Modeling And Characterizations Of New Notions In Life Testing With Statistical Applications

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MODELING AND CHARACTERIZATIONS OF NEW NOTIONS IN LIFE TESTING WITH STATISTICAL APPLICATIONS

by

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ABSTRACT

Knowing the class to which a life distribution belongs gives us an idea about the aging of the device or system the life distribution represents, and enables us to compare the aging properties of different systems. This research intends to establish several new nonparametric classes of life distributions defined by the concept of inactivity time of a unit with a guaranteed minimum life length. These classes play an important role in the study of reliability theory, survival analysis, maintenance policies, economics, actuarial sciences and many other applied areas.

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CHAPTER ONE: INTRODUCTION

1.0 Chapter Outline

In this chapter, we will review the most important aging notions and classes known in the literature. The first section introduces the notions of aging and presents their related probabilistic properties. These notions are applicable in both biostatistics and actuarial science among other areas. They also are useful in survival analysis studies when we are faced with left or right censored data. The second section considers the life distribution of a unit subjected to a sequence of shocks occurring randomly in time, and each shock causes a random amount of damage to the life of the unit.

1.1 Introduction

Univariate notions of aging and their related nonparametric classes of life distributions defined by aging properties play a central role in survival analysis, reliability theory, maintenance policies and many other actuarial science, engineering, economics, biometry and applied probability areas. They are also useful in obtaining fundamental inequalities of estimates and test procedures. In the last four decades, remarkable studies have been done on the different aspects of life time distributions, i.e., a distribution F such that $F(t) = 0, \text{ if } t < 0$. Although this area of research enjoys a very rich literature, there is still room for further development. The concept of inactivity time of a unit with a guaranteed minimum length life which is developed in this research among others is important in modeling life time data and in defining various new life classes. Based on this concept, we define several functions such as mean inactivity time guaranteed minimum length life. These functions are useful for example, in studying survival analysis when faced with such interval-censored data. They also enable us to better estimate premium amounts for clients by more accurately estimating value when encountering left or interval-censored data. Based on U-statistics, we establish new nonparametric test procedures to test these new classes against specific alternatives. These results are applicable in areas such as actuarial sciences, financial risk companies, and biostatistics. Multivariate extensions of the aging notions we develop are also discussed as well.

1.2 Background on Notions and Classes of Life Distributions

1.2.1 Notions of Failure

Barlow and Proschan (1975) introduced some important notions of aging and their related probabilistic properties. Ahmad et al. (2005), Asadi (2005), Kayid and Ahmad (2004), Nanda et al. (2003), Nanda et al. (2001) studied these notions and their related classes.

Let X be a nonnegative random variable representing the lifetime of a unit having distribution function, survival function, and density function F, $\bar{F} = 1 - F$, and f, respectively. Let the random variables $X_t = \left[X - t \mid X \geq t\right]$and $X_{(t)} = [t - X \mid X \leq t]$ denote the residual lifetime and the reversed residual lifetime (that is, the time elapsed since failure, given that failure has occurred at or before time t of a unit at age $t \geq 0$ (see also Li et al. 2005, Li et al. 2004, Chandra et al. 2001, and Block et al. 1998). The random variable $X_{(t)}$ is also known as "inactivity time" of a unit at age $t \geq 0$.

1.2.1.1 Hazard (Failure) Rate

Consider the conditional probability that a unit fails in an interval $[t, t+\delta_t)$, given that this unit is working without any failure, until time t. The hazard rate (or failure rate) function is defined as

$$r(t) = {f(t)}\Big/{\bar{F}(t)} = \lim_{\delta_t \to 0} \frac{\bar{F}(t) - \bar{F}(t+\delta_t)}{\delta_t \bar{F}(t)} = -\frac{d}{dt} \log \bar{F}(t), \text{ for } t \geq 0. \qquad (1.2.1)$$

This function uniquely characterizes the survival function and among its uses is its application in studies of right censored data.

1.2.1.2 Reversed Hazard (Failure) Rate

The reversed hazard rate of a random variable at the point t is defined by

$$\tau(t) = \frac{f(t)}{F(t)} = \lim_{\delta_t \to 0} \frac{F(t+\delta_t) - F(t)}{\delta_t F(t)} = \frac{d}{dt} \log F(t), \text{ for } t \geq 0 \qquad (1.2.2)$$

This is the conditional probability that a unit failed in an interval of width δ_t preceding t, given that it failed at or before time t. Reversed hazard rate function is useful among other ways in the estimation of the survival function for left censored lifetimes.

1.2.1.3 Mean Residual Life Function

The mean residual life function (*MRL*), which is also called expected remaining life or mean excess function, has received much attention in the literature. Ebrechts et al. (1997) showed that this function has an important role in many fields of applied probability such as industrial reliability, economics, biomedical science, actuarial science and maintenance policies. Mean residual life functions uniquely characterize the life distribution functions (Kotz and Shanbhag, 1980). It is simply the expected remaining lifetime given that the unit is survived up to age t, $i.e.$

$$\gamma(t) = E(X_t) = E(X - t \mid X \geq t) = \frac{1}{\overline{F}(t)} \int_t^\infty \overline{F}(u)du. \qquad (1.2.3)$$

In another words, it is the area under the survival curve to the right of time t divided by the survival distribution at time t.

1.2.1.4 Mean Inactivity Time

The mean inactivity time (MIT) of a unit at time t is defined by

$$E(X_{(t)}) = E[t - X \mid X \leq t] = \frac{1}{\overline{F}(t)} \int_0^t F(u)du \ . \qquad (1.2.4)$$

Ahmad et. al. (2005) and Kayid and Ahmad (2004) studied the class of life time distribution with increasing mean inactivity time and established a new nonparametric test procedure to test its specific alternative. Nanda, et al. (2003) have shown that a random variable X has decreasing reversed hazard rate ($DRHR$) if and only if $X_{(t_1)} \overset{ST}{\leq} X_{(t_2)}$, for all $0 < t_1 < t_2$, where $X \overset{ST}{\leq} Y$ means $\overline{F}_X(x) \leq \overline{G}_Y(x)$ for all x. Asadi (2005) obtained the mean inactivity components of a parallel system, so called mean past lifetime (MPL), where

$$\begin{aligned} M_n^r(t) &= E(\phi_n^r(t)) = E(t - T_{r:n} \mid T_{n:n} \leq t) \\ &= \sum_{i=r}^{n} \binom{n}{i} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} M_{i+j}(t), \end{aligned} \qquad (1.2.5)$$

where $M_k(t)=\dfrac{\int_0^t F^k(u)du}{F^k(t)}$ and components are ordered in terms of reversed hazard rate. He also showed that if $\tau_F(t)\ge\tau_G(t)$ then $M_n^r(t)\le K_n^r(t)$, where $\tau_F(t)$, $\tau_G(t)$ are hazard rate functions corresponding to the distribution functions F and G respectively, also $M_n^r(t)$ and $K_n^r(t)$ are mean past lifetimes of two parallel systems, respectively.

1.2.2 Characterizations of Aging Notions

The following theorems summarize the most familiar aging classes of distributions in the area of reliability and life testing.

Theorem 1.1: The following statements are equivalent

a) Life distribution function F is increasing failure rate (IFR)

b) $\bar{F}(x+t)/\bar{F}(t)$ is decreasing in $t\ge 0$ for all $x\ge 0$

c) $r(t)=f(t)/\bar{F}(t)$ is increasing in $t\ge 0$

d) $Log\bar{F}$ is concave

e) $\bar{F}$ is pólya frequency function(PF_2). (Function F is PF_2 if F has a density f such that $\dfrac{f(x+t)}{f(t)}$ is increasing in t for all $x\ge 0$).

f) The residual lifetime of a functioning unit of age t is stochastically decreasing in t.

Theorem 1.2: Life distribution F is Increasing Failure Rate in Average ($IFRA$) if and only if

a) $-t^{-1}\ln\bar{F}(t)$ is increasing in $t\ge 0$

b) $R(t) = t^{-1}\int_0^t r(u)du$ is increasing in $t \geq 0$

c) $-\log \overline{F}(t)$ is star-shaped (a nonnegative function g defined on $[0,\infty)$ with $g(0)=0$ is star-shaped if $x^{-1}g(x)$ is increasing on $(0,\infty)$, or if $g(\alpha x) \leq \alpha g(x)$ for $0 \leq \alpha \leq 1$, $x \in [0,\infty)$)

Theorem 1.3: Life distribution F belongs to the class of new better than used (NBU) if and only if

a) $\overline{F}(s+t) \leq \overline{F}(s)\overline{F}(t)$ for all $s,t \geq 0$

b) The lifetime of a new unit is stochastically greater than the residual lifetime of a functioning unit of age t.

Theorem 1.4: Distribution function F is new better than used in expectation($NBUE$) if and only if

a) $\int_t^\infty \overline{F}(u)du \leq \mu\overline{F}(t)$ for all $t \geq 0$

b) The expected lifetime of a new unit is greater than the expected residual lifetime of a functioning unit of age t.

Theorem 1.5: F belongs to the class of decreasing mean residual life ($DMRL$) if and only if

a) $\gamma(s) \geq \gamma(t)$ for $0 \leq s \leq t$.

b) The residual life of an un-failed unit at age t has a mean that is decreasing in t.

Hollander and Proschan (1975) showed that the exponential distribution is the boundary member for the above classes. Bryson and Siddiqui (1969) proved the existence of the following chain implication among these classes:

$$IFR \Rightarrow IFRA \Rightarrow NBU \Rightarrow NBUE \quad \text{and} \quad IFR \Rightarrow DMRL \Rightarrow NBUE .$$

1.2.3 Mean Residual Life Function in terms of Hazard Rate Function and Vice Versa

We can express a hazard rate function, $r(t)$, in terms of $\gamma(t)$ and $\gamma'(t)$ as $\gamma'(t) = 1 - \gamma(t)r(t)$, where $\gamma'(t)$ is the first derivative of $\gamma(t)$. Gupta and Bradley (2003) proposed an expression for the mean residual life function in terms of hazard rate function for a general family of distributions including the Pearson family by

$$\gamma(t) = \mu - t + g(t)r(t), \qquad t \geq 0$$

This family has the following property

$$\frac{f'(x)}{f(x)} = \frac{\mu - x}{g(x)} - \frac{g'(x)}{g(x)}, \tag{1.2.6}$$

where μ is constant and g satisfies the first order linear differential equation

$$g'(x) + \frac{f'(x)}{f(x)} g(x) = \mu - x. \tag{1.2.7}$$

They showed that if the hazard rate function is strictly increasing from some point onward, and under some certain conditions, the mean residual life function can be expressed in terms of normal probability function, $\Phi(t)$:

$$\gamma(t) = \sum_{k=0}^{n-1} b_k(t)\varphi_k(t) + \circ((r(t))^{-1-n\varepsilon}) \qquad \text{as} \quad t \to \infty, \tag{1.2.8}$$

where $\sum_{k=0}^{\infty} b_k x^k = \exp\{-\sum_{k=3}^{\infty} r^{(k-1)}(t)\frac{x^k}{k!}\}$, and

$$\varphi_k(t) = \int_0^\infty x^k \exp\{-xr(t) - \tfrac{1}{2}x^2 r'(t)\}dx$$

$$= (-1)^k \sqrt{\frac{2\pi}{r'(t)}} \left[\frac{\partial^k}{\partial p^k} \left(1 - \Phi\left(\frac{p}{\sqrt{r'(t)}} \right) \right) \exp\left\{ \frac{p^2}{2r'(t)} \right\} \Bigg|_{p=r(t)} \right].$$

1.2.4 Partial Ordering of Life Distributions (Two-Life-Distribution Cases)

Let X and Y be nonnegative absolutely continuous random variables with density functions *f, g* and corresponding survival functions $\overline{F}$ and $\overline{G}$, respectively. The random variable X is larger than Y in mean residual life ordering $X \overset{MR}{\geq} Y$ if $\gamma_F(t) \geq \gamma_G(t)$ or equivalently

$$X \overset{MR}{\geq} Y \qquad iff \qquad \frac{\int_t^\infty \overline{F}(u)du}{\int_t^\infty \overline{G}(u)du} \quad \text{increasing } in\ t \geq 0$$

Singh (1986) proposed the variance residual life (*VR*) ordering as follows

$$X \overset{VR}{\geq} Y \qquad if \quad v_F(t) \leq v_G(t) \quad \text{for all } t \geq 0$$

where $\quad v_F(t) = \dfrac{d}{dt}\left(-\log \displaystyle\int_t^\infty \int_u^\infty \overline{F}(v)dvdu\right),$

or equivalently

$$X \overset{VR}{\geq} Y \qquad iff \quad \frac{\int_t^\infty \int_u^\infty \overline{F}(v)dvdu}{\int_t^\infty \int_u^\infty \overline{G}(v)dvdu} \qquad \text{is increasing in } t \geq 0$$

Singh (1986) showed that the *VR* ordering is stronger than the *MRL* ordering, *i.e*,

$$X \overset{VR}{\geq} Y \Rightarrow X \overset{MRL}{\geq} Y$$

Deshpande et. al. (1986) showed that $X \overset{MRL}{\geq} X_t$ *iff* X is *DMRL* for all $t \geq 0$. Launer (1984) introduced the class of distribution having decreasing variance residual life (*DVRL*). A random variable X is said to be *DVRL* if the variance of residual life X_t is a non-increasing function of t on $[0,\infty)$. Gupta et. al. (1987) have shown that a random variable X is *DVRL* *iff*

$$\frac{E\left[(X-t)^2 \mid X \geq t\right]}{E\left[X-t \mid X \geq t\right]} \text{ is decreasing in } t \geq 0.$$

They also showed that the $DMRL$ class is contained in $DVRL$ class.

Theorem 1.6: Define $v_F(t) = \frac{d}{dt}\left(-\log \int_t^{\infty} \int_u^{\infty} \bar{F}(v)dvdu\right)$ and $u_F(t) = \frac{d}{dt}\left(-\log \int_t^{\infty} \bar{F}(u)du\right)$. Then

(i) $X \overset{VR}{\geq} X_t$ for all $t \geq 0$ *iff* X is $DVRL$

(ii) X is $DMRL$ *iff* $u_F(t)$ is a non-decreasing function of t

(iii) X is $DVRL$ *iff* $v_F(t)$ is a non-decreasing function of t

1.2.5 Estimation of Mean Residual Life

An empirical estimate of $\gamma(t)$, MRL, for a sample $X_1, X_2, ..., X_n$ with distribution function F, is given by

$$\gamma_n(t) = \frac{\int_t^{\infty} \bar{F}_n(u)du}{\bar{F}_n(t)}, \qquad (1.2.9)$$

where $\bar{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i - x)$ is the empirical survival function.

Csörgő, and Zitikis (1996) have studied the asymptotic behavior of the function $\gamma(t) - \gamma_n(t)$. The empirical function $\gamma_n(t)$ does not take into account the smoothness of $\gamma(t)$, which makes it a rough estimate of $\gamma(t)$. Several nonparametric estimates of $\gamma(t)$

have been proposed in the literature . For instance, Ruiz and Guillamon (1996) used a recursive kernel estimate and empirical survival function to estimate the denominator. To smooth both numerator and denominator in $\gamma(t)$, Chaubey and Sen (1999) applied Hille's Theorem (see Hille (1948)). Abdous and Berred (2005) used the following kernel estimator:

$$\hat{\gamma}_n(t) = \frac{\int_t^\infty \sum_{i=1}^n \bar{K}(\frac{u-X_i}{h})du}{\sum_{i=1}^n \bar{K}(\frac{t-X_i}{h})}, \tag{1.2.10}$$

where $\bar{K}(t) = \int_t^\infty k(u)du$ in which $k(\cdot)$ stands for an arbitrary probability density function. Also, the bandwidth $h = h_n > 0$ is a sequence of smoothing parameters. They also looked at $\gamma(t)$ as a function and then considered the following estimator:

$$\tilde{\gamma}_n(t) = \int_{-\infty}^{\infty} k_h(t-u)\gamma_n(u)du\,. \tag{1.2.11}$$

Using the fact that the MRL function $\gamma(\cdot)$ is continuously differentiable at t, by Taylor's Theorem, we have

$$\gamma(y) \approx \gamma(t) + (y-t)\gamma'(t)\,.$$

In other words, $\gamma(y)$ is approximately linear in a neighborhood of t. For this reason, they looked at a linear polynomial which minimized the following least squares problem:

$$\int_{-\infty}^{U_F} k_h(y-t)[\gamma_n(y) - a_0 - a_1(y-t)]^2 dy\,, \tag{1.2.12}$$

where U_F is assumed to be known (possibly infinite) and the kernel k is a symmetric probability density function with support $[-1, 1]$.

By comparing (1.2.12) with Taylor's Theorem, $\hat{a}_{0n}(t)$ and $\hat{a}_{1n}(t)$ are expected to estimate $\gamma(t)$ and $\gamma'(t)$, respectively. Here

$$\hat{a}_{in}(x) = \begin{cases} \frac{1}{h^i}\int_{-1}^{1} W_i(\upsilon,1)e_n(x+h\upsilon)d\upsilon & \text{if} \quad U_F = \infty \text{ or if } U_F < \infty \text{ and if } x < U_F - h \\ \frac{1}{h^i}\int_{-1}^{1} W_i(\upsilon,\Gamma)e_n(x+h\upsilon)d\upsilon & \text{if} \quad U_F < \infty \text{ and } x = U_F - \Gamma h \\ 0 & \text{if } U_F < \infty \text{ and } x \geq U_F + \text{h} \end{cases}$$

where $i = 0,1$, the constant Γ is fixed in $(-1,1]$ and the weight functions W_i are defined by

$$W_0(\upsilon,\eta) = \frac{\bar{K}(\upsilon)\int_{-1}^{\eta}(u^2 - u\upsilon)\bar{K}(u)du}{\int_{-1}^{\eta}\int_{-1}^{\eta}(u^2 - u\upsilon)\bar{K}(u)\bar{K}(\upsilon)dud\upsilon}1_{[-1,\eta]}(\upsilon)$$

and

$$W_1(\upsilon,\eta) = \frac{\bar{K}(\upsilon)\int_{-1}^{\eta}(u - \upsilon)\bar{K}(u)du}{\int_{-1}^{\eta}\int_{-1}^{\eta}(u^2 - u\upsilon)\bar{K}(u)\bar{K}(\upsilon)dud\upsilon}1_{[-1,\eta]}(\upsilon),$$

where η stands for either 1 or Γ.

They also showed that $\hat{a}_{0n}(t)$ is asymptotically unbiased for $\gamma(t)$ and its variance goes to zero as $n \to \infty$.

1.2.6 Tests for Decreasing Mean Residual Life

Hollander and Proschan (1975) proposed a test procedure for testing $DMRL$, against exponentiality. They considered the following integral as a measure of deviation, for a given F, from the null hypothesis of exponentiality:

$$\Delta(F) = \iint_{s<t} \bar{F}(s)\bar{F}(t)\left\{\gamma_F(s) - \gamma_F(t)\right\} dF(s)dF(t) . \qquad (1.2.13)$$

We can rewrite (1.2.13) as $\Delta(F) = E_F\left\{I(S \le T)D(S,T)\right\}$, where

$$D(s,t) = \bar{F}(s)\bar{F}(t)\left\{\gamma_F(s) - \gamma_F(t)\right\}$$

They used L-statistic to come up with the test statistic $V^* = \dfrac{V}{\bar{X}}$ where

$$V \simeq n^{-1}\sum_{i=1}^{n} J_1\left(\frac{i}{n}\right)X_{(i)} , \qquad (1.2.14)$$

where $J_1(u) = \dfrac{4}{3}u^3 - 4u^2 + 3u - \dfrac{1}{2}$ and $X_{(i)}$ is the i^{th} statistic of the sample.

They showed the limiting distribution of $n^{\frac{1}{2}}\left\{V^* - \mu\left(J_1, F\right)\right\} \big/ \mu(F)$ is $N\left\{0, \sigma^2(J_1,F) \big/ \mu^2(F)\right\}$

where $\mu(F)$ is the mean of the distribution F and

$$\mu\left(J_1, F\right) = \int xJ_1\{F(x)\}dF(x)$$

$$\sigma^2(J_1,F) = \iint J_1\{F(x)\}J_1\{F(y)\}[F\{\min(x,y) - F(x)F(y)]dxdy$$

Whereas significantly large values of V^* suggest decreasing mean residual life, significantly small values suggest increasing mean residual life.

