Polarimetry Of Random Fields

Jeremy Ellis
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Polarimetry of Random Fields

by

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B.S. Duke University, 2000

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for the degree of Doctor of Philosophy
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Aristide Dogariu
ABSTRACT

On temporal, spatial and spectral scales which are small enough, all fields are fully polarized. In the optical regime, however, instantaneous fields can rarely be examined, and, instead, only average quantities are accessible. The study of polarimetry is concerned with both the description of electromagnetic fields and the characterization of media a field has interacted with. The polarimetric information is conventionally presented in terms of second order field correlations which are averaged over the ensemble of field realizations.

Motivated by the deficiencies of classical polarimetry in dealing with specific practical situations, this dissertation expands the traditional polarimetric approaches to include higher order field correlations and the description of fields fluctuating in three dimensions.

In relation to characterization of depolarizing media, a number of fourth-order correlations are introduced in this dissertation. Measurements of full polarization distributions, and the subsequent evaluation of Stokes vector element correlations and Complex Degree of Mutual Polarization demonstrate the use of these quantities for material discrimination and characterization.

Recent advancements in detection capabilities allow access to fields near their sources and close to material boundaries, where a unique direction of propagation is not evident. Similarly, there exist classical situations such as overlapping beams, focusing, or diffusive
scattering in which there is no unique transverse direction. In this dissertation, the correlation matrix formalism is expanded to describe three dimensional electromagnetic fields, providing a definition for the degree of polarization of such a field. It is also shown that, because of the dimensionality of the problem, a second parameter is necessary to fully describe the polarimetric properties of three dimensional fields. Measurements of second-order correlations of a three dimensional field are demonstrated, allowing the determination of both the degree of polarization and the state of polarization.

These new theoretical concepts and innovative experimental approaches introduced in this dissertation are expected to impact scientific areas as diverse as near field optics, remote sensing, high energy laser physics, fluorescence microscopy, and imaging.
To my wife, Linda
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LIST OF SYMBOLS

$E$ Complex Electric Field vector, Jones Vector

$r$ Position vector

$\omega$ Frequency

$k$ Wave vector

$x, y, z$ Spatial Co-ordinate axis

$W$ Cross Spectral Density matrix

$\lambda$ Eigenvalue of Correlation Matrix
CHAPTER 1

INTRODUCTION

On small enough temporal, spatial, and spectral scales, all fields are fully polarized. In the optical regime, however, instantaneous fields can rarely be examined, and, instead, only average quantities are accessible. Any measurement may then be regarded as the result of an ensemble of independent field realizations, each of them being fully polarized. Deterministic fields are idealizations for which there is a single, unique realization of this ensemble.

The study of polarimetry is concerned with both the description of electromagnetic fields and the characterization of media a field has interacted with. In the optical regime, the polarimetric information is conventionally described in terms of second order field correlations which are averaged over the ensemble of field realizations. This is because the second order field correlations are the physically accessible parameters, not the field itself.

In Chapter 2 of this dissertation, I will give a brief overview of classical polarimetry, discussing the three main formalisms used to describe the polarimetric properties of transverse, propagating fields. These are (i) the Jones calculus, useful for describing the coherent superposition of deterministic fields as well as most interference phenomena, (ii) the Stokes-
Mueller calculus, capable of treating the incoherent superposition of random fields and the phenomenon of depolarization, and (iii) the correlation matrix formalism which can also describe the characteristics of both partially coherent and partially depolarized fields. The decomposition of both the Stokes vector and the correlation matrix is shown to provide an equivalent ensemble representation in terms of the second order field correlations. Typical means for polarimetric measurements are then briefly discussed, after which the Stokes-Mueller calculus is applied to practical imaging applications. Several benefits in improving the imaging depth in highly scattering media will be demonstrated, even when only a reduced set of Mueller matrix elements are accessible.

Examination of the experimental data, especially in the case of depolarizing media, suggests the need for a more comprehensive description of polarimetric properties. While polarimetry has traditionally been developed in terms of second order field correlations, it is possible that higher order correlations can provide additional information about physical processes affecting the polarization of an optical field. In Chapter 3, a number of fourth-order correlations will be examined in relation to the polarization properties of an ensemble of field realizations. The second order correlations of Stokes vector elements will be studied both theoretically and experimentally, and it will be demonstrated that these permit discriminating between different types of globally "unpolarized" light. The Complex Degree of Mutual Polarization will then be introduced, to provide a measure of the similarity between the states of polarization at two different points in a fully polarized field. Measurements of full polarization distributions, and the subsequent evaluation of Stokes vector element
correlations and Complex Degree of Mutual Polarization will demonstrate the use of these quantities for material discrimination and characterization.

While fourth-order field correlations, evaluated both in a single spatial point or involving a pair of points, provide substantially more information than the classical polarimetric descriptors, one should bare in mind that, in practice, the measured quantities are only projections of the actual fields. In other words, the traditional polarimetric description is limited to transverse electromagnetic fields. However, recent advancements in detection capabilities allow access to fields near their sources and close to material boundaries, where a unique direction of propagation is not evident. Similarly, there exist classical situations such as overlapping beams, focusing, or diffusive scattering in which there is no unique transverse direction. In these situations and others, a fully three dimensional optical field may develop. In Chapter 4, the correlation matrix formalism will be expanded to describe three dimensional electromagnetic fields. Specifically, it will be found that an equivalent ensemble description can be developed in terms of the second-order field correlations, and that the number of independent members of this equivalent ensemble is equal to the dimensions of the space in which the original field fluctuates. Using the fact that the unique polarized portion of a field is always two dimensional and oscillates on an ellipse, I will show that a degree of polarization can be defined which provides a measure of the relative intensity of the polarized component with respect to the entire intensity of the field. It will also be demonstrated that, due to the dimensionality of the problem, a second parameter is necessary to fully describe three dimensional fields. Because the physically significant information is contained
in the eigenvalues and the eigenvectors of the field correlation matrix, these quantities will be thoroughly examined for different situations. Furthermore, I will demonstrate that the second-order correlations of a three dimensional field can be measured experimentally and that the degree of polarization and the polarization ellipse of the polarized component can be practically determined.

Motivated by the deficiencies of classical polarimetry in dealing with specific practical situations, this dissertation expands the traditional polarimetric approaches to include higher order field correlations and the description of fields fluctuating in three dimensions. These new concepts are expected to impact scientific areas as diverse as near field optics, remote sensing, high energy laser physics, fluorescence microscopy, and imaging.
CHAPTER 2

CLASSICAL POLARIMETRY

The goal of polarimetry is to characterize and quantify the fluctuations of a field and the means and the extent to which a medium may alter these fluctuations [1]. The fluctuations of a field at a point are conventionally characterized by a state of polarization and a degree of polarization, both of which are measured via second-order field correlations. The effect of a medium is then described in terms of the changes in state and degree of polarization it introduces to a field with which it interacts. In this dissertation, I will make use of the Jones, Stokes-Mueller, and Correlation Matrix formalisms to describe the polarimetric properties of fields and media in a number of practical imaging and detection problems. The practical experimental task will require expanding these formalisms to describe situations beyond the scope of conventional polarimetry. In this Chapter, I will begin by considering the physical significance of these three formalisms.

To do this, one must first understand the physical concept of the state of polarization of a field. A field at a point is said to be polarized if the field oscillates in a particular way [2]. More specifically, if the field vector at that point sweeps out an ellipse (circle and line being
degenerate forms of an ellipse) with increasing time, then the field is polarized. The ellipse on
which the field vector oscillates is known as the polarization ellipse, and specifies the state of
polarization. Common polarimetric quantities of interest include ellipticity, orientation, and
a host of other equivalent parameters, all of which can be determined from measurements of
second order field correlations as will be discussed below [1, 3, 4, 5].

An equivalent definition for a polarized field is that its orthogonal components are fully
correlated. That is to say that there exists a fixed phase and amplitude relationship between
all orthogonal components of the complex field at that point [2]. This is essentially what
is meant by a deterministic field, and all such fields are fully polarized. Another important
polarimetric quantity is the degree of polarization, which specifies how much of the field at
that point is polarized [1, 3, 4, 5]. Partially polarized and unpolarized fields are examples
of random fields, because, in this case, there is a random or varying phase or amplitude
relationship between orthogonal components of the complex field vector.

The above is a rather simplistic, heuristic description of the concept of deterministic and
random fields as well as polarized and partially polarized fields. A more rigorous treatment
is given below, in which the formalisms and notations of the Jones, Stokes-Mueller, and
Correlation Matrix Formalisms are introduced.
2.1 Deterministic Fields

Let us examine an ensemble of field realizations \( \{E^\alpha(r, \omega)\} \), for a specific point \( r \) and frequency \( \omega \). The field at a point can be said to be deterministic if there exists only a single realization in the ensemble of fields at that point. Equivalently, the field is deterministic if all the realizations in the ensemble are identical. This then implies that there is a fixed phase and amplitude relationship between all orthogonal field components. As such, the field oscillates on an ellipse, and is fully polarized. If one associates the direction of propagation, \( k \), of such a field with the \( z \) direction of a Cartesian system of coordinates (as shown in Figure 2.1), a complete description of the field in terms of three parameters, two amplitudes and a phase difference, can be made as
\[
E(r, \omega) = \begin{pmatrix} E_x(r, \omega) \\ E_y(r, \omega) e^{i \delta(r, \omega)} \end{pmatrix} e^{i(\omega t - k z)},
\tag{2.1}
\]

where \(E_x, E_y,\) and \(\delta\) are real quantities that depend on position and frequency. When an optical system is entirely deterministic, it is most easily described in terms of the Jones formalism, discussed below.

### 2.1.1 Jones Formalism

The vector description of the electric field presented in Eq. (2.1) is known as the Jones Vector of the field. The polarization ellipse is found simply by solving for the real part of the field. Explicitly,

\[
\text{Re} \{ E(r, \omega) \} = |E_x(r, \omega)| \cos(\omega t) \hat{x} + |E_y(r, \omega)| \cos(\omega t + \delta(r, \omega)) \hat{y} \tag{2.2}
\]

\[
= \cos(\omega t + \beta) \mathbf{a} + \sin(\omega t + \beta) \mathbf{b}, \tag{2.3}
\]

which is just the parametric (in \(t\)) equation for an ellipse. This ellipse is known as the polarization ellipse, and is shown graphically in Figure 2.2. From Eq(2.2), the ellipticity and orientation of the polarization ellipse may be calculated, and as such, specifying the Jones vector of a field is sufficient to specify the state of polarization (ellipticity, orientation,
Figure 2.2: Polarization ellipse. The state of polarization of a deterministic field is completely specified by the orientation and ellipticity.

and sense of rotation) of the field at that point [1, 3, 4, 5]. As deterministic fields are fully polarized, the degree of polarization of this field is then, by definition, unity.

When dealing with a linear medium that does not introduce any randomness in the field’s properties (i.e. depolarization, decoherence, etc.), the transfer characteristics of the system are described in terms of a 2x2 Jones matrix. Explicitly, if an initial field \( E_i \) propagates through such a linear system, the output field is

\[
E' = JE_i
\]  

(2.4)

where \( J \) is the Jones matrix of the system.

The major benefits of the Jones calculus are its simplicity and its ability to handle the amplitude addition of fields. As all fields are fully deterministic, they are fully polarized and
fully coherent. It follows that amplitude addition is the appropriate means of superposition, and this allows for a simple description of interference phenomena. However, the drawback is that the Jones formalism is limited to deterministic fields, and therefore is not capable of describing random behaviors such as partially polarized or unpolarized fields.

In the description of polarized fields, I implicitly used the concept of orthogonal field components. This concept is most easily demonstrated with the use of Jones vectors. Specifically, two field components, \( E_i \) and \( E_j \), are orthogonal if the Jones vectors representing those field components satisfy the following relationship

\[
E_i^* \cdot E_j = 0, \tag{2.5}
\]

where * represents complex conjugation and \( \cdot \) represents the dot product. For fields transverse to the \( z \) direction, this implies orthogonal field components have the same ellipticity, opposite handedness, and complimentary orientation. A common example of a pair of orthogonal field components that are not simply perpendicular field components would be right and left circular components of the field. This definition of orthogonal field components, and its implications, will be further explored and used throughout this dissertation.

### 2.2 Random Fields

On small enough scales, temporal and spatial, all fields are fully polarized. In the optical regime, however, we can very rarely look at instantaneous field as that would require the
discrimination of about $10^{15}$ oscillations per second. Instead, we are generally concerned with average quantities [6]. A deterministic field has the property that the average values are the same as the instantaneous ones, i.e. there is only one unique realization in the field ensemble. However, in practice, this is an idealization. In general, a field will fluctuate as opposed to oscillate, as shown in Figure 2.3. This occurs in general because light is usually generated by a large number of random atomic oscillators. While their emissions will combine momentarily (on the order of the coherence time of the oscillators) to form a polarized wave, the state of polarization (that is the polarization ellipse) will fluctuate on time scales greater than the coherence time (which is usually much less than the measurement time). If the field is completely random, then the resulting field will be unpolarized in average. In practice, fields are most of the time partially polarized.
Any field may be viewed as the result of an ensemble of field realizations [5]. Each member \( \mathbf{E}^a (\mathbf{r}, \omega) \) of the ensemble is fully polarized and described by a unique Jones vector. It is the average quantities of this ensemble that determine the measured parameters of the field at the point of measurement. Polarimetry is primarily concerned with the second order field correlations, which are expressed in terms of the second order averages of the ensemble. There are two main formalisms fully capable of handling this regime, namely the Stokes-Mueller calculus, and the Correlation matrix.

### 2.2.1 Stokes-Mueller Formalism

In optics, the Stokes-Mueller formalism is developed around measurable quantities, i.e. intensities. It is worth noting that these intensities are ensemble averaged quantity. The Stokes vector fully describes the second order correlations of a field, and is defined as [1]

\[
\mathbf{S} (\mathbf{r}, \omega) = I \begin{pmatrix} 1 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} \langle E_x^* E_x + E_y^* E_y \rangle \\ \langle E_x^* E_x - E_y^* E_y \rangle \\ \langle E_y^* E_y + E_x^* E_x \rangle \\ i \langle E_x^* E_y - E_y^* E_x \rangle \end{pmatrix},
\]

where the angle brackets denote ensemble averaging, and the field is assumed to be transverse to the \( z \) direction.

The Stokes vector is capable of representing partially polarized fields. In this case, a polarized field is one for which the phase and amplitude relations between orthogonal field
components is the same for all realizations of the ensemble. This is to say that the end point of the electric field vector is confined to an ellipse. A completely unpolarized field would then be one for which there is no unique polarized portion. It is always possible to uniquely decompose the Stokes vector into a portion that represents the polarized contribution to the field and the portion that represents the unpolarized contribution to the field [7]

\[
S(\mathbf{r}, \omega) = I \begin{pmatrix} 1 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} = IP \begin{pmatrix} 1 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} + I (1 - P) \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},
\]

(2.7)

where

\[
P(\mathbf{r}, \omega) = \sqrt{s_1^2 + s_2^2 + s_3^2}
\]

(2.8)

is the degree of polarization, which measures the ratio between the polarized component and the total field and can vary \(0 \leq P \leq 1\). The first element of the Stokes vector specifies the average intensity of the field at a point. The last three elements of the Stokes vector completely specify the polarization ellipse (orientation, ellipticity, and sense of rotation) of the polarized component of the field. The polarized portion of the field oscillates on this ellipse, while the unpolarized portion of the field fluctuates uniformly over a two-dimensional space transverse to the \(z\) direction.

There are two particularly useful visualization tools associated with this formalism. These are the Poincare Sphere (PS) and the Observable Polarization Sphere (OPS) shown in Figures
Figure 2.4: Poincare sphere. Fully polarized fields are on the surface of the sphere while partially polarized fields lie in the interior.

2.4 and 2.5 respectively. It should be noted that these two representations are completely equivalent (the OPS is obtained by a simple rotation of the PS). In both cases, the last three components of the Stokes vector are mapped into the coordinate axis. The surface of the spheres represent fully polarized states, while any interior point represents a partially polarized field. The direction of the vector determines the state of polarization, and its magnitude specifies the degree of polarization, as shown in the Figures 2.4 and 2.5.

The so-called Mueller matrix is used to model light interaction with linear media, the Mueller matrix is used. This 4x4 matrix represents the linear transfer properties of a medium in exactly the same way that the Jones matrix does. Specifically, an initial field described by the Stokes vector \( \mathbf{S}_i \) and propagated through a linear system characterized by a Mueller matrix \( \mathbf{M} \) (which may depolarize it), becomes \[ \mathbf{S'} = \mathbf{M} \mathbf{S}_i. \] (2.9)
Figure 2.5: Observable Polarization Sphere. Completely equivalent to the Poincare sphere (differs only in a rotation of the axis).

The Stokes-Mueller formalism has the advantage that it can describe partially polarized light, depolarizing media, and incoherent fields, as well as fully polarized fields and polarization preserving linear systems. The subset of Mueller matrices which do not depolarize an incident beam has the property that there exists a one to one mapping onto the possible Jones matrices, and, similarly, there is a one to one mapping between all Jones vectors and all fully polarized Stokes vectors. Specifically, Jones vectors and Stokes vectors representing fully polarized fields are related by

\[
S = I \begin{pmatrix} 1 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} E_x^2 + E_y^2 \\ E_x^2 - E_y^2 \\ 2E_xE_y\cos(\delta) \\ 2E_xE_y\sin(\delta) \end{pmatrix}, \tag{2.10}
\]
Another interesting decomposition of the Stokes vector, is the one onto orthogonal polarization states. Specifically, any Stokes vector which does not represent a fully unpolarized field may be uniquely represented as the sum of two orthogonally polarized, uncorrelated fields as

\[ \mathbf{S} = I \begin{pmatrix} 1 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} = I \begin{pmatrix} \frac{1 + P}{2} \\ \frac{s_1}{P} \\ \frac{s_2}{P} \\ \frac{s_3}{P} \end{pmatrix} + I \begin{pmatrix} \frac{1 - P}{2} \\ -\frac{s_1}{P} \\ -\frac{s_2}{P} \\ -\frac{s_3}{P} \end{pmatrix} \] (2.11)

The two components of this decomposition are fully polarized. Furthermore, the two states of polarization are orthogonal. The Stokes-Mueller formalism is one of intensity addition, and as such, the two fields represented in this decomposition are uncorrelated. This decomposition then implies that any ensemble of field realizations which results in a partially polarized field may be represented as an equivalent ensemble containing only two unique realizations having orthogonal states of polarization. This concept will be further discussed here and also in Chapter 3.

