2020

Representations of Cuntz Algebras Associated to Random Walks

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REPRESENTATIONS OF CUNTZ ALGEBRAS ASSOCIATED TO RANDOM WALKS

by

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A thesis submitted in partial fulfillment of the requirements
for the degree of Honors in the Major Program in Mathematics
in the College of Sciences
in the Burnett Honors College
at the University of Central Florida

Spring Term, 2020

Thesis Chair: Dr. Dorin Ervin Dutkay

ABSTRACT

In the present thesis, we investigate representations of Cuntz algebras coming from dilations of row co-isometries. First, we give some general results about such representations. Next, we show that by labeling a random walk, a row co-isometry appears naturally. We give an explicit form for representations that come from such random walks. Then, we give some conditions relating to the reducibility of these representations, exploring how properties of a random walk relate to the Cuntz algebra representation that comes from it.

ACKNOWLEDGMENTS

Firstly, I would like to thank Dr. Dorin Dutkay for his guidance in writing this thesis, and for challenging me when I make mathematical assumptions. Secondly, I would like to thank Dr. Barry Griffiths for his helpful advice throughout my undergraduate career. Thirdly, thanks to all of the incredible professors I have had, that have shown me what communicating mathematics should be. A special thanks to Dr. Zi-Xia Song, who told me about the Honors in the Major program, and inspired me to write a thesis. Finally, thanks to the UCF Burnett Honors College for the opportunities that they have given me that have enriched my UCF College Experience.

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1 INTRODUCTION

Definition 1.1. The Cuntz algebra of order $N \in \mathbb{N}$, $\mathcal{O}_N$, is the C^*-algebra generated by $s_i : i = 0, \ldots, N-1$ with the Cuntz relations:

$$s_i^* s_j = \delta_i^j, \quad \sum_{i=0}^{N-1} s_i s_i^* = 1 \tag{1.1}$$

Definition 1.2. Let Λ be a finite index set with $|\Lambda| = N$. A representation of $\mathcal{O}_N$ is a set of N isometries $\{S_\lambda\}_{\lambda \in \Lambda}$ on a Hilbert space $\mathcal{H}$ that satisfy the Cuntz relations:

$$S_\lambda^* S_{\lambda'} = \delta_{\lambda'}^{\lambda} I_{\mathcal{H}}, \quad \sum_{\lambda \in \Lambda} S_\lambda S_\lambda^* = I_{\mathcal{H}} \tag{1.2}$$

and whose range is all of $\mathcal{H}$.

Definition 1.3. A set of linear operators $\{V_\lambda : \mathcal{K} \to \mathcal{K}\}_{\lambda \in \Lambda}$ on a Hilbert space $\mathcal{K}$ is a "row co-isometry" if

$$\sum_{\lambda \in \Lambda} V_\lambda V_\lambda^* = I_{\mathcal{K}} \tag{1.3}$$

We know that from Bratelli et. al. that each row co-isometry can be dilated to a representation of a Cuntz algebra. More precisely,

Theorem 1.4. *[1, Theorem 5.1] Let $\mathcal{K}$ be a Hilbert space and $(V_\lambda)_{\lambda \in \Lambda}$ be a row co-isometry on $\mathcal{K}$. Then $\mathcal{K}$ can be embedded into a larger Hilbert space $\mathcal{H} = \mathcal{H}_V$ carrying a representation $(S_\lambda)_{\lambda \in \Lambda}$ of the Cuntz algebra $\mathcal{O}_N$ such that, if $P : \mathcal{H} \to \mathcal{K}$*

is the projection onto $\mathcal{K}$, we have

$$V_\lambda^* = S_\lambda^* P, \tag{1.4}$$

(i.e., $S_\lambda^* \mathcal{K} \subset \mathcal{K}$ and $S_\lambda^* P = P S_\lambda^* P = V_\lambda^*$) and $\mathcal{K}$ is cyclic for the representation.

The system $(\mathcal{H}, (S_\lambda)_{\lambda\in\Lambda}, P)$ is unique up to unitary equivalence, and if $\boldsymbol{\sigma} : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{K})$ is defined by

$$\boldsymbol{\sigma}(A) = \sum_{\lambda\in\Lambda} V_\lambda A V_\lambda^*, \tag{1.5}$$

then the commutant of the representation $\{S_0, \dots, S_{N-1}\}'$ is isometrically order isomorphic to the fixed point set $\mathcal{B}(\mathcal{K})^{\boldsymbol{\sigma}} = \{A \in \mathcal{B}(\mathcal{K}) : \boldsymbol{\sigma}(A) = A\}$, by the map $A' \mapsto PA'P$. More generally, if $(W_\lambda)_{\lambda\in\Lambda}$ is another row co-isometry on the same space $\mathcal{K}$, and $(T_\lambda)_{\lambda\in\Lambda}$ is the corresponding Cuntz dilation, then there exists an isometric linear isomorphism between the intertwiners $U : \mathcal{H}_V \to \mathcal{H}_W$, i.e., operators satisfying

$$U S_\lambda = T_\lambda U, \tag{1.6}$$

and operators $V \in \mathcal{B}(\mathcal{K})$ such that

$$\sum_{\lambda\in\Lambda} W_\lambda V V_\lambda^* = V, \tag{1.7}$$

given by the map $U \mapsto V = PUP$.