Ahmad (1992) proposed a new test procedure based on U-statistic to test DMRL against exponentiality. The measurement of deviation from the null hypothesis (constant $\gamma(t)$) is defined as follow:

$$\delta_F = \int_0^\infty \{\bar{F}^2(x) - f(x)v(x)\}dF(x)$$

$$=\int_0^\infty \{2x\bar{F}(x)-v(x)\}dF(x), \tag{1.2.15}$$

where $v(x)=\int \bar{F}(u)du$ assuming $x\bar{F}^2(x)\to 0$ as $x\to\infty$.

Using the empirical distribution $\bar{F}_n(x)$, Ahmad (1992) estimated δ_F by

$$\hat{\delta}_F = \int_0^\infty \{2x\bar{F}_n(x)-\hat{v}(x)\}dF_n(x)$$

$$= n^{-2}\sum_{i=1}^n\sum_j^n (3X_i - X_j)I(X_j - X_i), \tag{1.2.16}$$

where

$$\hat{v}(x)=n^{-1}\sum_{j=1}^n (X_j - x)I(X_j - x) \quad \text{and } I(u)=1 \text{ if } u>0 \text{ and } I(u)=0 \text{ if } u\le 0.$$

The symmetrized U-statistic form of $\hat{\delta}_F$ is given by

$$U_n = (n(n-1))^{-1}\sum_{i<j}\{\phi(X_i,X_j)+\phi(X_j,X_i)\},$$

where $\phi(X_1,X_2)=(3X_1-X_2)I(X_2-X_1)$.

Asymptotic properties have been studied through the following theorem:

Theorem 1.7: (*i*) $U_n \to \delta_F$ with probability 1 as $n\to\infty$

(ii) As $n\to\infty, n^{\frac{1}{2}}(U_n-\delta_F)$ is asymptotically normal with mean 0 and variance

$$\sigma^2 = 4[E\{E\bar{\phi}(X_1,X_2)\,|\,X_1\}^2-\delta_F^2]$$
$$=4\operatorname{var}\left\{\frac{1}{2}(3-X_1)+2X_1\bar{F}(X_1)-2\int_{X_1}^\infty u dF(u)\right\}$$

where $\bar{\phi}(X_1,X_2)=\frac{1}{2}\{\phi(X_1,X_2)+\phi(X_2,X_1)\}$

(iii) $\mathrm{var}(U_n) = \binom{n}{2}^{-1} \sum_{r=1}^{2} \binom{2}{n} \binom{n-2}{2-r} \zeta_r$ where

$$\zeta_1 = \sigma^2$$

$$\zeta_2 = \mathrm{var}\{\bar{\phi}(X_1, X_2)\} = \mathrm{var}\{\frac{1}{2}(3X_2 - X_1) + 2(X_1 - X_2)I(X_2 - X_1)\}$$

(iv) Under the null hypothesis, H_o, $\delta_F = 0$ and $n^{\frac{1}{2}}U_n$ is asymptotically normal with mean 0 and variance $\sigma_0^2 = \frac{1}{3}$

(v) If F is continuous with decreasing mean residual life then the U_n test is consistent.

Ahmad (1992) also studied the Pitman asymptotic relative efficiency of U_n and V^* using a linear failure rate distribution, Makeham distribution and Weibull distribution. The Pitman asymptotic relative efficiency is defined as:

$$e(T_1, T_2) = \{\mu_1'(\theta)/\sigma_1(\theta)\} \Big/ \{\mu_2'(\theta)/\sigma_2(\theta)\}$$

where, for $i = 1, 2$, $\mu_i'(\theta)$ and $\sigma_i(\theta)$ respectively, denote the derivative of the limit of $E(T_i)$ and the limit of $\mathrm{var}(T_i)$ as $n \to \infty$ evaluated at the null hypotheses.

The results are, as follows:

Table 1.1

Pitman asymptotic relative efficiency of U_n and V^* using a linear failure rate distribution, Makeham distribution and Weibull distribution

Distribution	$e(U_n, V^*)$
$\bar{F}_1 = \exp\{-(x + \frac{1}{2}\theta x^2)\} \quad \theta, x \geq 0$	2.63
$\bar{F}_2 = \exp\{-(x + \frac{1}{2}\theta(x + e^x - 1))\} \quad \theta, x \geq 0$	4.2
$\bar{F}_3 = \exp(-x^\theta) \quad \theta, x \geq 0$	1.43

1.2.7 Aging Notions in Shock Models

Shock models provide realistic aging classes of life distributions in order to study a unit situated in a random environment. Esary et al. (1973) considered the life distribution of a unit subjected to a sequence of shocks occurring randomly in time. The probability $\bar{H}(t)$ for this unit to survive beyond time t is given by

$$\bar{H}(t) = \sum_{k=0}^{\infty} P(N(t) = k)\bar{P}(k)$$

where $N(t)$ denotes the number of shocks the unit is subjected to in the time interval $[0, t]$, and $\bar{P}(k)$ is the probability of surviving the first k shocks, $k = 0, 1, \ldots$. Here the shocks are governed by a Poisson process with intensity λ. Esary et al. (1973) have shown that models of the type $\bar{H}(t) = \sum_{k=0}^{\infty} P(N(t) = k)\bar{P}(k)$ governed by a Poisson counting process with the $\bar{P}(k)$'s satisfying a discrete $DMRL$ property (that means $\frac{\sum_{j=k}^{\infty}\bar{P}_j}{\bar{P}_k}$ is decreasing in $k = 0,1,...$) lead to the $DMRL$ property of the survival function $\bar{H}(t)$. A-Hameed and Proschan (1973) considered a unit subjected to shocks when shocks were governed by a non- homogeneous Poisson process. They also showed if $\bar{P}_k$ is discrete $DMRL$ and $\Lambda(t)$ is convex then $\bar{H}$ is $DMRL$. ($\Lambda(t)$ is a mean value function and event rate $\lambda(t) = \frac{d\Lambda(t)}{dt}$, both are defined on the domain $[0,\infty)$). Ghosh and Ebrahimi (1982) have extended this result to a bivariate case.

<u>1.2.7.1 Cumulative Damage Shock Models</u>

Suppose that the i^{th} shock of an item causes a random damage X_i. Damages accumulate additively, and the k^{th} shock is survived by the unit if $\sum_{i=1}^{k} X_i$ does not exceed the capacity or threshold x (which is not random) of the unit. Assuming $X_i^{'}s$ are independent with common distribution F. Here

$$\bar{P}_k = F^{(k)}(x)$$

where $F^{(k)}$ denotes the k^{th} convolution of F, $k = 1,2,...$, and $F^{(0)}$ is degenerate at 0. Alternatively, it may happen that successive shocks become increasingly effective in causing wear or damage, even though they are independent. That means the distribution of i^{th} damage, $F_i(z)$ is an increasing function in $i = 1,2,...$ for each z. In this situation

$$\bar{P}_0 = 1 \quad \text{and} \quad \bar{P}_k = F_1 * F_2 * ....F_k(x), \qquad k = 1,2,....,$$

where '*' denotes convolution. Esary et al. (1973) and A-Hameed et al. (1973) have studied the probabilistic properties of these models when the successive damages are dependent and the critical threshold is regarded as a random variable (Y) with distribution G, and in addition the $X_i's$ are independent of the threshold. In this case, the probability of surviving k shocks is

$$\overline{P}_k = P[\sum_{i=1}^{k} X_i \leq Y] \quad k = 1, 2, ..., \text{ and } \overline{P}_0 = 1$$

It follows that

$$\overline{P}_k = \int_0^\infty F_1 * F_2 * ....F_k(x) dG(x), \quad k = 1, 2, ...$$

$$= E\overline{G}(X_1 + X_2 + ... + X_k).$$

A-Hameed et al. (1973) considered the i^{th} shock causing damage X_i having distribution F_i with the gamma density $f_i(x) = \frac{b^{a_i} x^{a_i - 1}}{\Gamma(a_i)} e^{-bx} \quad x \geq 0,\ a_i > 0$. They showed that if $\sum_{j=1}^{k} a_j = k$ and $\overline{G}$ is $DMRL$ then $\overline{H}$ is $DMRL$. (In the case of shocks from a non-homogeneous Poisson process, if $\sum_{j=1}^{k} a_j = k$, $\Lambda(t)$ is convex and G is $DMRL$ then $\overline{H}$ is $DMRL$). Esary et al. (1973) mentioned that $\frac{\sum_{j=k}^{\infty} \overline{P}_j}{\overline{P}_k}$ need not be decreasing in $k = 1, 2, ...$, when G is $DMRL$.

1.2.7.2 Partial Ordering under Poisson Shock Models

Consider two devices subjected to the shocks occurring randomly as events in a Poisson process with the same constant intensity λ and the probability of surviving k shocks given by $\overline{P}_k$ and

$\bar{Q}_k$, respectively. The survival functions of these devices are $\bar{F}$ and $\bar{G}$. Singh and Jain (1989) showed that if $\bar{P}_k \overset{MRL}{\geq} \bar{Q}_k$ then $\bar{F} \overset{MRL}{\geq} \bar{G}$. In other words, if $\frac{\sum_{j=k}^{\infty} \bar{P}_j}{\bar{P}_k} \geq \frac{\sum_{j=k}^{\infty} \bar{Q}_j}{\bar{Q}_k}$ $k = 1, 2, \ldots$, then $\frac{1}{\bar{F}(t)} \int_t^{\infty} \bar{F}(u)du \geq \frac{1}{\bar{G}(t)} \int_t^{\infty} \bar{G}(u)du$ for all $t \geq 0$. They also proved that if the underlying counting process is a homogeneous Poisson process, the stochastic comparison $M_1 \overset{*}{\leq} M_2$ is preserved by the corresponding comparison $T_1 \overset{*}{\leq} T_2$, where T_i $i = 1, 2$ is the random lifetime of the i^{th} device subjected to the shocks and M_i, $i = 1, 2$ is random number of shocks that cause the failure of the i^{th} device. The ordering (*) can be the likelihood ratio (*LR*) ordering, the failure rate (*FR*) ordering, the mean residual life (*MRL*) ordering, the usual stochastic (*ST*) ordering or the increasing convex (*ICX*) ordering. In addition, under suitable assumptions on the counting process $\mathrm{N}(\cdot)$, Fagiuoli and Pellerey (1994) have shown that if the interarrival times of the counting process are independent, identically distributed and are *IFR*, then the inequality $M_1 \overset{MRL}{\leq} M_2$ implies $T_1 \overset{MRL}{\leq} T_2$.

1.2.7.3 Partial Ordering Under Cumulative Damage Shock Models

Next, we consider cumulative damage models with random thresholds $\tilde{X}$ and $\tilde{Y}$ having distributions F and G, respectively and common damage distribution. Suppose that the damages and successive shocks are independent. Pellerey (1993) showed that if the interarrival times have PF_2 densities (*IFR* or *DMRL* property) and if the damage has *IFR* property, then $\tilde{X} \overset{HR}{\leq} \tilde{Y}$

implies $T_1 \overset{MRL}{\leq} T_2$. Also, in the case of different damages X_i and Y_i with distributions F and G, respectively but fixed identical thresholds, if interarrival times have PF_2 and damages are $DRHR$ distributed then $X_i \overset{RHR}{\geq} Y_i$ implies $T_1 \overset{MRL}{\leq} T_2$.

CHAPTER TWO: PROBABILISTIC PROPERTIES OF INACTIVITY TIME WITH A GUARANTEED MINIMUM LENGTH OF LIFE

2.0 Chapter Plan

The first two sections of this chapter introduce the concept of inactivity time guaranteed minimum life length and its related aging classes. In the third section we will study the characterizations and probabilistic properties of these new classes of distributions. Stochastic comparisons between probability distributions for these new notions will be studied in Section 4. Sections 5 and 6 will reveal some preservation results for these new notions under typical reliability operations, and will introduce some nonparametric procedures for testing these classes against the specific alternatives. The last section offers some applications and simulation results.

2.1 Introduction

In several survival analysis applications, individuals are followed over time for the occurrence of a specific event. If the occurrence of an event is observed, data is recorded as the time the event occurred. In some situations, however, the times of the events of interest may only be known to have occurred within some interval of time. In clinical trials, for example, patients are often seen at pre-scheduled visits but the event of interest may occur in between visits. This chapter introduces the concept of inactivity time of a unit with a guaranteed minimum life length. This is the time that has elapsed from failure to the last visit. Since the unit has already met the visit before failure time, it has a minimum length of life. These notions are useful in studying areas of science such as clinical trials where exact time of failure of a unit with minimum length is of importance. Also, they are useful for estimation of the survival function for left- censored data coming from a unit with minimum life length.

The purpose of the current research is to establish some realistic univariate aging classes and to extend these new life classes to multivariate cases as done in Chapter 4.

2.2 Definitions and Notions

Let X be a non negative random variable that denotes the lifetime of a unit having an absolutely continuous distribution function $F(\cdot)$ and the density function $f(\cdot)$, with respect to the Lebesgue measure on the positive half of the real line R^+. For $\zeta \geq t \geq 0$, let the random variable $X_{(\zeta,t)} = [\zeta - X \mid t \leq X \leq \zeta],\ t, \zeta \in \mathrm{R}^+$ denote the time elapsed from failure to the time ζ, given that the unit was guaranteed to have a minimum length of life. We call the random variable $X_{(\zeta,t)}$, "Inactivity Time Guaranteed Minimum Life Length" ($ITGML$). In this chapter, we will study some of the probabilistic and statistical properties related to the random variable $X_{(\zeta,t)}$.

Define the random variable $X^* = X - t$ as the residual lifetime of the unit after time t with distribution function $F^*(x^*) = F(t + x^*)$ and density function $f^*(x^*) = f(t + x^*)$, where x^* belongs to the support of $f^*(\cdot)$, *i.e.*, R^+, $t^* = \zeta - t$ and $F^*(0) = F(t)$. The random variable inactivity time guaranteed minimum length t, $X_{(\zeta,t)}$ can be rewritten as

$$X^*_{(t^*)} = [t^* - X^* \mid 0 \leq X^* \leq t^*],\ t^* \in \mathrm{R}^+.$$

For convenience, we define $H^*(x) = F^*(x) - F^*(0)$ and $H(x) = F(x+t) - F(t)$. Also, $h^*(x) = f^*(x)$ and $h(x) = f(x+t)$

Definition 2.1: The reversed hazard rate function guaranteed minimum length of life ($RHRGML$) is defined

$$\tau(t,\zeta) = \tau_*(x^*) = \lim_{\varepsilon \to 0} \frac{p(x^* - \varepsilon \le X^* \le x^* \mid 0 \le X^* \le x^*)}{\varepsilon}$$

$$= \frac{f^*(x^*)}{F^*(x^*) - F^*(0)} \quad = \frac{h^*(x^*)}{H^*(x^*)} = \frac{f(t+x^*)}{F(t+x^*) - F(t)}$$

$$= \left(\tau(t+x^*)^{-1} - \frac{F(t)}{f(x^*+t)} \right)^{-1}, \quad for\ x^*, t \in R^+ . \quad (2.2.1)$$

For $t = 0$, $\tau_*(x^*) = \tau(x^*)$, where $\tau(x^*)$ is the reversed hazard rate function defined by $\tau(x^*) = \dfrac{f(x^*)}{F(x^*)}$.

Definition 2.2: The mean inactivity time guaranteed minimum length ($MITGML$) is defined as

$$\mu(t,\zeta) = \mu_*(t^*) = E(X^*_{(t^*)}) = E[t^* - X^* \mid 0 \le X^* \le t^*]$$

$$= \frac{\int_0^{t^*} [F^*(z) - F^*(0)]dz}{F^*(t^*) - F^*(0)}$$

$$= \frac{\int_0^{t^*} H^*(z)dz}{H^*(t^*)} = \frac{\int_t^{\zeta} H(z)dz}{H(\zeta)} , \qquad t^*, t, \zeta \in \mathrm{R}^+ . \quad (2.2.2)$$

Definition 2.3: The variance inactivity time guaranteed minimum length ($VITGML$) is defined

$$\sigma^2(t,\zeta) = \sigma^2{}_*(t^*) = Var(X^*_{(t^*)}) = Var[t^* - X^* \mid 0 \le X^* \le t^*]$$

$$=2t^*\mu_* - \mu_*^2 - \frac{2\int_0^{t^*} zH^*(z)dz}{H^*(t^*)}$$

$$=2t^*\mu_* - \mu_*^2 - \frac{2\int_t^{\zeta} (u-t)H(u)du}{H(\zeta)}, \qquad (2.2.3)$$

where $t^* = \zeta - t$.

Remark 2.1: One can show that

$$\mu_*(t^*)\tau_*(t^*) = 1 - \mu_*'(t^*) \qquad (2.2.4)$$

where $\mu_*'(t^*)$ denotes the derivative of μ_* with respect to t^*.

Definition 2.4: Inactivity coefficient of variation guaranteed minimum length ($ICVGML$) is define

$$CV(t,\zeta) = CV_*(t^*) = \frac{\sigma_*(t^*)}{\mu_*(t^*)}$$

$$= \left(\frac{2t^*}{\mu_*} - \frac{2\int_0^{t^*} z[F^*(z) - F^*(0)]dz}{\mu_* \int_0^{t^*} [F^*(z) - F^*(0)]dz} - 1 \right)^{\frac{1}{2}}, \quad t^* \in R^+. \qquad (2.2.5)$$

Remark 2.2: Applying Remark 2.1, one can prove the following

$$\sigma_*^{2'} = \tau_*(\mu_*^2 - \sigma_*^2)$$

where $\sigma_*^{2'}$ denotes the derivative of σ_*^2 with respect to t^*.

2.3 New Classes of Distributions and their Characterizations

Here, we define new classes of distributions for random life variable inactivity time guaranteed minimum life length, $ITGML$.

Definition 2.5: A nonnegative random variable X, having distribution $F(\cdot)$, is said to be:

a) Decreasing reversed hazard rate guaranteed minimum length ($DRHRGML$) if $\tau_*(t^*)$ is decreasing in $t^* \geq 0$.

b) Increasing mean inactivity time guaranteed minimum length ($IMITGML$) if $\mu_*(t^*)$ is increasing in $t^* \geq 0$.

c) Increasing variance inactivity time guaranteed minimum length ($IVITGML$) if $\sigma_*(t^*)$ is increasing in $t^* \geq 0$.

d) Increasing coefficient of variation inactivity guaranteed minimum length ($ICVIGML$) if $CV_*(t^*)$ is increasing in $t^* \geq 0$.

Theorem 2.1: There exists no nonnegative random variable X for which $\tau_*(t^*)$ increases over the entire domain $[0,\infty)$.

Proof: Suppose $\tau_*(t^*)$ is increasing function in t^*, i.e., for $t^* > 0$

$$
\begin{aligned}
\frac{f^*(t^*)}{F^*(t^*)-F^*(0)} &\geq \lim_{t^*\to 0}\frac{f^*(t^*)}{F^*(t^*)-F^*(0)} \\
&= \lim_{t^*\to 0}\frac{\partial}{\partial t^*} Ln\{F^*(t^*)-F^*(0)\} \\
&= \frac{\partial}{\partial t^*}\lim_{t^*\to 0} Ln\{F^*(t^*)-F^*(0)\} \\
&= \infty,
\end{aligned}
$$

which is a contradiction. ■

Remark 2.3: Using the same arguments, one can prove that there exists no nonnegative random variable X for which $\mu_*(t^*)$ decreases over the entire domain $[0,\infty)$.