While the Stokes-Mueller formalism is capable of describing both polarization preserving and depolarizing media, it is limited to intensity additions and is thus incapable of describing the phenomena of interference.
2.2.2 Correlation Matrix Formalism

Another formalism capable of dealing with random fields is that of the correlation matrix \([3]\). The correlation matrix \(W(\mathbf{r}, \omega)\) is simply the cross-spectral density matrix evaluated at a single point

\[
W(\mathbf{r}, \omega) = \begin{bmatrix}
\langle E_i^* E_i \rangle & \langle E_i^* E_j \rangle \\
\langle E_j^* E_i \rangle & \langle E_j^* E_j \rangle
\end{bmatrix}
\tag{2.12}
\]

where the spatial and frequency dependence has been suppressed, the angle brackets denote ensemble averaging, and the field is assumed to be transverse to the \(z\) direction. This matrix fully characterizes the second order correlations of a field at a point \(\mathbf{r}\). It is worth noting that the subscripts \(i\) and \(j\) do not necessarily denote perpendicular directions in space, rather, they denote specific, orthogonal states of polarization transverse to the \(z\) direction.

While in most cases it is convenient to think in terms of perpendicular linear states, it is occasionally convenient to work for instance in the circular polarization basis. Furthermore, the so-called "natural" basis for a particular field, i.e. that basis in which the correlation matrix is diagonal, will in general be orthogonal elliptical states of polarization \([6]\). This "natural" basis is specified by the eigenvectors of \(W\), and the fact that such a basis exists is due to the fact that \(W\) is Hermitian. Explicitly, as \(W\) is a non-negative definite, Hermitian matrix, it can be written uniquely as the sum of two uncorrelated fields with orthogonal states of polarization, given by
where $\lambda_1 \geq \lambda_2 \geq 0$ are the ordered eigenvalues corresponding to the eigenvectors $\mathbf{\nu}_1 = [\mathcal{E}_x^*, \mathcal{E}_y^*]^T$ and $\mathbf{\nu}_2 = [-\mathcal{E}_y, \mathcal{E}_x]^T$. It is worth noting that the elements of both matrices in the decomposition of Eq(2.13) explicitly factor, a property not present in the general matrix of Eq(2.12). It is, in fact, this factorization that implies the two matrices represent fully polarized fields. The eigenvectors of the correlation matrix may be viewed as the complex conjugate of the Jones vectors of the fields represented by these two matrices. Orthogonality of fully polarized states of polarization is defined in terms of the corresponding Jones vector. Hence, orthogonality of the eigenvectors implies orthogonality of the two states in the decomposition of Eq(2.13). The eigenvectors then provide the "natural" basis, while the eigenvalues are proportional to the intensities of the field components in these states. It is worth emphasizing that this decomposition is entirely equivalent to that given in Eq(2.11).

There are a number of equivalent definitions for a polarized field. In terms of the correlation matrix, a field is polarized if every element of the matrix factorizes. This is equivalent to a deterministic relationship between orthogonal field components, which, in turn, is equivalent to the real part of the complex field vector being confined to an ellipse [2]. For the case of a two-dimensional field, another equivalent property is that the determinant of the correlation matrix is zero [3]. As the correlation matrix formalism can describe partially
polarized fields, it can be uniquely decomposed into a portion representing polarized light and a portion representing unpolarized light

\[ W(\mathbf{r}, \omega) = I \left\{ P \begin{bmatrix} \mathcal{E}^*_x \mathcal{E}_x & \mathcal{E}^*_x \mathcal{E}_y \\ \mathcal{E}^*_y \mathcal{E}_x & \mathcal{E}^*_y \mathcal{E}_y \end{bmatrix} + (1 - P) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}, \quad (2.14) \]

where \( I \) is the total intensity of the field at the point

\[ I = \text{Tr} \{ W \}, \quad (2.15) \]

\( P \) is the degree of polarization given by

\[ P(\mathbf{r}, \omega) = \sqrt{1 - \frac{4 \text{Det} \{ W \}}{\text{Tr} \{ W \}^2}} = \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}, \quad (2.16) \]

and \( \text{Det} \) and \( \text{Tr} \) represent the determinant and trace respectively.

It is also possible to decompose the correlation matrix onto the basis of the Pauli-spin matrices. In fact, this decomposition illustrates the direct relationship between the correlation matrix formalism and the description in terms of the Stokes vector. The projections of the correlation matrix onto the Pauli-spin matrices yield the Stokes vector elements \[3\]. Specifically,

\[ W = \rho_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \rho_1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \rho_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \rho_3 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad (2.17) \]
with the equivalent Stokes vector being

\[
\mathbf{S} = 2 \begin{pmatrix}
\rho_0 \\
\rho_1 \\
\rho_2 \\
\rho_3
\end{pmatrix}
\]

In this formalism, the interaction of the field with a linear medium is described by

\[
\mathbf{W}' = \langle \mathbf{J}' \mathbf{W} \mathbf{J} \rangle
\]

where \( \mathbf{J} \) is the Jones matrix associated with one realization of the ensemble of the linear medium, the dagger denotes Hermitian conjugate, and the angle brackets denote ensemble averaging over both field and medium realizations \[3\].

The elements of the correlation matrix are all in terms of physically accessible parameters, intensities \[5\]. However the formalism allows for the addition of fields in terms of amplitudes and phases, and as such is capable of handling both partially polarized fields, and partially coherent and partially correlated fields. As opposed to both the Jones or Stokes-Mueller formalisms, this permits describing both depolarization phenomena and interference effects. Furthermore, this theory has important counterparts in the quantum treatment of polarization \[8, 9, 10, 11, 12, 13\].
2.2.3 Measurables and the Degree of Polarization

From the standpoint of the second order correlations of a transverse field, there are four measurable parameters [1, 3, 4, 5]. These can be expressed as the second order field correlations themselves, or as more tractable physical quantities. In other words, a field can be described either by the Correlation Matrix or by a set of parameters based on linear combinations of the elements of the Correlation Matrix. One such set of parameters is of particular interest. This set is comprised of the total intensity $I$, the degree of polarization $P$, and the state of polarization (polarization ellipse) of the polarized component of the field (ellipticity $\varepsilon$ and orientation $\phi$ with regards to a specified axis). As a set of four independent parameters, it is possible to express any quantity of interest in terms of these four parameters.

The above set of parameters is of particular interest because they represent physically meaningful characteristics of the field under study. As such, independent of which formalism is used to describe the field, the final result of these parameters remain the same. Careful study of the correlation matrix provides some insight into these quantities. The reason for examination of the correlation matrix formalism is that it fully encompasses both the Jones and the Stokes-Mueller formalisms.

As we are interested in physical parameters of the field, we should look at the invariants in the correlation matrix. An important note is that the intensity, degree of polarization, and ellipticity of the polarized component must be independent of the orientation of the reference frame. The orientation of the polarized component is by its very nature dependent on the
reference frame. So, to describe the physical parameters of a field, we need to examine the invariants of the correlation matrix. Specifically, the physically meaningful parameters are specified by the eigenvalues and eigenvectors of the correlation matrix. The total intensity of the field is given by the sum of the eigenvalues, while the degree of polarization is given by the difference normalized to the total intensity. The state of polarization (the polarization ellipse, its orientation and ellipticity) is determined by the complex conjugate of the eigenvector of the maximal eigenvalue. Of course, while the complex conjugate of the eigenvector of the minimal eigenvalue specifies the state of polarization that is orthogonal to the polarized component [6].

2.3 Classical Methods of Measurement

There are a number of standard methods for measuring the polarimetric information pertaining to both fields and media [1, 3, 4, 5]. They generally fall into one of two classes. The first is a minimal set of measurements necessary to specify the polarimetric information, while the second is an overdetermination of the parameters of interest.

When dealing with random fields, a minimum of four measurements must be made to specify the field, while sixteen measurements are required to specify the polarimetric properties of a medium. There are various assumptions that can be made to reduce the number of necessary measurements. If the field is deterministic and the medium does not introduce any depolarization, then the Jones formalism is sufficient and three field measurements and
seven measurements of a medium will completely specify the system. Other assumptions are possible as well, resulting in a range of one to 4 measurements for field and 1 to 16 measurements for a medium.

For the general case, four measurements are required to determine the state of polarization of a field. The standard set of measurements makes use of a polarizer oriented at 0, 45, and 90 degrees, and then the inclusion of a quarter wave plate before the polarizer [1]. This is shown in Figure 2.6. The detected intensity, in terms of the Stokes vector elements of the source, is given by

$$I_{\text{det}} = \frac{1}{2} \left[ S_0 + S_1 \cos (2\theta) + S_2 \cos (\phi) \sin (2\theta) - S_3 \sin (\phi) \sin (2\theta) \right]$$  \hspace{1cm} (2.20)

where $\phi$ is the total phase shift between perpendicular field components induced by the waveplate, and $\theta$ is the orientation of the polarizer with respect to the axis of the waveplate.
This fully specifies the field, and is the simplest (mathematically/conceptually) set of measurements to acquire either the Stokes vector or the correlation matrix. To describe a medium, these same measurements are performed four times, with four different incident states of polarization. Other combinations of waveplate and polarizer orientations may be used, and, from an experimental standpoint, it is possible to minimize errors by making more elaborated choices of measured states. Practically, a choice that maximizes the determinant of the transfer matrix of the polarimeter will minimize the experimental error in the measured state of polarization. This can also be accomplished via an overdetermination of the system [14].

One possible method for over-determining the system is to encode the information into an oscillating signal and perform a Fourier decomposition. A practical means of doing this is using a rotating waveplate in front of a fixed polarizer [1]. While the waveplate must be chosen to provide four independent Fourier components; this requirement is easily satisfied by a quarterwaveplate.

The detected intensity as a function of the angle of rotation of the quarterwave plate with respect to the fixed polarizer is given by

\[
I_{\text{det}} = \frac{1}{2} \left\{ S_0 + S_1 \cos^2 (2\theta) + S_2 \cos (2\theta) \sin (2\theta) + S_3 \sin (2\theta) \right\}, \quad (2.21)
\]

equivalently,
Figure 2.7: The Stokes vector of an unknown source can be determined through Fourier analysis of the signal resulting from a rotating quarterwave plate and a fixed polarizer.

\[ I_{\text{det}} = \frac{1}{2} \left\{ A + B \cos (2\theta) + C \cos (4\theta) + D \sin (4\theta) \right\}, \quad (2.22) \]

from which we find

\[
\begin{align*}
S_0 &= A - C \\
S_1 &= 2C \\
S_2 &= 2D \\
S_3 &= B.
\end{align*}
\]

Thus measuring the Fourier components of the detected signal as a function of rotation angle allows for the reconstruction of the Stokes vector of the unknown field.
In all of the above cases, a measurement is performed on a two-dimensional detector that is assumed to be oriented normal to the direction of propagation. Under the formalism of the correlation matrix, the measured signal at the detector is \[ I_{\text{det}} = \text{Tr} \{ J^t W J \} \] (2.24)

where \( J \) is the Jones matrix of the polarimeter, and \( \text{Tr} \) denotes the trace of a matrix. If we are attempting to characterize the correlation matrix of a field, then at least four realizations of \( J \) are necessary. In the case of a medium, we find that the detected signal is given by

\[
I_{\text{det}} = \text{Tr} \{ J_{\text{pol}}^t J_{\text{med}}^t W J_{\text{med}} J_{\text{pol}} \} \] (2.25)

and characterization of \( J_{\text{med}} \) requires at least four realizations of \( W \) paired with four realizations of \( J_{\text{pol}} \) to provide sixteen independent measurements. It is possible that full characterization of a field or medium is not necessary, in which case a reduced set of measurements may suffice.

### 2.4 Application to Imaging

One application of practical interest is polarimetric imaging. There are multiple ways in which polarimetry may be used to improve imaging [[15]-[19]]. A full polarimetric analysis of a scene provides on the order of sixteen times more information as a single intensity image.
In practice, however, much of this is effectively redundant information [20]. It is possible that this duplication of information could be used to improve the fidelity of imaging obscured objects. Conversely, an understanding of the scattering medium and object under investigation may allow for a reduced set of measurements. Another practical problem of interest is polarimetric sensing in a backscattering geometry [[21]-[27]]. In this regime, interesting phenomena incorporating both low order and high order scattering occur, and it has been found that polarimetric measurements may serve to greatly increase the optical density through which an object can be detected [28, 29, 30]. In the following, I will examine the use of polarimetric measurements as applied to material discrimination and imaging objects imbedded in a dense scattering medium. The main goal is to improve object/material discrimination with a minimum of additional measurements over the standard intensity imaging. While in many cases, the polarimetric measurements serve to improve the available information, these practical situations will also illustrate deficiencies in the current formalism.

2.4.1 Polarization Difference Imaging

The simplest experimental implementation of polarimetric measurements in an imaging geometry would be that of polarization difference imaging. More specifically, the target is illuminated with a pure state of polarization, and images of the scene are acquired in the co- and cross-polarized channels. It has been shown that the combination of intensity and polarization difference imaging can be used to improve material recognition [31]. For exam-
ple, absorbent objects are relatively obscured under straight intensity, but due to the fact
that their major contribution to the image comes from single scattering, they are "bright"
under polarization difference imaging. Similarly, metallic structures can be distinguished
from diffusive scatterers, as a typical scatterer will be "dark" under polarization difference,
unlike a smooth metal surface which does not depolarize the incident illumination.

Polarization difference imaging can also be applied to objects imbedded in a dense scat-
tering medium [32, 33, 34, 35]. I have investigated the use of polarization difference imaging
as it applies to this regime. The improvement in imaging depth is related to the characteris-
tics of both the scattering medium and the object under investigation. The object tends to
have a larger effect on the improvement than the medium, simply because a dense random
medium will tend to depolarize the light. When the object tends to maintain the polariza-
tion of the illuminating beam, an improvement of over twice the depth can be achieved via
polarization difference imaging.

A backscattering experiment has been conducted to examine the usefulness of polariza-
tion difference imaging to object detection and material discrimination. The object was
placed inside a cloud chamber, where the optical density was varied both by the volume
of scatterers initially introduced, and by the natural settling of the cloud. This scene was
then illuminated with linearly polarized radiation and the co- and cross- polarized images
were recorded. From these, the Intensity and Polarization Difference images were calculated.
Three different cases are illustrated here. First the detection of an absorbing obscured by a
random scattering medium was examined. Then the differentiation between a metallic and a
diffusive object obscured by a random scattering medium was studied. Finally, the detection of a diffusive object obscured by a random scattering medium was examined. Typical images for Polarization Difference and Intensity are shown in Figure 2.8 for an absorbing object and Figure 2.9 for the complex scene consisting of a metallic and diffusive cube obscured by a scattering medium. The calculated contrast ratios as a function of optical density for these three cases are given in Figures 2.10, 2.11 and 2.12 respectively.

Both the absorbent and metallic objects tend to maintain the incident polarization state, as their primary contribution to the backscattered beam arises from single scattering. This is in contrast to the diffusive surface and the random scattering medium, both of which tend to depolarize the backscattered light. The resulting effect is that both the absorbent object and the metallic surface are bright under polarization difference imaging, while areas that contain only the scattering medium are dark. This provides the measured enhancement in
Figure 2.9: The same scene under polarization difference and intensity imaging. The scene is comprised of a metallic surface next to a diffusive surface, both at the same angle to the incident illumination.

Figure 2.10: Contrast ratio as a function of optical depth for an absorbant target obscured by a random scattering medium. The red line represents the contrast in the Polarization difference image, while the blue line is the contrast calculated for the Intensity images.
Figure 2.11: Contrast ratio as a function of optical depth for a metallic and diffusive surface obscured by a random scattering medium. The red line represents the contrast in the Polarization difference image, while the blue line is the contrast calculated for the Intensity images.

Figure 2.12: Contrast ratio as a function of optical depth for a diffusive surface obscured by a random scattering medium. The red line represents the contrast in the Polarization difference image, while the blue line is the contrast calculated for the Intensity images.
contrast ratio over the intensity images. However, there is no improvement in imaging of the diffusive target, as the behavior of both the diffuser and the scattering medium result in unpolarized light. While there is a distinct difference between a system with a diffusive solid in a random scattering medium and one without, classical polarimetric measurements are not capable of differentiating between different globally unpolarized ensembles, and so they are not capable of discriminating these two systems.

2.4.2 Off-diagonal Mueller Matrix Imaging

Another application of practical interest for polarimetric imaging is material discrimination [31, 36]. While it is difficult in general to model and predict the behavior of the Mueller matrix for arbitrary systems, the simple case of a diffusive surface measured in a backscattering geometry is somewhat tractable. If the illumination is normal to the surface, then the Mueller matrix will be diagonal, and the elements will have contributions from scattering events of all orders. If the illumination is not normal to the surface, then the off-diagonal elements will be non-zero, and their dependence on both material properties and angle of illumination may be used to discriminate between various diffusive media.

The off-diagonal Mueller matrix elements may be obtained via a setup in which the scene is illuminated sequentially with two orthogonally polarized fields, and the backscattered co- and cross-polarized channels for each illumination configuration are measured. Specifically, the off-diagonal elements that pertain to linear horizontal and vertical polarized light, were
investigated. The target is rotated about the vertical axis in increments of 10 degrees between measurements and values of the reduced Mueller matrix are generated as a function of the angle of incidence. The propagation direction of the incident and the measured beams remain fixed so that we explore only the backscattering geometry.

Various inhomogenous samples from a wide range of categories were subjected to the systematic measurements described above. The samples can be broken down into two broad categories: (1) rough surfaces (dielectric and/or metallic) and (2) inhomogenous dielectric diffusers. Typical results for the strongly scattering, diffusing samples are summarized in Figure 2.13. Sample A is a nonabsorbent, anisotropically fibrous diffuser with a characteristic length scale of inhomogeneities on the order of 1 micron. Sample B is a structurally similar diffuser that is also highly absorbent at the wavelength of illumination (633 nm). Sample C is a diffusive material with a totally different morphology: spherical CaCO$_3$ pigments with a narrow size distribution centered on 500 nm in diameter were densely packed to make a 1 mm thick diffusing layer.