Definition 1.5. Given a row co-isometry $(V_\lambda)_{\lambda\in\Lambda}$ on the Hilbert space $\mathcal{K}$, we call the representation $(S_\lambda)_{\lambda\in\Lambda}$ of the Cuntz algebra $\mathcal{H}$ in Theorem 1.4, *the Cuntz dilation of the row co- isometry* $(V_\lambda)_{\lambda\in\Lambda}$.

Definition 1.6. A representation $\{S_\lambda : \mathcal{H} \to \mathcal{H}\}_{\lambda\in\Lambda}$ of $\mathcal{O}_N$ is irreducible if

$$(S_\lambda)' \triangleq \left\{A \in B(\mathcal{H}) : AS_\lambda = S_\lambda A, \forall\lambda \in \Lambda\right\} = \{cI_{\mathcal{H}} \,:\, c \in \mathbb{C}\}. \tag{1.8}$$

Otherwise, we say that the representation is reducible.

Remark 1.7. If $\{S_\lambda\}_{\lambda\in\Lambda}$ is a representation of a Cuntz algebra, and if $A : \mathcal{H} \to \mathcal{H}$ commutes with the S_λ, then it also commutes with the S_λ^*.

The proof is a simple result of taking the adjoint of both sides of the equation $AS_\lambda = S_\lambda A$

Definition 1.8. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a directed graph, where the set of vertices $\mathcal{V}$ is finite or countable. For each edge $e \in \mathcal{E}$ we assume we have a label $\lambda(e)$ chosen from a finite set of labels Λ, $|\Lambda| = N$.

We assume in addition, that given a vertex i, two different edges $e_1 \neq e_2$ from i, will have different labels $\lambda(e_1) \neq \lambda(e_2)$ and their end vertices are different. We use the notation $i \xrightarrow{\lambda} j$ to indicate that there is an edge from i to j with label λ, and, in this case we also write $i \cdot \lambda = j$. Thus, for each vertex i, there is at most one edge leaving i with label λ.

We also assume that for each vertex i there is at most one edge coming into the vertex i with label λ.

For each vertex i and each label λ we assume that we have an associated complex number $\alpha_{i,\lambda}$, $\alpha_{i,\lambda} = 0$ in case there is no edge from i with label λ, and we assume that

$$\sum_{\lambda\in\Lambda}|\alpha_{i,\lambda}|^2 = 1, \quad (i \in \mathcal{V}). \tag{1.9}$$

Thus, we have a random walk on the graph $\mathcal{G}$, the probability of transition from i to $i \cdot \lambda$ with label λ, being $|\alpha_{i,\lambda}|^2$.

Here is the way we define the representation of the Cuntz algebra $\mathcal{O}_N$.

First, we define the row co-isometry $(V_\lambda)_{\lambda\in\Lambda}$ on $\mathcal{K} := \mathbb{C}[\mathcal{V}]$,

$$V_\lambda^*(\vec{i}) = \begin{cases} \alpha_{i,\lambda}\vec{j}, & \text{if } i \xrightarrow{\lambda} j \\ 0, & \text{otherwise.} \end{cases} \tag{1.10}$$

(Here, the vertices i in V are considered as the canonical basis vectors $\vec{i} := \delta_i$ for $\mathbb{C}[V]$).

Then, we use Theorem 1.4, to construct the Cuntz dilation $(S_\lambda)_{\lambda\in\Lambda}$. We call it *the representation of the Cuntz algebra associated to our random walk.*

Definition 1.9. We denote by Ω the set of all finite words with letters in Λ,

$$\Omega = \{\lambda_1 \cdots \lambda_m : \lambda_1, \dots, \lambda_m \in \Lambda, m \geq 0\},$$

For $\lambda = \lambda_1 \cdots \lambda_m \in \Omega$, we denote by $|\lambda| = m$, the length of λ.

We use the notation, for $\lambda_1 \cdots \lambda_m \in \Omega$,

$$\begin{gathered} V_{\lambda_1\lambda_2\cdots\lambda_m} = V_{\lambda_1}V_{\lambda_2}\dots V_{\lambda_m}, \quad S_{\lambda_1\cdots\lambda_m} = S_{\lambda_1}\cdots S_{\lambda_m} \\ \alpha_{i,\lambda} = \alpha_{i,\lambda_1}\alpha_{i\cdot\lambda_1,\lambda_2}\cdots\alpha_{i\cdot\lambda_1\cdot\lambda_2\cdot\dots\cdot\lambda_{m-1},\lambda_m} \end{gathered} \tag{1.11}$$

2 GENERAL RESULTS

Let $\{V_\lambda : \mathcal{K} \to \mathcal{K}\}_{\lambda\in\Lambda}$ be a row co-isometry and $\{S_\lambda : \mathcal{H} \to \mathcal{H}\}_{\lambda\in\Lambda}$ be the corresponding Cuntz dilation. Define

$$\mathcal{K}_m = \overline{\operatorname{span}} \left\{ S_{\lambda_1\cdots\lambda_m} k \mid k \in \mathcal{K}, \lambda_1, \ldots, \lambda_m \in \Lambda \right\} \tag{2.1}$$

Claim 2.1. *$\mathcal{K}_m$ is an increasing sequence and*

$$\overline{\bigcup_{m\in\mathbb{N}} \mathcal{K}_m} = \mathcal{H}$$

Proof. Firstly, let $S_{\lambda_1\cdots\lambda_m} k \in \mathcal{K}_m$ $(k \in \mathcal{K})$. Then we see that

$$S_{\lambda_1\cdots\lambda_m} k = S_{\lambda_1\cdots\lambda_m} \sum_{\lambda\in\Lambda} S_\lambda S_\lambda^* k = \sum_{\lambda\in\Lambda} S_{\lambda_1\cdots\lambda_m} S_\lambda (S_\lambda^* k)$$