Proposition 2.1: For $0 \le t_1^* \le t_2^*$, the following statements are equivalent.

a) X has decreasing reversed hazard rate guaranteed minimum length

b) $X^*_{(t_1^*)}$ is smaller than $X^*_{(t_2^*)}$ in stochastic ordering

c) $X^*_{(t_1^*)}$ is smaller than $X^*_{(t_2^*)}$ in hazard rate ordering

d) The distribution function of X is log –concave

Proof: Following the above definitions we prove the desired results.

$$a \Leftrightarrow b : X \in DRHRGML \Leftrightarrow X_{(\zeta_1,t_1)}(x) \overset{St}{\le} X_{(\zeta_2,t_2)}(x), \quad 0 \le t_1 \le t_2 \ \in R^+$$

where $X_{(\zeta,t)} = [\zeta - X \mid t \le X \le \zeta], \ t,\zeta \in \mathrm{R}^+$.

Let

$$P\left(X \le \zeta_1 - x \,|\, t_1 \le X \le \zeta_1\right) \le P\left(X \le \zeta_2 - x \,|\, t_2 \le X \le \zeta_2\right)$$

$$\Leftrightarrow \frac{F(\zeta_1)-F(\zeta_1-x)}{F(\zeta_1)-F(t_1)} \ge \frac{F(\zeta_2)-F(\zeta_2-x)}{F(\zeta_2)-F(t_2)}$$

$$\Leftrightarrow \lim_{x\to 0}\frac{1}{x}\frac{F(\zeta_1)-F(\zeta_1-x)}{F(\zeta_1)-F(t_1)} \ge \lim_{x\to 0}\frac{1}{x}\frac{F(\zeta_2)-F(\zeta_2-x)}{F(\zeta_2)-F(t_2)}$$

$$\Leftrightarrow \frac{f(\zeta_1)}{F(\zeta_1)-F(t_1)} \ge \frac{f(\zeta_2)}{F(\zeta_2)-F(t_2)}$$

$$\Leftrightarrow X \in DRHRGML$$

■

$$a \Leftrightarrow c : X \in DRHRGML \Leftrightarrow X_{(\zeta_1,t_1)}(x) \overset{RH}{\le} X_{(\zeta_2,t_2)}(x), \quad 0 \le t_1 \le t_2 \ \in R^+$$

Let $\dfrac{\overline{F}_{(\zeta_1,t_1)}(x)}{\overline{F}_{(\zeta_2,t_2)}(x)}$ be decreasing in x, then this equivalent to

$$\frac{P\left(X \le \zeta_1 - x \mid t_1 \le X \le \zeta_1\right)}{P\left(X \le \zeta_2 - x \mid t_2 \le X \le \zeta_2\right)} \quad \text{is decreasing in } x$$

$$\Leftrightarrow f(\zeta_2 - x)\left[F(\zeta_2 - x) - F(t_2)\right] \le f(\zeta_1 - x)\left[F(\zeta_1 - x) - F(t_1)\right]$$

$$\Leftrightarrow X \in DRHRGML$$

■

$$c \Leftrightarrow d: \quad X^*_{(t_1^*)} \overset{HR}{\le} X^*_{(t_2^*)} \Leftrightarrow F \in \text{Log- Concave}$$

Let's define $\Delta = \dfrac{p(t_2 \le X \le \zeta_2)}{p(t_1 \le X \le \zeta_1)} > 0$ and assume $\zeta_1 \le \zeta_2,\ x_1 \le x_2,\ t_1 \le t_2$, by

definition $X^*_{(t_1^*)} \overset{HR}{\le} X^*_{(t_2^*)}$ iff $\dfrac{\overline{F}_{(\zeta_1,t_1)}(x)}{\overline{F}_{(\zeta_2,t_2)}(x)}$ is decreasing in x

$$\Leftrightarrow \Delta \times \frac{F(\zeta_1 - x_1) - F(t_1)}{F(\zeta_2 - x_1) - F(t_2)} \ge \Delta \times \frac{F(\zeta_1 - x_2) - F(t_1)}{F(\zeta_2 - x_2) - F(t_2)}$$

$$\Leftrightarrow \begin{vmatrix} F(\zeta_1 - x_1) - F(t_1) & F(\zeta_2 - x_1) - F(t_2) \\ F(\zeta_1 - x_2) - F(t_1) & F(\zeta_2 - x_2) - F(t_2) \end{vmatrix} \ge 0$$

$$\Leftrightarrow F \in PF_2 ,$$

which completes the desired result. ■

Clearly, if a function is log-concave, so will be any linear transformation of that function. Therefore, we can claim the following.

Theorem 2.2: $F^*(\cdot)$ is log-concave if and only if $F(\cdot)$ is log-concave.

Corollary 2.1: $F^*(\cdot)$ is $DRHRGML$ if and only if $F(\cdot)$ is $DRHRGML$.

Remark 2.4: X has $DRHR$ property, then it has $DRHRGML$ property.

Remark 2.5: $F(\cdot)$ belongs to $DRHRGML$ class if and only if $\dfrac{F^*(t^*-x)-F^*(0)}{F^*(t^*)-F^*(0)}$ is increasing in $t^*, \forall\, x \geq 0$.

Theorem 2.3: If X has the $DRHRGML$ property then X has the $IMITGML$ property.

Proof: Remark 2.5 implies that $\tau_*(t^*)$ is decreasing in t^*, if and only if $\dfrac{F^*(t^*-x)-F^*(0)}{F^*(t^*)-F^*(0)}$ is increasing in t^* for all $x \geq 0$. Since, for $x > t^*$, $F^*(t^*-x)=0$. The above statement implies that $\mu_*(t^*)=\dfrac{\int_0^{t^*}[F^*(t^*-x)-F^*(0)]dx}{F^*(t^*)-F^*(0)}$ is increasing in t^*, which proves the required result.

■

Theorem 2.4: If μ_* is increasing function in t^*, then

$$\sigma_*^2(t^*) \leq \mu_*^2(t^*). \tag{2.3.1}$$

Proof: We know that

$$\int_0^{t^*}\mu_*(x)\left[F^*(x)-F^*(0)\right]dx=\int_0^{t^*}\int_0^{x}\left[F^*(z)-F^*(0)\right]dzdx$$

$$=\int_0^{x}\int_0^{t^*}\left[F^*(z)-F^*(0)\right]dxdz=\int_0^{x}t^*\left[F^*(z)-F^*(0)\right]dz$$

Also, from the definition of $VITGML$ we have the following

$$\sigma_*^2+\mu_*^2=2t^*\mu_*-\frac{2\int_0^{t^*}z\left[F^*(z)-F^*(0)\right]dz}{F^*(t^*)-F^*(0)}$$

$$=\frac{2}{F^*(t^*)-F^*(0)}\int_0^{t^*}\left[F^*(z)-F^*(0)\right]\mu_*(z)dz$$

Therefore, by assuming that μ_* is increasing in t^*, we have the following

$$\sigma_*^2 - \mu_*^2 = \frac{2}{F^*(t^*) - F^*(0)} \int_0^{t^*} \left[F^*(z) - F^*(0) \right] \left(\mu_*(z) - \mu_*(t^*) \right) dz \leq 0 \ .$$

Thus, the desired result holds. ■

The following theorem characterizes the $IVITGML$.

Theorem 2.5: The following statements are equivalent

a) $\sigma_*^2(t^*)$ is increasing in t^*

b) $CV_*(t^*) \leq 1$ for all t^*

c) $E[(t^* - X^*)^2 \mid 0 \leq X^* \leq t^*] / E[(t^* - X^*) \mid 0 \leq X^* \leq t^*]$ is an increasing function in t^*

Proof: The equivalence of parts (a) and (b) follows from Remark 2.2 and Theorem 2.4. To show the equivalence of (b) and (c), one might consider the following

$$\begin{aligned} \Lambda(t^*) &= E[(t^* - X^*)^2 \mid 0 \leq X^* \leq t^*] / E[(t^* - X^*) \mid 0 \leq X^* \leq t^*] \\ &= \frac{\sigma_*^2 + \mu_*^2}{\mu_*^2}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial t^*} \Lambda(t^*) &= \frac{\sigma_*^{2'} \mu_* - \mu_*' \sigma_*^2}{\mu_*^2} + \mu_*' \\ &= \frac{\sigma_*^{2'}}{\mu_*} + \mu_*' \left(1 - cv_*^2 \right) \ , \end{aligned}$$

which provides the desired result. ■

Theorem 2.6: X has $ICVIGML$ if and only if $\dfrac{\int_0^{t^*} \left[F^*(z) - F^*(0) \right] \mu_*(z) dz}{\int_0^{t^*} \left[F^*(z) - F^*(0) \right] dz}$ is increasing in t^* .

Proof: Let us define $A(t^*)$ as

$$A(t^*) = \frac{\int_0^{t^*} \left[F^*(z) - F^*(0) \right] \mu_*(z) dz}{\int_0^{t^*} \left[F^*(z) - F^*(0) \right] dz}. \tag{2.3.2}$$

Using Theorem 2.5, one can find $A(t^*)$ is increasing in t^* if and only if $\frac{\sigma_*^2 + \mu_*^2}{\mu_*^2}$ is increasing in t^* which implies that $CV_*(t^*)$ is an increasing function in t^*. This finishes the proof.

■

Remark 2.6: X has $ICVIGML$ property if and only if

$$\mu_*(t^*) \geq \frac{\int_0^{t^*} \left[F^*(z) - F^*(0) \right] \mu_*(z) dz}{\int_0^{t^*} \left[F^*(z) - F^*(0) \right] dz}. \tag{2.3.3}$$

By using Theorems 2.4, 2.5 and 2.6, the following chain of implications holds immediately.

Corollary 2.2: $DRHRGML \Rightarrow IMITGML \Rightarrow IVIGML \Rightarrow ICVIGML$

Definition 2.6: A life distribution $F(\cdot)$ belongs to the class of "Used has Shorter Inactivity time than New Guaranteed Minimum length" ($USINGML$) if $X^*_{(0)} \overset{St}{\geq} X^*_{(t^*)}$,

where $X^*_{(0)} = [t - X \mid X \leq t]$ and $X^*_{(t^*)} = [\zeta - X \mid t \leq X \leq \zeta]$

Definition 2.7: A life distribution $F(\cdot)$ belongs to the class of "Used has Longer Inactivity time than New Guaranteed Minimum length" ($ULINGML$) if $X^*_{(0)} \overset{St}{\leq} X^*_{(t^*)}$.

Remark 2.7: A life distribution $F(\cdot)$ is said to be $USINGML$ if and only if

$$F(t-u)[F(\zeta) - F(t)] \geq F(t)[F(\zeta - u) - F(t)] \qquad t, \zeta, u \in \mathrm{R}^+,$$

Definition 2.8: A life distribution $F(\cdot)$ belongs to the class of "Used has Shorter Mean Inactivity Time than New Guaranteed Minimum length" ($USMINGML$) if $EX^*_{(0)} \overset{St}{\geq} EX^*_{(t^*)}$.

Remark 2.8: A life distribution $F(\cdot)$ is said to be $USMINGML$ if and only if

$$[F(\zeta)-F(t)]\int_0^t F(t-u)du \geq F(t)\int_t^\zeta [F(\zeta-u)-F(t)]du \ , \qquad t,\zeta \in \mathrm{R}^+ .$$

2.4 Orderings and Characterizations of New Life Distributions

Stochastic comparisons between probability distributions play a fundamental role in probability, statistics and some related areas, such as reliability theory, survival analysis, economics and actuarial science. Here, we study some of these orders for our new notions by placing emphasis on having a life distribution with a guaranteed minimum length.

Let X and Y be two nonnegative random variables having absolutely continuous distributions $F(\cdot)$ and $G(\cdot)$ with densities $f(\cdot)$ and $g(\cdot)$ respectively.

Definition 2.9:

a) X is said to be smaller than Y in coefficient of variation inactivity guaranteed minimum length ($ICVGML$) order ($X \overset{CVIGML}{\leq} Y$) if

$$CV_*^X(t^*) \geq CV_*^Y(t^*), \ \ t^* \in \mathrm{R}^+$$

b) $X^*_{(t_1^*)}$ is smaller than $X^*_{(t_2^*)}$ in coefficient of variation inactivity guaranteed minimum length order ($X^*_{(t_1^*)} \overset{CVIGML}{\leq} X^*_{(t_2^*)}$) if

$$CV_*(t_1^*) \leq CV_*(t_2^*), \ \ t_1^*, t_2^* \in \mathrm{R}^+$$

c) X is said to be larger than Y in reversed hazard rate guaranteed minimum length ($RHRGML$) order ($X \overset{RHRGML}{\geq} Y$) if $\tau_*^X(x) \geq \tau_*^Y(x)$, $x \in \mathrm{R}^+$

d) X is said to be larger than Y in mean inactivity time guaranteed minimum length ($MITGML$) order ($X \overset{MITGML}{\geq} Y$) if $\mu_*^X(x) \leq \mu_*^Y(x)$, $x \in \mathrm{R}^+$

e) X is less than Y in right spread order ($X \overset{RS}{\leq} Y$) if

$$\int_{F^{-1}(p)}^{\infty} \bar{F}_X(x)dx \leq \int_{G^{-1}(p)}^{\infty} \bar{G}_Y(x)dx \qquad \text{where}$$

$$F^{-1}(p) = Inf\{t : F(t) \geq p\} \text{ and } p \in (0,1)$$

f) X is less than Y in Laplace-Stieljes transform ($X \overset{LT}{\leq} Y$) if $L_X(S) \leq L_Y(S)$

where $L_X(S) = \int_0^{\infty} e^{-su} \bar{F}_X(u)du$, $s \in R^+$

g) X is said to be smaller than Y in the increasing concave order ($X \overset{IC}{\leq} Y$) if

$$\int_{-\infty}^{x} F_X(u)du \geq \int_{-\infty}^{x} G_Y(u)du \qquad \text{for all } x$$

Provided the integrals exist.

h) X is said to be larger than Y in variance inactivity guaranteed minimum length ($VIGML$) order ($X \overset{VIGML}{\geq} Y$) if

$$\frac{\int_0^t \int_0^x H_X^*(u)dudx}{\int_0^t H_X^*(x)dx} \leq \frac{\int_0^t \int_0^x H_Y^*(u)dudx}{\int_0^t H_Y^*(x)dx} \text{ where } H^*(t) = F^*(t) - F^*(0)$$

Remark 2.9: $X \overset{RHRGML}{\geq} Y$ if and only if $\dfrac{F^*(x) - F^*(0)}{G^*(x) - G^*(0)}$ is increasing in x, $x \in [0, \zeta - t]$, $t \leq \zeta \in R^+$.

Proposition 2.2: $\dfrac{\int_0^x [F^*(z)-F^*(0)]dz}{\int_0^x [G^*(z)-G^*(0)]dz}$ is an increasing function in x, if and only if $X \overset{MITGML}{\geq} Y$.

Proof: The desired results follow from the definition of increasing function.

Theorem 2.7: $X \overset{RHRGML}{\geq} Y \Rightarrow X \overset{MITGML}{\geq} Y$

Proof: Suppose $X \overset{RHRGML}{\geq} Y$, i.e., $\dfrac{F^*(x)-F^*(0)}{G^*(x)-G^*(0)}$ is increasing in $x \geq 0$. For fixed real valued λ, the function $A(x) = \{F(t+x)-F(t)\} - \lambda\{G(t+x)-G(t)\}$ will have at most one change of sign and if one such change does occur, it occurs from – to +. This implies that, for fixed λ, $\int_0^x A(u)du$ will have at most one change of sign from – to +, which means $\dfrac{\int_0^x \{F(t+u)-F(t)\}du}{\int_0^x \{G(t+u)-G(t)\}du}$ is increasing in $x \geq 0$. ■

Theorem 2.8: X is said to be smaller than Y in reversed hazard rate guaranteed minimum length ($RHRGML-t$) order ($X \overset{RHRGML}{\leq} Y$) if and only if $X^*_{(t^*)} \overset{St}{\geq} Y^*_{(t^*)}$.

Proof: Let $X^*_{(t^*)} \overset{St}{\geq} Y^*_{(t^*)}$. This is equivalent to

$$P(X^* \geq t^* - u \mid 0 \leq X^* \leq t^*) \leq P(Y^* \geq t^* - u \mid 0 \leq Y^* \leq t^*)$$

$$\Leftrightarrow \quad \frac{F^*(t^*)-F^*(t^*-u)}{F^*(t^*)-F^*(0)} \leq \frac{G^*(t^*)-G^*(t^*-u)}{G^*(t^*)-G^*(0)}$$

$$\Leftrightarrow \lim_{u\to 0}\frac{1}{u}\frac{F^*(t^*)-F^*(t^*-u)}{F^*(t^*)-F^*(0)} \leq \lim_{u\to 0}\frac{1}{u}\frac{G^*(t^*)-G^*(t^*-u)}{G^*(t^*)-G^*(0)}$$

$$\Leftrightarrow \frac{f^*(t^*)}{F^*(t^*)-F^*(0)} \leq \frac{g^*(t^*)}{G^*(t^*)-G^*(0)}$$

$$\Leftrightarrow X \overset{RHRGML}{\le} Y \, ,$$

which is the desired result. ■

Theorem 2.9: $X \in IMITGML \Leftrightarrow X^*{}_{(t^*)} \overset{RS}{\le} X^*{}_{(s^*)}, \quad 0 \le t^* \le s^*$

Proof: Note that

$$X \in IMITGML \Leftrightarrow \int_0^{t^*} \overline{F}_{X^*_{(t^*)}}(x)dx \le \int_0^{s^*} \overline{F}_{X^*_{(s^*)}}(x)dx \quad \text{for } 0 \le t^* \le s^*$$

$$\Leftrightarrow \int_{F^{-1}_{X^*_{(t^*)}}(p)}^{t^*} \overline{F}_{X^*_{(t^*)}}(x)dx \le \int_{F^{-1}_{X^*_{(s^*)}}(p)}^{s^*} \overline{F}_{X^*_{(s^*)}}(x)dx$$

where $F^{-1}_{X^*_{(t^*)}}(p) = Inf\{t^* : F_{X^*_{(t^*)}}(t^*) \ge p\}$ ■

Proposition 2.3: For all $t^*, s \ge 0$;

$$X^*_{(t*)} \overset{Lt}{\ge} Y^*_{(t*)} \Leftrightarrow \frac{\int_0^{t^*} e^{su}[F^*(u) - F^*(0)]du}{\int_0^{t^*} e^{su}[G^*(u) - G^*(0)]du} \text{ is decreasing in } t^* . \tag{2.4.1}$$

Proof: First consider

$$L_{X^*_{(t^*)}}(s) = \int_0^{\infty} e^{-su} \frac{\left[F^*(t^* - u) - F^*(0)\right]}{F^*(t^*) - F^*(0)} du$$

$$= \int_0^{t^*} \frac{e^{sz}\left[F^*(z) - F^*(0)\right]}{e^{st^*}\left[F^*(z) - F^*(0)\right]} dz$$

$$= \frac{\int_0^{t^*} e^{sz}\left[F^*(z) - F^*(0)\right]dz}{\frac{\partial}{\partial t^*}\int_0^{t^*} e^{sz}\left[F^*(z) - F^*(0)\right]dz}$$

Now, $X^*_{(t^*)} \overset{Lt}{\geq} Y^*_{(t^*)}$ for all $t^* \geq 0$ if and only if $L_{X^*_{(t^*)}}(s) \geq L_{Y^*_{(t^*)}}(s)$ for all $s, t^* \geq 0$

$$\Leftrightarrow \frac{\int_0^{t^*} e^{sz}\left[F^*(z)-F^*(0)\right]dz}{\frac{\partial}{\partial t^*}\int_0^{t^*} e^{sz}\left[F^*(z)-F^*(0)\right]dz} \geq \frac{\int_0^{t^*} e^{sz}\left[G^*(z)-G^*(0)\right]dz}{\frac{\partial}{\partial t^*}\int_0^{t^*} e^{sz}\left[G^*(z)-G^*(0)\right]dz}$$

$$\Leftrightarrow \frac{\int_0^{t^*} e^{sz}\left[F^*(z)-F^*(0)\right]dz}{\int_0^{t^*} e^{sz}\left[G^*(z)-G^*(0)\right]dz} \quad \text{is decreasing in t}^*$$

The desired result holds. ■

Theorem 2.10: $X \overset{RHRGML}{\leq} Y \Leftrightarrow X^*_{(t*)} \overset{Lt}{\geq} Y^*_{(t*)}$.