As can be seen, the off diagonal elements are always equal, within the accuracy of the measurements, and increase with the angle of incidence as shown. The non-zero off-diagonal elements relate to the behavior of a typical polarizing transfer system. This simply means that, at large angles of incidence, a polarization tendency appears to be significant despite the overall diffusive behavior of the scattering media. This behavior can be described by examining contributions to the Mueller matrix from scattering events of all orders. Given the chosen configuration, certain simplifications to this expansion exist.
Figure 2.13: Theoretical fits of the experimental data corresponding to different values of N.
Sample A: \( N = 0.120, r^2 = 0.75 \); sample B: \( N = 0.150, r^2 = 0.92 \); sample C: \( N = 0.374, r^2 = 0.90 \). See text for a description of the samples.

While the backscattering geometry was chosen for its relevance to active imaging, it also provides a simplification in the scattering order. This is due to the fact that in the backscattering geometry, the contribution of single scattering in the off-diagonal elements is negligible \([20, 37, 38]\). Similarly, high order multiple scattering will have negligible contribution due to the fact that it will in general depolarize the light. This leaves low order multiple scattering with the most significant contribution \([31]\).

A simple model based on second order dipole scattering in a plane is used to describe the angular dependence on the orientation of the object of the off-diagonal elements of the reduced Mueller matrix. The geometry is shown in Figure While neglecting the effects of the higher-order scattering events essentially ignores a fraction of the scattered energy, this first order approximation is sufficient to explain the polarization behavior. If we consider
the scattering events to be small particles with polarizability tensor \( \alpha \), then tedious but straightforward calculation [36] yields a functional form for the off-diagonal elements in terms of the angle of incidence, \( \theta \), and a material parameter \( N \)

\[
N = \frac{\alpha_1\alpha_2^* + \alpha_2\alpha_3^* + \alpha_3\alpha_1^* + \alpha_2\alpha_1^* + \alpha_3\alpha_2^* + \alpha_1\alpha_3^*}{\alpha_1\alpha_1^* + \alpha_2\alpha_2^* + \alpha_3\alpha_3^*} \tag{2.26}
\]

as

\[
M_{12} = C (2 + 3N) \sin^2 (\theta) \left[ 4 + N + \frac{(2 + 3N) [1 - 3 \cos^2 (\theta)]}{4} \right]. \tag{2.27}
\]

The monotonic relationship between the angle of incidence and the off-diagonal elements of a reduced Mueller matrix were observed for a number of diffusive samples. The off-diagonal

---

Figure 2.14: Scattering geometry for the double scattering model. The plane of the surface is the \( xy \)-plane. Without loss of generality, we consider only rotations about the \( y \)-axis. The incident beam \( \mathbf{k} \) is confined to the \( xz \)-plane, and makes an angle \( \theta \) with the \( z \)-axis.
elements were found to be equal, and increase to a maximum at a near-grazing incidence. The non-zero presence of these off-diagonal elements indicates an effective anisotropy, induced by the scattering geometry. This anisotropy and its effects can be modeled in terms of second order scattering events in plane. Based on this simple model, material discrimination can be accomplished through an angular measurement. However, the material discrimination provided by this model relies on the fact that fields under consideration have not been completely depolarized by the diffusive medium under consideration. As in the case of the polarization difference imaging, if the depolarizing nature of the medium dominates the interaction, little information about the sample under consideration can be gained through polarimetric measurements.

2.5 Conclusions

Quantitative polarimetric information has traditionally been described in terms of ensemble averages of second order field correlations. In this Chapter, three different formalisms were described which will be used throughout this dissertation. These are the Jones Calculus, pertinent to the case of coherent, fully polarized fields; the Stokes-Mueller calculus applicable to fully incoherent, partially polarized fields, and the Correlation Matrix formalism, capable of describing both partially coherent and partially polarized fields. Two classical measurement techniques were also briefly described. A method based on minimal measurements with a simple choice for the 4 orientations of quarter wave plate, half-wave plate and polarizer,
and the Fourier approach which makes use of a time varying quarterwave plate and fixed polarizer.

The concept of orthogonal fields (equivalently, orthogonal states of polarization) was introduced in terms of the Jones vector. The unique decomposition of either the Stokes vector or the Correlation matrix into two fully polarized, orthogonal and uncorrelated fields was shown to indicate an equivalent ensemble representation of the fields. The decomposition of either the Stokes vector or the Correlation matrix into a unique fully polarized component and a fully unpolarized component was used to calculate the degree of polarization, the state of polarization, and the polarization ellipse for a partially polarized field.

The general theory of polarimetric measurements was then briefly discussed, after which polarimetric information was examined in a practical imaging situation. It was found that while full polarimetric characterization of a scene may provide a greater degree of information, polarization difference imaging (i.e., a reduced set of polarimetric properties of the scene) was sufficient to extend the distance into a scattering medium at which a polarization preserving target could be identified. It is worth noting that polarimetric measurements are primarily useful when intensity measurements are degenerate. This is the case for targets with similar intensity profiles, as well as in the case of targets embedded in dense scattering media. For the case of objects embedded in scattering media, polarimetric measurements are useful if the object is polarization preserving. However, the standard polarimetric measurements provide no extra information if both the medium and the object embedded in it are diffusive and depolarizing. Similarly, while a reduced set of polarimetric measurements
can provide a means of material discrimination as shown in Section 1.4.2, if the materials under consideration are dominated by a depolarizing effect, the discriminatory capabilities are greatly reduced. It is interesting to note that, in these experiments, the averaging implied in the classical theory of polarimetry of random fields occurs both in time and space. The spatial averaging was necessary due to the speckled nature of the imaged scenes and is a result of the coherent illumination. Investigation of the spatial statistics of these speckled fields may lead to an interesting extension of polarimetric approaches, especially in the difficult case of a depolarizing object obscured by a depolarizing random medium.

This suggests that information pertinent to the discrimination of material, shape, and orientational information may be obtained without using a full imaging system. The polarimetric distributions associated with these speckle patterns resulting from coherent illumination, may contain information pertinent to object’s shape and orientation. This is of particular interest when the average values of the polarimetric information are not sufficient to discriminate between objects and scattering media. Practical examples of this lack of discrimination were presented in the previous sections. In the following Chapter, I will examine an extension of the quantities of interest in polarimetric investigations especially for situations involving globally unpolarized field. New measurable quantifiers of the field will be introduced and a thorough examination of their applicability will be presented.
In the previous chapter, the classical formalisms in the practice of polarimetry were reviewed. Specifically, the Jones Calculus which is applicable to deterministic fields, as well as the Stokes-Mueller Calculus and Correlation Matrix Formalisms applicable to random fields were examined. A reduced set of classical polarimetric measurements pertinent to the Stokes-Mueller formalism was applied to the case of active imaging. It was shown that the available information as well as the depth of imaging inside a scattering medium can be improved in certain object-medium configurations. For the case of coherent active imaging, both the intensity and polarimetric images are degraded by speckle noise, and spatial averaging is necessary for analyzing the information content of the images. This is consistent with classical polarimetry, which is primarily concerned with determining the ensemble average of the second order correlations of field components. In the case of active coherent imaging systems, the speckled field constitutes a spatial ensemble of realizations over which the polarimetric information is averaged.

The statistics of random complex fields give rise to a number of measurable distributions, including intensity, phase difference, ellipticity, and states of polarization. Considering a specific direction of propagation, all of the above can also have angular characteristics. Random
complex fields arise in many practical situations, such as the speckle phenomena resulting from the coherent illumination of diffusive random media and scattering from rough surfaces, as examined in the previous Chapter. While common assumptions in the case of speckle phenomena imply a circular Gaussian distribution for the complex fields [39], greatly simplifying the mathematics and leading to an equal probability for any state of polarization, these assumptions are not always valid in practice. Furthermore, as Gaussian statistics are completely specified by the second order moments, they provide little information about the underlying scattering system. Non-Gaussian distributions, on the other hand, are not subject to this restriction and may provide specific information about the physical system [40, 41, 42, 43], even in the highly scattering regime.

Non-Gaussian statistics have been encountered in surface roughness measurements [40, 41, 44], detection of particles on a surface and their surface distribution [42], as well as scattering from small particles and particle shape determination [43]. In all these cases, the statistics of underlying fields had a non-Gaussian behaviors, thus allowing meaningful information pertaining to the scattering system to be derived from the higher order statistics of the scattered light. The assumptions leading to a Gaussian distributed ensemble of field realizations can fail for a number of reasons such as a small number of scatterers arising due to imaging geometry [45], a small number of scatterers present in the scattering volume [42, 43], or any other means of coupling between the underlying random fields. It is the relationship between this coupling and system parameters that allows the discrimination of characteristics such as particle size, orientation, and surface roughness.
In the previous chapter, speckle patterns for various samples obscured by a dense scattering medium were examined. The speckle patterns shown in Figures 2.8 and 2.9 are composed of locally polarized speckles with some spatial distribution. When ensemble averaging, in this case equivalent to spatial averaging, is performed, the various samples result in different degrees of polarization. For the diffusive surface and areas which contain only the dense scattering medium, the ensemble average results in an unpolarized field. It is, however, only the ensemble of field realizations that is unpolarized, as locally each speckle is fully polarized. Thus, both the diffusive surface and the scattering medium are examples of systems which produce globally unpolarized ensembles. Globally unpolarized ensembles of field realizations are of particular interest in the case of speckle patterns arising from coherent scattering through a diffusive medium. In the case of globally unpolarized ensembles, the average Stokes vector is that of unpolarized light and hence independent of coordinate frame or the introduction of an arbitrary retardance. When there is a negative exponential probability distribution for the intensity components in perpendicular directions, it is common to assume a circular Gaussian distribution for the underlying ensemble of field realizations. This results in a uniform distribution of polarization states, the average intensity determines all moments of the distribution, and little information is available about the scattering system. This is an example of Type I unpolarized light which will be discussed in this Chapter. However, this is not the only possibility for globally unpolarized fields with negative exponential distributions in intensity [46], as will be demonstrated in the following Sections.
It will be shown that by resolving individual speckles in terms of Stokes vector elements, even when the global properties of the ensemble are such that the field is unpolarized in average, one has access to the underlying distribution of ensemble realizations. The use of the correlations between the Stokes vector elements to discriminate between different types of globally unpolarized fields will also be demonstrated. First, theoretical examples of the different types of globally unpolarized fields will be presented. The OPS [1] will be used to display the various polarization state distributions. A simple dipole-like scattering model will be developed to examine aspects of the statistics of speckle fields which result from multiple light scattering. Numerical simulations will demonstrate that subtle mophological details such as shape and orientation of scattering centers influence the resulting speckle statistics. Experimental measurements on various scattering systems will also be presented, demonstrating the influence of particle shapes on both the Stokes vector element correlations and the joint probability distribution of the polarization states. This constitutes the experimental verification of the existence of various types of globally unpolarized light.

The Stokes vector element correlations, while fourth order field correlations, are still global parameters in the same sense that the degree and state of polarization are global parameters. This is to say that they provide little information pertinent to length scales in the distribution, how similar various members of the ensemble of field realizations are. A greater level of detail can be obtained by considering a resolved measure of the polarization similarity, providing insight into the structure of the field ensemble. To examine these structures, the Complex Degree of Mutual Polarization (CDMP) will be introduced. This
serves as a resolved measure of how similar the states of polarization in two different points are. Theoretical and experimental examples will be presented indicating the information content available in such a polarization similarity measure.

### 3.1 Resolution of ensembles

A field can fluctuate in many different ways. While measurements are usually considered to be point-like, in fact there is a finite extent of any detector, and therefore it is possible that the field fluctuates spatially over its area. Similarly, it is possible that the field fluctuates temporally as well. What all these situations have in common is that the detected signal is, in essence, the average of an ensemble of field realizations.

If we had access to the individual members of the ensemble, we would find that the field at a point was composed of the superposition of independent, fully polarized realizations. The ensemble of field realizations could then be described in terms of the distribution of states of polarization. Of course, when we are interested only in the second order statistics of the field, the full ensemble is replaced with an equivalent one containing only two independent realizations. While the second order statistics describe the polarimetric properties of the field at a point, there are practical situations in which they do not fully discriminate between different ensembles of field realizations. This is the case for globally unpolarized fields. A common assumption for globally unpolarized fields is Gaussian distributed members of the ensemble of field realizations. Gaussian distributions are fully specified by their second order
moments, but, in practice, to discriminate between Gaussian and non-Gaussian distributions, or simply to verify that the distribution is in fact Gaussian, one must examine the higher order moments of the distribution. It is worth mentioning for the case of non-Gaussian field distributions, it is possible to design measurements that can provide higher order correlations, such as intensity-intensity correlations or fourth order field statistics. In certain instances, it is possible to recover the actual members of the full ensemble of field realizations, which then permits evaluating the higher order statistics in order to have a greater specificity in characterizing a random field. This is of particular interest in cases where second order statistics do not discriminate between the various distributions. This situation is typical for globally unpolarized fields arising from coherent illumination of diffusive materials.

### 3.1.1 Globally unpolarized fields

One practical instance in which the second order statistics does not discriminate between various forms of underlying fields is the case of fully resolved speckle fields arising from diffusive scattering. These fields tend to be globally depolarized; however, the individual speckles are fully polarized. Thus, by resolving the spatial speckle pattern, one has direct access to the ensemble itself. In the case when a globally unpolarized field is composed of locally polarized states, one can differentiate between various types of globally unpolarized light, and, therefore, infer additional information about the underlying random fields and the systems that gave rise to them [46, 47, 48, 49, 50].
By considering the OPS, it is easy to visualize a virtually infinite number of distributions of states of polarization that result in globally unpolarized light. Any distribution that has an equal probability of polar pairs on the OPS, that is to say any distribution \( p(s_1, s_2, s_3) \) such that \( p(s_1, s_2, s_3) = p(-s_1, -s_2, -s_3) \), will average to a completely depolarized field in the global sense. This is only a sufficient, and rather weak, condition for globally unpolarized light. These distributions can be as simple as two polar points with equal probability or a uniform covering over the entire sphere. Much more complicated distributions of states of polarization could also lead to a globally depolarized field. For instance, we can consider various banded structures, or even amorphously shaped distributions on the surface of the OPS, as long as they have polar symmetry. It is not even necessary that every element in the ensemble be a pure state of polarization, but since any partially polarized state can be decomposed into a pure state of polarization and an unpolarized component, or equivalently a unique pair of uncorrelated, polarized realizations, I will limit the discussion to random fields comprised of locally pure states of polarization.

When dealing with globally unpolarized fields composed of locally pure states of polarization, one can classify them based upon their invariance to (i) the introduction of a half-wave plate at an arbitrary angle, (ii) a reversal in the direction of propagation, and (iii) an introduction of an arbitrary retardation [46, 49, 50]. In terms of polarization states distributed on the surface of the OPS, these conditions are equivalent to the distributions of states of polarization being (i) invariant under rotation about the \( s_3 \) axis, (ii) symmetric about the \( s_1, s_2 \) plane, equivalently invariant under the operation of \( s_3 \) to \(-s_3\), and (iii) in-
variant under an arbitrary rotation of the three axes \( s_1, s_2, s_3 \) to \( s'_1, s'_2, s'_3 \). Type I unpolarized light has statistics which are invariant to all three, while the statistical properties of Type II unpolarized light obey only the conditions (i) and (ii). Other types of unpolarized light can be considered by examining distributions whose statistical properties are invariant under different combinations, such as a third type of unpolarized light, denoted here as Type III, which has statistical properties that are only invariant under the operation of \( s_3 \) to \(-s_3\).

The only distribution capable of conforming to Type I unpolarized light is a uniform distribution of polarization states. This is the distribution that arises when the complex field vector has a circular Gaussian distribution. Such a distribution is shown in Figure 3.1. While Gaussian distributed fields are not the only field distribution that results in a uniform covering of the OPS, the added constraint of a negative exponential distribution in scalar intensity (equivalently a gamma distribution in total intensity) results uniquely in a Gaussian field distribution. Specifically, when the underlying fields are Gaussian distributed, the joint probability of the complex fields is given by [51]

\[
p(E'_x, E'_y, E''_x, E''_y) = \frac{1}{4\pi\sigma^4} \exp \left\{ -\frac{(E'_x)^2 + (E'_y)^2 + (E''_x)^2 + (E''_y)^2}{2\sigma^2} \right\}, \tag{3.1}
\]

which implies [51, 52, 53]

\[
p(2\alpha, \Delta) = \frac{1}{2\pi} \frac{\sin (2\alpha)}{2}, \tag{3.2}
\]
Figure 3.1: Type I unpolarized light. This type of unpolarized light is unique in that the only distribution invariant to all possible retardances/co-ordinate transforms is a uniform distribution.

where $2\alpha \in [0, \pi)$ is the polar angle measured from the $s_1$-axis of the OPS, and $\Delta \in [0, 2\pi)$ is the azimuthal angle measured from the $s_2$-axis.

As the Gaussian field distribution results in Type I unpolarized fields, this implies that all Type II and III globally unpolarized fields arise from non-Gaussian statistics. As noted above, the coupling resulting in non-Gaussian fields can arise from a number of physically pertinent configurations such as a small number of scatterers arising due to imaging geometry [45], a small number of scatterers present in the scattering volume [42, 43], and others. Figures 3.2 and 3.3 illustrate two theoretical couplings in the complex field distribution resulting in Type II and Type III unpolarized light, respectively.

The distribution in Figure 2 is given by

$$p(\theta, \phi) = \frac{1}{2\pi} \exp \left\{ \frac{\theta - \frac{\pi}{2}}{2\sigma^2} \right\} \frac{1}{\text{erf} \left\{ \frac{\pi \sqrt{2}}{4\sigma} \right\}} \frac{1}{\sqrt{2\pi\sigma^2}},$$

(3.3)
Figure 3.2: Theoretical example of Type II unpolarized light. While the moments of the distribution are independent of a rotation of reference frame, the 4th order field correlations are not independent of the introduction of an arbitrary retardance.