Since $S_\lambda^* k = V_\lambda^* k \in \mathcal{K}$ for each $\lambda \in \Lambda$, we have that $S_{\lambda_1\cdots\lambda_m} k \in \mathcal{K}_{m+1}$, so $\mathcal{K}_m \subseteq \mathcal{K}_{m+1}$ Further, since $\mathcal{K}$ is cyclic for the representation, we have that

$$\overline{\bigcup_{m\in\mathbb{N}} \mathcal{K}_m} = \mathcal{H}$$

□

Claim 2.2. *For each $m \geq 0$, $\lambda \in \Lambda$,*

$$S_\lambda^* \mathcal{K}_{m+1} \subseteq \mathcal{K}_m$$

Proof. Let $S_{\lambda_1 \cdots \lambda_{m+1}} k \in \mathcal{K}_{m+1}$. Then

$$S_\lambda^* S_{\lambda_1 \cdots \lambda_{m+1}} k = \delta_{\lambda_1}^{\lambda} S_{\lambda_2 \cdots \lambda_{m+1}} k \in \mathcal{K}_m$$

□

Claim 2.3. *For $k \in \mathcal{K}_m$, and each $n \leq m$, we have that the representation*

$$k = \sum_{|\lambda|=n} S_\lambda k_\lambda$$

is unique, and given by

$$k_\lambda = S_\lambda^* k \in \mathcal{K}_{m-n}$$

Proof. Assume that you have two such representations,

$$k = \sum_{|\lambda|=n} S_\lambda k_\lambda, \quad k = \sum_{|\lambda|=n} S_\lambda k'_\lambda$$

Then we have that

$$0 = ||k - k||^2 = \Big|\Big| \sum_{|\lambda|=n} S_\lambda (k_\lambda - k'_\lambda) \Big|\Big|^2 = \sum_{|\lambda|=n} ||k_\lambda - k'_\lambda||^2.$$

Therefore the representation is unique. Further, for $n \leq m$, $|\lambda'| = n$,

$$S_{\lambda'}^* k = \sum_{|\lambda|=n} S_{\lambda'}^* S_\lambda k_\lambda = \sum_{|\lambda|=n} \delta_{\lambda'}^{\lambda} k_\lambda = k_{\lambda'},$$

and by 2.2, we have that $k_\lambda \in \mathcal{K}_{m-n}$.

□

Claim 2.4. *Let $P_{\mathcal{K}_m}$ be the projection onto $\mathcal{K}_m$. Then*

$$P_{\mathcal{K}_m} = \sum_{|\lambda|=m} S_\lambda P_{\mathcal{K}_0} S_\lambda^*$$

Proof. Denote

$$T = \sum_{|\lambda|=m} S_\lambda P_{\mathcal{K}_0} S_\lambda^*$$

If $h \in \mathcal{K}_m$, then by 2.2, we have that $S_\lambda^* h \in \mathcal{K}_0$, so then

$$P_{\mathcal{K}_0} S_\lambda^* h = S_\lambda^* h$$

Therefore,

$$Th = \sum_\lambda S_\lambda \left(P_{\mathcal{K}_0} S_\lambda^* h\right) = \sum_\lambda S_\lambda S_\lambda^* h = h$$

Now, if $h \perp \mathcal{K}_m$, then we have that for each $k \in \mathcal{K}_0 = \mathcal{K}$

$$\langle S_\lambda^* h, k \rangle = \langle h, S_\lambda k \rangle = 0$$

So then $P_{\mathcal{K}_0} S_\lambda^* h = 0$, and therefore $Th = 0$ for $h \perp \mathcal{K}_m$, and $Th = h$ for $h \in \mathcal{K}_m$, so $T = P_{\mathcal{K}_m}$. □

3 RANDOM WALKS

3.1 CONSTRUCTING THE CUNTZ DILATION

Consider a random walk as with 1.8. We want to find an explicit form for the Cuntz dilation of the row co-isometry $\{V_\lambda\}_{\lambda\in\Lambda}$ coming from these random walks. To this end, we need a bit more notation.

For each fixed vertex i, define

$$\Lambda_i = \{\lambda \in \Lambda \mid \exists j : i \xrightarrow{\lambda} j\},$$

and for each fixed vertex j, denote:

$$\Lambda^j = \{\lambda \in \Lambda \mid \exists i : i \xrightarrow{\lambda} j\}.$$

Further, we define $n_i := |\Lambda_i|$ and $n^j := \left|\Lambda^j\right|$.

Since for each vertex i, we have:

$$\sum_{\lambda\in\Lambda_i} \left|\alpha_{i,\lambda}\right|^2 = 1,$$

we may create the following unitary matrices (not necessarily unique)

$$C_i = \begin{bmatrix} \alpha_{i,\lambda_0} & c^1_{i,\lambda_0} & c^2_{i,\lambda_0} & \cdots & c^{n-1}_{i,\lambda_0} \\ \alpha_{i,\lambda_1} & c^1_{i,\lambda_1} & c^2_{i,\lambda_1} & \cdots & c^{n-1}_{i,\lambda_1} \\ \vdots & \vdots & \vdots & \ddots & \\ \alpha_{i,\lambda_{n_i-1}} & c^1_{i,\lambda_{n_i-1}} & c^2_{i,\lambda_{n_i-1}} & \cdots & c^{n-1}_{i,\lambda_{n_i-1}} \end{bmatrix} = \left[c^k_{i,\lambda}\right]^{k=0,\ldots,n_i-1}_{\lambda\in\Lambda_i},$$

where we adopt the notation that $c^0_{i,\lambda_j} = \alpha_{i,\lambda_j}$.