Proof: Using the same arguments in Theorem 2.7, one can get the 'if' part, i.e., $X \overset{RHRGML}{\leq} Y \Rightarrow X^*_{(t*)} \overset{Lt}{\geq} Y^*_{(t*)}$. We need to prove the implication $X^*_{(t*)} \overset{Lt}{\geq} Y^*_{(t*)} \Rightarrow X \overset{RHRGML}{\leq} Y$. For fixed $s, t^* \geq 0$, we have

$$\begin{aligned}\int_0^{t^*} e^{su}\left[F^*(u)-F^*(0)\right]du &= \frac{1}{s}e^{su}\left[F^*(t^*)-F^*(0)\right] - \int_0^{t^*}\frac{1}{s}e^{su}dF(u) \\ &= \frac{1}{s}e^{st}\left[\left[F^*(t^*)-F^*(0)\right] - e^{-st}\int_0^{t^*} e^{su}dF(u)\right].\end{aligned}$$

Then

$$\frac{\int_0^{t^*} e^{sz}\left[F^*(z)-F^*(0)\right]dz}{\int_0^{t^*} e^{sz}\left[G^*(z)-G^*(0)\right]dz} = \frac{\frac{1}{s}e^{st}\left[\left[F^*(t^*)-F^*(0)\right] - e^{-st}\int_0^{t^*} e^{su}dF(u)\right]}{\frac{1}{s}e^{st}\left[\left[G^*(t^*)-G^*(0)\right] - e^{-st}\int_0^{t^*} e^{su}dG(u)\right]}.$$

Using the dominated convergence theorem along with Proposition 2.3, the result follows.

■

Remark 2.10: The converse of Theorem 2.7 is not necessarily true.

The next theorem, gives a condition on which $X \overset{RHRGML}{\leq} Y$ if and only if $X \overset{MITGML}{\leq} Y$.

Theorem 2.11: Suppose that $\dfrac{\mu_*^X(t^*)}{\mu_*^Y(t^*)}$ is increasing in t^* then we have the following

$$X \overset{MITGML}{\leq} Y \Leftrightarrow X \overset{RHRGML}{\leq} Y \tag{2.4.2}$$

Proof:

$$X \overset{MITGML}{\leq} Y \Leftrightarrow \frac{1}{\mu_*^X} \leq \frac{1}{\mu_*^Y}$$

applying Remark 2.1 and the assumption above we have

$$\Leftrightarrow \frac{1}{\mu_*^X} - \frac{\mu'^X_*}{\mu_*^X} \leq \frac{1}{\mu_*^Y} - \frac{\mu'^X_*}{\mu_*^X}$$
$$\Leftrightarrow \tau_*^X \leq \tau_*^Y$$

■

Remark 2.11: $X \overset{VIGML}{\geq} Y$ if and only if $\dfrac{\int_0^t \int_0^x H_X^*(u)dudx}{\int_0^t \int_0^x H_Y^*(u)dudx}$ is an increasing function in t.

Theorem 2.12: $X \overset{MITGML}{\geq} Y \Rightarrow X \overset{VIGML}{\geq} Y$

Proof: Using Proposition 2.2, we have $X \overset{MITGML}{\geq} Y \Leftrightarrow \dfrac{\int_0^x H_X^*(z)dz}{\int_0^x H_Y^*(z)dz}$ and following the same fashion as in Theorem 2.7 the desired result follows. ■

2.5 Closure Properties

In this section, we present some preservation results for mean inactivity time guaranteed minimum length t order under typical reliability operations like convolution and mixture.

2.5.1 Convolution

Before we state the result of convolution, we need to have the following definition and lemma (Ahmad et al. 2005)

Definition 2.10: A nonnegative function h defined on $\chi \times \gamma$ (χ, γ are two subsets of real line) is said to be totally positive of order two (TP_2) if

$h(x,y) \times h(x',y') \geq h(x,y') \times h(x',y)$, where $x \leq x'$, $y \leq y'$ and $x, x' \in \chi$, $y, y' \in \gamma$.

Lemma 2.1: Let $\phi(\theta,x)$ be any TP_2 function and $\theta \in \chi$, $x \in \gamma$, and let $F_i(\theta)$ be a distribution function for each i. Let,

$$H_i(x) = \int_{\chi} \phi(\theta,x) dF_i(\theta). \qquad (2.5.1)$$

If $F_i(\theta)$ is TP_2 in $i \in \{1,2\}$ and if $\phi(\theta,x)$ is increasing in θ for every x, then $H_i(x)$ is TP_2 in $x \in \gamma$ and $i \in \{1,2\}$.

Theorem 2.13: Let X_1, X_2 and Y be three nonnegative random variables, where Y is independent of both X_1 and X_2. If $X_1 \overset{MITGML}{\leq} X_2$ and the distribution function of Y belongs to class of $DRHRGML$ then $X_1 + Y \overset{MITGML}{\leq} X_2 + Y$.

Proof: we need to show that $\dfrac{\Phi(1,x)}{\Phi(2,x)}$ is a decreasing function in x, where for $i=1,2$

$$\Phi(i,x)=\int_0^x\left[F^*_{X_i+Y}(u)-F^*_{X_i+Y}(0)\right]du$$

$$=\int_0^\infty\int_0^\infty\left[F^*_{X_i}(x-v-u)-F^*_{X_i}(x-v)\right]f^*_Y(v)dvdu$$

$$=\int_0^\infty f^*_Y(v)\left(\int_0^\infty\left[F^*_{X_i}(x-v-u)-F^*_{X_i}(x-v)\right]du\right)dv$$

$$=\int_0^\infty f^*_Y(x-z)\left(\int_0^\infty\left[F^*_{X_i}(z-v)-F^*_{X_i}(z)\right]dv\right)dz$$

$$=\int_0^x f^*_Y(x-z)\left(\int_0^z\left[F^*_{X_i}(v)-F^*_{X_i}(0)\right]dv\right)dz$$

$$=\int_0^x\Phi(i,z)d_z\bar{F}^*_Y(x-z)$$

Since Y is $DRHGML$, then $\bar{F}^*_Y$ has the property of TP_2 in $(x,z)\in R^2$. Also, since $X_1\overset{MITGML}{\leq}X_2$ implies that $\Phi(i,z)$ is TP_2 in $(i,z)\in\{1,2\}\times R^2$ and $\Phi(i,z)$ is an increasing function in z for fixed $i=1,2$. Having Lemma 2.1, the desired result holds. ■

Corollary 2.3: Let (X_i,Y_i) $i=1\ldots m$, be independent pairs of random variables such that $X_i\overset{MITGML}{\leq}Y_i$ for $i=1\ldots m$. If the distribution function of X_i, Y_i are all belong to $DRHRGML$ then

$$\sum_{i=1}^m X_i\overset{MITGML}{\leq}\sum_{i=1}^m Y_i \qquad (2.5.2)$$

Proof: It follows from Theorem 2.13, that the following chain holds

$$X_1+X_2\overset{MITGML}{\leq}X_1+Y_2\overset{MITGML}{\leq}Y_1+Y_2$$

Now, having induction procedure, the result follows. ■

2.5.2 Mixture

Consider a family of distributions $\left\{F_\theta^*, \theta \in \chi\right\}$ where χ is a subset of the real line R. Let $X^*(\theta)$ denotes a random variable with distribution function F_θ^* and let Θ be a random variable having distribution function Λ, then $X^*(\Theta)$ denotes a random variable with distribution function G^* given by

$G^*(y) = \int_\chi F_\theta^*(y) d\Lambda(\theta)$, $\bar{G}^*(y) = \int_\chi \bar{F}_\theta^*(y) d\Lambda(\theta)$ where $y \in R$, $F_\theta^*(y) = F(y+t)$ and $\bar{F}_\theta^*(y) = 1 - F_\theta^*(y)$. In this case, $X^*(\Theta)$ is called mixture of $X^*(\theta)$ with respect to the Θ.

Theorem 2.14: Consider a family of distribution functions $\left\{F_\theta^*, \theta \in \chi\right\}$ corresponding to the family of random variables $\left\{X^*(\theta), \theta \in R^+\right\}$ which are independent of Θ_1 and Θ_2. If $X^*(\theta) \overset{MITGML}{\leq} X^*(\theta')$ when $\theta \leq \theta'$ and if $\Theta_1 \overset{RHRGML}{\leq} \Theta_2$ then $X^*(\Theta_1) \overset{MITGML}{\leq} X^*(\Theta_2)$.

Proof: We need to prove that $\dfrac{\int_0^x \left[G_{X^*_{(\Theta_1)}}^*(z) - G_{X^*_{(\Theta_1)}}^*(0)\right] dz}{\int_0^x \left[G_{X^*_{(\Theta_2)}}^*(z) - G_{X^*_{(\Theta_2)}}^*(0)\right] dz}$ is decreasing in x. Now , let us consider the following

$$\Phi(i,x) = \int_0^\infty \left[G_{X^*_{(\Theta_i)}}^*(x-z) - G_{X^*_{(\Theta_i)}}^*(0)\right] dz$$

$$= \int_0^\infty \int_0^\infty \left[F_\theta^*(x-z) - F_\theta^*(0)\right] d\Lambda_i(\theta) dz$$

$$= \int_0^\infty g_i^*(\theta) \left(\int_0^x \left[F_\theta^*(z) - F_\theta^*(0)\right] dz\right) d\theta$$

$$= \int_0^{\infty} g_i^*(\theta)\varphi(\theta, x)d\theta$$

$$= \int_0^{\infty} \varphi(\theta, x)d\Lambda_i(\theta).$$

But by the assumption that $X^*(\theta) \overset{MITGML}{\leq} X^*(\theta')$ where $\theta \leq \theta'$, $\varphi(\theta, x)$ is TP_2, *i.e.*, an increasing function in θ for fixed x. Also, $\Theta_1 \overset{RHRGML}{\leq} \Theta_2$ implies that $\Lambda_i(\theta)$ is TP_2. Applying Lemma 2.1, the desired result holds. ■

Theorem 2.15: Let $\{F_\theta^*, \theta \in \chi\}$ be a family of life distributions and $F_\theta^* \in IIVGML$, then the mixture $G^*(t)$ of family $\{F_\theta^*\}$ with respect to an arbitrary mixing distribution Λ belongs to the same class of $IIVGML$.

Proof: We need to show that $\sigma_{*G^*}^2 \leq \mu_{*G^*}^2$ or equivalently one must show that

$$\left(G^*(t^*) - G^*(0)\right)\int_0^{t^*}\int_0^z \left[G^*(u) - G^*(0)\right]dudz \geq \left(\int_0^{t^*}\left[G^*(t^*) - G^*(0)\right]du\right)^2$$

or

$$H_{G^*}^*(t^*)\int_0^{t^*}\int_0^z \left[H_{G^*}^*(u)\right]dudz \geq \left(\int_0^{t^*}\left[H_{G^*}^*(u)\right]du\right)^2$$

or $I \geq II$, say.

Starting with the left hand side:

$$I = \int_0^{\infty} H_{F_\theta^*}^*(t^*)d\Lambda(\theta)\int_0^{t^*}\int_0^z\int_0^{\infty}\left[H_{F_\theta^*}^*(u)\right]dudzd\Lambda(\theta)$$

$$= \int_0^{\infty} H_{F_\theta^*}^*(t^*)\int_0^{t^*}\int_0^z H_{F_\theta^*}^*(u)dudzd\Lambda(\theta)$$

$$\geq \int_0^{\infty}\left(\int_0^{t^*}\left[H^*_{F^*_\theta}(u)\right]du\right)^2 d\Lambda(\theta)$$

$$\geq \int_0^{t^*}\left(\int_0^{\infty}\left[H^*_{F^*_\theta}(u)\right]d\Lambda(\theta)\right)^2 du$$

$$\geq \left(\int_0^{t^*}\int_0^{\infty} H^*_{F^*_\theta}(u)d\Lambda(\theta)du\right)\left(\int_0^{t^*}\int_0^{\infty} H^*_{F^*_\theta}(w)d\Lambda(\theta)dw\right)$$

$$= \left(\int_0^{t^*}\left[H^*_{G^*}(u)\right]du\right)\left(\int_0^{t^*}\left[H^*_{G^*}(w)\right]dw\right)$$

$$= \left(\int_0^{t^*}\left[H^*_{G^*}(u)\right]du\right)^2 = II \ ,$$

which finishes the proof. ■

2.6 A Nonparametric Procedure for Testing Decreasing Reversed Hazard Rate Guaranteed Minimum Length of Life

In order to do testing in the *DRHRGML* class, one might observe that there is no boundary distribution for this class, i.e. no distribution with constant *RHRGML*. It is easy to check that the exponential distribution has decreasing reversed hazard rate guaranteed minimum length. In general, the testing hypothesis would be $H_0 : F = F_0$ against $H_1 : F$ is *DRHRGML* and not F_0, where F_0 is known (up to a set of parameter). Figure 2.1 shows that one possible choice of F_0 is the exponential distribution which its *DRHRGML* has the lowest value among the others . In this case the null hypothesis is $H_0 : F =$ exponential(λ) against $H_1 : F$ is *DRHRGML* and not exponential.

Let $X_1, X_2, ..., X_n$ be a random sample from a life distribution F with the reversed hazard rate function guaranteed minimum length $\tau_*(x),\ x \in [0, t^*],\ t^* = (\zeta - t) \in R^+$. We wish to test $H_0 : \tau_*(x) = \tau_{F_0^*}(x)$ for some known distribution function F_0 against $H_1 : \tau_*(x)$ is decreasing in $x \geq 0$ and not equal to $\tau_{F_0^*}(x)$. Note that the distribution function F belongs to *DRHRGML* if and only if $\dfrac{F^*(t^* - x)}{F^*(t^*) - F^*(0)}$ is increasing in $t^*, \forall\, x \geq 0$, which is equivalent to $f^*(t^* - x)[F^*(t^*) - F^*(0)] - f^*(t^*)F^*(t^* - x) \geq 0$. A measure of departure from H_0 in favor of H_1 can be given by

$$\delta = \int_0^\infty \int_0^\zeta \int_0^{\zeta - t} \{ f(\zeta - x)[F(\zeta) - F(t)] - f(\zeta)F(\zeta - x)\} f(t) f(x) dx dt d\zeta - \frac{1}{12}$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty I(y - t)\left\{\left[\bar{F}(t) - \bar{F}(y + x) - I(y - t - x)F(y - x)\right]\right\} dF(x) dF(t) dF(y) - \frac{1}{12} \tag{2.6.1}$$

under H_0, $\delta = \delta_0$ a known constant value and under $H_1 : \delta \geq \delta_0$. If F_0 is exponential with parameter $\lambda > 0$ then $\delta_0 = 0$. Now, if $F_n(x) = \frac{1}{n}\sum_{i=1}^n I(x - X_i)$ and $\bar{F}_n(x) = \frac{1}{n}\sum_{i=1}^n I(X_i - x)$ denote the empirical distribution function and survival function corresponding to $F(x)$, then one might find an unbiased estimate of δ in the form of

$$\hat{\delta} = \int_0^\infty \int_0^\infty \int_0^\infty I(y - t)\left\{\left[\bar{F}_n(t) - \bar{F}_n(y + x) - I(y - t - x)F_n(y - x)\right]\right\} dF_n(x) dF_n(t) dF_n(y) - \frac{1}{12} \tag{2.6.2}$$

Here, $I(x) = 1$ if $x \geq 0$ and zero otherwise. Equivalently, $\hat{\delta}$ can be written as

$$\hat{\delta} = \frac{1}{(n)_4} \sum\sum\sum\sum_{i \neq j \neq k \neq l} I(X_l - X_j)\left\{I(X_i - X_j) - I(X_i - X_l - X_k) - I(X_l - X_j - X_k) I(X_l - X_k - X_i)\right\}$$

where $(n)_4 = n(n-1)(n-2)(n-3)$.

Setting

$\phi(X_1, X_2, X_3, X_4) = I(X_4 - X_2)\{I(X_1 - X_2) - I(X_1 - X_4 - X_3) - I(X_4 - X_2 - X_3) I(X_4 - X_3 - X_1)\}$ and

$\Phi(X_1, X_2, X_3, X_4) = \frac{1}{4!}\sum \phi(X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4})$ where the sum is extended over all permutations (i_1, i_2, i_3, i_4) of $\{1,2,3,4\}$. One can simplify the above symmetric kernel in the form of

$$\Phi(X_1, X_2, X_3, X_4) \quad = \frac{1}{4}\{\phi(X_1, X_2, X_3, X_4) + \phi(X_2, X_1, X_3, X_4) + \phi(X_2, X_3, X_1, X_4) + \phi(X_2, X_3, X_4, X_1)\}$$

An equivalent U-Statistic type of $\hat{\delta}$ is then of the form

$$U = \binom{n}{4}^{-1} \sum_{i \leq j \leq k \leq l} \Phi(X_i, X_j, X_k, X_l). \tag{2.6.3}$$

Theorem 2.16: as $n \to \infty$, $n^{\frac{1}{2}}(U - \delta)$ is asymptotically distributed normal with zero mean and variance σ^2 where

$$\sigma^2 = Var\{F(X_1) - (\int_0^{X_1} F(X_1 - x) dF(x))(\frac{1}{2} + \int_0^{X_1} F(X_1 - x) dF(x)) - (\int_0^{\infty}\int_0^{\infty} \bar{F}(x+y) dF(x) dF(y))$$

$$\times(\frac{1}{2} + \bar{F}(X_1)) \times \int_0^{\infty} \bar{F}(X_1 + x) dF(x) - \frac{1}{2}\int_0^{\infty} \bar{F}(X_1 + x) dF(x)(1 + \int_0^{\infty} \bar{F}(X_1 + x) dF(x) + F(X_1))$$

$$+ \bar{F}^2(X_1) - \bar{F}(X_1)\int_0^{\infty}\int_0^{x} \bar{F}(x - y) dF(x) dF(y) + \frac{1}{4}\} \quad . \tag{2.6.4}$$

Proof: Using the general theory of standard U-statistics, cf. Lee (1990), the asymptotic variance is equal to

$$\sigma^2 = Var[E\{\Phi(X_1, X_2, X_3, X_4) \mid X_1\} + E\{\Phi(X_2, X_1, X_3, X_4) \mid X_1\} + E\{\Phi(X_2, X_3, X_1, X_4) \mid X_1\}$$

$$+ E\{\Phi(X_2, X_3, X_4, X_1) \mid X_1\}]$$

$$= Var(\sum_{i=1}^{4} \phi_i(X_1))$$

where

$$\phi_1(X_1) = \frac{1}{2}\{F(X_1) - \int_0^{X_1} F(X_1 - x)dF(x) - (\int_0^{\infty}\int_0^{\infty} \overline{F}(x+y)dF(x)dF(y))(\int_0^{\infty} \overline{F}(X_1 + x)dF(x))\}$$

$$\phi_2(X_1) = \overline{F}(X_1)\{\overline{F}(X_1) - \int_0^{\infty}\int_0^{x} \overline{F}(x-y)dF(x)dF(y) - (\int_0^{\infty}\int_0^{\infty} \overline{F}(x+y)dF(x)dF(y))(\int_0^{\infty} \overline{F}(X_1 + x)dF(x))\}$$

$$\phi_3(X_1) = \frac{1}{2}\{\frac{1}{2} - \int_0^{\infty} \overline{F}(X_1 + x)dF(x) - (\int_0^{\infty} \overline{F}(X_1 + x)dF(x))^2\}$$

and

$$\phi_4(X_1) = F(X_1)\{\frac{1}{2} - \int_0^{\infty} \overline{F}(X_1 + x)dF(x) - (\int_0^{\infty} F(X_1 - x)dF(x))^2\}$$

Direct calculations give the result. ■

under H_0, plug in exponential distribution function F_0 with parameter λ, and we have

$\sigma_{0,\lambda}^2 = 0.0158 \quad \forall \lambda \geq 0$.

Now, we reject H_0 in favor of H_1 if $\sigma_{0,\lambda}^{-1} n^{\frac{1}{2}} U \geq z_\alpha$, where z_α is the α quantile of the standard normal distribution.