Figure 3.3: Theoretical distribution resulting in Type III unpolarized light. The 4th order field correlations are not independent of a rotation of the reference frame.
where the alternate angles \( \theta \in [0, \pi) \), the polar angle measured from the \( S_3 \)-axis, and \( \phi \in [0, 2\pi) \), the azimuthal angle measured from the \( S_1 \)-axis, have been used for convenience. All moments of this distribution are independent of the reference frame as seen by the independence to the angle \( \phi \) and it is a representative of Type II unpolarized light. The distribution of Figure 3 corresponds to

\[
p(2\alpha, \Delta) = \frac{1}{2\pi} \delta \left( 2\alpha - \frac{\pi}{2} \right),
\]

which is a distribution for which the moments are not independent of the reference frame, as it depends on the angle \( 2\alpha \), or the introduction of an arbitrary retardance. These examples are related in that they differ only by an introduced coupling factor in the complex fields. Specifically, the three distributions of states of polarization come from the ensemble of complex field realizations shown in Eq. (3.5)
While the full polarization distribution serves to differentiate various underlying field configurations, the characterization of these distributions result in an infinite number of parameters; specifically, the moments of the polarization distribution. Beyond the simple moments, be they of the polarization quantities or the fields themselves, there are also global measures such as the area occupied by the distribution on the OPS [12, 13], the entropy represented by the equivalent ensemble, or the entropy represented by the full field distribution. The most common are the average intensity, degree of polarization, and state of polarization, as discussed in the previous chapter. These are equally well defined in terms of the Stokes and Correlation Matrix formalisms. A lesser used parameter would be the von-Neumann entropy [38].

\[ N = -\lambda_1 \log(\lambda_1) - \lambda_2 \log(\lambda_2), \]  

(3.7)

defined in terms of the eigenvalues \( \lambda_i \) of the Correlation Matrix. In the case of two-dimensional fields, this quantity is directly related to the degree of polarization, and knowledge of one implies knowledge of the other. It is worth noting that the von-Neumann entropy is actually the Gibbs entropy of the equivalent ensemble representation[54]. This leaves the entropy of the actual ensemble as yet another global parameter, which can not be expressed in terms of the measures and formalisms described in Chapter 1. It is expressed in terms of the area the polarization distribution occupies on the OPS as [54].
\[ G = \frac{1}{4\pi} \int p(\vec{s}) \log[p(\vec{s})], \quad (3.8) \]

with equivalent parameters finding use in fields such as non-linear and quantum optics.

For the case of globally unpolarized field distributions, only the entropy (equivalently area) serves to discriminate between various distributions. However, we are not limited to these global quantities alone. One can also examine the moments of any order of the distribution. For cases in which the average state of polarization of two different ensembles of field realizations is identical, it is possible that the fourth order field correlations (i.e. the second moment of the equivalent polarization distribution) can be examined and may serve to differentiate between the states. In the following, the discussion will be limited to fourth order field correlations.

### 3.2 Fourth order point statistics

There are many various combinations of fourth order field correlations that can be examined. The relevant form of such a correlation can be expressed in terms of a sum of correlations of the form

\[ \langle E_i^* E_j^* E_k^* E_l \rangle. \quad (3.9) \]

In the case of Gaussian distributed fields, the fourth order correlations expressed in equation Eq. (3.9) factorize and can be fully expressed in terms of the second order correlations.
If they do not factorize, then the field distribution is non-Gaussian and these correlations carry specific information pertaining to the field distribution. In the following, I will examine those fourth order field correlations which can be expressed in terms of the Stokes vector elements, specifically

\[ \langle s_i s_j \rangle \]  

(3.10)

as these relate most directly to the shape information of the polarization distribution on the OPS.

### 3.2.1 Theoretical Results

By examining the three distributions discussed above, one finds that the Stokes vector element correlations differentiate between the three types of Globally unpolarized polarization state distributions. Specifically, all distributions are characterized by no cross-correlation between Stokes vector elements, but differ in the individual correlation of the elements. Due to the fact that all the field realizations are taken to be fully polarized, the sum of the three Stokes vector element correlations for a given ensemble must be unity. For the uniform distribution, this implies that all Stokes vector element correlations are identical and equal to 1/3. Simple calculations based on the underlying field distributions result in the values summarized in Figure 3.4.
Figure 3.4: Statistical characteristics of globally unpolarized light: (a) the average Stokes vector, (b) the average intensity along the two orthogonal directions defining the reference frame and the average total intensity (σ is an arbitrary constant), (c) the Stokes vector element auto-correlations, and (d) the Stokes vector element cross-correlations. The values of the quantities in (a) and (b) are independent of the choice of reference frame, the choice of right versus left handed circular polarization, and they are invariant to the introduction of an arbitrary retardance. The values in (c) and (d) for Type II and III unpolarized light depend on the specific distributions chosen. Under an arbitrary retardance, introduction of half-wave plate, or reversal of direction of propagation, the resulting values of these six correlations in the new coordinates of the OPS will be linear combinations of the original values.

\[
\begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix}
\]

\[
\begin{array}{l}
\langle I_x \rangle = 2\sigma^2 \\
\langle I_y \rangle = 2\sigma^2 \\
\langle I_t \rangle = 4\sigma^2 \\
\langle s_1^2 \rangle = \frac{1}{4} \\
\langle s_2^2 \rangle = \frac{1}{4} \\
\langle s_3^2 \rangle = \frac{1}{4} \\
\end{array}
\]

\[
\begin{array}{l}
\langle s_1 s_2 \rangle = 0 \\
\langle s_1 s_3 \rangle = 0 \\
\langle s_2 s_3 \rangle = 0
\end{array}
\]
3.2.2 Numerical Results

A simple model can describe some of the specific features of scattering within a highly diffusive medium. It considers the physical system of coherent light diffused through a bulk scattering medium and incident on a "last surface layer" creating a speckle pattern that is subsequently imaged onto a detector. The bulk of the scattering medium provides a random distribution of incident fields in terms of polarization, intensity, and incident angles. Within the last scattering layer, small, independent particle scattering occurs. By small particle, I mean a particle that is small compared to the wavelength and can be described by its polarizability tensor. The last scattering layer is then modeled as a layer of independent particles with specific polarizabilities (related to the effective shape of the particle) and possible restrictions on the orientation of each such scatterer.

The numerical simulation generates a large number (~50,000) of particles with the same polarizabilities, which are then oriented according to various constraints. For each particle, it is assumed that light is incident within a solid angle of $2\pi$ steradians, and that along each incident direction, the light is fully polarized. This is modeled via discretization of the incident directions (~720 directions for each particle), and randomly generating the field along each incident direction for each particle assuming circular Gaussian distributed fields. In order to facilitate comparison with experiments, for each particle, the light scattered from all incident directions into the optical axis is coherently summed. The contributions from the various scatterers yield the distribution of underlying fields for the particular arrangement.
(polarizability and orientational confinement of the scatterers), and it is considered to be equivalent to a spatially resolved speckle pattern. The joint distribution of Stokes vector elements is then calculated from which the Stokes vector element correlations are obtained.

Different constraints on the particles’ orientations were examined, such as perfect alignment, restrictions to a plane, and restrictions to a uniform distribution of planes with a certain angular extent around the optical axis, and uniform distribution of orientation in all three dimensions, all with the assumption of a Gaussian distribution being produced by the bulk diffuser. It was found, as expected, that for the spherical particles no extra information was imparted onto the underlying distribution. However, for particles with cylindrical or disk-like polarizabilities, absolute confinement to a given orientation has a polarizing effect. This is to be expected as well, as long particles preferentially aligned act as a polarizer. The more interesting results were for the case of particles uniformly distributed in a plane or confined to a select range of planes. The polarizing effects were still present, however, given the random alignment of the particles, the global distribution itself was unpolarized. While the distributions resulted in globally unpolarized light, the polarizing nature of the cylindrical particles was evident in the Stokes vector element correlations as a reduction in the value of \( \langle s_3^2 \rangle \); i.e. a selection against circularly polarized light. Some of the results of the numerical simulations are summarized in Figure 3.5.

For the uniform orientation in three dimensions, the resulting correlations are essentially identical to those of the incident uniform distribution \( \langle s_1 \rangle = \langle s_2 \rangle = \langle s_3 \rangle = 0, \langle s_1^2 \rangle = \langle s_2^2 \rangle = \langle s_3^2 \rangle = 1/3 \) with little to no deviation
Figure 3.5: Results of numerical simulations (see text for details). Filled symbols represent particles with uniform orientation in three dimensions, while open symbols represent particles uniformly oriented in a plane. The deviation of the autocorrelations of Stokes vectors from the value $1/3$ are indications of an inherent coupling of fields produced by this "last scattering layer" of confined cylindrical particles. The size of the symbols is larger than the error between different numerical runs of the same number of particles.
based on particle shape. For the cylindrical particles confined to a plane, there is noticeable deviation from the incident uniform distribution. This simple model provides an indication that the last scattering events can induce specific couplings in the field based on shape and orientational restrictions. These couplings are evident in the deviation from the uniform distribution of polarization states in the numerical model. The existence of such non-Gaussian distributions and the possible role of the last scattering layer is further supported by the experimental results shown below.

### 3.2.3 Experimental Results

To confirm the usefulness of the Stokes vector element correlations as they relate to morphological properties of diffusive scattering systems, various samples were experimentally examined. The experimental setup consists of a coherent, polarized light source (HeNe laser), a polarization state generator, a sample illuminated in transmission geometry, a microscope objective and collimating optics, a polarization state analyzer, imaging optics, and a CCD camera, as shown in Figure 3.6. The polarization analyzer makes use of the Fourier approach discussed in Chapter 1. Samples under consideration included an amorphous diffuser with various spherical and anisotropic powders placed in contact with its surface, sheets of diffusive material in various thicknesses, and an optical depolarizer. This allowed us to examine both surface and bulk effects on the Stokes vector element correlations. The polarization analysis was performed with a rotating quarter wave plate and fixed polarizer. The
Figure 3.6: Experimental setup comprised of coherent illumination, polarization state generator, sample, and polarization state analyzer.

magnification was sufficient to provide 100 pixels per speckle, with ~1800 speckles per image. This allowed for the complete recovery of the underlying spatial distribution of polarization states. The intensity of the illuminating laser was adjusted such that the average intensity was the same for all samples. A negative exponential distribution in the scalar intensity was recovered for all samples except the optical depolarizer.

Figure 3.7 illustrates some typical experimental results. The measured polarization state distributions for spherical particles, anisotropic particles, and the optical depolarizer, along with their corresponding Stokes vector element correlations are displayed. For all samples, the global degree of polarization was zero, the average Stokes vector elements, and Stokes vector element cross-correlations were also negligible. The size of the symbol is representative of experimental error. The deviation from Gaussian statistics is evident in the Stokes vector element correlations for all three samples. While deviation was not expected for the spherical particles on the surface of a bulk diffuser, inspection of the particles revealed distinct asphericity due to the compression mechanism. The platelet particles demonstrated distinct anisotropy, and the suppression of the circular state of polarization is indicative of randomly
Figure 3.7: Experimental results for three samples. The first sample is comprised of spherical particles on a bulk diffuser. The second is anisotropically shaped particles on the same bulk diffuser. The third is an optical depolarizer. Both the full ensemble shown on the OPS and the Stokes Vector Element Correlations serve to differentiate these globally unpolarized fields.
oriented partial linear polarizers. For both of these samples, the resulting distribution was independent of the incident illumination. For the optical depolarizer, the distribution depended greatly on the incident state of polarization, though the global degree of polarization seemed to be independent of initial state.

The results for the particles on the diffusive surface were similar to those generated numerically with the simple model of treating the bulk as a randomizer of an incident field on a last scattering surface. The spherical particles, although they demonstrate slight asphericity, do not duplicate a uniform distribution of polarization states as would be expected. It is possible then that the bulk diffuser plays a role in the statistics as well. The results for the bulk diffuser are shown in Figure 3.8. The top row indicates the initial state of illumination while the bottom row indicates the measured polarization state distribution after transmission through the bulk diffuser. The invariance to the initial state is a check on the diffusive regime.

While it is possible to describe the deviation from Gaussian statistics in terms of surface phenomena, it is interesting to consider the effect of the bulk scattering. For the numerical simulation, it was assumed that bulk diffuser acted as a source of Gaussian distributed fields. To test the effects of the bulk, statistics of samples with the same last scattering layer but with different bulk diffusers were examined. Specifically, diffusive samples formed from multiple layers were examined. The results for two materials are shown in Figures 3.9 and 3.10 for three different sample thicknesses. In each case, increasing the thickness of the sample reduced, but did not remove, the deviation from Gaussian statistics.
Figure 3.8: The top row depicts four distributions of states of polarization incident on the first surface of the bulk diffuser. The spread of the distributions is equivalent to the uncertainty in the state of polarization across the incident beam. The second row indicates the corresponding measured distribution across the second surface of the diffuser. The distributions are roughly uniform and there is no dependence on the incident state of polarization, confirming the assumption of high multiple scattering and diffusion-like illumination of the last surface.

Figure 3.9: Stokes vector element correlations for a sample with increasing thickness. The deviation from Gaussian statistics is reduced with sample thickness, but remains present in all the situations examined.
Figure 3.10: Stokes vector element correlations for a sample with increasing thickness. The deviation from Gaussian statistics is reduced with sample thickness, but remains present.

### 3.3 Fourth order point-pair statistics

While the Stokes vector element correlations provide a means to discriminate between various types of globally unpolarized fields, another forth order field correlator can be introduced to provide additional information pertinent to the structure of the random field. Forth order point-pair statistics serve to characterize the similarity in polarization information at two points in a distribution, and provide typical length scales in the structure of the field. This length may be in terms of physical distance, temporal distance, or any other mechanism that gives rise to the ensemble of field realizations under consideration. This polarization similarity length can be used to characterize various structures as well as optical materials. While the degree of coherence measures how possible it is for the field at two points to interfere, it is also interesting to ask how similar the state of polarization is at those
two points. In an attempt to answer this question, I have generalized the mathematical framework of the degree of polarization to account for point pairs. The analysis and physical significance of this measure are discussed below.

A generalized degree of polarization has been recently introduced [55] by Movilla et al. which takes into account the spatial variation of the state of polarization across a beam. While this quantity characterizes globally how similar the state of polarization across a beam is, it would also be useful to have a more comprehensive quantity that could measure the similarity between the states of polarization at any two arbitrary points. With this in mind, I introduce a two point quantity that is obtained from a direct generalization of the concept of degree of polarization [56]. It should be invariant to the choice of coordinate frame and it should also reduce to the classical definition of the degree of polarization when \( r_1 = r_2 \). Following the decomposition of the correlation matrix onto the basis formed by the identity matrix and the Pauli spin matrices

\[
W (r) = \rho_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \rho_1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \rho_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \rho_3 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},
\]

the electric field coherence matrix can be decomposed onto this same basis. For the two point electric field correlation matrix, a similar decomposition yields:

\[
W (r_1, r_2) = \nu_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \nu_1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \nu_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \nu_3 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},
\]
\[ \nu_1 = \frac{W_{11}(r_1, r_2, \tau) - W_{22}(r_1, r_2, \tau)}{2}, \]  
\[ \nu_2 = \frac{W_{12}(r_1, r_2, \tau) + W_{21}(r_1, r_2, \tau)}{2}, \]  
\[ \nu_3 = i \frac{W_{12}(r_1, r_2, \tau) - W_{21}(r_1, r_2, \tau)}{2}. \]  

(3.13)

where the \( \rho_i \) are obtained via intensity measurements \([1]\) and the \( \nu_i \) are obtained via visibility measurements. When \( r_1 = r_2 \), the \( \nu_i \) in Eq. (3.13) reduce to the \( \rho_i \) of Eq. (3.11), which are equivalent to the conventional Stokes vector elements.

In the traditional theory of polarization, i.e. at a particular point in space, the degree of polarization is defined in terms of measurable intensities as the ratio

\[ P^2 = \frac{\rho_1^2 + \rho_2^2 + \rho_3^2}{\rho_0^2} = 4 \frac{\rho_1^2 + \rho_2^2 + \rho_3^2}{I^2}, 0 \leq P \leq 1. \]  

(3.14)

Continuing the analogy, one can start from Eq. (3.13) and define a complex degree of mutual polarization (CDMP) between the light fluctuations at \( r_1 \) and \( r_2 \) as

\[ V^2(r_1, r_2, \tau) = 4 \frac{\nu_1^2 + \nu_2^2 + \nu_3^2}{I(r_1, t) I(r_2, t + \tau)}. \]  

(3.15)

I would like to emphasize that the complex degree of mutual polarization, \( V(r_1, r_2, \tau) \), expresses the relation between polarimetric quantities at two points in terms of measurable visibilities. Furthermore, it is a complex quantity, containing information in both its phase and magnitude.
It is worth considering other potential measures of polarization similarity. The Stokes vector has recently been generalized to a multi-point quantity \[57\]. This generalization is almost identical to the decomposition provided in Eq. (3.13). One can then define a degree of polarization similarly to the degree of polarization as defined from the standard Stokes vector. If we follow this analogy and define a generalized degree of polarization in the same manner as is done for the standard Stokes vector, the only difference from the complex degree of mutual polarization defined above is that the normalization is in terms of the visibility of the fields' interference instead of the intensity of the field at the two points. This difference in normalization has a rather important effect. While it is not possible for that sum of the squares of the last three generalized Stokes vector elements to exceed the intensity of the fields at the two points, it is possible for the sum to exceed the potential visibility of the interference of the two fields. Therefore, normalization by intensity results in a parameter that is bounded \(0 \leq V(r_1, r_2) \leq 1\), normalization to the visibility results in an unbounded quantity. In this regards the CDMP provides a truly normalized degree of mutual polarization.

For the sake of completeness, one should mention that other potential polarization similarity measures can be defined such as the inner product of Correlation matrices,

\[
Tr \{W(r_1) W(r_2)\}
\]  

(3.16)

and the dot product of Stokes vector elements
All of these measures have been shown to quantify in some regard the similarity between the states of polarization at two different points. It is interesting to note that all these three measures are identical under the condition that "fully correlated" fields are examined, i.e. that the time average in the correlation matrix may be neglected or, equivalently, that all the elements of the cross spectral density matrix factorize. This is the case for so-called fully developed speckle patterns where the field at any two points is fully coherent and fully polarized. When this is the case, the above measures may be expressed in terms of the eigenvalues and eigenvectors of the correlation matrices of the fields at the two points. In all three cases, the result is

$$ S(\mathbf{r}_1) \bullet S(\mathbf{r}_2). $$

(3.17)

where \( \lambda_{ij} \) is the \( j \text{th} \) eigenvalue of the correlation matrix at point \( i \) and \( \mathbf{v}_{ij} \) is the eigenvector associated with \( \lambda_{ij} \). As all of these polarization parameters are identical for the case under consideration, I will examine only the CDMP.