Definition 3.1. For $N \geq 2$, we define Ω_N as the set of finite words of $0,\ldots,N-1$ not ending in 0, including the empty word, $\emptyset$.

For a digit $k \in \{0,\ldots,N-1\}$ and word $w \in \Omega_N$, we let $kw \in \Omega_N$ denote the concatenation of k and w. As an **important** convention, we define

$$0\emptyset = \emptyset.$$

Definition 3.2. We define "inverse concatenation"

$$\backslash : \Omega_N \times \{0,\ldots,N-1\} \to \Omega_N \cup \{\mathbf{null}\}$$

$$\backslash(w,k) \triangleq w \setminus k = \begin{cases} w' & kw' = w \\ \mathbf{null} & else \end{cases}.$$

It is important to note that **null** is not the same as the empty word, $\emptyset$.

Remark 3.3. For fixed $w \in \Omega_N$, $w \setminus k \neq \mathbf{null}$ for exactly one digit $k \in \{0,\ldots,N\}$. Equivalently, any non-null word w can be uniquely written as $w = kw'$, for some digit k and $w' \in \Omega_N$.

Remark 3.4.

$$\delta_{kw}^{w'} = \delta_w^{w' \setminus k}$$

$$\delta_i^{i'} = \delta_{i \cdot \lambda}^{i' \cdot \lambda}, \forall \lambda \in \Lambda_i, \text{ and } \delta_j^{j'} = \delta_{\lambda \cdot j}^{\lambda \cdot j'}, \forall \lambda \in \Lambda^j$$

We define the Hilbert space of the dilation $\mathcal{H} = \overline{\text{span}} \left\{ (i, w) : i \,\text{vertex},\, w \in \Omega_N \right\}$. We identify $\mathcal{K}$ with $\overline{\text{span}} \left\{ (i, \emptyset) : i \,\text{vertex} \right\}$. In either case, we define the inner product as the following:

$$\langle (i, w), (i', w') \rangle = \delta_i^{i'} \delta_w^{w'},$$

so that the generators of $\mathcal{H}, \mathcal{K}$ form respective orthonormal bases.

Definition 3.5. For convention, we take

$$(\mathbf{null}, w) = (i, \mathbf{null}) = 0, \quad \delta_i^{\mathbf{null}} = \delta_w^{\mathbf{null}} = \delta_{\mathbf{null}}^{\mathbf{null}} = 0$$

for all vertices i and words $w \in \Omega_N$.

Lemma 3.6. *If we have that*

$$\sum_{i \,\text{vertex}} (N - n_i) = \sum_{j \,\text{vertex}} (N - n^j)$$

then the sets

$$\left\{ (j, \lambda) : j \textit{ vertex, } \lambda \in \Lambda \setminus \Lambda^j \right\}, \left\{ (i, k) : i \textit{ vertex, } k \geq n_i \right\}$$

Have the same cardinality and thus there exists a bijection between them, call it φ*. We will define*

$$F = \pi_1 \circ \varphi,\ G = \pi_2 \circ \varphi,\ \tilde{F} = \pi_1 \circ \varphi^{-1},\ \tilde{G} = \pi_2 \circ \varphi^{-1}.$$

Where π_1, π_2 *denote the usual first and second coordinate projections.*

Theorem 3.7. *If we have that*

$$\sum_{i \text{ vertex}} (N - n_i) = \sum_{j \text{ vertex}} (N - n^j)$$

then the row-co-isometries $\left\{V_\lambda : \mathcal{K} \to \mathcal{K} \mid \lambda \in \Lambda\right\}$ *defined by*

$$V_\lambda^*(i, \emptyset) = \begin{cases} \alpha_{i,\lambda}(j, \emptyset) & i \xrightarrow{\lambda} j \\ 0 & else \end{cases}$$

Have a dilation of the form:

$$S_\lambda : \mathcal{H} \to \mathcal{H}$$

$$S_\lambda(j, w) = \begin{cases} \sum_{k=0}^{n_i - 1} \overline{c_{i,\lambda}^k}(i, kw) & i \xrightarrow{\lambda} j \\ (F(j, \lambda), G(j, \lambda)w) & else, \end{cases}$$

where F *and* G *are defined by lemma 3.6*

Proof. We first calculate the adjoint. Case 1: $\lambda \in \Lambda^j$, then denote $i \xrightarrow{\lambda} j$

$$\langle (j,w), S_\lambda^*(i',w')\rangle = \langle S_\lambda(j,w),(i',w')\rangle = \left\langle \sum_{k=0}^{n_i-1} \overline{c_{i,\lambda}^k}(i,kw),(i',w') \right\rangle$$

$$= \sum_{k=0}^{n_i-1} \overline{c_{i,\lambda}^k}\delta_i^{i'}\delta_{kw}^{w'} = \sum_{k=0}^{n_i-1} \overline{c_{i,\lambda}^k}\delta_j^{i'\cdot\lambda}\delta_w^{w'\setminus k} = \left\langle\ (j,w), \sum_{k=0}^{n_i-1} c_{i,\lambda}^k (i'\cdot\lambda, w'\setminus k) \right\rangle$$

