psi'(b_1)(b_1-1)} e^{\frac{T_1}{\Gamma(b_1)}} \lambda_1^{n_1+2} x^{B_1 b_1-1} \left( T_1 + x^{\lambda_1} \right)^{b_1 n_1-1} \frac{1}{\lambda_1} \, f \, z^{b_1 n_1+b_1-1} e^{-z} \, dz
\]

\[
= \int \sqrt{\Psi'(b_1)(b_1-1)} \frac{e^{\frac{T_1}{\Gamma(b_1)}} \lambda_1^{n_1+2} x^{B_1 b_1-1} \left( T_1 + x^{\lambda_1} \right)^{b_1 n_1-1} \Gamma(n_1 b_1 + b_1) \, db_1.
\]

Likewise,

\[
F(y, Y, \lambda_2)
\]

\[
= \int \sqrt{\Psi'(b_2)(b_2-1)} e^{\frac{T_2}{\Gamma(b_2)}} \lambda_2^{n_2+1} y^{\lambda_2 b_2-1} \left( T_2 + y^{\lambda_2} \right)^{-b_2 n_2-2} \frac{1}{\lambda_2} \, f \, y^{\lambda_2 b_2-1} \, db_2.
\]

Now, \( \hat{R} \) can be presented as

\[
\hat{R} = \frac{R(X, Y, \lambda_1, \lambda_2)}{G(Y, \lambda_1)G(Y, \lambda_2)}
\]

(5.18)
where $G(X, \lambda_1)$ and $G(Y, \lambda_2)$ are given by formulae (5.16) and (5.17) and

$$R(X, Y, \lambda_1, \lambda_2) = \int_0^\infty \int_0^\infty I(x < y)F(x, X, \lambda_1)F(y, Y, \lambda_2)\,dx\,dy. \quad (5.19)$$

Changing the order of integration in (5.19) we obtain

$$R(X, Y, \lambda_1, \lambda_2) = \int_0^\infty \int_0^\infty U(X, Y, \lambda_1, \lambda_2, b_1, b_2)J(b_1, b_2, \lambda_1, \lambda_2)\,db_1\,db_2 \quad (5.20)$$

where

$$U(X, Y, \lambda_1, \lambda_2, b_1, b_2) = \frac{\Gamma(n_1 b_1 + b_1)\Gamma(n_2 b_2 + b_2)}{\Gamma(b_1)\Gamma(b_2)^{n_1+1}} \sqrt{\Psi'(b_1)\Psi'(b_2)}(b_1 - 1)(b_2 - 1)e^{t_1(b_1 \lambda_1 - 1) + t_2(b_2 \lambda_2 - 1)}$$

and

$$J(b_1, b_2, \lambda_1, \lambda_2) = \int_0^\infty \int_0^\infty I(x < y)x^{\lambda_1 b_1 - 1}(x^{\lambda_1 + T_1})^{-b_1 n_1 - b_1}y^{\lambda_2 b_2 - 1}(y^{\lambda_2 + T_2})^{-b_2 n_2 - b_2}\,dx\,dy.$$

The last expression can be re-written as

$$J(b_1, b_2, \lambda_1, \lambda_2) = \int_0^\infty Q(y)y^{\lambda_2 b_2 - 1}(y^{\lambda_2 + T_2})^{-b_2 n_2 - b_2}\,dy \quad (5.21)$$

where

$$Q(y) = \int_0^y x^{\lambda_1 b_1 - 1}(x^{\lambda_1 + T_1})^{-b_1 n_1 - b_1}\,dx.$$

Using change of variables $z = x^{\lambda_1}$, we obtain

$$Q(y) = T_1^{-b_1(n_1 + 1)} \int_0^y x^{\lambda_1 b_1 - 1} \left(\frac{x^{\lambda_1}}{T_1} + 1\right)^{-b_1 n_1 - b_1}\,dx = \frac{T_1^{-b_1(n_1 + 1)}}{\lambda_1} \int_0^y z^{\lambda_1 b_1 - 1} \left(\frac{z}{T_1} + 1\right)^{-b_1(n_1 + 1)}\,dz.$$  