The CDMP does, however, have some deficiencies. These stem, essentially, from the semantic confusion in the current literature as to the nature of coherence, polarization, and correlations of fields [56, 58]. Classically, the degree of polarization of a field is a point quantity related to the maximal correlation between orthogonal field components [59], the
degree of coherence of a field is related only to the correlation of parallel field components and may be associated with the visibility of interference fringes [60], and the degree of correlation of a field measures in general how correlated all the components of a field are. Specifically, it is possible for a pair of fully correlated fields to have a zero degree of coherence, if they have orthogonal states of polarization. Taking these semantic definitions into consideration, one finds that while the CDMP does measure the similarity between the state of polarization at two points, it is bounded by the correlation of the fields at those points. While for the application of a fully correlated speckle field this is not a problem, it may be useful to define a polarization similarity measure in terms of point quantities without regard to spatial/temporal correlations. In this regard, there are two potential polarization state comparison measures that may be of interest. The first is that expressed in Eq. (3.18), which the CDMP reduces to in the case of fully correlated fields. This measure indicates how similar the independent realizations of the equivalent ensemble representations of the fields at the two points are. Another possible measure would be the ordered comparison of polarization states, given by

$$\sum \lambda_{1i} \lambda_{2i} \nu_{1i} \cdot \nu_{2i}$$

(3.19)

which provides a measure of how similar the ordered pairs of independent realizations are weighted by their relative intensities. Both of these eigenvector comparisons have the benefit that they are independent of the frame of reference in which the field is described. Furthermore, they are fully generalizable to any number of points or fields of any dimension. The value of the second measure over the first is that it will discriminate between different
types of unpolarized light. This is to say that if the fields at two points under consideration are unpolarized, but are formed from the incoherent superposition of different pairs of orthogonal states of polarization, the similarity as measured by Eq. (3.18) would be greater than that measured by Eq. (3.19). As I only consider global distributions comprised of locally pure states of polarization, these two measures are identical, and I will refer only to the CDMP in the following sections.

3.3.1 Theoretical Considerations

I will now examine the complex degree of mutual polarization corresponding to various cases of fully correlated light; that is light for which the elements of the cross spectral density matrix factorize and, as a consequence, is fully polarized at the two points \( r_1 \) and \( r_2 \). Furthermore, as I consider fully correlated fields, the time dependence for the electric fields at the two points differ by a constant \( \arg(\mathbf{E}(r_2)) = \arg(\mathbf{E}(r_1)) + \beta \). This phase difference appears in both the cross spectral density matrix, \( W \), and the complex degree of mutual polarization, \( V \). Figure 3.11 summarizes the details for different cases for the state of polarization at two points. The first row of Figure 3.11 illustrates the case when the state of polarization at the two points is the same linear state with some constant phase difference \( \beta \). The magnitude of \( V(r_1, r_2, \tau) \) is unity, as one would expect for the same state of polarization, while the phase of \( V(r_1, r_2, \tau) \) is exactly the fixed phase difference \( \beta \). The second example represents the case of different linear polarizations in \( r_1 \) and \( r_2 \), again related by the constant
Figure 3.11: Results of applying CDMP to various field configurations. The first two columns represent fields at two separate points. The third column gives the cross spectral density matrix of the fields, while the last column gives the value of the CDMP. As can be seen, CDMP provides a measure of how similar the state of polarization of two correlated fields are.

| $\mathbf{E}(\mathbf{r}_1)e^{-i\phi(t)}$ | $\mathbf{E}(\mathbf{r}_2)e^{-i(\beta+\phi(t))}$ | $w(\mathbf{r}_1, \mathbf{r}_2)e^{-i\beta}$ | $|v^2(\mathbf{r}_1, \mathbf{r}_2)|$ |
|--------------------------------|--------------------------------|--------------------------------|------------------|
| $E_{x1}\hat{x}$ | $E_{x2}\hat{x}$ | $\begin{bmatrix} E_{x1} & 0 \\ 0 & 0 \end{bmatrix}$ | 1 |
| $E_{x1}\hat{x}$ | $E_{x2}\hat{x} + E_{y2}\hat{y}$ | $\begin{bmatrix} E_{x1} & E_{x1}E_{y2} \\ 0 & 0 \end{bmatrix}$ | $\frac{E_{x1}^2}{E_{x1}^2 + E_{y2}^2}$ |
| $E_{x1}\hat{x}$ | $E_{y1}\hat{x} + iE_{x2}\hat{y}$ | $\begin{bmatrix} E_{x1} & iE_{x1}E_{y1} \\ 0 & 0 \end{bmatrix}$ | $\frac{1}{2}$ |
| $E_{x1}\hat{x}$ | $E_{y1}\hat{x} + E_{y2}\hat{y}$ | $\begin{bmatrix} E_{x1} & -E_{x1}e^{ia} \\ E_{y1}e^{-ia} & -E_{x1}E_{y1} \end{bmatrix}$ | 0 |
| $E_{x1}\hat{x}$ | $E_{x2}\hat{x} + E_{y2}\hat{y}$ | $\begin{bmatrix} E_{x1}E_{x2} & E_{x1}E_{y2}e^{ia2} \\ E_{y1}e^{-ia1} & E_{y1}E_{y2} \end{bmatrix}$ | $\frac{(E_{x1}E_{x2} - E_{x1}E_{y2})^2}{(E_{x1}^2 + E_{y1}^2)(E_{x2}^2 + E_{y2}^2)} + \frac{4E_{x1}E_{x2}E_{x1}E_{y2}e^{ia2 - ia1}}{(E_{x1}^2 + E_{y1}^2)(E_{x2}^2 + E_{y2}^2)}$ |

The last example corresponds to the case of general elliptical states at both points $\mathbf{r}_1$ and $\mathbf{r}_2$. As can be seen, $|V|$ is unity when the two states are the same and 0 when the states are orthogonal, while the phase of $V$ is indicative of the relative phase of the fields at the two points.

phase difference $\beta$. The magnitude of $V(\mathbf{r}_1, \mathbf{r}_2, \tau)$ is now a ratio related to how similar the second linear state is to the first. The third row demonstrates the relationship between linear and circular polarizations, while the fourth row shows that, for orthogonal states of polarization with a constant phase difference, the degree of mutual polarization $V$ is zero.
To gain more insight into the meaning of this degree of polarization similarity, let us examine the interference of two fully correlated beams. While the intensity at a given point

\[
I(\mathbf{r}) = I_1(\mathbf{r}) + I_2(\mathbf{r}) + 2 \text{Re} \{ \mathbf{E}_1 \cdot \mathbf{E}_2^* \exp[i(\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{r}] \}
\]  

(3.20)

is proportional to both the dot product of the states of polarization with a periodicity that depends on the relative angle between the two beams, the CDMP

\[
V(0, \mathbf{r}) = F(\mathbf{E}_1 + \mathbf{E}_2, \mathbf{E}_1 \exp[i\mathbf{k}_1 \cdot \mathbf{r}] + \mathbf{E}_1 \exp[i\mathbf{k}_2 \cdot \mathbf{r}])
\]  

(3.21)

has a much more complicated functional form, though the periodicity is the same as for the intensity measurement. Let us first consider the case of arbitrarily polarized, fully correlated beams propagating at an angle \(\theta\) with respect to each other. The intensity modulation in this case is not unity due to the fact that the beams do not have the same state of polarization. The period of modulation in intensity and CDMP is identical, and provides information concerning the angle between the interfering beams. However, the CDMP has a higher information content in the shape of the modulated portion. Specifically, there is information pertinent to the relative ellipticities in the CDMP that is absent in the intensity pattern.

The second case illustrated is that of the interference between two orthogonally polarized, fully correlated beams. While there is no intensity fluctuation, there is a fluctuation in the state of polarization. This can be seen by choosing a fixed point and examining how the state
of polarization varies spatially in regards to this point. The CDMP recovers the modulation that would have been present if the state of polarization for the two beams were identical. That is to say that the angle between the two beams is recovered in the modulation of the CDMP, even though there is no intensity modulation. Figure 3.12 depicts the calculated intensity (red) and CDMP (blue) for these two cases.

3.3.2 Experiment

The same experimental setup, shown in Figure 3.6, used to measure the Stokes vector element correlations can also be used to calculate the CDMP for various samples. While the Stokes vector element correlations indicate global morphological structure of different scattering media, further insight can be gained by examining the CDMP [61]. For instance, in the case of media consisting of compressed particles, the CDMP variation followed closely the fluctuations of intensity, demonstrating that while the speckles were polarized, there was
very little correlation between the state of polarization from speckle to speckle. This is to be expected given that spatially random nature of the distribution of particles on the last surface layer. A totally different structure was revealed when an optical depolarizer, comprised of two crystal quartz wedges, was examined. The results shown in Figure 3.13 reveal the measured value of $V(0, r)$ for this sample.

While the intensity across the polarizer is constant, the state of polarization varies spatially. As can be seen, the structure of the depolarizer is revealed under CDMP analysis. Specifically, the state of polarization is constant along one direction, while in the perpendicular direction, the state of polarization varies continuously through a great circle on the OPS, providing a unity modulation depth in CDMP.
3.4 Conclusions

Motivated by the practical limitations of classical polarimetry illustrated in Chapter 1, the traditional polarimetric approaches were expanded to include higher order field correlations and the description of the ensemble in terms of the full distribution of polarization states. This was necessitated by the lack of a unique equivalent ensemble representation in the case of globally unpolarized fields. Various global and point characterizations of globally unpolarized ensembles of field realizations have been examined in this Chapter, both theoretically and experimentally. Experimentally accessible global measures include the classical parameters of degree of polarization, the classical state of polarization, the entropy of the equivalent ensemble representation, and the actual entropy of the full ensemble. The degree of polarization and state of polarization are global, point-like measures of the polarimetric properties of the field. When these second order field correlations can not discriminate between systems of interest, higher order correlations should, and can, be examined.

Polarimetric investigation of fully resolved speckle patterns provides a direct measure of the joint distribution of Stokes vector elements. In the case of highly diffusive media, Gaussian statistics are usually invoked to describe the ensemble of field realizations and subsequently determine the polarization state distribution. However, it has been shown that various types of globally unpolarized ensembles of fields can arise in practice, and that second order polarization correlations (i.e. forth order field correlations) can be used as a means for
discriminating between various ensembles of field realizations. Specifically, the Stokes vector element correlations were introduced to characterize these various globally unpolarized fields.

A simple model describing specific surface interactions was introduced. Numerical simulations based on this model indicate the possibility to observe material dependent correlations between components of the scattered field. Experimental measurements for various diffusive, scattering, and depolarizing samples were obtained. The experimental results for various scatterers show general agreement with the trends predicted by the numerical calculations, demonstrating the usefulness of these forth order correlations for discriminating between different globally unpolarized regimes and as descriptors of subtle morphological properties of materials.

To gain greater insight into the structure of these random fields, the Complex Degree of Mutual Polarization was introduced. This measure serves as a means for characterizing the polarization similarity between two different points in a field. For completeness, various other polarization similarity measures were considered, and it was noted that in the case of fully polarized fields, they provided identical results. Both theoretical and experimental examination demonstrated the effectiveness of the CDMP in regards to structural characterization of the systems under consideration.

In all the polarimetric treatments so far, we were concerned only with projections of optical fields onto the plane of a sensing system. The general theory does not consider this, as it is formulated under the assumption of propagating, transverse electromagnetic fields. While it is common practice in classical polarimetric measurements to ignore any component
of the field normal to the detection system, there are many practical situations of current interest where this component is non-negligible. Such situations range from focusing of high energy systems, to near field optics, and fields inside fluorescing and scattering volumes. In all these cases, the fields are not confined to the plane of the detection system, and, furthermore, it is possible for these fields to have a fully three-dimensional nature. This is to say, it is possible for these fields to fluctuate independently in three perpendicular directions. If classical measurement techniques are applied, only a projection of the actual field will be recovered. This is a major limitation of classical polarimetry: it can not be applied to non-transverse fields or fields for which the direction of propagation is not known a priori. In the following Chapter, the steps necessary to expand the polarimetric principles to handle three dimensional fields and the necessary experimental procedures to measure them will be considered.
CHAPTER 4
3D POLARIMETRY

Classical polarimetry is concerned with propagating electromagnetic fields. This is because polarimetric measurements of the field’s properties are usually done by propagating the field through optical elements towards a detector. While it is true that a propagating electromagnetic field is transverse, a broader scope of polarimetry can be thought of where the fields are not necessarily propagating.

In the last Chapter, the measurement of the full ensemble of field realizations and the resulting characterization of highly diffusive and random media in terms of higher order field correlations were considered. The very nature of a highly diffusive material gives rise to a field which has no unique direction of propagation. Other practical examples of such fields include numerous classical and novel configurations in \cite{2, 6, 62, 63, 64}, such as high energy fields, tight focusing, fields inside fluorescing volumes, evanescent fields, and others.

In these situations, it is necessary to expand the classical concepts and formalisms of polarimetry. This is a non-trivial task which has received much recent attention \cite{38, 65, 66, 67}. In this Chapter, a generalization for both deterministic and random fields will be introduced. The mathematical generalization maintains the physical significance of the polarimetric properties. It is the physical interpretation of the formalism that is of greatest
interest. Specific attention will be given to the form of polarized fields, the meaning of the degree of polarization, and it will be demonstrated that higher dimensional fields require more parameters to fully describe them. Various possible measures for specifying the three dimensional fluctuating fields will be examined. Finally, the possibility to measure these quantities will be demonstrated, and several potential applications will be reviewed.

4.1 Deterministic Fields

A field is said to be deterministic if a fixed phase relationship between all orthogonal field components at a point exists. Such a field can be described completely in terms of five parameters; three amplitudes and two phase differences

\[
\mathbf{E}(\mathbf{r}, \omega) = \begin{pmatrix}
E_x(\mathbf{r}, \omega) \\
E_y(\mathbf{r}, \omega) e^{i\phi_y(\mathbf{r}, \omega)} \\
E_z(\mathbf{r}, \omega) e^{i\phi_z(\mathbf{r}, \omega)}
\end{pmatrix} e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})},
\]

(4.1)

where \(E_x, E_y, E_z, \phi_y\), and \(\phi_z\) are real quantities that depend on position and frequency. It is interesting to note that while this field appears to have two more degrees of freedom than the deterministic case of classical polarimetry, the electric field vector is still confined to an ellipse and can thus be considered fully polarized. These extra degrees of freedom merely specify the orientation of the plane of polarization in three dimensions. It is important to realize that a deterministic field, no matter the dimensionality of the reference frame, is inherently a two-dimensional, polarized field. This is easily seen via the Jones calculus [2].

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Figure 4.1: Polarization ellipse for a deterministic field in 3 Dimensions. While a polarized field oscillates on an ellipse, the plane of the ellipse now has an arbitrary but fixed orientation in three dimensions.

4.1.1 Jones Calculus

A deterministic field is most easily described via the generalized Jones vector, given in Eq. (4.1). The polarization ellipse, shown in Figure 4.1, can be found, as in the case of the two dimensional field, by solving for the real electric field vector. An equivalent procedure for calculating the polarization ellipse of an arbitrary deterministic field in three dimension can be found in Section 1.4.3 of reference [3]. A brief outline of this procedure is presented here.

Any complex electric field of the form

$$E(r, \omega) = E(r, \omega) e^{i(\omega t - k r)}$$

(4.2)

can be expressed as the sum of two fields
\[
E(r, \omega) = (p + i q) e^{i(\omega t - k \cdot r)} = \left[ \text{Re} \{ E(r, \omega) \} + \text{Im} \{ E(r, \omega) \} \right] e^{i(\omega t - k \cdot r)}. \tag{4.3}
\]

The major and minor axis of the polarization ellipse in Figure 4.1 are then given in terms of

\[
\begin{align*}
a &= p \cos(\varepsilon) + q \sin(\varepsilon), \\
b &= -p \sin(\varepsilon) + q \cos(\varepsilon), \\
\tan(2\varepsilon) &= \frac{2p \cdot q}{p^2 - q^2}.
\end{align*}
\tag{4.4}
\]

The most important observation from this argument is that a polarized field is one for which the field vector is confined to an ellipse, and that this is equivalent to the existence of a fixed phase and amplitude relationship between the orthogonal components of the field vector. The nature of a polarized field will be further examined in terms of the Correlation matrix formalism below.

### 4.2 Random Fields

While the generalization of a polarized field to three dimensions is straightforward, the generalization of random fields is somewhat more complicated. Access to the individual members of the ensemble of field realizations fully specifies all properties of a random field. However, it is often more convenient to work only with the equivalent ensemble representation.
The focus of the following discussion will be restricted to the discussion to this equivalent ensemble described in terms of the second order field correlations, as presented in Chapter 1. In two dimensions, it was shown that a field at a point can be uniquely represented as the sum of a fully polarized field and an unpolarized field. More fundamentally a field can be represented as an ensemble composed of two uncorrelated, fully polarized, orthogonal field realizations. To examine the generalization of this equivalent ensemble to three dimensional random fields, we will start by examining the extension of a polarized field to three dimensions.

In general, it is considered that light whose electric field oscillates in a particular way is polarized. More specifically, an electric field at a point \( \mathbf{r} \) is said to be polarized if and only if the electric field vector traces out an ellipse with increasing time. It is clearly understood that a monochromatic wave is fully polarized at every point \([68, ?]\). However, this is merely a sufficient and not a necessary condition as we will show later. If \( \mathbf{e}(\mathbf{r}, \omega) \) is a complex unit amplitude vector and \( U(\mathbf{r}, \omega) \) an amplitude weighting factor, the polarized nature of a monochromatic field written as

\[
\mathbf{E}(\mathbf{r}, \omega, t) = \mathbf{e}(\mathbf{r}, \omega) U(\mathbf{r}, \omega) e^{i\omega t}, \tag{4.5}
\]

follows from the fact that, at each point, the orthogonal field components, \( E_x, E_y, \) and \( E_z \), have a fixed phase relationship and oscillate at the same frequency. This implies that the end point of the electric field vector is confined to an ellipse (cf. \([3, \text{ Sec. 1.4.3}]\) with the state of polarization depending only on \( \mathbf{e}(\mathbf{r}, \omega) = \mathbf{p}(\mathbf{r}, \omega) + \mathbf{i} \mathbf{q}(\mathbf{r}, \omega) \). The plane of this
polarization ellipse is not, in general, confined to the xy-plane, and is depicted in Figure 4.1. One concludes that for three-dimensional fields which are monochromatic, the electric field vector traces always an ellipse with increasing time and hence they are polarized. Of course, the two vectors \( p(\mathbf{r}, \omega) \) and \( q(\mathbf{r}, \omega) \) depend on the spatial position and, therefore, they define a plane of polarization which, in general, can vary from point to point.