Case 2: $\lambda \notin \Lambda^j$. Then we define F and G as in 3.6. We use remark 3.3 to write an arbitrary word as kw'

$$\langle (j,w), S_\lambda^*(i',kw')\rangle = \langle S_\lambda(j,w),(i',kw')\rangle = \langle (F(j,\lambda), G(j,\lambda)w),(i',kw')\rangle$$

$$= \delta_\lambda^{\tilde{G}(i,k')}\langle (F(j,\lambda),w),(i',w')\rangle = \delta_\lambda^{\tilde{G}(i,k')}\langle (j,w),(\tilde{G}(i',k),w')\rangle$$

Therefore, we may write:

$$S_\lambda^*(i,k'w) = \begin{cases} \sum\limits_{k=0}^{n_i-1} c_{i,\lambda}^k (j, k'w\setminus k) & i \xrightarrow{\lambda} j \\ (\tilde{F}(i,k'),w) & \lambda = \tilde{G}(i,k') \\ 0 & else \end{cases}$$

Now, we show that the S_λ are isometries. Let $(j,w) \in \mathcal{H}$. Case 1: $\lambda \in \Lambda^j$. Then we will take i such that $i \xrightarrow{\lambda} j$.

$$S_{\lambda'}^* S_\lambda(j,w) = S_{\lambda'}^* \sum_{k=0}^{n_i-1} \overline{c_{i,\lambda}^k}(i,kw) = \sum_{k'=0}^{n_i-1}\sum_{k=0}^{n_i-1} \overline{c_{i,\lambda}^k} c_{i,\lambda'}^{k'}(i\cdot\lambda', kw\setminus k')$$

$$= \sum_{k=0}^{n_i-1} \overline{c_{i,\lambda}^k} c_{i,\lambda'}^k (i \cdot \lambda', w) = \delta_\lambda^{\lambda'}(j, w)$$

Case 2: $\lambda \neq \Lambda^j$

$$S_{\lambda'}^* S_\lambda(j, w) = S_{\lambda'}^*(F(j,\lambda), G(j,\lambda)w) = \delta_\lambda^{\lambda'}(j, w)$$

So they are isometries. Now we prove that they are row co-isometries. Let $(i, kw) \in \mathcal{H}$. Case 1: $k \geq n_i$,

$$\sum_{\lambda \in \Lambda} S_\lambda S_\lambda^*(i, kw) = \sum_{\lambda \in \Lambda_i} S_\lambda S_\lambda^*(i, kw) + \sum_{\lambda \notin \Lambda_i} S_\lambda S_\lambda^*(i, kw)$$

$$= \sum_{\lambda \in \Lambda_i} S_\lambda \sum_{k'=0}^{n_i-1} c_{i,\lambda}^{k'}(i \cdot \lambda, kw \setminus k') + \sum_{\lambda \notin \Lambda_i} S_\lambda S_\lambda^*(i, kw) = \sum_{\lambda \notin \Lambda_i} S_\lambda \delta_\lambda^{\tilde{G}(i,k)}(\tilde{F}(i,k), w)$$

$$= S_{\tilde{G}(i,k)}(\tilde{F}(i,k), w) = (i, kw)$$

Case 2: $k < n_i$,

$$\sum_{\lambda \in \Lambda} S_\lambda S_\lambda^*(i, kw) = \sum_{\lambda \in \Lambda_i} S_\lambda S_\lambda^*(i, kw) + \sum_{\lambda \notin \Lambda_i} S_\lambda S_\lambda^*(i, kw)$$

$$= \sum_{\lambda \in \Lambda_i} S_\lambda \sum_{k'=0}^{n_i-1} c_{i,\lambda}^{k'}(i \cdot \lambda, kw \setminus k') = \sum_{\lambda \in \Lambda_i} c_{i,\lambda}^k S_\lambda(i \cdot \lambda, w)$$

$$= \sum_{\lambda \in \Lambda_i} \sum_{k'=0}^{n_i-1} \overline{c_{i,\lambda}^{k'}} c_{i,\lambda}^k (i, k'w) = \sum_{k'=0}^{n_i-1} \delta_k^{k'}(i, k'w) = (i, kw)$$

So they are row co-isometries. Now we show that it is cyclic for the representation.

We do so by inducting on the length of words. For the base case, we trivially have that $\mathcal{K} \subseteq \overline{\text{span}}_{p\geq 0} \left\{S_{\lambda_1\cdots\lambda_p}\mathcal{K}\right\}$.

Now, assume that for all words w of length m, and for all vertices i, $(i,w) \in \overline{\text{span}}_{p\geq 0} \left\{S_{\lambda_1\cdots\lambda_p}\mathcal{K}\right\}$. Now let kw be an arbitrary word of length $m+1$, and i some vertex. Case 1: $k \geq n_i$.

Take $(j,\lambda) = \varphi^{-1}(i,k)$. Then

$$(i,kw) = (F(j,\lambda), G(j,\lambda)w) = S_\lambda(j,w) \in \overline{\text{span}}_{p\geq 0} \left\{S_{\lambda_1\cdots\lambda_p}\mathcal{K}\right\}$$

Case 2: $k < n_i$.

$$(i,kw) = \sum_{\lambda\in\Lambda_i}\sum_{k'=0}^{n_i-1} c_{i,\lambda}^k \overline{c_{i,\lambda}^{k'}}(i,k'w) = \sum_{\lambda\in\Lambda_i}\sum_{k'=0}^{n_i-1} c_{i,\lambda}^k S_\lambda(i\cdot\lambda, w) \in \overline{\text{span}}_{p\geq 0} \left\{S_{\lambda_1\cdots\lambda_p}\mathcal{K}\right\}$$

□

3.2 HARMONIC FUNCTIONS AND FIXED POINTS OF B^σ

Definition 3.8. We say that a random walk ***separates*** if for all vertices $i \neq i'$, there exists $m \in \mathbb{N}$ such that for all words of labels $\lambda = \lambda_1\lambda_2\cdots\lambda_m$, either $i \cdot \lambda = \textbf{null}$ or $i' \cdot \lambda = \textbf{null}$.