Then, formula 3.194.1 of Gradshteyn and Ryzhik (1980) with $u = y^{\lambda_1}$, $\mu = b_1$, $\nu = b_1(n_1 + 1)$ and $\beta = \frac{1}{\lambda_1}$ yields

$$Q(y) = \frac{(y^{\lambda_1})^{b_1}T_1^{-b_1(n_1 + 1)}}{\lambda_1 b_1} \, _2F_1 \left(b_1(n_1 + 1), b_1, b_1 + 1, 1 - \frac{y^{\lambda_1}}{T_1}\right).$$

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Here,

\[ _2F_1(\alpha, \beta; \gamma; z) = 1 + \frac{\alpha \beta}{\gamma 1!} z + \frac{\alpha(\alpha + 1) \beta(\beta + 1)}{\gamma(\gamma + 1) 2!} z^2 + \ldots \]

is the hypergeometric series (see the definition of the hypergeometric series in formula 9.100 of Gradshtein and Ryzhik (1980)).

Substituting \( Q(y) \) back into (5.21), we derive

\[
J(b_1, b_2, \lambda_1, \lambda_2) = \frac{T_1^{-b_1(n_1+1)}}{\lambda_1 b_1} \int_0^\infty \frac{y^{\lambda_1 b_1 + \lambda_2 b_2 - 1}}{(y^{\lambda_2} + T_2)^{b_2(n_2+1)}} \ _2F_1 \left( b_1(n_1 + 1), b_1, b_1 + 1, 1 - \frac{y^{\lambda_1}}{T_1} \right) dy.
\]

(5.22)

If \( \lambda_1 \neq \lambda_2 \), then the value of \( J(b_1, b_2, \lambda_1, \lambda_2) \) can be calculated only by application of numerical integration.

However, if \( \lambda_1 = \lambda_2 = \lambda \), \( J(b_1, b_2, \lambda_1, \lambda_2) = J(b_1, b_2, \lambda) \) can be further simplified by using change of variables \( v = y^\lambda \):

\[
J(b_1, b_2, \lambda) = \frac{T_1^{-b_1(n_1+1)}}{\lambda^2 b_1} \int_0^\infty y^{\lambda_1 b_1 + b_2 b_2 - 1} (y^\lambda + T_2)^{-b_2 n_2 - b_2} \ _2F_1 \left( b_1(n_1 + 1), b_1, b_1 + 1, 1 - \frac{y^\lambda}{T_1} \right) dy
\]

\[
= \frac{T_1^{-b_1(n_1+1)}}{\lambda^2 b_1} \int_0^\infty y^{b_1 + b_2 - 1} (T_2 + v)^{-b_2(n_2+1)} \ _2F_1 \left( b_1(n_1 + 1), b_1, b_1 + 1, 1 - \frac{v}{T_1} \right) dv.
\]

Using formula 9.132.1 of Gradshtein and Ryzhik (1980) we obtain

\[
_2F_1(b_1(n_1 + 1), b_1; 1 + b_1, -\frac{v}{T_1})
\]

\[
= \left(1 + \frac{v}{T_1}\right)^{-b_1(n_1+1)} \frac{\Gamma(1 + b_1)\Gamma(-b_1 n_1)}{\Gamma(b_1)\Gamma(1 - b_1 n_1)} \ _2F_1(b_1(n_1 + 1), b_1 n_1 + 1; 1 + \frac{v}{T_1})
\]

\[
= \left(1 + \frac{v}{T_1}\right)^{-b_1(n_1+1)} \frac{\Gamma(1 + b_1)\Gamma(b_1 n_1)}{\Gamma(b_1 n_1 + b_1)\Gamma(1)} \ _2F_1(b_1, 1 - b_1 n_1; 1 - b_1 n_1; 1 + \frac{v}{T_1})
\]

\[
= -\frac{1}{n_1} \left(1 + \frac{v}{T_1}\right)^{-b_1(n_1+1)} \ _2F_1(b_1(n_1 + 1), b_1 n_1 + 1; 1 + \frac{v}{T_1})
\]

\[
= \left(1 + \frac{v}{T_1}\right)^{-b_1} b_1 B(b_1 n_1, b_1) \ _2F_1(b_1, 1 - b_1 n_1; 1 - b_1 n_1; 1 + \frac{v}{T_1})
\]

where \( B(x, y) \) is the Beta-function (see Section 8.38 of Gradshtein and Ryzhik (1980) for definition and properties).
Using series representation 9.100 of the hypergeometric function, we derive