A completely monochromatic field is an idealization. Real optical fields are fluctuating and can only be represented by statistical ensemble of realizations. In this context, the question at hand is: can fluctuating fields be polarized? In order to answer it, let us first consider a statistically stationary (in the wide sense) ensemble \( \{ E(\mathbf{r}, \omega, t) \}_\alpha \) (cf. [5], Sec. 4.7.2) of realizations of the electric field at the point \( \mathbf{r} \) and at frequency \( \omega \). Each realization \( E^\alpha \) can be written as

\[
E^\alpha(\mathbf{r}, \omega, t) = e^\alpha(\mathbf{r}, \omega) U^\alpha(\mathbf{r}, \omega) e^{i\omega t}
\]

(4.6)

where the \( e^\alpha(\mathbf{r}, \omega) \)'s are complex, unit amplitude vectors which depend on position and represent the specific state of polarization of the realization, while the \( U^\alpha(\mathbf{r}, \omega) \)'s are the corresponding amplitude weighting functions. It should be noted that both \( e^\alpha(\mathbf{r}, \omega) \) and \( U^\alpha(\mathbf{r}, \omega) \) are random functions of the realization \( \alpha \). Eq.(4.6) represents the most general statistical ensemble, each realization being a fully polarized field having a certain amplitude. In the case of three-dimensional field distributions, any polarized field can be written as the superposition of three orthogonally polarized fields with specific phase and amplitude relationships (this is just a more general version of the decomposition of a vector into its
Cartesian components). We can therefore decompose all members of the statistical ensemble into the same basis \((i, j, k)\) as

\[
\mathbf{E}^\alpha (\mathbf{r}, \omega, t) = E_i^\alpha (\mathbf{r}, \omega, t) + E_j^\alpha (\mathbf{r}, \omega, t) + E_k^\alpha (\mathbf{r}, \omega, t) =
\]

\[
= \left[ e_i (\mathbf{r}, \omega) U_i^\alpha (\mathbf{r}, \omega) + e_j (\mathbf{r}, \omega) U_j^\alpha (\mathbf{r}, \omega) + e_k (\mathbf{r}, \omega) U_k^\alpha (\mathbf{r}, \omega) \right] e^{i\omega t},
\]

where the \(e_l\)'s \((l = i, j, k)\) are the complex, orthogonal basis vectors of unit magnitude, \(E_l^\alpha (\mathbf{r}, \omega)\) is the projection of \(\mathbf{E}^\alpha (\mathbf{r}, \omega, t)\) onto \(e_l (\mathbf{r}, \omega)\), and the \(U_l^\alpha\)'s are amplitude weighting functions as in Eq. (4.6). As the \(e_l\)'s represent an arbitrary basis for the states of polarization, the corresponding amplitude weighting functions, \(U_l^\alpha\), will depend on the specific choice for this basis. With this notation, one can define for each realization \(\alpha\) the coupling coefficients

\[
E_i^\alpha (\mathbf{r}, \omega, t) = c_{i,j}^\alpha (\mathbf{r}, \omega) E_j^\alpha (\mathbf{r}, \omega, t) \tag{4.9}
\]

that describe the deterministic relationship between the phases and amplitudes of the orthogonal components. It is also apparent that, in general, the average magnitudes of the components are not equal, i.e. \(\langle |E_i (\mathbf{r}, \omega, t)| \rangle_\alpha \neq \langle |E_j (\mathbf{r}, \omega, t)| \rangle_\alpha\).

The statistical ensemble of Eq. (4.6) can be reduced to the case of polarized light by applying the simple restriction \(e^\alpha (\mathbf{r}, \omega) = e (\mathbf{r}, \omega)\). That is to say that the state of polarization is the same for each realization \(\alpha\) of the field and only the weighting functions are random. In these conditions, it follows at once from Eq.(4.6) that the ratio
is a deterministic constant independent of the realization $\alpha$. This is true for any pair $(i, j)$ of orthogonal vectors of the basis for the state of polarization. Two realizations for which the ratios $E_i^\alpha / E_j^\alpha$ have the same (deterministic) value for all pairs of orthogonal basis vectors $i$ and $j$ are said to be "statistically similar", i.e. they have the same fixed phase relationship between any pair of orthogonal field components. Using these deterministic relationships between the vector components one can define an "equivalent electromagnetic field"

$$\mathcal{E}(\mathbf{r}, \omega, t) = \langle \mathbf{E}(\mathbf{r}, \omega, t) \rangle _\alpha = \mathbf{e}(\mathbf{r}, \omega) \sqrt{\langle |U^\alpha(\mathbf{r}, \omega)|^2 \rangle _\alpha} e^{i\omega t}$$

which is similar to the expression in Eq. (4.5) and will therefore be fully polarized at each point $\mathbf{r}$.

I conclude that the necessary condition for both two- and three-dimensional fluctuating fields to be polarized is that their orthogonal field components have a deterministic phase and amplitude relationship, i.e. that the ensemble is composed of statistically similar realizations. In this case, one can define an equivalent field which is fully polarized. As in the case of monochromatic fields, the plane and the shape of the polarization ellipse can depend on the spatial position.

Now that the nature of a polarized field in three dimensions in understood, the characteristics of fluctuating three dimensional fields with no unique polarized component must
be described. Let us examine now the concept of **unpolarized** fields. To examine what is actually meant by an unpolarized field, we return to the statistical ensemble represented in Eq. (4.6) where each member is itself polarized because they all have the form of Eq. (4.5). An unpolarized ensemble of field realizations will be one for which there is no polarized component.

The equivalent field, $\mathcal{E}(r, \omega, t)$ of Eq. (4.11), can be viewed as the simplest equivalent ensemble which fully reconstructs the second order average properties of the original field. In the case of a polarized field, this simplest ensemble has only a single realization. To replicate the second order average properties of a general ensemble $\{\mathbf{E}(r, \omega, t)\}_\alpha$ composed of fully polarized realizations oriented in a three dimensional space, a minimum of three polarized members are necessary. The fact that any ensemble of three dimensional fields can be represented by an equivalent ensemble comprised of three independent realizations in orthogonal states of polarization can easily been seen using the correlation matrix formalism as described below. This formalism provides the means for isolating the basis $(i, j, k)$ in which in which the fluctuations of the orthogonal field components $\mathbf{E}_l$ are independent. Being members of an ensemble, each of the three realizations are statistically independent, hence their cross-correlations will be zero, while the autocorrelations of the three polarized members of the equivalent ensemble will be given by the average intensities of the original ensemble. The equivalent ensemble can then be expressed as

\[
\begin{align*}
\left\{ \mathbf{E}_l = c_l (r, \omega) \sqrt{\langle |U_l^\alpha (r, \omega)|^2 \rangle} e^{i\omega t}, l = (i, j, k) \right\} & \leftrightarrow \{ \mathbf{E}(r, \omega, t) \}_\alpha . \quad (4.12)
\end{align*}
\]
In general, the intensities of the three polarized members of the ensemble will differ, and they can be uniquely ordered as

\[ \langle |E_i(r, \omega)|^2 \rangle_\alpha \geq \langle |E_j(r, \omega)|^2 \rangle_\alpha \geq \langle |E_k(r, \omega)|^2 \rangle_\alpha \geq 0. \quad (4.13) \]

The polarized component of the field will be given by the portion of the realization that is in excess of the other two. That is to say, the polarized component can be written as

\[ E_p(r, \omega, t) = e_i(r, \omega) \sqrt{\langle |U_i(r, \omega)|^2 \rangle_\alpha - \langle |U_j(r, \omega)|^2 \rangle_\alpha} e^{i\omega t}. \quad (4.14) \]

An unpolarized field may then be defined as a field which has no polarized component, i.e. one for which the average intensities of the two maximal members of the equivalent ensemble are equal. Of course, polarized and unpolarized light are two extreme cases, and a real random field will in general have components of both. In other words, one can say that in general, there will be a basis \((i, j, k)\) such that the fluctuations are not statistically similar and their average magnitudes are different. Following the procedure outlined here, the unique polarized component of the equivalent ensemble can always be isolated. One possible way to characterize such a field is to answer the question as to what part of the total intensity of the field is contained in the polarized component. Practically this means to define a degree of polarization \(P\) of the field which a measure of the relative intensity of the polarized component to the intensity of the total field. Using the results of the field
decomposition outlined here, this measure can be defined as

\[
P = \frac{\langle |E_i(\mathbf{r}, \omega)|^2 \rangle_\alpha - \langle |E_j(\mathbf{r}, \omega)|^2 \rangle_\alpha}{\langle |E_i(\mathbf{r}, \omega)|^2 \rangle_\alpha + \langle |E_j(\mathbf{r}, \omega)|^2 \rangle_\alpha + \langle |E_k(\mathbf{r}, \omega)|^2 \rangle_\alpha}.
\]  

(4.15)

The procedure for determining the three members of the equivalent ensemble, and hence the unique polarized component, is greatly simplified by the use of the correlation matrix formalism. The next section will discuss this approach.

4.2.1 Correlation Matrix

The difficulty in extending the correlation matrix formalism to three dimensions does not lay in the mathematics, but rather in maintaining the physical significance. The definition of the matrix elements do not change at all, and we have

\[
W_{i,j} = \langle E^*_i E_j \rangle,
\]

(4.16)

where \(i\) and \(j\) again denote polarization basis vectors, with the generalization occurring from the fact that these basis vectors now span a three dimensional space.

While decompositions of this generalized correlation matrix have been previously introduced, my contribution to the formalism is in the interpretation of the possible decompositions and the isolation of physically significant parameters. While most generalizations start by considering a three-dimensional equivalent of an unpolarized field, this leads to problems
as the generalization is not unique. Therefore, starting with the polarized portion of the field provides greater insight.

As shown above, a polarized field is one for which the field vector moves on an ellipse. This is equivalent to the statement that every element of the correlation matrix factorizes [2]. Factorization is equivalent to the existence of a single non-zero eigenvalue, which is proportional to the intensity of the field. The eigenvector associated with this non-zero eigenvalue is the complex conjugate of the Jones vector of the field, and as such, specifies the state of polarization. To see this, let us first examine a correlation matrix for which all elements factor,

\[
W_{ij} (r, \omega) = E_i^* (r, \omega) E_j (r, \omega). \tag{4.17}
\]

The eigenvalues are solutions of the equation

\[
\det \left[ W (r, \omega) - \lambda I \right] = 0, \tag{4.18}
\]

where \( \det \) denotes the determinant and \( I \) is the unit matrix. On substituting from Eq. (4.17) into Eq. (4.18) we find that:

\[
(|E_x|^2 - \lambda) \left[ (|E_y|^2 - \lambda) (|E_z|^2 - \lambda) - |E_y|^2 |E_z|^2 \right] - E_x^* E_y^* E_x \left( |E_z|^2 - \lambda \right) - |E_z|^2 E_y^* E_x
\]

\[
+ E_x^* E_z \left[ |E_y|^2 E_z^* E_x - E_z^* E_x \left( |E_y|^2 - \lambda \right) \right] = 0, \tag{4.19}
\]

where the arguments of \( E_x, E_y, E_z \) are of course \( r \) and \( \omega \).
After lengthy but straightforward calculation, Eq. (4.19) can be shown to imply that

\[
\lambda^2 \left[ \lambda - \text{Tr} \hat{W}(\vec{r}, \omega) \right] = 0,
\] (4.20)

where \(\text{Tr}\) denotes the trace. Eq. (4.20) shows that the matrix \(\hat{W}(\vec{r}, \vec{r}, \omega)\) has only one non-zero eigenvalue which is proportional to the average electric energy density of the field at the point \(\vec{r}\) because

\[
\lambda_1 = \text{Tr} \hat{W}(\vec{r}, \omega) = \langle E^*_x(\vec{r}, \omega) E_x(\vec{r}, \omega) \rangle + \langle E^*_y(\vec{r}, \omega) E_y(\vec{r}, \omega) \rangle + \langle E^*_z(\vec{r}, \omega) E_z(\vec{r}, \omega) \rangle.
\] (4.21)

To explicitly determine the state of polarization represented by a cross-spectral density matrix each element of which factorizes, it is necessary to transform (rotate) the coordinate system appropriately.

In general, the electric field at the point \(\vec{r}\) will be elliptically polarized, and the eigenvector \(\vec{v}_1\) corresponding to the non-zero eigenvalue \(\lambda_1\), is a complex vector. Just as the eigenvalue is associated with the electric energy density of the field at the point, the eigenvector \(\vec{v}_1\) can be associated with the equivalent electric field. Up to a constant, \(\vec{v}_1 = \left[ E^*_x, E^*_y, E^*_z \right]^T\),

where \(\overrightarrow{\text{T}}\) denotes the transpose, which is just the complex conjugate of the equivalent electric field at the point \(\vec{r}\). A treatment similar to that given in Sec. 1.4.2 of reference [3] provides the polarization ellipse of the electric field at the point \(\vec{r}\). Presented here is an equivalent argument via real rotations defined in terms of the component of the eigenvector.
(equivalently, in terms of the electric field components). This rotation is given by a matrix of the form

\[
R = \begin{bmatrix}
\cos (\theta) & 0 & \sin (\theta) \\
0 & 1 & 0 \\
-\sin (\theta) & 0 & \cos (\theta)
\end{bmatrix}
\begin{bmatrix}
\cos (\alpha) & \sin (\alpha) & 0 \\
-\sin (\alpha) & \cos (\alpha) & 0 \\
0 & 0 & 1
\end{bmatrix},
\]

where the angles of rotation are

\[
\theta (r, \omega) = \arctan \left( \frac{\mathcal{E}_z (r, \omega)}{\mathcal{E}_x (r, \omega) \cos (\alpha) + \mathcal{E}_y (r, \omega) \sin (\alpha)} \right)
\]

\[
\alpha (r, \omega) = \arctan \left( \frac{|\mathcal{E}_x (r, \omega)| \sin [\phi_z (r, \omega) - \phi_x (r, \omega)]}{|\mathcal{E}_y (r, \omega)| \sin [\phi_y (r, \omega) - \phi_z (r, \omega)]} \right).
\]

(4.22a)

The cross-spectral density matrix in the new coordinate system is:

\[
\tilde{W} (r, \omega) = R W (r, \omega) R^{-1} = 
\begin{bmatrix}
\mathcal{E}_{x'}^* (r, \omega) \mathcal{E}_{x'} (r, \omega) & \mathcal{E}_{y'}^* (r, \omega) \mathcal{E}_{y'} (r, \omega) & \mathcal{E}_{z'}^* (r, \omega) \mathcal{E}_{z'} (r, \omega) \\
\mathcal{E}_{y'} (r, \omega) \mathcal{E}_{x'} (r, \omega) & \mathcal{E}_{y'}^* (r, \omega) \mathcal{E}_{y'} (r, \omega) & \mathcal{E}_{z'}^* (r, \omega) \mathcal{E}_{z'} (r, \omega) \\
\mathcal{E}_{z'} (r, \omega) \mathcal{E}_{x'} (r, \omega) & \mathcal{E}_{z'}^* (r, \omega) \mathcal{E}_{y'} (r, \omega) & \mathcal{E}_{z'}^* (r, \omega) \mathcal{E}_{z'} (r, \omega)
\end{bmatrix},
\]

(4.23)

where
\[ E_x'(r, \omega) = E_x(r, \omega) \cos(\theta) \cos(\alpha) + E_y(r, \omega) \cos(\theta) \sin(\alpha) - E_z(r, \omega) \sin(\theta), \]  

(4.24a)

\[ E_y'(r, \omega) = -E_x(r, \omega) \sin(\alpha) + E_y(r, \omega) \cos(\alpha), \]  

(4.24b)

\[ E_z'(r, \omega) = -E_x(r, \omega) \sin(\theta) \cos(\alpha) - E_y(r, \omega) \sin(\theta) \sin(\alpha) + E_z(r, \omega) \cos(\theta). \]  

(4.24c)

On substituting for \( \theta \) and \( \alpha \) from Eq. (4.22) into Eq. (4.24), one finds that \( E_z'(r, \omega) = 0 \), which implies that the electric field is confined to the \( x', y' \) plane. It should be noted that while the ratio \( E_z(r, \omega) / [E_x(r, \omega) \cos(\alpha) + E_y(r, \omega) \sin(\alpha)] \) appears to be a complex quantity, its imaginary component is identically zero, and hence both \( \theta \) and \( \alpha \) are real angles. I have thus reduced the matrix representation of the three-dimensional field at a point \( r \) to that of a locally two-dimensional field represented by the matrix

\[
\begin{bmatrix}
E_{xx}^*(r, \omega) E_{xy}^*(r, \omega) & E_{yx}^*(r, \omega) E_{yy}^*(r, \omega) & 0 \\
E_{yx}^*(r, \omega) E_{xy}^*(r, \omega) & E_{yy}^*(r, \omega) E_{yy}^*(r, \omega) & 0 \\
0 & 0 & 0 
\end{bmatrix}.
\]  

(4.25)

The \( 2 \times 2 \) submatrix in Eq. (4.25), with the factorized terms, will be recognized as a correlation matrix of a completely polarized field confined to the \( x', y' \) plane through the point \( r \) [3].

Having understood the form of a polarized field, it is then possible to further investigate the information contained in the correlation matrix. As the correlation matrix is Hermitian, it can be diagonalized via a unitary transformation. The unitary matrix which
diagonalizes the correlation matrix is, in fact, the matrix of eigenvectors of the correlation matrix, \( \mathbf{V}(r, \omega) \). Because the eigenvalues are all real, and positive \([63]\), they can be ordered \( 1 \geq \lambda_2 \geq \lambda_3 \geq 0 \). The correlation matrix can then be expressed in terms of its "natural" basis, i.e. the three orthogonal states of polarization which are uncorrelated,

\[
\mathbf{W} = \lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + \lambda_3 \mathbf{w}_3 = \mathbf{V}^\dagger \left( \lambda_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \mathbf{V}
\]  

(4.26)

\[
\mathbf{W} = \sum \lambda_i w_i
\]  

(4.27)

where the \( \lambda_i \) are the ordered eigenvalues of the correlation matrix, and the \( w_i \) are correlation matrices of unit intensity polarized fields. Examining the form of these matrices, we find that the eigenvalues are proportional to the intensities of the orthogonal states, and the eigenvectors represent the orthogonal states (they are, in fact, the complex conjugate of the Jones vectors of the three orthogonal states).