Definition 3.9. A function h on a random walk is harmonic if

$$h(i) = \sum_{\lambda\in\Lambda} |\alpha_{i,\lambda}|^2 h(i\cdot\lambda) = \sum_{\lambda\in\Lambda_i} |\alpha_{i,\lambda}|^2 h(i\cdot\lambda)$$

Remark 3.10. If T is a fixed point of σ, then $h_T(i) = \langle T(i,\emptyset),(i,\emptyset)\rangle$ is a harmonic function.

Theorem 3.11. *On a separating random walk, let $T \in B^\sigma$. If $h_T \equiv 0$, then $T \equiv 0$.*

Proof. Let $T \in B^\sigma$. Assume that $h_T \equiv 0$. To show that $T \equiv 0$, it suffices to show that

$$\left\langle T(i,\emptyset),(j,\emptyset)\right\rangle = 0$$

for all $i \neq j$. Since the random walk separates, for each $i \neq j$, there exists $m \in \mathbb{N}$ such that for all λ with $|\lambda| = m$, either $i \cdot \lambda = \textbf{null}$ or $j \cdot \lambda = \textbf{null}$. But then since T is a fixed point of σ, we have that

$$\left\langle T(i,\emptyset),(j,\emptyset)\right\rangle = \left\langle \sum_{|\lambda|=m} V_\lambda T V_\lambda^*(i,\emptyset),(j,\emptyset)\right\rangle = \sum_{|\lambda|=m} \left\langle T V_\lambda^*(i,\emptyset), V_\lambda^*(j,\emptyset)\right\rangle.$$

$$= \sum_{|\lambda|=m} \alpha_{i,\lambda}\overline{\alpha_{j,\lambda}} \left\langle T(i\cdot\lambda,\emptyset),(j\cdot\lambda,\emptyset)\right\rangle = 0.$$

□

Corollary 3.12. *On a separating random walk, if all harmonic functions are constant, then the representation is irreducible.*

Proof. Let $A \in (S_\lambda)'$. Take $T = P_\mathcal{K} A P_\mathcal{K} \in B^\sigma$. Then h_T is harmonic and by hypothesis, constant. Denote $T = cI_\mathcal{K}$.

So then

$$h_{P_\mathcal{K}(A-cI_\mathcal{H})P_\mathcal{K}} = h_{T-cI_\mathcal{K}} = 0$$

From Bratteli et. al. [1], there is a one-to-one correspondence between fixed points of

σ and elements in the commutant. Therefore, it must be that $A = cI_{\mathcal{H}}$. Therefore, the representation is irreducible. $\square$

Remark 3.13. For random walks,

$$\mathcal{K}_m = \text{span}\{(i, w) : i \,\text{vertex}, |w| \leq m\}$$

Definition 3.14. For two vertices i, i', we define

$$\text{traj}(i, i') = \left\{(j, j') \,:\, \exists \lambda \in \Omega \text{ s.t. } i \cdot \lambda = j,\, i' \cdot \lambda = j'\right\} \tag{3.1}$$

to be the *trajectory* of (i, i').

Claim 3.15. *Let $T \in B^{\sigma}$, and i, i' vertices. Assume that $|T_{i,i'}|$ is at a max on its trajectory. More precisely, $|T_{i,i'}| \geq |T_{j,j'}|$ for each $(j, j') \in \text{traj}(i, i')$. Then for each $\lambda \in \Omega$ such that $\alpha_{i',\lambda} \neq 0$,*

$$|T_{i,i'}| = |T_{i\cdot\lambda, i'\cdot\lambda}|.$$

Further, the following vectors

$$\left(\alpha_{i,\lambda}\right)_{\lambda},\, \left(\alpha_{i',\lambda}\overline{T_{i,i'}}\right)_{\lambda}$$

are proportional.

Proof. Let $T \in B^{\sigma}$.

$$T_{i,i'} = \langle T(i, \emptyset), (i', \emptyset)\rangle = \sum_{\lambda} \langle TV_{\lambda}^{*}(i, \emptyset), V_{\lambda}^{*}(i', \emptyset)\rangle = \sum_{\lambda} \langle \alpha_{i,\lambda} T(i \cdot \lambda, \emptyset), \alpha_{i',\lambda}(i' \cdot \lambda\, \emptyset)\rangle$$

$$= \sum_{\lambda} \alpha_{i,\lambda} \overline{\alpha_{i',\lambda}} T_{i\cdot\lambda, i'\cdot\lambda}$$

Using the Cauchy-Schwartz inequality, we have that

$$|T_{i,i'}| \leq \left(\sum_{\lambda} |\alpha_{i,\lambda}|^2 \right)^{1/2} \left(\sum_{\lambda} |\alpha_{i',\lambda}|^2 |T_{i\cdot\lambda,i'\cdot\lambda}|^2 \right)^{1/2} = \left(\sum_{\lambda} |\alpha_{i',\lambda}|^2 |T_{i\cdot\lambda,i'\cdot\lambda}|^2 \right)^{1/2}$$