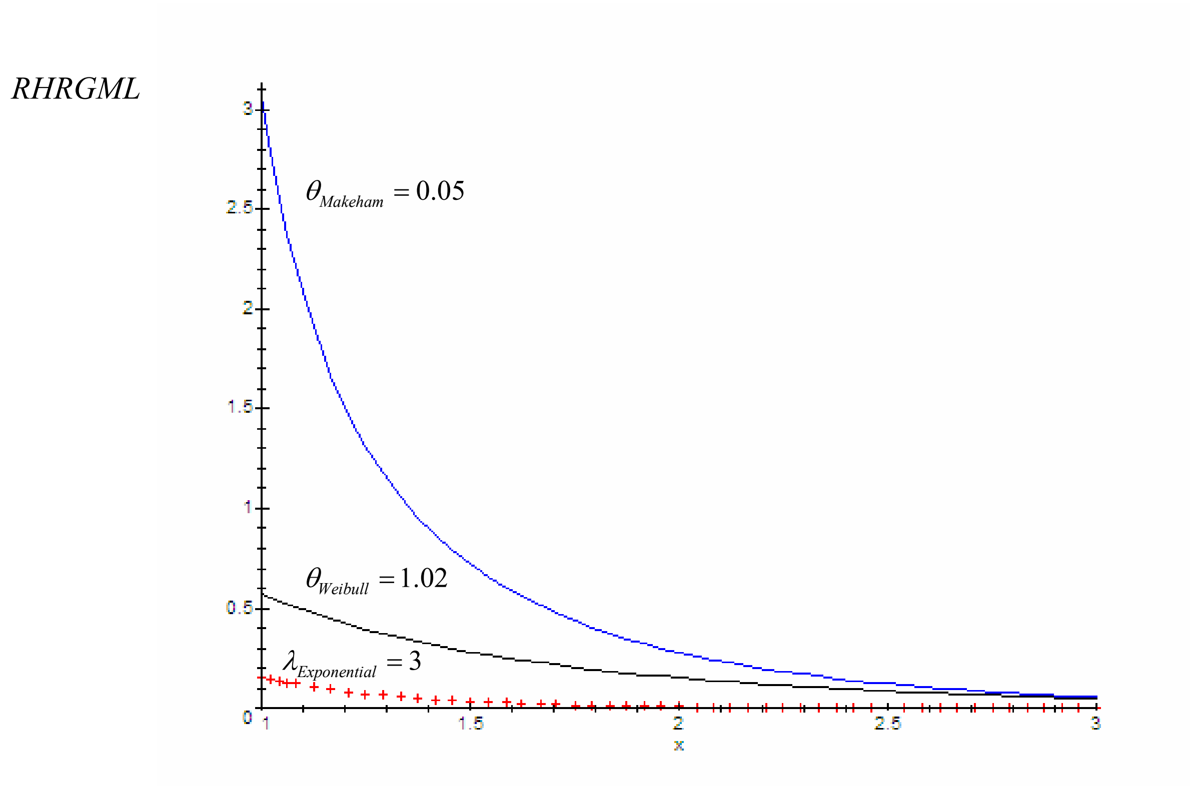

Figure 2.1: Comparing the reversed hazard rate guaranteed minimum length for different distributions functions

2.7 Testing Used has Shorter Inactivity Time than New Guaranteed Minimum Length

In this section, we will establish a new test procedure for testing $H_0 : F = F_0$ against $H_1 : F$ is *USINGML* and not F_0 , where F_0 is known. It is easy to check that there is no boundary member for this class of distribution; one possible choice of F_0 could be exponential distribution. The distribution function F has the property of *USINGML* if and only if

$$F(t-u)[F(\zeta)-F(t)] \geq F(t)[F(\zeta-u)-F(t)] \qquad t,\zeta,u \in \mathrm{R}^{+}.$$

Now, we consider the following as a measure of departure from H_0 in favor of H_1

$$\Delta = \int_0^\infty \int_0^\zeta \int_0^t \{F(t-u)[F(\zeta)-F(t)] - F(t)[F(\zeta-u)-F(t)]\}\, dF(u)dF(t)dF(\zeta) - \frac{1}{720} \tag{2.7.1}$$

under $H_1 : \Delta \geq 0$. If F_0 is exponential distribution with parameter $\lambda > 0$ then $\Delta_0 = 0$. As before let $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(x - X_i)$ and $\overline{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i - x)$ denote the empirical distribution function and survival function corresponding to $F(x)$, respectively, then an unbiased estimate of Δ is written in the form of

$$\hat{\Delta} = \int_0^\infty \int_0^\zeta \int_0^t \{F_n(t-u)[F_n(\zeta)-F_n(t)] - F_n(t)[F_n(\zeta-u)-F_n(t)]\}\, dF_n(u)dF_n(t)dF_n(\zeta) - \frac{1}{720}. \tag{2.7.2}$$

The expression above can be written as

$$\hat{\Delta} = \frac{1}{(n)_4} \sum\sum\sum\sum_{i\neq j\neq k\neq l} I(X_l - X_j - X_k) I(X_l - X_j)\{I(X_j - X_k - X_i)[I(X_l - X_i) - I(X_j - X_i)]$$
$$- I(X_j - X_i)[I(X_l - X_k - X_i) - I(X_j - X_i)]\}$$

where $(n)_4 = n(n-1)(n-2)(n-3)$.

Setting

$$\phi(X_1, X_2, X_3, X_4) = I(X_4 - X_2 - X_3) I(X_4 - X_2)\{I(X_2 - X_3 - X_1)[I(X_4 - X_1) - I(X_2 - X_1)]$$
$$- I(X_2 - X_1)[I(X_4 - X_3 - X_1) - I(X_2 - X_1)]\}$$

and define $\Phi(X_1, X_2, X_3, X_4) = \frac{1}{4!}\sum \phi(X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4})$ where the sum is extended over all permutations (i_1, i_2, i_3, i_4) of $\{1,2,3,4\}$. A simplified form of the symmetric kernel above is in form of

$$\Phi(X_1, X_2, X_3, X_4) = \frac{1}{4}\{\phi(X_1, X_2, X_3, X_4) + \phi(X_2, X_1, X_3, X_4) + \phi(X_2, X_3, X_1, X_4)$$
$$+ \phi(X_2, X_3, X_4, X_1)\}$$

An U-statistic equivalent type of $\hat{\Delta}$ is then of the form

$$U = \binom{n}{4}^{-1} \sum_{i\leq j\leq k\leq l} \Phi(X_i, X_j, X_k, X_l). \qquad (2.7.3)$$

Having the standard U-statistics theory, one can get the following result easily.

Theorem 2.17: As $n \to \infty$, $n^{\frac{1}{2}}(U - \Delta)$ is asymptotically distributed normal with zero mean and variance σ^2 where

$$\sigma^2 = Var\{\frac{1}{2}\bar{F}(X_1)\int_0^\infty\int_0^\infty \bar{F}(x+y)dF(x)dF(y)(\bar{F}(X_1) - \int_0^\infty \bar{F}(X_1 + x)dF(x))$$

$$+\overline{F}(X_1)\int_0^{\infty}\overline{F}(X_1+x)dF(x)[\int_0^{X_1}F(X_1-x)dF(x)(\frac{1}{2}-F(X_1))-F(X_1)$$

$$\times(\int_0^{\infty}\int_0^{\infty}\overline{F}(x+y)dF(x)dF(y)-F(X_1))]+\frac{1}{4}\int_0^{\infty}\overline{F}(X_1+x)dF(x)[\frac{1}{2}-\int_0^{\infty}\overline{F}(X_1+x)dF(x)]$$

$$+F(X_1)\int_0^{\infty}\overline{F}(X_1+x)dF(x)[\int_0^{\infty}\int_0^{\infty}\overline{F}(x+y)dF(x)dF(y)(F(X_1)-\frac{1}{2})$$

$$-\frac{1}{2}\int_0^{X_1}F(X_1-x)dF(x)-\frac{1}{2})]\} \tag{2.7.4}$$

Proof: Applying the same method of proof in Theorem 2.16 provides the desired result. ■

Under H_0, plug in exponential distribution function F_0 with parameter λ, we have

$\sigma_{0,\lambda}^2=0.00006 \quad \forall\lambda>0$. Now, we reject H_0 in favor of H_1 if $\sigma_{0,\lambda}^{-1}n^{\frac{1}{2}}U\geq z_{\alpha}$, where z_{α} is the α quantile of the standard normal distribution.

2.8 Nonparametric Procedure for Testing Increasing Variance Inactivity Guaranteed Minimum Life Length

Based on a sample $X_1,X_2,\cdots,X_n$ with distribution function F, One might test the null hypothesis $H_0:CV_*=1$ against $H_1:CV_*<1$, or, equivalently $H_0:F$ is exponential against $H_1:F$ is $IVIGML$. Note that F is $IVIGML$ if and only if $\sigma_*^2\leq\mu_*^2$ (Theorem 2.4) which means $\int_0^{t^*}\int_0^{x}H^*(u)dudx-\mu_*(t^*)\int_0^{t^*}H^*(u)du$, say I, can be written in form of

$$I=\int_0^{t^*} v(x)dx - \mu_*(t^*)v(t^*)$$

$$=\int_0^{t^*} v(x)dx - \frac{\left(\int_0^{t^*} H^*(u)du\right)^2}{H^*(t^*)}$$

$$=\int_0^{t^*} v(x)dx - \frac{v^2(t^*)}{H^*(t^*)} \le 0 \ ,$$

where $v(t)=\int_0^t H^*(u)du$ and $H^*(t)=v'(t)=F^*(t)-F^*(0).$

As a measure of departure from the null hypothesis, one can consider the following

$$\delta=\int_0^\infty [v^2(t^*)-v'(t^*)\int_0^{t^*} v(x)dx]dH^*(t^*) . \qquad (2.8.1)$$

Also, one can notice that

$$\int_0^{t^*} v(x)dx = t^* v(t^*) - \frac{1}{2}t^{*2}H^*(t^*) + \frac{1}{2}\int_0^{t^*} x^2 dH^*(x)$$

Therefore

$$\delta=\int_0^\infty [v^2(t^*)-t^*v(t^*)H^*(t^*)+\frac{1}{2}t^{*2}H^{*2}(t^*)-\frac{1}{2}H^*(t^*)\int_0^\infty x^2 I(t^*-x)dH^*(x)]dH^*(t^*) ,$$

where $I(t^*-x)=\begin{cases}1 & if \quad t^*\text{-}x\ge 0\\ 0 & o.w.\end{cases}$.

Note that under $H_1:\delta\ge 0$. Using the empirical distribution function, one can find an unbiased estimate of δ in the form of

$$\hat{\delta} = \frac{1}{(n)_4} \sum\sum\sum\sum_{i \neq j \neq k \neq l} \{ (\frac{1}{2} X_j^2 - 2X_i X_j + 3X_i^2 + X_j X_k - 2X_i X_k + X_k X_l - 2X_i X_l + \frac{1}{2} X_l^2)$$

$$\times I(X_j - X_k) I(X_j - X_l)$$

$$+ \frac{3}{2}(X_j - X_i)^2 I(X_i - X_k) I(X_j - X_l) + (X_j - X_i)(X_k + \frac{1}{2} X_i + \frac{1}{2} X_j) I(X_j - X_k) I(X_i - X_l)$$

$$+ (\frac{3}{2} X_l^2 - 2X_i X_k - X_j X_l + 2X_i X_j - \frac{1}{2} X_j^2 - X_j + X_i) I(X_i - X_k) I(X_i - X_l) \}. \qquad (2.8.2)$$

Setting

$$\phi(X_1, X_2, X_3, X_4) = (\frac{1}{2} X_2^2 - 2X_1 X_2 + 3X_1^2 + X_2 X_3 - 2X_1 X_3 + X_3 X_4 - 2X_1 X_4 + \frac{1}{2} X_4^2)$$

$$\times I(X_2 - X_3) I(X_2 - X_4) + \frac{3}{2}(X_2 - X_1)^2 I(X_1 - X_3) I(X_2 - X_4)$$

$$+ (X_2 - X_1)(X_3 + \frac{1}{2} X_1 + \frac{1}{2} X_2) I(X_2 - X_3) I(X_1 - X_4)$$

$$+ (\frac{3}{2} X_4^2 - 2X_1 X_3 - X_2 X_4 + 2X_1 X_2 - \frac{1}{2} X_2^2 - X_2 + X_1) I(X_1 - X_3) I(X_1 - X_4)$$

and define $\Phi(X_1, X_2, X_3, X_4) = \frac{1}{4!} \sum \phi(X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4})$ where the sum is extended over all permutations (i_1, i_2, i_3, i_4) of $\{1, 2, 3, 4\}$, a simplified form of the above symmetric kernel is in the form of

$$\Phi(X_1, X_2, X_3, X_4) = \frac{1}{4} \{ \phi(X_1, X_2, X_3, X_4) + \phi(X_2, X_1, X_3, X_4) + \phi(X_2, X_3, X_1, X_4)$$

$$+ \phi(X_2, X_3, X_4, X_1) \}$$

Then, an equivalent U-Statistic type of $\hat{\delta}$ is

$$U = \binom{n}{4}^{-1} \sum_{i \leq j \leq k \leq l} \Phi(X_i, X_j, X_k, X_l). \qquad (2.8.3)$$

Having the standard U-statistics theory, one can get the following result easily.

Theorem 2.18: as $n \to \infty$, $n^{\frac{1}{2}}(U-\delta)$ is asymptotically distributed normal with zero mean and variance σ^2 where

$$\sigma^2 = Var[E\{\Phi(X_1,X_2,X_3,X_4)\mid X_1\} + E\{\Phi(X_2,X_1,X_3,X_4)\mid X_1\} + E\{\Phi(X_2,X_3,X_1,X_4)\mid X_1\}$$

$$+E\{\Phi(X_2,X_3,X_4,X_1)\mid X_1\}]$$

$$= Var(\sum_{i=1}^{4}\phi_i(X_1))$$

Under the null hypothesis, standard exponential distribution, the variance is $\sigma_0^2 = 78.21$.

To perform a test hypothesis, we reject H_0 if $(78.2)^{-1/2} n^{\frac{1}{2}} U_n \geq Z_\alpha$, where Z_α is the standard normal variate.

2.9 A Test Procedure of the Increasing Coefficient Variation of Inactivity Guaranteed Minimum Length

Here, we want to establish a new test procedure to test $H_0: F$ is exponential with mean $\mu < \infty$ against $H_1: F$ is $ICVIGML$ and not exponential. Using Remark 2.6, F is $ICVIGML$, if

$$\mu_*(t^*)\int_0^{t^*}\left[H^*(z)\right]dz \geq \int_0^{t^*}\left[H^*(z)\right]\mu_*(z)dz \tag{2.9.1}$$

As a measure of departure from the null hypothesis, one might consider the following

$$\delta = \int_0^{\infty}\left\{H^*(z)\left[\mu_*(t^*)-\mu_*(z)\right]I(t^*-z)\right\}dH^*(z) \tag{2.9.2}$$

Using the empirical distribution function, we have an unbiased estimate of δ, $\hat{\delta}$ in the form of

$$\hat{\delta} = \int_0^\infty \left\{ H_n^*(z) \left[E\left[X_{(t^*)}^* - X_{(z)}^* \right] \right] I(t^* - z) \right\} dH_n^*(z)$$

where $H_n^*(z) = F_n^*(z) - F_n^*(0)$. Also, it can be written in the following form

$$\hat{\delta} = \frac{1}{n^4} \sum_i \sum_j \sum_k \sum_l [I(X_j + X_k - X_i) - I(X_k - X_i)][\left(X_l - X_i\right) I(X_l - X_i) - \left(X_j + X_k - X_i\right)$$

$$\times I(X_j + X_k - X_i)] I(X_l - X_k - X_j) \qquad (2.9.3)$$

Now, setting

$$\phi(X_1, X_2, X_3, X_4) = [I(X_2 + X_3 - X_1) - I(X_3 - X_1)][\left(X_4 - X_1\right) I(X_4 - X_1) - \left(X_2 + X_3 - X_1\right)$$

$$\times I(X_2 + X_3 - X_1)] I(X_4 - X_3 - X_2)$$

and define $\Phi(X_1, X_2, X_3, X_4) = \frac{1}{4!} \sum \phi(X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4})$ where the sum is extended over all permutations (i_1, i_2, i_3, i_4) of $\{1, 2, 3, 4\}$, a simplified form of the above symmetric kernel is in form of

$$\Phi(X_1, X_2, X_3, X_4) \quad = \frac{1}{4} \{\phi(X_1, X_2, X_3, X_4) + \phi(X_2, X_1, X_3, X_4) + \phi(X_2, X_3, X_1, X_4)$$

$$+ \phi(X_2, X_3, X_4, X_1)\}$$

Then, an equivalent U-Statistic type of $\hat{\delta}$ is

$$U = \binom{n}{4}^{-1} \sum_{i \le j \le k \le l} \Phi(X_i, X_j, X_k, X_l). \qquad (2.9.4)$$

Theorem 2.19: As $n \to \infty$, $n^{\frac{1}{2}}(U_n - \delta)$ is asymptotically normal with zero mean and variance σ^2, where

$$\sigma^2 = Var\{(2\int_{X_1}^{\infty} \overline{F}(x - X_1)dF(x) - \frac{1}{2} - F(X_1))(\iint_{y \le x}(x - y)dF(x)dF(y)$$

$$- \iint_{y \le x + X_1}(X_1 + x - y)dF(x)dF(y) \times (\int_0^{\infty} \overline{F}(X_1 + x)dF(x))$$

$$+ (\int_0^{X_1} \overline{F}(X_1 - x)dF(x) - \overline{F}(X_1))(\int_{X_1}^{\infty} \overline{F}(x - X_1)dF(x) - \iint_{y \ge X_1 - x}(x + y - X_1)dF(x)dF(y))$$

$$\times (\iint \overline{F}(x + y)dF(x)dF(y)) + (\iint_{x \ge y} \overline{F}(x - y)dF(x)dF(y) - \frac{1}{2})(\int_0^{X_1}(X_1 - x)dF(x)$$

$$- \iiint_{x + y - z \ge 0}(x + y - z)df(x)dF(y)dF(z))(\int_0^{X_1} F(X_1 - x)dF(x))\} \qquad (2.9.5)$$

Under the null hypothesis, standard exponential distribution, the variance is 0.014.

Proof: Applying the same method in the proof of Theorem 2.16 provides the desired result.

To conduct the test, we reject the null hypothesis, exponential distribution, in favor of H_1 if $(0.014)^{-1/2} n^{\frac{1}{2}} U_n \ge Z_\alpha$, where Z_α is the standard normal variate.

2.10 Applications and Simulation Results

2.10.1 Example with Uncensored Data

In a standard survival analysis application, individuals are followed over time for the occurrence of a specific event. If the event is observed to occur, the data is recorded as the time the event occurred, x^*. In some situations, however, the times of the events of interest may only be known to have occurred within an interval of time, [L, R], where $L \le X^* \le R$. This can occur in a clinical trial, for example, when patients are assessed often only at pre-scheduled visits, but the event of interest may occur in between visits. These data are known as interval-censored data.

Listed below are the survival times of 17 patients from a myelogenous leukemia study conducted by Feigl and Zelen (1965). The survival times, say x_i^*, are given in weeks from date of diagnosis. They are listed sequentially in order of entry into the study.

65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 0, 1, 5, 65

Here, '0' indicates the survival time for a patient who died at the same day of diagnosis. Using this data, the empirical estimate of the mean inactivity time ($\hat{\mu}_*(t^*)$) for a person who was found dead 20 weeks after his/her diagnosis (t^*), is 11 weeks. Having this result, the estimate of variance inactivity time is $\hat{\sigma}_*^2(t^*)$ =104.5. Using Theorem 2.4, it is easy to conclude that this data belongs to the class of increasing mean inactivity time guaranteed minimum length. Also, the 95% confidence interval of μ_* is [6.94, 15.1].