\[ 2F_1(b_1(n_1 + 1), b_1; 1 + b_1, -\frac{n}{n_1}) = \frac{1}{n_1} \sum_{i=1}^{\infty} C_1(n_1, b_1) (1 + \frac{n}{n_1})^{-i} \]

\[ + b_1 \Gamma(b_1 n_1, b_1) (1 + \frac{n}{n_1})^{-b_1} \sum_{i=10}^{\infty} C_2(n_1, b_1) (1 + \frac{n}{n_1})^{-i} \]

where

\[ C_1(n_1, b_1) = \frac{b_1(n_1 + 1)[b_1(n_1 + 1) + 1] \ldots [b_1(n_1 + 1) + i - 1]b_1(b_1 + 1) \ldots (b_1 + i - 1)}{(1 + b_1)(1 + b_1 + 1) \ldots (1 + b_1 + i - 1)} \]

\[ C_2(n_1, b_1) = \frac{b_1(b_1 + 1) \ldots (b_1 + i - 1)}{i!} \]

For convenience introduce the following integral

\[ I(s) = \int_0^{\infty} v^{b_1+b_2-1}(T_2 + v)^{-b_2(n_2+1)} \left(1 + \frac{v}{T_1}\right)^{-s} dv. \quad (5.23) \]

Using notation (5.23) we can further simplify \( J(b_1, b_2, \lambda) \) as

\[ J(b_1, b_2, \lambda) = \frac{\lambda^{-b_1(n_1+1)}}{\lambda^2 b_1} \left[ -\frac{1}{n_1} \left( I(b_1(n_1 + 1) + \sum_{i=1}^{\infty} C_1(n_1, b_1) I(b_1(n_1 + 1) + i) \right) \right] + b_1 \Gamma(b_1 n_1, b_1) \left( I(b_1) + \sum_{i=1}^{\infty} C_2(n_1, b_1) I(b_1 + i) \right) \]

where coefficients \( C_1(n_1, b_1) \) and \( C_2(n_1, b_1) \) are defined in (5.23) and (5.23) respectively.

To complete the calculation, we need to provide a way for calculating the integral (5.23). Transformation \( x = \frac{v}{T_1} \) reduces \( I(s) \) to

\[ I(s) = \int_0^{\infty} (T_1 x)^{b_1+b_2-1}(T_2 + T_1 x)^{-b_2(n_2+1)}(1 + x)^{-s} T_1 dx, \]

so that

\[ I(s) = T_1^{b_1+b_2} T_2^{-b_2(n_2+1)} \int_0^{\infty} x^{b_1+b_2-1} \left(1 + \frac{T_1}{T_2} x\right)^{-b_2(n_2+1)} (1 + x)^{-s} dx. \]

Then, using formula 3.197.5 of Gradshtein and Ryzhik (1980) with \( \lambda = b_1 + b_2 \), \( \mu = -b_2(n_2 + 1), \nu = -s \) and \( \alpha = T_1/T_2 \), we arrive at

\[ I(s) = \frac{T_1^{b_1+b_2}}{T_2^{b_2(n_2+1)}} B(b_1+b_2, b_2n_2+s-b_1) 2F_1(b_2(n_2+1); b_1+b_2; s+b_2(n_2+1); 1-T_1/T_2). \]

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5.2 Discussion

In the present paper we studied point estimation of $R = P(X < Y)$ in the case when $X$ and $Y$ have generalized gamma distributions. We constructed the MLE of $R$ when all three parameters of the generalized gamma distributions are unknown or when the shape parameters are known, and the objective Bayes estimators based on Jeffreys prior when the shape parameters are known. We also derive uniform minimum variance unbiased estimators (UMVUE) in the case when only the scale parameters are unknown.

Although in the present thesis, the estimators are not compared to each other, we are planning to carry out such a comparison in future. We shall generate data via Monte Carlo simulations and compare the MLE, the UMVUE and the Bayes estimators of $R$ with both the true and the estimated values for the known parameters. This comparison will allow one to see how sensitive various estimation techniques are to mis-specification of the parameter values. We shall also conduct study of those estimators for various popular distribution families (Weibull, Gamma and exponential pdfs).
References


*Technometrics*, 16, 625.