Now, if we assume that $|T_{i,i'}|$ is a maximum on a trajectory, we have that

$$|T_{i,i'}| \leq \left(\sum_{\lambda} |\alpha_{i',\lambda}|^2 |T_{i\cdot\lambda,i'\cdot\lambda}|^2 \right)^{1/2} \leq \left(\sum_{\lambda} |\alpha_{i',\lambda}|^2 |T_{i,i'}|^2 \right)^{1/2} = |T_{i,i'}|$$

So namely, we must have equality, that is: for each λ such that $\alpha_{i',\lambda} \neq 0$

$$|T_{i,i'}| = |T_{i\cdot\lambda,i'\cdot\lambda}|$$

Further, the following vectors

$$\left(\alpha_{i,\lambda}\right)_{\lambda}, \left(\alpha_{i',\lambda} \overline{T_{i,i'}}\right)_{\lambda}$$

must be proportional. □

Remark 3.16. On a random walk, assume the transition probabilities $\alpha_{i,\lambda}$ are independent of i, T is a fixed point of σ, and (i, i') is a pair such that $|T_{i,i'}|$ is maximum on a trajectory. Then $T_{i,i'}$ is constant on the trajectories.

Proof. We may assume that $T_{i,i'}$ is real and non-negative by multiplying by a phase angle.

$$T_{i,i'} = \langle T(i,\emptyset),(i',\emptyset)\rangle = \sum_{\lambda\in\Lambda}\langle TV_\lambda^*(i,\emptyset),V_\lambda^*(i',\emptyset)\rangle$$

$$= \sum_{\lambda\in\Lambda}\alpha_{i,\lambda}\overline{\alpha_{i',\lambda}}\langle T(i\cdot\lambda,\emptyset),(i'\cdot\lambda,\emptyset)\rangle = \sum_{\lambda\in\Lambda}|\alpha_{i,\lambda}|^2 T_{i\cdot\lambda,i'\cdot\lambda}$$

$$= \sum_{\lambda\in\Lambda}|\alpha_{i,\lambda}|^2\Re(T_{i\cdot\lambda,i'\cdot\lambda}) \leq \sum_{\lambda\in\Lambda}|\alpha_{i,\lambda}|^2|T_{i\cdot\lambda,i'\cdot\lambda}| \leq |T_{i,i'}| = T_{i,i'}.$$

Thus we must have equality. Namely, that $T_{i\cdot\lambda,i'\cdot\lambda}$ must be real whenever $\alpha_{i,\lambda}\neq 0$. Therefore, for $\alpha_{i,\lambda}\neq 0$, $T_{i\cdot\lambda,i'\cdot\lambda} = T_{i,i'}$. □

3.3 MINIMAL INVARIANTS

Definition 3.17. On a random walk, we say that a set of vertices M is invariant if all possible transitions stay within M. I.e.

$$i\in M \implies i\cdot\lambda\in M, \quad \forall\lambda\in\Lambda$$

Definition 3.18. We say that an invariant set is minimally invariant if it contains no proper, invariant subsets.

Definition 3.19. For a minimal invariant M, define

$$\mathcal{K}(M) = \overline{\text{span}}\left\{(m,\emptyset) : m\in M\right\}$$

$$\mathcal{H}(M) = \overline{\text{span}}\left\{S_\lambda(m,\emptyset) :\, m\in M;\, \lambda\in\Omega\right\}$$

Remark 3.20. $\mathcal{H}(M)$ is invariant for S_λ^* since

$$S_\lambda^* S_{\lambda'}(m,\emptyset) = \begin{cases} (m,\emptyset) & \text{if } \lambda' = \lambda \\ 0 & \text{else} \end{cases} \in \mathcal{H}(M)$$

Remark 3.21. Each finite random walk has at least one minimal invariant.

Proof. Since $\mathcal{V}$ itself is invariant, we know that the set of invariants is nonempty. Then we may pick one such invariant with the smallest cardinality. By construction, it must be a minimal invariant. □

Theorem 3.22. *Given a finite random walk with labels with minimal invariants* $\{M_l\}_{l=0}^L$*. Denote the Cuntz dilation of the* $V_\lambda : \mathcal{K} \to \mathcal{K}$ *as* $S_\lambda : \mathcal{H} \to \mathcal{H}$*.Then the dilation of* $V_\lambda\big|_{\mathcal{K}(M_l)} : \mathcal{K}(M_l) \to \mathcal{K}(M_l)$ *is* $S_\lambda\big|_{\mathcal{H}(M_l)}$ *and we have that*

$$\mathcal{H} = \oplus_l \mathcal{H}(M_l)$$

In other words, the Cuntz dilation of row co-isometries coming from a finite random walk can be decomposed in terms of the minimal invariants of the random walk.

Proof. Firstly, we show that for $l \neq p$, $\mathcal{H}(M_l) \perp \mathcal{H}(M_p)$. Let $\lambda, \lambda' \in \Omega$ with $|\lambda| \leq |\lambda'|$, and let

$$S_\lambda(m,\emptyset) \in M_l, \quad S_{\lambda'}(m',\emptyset) \in M_p.$$

If λ is a prefix of λ', then we have that

$$\langle S_\lambda(m,\emptyset), S_{\lambda'}(m',\emptyset)\rangle = \langle S_{\lambda'}^* S_\lambda(m,\emptyset), (m',\emptyset)\rangle = \langle S_{\lambda'\setminus\lambda}^*(m,\emptyset), (m',\emptyset)\rangle$$

$$= \langle P_{\mathcal{K}} S^*_{\lambda' \setminus \lambda}(m, \emptyset), (m', \emptyset) \rangle = \langle V^*_{\lambda' \setminus \lambda}(m, \emptyset), (m', \emptyset) \rangle = 0.$$

If λ is not a prefix of λ', then we have that

$$\langle S_\lambda(m, \emptyset), S_{\lambda'}(m', \emptyset) \rangle = \langle (m, \emptyset), S^*_\lambda S_{\lambda'}(m', \emptyset) \rangle = 0.$$

In either case, $S_\lambda(m, \emptyset) \perp S_{\lambda'}(m', \emptyset)$, and thus $\mathcal{H}(M_l) \perp \mathcal{H}(M_p)$.