2.10.2 Example with Interval-Censored Data

In this example, the study is on the development of drug resistance (measure using a plaque reduction assay) to zidovudine on the patients enrolled in four clinical trials for the treatment of AIDS (Lindsey and Ryan, 1998). Samples were collected on the patients at a subset of the scheduled visit times dictated by the four protocols. Since the resistance assays were very expensive, there were few assessments on each patient, resulting in very wide interval observations. Listed below are the sample data,

16, 15, 12, 17, 13, 24, 6, 15, 14, 12, 13, 12, 12, 18, 14, 17, 15, 3, 4, [1, 11], [13, 19], 6, 11, 6, 6, [2, 12], [1, 17], 14, 25, [2, 11], 14

To find a point estimate for the time of resistance for each of these intervals, we shall use the empirical estimate for mean inactivity time of each interval individually. For instance, the mean inactivity time for interval [1, 11] is 4.83; therefore, the estimate time of resistance would be 6.17 months and so on. Here, the estimate of variance inactivity time is $\hat{\sigma}_*^2(t^*)$ =1.44. Using Theorem 2.5, $CV_*(t^*) \le 1$ therefore this data belongs to the class of increasing variance inactivity time. The 95% confidence interval for μ_* is [4.5, 5.16]. Using Section 2.8 and comparing the test statistic, $(78.2)^{-1/2} n^{\frac{1}{2}} U_n = 40.39,\ n = 31$ with the $\alpha = 0.05$ quantile of the standard normal distribution $Z_{\alpha/2} = 1.96$, reveals that this data belongs to the class of $IVIGMLG$.

2.10.3 Empirical Power and Critical values

For empirical studies on the performance of $DRHRGML$ test procedure, we carried out a series of 1000 simulations of size n=10, n= 20 from the following alternative distributions:

a) The Weibull distribution with distribution function

$$F(x) = -e^{-x^{\theta}} + 1, \qquad \theta > 1,\ x \ge 0;$$

b) The Lomax distribution with distribution function

$$F(x) = -(1+\theta x)^{-\theta^{-1}} + 1 \qquad \theta > 0,\ \ x \ge 0$$

These distributions are commonly considered in power studies of tests for exponentiality. Figure 2.2 illustrates that the Lomax distribution belongs to the class of $DRHRGML$.

For small to moderate sample sizes, we use Monte Carlo methods with 1000 replicates to obtain empirical critical values and power estimates of our procedure. The results of simulation for this test procedure are shown in Table 2.1, Table 2.2 and Table 2.3.

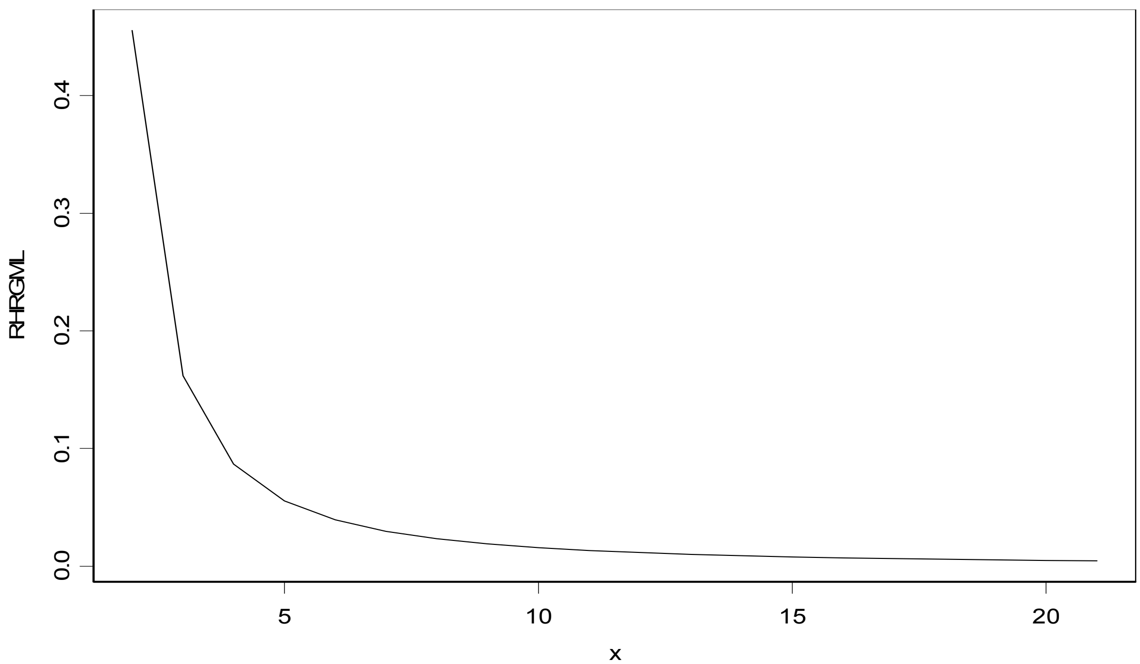

Figure 2.2: Reversed hazard rate guaranteed minimum length function for the Lomax distribution.

Table 2.1

Empirical Critical values of $\sigma_{0,\lambda}^{-1} n^{\frac{1}{2}} U$ for testing the *DRHRGML* (Weibull distribution)

n	θ	Critical values for the following percentage points: 0.90	0.95	0.975	0.99
10	1.25	2.69543	3.354311	4.01619	4.732876
	1.50	3.35778	4.044641	4.67282	5.751747
	1.80	4.07608	5.003015	5.66040	6.290233
	2.00	4.67507	5.362407	5.93144	6.82872
20	1.25	2.52808	3.06836	3.54617	4.065586
	1.50	3.67435	4.14248	4.46549	5.017303
	1.80	4.71967	5.17191	5.67561	6.069255
	2.00	5.03377	5.495016	5.82597	6.692087

Table 2.2

Empirical Critical values of $\sigma_{0,\lambda}^{-1} n^{\frac{1}{2}} U$ for testing the *DRHRGML* (Lomax distribution)

n	θ	Critical values for the following percentage points: 0.90	0.95	0.975	0.99
10	1.25	0.44924	1.019771	1.38066	1.946998
	1.50	0.29949	0.688832	1.07892	1.437562
	1.80	0.08984	0.479187	0.89923	1.258466
	2.00	0.02995	0.449238	0.77868	1.168019
20	1.25	-0.39634	-0.082519	0.27184	0.492913
	1.50	-0.67925	-0.291342	0.00941	0.255323
	1.80	-0.76516	-0.539451	-0.27881	-0.020120
	2.00	-0.86375	-0.557717	-0.38157	-0.110001

Table 2.3

Power estimates of the test against *DRHRGML*

	θ	*Power estimate for the following sample size:* *n=10*	*n=20*
Weibull	1.25	0.226	0.211
	1.50	0.326	0.513
	1.80	0.485	0.769
	2.00	0.595	0.893
Lomax	1.25	0.287	0.397
	1.50	0.28	0.422
	1.80	0.325	0.502
	2.00	0.343	0.554

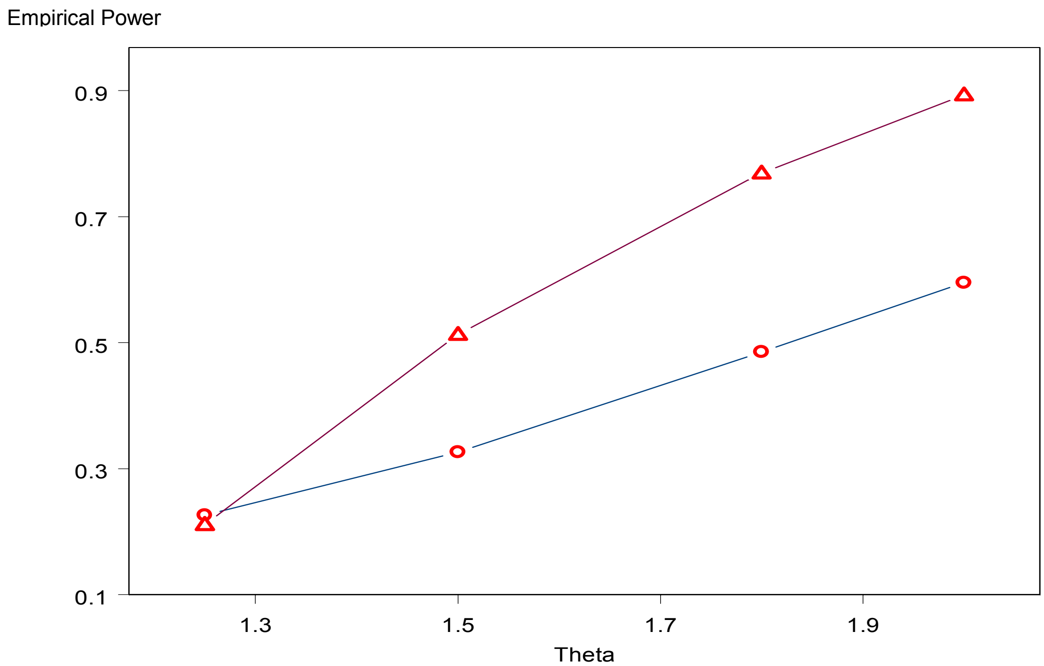

Figure 2.3: Comparing the empirical power of test DRHRGML for sample size n=10 and n=20 (Weibull distribution)

CHAPTER THREE: NEGATIVE MOMENT AND LAPLACE TRANSFORM INEQUALITIES AS THE CHARACTERISTICS OF INACTIVITY TIME CLASSES

3.0 Chapter Plan

The first part of this chapter uses negative moment and Lapace Transform inequalities in order to establish some new inequalities for these new notions. The second part introduces the applications of these inequalities in establishing some nonparametric procedures for testing against specific alternatives.

3.1 Introduction

In this chapter some new negative moments and Laplace transform inequalities of the families of life distributions we discussed in Chapter 2, are investigated. They are useful in constructing test statistics for these classes. Since these tests are based on sample negative moments and sample Laplace transforms of these aging classes; they are simple to devise, calculate and study relative to other more complicated tests.

3.2 Negative Moment Inequalities

The first result provides a negative moment inequality for the $DRHR$ distributions. In this result, as well as subsequent results, all moments are assumed to exist and are finite.

Theorem 3.1**:** If F belongs to the decreasing reversed hazard rate ($DRHR$) class, then for all integers $r, m > 1$

$$(r-1)E(X_1^{1-m}X_2^{-r}) \geq (m-1)E(X_2^{1-r}X_1^{-m}), \tag{3.2.1}$$

where X_1, X_2 are two nonnegative independent copies of random variables with distribution function F.

Proof: Since F is $DRHR$ then

$$\iint_{x\leq t} t^{-m}(t-x)^{-r} f(t-x)F(t)dxdt \geq \iint_{x\leq t} t^{-m}(t-x)^{-r} f(t)F(t-x)dxdt. \tag{3.2.2}$$

Assuming $0\times\infty = 0$ and taking $(t-x)=w, x=v$, we have the following

$$\int_0^\infty \int_0^t t^{-m}(t-x)^{-r} f(t-x)F(t)dxdt$$

$$= \int_0^\infty \int_0^\infty (w+v)^{-m} w^{-r} f(w)F(w+v)dvdw$$

$$= \int_0^\infty w^{-r} f(w)\left\{\int_0^\infty (w+v)^{-m} F(w+v)dv\right\} dw$$

$$= \int_0^\infty w^{-r} f(w)\left\{E\int_{X_1-w}^\infty (w+v)^{-m} dv\right\} dw$$

$$= \frac{1}{m-1} E(X_1^{1-m}X_2^{-r})$$

Also,

$$\int_0^\infty \int_0^t t^{-m}(t-x)^{-r} f(t)F(t-x)dxdt$$

$$= \int_0^\infty \int_0^\infty (v+w)^{-m} w^{-r} f(w+v)F(w)dvdw$$

$$= \int_0^\infty w^{-r} F(w)\left\{\int_0^\infty (v+w)^{-m} f(w+v)dv\right\} dw$$

$$= \frac{1}{r-1} E(X_1^{-m}X_2^{1-r})$$

The desired result follows immediately. ■

Corollary 3.1: Let $r = 2, m = 3$, then

$$E(X_1^{-2}X_2^{-2}) \geq 2E(X_2^{-1}X_1^{-3}), \qquad (3.2.3)$$

Next, we have the negative moment inequality for the increasing mean inactivity time ($IMIT$) class.

Theorem 3.2: If F belongs to the increasing mean inactivity time ($IMIT$) class, then for all integers $r > 1$,

$$E\left\{(\max(X_1, X_2))^{1-r}\right\} \geq (r-1)\left\{E(X_1^{1-r}) - E(X_1X_2^{-r})\right\}, \qquad (3.2.4)$$

Proof: Since F belongs to the $IMIT$ class, then $F^2(t) \geq f(t)\int_0^t F(u)du$. This is equivalent to saying that F belongs to the $IMIT$ if $F^2(t) \geq f(t)(t - E(X_1))$. Thus, for any integer $r > 2$,

$$\int_0^\infty t^{-r}F^2(t)dt \geq \int_0^\infty t^{-r}f(t)(t - E(X_1))dt$$

Now,

$$\int_0^\infty t^{-r}F^2(t)dt = E\int_{\max(X_1,X_2)}^\infty t^{-r}dt = \frac{1}{r-1}E(\max(X_1, X_2)^{1-r})$$

Using the same method, we have

$$\int_0^\infty t^{-r}f(t)(t - E(X_1))dt = E(X_1^{1-r}) - E(X_1X_2^{-r})$$

The result follows immediately. ■

Corollary 3.2: Let $r = 2$, then

$$E\left\{(\max(X_1, X_2))^{-1}\right\} \geq \left\{E(X_1^{-1}) - E(X_1X_2^{-2})\right\}, \qquad (3.2.5)$$

We now extend the inequalities above to the cases with guaranteed minimum life.

Theorem 3.3: If F belongs to the reversed hazard rate guaranteed minimum life length ($DRHRGML$) class, then for all integers $r,q \geq 2, p \geq 1$,

$$\frac{1}{(q-1)}(E(X_1^{-r}X_2^{-p})E(\max(X_1,X_2)^{1-q}) - E(X_1^{-r}X_2^{1-q})E(\max(X_1,X_2)^{-p}) \geq \frac{1}{(r-1)}E(X_2^{-p})$$

$$\times E(\max(X_1,X_2)^{1-q-r})(E(X_1^{1-r})-1), \qquad (3.2.6)$$

Proof: F belongs to the class of $DRHRGML$, then

$$f(\zeta-x)F(\zeta) - f(\zeta-x)F(t) \geq f(\zeta)F(\zeta-x), \text{ say, } I - II \geq III, \qquad (3.2.7)$$

Taking $(\zeta - x) = w, x = v$, we have the following

$$I = \int_0^\infty \int_0^\zeta \int_0^\zeta t^{-p}\zeta^{-m}(\zeta-x)^{-r} f(t)f(\zeta-x)F(\zeta)dxdtd\zeta$$

$$= \int_0^\infty \int_0^w \int_0^\infty t^{-p}(v+w)^{-m}w^{-r} f(t)f(w)F(v+w)dvdtdw$$

$$= \int_0^\infty \int_0^w t^{-p}w^{-r} f(t)f(w)\left\{\int_0^\infty (v+w)^{-m}F(v+w)dv\right\}dtdw$$

$$= \int_0^\infty \int_0^w t^{-p}w^{-r} f(t)\left\{E\int_{X_1-w}^\infty (v+w)^{-m}dv\right\}dtdw$$

$$= \frac{1}{m-1}E(X_1^{1-m})\int_0^\infty \int_0^w t^{-p}w^{-r} f(t)f(w)dtdw$$

$$= \frac{1}{m-1}E(X_1^{1-m}X_2^{-p}X_3^{-r})$$

and,

$$II=\int_0^\infty\int_0^\zeta\int_0^\zeta t^{-p}\zeta^{-m}(\zeta-x)^{-r}f(t)f(\zeta-x)F(t)dxdtd\zeta$$

$$=\int_0^\infty\int_0^w\int_0^\infty t^{-p}(v+w)^{-m}w^{-r}f(t)f(w)F(t)dvdtdw$$

$$=\int_0^\infty\int_0^w t^{-p}w^{-r}f(t)f(w)F(t)\left\{\int_0^\infty(v+w)^{-m}dv\right\}dtdw$$

$$=\int_0^\infty\int_0^w t^{-p}w^{-r}f(t)F(t)\left\{E\int_0^\infty(v+w)^{-m}dv\right\}dtdw$$

$$=\frac{1}{m-1}\int_0^\infty w^{-(r+m)+1}f(w)\left\{E\int_{X_2}^w t^{-p}f(t)dt\right\}dw$$

$$=\frac{1}{m-1}\{E\int_{X_3}^\infty w^{-(r+m+p)+1}f(w)dw-\frac{1}{2}EX_2{}^{-p}[\int_0^\infty(w^{-(r+m)+2}-w^{-(m+r)+1})f(w)dw]$$

$$-\int_0^\infty w^{-(r+m+p)+1}f(w)dw\}$$

$$=\frac{1}{m-1}\{\frac{1}{(m+r+p)-2}E(X_3^{2-(m+r+p)})+\frac{1}{2}E(X_2^{-p}X_3^{2-(r+m)}-X_3^{1-(m+r+p)}$$

$$+X_2^{-p}X_3^{1-(r+m)}\}$$

Also,

$$III=\int_0^\infty\int_0^\zeta\int_0^\zeta t^{-p}\zeta^{-m}(\zeta-x)^{-r}f(t)f(\zeta)F(\zeta-x)dxdtd\zeta$$

$$=\int_0^\infty\int_0^w\int_0^\infty t^{-p}(v+w)^{-m}w^{-r}f(t)f(w+v)F(w)dvdtdw$$

$$=\int_0^\infty\int_0^w t^{-p}w^{-r}f(t)F(w)\left\{\int_0^\infty(v+w)^{-m}f(v+w)dv\right\}dtdw$$

$$=\int_0^\infty\int_0^w t^{-p}w^{-r}f(t)F(w)\left\{EX_1^{-m}\right\}dtdw$$

$$=\frac{1}{r-1}E(X_1^{-m}X_2^{-p}X_3^{-r})$$

Simple calculation provides the desired result. ■

Corollary 3.3: Let $m = r = 2,\ p = 1$, then

$$3E(X_2^{-1}X_3^{-2} + X_3^{-4} - X_2^{-1}X_3^{-3}) \geq 2E(X_3^{-3}), \qquad (3.2.8)$$

Theorem 3.4: If F belongs to the class of increasing mean inactivity guaranteed minimum life length ($IMIGML$), then for all integers $m > 2$,

$$E(X_2^{1-m} - X_2^{2-m}) \geq E(X_1)E(X_2^{-m} - X_2^{1-m}) \qquad (3.2.9)$$

Proof: F belongs to the class of $IMIGML$, then $F(t) \geq \int_0^{\zeta - t} F(\zeta - x)dx$. Equivalently,

$$\int_0^{\infty}\int_0^{\zeta} \zeta^{-m} f(\zeta)F(t)dtd\zeta \geq \int_0^{\infty}\int_0^{\zeta} \zeta^{-m} f(\zeta)[\zeta - E(X_1)]dtd\zeta, \qquad (3.2.10)$$

then ,

$$LHS = \int_0^{\infty} \zeta^{-m} f(\zeta)(E\int_0^{\zeta} dt)d\zeta$$

$$= \int_0^{\infty} \zeta^{-m} f(\zeta)d\zeta - EX_1\int_0^{\infty} \zeta^{-m} f(\zeta)d\zeta$$
$$= E(X_2^{1-m}) - E(X_1)E(X_2^{-m})$$

and,

$$RHS = \int_0^{\infty} \zeta^{-m+1} f(\zeta)[\zeta - E(X_1)]d\zeta$$

$$= \int_0^{\infty} \zeta^{-m+2} f(\zeta) - EX_1\int_0^{\infty} \zeta^{-m+1} f(\zeta)d\zeta$$
$$= E(X_2^{-m+2}) - E(X_1)E(X_2^{-m+1})$$

which is the desired result. ■

Corollary 3.4: Let $m = 3$, then

$$E(X_2^{-2} - X_2^{-1}) \geq E(X_1)E(X_2^{-3} - X_2^{-2}), \qquad (3.2.11)$$

3.3 Applications of Negative Moment Inequalities in Hypotheses Testing

3.3.1 Testing Against *DRHR* Alternatives

Here, we want to establish a test procedure to test $H_0 : F$ is exponential with mean $\mu \in (0,\infty)$ against $H_1^{(1)} : F$ is $DRHR$ and not exponential. Using Corollary 3.1, we use the measure of departure from H_0 given by

$$\delta^{(1)} = E(X_1^{-2} X_2^{-2} - 2X_2^{-1} X_1^{-2}), \qquad (3.3.1)$$

This is estimated by

$$\hat{\delta}^{(1)} = \frac{1}{n(n-1)} \sum\sum_{i \neq j} (X_i^{-2} X_j^{-2} - 2X_j^{-1} X_i^{-2}) \qquad (3.3.2)$$

Now, we set $\phi^{(1)} = \phi(X_1, X_2) = X_1^{-2} X_2^{-2} - 2X_2^{-1} X_1^{-2}$ and define

$$\Phi^{(1)} = \Phi^{(1)}(X_1, X_2) = \frac{1}{2!} \sum \phi^{(1)}(X_{i_1}, X_{i_2})$$

where the sum is extended over all permutations (i_1, i_2) of $\{1,2\}$.