This decomposition then provides insight into a determination of the polarized component of an arbitrary random field. A polarized field is one for which the correlation matrix factorizes. We can uniquely identify that portion of the maximal field component which is in excess of the other two. This portion represents the unique polarized portion of the field, i.e.
\[ \mathbf{W} = (\lambda_1 - \lambda_2) w_1 + (2\lambda_2 + \lambda_3) w_u \] (4.28)

where \( w_1 \) is the correlation matrix of a polarized field as above, and \( w_u \) is a correlation matrix representing the unit intensity unpolarized contribution. While the form of a correlation matrix pertaining to a polarized field can be uniquely expressed (i.e. all elements factorize), the treatment unpolarized portion is more complicated. In fact, this component can be decomposed into two terms, one representing a two dimensional, unpolarized field and one representing a three dimensional unpolarized field. This yields the following decomposition of the correlation matrix

\[ \mathbf{W} = (\lambda_1 - \lambda_2) w_1 + (\lambda_2 - \lambda_3) (w_1 + w_2) + \lambda_3 (w_1 + w_2 + w_3). \] (4.29)

The polarized portion of the field is that portion which is confined to an ellipse, or equivalently is specified by a correlation matrix for which all elements factorize. The two-dimensional unpolarized portion of the field is characterized by the incoherent superposition of two orthogonally polarized fields with equal intensities, while the three dimensional unpolarized portion is characterized by the incoherent superposition of three orthogonally polarized fields with equal intensities. This decomposition is illustrated graphically in Figure 4.2. The two dimensional and three dimensional unpolarized fields share the property that there is no unique polarized component that can be removed from the field, however, they are clearly different. A second polarimetric quantity beyond the degree of polarization is
Figure 4.2: Graphical representation of the decomposition of any three dimensional field into a polarized component, a 2D unpolarized component, and a 3D unpolarized component. It should be noted that the axis do not refer to directions in space, but rather to orthogonal states of polarization.

necessary to quantify this difference. Some possible quantities capable of this discrimination will be discussed in the following Section.

4.2.2 Measurable Quantities and the Degree of Polarization

The decompositions of Eqs. (4.27) and (4.29) have the benefit that they represent the field under consideration as the sum of physically significant components. More specifically, Eq. (4.27) provides a the equivalent ensemble representation for a three dimensional field, in analogy with that discussed in Chapter 1. This is to say, the "natural" basis weighted by the eigenvalues specifies the three members of an ensemble whose second order correlations are identical to those of the original field ensemble under consideration \[6\]. The decomposition of Eq. (4.29) can be used to uniquely identify the polarized portion of the field, from which the degree of polarization can then be calculated. It is worth noting that this decomposition which I have introduced is the first to permit the unique determination of the polarized
portion of a field fluctuating in three dimensions. The degree of polarization is then expressed in terms of the eigenvalues as

$$P = \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2 + \lambda_3},$$  \hspace{1cm} (4.30)

and reduces to the expected value when this formalism is applied to a transverse field. While the degree of polarization uniquely identifies the polarized portion of the field, it is not sufficient to uniquely specify the polarimetric properties of a three dimensional field. This is because the unpolarized portion is no longer uniquely specified by the degree of polarization, as it was in the two-dimensional case. Another parameter is necessary, and it would be desirable that such a parameter retain physical meaning independent of the dimension of the reference frame in a manner similar to the degree of polarization presented here.

Of course, other measures for the field properties can be immediately derived. For instance, when no distinction is made between different types of unpolarized components, one can find that the relative intensity in the non-polarized components is

$$1-P = \frac{2\lambda_2 + \lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}.$$  

It is evident that the degree of polarization $P$ does not distinguish between different types of unpolarized light simply because, in both these extreme cases, there is no polarized component and $P = 0$. One should note that the conventional results corresponding to two-dimensional fields can be easily recovered by simply putting $\lambda_3 = 0$. 
Another measure of the field's properties could be the degree of three-dimensional unpolarized light defined as the ratio between the intensity in this component and the total intensity, which from Eq. (4.29) can be seen to be:

\[ R = \frac{3\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}. \] (4.31)

It is worth noting that while the meanings of \( P \) and \( R \) remain the same in three dimensions as they are in two, there is no longer a deterministic relationship between them. In fact, the two degrees \( P \) and \( R \) provide a description of any possible combination of eigenvalues, as illustrated in Figure 4.3. Thus, they will determine the polarization properties of the three-dimensional field (except for the exact description of its state of polarization). As can be seen in Figure 4.3, any possible configuration of a three-dimensional field can be uniquely identified as a point in the \( PR \)-plane. For instance, in this representation, the unpolarized light states lie along the \( P=0 \) axis, while all the two-dimensional fields lie along the \( R=0 \) axis. The upper bound of the allowable region represents the combination between of polarized and three-dimensional unpolarized light.

I would like to stress again that the full description of a three-dimensional field is provided only by the entire correlation matrix \( \mathbf{W} \). The two degrees exemplified here are not the only possible measures and, in fact, nine independent measures are necessary to fully specify all the properties of a three-dimensional fluctuating field. When using one degree or another, its physical significance and its limited characterization capability should be well understood.
Figure 4.3: All possible states of polarization of three-dimensional fields lie in the bounded region in the $PR$ plane. While two dimensional fields are confined the the $P$-axis, three dimensional fields require both of the scalar invariants of the $W$ matrix to be uniquely specified.
In this context, I will mention two other possible measures of order/disorder of a random field that have been used in the past and interpret them in terms of the general decomposition outlined in Eq. (4.29). The first one is the von-Neumann entropy \[ S \] which has the physical meaning of a measure of total disorder of the random field. In terms of our decomposition in Eq. (4.26) it is defined as

\[
S = - \left[ \lambda_1 \ln(\lambda_1) + \lambda_2 \ln(\lambda_2) + \lambda_3 \ln(\lambda_3) \right]. \tag{4.32}
\]

The second quantity was recently proposed [65] and can be regarded as an "intensity weighted" measure of the average cross-correlation between the orthogonal elements of the correlation matrix \( \mathbf{W} \). Using the results of our general representation of a random field Eq. (4.26), this measure is

\[
C = \sqrt{\frac{3}{2} \left[ \frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}{(\lambda_1 + \lambda_2 + \lambda_3)^2} - \frac{1}{3} \right]} . \tag{4.33}
\]

In is worth mentioning again that, in the case of two-dimensional fields, all these measures of the field’s properties can be directly related to the degree of polarization \( P \). However, as I have explained here, for three-dimensional fields, no direct connection between these measures and the polarized component of the field can be made. This is a direct consequence of the fact that a full characterization of a three-dimensional field can be made only by the full set of nine correlation factors included in the correlation matrix \( \mathbf{W} \). None of those, \( P \) included, can provide a complete description of the three dimensional random field. For a
two dimensional random field, the correlation matrix has one scalar invariant beyond the intensity. This scalar invariant can be expressed in terms of the degree of polarization. For three dimensional fields, the correlation matrix has two scalar invariants, hence two parameters are necessary to specify the three dimensional field.

4.3 Measurement of the Correlation Matrix

The act of generalizing the theory of polarization to three dimensions is mathematically simple. The important contribution made in this dissertation is not the elegant mathematical formalism, but rather the retention of physical significance. Because the description of the state of polarization is contained in the eigenvalues and eigenvectors, meaningful decompositions and a greater understanding of polarized and unpolarized fields can be achieved. While this is, in itself, a significant advancement, from a practical standpoint, it is the measurement of these quantities that would be of greatest interest.

As there are nine independent parameters in the $3\times3$ correlation matrix, the minimal number of measurements necessary to fully specify the second order correlations of a three dimensional, fluctuating field is nine. There are certain assumptions that could reduce the number of necessary measurements. For example, if the field is fully polarized, five measurements suffice; two measurements to determine the plane of polarization, and three to determine the state of the polarization ellipse in that plane. Similarly, if the field is assumed to be fully uncorrelated, three measurements would suffice, specifically measurements of
intensity in three orthogonal directions. The most common assumption is knowledge of a unique direction of propagation for the field, in which case the field is inherently two-dimensional, and only four measurements are sufficient to specify it; this is the reduction to the two-dimensional case.

While polarization is inherently a point quantity, that is, it describes the second order field correlations at a point, the measurement of these quantities is effectively achieved by measurement of a propagating, beam-like field. The measurement of the nine independent second order field correlations could be realized by using, for instance, three orthogonal dipole-like probes which are detected simultaneously. In the optical domain however, this approach cannot be implemented because an ensemble of three dipoles which can be read independently simply does not exist. This is not, however, the only means by which the polarimetric information can be obtained. Instead of superposing three dipole-like detectors, one could envision placing in the point of interest a probe which couples all three components of the field and then re-emits the radiation. This probe would act as a secondary source for the radiation which will eventually be sensed by a conventional detector, placed away from the point where measurements are made. The result will be a linear combination of the measurements possible with three independent dipoles and, in order to determine all the nine elements of the field correlation matrix, one would need to perform measurements with nine different probe configurations. It is important to emphasize that, due to propagation from the probe to the detector, the field becomes practically transverse having a propagation vector pointed toward the two-dimensional detector.
The entire operation of detecting the intensity $I_{\text{det}}$ by the two-dimensional detector can be formally written as

$$I_{\text{det}} = Tr \{ J^t_{2x3} \alpha^t_{3x3} W_{3x3} \alpha_{3x3} J_{3x2} \}$$

(4.34)

where the $J_{3x2}$ provides the propagation to the detector, $\alpha_{3x3}$ represents the coupling by the probe of the field of interest, $Tr$ denotes the trace (and is included due to detectors yielding intensity information), $W_{3x3}$ characterizes the field at the point of interest, and the subscripts indicate dimensionality of the matrices. Examination of Eq.(4.34) implies that a full field characterization can not be accomplished by altering only $J_{2x3}$, as was the case for two dimensional fields. To fully characterize the field, nine independent realizations of $\alpha_{3x3} J_{2x3}$ must be achieved, and $J_{3x2}$ alone provides only four. Hence, to fully measure the correlation matrix at a point requires altering the coupling of the field to the detector, as well as the effects of propagation to the detector. It is also possible to fix the propagation matrix, and alter only the coupling matrix as it is possible to realize nine independent forms of the coupling matrix due to its dimensionality.

4.4 Application of 3D Polarimetry

While we understand the theoretical formalism of measurement, it is of interest to determine how this new theory can be applied. There are several applications of interest, a select few where we expect this formalism and measurement to be particularly useful will be examined.
Classically, polarimetry has been applied to two-dimensional fields. This is due, in part, to the transverse nature of propagating electromagnetic fields. As the field component parallel to the direction of propagation is essentially negligible, there was no need to incorporate it into the mathematical framework for describing a fluctuating field. However, there are a number of regimes in which the assumption of transversallity fails. This is not to say that Maxwell’s equations are in error (as they imply transversallity of propagation fields), rather it indicates regimes in which the direction of propagation is indeterminate. Some examples of these regimes include fields near the surface of sources, inside scattering volumes, evanescent fields, and, more classically, areas in which propagating fields with differing directions of propagation may overlap. Below, I examine certain cases of general interest such as the application to near field optics and classical polarimetry, followed by the description of a proof of concept experiment in which three dimensional fields are generated and measured, confirming the validity of the theory and the subsequent decompositions.

4.4.1 Near Field Optics

One particularly interesting domain in which three dimensional field effects are of particular interest is that of near field optics [69, 70, 71]. It has been shown that near field probes are sensitive to three dimensional field fluctuations [70]. Their sensitivity, along with certain assumptions about the coupling properties of the probe, can be used to examine some of the field fluctuations. However, a single probe measurement is insufficient to fully determine
the structure of the three dimensional field fluctuations. These fluctuations are of particular interest in the determination of the shape of nano-particles. Below is a theoretical investigation of a well characterized problem. It is possible that this may find use in the measurement and control of field fluctuations at a surface.

Let us examine the problem of a small spherical scattering particle placed in contact with a semi-infinite slab and illuminated in the total internal reflection geometry shown in Figure 4.4. The problem is discussed in the electrostatic approximation which is applicable when all distances in the problem are much smaller than the wavelength of light, i.e. we are concerned only with the \(1/r^3\) term of dipole radiation. The origin of coordinates, \(\mathbf{r} = (0, 0, 0)\), is defined such that the center of the particle of radius \(a\) is at \(\mathbf{a} = (0, 0, a)\). In the first Born approximation, the total field at a point \(r\) is then given by

\[
\mathbf{E}(\mathbf{r}) = \mathbf{E}_0(\mathbf{r}) + \mathbf{S}(r) \alpha_{eff}(a) \mathbf{E}_0(\mathbf{a})
\]

(4.35)

where \(\mathbf{S}(r)\) is the nonretarded propagator which, in the near field zone, can be expressed as [72]

\[
\mathbf{S}(\mathbf{r}) = \frac{3 (\mathbf{r} - \mathbf{a}) (\mathbf{r} - \mathbf{a}) - |\mathbf{r} - \mathbf{a}|^2 \mathbf{I}}{|\mathbf{r} - \mathbf{a}|^5},
\]

(4.36)

with \(\mathbf{I}\) being the unit matrix.

The polarizability of the particle is given by [20]
A spherical particle of radius $a$ and index $m$ is placed in contact with the surface of a planar dielectric of index $n$ illuminated by two s-polarized beams under conditions of total internal reflection ($\phi_c$ is the critical angle). The beams are "perpendicular" in the sense that the components of their $k$ vectors parallel to the surface of the substrate are perpendicular. The intensity of the scattered field, its state of polarization, and the degree of polarization are then calculated in a plane situated at a height of $z_0$ from the substrate.

$$\vec{\alpha}_0 = \alpha_0 \vec{I} = a^3 \frac{m^2}{m^2 + 2} \vec{I}$$

(4.37)

where $m$ is the particle’s index of refraction. As the particle’s polarizability is initially isotropic, the symmetry of the effective polarizability tensor $\alpha^{eff}$ is governed mainly by the symmetry of the dielectric slab [73]. In the case of a spherical particle on a planar surface, the polarizability was shown to be [73].

$$\alpha^{eff} (\alpha_0) = \begin{bmatrix} \alpha^{eff}_\parallel & 0 & 0 \\ 0 & \alpha^{eff}_\parallel & 0 \\ 0 & 0 & \alpha^{eff}_\perp \end{bmatrix}$$

(4.38)

with
\[ \alpha_{\parallel}^{\text{eff}}(\alpha_0) = \frac{8(n^2 + 1)a^3\alpha_0}{8(n^2 + 1)a^3 + (n^2 - 1)\alpha_0} \]  

(4.39)

and

\[ \alpha_{\perp}^{\text{eff}}(\alpha_0) = \frac{4(n^2 + 1)a^3\alpha_0}{4(n^2 + 1)a^3 + (n^2 - 1)\alpha_0}. \]  

(4.40)

Following the model outlined here, the scattered field can be evaluated for any excitation field \( \vec{E}_0 \) and, therefore, any kind of illumination can be accommodated. This includes any number of beams with different orientations and different states of polarization. Using this model, the cross spectral density matrix of the total field at a point \( \vec{r} \), \( W_{i,j}(\vec{r}) = \langle E_i^*(\vec{r}) E_j(\vec{r}) \rangle \), can then be calculated.

I examine in detail an example where two perpendicular (\( \vec{k}_1 \parallel \vec{k}_2 = 0 \)) beams, each TE polarized, are illuminating the scattering particle as illustrated in Figure 4.4. The superposition of these beams constitute the excitation field \( \vec{E}_0 \) and by applying Eq. (4.35), the total field \( \vec{E}(\vec{r}) \) can be calculated. From this field calculation, the cross spectral density matrix can then be computed,

\[ W_{i,j} = \langle E_i^* E_j \rangle = \langle E_1^* E_1 \rangle + \langle E_2^* E_2 \rangle + \langle E_1^* E_2 \rangle + \langle E_2^* E_1 \rangle \]

with the model permitting any possible correlation between the two exciting beams.
When the fields arising from the two beams are incoherent, i.e. \( \langle E_1^* E_2 \rangle = 0 \), the resulting cross-spectral density matrix is simply the summation of two independent correlation matrices. Explicitly

\[
W_{i,j} = \langle E_i^* E_j \rangle = \langle E_1^* E_1 \rangle + \langle E_2^* E_2 \rangle
\]

The eigenvalues and eigenvectors of the cross-spectral density matrix \( W_{i,j} \) can then be calculated, from which the intensity, degree of polarization and state of polarization are evaluated according to the treatment of Section 3.2. Figure 4.5 depicts the theoretically calculated degree of polarization and intensity in a plane \( z_0 = 2a \). Directly above the center of the particle, the degree of polarization is identically 0. It is interesting to note that while there are components of the field in all three orthogonal directions, the field itself at any point is composed of only two uncorrelated, orthogonal states of polarization, as opposed to being a fully isotropic field composed of three uncorrelated, orthogonal states of polarization. This can be seen in Figure 4.6, where the eigenvalues of the decomposition presented in Eq. (4.29) have been calculated. These eigenvalues provide the relative intensity in each of the three orthogonal states of polarization [63].

By controlling the incident illumination, full control over the three dimensional field near the particle is achievable. The spherical particle acts as a coupler between the two incident beams, resulting in a controlled distribution of states of polarization. With the use of partially polarized illumination beams, a fully three dimensional field can be generated.
Figure 4.5: Calculated degree of polarization and intensity at a height $2a$ above the surface.