Second, we see that $\mathcal{K} \subset \overline{\oplus_l \mathcal{H}(M_l)}$. Let $(i, \emptyset) \in \mathcal{K}$. Then we have that for $n \in \mathbb{N}$,

$$\begin{aligned}
&\inf_{c_{m,\lambda}} \left|\left|(i, \emptyset) - \sum_{|\lambda|=n} \sum_{m \in \cup_l M_l} c_{m,\lambda} S_\lambda(m, \emptyset)\right|\right|^2 \\
&= \inf_{c_{m,\lambda}} \left|\left| \sum_{|\lambda|=n} S_\lambda S^*_\lambda(i, \emptyset) - \sum_{|\lambda|=n} \sum_{m \in \cup_l M_l} c_{m,\lambda} S_\lambda(m, \emptyset)\right|\right|^2 \\
&= \inf_{c_{m,\lambda}} \sum_{|\lambda|=n} \left|\left|S^*_\lambda(i, \emptyset) - \sum_{m \in \cup_l M_l} c_{m,\lambda}(m, \emptyset)\right|\right|^2 \\
&\leq \sum_{\substack{|\lambda|=n \\ V^*_\lambda(i,\emptyset) \notin \cup_l M_l}} \left|\left|S^*_\lambda(i, \emptyset)\right|\right|^2 \\
&= \sum_{\substack{|\lambda|=n \\ V^*_\lambda(i,\emptyset) \notin \cup_l M_l}} |\alpha_{i,\lambda}|^2 =: P(i, n),
\end{aligned}$$

where $P(i, n)$ is exactly the probability of staying outside of a minimal invariant after n steps from i. Note that $P(i, n)$ is decreasing since the probability of transitioning from inside a minimal invariant to outside is zero. Now, for each vertex i, note that there is a smallest number steps after which you are in a minimal invariant with non-zero probability (otherwise take all paths from i and this is a new invariant). Since we are working on a finite graph, then denote L to be the maximum such number of steps for vertices on the graph. Then, define p_i to be the probability that you enter a minimal invariant after L steps. $p_i > 0$ for each vertex i, and since we have a finite graph

$$p = \min_{i\,\text{vertex}} p_i > 0.$$

Then by induction we have that

$$\begin{aligned}
&\lim_{n\to\infty} \inf_{c_{m,\lambda}} \left|\left|(i,\emptyset) - \sum_{|\lambda|=n} \sum_{m\in\cup_l M_l} c_{m,\lambda} S_\lambda(m,\emptyset)\right|\right|^2 \\
&\leq \lim_{n\to\infty} P(i,n) = \lim_{n\to\infty} P(i,nL) \\
&\leq \lim_{n\to\infty} \sum_{\substack{|\lambda|=L \\ i\cdot\lambda\neq\cup_l M_l}} |\alpha_{i,\lambda}|^2 P(i\cdot\lambda,(n-1)L) \\
&\leq \lim_{n\to\infty} (1-p)^n = 0.
\end{aligned}$$

Thus we have that $\mathcal{K} \subseteq \overline{\oplus_l \mathcal{H}(M_l)}$. Now if we have some element $(i,w) \in \mathcal{H}$, we know we may write

$$(i,w) = \sum_{|\lambda|=|w|} S_\lambda S_\lambda^*(i,w),$$

By remark 3.13, $S_\lambda^*(i,w) \in \mathcal{K}_0 = \mathcal{K}$. Therefore

$$(i,w) = \sum_{|\lambda|=|w|} S_\lambda S_\lambda^*(i,w) \in \oplus_l \mathcal{H}(M_l).$$

Therefore $\mathcal{H} \subseteq \oplus_l \mathcal{H}(M_l)$, and the other inequality is trivial thus

$$\mathcal{H} = \oplus_l \mathcal{H}(M_l)$$

Further, the Cuntz relations on S_λ immediately imply that

$$S_\lambda\big|_{\mathcal{H}(M_l)} : \mathcal{H}(M_l) \to \mathcal{H}(M_l)$$

satisfies the Cuntz relations and it is a Cuntz dilation of

$$V_\lambda\big|_{\mathcal{K}(M_l)} : \mathcal{K}(M_l) \to \mathcal{K}(M_l).$$

□

Corollary 3.23. *If a random walk with labels has more than one minimal invariant, then the associated Cuntz dilation is reducible.*

Proof. For minimal invariants $M_1 \neq M_2$,

$$A = I_{\mathcal{H}(M_1)} + 0_{\mathcal{H}(M_2)} \in \big(S_\lambda\big)'$$

□

LIST OF REFERENCES

[1] Ola Bratteli, Palle Jorgensen, Aki Kishimoto, and Reinhard Werner. Pure states on $\mathcal{O}_d$. *J. Operator Theory*, 43, 04 1999.