An equivalent U-Statistic type of $\hat{\delta}^{(1)}$ is of the form

$$U^{(1)} = \binom{n}{2} \sum_{i \le j} \Phi^{(1)}(X_i, X_j), \qquad (3.3.3)$$

Using the standard U-Statistics theorem, the following is provided.

Theorem 3.5: As $n \to \infty$, $n^{\frac{1}{2}}(U^{(1)} - \delta^{(1)})$ is asymptotically normal with mean zero and variance

$$\sigma^2 = 4Var\{(X_1^{-2} - X_1^{-1}) \int_0^\infty x^{-2} dF_{X^{-1}}(x) - X_1^{-2} \int_0^\infty x^{-1} dF_{X^{-1}}(x)\}, \qquad (3.3.4)$$

where $dF_{X^{-1}}(x) = x^{-2} dF_X(x^{-1})$.

Proof: Using the general theory of standard U-statistics, cf. Lee (1989), the asymptotic variance is equal to $\sigma^2 = 4Var\{E(\Phi^{(1)}(X_1, X_2) \mid X_1)\}$. Simple calculation gives us the desired result.

Under the null hypothesis, standard exponential distribution, the null variance is $\sigma_0^{\ 2}$ =32. We reject the null hypothesis in favor of H_1 if $(5.66)^{-1} n^{1/2} U^{(1)} \geq Z_\alpha$, where Z_α is the standard normal variate. ∎

3.3.2 Testing Against *IMIT* Alternatives

Here, we want to establish a test procedure to test $H_0 : F$ is exponential with mean $\mu \in (0, \infty)$ against $H_1^{(2)} : F$ is *IMIT* and not exponential. Using the Corollary 3.2, we use the measure of departure from H_0 given by

$$\delta^{(2)} = E\left\{(\max(X_1, X_2))^{-1}\right\} - \left\{E(X_1^{\ -1}) - E(X_1 X_2^{\ -2})\right\} \qquad (3.3.5)$$

This is estimated by

$$\hat{\delta}^{(2)} = \frac{1}{n(n-1)} \sum\sum\nolimits_{i \neq j} (\max(X_i, X_j))^{-1} - X_i^{\ -1} + X_i X_j^{\ -2}) \qquad (3.3.6)$$

Now, we set $\phi^{(2)} = \phi^{(2)}(X_1, X_2) = \max(X_1, X_2))^{-1} - X_1^{\ -1} + X_1 X_2^{\ -2}$ and define

$$\Phi^{(2)} = \Phi^{(2)}(X_1, X_2) = \frac{1}{2!} \sum \phi^{(2)}(X_{i_1}, X_{i_2})$$

where the sum is extended over all permutations (i_1, i_2) of $\{1, 2\}$.

An equivalent U-Statistic type of $\hat{\delta}^{(2)}$ is of the form

$$U^{(2)} = \binom{n}{2} \sum_{i \leq j} \Phi^{(2)}(X_i, X_j)$$

Using the standard U-Statistics theorem, the following is provided.

Theorem 3.6: As $n \to \infty$, $n^{\frac{1}{2}}(U^{(2)} - \delta^{(2)})$ is asymptotically normal with mean zero and variance

$$\sigma^2 = Var\{2X_1^{-1}F(X_1) + 2\int_{X_1}^{\infty} xdF(x) - X_1^{-1} + (X_1 - 1)\int_0^{\infty} x^{-1}dF_{X_1^{-1}}(x) + X_1^{-1}\int_0^{\infty} xdF(x)\}$$

(3.3.7)

Proof: The same method in Theorem 3.5, provides the desired results. ■

Under the null hypothesis, standard exponential distribution, the null variance, ${\sigma_0}^2$,is 0.175. we reject the null hypothesis in favor of H_1 if $(0.42)^{-1}n^{1/2}U^{(2)} \geq Z_\alpha$, where Z_α is the standard normal variate.

3.3.3 Testing Against *DRHRGML* Alternatives

Here, we want to establish a test procedure to test $H_0 : F$ is exponential with mean $\mu \in (0,\infty)$ against $H_1^{(3)} : F$ is *DRHRGML* and not exponential. Using Corollary 3.3, we use the measure of departure from H_0 given by

$$\delta^{(3)} = E[3(X_2^{-1}X_3^{-2} + X_3^{-4} - X_2^{-1}X_3^{-3}) - 2X_3^{-3}] \qquad (3.3.8)$$

This is estimated by

$$\hat{\delta}^{(3)} = \frac{1}{n(n-1)}\sum\sum_{i \neq j}[3(X_i^{-1}X_j^{-2} + X_j^{-4} - X_i^{-1}X_j^{-3}) - 2X_j^{-3}] \qquad (3.3.9)$$

Now, we set $\phi^{(3)} = \phi^{(3)}(X_1, X_2) = 3(X_1^{-1}X_2^{-2} + X_2^{-4} - X_1^{-1}X_2^{-3}) - 2X_2^{-3})$ and define

$$\Phi^{(3)} = \Phi^{(3)}(X_1, X_2) = \frac{1}{2!}\sum\phi^{(3)}(X_{i_1}, X_{i_2})$$

where the sum is extended over all permutation (i_1, i_2) of $\{1,2\}$.

An equivalent U-Statistic type of $\hat{\delta}^{(3)}$ is of the form

$$U^{(3)} = \binom{n}{2} \sum_{i \le j} \Phi^{(3)}(X_i, X_j)$$

Using the standard U-Statistics theorem, the following is provided.

Theorem 3.7: As $n \to \infty$, $n^{\frac{1}{2}}(U^{(3)} - \delta^{(3)})$ is asymptotically normal with mean zero and variance

$$\sigma^2 = 4Var\{3X_1^{-1} \int_0^\infty (x^{-2} - x^{-3}) dF_{X^{-1}}(x) + 3(X_1^{-2} - X_1^{-3}) \int_0^\infty x^{-1} dF_{X^{-1}}(x) - 2X_1^{-3}\}$$

(3.3.10)

Proof: We shall use the same method in Theorem 3.5 to get the desired result.

3.3.4 Testing Against *IMIGML* Alternatives

Here, we want to establish a test procedure to test $H_0 : F$ is exponential with mean $\mu \in (0, \infty)$ against $H_1^{(4)} : F$ is $IMIGML$ and not exponential. Using Corollary 3.4, we use the measure of departure from H_0 given by

$$\delta^{(4)} = E(X_2^{-2} - X_2^{-1} - X_1(X_2^{-3} - X_2^{-2})) \qquad (3.3.11)$$

This is estimated by

$$\hat{\delta}^{(4)} = \frac{1}{n(n-1)} \sum \sum_{i \ne j} (X_j^{-2} - X_j^{-1} - X_i(X_j^{-3} - X_j^{-2})) \qquad (3.3.12)$$

Now, we set $\phi^{(4)} = \phi^{(4)}(X_1, X_2) = X_2^{-2} - X_2^{-1} - X_1(X_2^{-3} - X_2^{-2})$ and define

$$\Phi^{(4)} = \Phi^{(4)}(X_1, X_2) = \frac{1}{2!} \sum \phi^{(4)}(X_{i_1}, X_{i_2})$$

where the sum is extended over all permutations (i_1, i_2) of $\{1, 2\}$.

An equivalent U-Statistic type of $\hat{\delta}^{(4)}$ is of the form

$$U^{(4)} = \binom{n}{2} \sum_{i \le j} \Phi^{(4)}(X_i, X_j) \tag{3.3.13}$$

Using the standard U-Statistics theorem, the following is provided.

Theorem 3.8: As $n \to \infty$, $n^{\frac{1}{2}}(U^{(4)} - \delta^{(4)})$ is asymptotically normal with mean zero and variance

$$\sigma^2 = 4Var\{X_1^{-2} - X_1^{-1} - X_1 \int_0^\infty (x^{-3} - x^{-2}) dF_{X^{-1}}(x) + X_1^{-2} - X_1^{-1} - (X_1^{-3} - X_1^{-2}) \int_0^\infty x dF_X(x)\}$$

(3.3.14)

Proof: The same fashion as Theorem 3.5 provides the desired result.

3.4 Laplace Transform Inequalities

Theorem 3.9: If X has decreasing reversed hazard rate($DRHR$) life distribution, then $\phi_Y^*(u+1) \ge \gamma \phi_X^*(u+1)$,where $Y = \max(X_1, X_2)$, $\gamma = \frac{(u-2)(u+1)}{u^2}$, $u \in (0,\infty)$ and X_1, X_2 are two nonnegative independent copies of random variables with distribution function F .

Here, the Laplace transform, $\phi_X^*(u)$, is defined as $\phi_X^*(u) = E(e^{-uX})$.

Proof: Since F is $DRHR$, then we have

$$\int_0^\infty \int_0^t e^{-u(t-x)-t} f(t-x) F(t) dx dt \ge \int_0^\infty \int_0^t e^{-u(t-x)-t} f(t) F(t-x) dx dt \tag{3.4.1}$$

but,

$$LHS = \int_0^{\infty}\int_0^{t} e^{-u(t-x)-t} f(t-x)F(t)dxdt$$

$$= \int_0^{\infty} e^{-t}F(t)\{\int_0^{t} e^{-u(t-x)} f(t-x)dx\}dt$$

$$= E\int_{\max(X_1,X_2)}^{\infty} e^{-(u+1)t}dt + \frac{1}{u^2}E\int_X^{\infty} e^{-(u+1)t}dt + \frac{1}{u^2}E(e^{-(u+1)X})$$

$$= \frac{1}{u+1}\phi_Y^*(u+1) + \frac{2+u}{u^2(u+1)}\phi_X^*(u+1)$$

$$RHS = \int_0^{\infty}\int_0^{t} e^{-u(t-x)-t} f(t)F(t-x)dxdt$$

$$= \int_0^{\infty} e^{-t}f(t)\{\int_0^{t} e^{-u(t-x)}F(t-x)dx\}dt$$

$$= \frac{-1}{u}\int_0^{\infty} e^{-(u+1)t}f(t)dt + \frac{1}{u}E(e^{-Xu})\int_0^{\infty} e^{-t}f(t)dt$$

$$= \frac{1}{u+1}E(e^{-(u+1)X})$$

$$= \frac{1}{u+1}\phi_X^*(1+u)$$

Simple calculation provides the desired result. ■

Theorem 3.10: If F has increasing mean inactivity time ($IMIT$) property, then $\phi_Y^*(\alpha)(1-\phi_Y^*(\alpha)) \geq \alpha\phi_Y^*(2\alpha),\ \ \alpha > 0$, where $Y = \max(X_1, X_2)$ and X_1, X_2 are two nonnegative independent copies of random variables with distribution function F.

Proof: Since F belongs to the class of $IMIT$, then

$$\int_0^{\infty}\int_0^{t} e^{-\alpha(t+u)}F^2(t)dudt \geq \int_0^{\infty}\int_0^{t} e^{-\alpha(t+u)} f(t)F(u)dudt \qquad (3.4.2)$$

But,

$$LHS = E\int_{\max(X_1,X_2)}^{\infty} \frac{-1}{\alpha} e^{-2\alpha t} + \frac{1}{\alpha} e^{-\alpha t} dt$$

$$= \frac{1}{\alpha^2} e^{-\alpha Y}(1 - e^{-\alpha Y})$$

$$= \frac{1}{\alpha^2} \phi_Y^*(\alpha)(1 - \phi_Y^*(\alpha))$$

and

$$RHS = E\{\int_{\max(X_1,X_2)}^{\infty} e^{-2\alpha u} du + \int_{\max(X,u)}^{\infty} e^{-2\alpha u} du\}$$

$$= \frac{1}{\alpha} E(e^{-2Y}) = \frac{1}{\alpha} \phi_Y^*(2\alpha)$$

Simple calculation provides the desired result. ■

3.5 Application of Laplace Transformation Inequalities in Hypotheses Testing

3.5.1 Testing Against *DRHR* Alternatives

Here, we want to establish a test procedure to test $H_0 : F$ is exponential with mean $\mu \in (0, \infty)$ against $H_1^{(1)} : F$ is *DRHR* and not exponential. Using the Theorem 3.9 and for fixed $\gamma = \frac{(u-2)(u+1)}{u^2}$, one might use the measure of departure from H_0 by

$$\Delta^{(1)} = \phi_Y^*(u+1) - \gamma \phi_X^*(u+1) \tag{3.5.1}$$

where $Y = \max(X_1, X_2)$, $\phi_X^*(u) = E(e^{-\alpha X})$ and X_1, X_2 are two nonnegative independent copies of random variables with distribution function F.

$\Delta^{(1)}$ is estimated by

$$\hat{\Delta}^{(1)} = \frac{1}{n(n-1)} \sum_{i \neq j=1}^{n} \sum (e^{-(u+1)\max(X_i, X_j)} - \gamma e^{-(u+1)X_i}) \qquad (3.5.2)$$

Now, we set $\varphi^{(1)} = e^{-(u+1)\max(X_1, X_2)} - \gamma e^{-(u+1)X_1}$ and define

$$\rho^{(1)}(X_1, X_2) = \frac{1}{2!} \sum \varphi^{(1)}(X_{i_1}, X_{i_2})$$

where the sum is extended over all permutations (i_1, i_2) of $\{1, 2\}$.

An equivalent U-Statistic type of $\hat{\Delta}^{(1)}$ is of the form

$$U^{(1)} = \binom{n}{2} \sum_{i \leq j} \rho^{(1)}(X_i, X_j)$$

Using the standard U-Statistics theorem, the following is provided.

Theorem 3.11: As $n \to \infty$, $n^{\frac{1}{2}}(U^{(1)} - \hat{\Delta}^{(1)})$ is asymptotically normal with mean zero and variance

$$\sigma^2 = 4Var\{2e^{-(u+1)X_1}F(X_1) + 2\int_{X_1}^{\infty} e^{-(u+1)y} dF(y) - \gamma e^{-(u+1)X_1} - \int_0^{\infty} e^{-(u+1)y} dF(y)\}$$

$$(3.5.3)$$

Proof: Using the same method in the proof of Theorem 3.5 leads us to the desired result for this theorem. ■

Under the null hypothesis, standard exponential and fixed value $u = 3$, one can get $\sigma_0{}^2 = 0.05$.

3.5.2 Testing Against *IMIT* Alternatives

Here, we want to establish a test procedure to test $H_0 : F$ is exponential with mean $\mu<\infty$ against $H_1^{(2)} : F$ is *IMIT* and not exponential. Using Theorem 3.10 and for fixed $\alpha > 0$, we use the measure of departure from H_0 as

$$\Delta^{(2)} = \phi_Y^*(\alpha)(1-\phi_Y^*(\alpha)) - \alpha\phi_Y^*(2\alpha) \tag{3.5.4}$$

This is estimated by

$$\hat{\Delta}^{(2)} = \frac{1}{n(n-1)} \sum_{i \neq j=1}^{n} \sum \{e^{-\alpha \max(X_i, X_j)}(1 - e^{-\alpha \max(X_i, X_j)}) - \alpha e^{-2\alpha \max(X_i, X_j)}\} \tag{3.5.5}$$

Now, we set $\rho^{(2)}(X_1, X_2) = e^{-\alpha \max(X_1, X_2)}(1 - e^{-\alpha \max(X_1, X_2)}) - \alpha e^{-2\alpha \max(X_1, X_2)}$.

An equivalent U-Statistic type of $\hat{\delta}^{(4)}$ is of the form

$$U^{(2)} = \binom{n}{2} \sum_{i \leq j} \rho^{(2)}(X_i, X_j)$$

Using the standard U-Statistics theorem, the following is provided.

Theorem 3.12: As $n \to \infty$, $n^{\frac{1}{2}}(U^{(2)} - \hat{\Delta}^{(2)})$ is asymptotically normal with mean zero and variance

$$\sigma^2 = 4Var\{(e^{-\alpha X_1}F(X_1) + \int_{X_1}^{\infty} e^{-\alpha y} dF(y))(1 - e^{-\alpha X_1}F(X_1) - \int_{X_1}^{\infty} e^{-\alpha y} dF(y))$$

$$- \alpha(e^{-2\alpha X_1}F(X_1) + \int_{X1}^{\infty} e^{-2\alpha y} dF(y))\} \tag{3.5.6}$$

Proof: The same fashion as Theorem 3.10 provides the desired result. ■

Under the null hypothesis, standard exponential and fixed value $\alpha = 1$, one can get $\sigma_0{}^2 = 0.06$.

3.6 Further Inequalities of Life Distributions Based on the Negative Moments and Laplace Transforms

In this section we introduce more inequalities based upon the negative moment and Laplace transform methods. These inequalities are useful to devise new testing procedures for exponentiality against an alternative among the classes indicated in the Chapter 2. These tests are simpler, more competitive tests than those from Chapter 2 with very good efficiencies.

Theorem 3.13: If $f(t-x)F(t) \geq f(t)F(t-x)$, $0 \leq x \leq t$, then for all integers $r > 1$,

$$rE(X_1 + X_2)^{-r} \leq E(X_1 + X_2)^{-r+1} \qquad (3.6.1)$$

Proof: Since F belongs to the class of $DRHR$, then

$$\int_0^\infty \int_0^t (2t-x)^{-r} f(t-x)F(t)dxdt \geq \int_0^\infty \int_0^t (2t-x)^{-r} f(t)F(t-x)dxdt$$

$$LHS = E\int_{X_1}^\infty \int_0^t (2t-x)^{-r} f(t-x)dxdt$$

$$= E\int_{X_1}^\infty \int_0^\infty (v+w)^{-r} f(w)dwdt$$

$$= \frac{1}{r(r-1)} E(X_1 + X_2)^{-r+1}$$

also,

$$RHS = \int_0^\infty \int_0^t (2t-x)^{-r} f(t)F(t-x)dxdt$$

$$= \int_0^\infty \int_0^\infty (v+w)^{-r} f(v)F(w)dwdv$$

$$= E\int_0^\infty (\int_{X_1}^\infty (v+w)^{-r} dw) f(v)dv$$

$$= \frac{1}{(r-1)} E(X_1 + X_2)^{-r}$$

the result follows immediately. ■

Corollary 3.5: Let $r = 2$, then

$$2E(X_1 + X_2)^{-2} \le E(X_1 + X_2)^{-1}. \qquad (3.6.2)$$

Theorem 3.14: If $F(t)\int_0^t f(t-x)dx \ge f(t)\int_0^t F(t-x)dx$ for $0 \le x \le t$, then for all integer $r > 2$,

$$E(X_1^{-r+2} - (X_1 + X_2)^{-r+2}) \ge (r-2)(1-2^{-r+1})E(X_1^{-r+2} - X_1X_2^{-r+1}), \qquad (3.6.3)$$

Proof: Define $v(t) = \int_0^\infty F(t-x)dx$. Since F belongs to $IMIT$, we have the following inequality,

$$\int_0^\infty \int_0^t (2t-x)^{-r} F(t)F(t-x)dxdt \ge \int_0^\infty \int_0^t (2t-x)^{-r} f(t)v(t)dxdt$$

$$LHS = E\int_{X_1}^\infty (\int_0^t (2t-x)^{-r} F(t-x)dx)dt$$

$$= E\int_{X_1}^\infty (\int_{t-X_2}^t (2t-x)^{-r} dx)dt$$

$$= (r-1)^{-1}(r-2)^{-1}E(X_1^{-r+2} - (X_1 + X_2)^{-r+2})$$

and,

$$RHS = \int_0^\infty f(t)v(t)(\int_0^t (2t-x)^{-r} dx)dt$$

$$= (r-1)^{-1}(1-2^{-r+1})\int_0^\infty t^{-r+1} f(t)(t - E(X_1))dt$$

$$= (r-1)^{-1}(1-2^{-r+1})(\int_0^\infty t^{-r+2} f(t)dt - E(X_1)\int_0^\infty t^{-r+1} f(t)dt)$$

$$= (r-1)^{-1}(1-2^{-r+1})\{E(X_2^{-r+2}) - E(X_1X_2^{-r+1})\}$$

which provides the results. ■

Corollary 3.6: Let $r = 3$, then

$$E(X_1^{-1}) \ge 4E(X_1 + X_2)^{-1} - 3E(X_1X_2^{-2}) \qquad (3.6.4)$$

Theorem 3.15: If $f(\zeta - x)[F(\zeta) - F(t)] \geq f(\zeta)F(\zeta - x)$, then for all integer $r > 2$,

$$(2^{-r+1} - 3^{-r+1})E(\tilde{X}_3)^{-r+2} + 3^{-r+1}E(X_1^{-r+2} - \tilde{X}_2^{-r+2}) - E(X_3 + X_2 + \tilde{X}_2)^{-r+1} \geq \frac{1}{2}\{E(2\tilde{X}_2 + X_2)^{-r+2}$$
$$- E(2X_1 + X_3)^{-r+2} - E(2X_1 + X_2)^{-r+2}\} \tag{3.6.5}$$

where $\tilde{X}_n = \max(X_1, X_2, \ldots, X_n)$.