Figure 4.6: Theoretical calculations of the degree of polarization and intensity in the orthogonal polarization states as a function of height $z$ from the surface.
4.4.2 Beam Direction

One practical problem with current polarimeters is the necessity for alignment with the direction of propagation of the measured radiation. When the polarimeter is not normal to the direction of propagation, an error is introduced into the determination of the state of polarization. This error results from an effective projection of the real field onto the 2D surface of the detection device. This projection is easily seen when the problem is posed in the formalism of three dimensional fields. To demonstrate this, we will examine a partially polarized beam whose direction of propagation is at an angle $\theta$ to the normal of the polarimeter. As we are considering a two dimensional field described in the three dimensional formalism, the equivalent ensemble will be composed of three orthogonal, independent, fully polarized realizations. However, one as we have restricted ourselves to a beam, one of those realizations is identically zero, while the relative intensities of the other two are $\alpha^2$ and $\beta^2$, with the constraint that $(\alpha^2 + \beta^2 = 1)$. The degree of polarization is then given by $|\alpha^2 - \beta^2|$, and the correlation matrix is
The field measured by the classical polarimeter will be

\[ \mathbf{W} = \begin{bmatrix} \langle E_x^* E_x \rangle & \langle E_x^* E_y \rangle & \langle E_x^* E_z \rangle \\ \langle E_y^* E_x \rangle & \langle E_y^* E_y \rangle & \langle E_y^* E_z \rangle \\ \langle E_z^* E_x \rangle & \langle E_z^* E_y \rangle & \langle E_z^* E_z \rangle \end{bmatrix} \]

(4.41)

\[ \mathbf{W} = \alpha^2 \begin{bmatrix} \mathcal{E}_a \mathcal{E}_a \cos^2 \theta & \mathcal{E}_a^* \mathcal{E}_b \cos \theta & -\mathcal{E}_a^* \mathcal{E}_a \cos \theta \sin \theta \\ \mathcal{E}_b \mathcal{E}_a \cos \theta & \mathcal{E}_b^* \mathcal{E}_b & -\mathcal{E}_b^* \mathcal{E}_a \sin \theta \\ -\mathcal{E}_a^* \mathcal{E}_a \cos \theta \sin \theta & -\mathcal{E}_a^* \mathcal{E}_b \sin \theta & \mathcal{E}_a \mathcal{E}_a \sin^2 \theta \end{bmatrix} + \beta^2 \begin{bmatrix} \mathcal{E}_b^* \mathcal{E}_b \cos^2 \theta & -\mathcal{E}_b \mathcal{E}_a^* \cos \theta & -\mathcal{E}_b^* \mathcal{E}_b \cos \theta \sin \theta \\ -\mathcal{E}_a \mathcal{E}_b^* \cos \theta & \mathcal{E}_a \mathcal{E}_a^* & \mathcal{E}_a \mathcal{E}_b^* \sin \theta \\ -\mathcal{E}_b \mathcal{E}_b^* \cos \theta \sin \theta & \mathcal{E}_b \mathcal{E}_a^* \sin \theta & \mathcal{E}_b \mathcal{E}_b^* \sin^2 \theta \end{bmatrix}. \]

and one finds

\[ \mathbf{W} = \begin{bmatrix} \langle E_x^* E_x \rangle & \langle E_x^* E_y \rangle \\ \langle E_y^* E_x \rangle & \langle E_y^* E_y \rangle \end{bmatrix} = \alpha^2 \begin{bmatrix} \mathcal{E}_a \mathcal{E}_a \cos^2 \theta & \mathcal{E}_a^* \mathcal{E}_b \cos \theta \\ \mathcal{E}_b \mathcal{E}_a \cos \theta & \mathcal{E}_b^* \mathcal{E}_b \end{bmatrix} + \beta^2 \begin{bmatrix} \mathcal{E}_b \mathcal{E}_b^* \cos^2 \theta & -\mathcal{E}_b \mathcal{E}_a^* \cos \theta \\ -\mathcal{E}_a \mathcal{E}_b^* \cos \theta & \mathcal{E}_a \mathcal{E}_a^* \end{bmatrix}, \]

(4.42)

and one finds

\[ P_{\text{measured}} = \sqrt{1 - \frac{4\alpha^2 \beta^2 (\mathcal{E}_a^* \mathcal{E}_a + \mathcal{E}_b^* \mathcal{E}_b)^2 \cos^2 (\theta)}{[\mathcal{E}_a^* \mathcal{E}_a (\alpha^2 \cos^2 (\theta) + \beta^2) + \mathcal{E}_b^* \mathcal{E}_b (\beta^2 \cos^2 (\theta) + \alpha^2)]^2}. \]

(4.43)
The error in the measured degree of polarization depends on the initial state of polarization as well as the error in angular alignment. An error of 5 degrees for a slightly elliptical beam that is primarily polarized \( (P = 0.9, |\mathcal{E}_a| = 0.9) \) can result in an error of as much as 30\% in the measured degree of polarization.

When we examine the full measurement of the three dimensional field, this error is removed. Furthermore, for the case of a beam-like field, the unique direction of propagation can be recovered from the measurement (except for the degenerate case of a fully linearly polarized beam). To see the recovery of beam direction, we return to the example above. Applying a simple rotation, on finds

\[
\begin{bmatrix}
\cos(\theta) & 0 & -\sin(\theta) \\
0 & 1 & 0 \\
\sin(\theta) & 0 & \cos(\theta)
\end{bmatrix}
\begin{bmatrix}
\langle E_x^* E_x \rangle & \langle E_x^* E_y \rangle & \langle E_x^* E_z \rangle \\
\langle E_y^* E_x \rangle & \langle E_y^* E_y \rangle & \langle E_y^* E_z \rangle \\
\langle E_z^* E_x \rangle & \langle E_z^* E_y \rangle & \langle E_z^* E_z \rangle
\end{bmatrix}
\begin{bmatrix}
\cos(\theta) & 0 & \sin(\theta) \\
0 & 1 & 0 \\
-\sin(\theta) & 0 & \cos(\theta)
\end{bmatrix}
\]

\[
= \alpha^2 \begin{bmatrix}
\mathcal{E}_a^* \mathcal{E}_a & \mathcal{E}_a^* \mathcal{E}_b & 0 \\
\mathcal{E}_b^* \mathcal{E}_a & \mathcal{E}_b^* \mathcal{E}_b & 0 \\
0 & 0 & 0
\end{bmatrix} + \beta^2 \begin{bmatrix}
\mathcal{E}_b \mathcal{E}_b^* & -\mathcal{E}_b \mathcal{E}_a^* & 0 \\
-\mathcal{E}_a \mathcal{E}_b^* & \mathcal{E}_a \mathcal{E}_a^* & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

The degree of polarization is given by

\[
P = \alpha^2 - \beta^2, \quad (4.45)
\]
exactly what it was assumed to be when the exercise was started, and the unique transverse direction given, under the current rotation, is given by

\[ k' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \]  \hspace{1cm} (4.46)

Returning to the original coordinate system, we find the transverse direction is given by

\[ k' = \begin{pmatrix} \sin(\theta) \\ 0 \\ \cos(\theta) \end{pmatrix}. \]  \hspace{1cm} (4.47)

A completely equivalent procedure for determining the direction of propagation would be to take the cross product of the real and imaginary portions of the eigenvector associated with the non-zero eigenvalue. Thus, the three dimensional formalism can be used to correctly identify both the state of polarization and the direction of propagation of beam-like fields. No a priori knowledge of the beam-like fields direction of propagation is necessary, as long as the field is not degenerate in the sense of being fully polarized in a linear state. Experimental confirmation of beam direction measurement will be discussed below.
4.4.3 Proof of Concept Measurement

For the proof of concept measurement, it will be necessary to both generate three dimensional random fields, and then measure the second order field correlations at a point there-in. While it is possible to generate such a field by taking advantage of the three dimensional nature of optical near fields and scattering volumes, it is more convenient to work in a classical regime.

4.4.3.1 Generation of 3D Fields

The generation of a three dimensional field was achieved by overlapping three beams propagating in orthogonal directions as shown in Figure 4.7. By controlling the individual states of polarization and the correlation between these three beams, it is possible to produce any desired three dimensional fluctuating field in the area of intersection.
4.4.3.2 Measurement of 3D Fields

The field probe used in this experiment is a single mode silica fiber which is conically shaped at one end and is coupled to a photomultiplier at the other. To realize nine different probe configurations, the alteration in coupling properties resulting from proximity to a secondary tip was exploited. By adjusting the relative position of additional tapered fibers placed in the probe’s proximity, nine independent probes were realized. It is worth noting that the polarizabilities of these nine probe configurations do not need to be known a priori (i.e. we do not need nine specifically shaped probes but just nine independent polarizabilities), because the coupling may be calibrated for by using a set of at least nine know generated fields. Once the nine independent configurations have been realized and the calibration made, it is possible to reconstruct all the second-order field correlations, i.e., the nine elements of the \( W \) matrix, of a given field.

4.4.3.3 Experimental Confirmation

To verify both the theory of measurement, applicability of calculated parameters, and the measurement technique I examined a number of states of interest. The first two states examined were elliptical beams, one propagating along a calibration direction, and one propagating at an arbitrary angle. These two states were used to examine the beam direction measurement. The results are shown in Figure 4.8. The results are shown graphically in direction.
Figure 4.8: Two examples of Beam Direction Measurement shown in direction space. The black dots represent the known directions to be recovered, while the red dots represent measured values of the direction.

space, with the two black points being the actual beam direction, and the red points being the calculated direction of propagation based on the measured correlation matrices.

In both cases, the direction and state of polarization was recovered. The measured points are obtained from different sets of probe configurations and calibrations. Calibration of the probes were conducted with known states which propagated along the coordinate axis. It is not surprising that the error in measured direction along one of the calibration directions was less than the error along an arbitrary direction, even though neither of the states were used for the calibration of the probe. This error can be expressed in the solid angle represented by the spread of measured values. For the beam which was colinear to a calibration axis, the error is less than 0.01 steradians, while for the off-axis case the error is less than 0.15 steradians. It is worth noting that for the degenerate case of a fully polarized beam in a
linear state of polarization, a unique solution to the polarization plane does not exist, and in this case, it is not possible to determine the direction of propagation from a measurement of the field at a single point. However, any amount of partial depolarization or ellipticity is sufficient to uniquely determine a plane of polarization.

The recovery of state of polarization has been confirmed with the measurement of a number of three dimensional field configurations. Four of these fields are illustrated in Figure 4.9, in terms of the decomposition into the natural basis of uncorrelated, fully polarized fields. The black curves are obtained from decomposing the correlation matrix resulting from the superposition of the three independent beams in known states of polarization, while the red curves are obtained following similar calculations based on $W$ measured by using the independent probe configurations. The orthogonal field components have been represented as polarization ellipses with appropriate magnitudes. The scale on the graphs is arbitrary, but the same for the entire figure, allowing for a direct comparison. The first is an elliptically polarized field. The second row is a two-dimensional, unpolarized field comprised of the incoherent superposition of the first field with a linearly polarized field of equal intensity normal to the polarization ellipse of the first. The third state is three-dimensional unpolarized light resulting from the incoherent superposition of three perpendicularly oriented, linearly polarized fields of equal magnitude. This is an interesting case, as there is no unique equivalent ensemble of three states (in fact, any three orthogonal fields with equal intensities results in this distribution). The last row represents an interesting partially polarized, three dimensional field. It is the result of the superposition of three right circular states with mu-
Figure 4.9: Black curves represent the field decomposition based on the known states of the three independent lasers, while red curves indicate the experimentally measured field decomposition. The representations are in terms of the polarization ellipses of the three independent field components.

tually perpendicular directions of propagation. Reconstruction of the states of polarization is good, with experimental errors being mainly present in the intensities.

4.4.4 Other applications

It is expected that the ordered portion of a measured three-dimensional field - the polarized local density of states - will relate to both bulk structural and morphological properties of the sources. In this context, three-dimensional polarimetry is directly applicable to the fields of fluorescence and multiphoton microscopy. For instance, fluorescence measurements
performed in the far-field provide only limited information about the structural and dynamic properties of the fluorophore centers. Determining the field inside the emission volume should offer a greater level of detail regarding the structural and dynamic properties of the fluorophores. Also, because the measurement technique demonstrated uniquely identifies and fully describes the polarized component, it is expected that the use of such a 3D polarimeter will result in an improvement of imaging objects embedded in dense scattering media.

While it is impossible to enumerate all areas in which this theory may have impact, some areas of current interest worth noting and not considered above include tight focusing and high energy lasers, as well as the application of the three dimensional theory as it applies to speckle fields generated from random scattering.
CHAPTER 5
SUMMARY OF ORIGINAL CONTRIBUTIONS AND CONCLUSIONS

The goal of polarimetry is to describe and quantify the fluctuations of a field and also the ways in which a medium can alter those fluctuations. On small enough temporal and spatial scales, any field is deterministic and hence fully polarized. In practice, there is no direct access to this instantaneous field, and one deals with ensembles of field realizations. The independent realizations of the ensemble are fully polarized, and it is only in a global sense that an ensemble of field realizations can be partially polarized. The standard formalisms of polarimetry rely only on the second order correlations of the field components averaged over this ensemble. For transverse electromagnetic fields, this procedure defines an equivalent representation of the ensemble in terms of the superposition of two uncorrelated fields in orthogonal states of polarization.

The basic formalisms, the Jones calculus, the Stokes-Mueller calculus, and the field correlation matrix, as well as standard measurement techniques were detailed in Chapter 1. These concepts were then applied to practical imaging applications. In many situations, the polarimetric measurements were shown to greatly enhance the information available, providing means for material discrimination and increasing the depth of imaging through a dense scattering medium. However, in the case of highly scattering media which result in
globally unpolarized fields, very little information pertaining to the system or the sample can be gained from classical polarimetric measurements. Moreover, the classical formalisms and measurements rely on the assumption of transverse fields propagating normal to the detection system, and can not be used to describe fields for which the direction of propagation is not unique. The research included in this dissertation expanded the classical polarimetric formalisms and measurement techniques to handle these situations.

In the case of globally unpolarized fields, there is no unique equivalent ensemble of field realizations. A common assumption then is that the independent members of this ensemble obey Gaussian statistics. This results in a uniform distribution of the polarization states. However, this uniform distribution is not the only one capable of describing a globally unpolarized field. Furthermore, the second order field correlations are not capable of discriminating between Gaussian and non-Gaussian distributed ensembles of field realizations. In order to deal with this limitation of the classical theory, Chapter 2 of this dissertation provided a generalization of the polarimetric theory to higher order field correlations and the description of the ensemble of field realizations in terms of the full distribution of states of polarization.

The classical parameters of polarimetry are quantities averaged over the ensemble of field realizations in space or time or both. As such, their ability to discriminate between various ensembles is inherently reduced, a limitation which is evident in the practical case of globally unpolarized fields. Three theoretical examples of globally unpolarized light that differ in terms of their invariances of fourth order correlations were presented in Chapter 2. It is interesting to note that for all three distributions, the intensity statistics are identical.
Differences in the polarization statistics present in these examples result from correlations of the field components in the underlying ensemble. When these field correlations arise due to material characteristics, it is then possible to discriminate between different materials even when the scattered light is globally unpolarized.

In Chapter 2 a simple model describing specific surface interactions was also introduced. Numerical simulations based on this model indicate the possibility to observe material dependent correlations between components of the scattered field. The experimental results for spherical and aspherical scatterers show general agreement with the trends predicted by the numerical calculations, demonstrating the usefulness of these fourth order field correlations for (i) discriminating between different globally unpolarized regimes and (ii) as descriptors of subtle material properties. While the present treatment refers to spatial distributions, the theory is completely general and applicable to temporal distributions as well.

In certain situations, another forth order field correlator can be introduced to provide additional information pertinent to the structure of the random field. Forth order point-pair statistics characterize the similarity in polarization information at two points in a distribution. To examine this "polarization-similarity", the Complex Degree of Mutual Polarization (CDMP) was introduced. Both theoretical and experimental examinations demonstrated the effectiveness of the CDMP in regards to structural characterization of systems under consideration.

Another important limitation of classical polarimetry is the assumption of paraxial, transverse electromagnetic fields. There are many practical instances where this approximation
is not valid. In the case of near field optics, in areas of strong focusing, inside scattering and fluorescing volumes, and even in non-collinear interference of classical beams, the direction of propagation is not unique. Chapter 3 described the generalization of the classical polarimetric formalisms to the case of three dimensional fields. The physically meaningful parameters were then explored, and it was noted that a higher dimensional field requires more parameters to fully describe its polarimetric properties. A theory for the measurement of three dimensional fields was discussed along with experimental confirmation. Various applications of this generalized theory, including beam direction measurement and near-field statistics, were presented.

The formal mathematical generalization to higher dimensional fields is straightforward. However, retaining the physical significance of the polarimetric characteristics requires careful consideration. It is important to note that the generalization of a deterministic field to higher dimensions does not alter any of its properties. While the polarization ellipse may vary from point to point, a deterministic field in three dimensions is still fully described by the ellipse on which it oscillates.

The generalization of an unpolarized field, however, is not as straightforward. The most common description of an unpolarized field requires that there is no phase relationship between orthogonal components, while the average intensity in any given direction is the same. This constitutes a three-dimensional unpolarized field. There is, however, a second possible generalization to a field which is not completely isotropic in three dimensions. This
second type of unpolarized field can be viewed as a two-dimensional unpolarized field which is described in a three dimensional space.

Having understood for the first time the physical significance of generalizing to higher dimensions the concepts of polarized and unpolarized fields, it was then possible to specify the measurable parameters (specifically the second order field correlations) and the derived, physically meaningful, descriptors such as the equivalent ensemble representation, the degree of polarization, the state of polarization, etc. These various parameters and their usefulness in characterizing a three dimensional random field were carefully examined. It is important to note that while a two dimensional random field can in essence be described by two invariant parameters, specifically the intensity and degree of polarization, a third parameter is necessary for a three dimensional field. This should be expected from simple dimensional arguments.

Finally, the general theory of the complete measurement of a three dimensional random field was presented. It was emphasized that while propagation serves to reduce the random field to a two-dimensional, transverse one, the interaction with certain media will in general depend on the complete three dimensional structure of both the material and the field. Making use of this, an experimental procedure for measuring the second order field correlations was for the first time proposed and demonstrated. The results of a proof of concept experiment were presented, and the retrieval of various three dimensional fields was illustrated.
This dissertation presented several novel theoretical and experimental contributions to the polarimetry of random fields. Theoretical advancements and innovative experimental techniques were introduced for the study of globally unpolarized fields and random fields fluctuating in three dimensions. Notably, this dissertation contains the first experimental measurement of the polarization characteristics of a three dimensional optical field. These contributions to the field of polarimetry are expected to impact scientific areas as diverse as near field optics, remote sensing, high energy laser physics and fluorescence microscopy.
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