Proof: Since F belongs to the class of $DRHRGML$, therefore

$$\int_0^\infty \int_0^\zeta \int_0^\zeta (t + 2\zeta - x)^{-r} f(\zeta - x)F^2(\zeta)dxdtd\zeta - \int_0^\infty \int_0^\zeta \int_0^\zeta (t + 2\zeta - x)^{-r} f(\zeta - x)F(t)F(\zeta)dxdtd\zeta$$
$$\geq \int_0^\infty \int_0^\zeta \int_0^\zeta (t + 2\zeta - x)^{-r} f(\zeta)F(\zeta - x)F(\zeta)dxdtd\zeta$$

say, $I - II \geq III$.

$$I = \int_0^\infty \int_0^\zeta \int_0^\zeta (t + 2\zeta - x)^{-r} f(\zeta - x)F^2(\zeta)dxdtd\zeta$$

$$= \int_0^\infty F^3(\zeta)\{\int_0^\infty (t + 2\zeta)^{-r} dt\}d\zeta + \int_0^\infty F^2(\zeta)\{\int_0^\zeta [(t + \zeta + X_2)^{-r} - (t + 2\zeta)^{-r}$$

$$= (r-1)^{-1}(r-2)^{-1}(3^{-r+1} - 2^{-r+1})E(\tilde{X}_3)^{-r+2} + (-r+1)E\int_{\tilde{X}_2}^\infty [(2\zeta + X_3)^{-r+1} - (\zeta + X_3)^{-r+1} - (3\zeta)^{-r+1}$$
$$+ (2\zeta)^{-r+1}]d\zeta$$

$$= (r-1)^{-1}(r-2)^{-1}\{(3^{-r+1} - 2^{-r+1})E(\tilde{X}_3)^{-r+2} - \frac{1}{2}E(2\tilde{X}_2 + X_3)^{-r+2} + E(\tilde{X}_2 + X_3)^{-r+2}$$
$$+ (3^{-r+1} - 2^{-r+1})E(\tilde{X}_2)^{-r+2}\},$$

$$II = \int_0^\infty \int_0^\zeta \int_0^\zeta (t+2\zeta - x)^{-r} f(\zeta - x) F(t) F(\zeta) dx dt d\zeta$$

$$= E\int_0^\infty \int_0^\zeta F(t)F(\zeta)\{(t+2\zeta)^{-r} F(\zeta) + (t+\zeta + X_3)^{-r} - (t+2\zeta)^{-r}\} dt d\zeta$$

$$= E\int_0^\infty F(\zeta)\{\int_{X_2}^\zeta [(t+2\zeta)^{-r} F(\zeta) + (t+\zeta + X_3)^{-r} - (t+2\zeta)^{-r}] dt\} d\zeta$$

$$= (-r+1)^{-1} E\int_0^\infty F(\zeta)[(3\zeta)^{-r+1}\overline{F}(\zeta) + (2\zeta + X_3)^{-r+1} - (X_2 + 2\zeta)^{-r+1} F(\zeta)$$

$$- (X_2 + X_3 + \zeta)^{-r+1} + (X_2 + 2\zeta)^{-r+1}] d\zeta$$

$$= (r-2)^{-1}(-r+1)^{-1} E\{3^{-r+1}(X_1^{-r+2} - \tilde{X}_2^{-r+2}) - \frac{1}{2}(X_2 + 2\tilde{X}_2)^{-r+2} + \frac{1}{2}(2X_1 + X_2)^{-r+2}$$

$$+ \frac{1}{2}(2X_1 + X_3)^{-r+2} - (X_1 + X_2 + X_3)^{-r+2}\}$$

also,

$$III = \int_0^\infty \int_0^\zeta \int_0^\zeta (t+2\zeta - x)^{-r} f(\zeta) F(\zeta - x) dx dt d\zeta$$

$$= \int_0^\infty \int_0^\zeta f(\zeta)F(\zeta)\{\int_0^\zeta (t+2\zeta - x)^{-r} F(\zeta - x) dx\} dt d\zeta$$

$$= \int_0^\infty \int_0^\zeta f(\zeta)F(\zeta)\{E\int_0^{\zeta - X_3} (t+2\zeta - x)^{-r} dx\} dt d\zeta$$

$$= (r-1)^{-1} E\int_0^\infty [\int_0^\zeta \{(t+\zeta + X_3)^{-r+1} - (t+2\zeta)^{-r+1}\} dt]$$

$$= (r-1)^{-1}(-r+2)^{-1} E\int_0^\infty [(2\zeta + X_3)^{-r+2} - (3\zeta)^{-r+2} - (\zeta + X_3)^{-r+2} + (2\zeta)^{-r+2}] F(\zeta) dF(\zeta)$$

$$= \frac{1}{2}(r-1)^{-1}(-r+2)^{-1} E[(2\tilde{X}_2 + X_3)^{-r+2} - (3^{-r+2} - 2^{-r+2})\tilde{X}_2^{-r+2} - (\tilde{X}_2 + X_3)^{-r+2}]$$

This provides the desired result. ■

Corollary 3.7: Let $r = 3$, then

$$2^{-2}E(\tilde{X}_3)^{-1} + 3^{-2}E(X_1^{-1} - \tilde{X}_3^{-1} - \tilde{X}_2^{-1}) - E(X_3 + X_2 + \tilde{X}_2)^{-1} \geq \frac{1}{2}\{E(2\tilde{X}_2 + X_2)^{-1} - E(2X_1 + X_3)^{-1}$$

$$-E(2X_1 + X_2)^{-1}\}, \quad (3.6.6)$$

CHAPTER FOUR: MULTIVARIATE EXTENSIONS OF NEW NOTIONS IN LIFE DISTRIBUTINS

4.0 Chapter Plan

The concept of multivariate inactivity time guaranteed minimum life length and its related aging classes will shape the first section of this chapter. In the second section we will study the characterizations and probabilistic properties of these new multivariate classes of distributions. Some preservation results under typical reliability operations such as mixture and convolution will be taken under consideration. The last section introduces new inequalities in multivariate cases based on the extension of the univariate notions.

4.1 Introduction

In the previous chapters we introduced some new univariate notions and classes of life distributions. Also, we studied the probabilistic and statistical properties of these classes. In this chapter by implementing different approaches, we are interested in obtaining multivariate extensions of these classes and notions. The most important feature of univariate investigations is that the components comprising the system are assumed to function independently. Hence, the univariate life distribution classes are sufficient to study and to describe a multicomponent system. However, due to complexity in the system, the assumption of independence sometimes becomes untenable. Thus, systems comprising components that have positive dependence due to linkage to common power sources and subject to common environmental stresses are to be considered. This motivates us to consider the multivariate aging classes.

The multivariate reversed hazard rate, multivariate mean inactivity time, multivariate reversed hazard rate guaranteed minimum life length and the multivariate mean inactivity time guaranteed minimum life length are generalizations of the univariate concepts with regard to the motivations above.

4.2 Notations and Definitions

Let F be the joint distribution function of the random vector $\mathbf{X} = (X_1, \ldots, X_p)'$ defined on R_p^+ (the p-dimensional nonnegative half space of R_p), $p \geq 1$. The random vector

$\mathbf{X}_{(\mathbf{t},\boldsymbol{\zeta})} = [\boldsymbol{\zeta} - \mathbf{X} \mid \mathbf{t} \leq \mathbf{X} \leq \boldsymbol{\zeta}]$ denotes the time elapsed after failure until time $\boldsymbol{\zeta}$, given that the unit has a guaranteed minimum length life $\mathbf{t}$. This random vector variable is called the multivariate inactivity time guaranteed minimum life length, ($MITGML$).

Definition 4.1: The multivariate reversed hazard rate function ($MRHR$) is defined as

$$\boldsymbol{\tau}(\mathbf{t}) = (\tau_1(\mathbf{t}), \ldots \tau_p(\mathbf{t}))', \qquad (4.2.1)$$

where

$$\tau_i(\mathbf{t}) = \frac{\partial}{\partial t_i} \log F(\mathbf{t}) = \frac{\frac{\partial}{\partial t_i} F(\mathbf{t})}{F(\mathbf{t})}, \quad \mathbf{t} \in R_p^+$$

with $F(\mathbf{t}) = P(\mathbf{X} \leq \mathbf{t}) > 0;\ \ i = 1(1)p$.

Definition 4.2: The multivariate mean inactivity time ($MMIT$) is defined as

$$\boldsymbol{\mu}(\mathbf{t}) = (\mu_1(\mathbf{t}), \ldots \mu_p(\mathbf{t}))' \qquad (4.2.2)$$

where

$$\mu_i(\mathbf{t}) = E(t_i - X_i \mid \mathbf{X} \le \mathbf{t})$$
$$= \frac{1}{F(\mathbf{t})}\int_0^{t_i} F(\mathbf{t}_{(u_i)})du_i$$

and

$$\mathbf{t}_{(u_i)} = (t_1, \ldots, t_{i-1}, u_i, t_{i+1}, \ldots, t_p)', \quad i = 1(1)p.$$

Definition 4.3: The multivariate reversed hazard rate guaranteed minimum life length ($MRHRGML$) is defined as

$$\boldsymbol{\tau}_*(\mathbf{x}) = (\tau_{*_1}(\mathbf{x}), \ldots \tau_{*_p}(\mathbf{x}))', \tag{4.2.3}$$

where

$$\tau_{*_i}(\mathbf{x}) = \frac{\partial}{\partial x_i}\log[F^*(\mathbf{x}) - F^*(\mathbf{0})] = \frac{\frac{\partial}{\partial x_i}F^*(\mathbf{x})}{F^*(\mathbf{x}) - F^*(\mathbf{0})}, \quad \mathbf{x} \in R_p^+, \ i = 1(1)p$$

with $F^*(\mathbf{x}) = P(\mathbf{X}^* \le \mathbf{x}) = F_{\mathbf{X}}(\mathbf{t} + \mathbf{x}) > 0;\ \mathbf{X}^* = \mathbf{X} - \mathbf{t},\ \mathbf{t} \le \mathbf{x} \le \boldsymbol{\zeta}$.

Definition 4.4: The multivariate mean inactivity time guaranteed minimum life length ($MMITGML$) is defined as

$$\boldsymbol{\mu}_*(\mathbf{x}) = (\mu_{*_1}(\mathbf{x}), \ldots \mu_{*_p}(\mathbf{x}))' \tag{4.2.4}$$

where

$$\mu_{*_i}(\mathbf{x}) = E(x_i - X_i^* \mid \mathbf{0} \le \mathbf{X}^* \le \mathbf{x})$$
$$= \frac{1}{F^*(\mathbf{x}) - F^*(\mathbf{0})}\int_0^{x_i} [F^*(\mathbf{x}_{(x_i - u_i)}) - F^*(\mathbf{0})]du_i, \ i = 1(1)p.$$

Definition 4.5: Multivariate reversed hazard rate function corresponding to the equilibrium distribution function of F is defined as

$$G(\mathbf{x}) = \frac{\int_0^{x_i} F(\mathbf{x}_{(u_i)}) du_i}{\int_0^{\infty} F(\mathbf{x}_{(u_i)}) du_i}. \qquad (4.2.5)$$

Definition 4.6: Multivariate failure function corresponding to the equilibrium distribution function of F^* is defined as

$$G^*(\mathbf{x}) = \frac{\int_0^{x_i} [F^*(\mathbf{x}_{(u_i)}) - F^*(\mathbf{0})] du_i}{\int_0^{\infty} F^*(\mathbf{x}_{(u_i)}) du_i}. \qquad (4.2.6)$$

4.3 New Multivariate Classes of Distributions and their Characterizations

In the following section, we are going to introduce the multivariate decreasing reversed hazard rate class, the multivariate increasing mean inactivity time class, the multivariate decreasing reversed hazard rate guaranteed minimum length life ($MDRHRGML$) class, and multivariate increasing mean inactivity time guaranteed minimum length life ($MIMITGML$) class, respectively. We also will study some characteristics properties of these classes as well.

Definition 4.5: A distribution function F is said to be

a) Multivariate decreasing reversed hazard rate ($MDRHR$) if

$\tau_i(\mathbf{t})$ is a decreasing function in t_i, $\mathbf{t} \in R_p^+$, $i = 1, \ldots, p$

b) Multivariate decreasing reversed hazard rate guaranteed minimum length life ($MDRHRGML$) if

$\tau_{*_i}(\mathbf{x})$ is a decreasing function in x_i, $\mathbf{x} \in R_p^+$, $i = 1, \ldots, p$

c) Multivariate increasing mean inactivity time ($MIMIT$) if

$\mu_i(\mathbf{t})$ is an increasing function in t_i, $\mathbf{t} \in R_p^+$, $i = 1, \ldots, p$

d) Multivariate increasing mean in activity time guaranteed minimum length life ($MIMITGML$) if

$\mu_{*_i}(\mathbf{x})$ is an increasing function in x_i, $\mathbf{x} \in R_p^+$, $i = 1, \ldots, p$

Remark 4.1: F belongs to $MDRHR$ if and only if $\dfrac{F(\mathbf{x}_{(x_i - h)})}{F(\mathbf{x})}$ is increasing in x_i, $i = 1, \ldots, p$

Remark 4.2: F belongs to $MDRHRGML$ if and only if $\dfrac{F^*(\mathbf{x}_{(x_i - h)}) - F^*(\mathbf{0})}{F^*(\mathbf{x}) - F^*(\mathbf{0})}$ is increasing in x_i, $i = 1, \ldots, p$

Theorem 4.1: If $\mathbf{X}$ has the $MDRHR$ property then $\mathbf{X}$ has the $MIMIT$ property.

Proof: Observing that

$$\mu_i(\mathbf{t}) = E(t_i - X_i \mid \mathbf{X} \le \mathbf{t}) = \frac{1}{F(\mathbf{t})} \int_0^{t_i} F(\mathbf{t}_{(u_i)}) du_i$$

and implementing the Remark 4.1, the desired result follows by direct integration. ■

Theorem 4.2: If $\mathbf{X}$ has the $MDRHRGML$ property then $\mathbf{X}$ has the $MIMITGML$ property.

Proof: Using the same arguments provide the desired result. ■

Theorem 4.3: $G(\mathbf{x})$ is $MDRHR$ iff $F(\mathbf{x})$ is $MIMIT$.

Proof: Observe that

$\tau_{iG}(\mathbf{x}) =$ the i^{th} component of reversed hazard rate gradient of G

$$= \frac{\frac{\partial}{\partial x_i} G(\mathbf{x})}{G(\mathbf{x})}$$

$$= \frac{\dfrac{1}{\int_0^\infty F(\mathbf{x}_{(u_i)})du_i} F(\mathbf{x}_{(x_i)})}{\dfrac{1}{\int_0^\infty F(\mathbf{x}_{(u_i)})du_i} \int_0^{x_i} F(\mathbf{x}_{(u_i)})du_i}$$

$$= \frac{1}{\int_0^{x_i} F(\mathbf{x}_{(u_i)})du_i} F(\mathbf{x})$$

$$= \mu_{iF}^{-1}(\mathbf{x})$$

$= 1/i^{th}$ Component of the multivariate mean inactivity time function of F .

Hence,

$\tau_{iG}(\mathbf{x})$ is a decreasing function in $x_i \Leftrightarrow \mu_{iF}(\mathbf{x})$ is an increasing function in x_i , $i = 1, \ldots, p$

Theorem 4.4: $G^*(\mathbf{x})$ is $MDRHRGML$ iff $F^*(\mathbf{x})$ is $MIMITGML$.

Proof: The proof holds by similar arguments in the proof of Theorem 4.3.

4.4 Closure under Mixture

Let us define the survival function corresponding to the multivariate mixture distribution as $F(\mathbf{x}) = \int F_{\boldsymbol{\alpha}}(\mathbf{x})dP(\boldsymbol{\alpha})$, where $P(\boldsymbol{\alpha})$ is the multivariate probability measure of the mixing distribution and the integral is a multiple integral over the domain of $\boldsymbol{\alpha}$.

Theorem 4.5: If $F_{\boldsymbol{\alpha}}(\mathbf{x})$ is $MDRHR$ then so is $F(\mathbf{x})$, provided that the first and the second derivatives of $F_{\boldsymbol{\alpha}}(\mathbf{x})$ respect to the x_i are bounded, for $i = 1(1)p$ and $\forall \boldsymbol{\alpha}$.

Proof: By definition

$\tau_{iF}(\mathbf{x}) =$ the i^{th} component of the reversed hazard rate gradient of F

by having the Dominated Convergence Theorem, in view of the assumption,

$$= \frac{\frac{\partial}{\partial x_i} F(\mathbf{x})}{F(\mathbf{x})} = \frac{\int \frac{\partial}{\partial x_i} F_{\boldsymbol{\alpha}}(\mathbf{x}) dP(\boldsymbol{\alpha})}{\int F_{\boldsymbol{\alpha}}(\mathbf{x}) dP(\boldsymbol{\alpha})}.$$

We are to show that $\tau_{iF}(\mathbf{x})$ is decreasing in x_i, $\mathbf{x} \in R_p^+, i = 1(1)p$. Thus, we need to show

$$\frac{\partial}{\partial x_i} \tau_{iF}(\mathbf{x}) = \frac{1}{[\int F_{\boldsymbol{\alpha}}(\mathbf{x}) dP(\boldsymbol{\alpha})]^2} \left[\int F_{\boldsymbol{\alpha}}(\mathbf{x}) dP(\boldsymbol{\alpha}) \times \int \frac{\partial^2}{\partial x_i^2} F_{\boldsymbol{\alpha}}(\mathbf{x}) dP(\boldsymbol{\alpha}) \right.$$

$$\left. - \left(\int \frac{\partial}{\partial x_i} F_{\boldsymbol{\alpha}}(\mathbf{x}) dP(\boldsymbol{\alpha}) \right)^2 \right] \leq 0, \; \mathbf{x} \in R_p^+.$$

by the Dominated Convergence Theorem, in view of the assumption.

To do so, the following inequality must be held

$$\int F_{\boldsymbol{\alpha}}(\mathbf{x}) dP(\boldsymbol{\alpha}) \times \int \frac{\partial^2}{\partial x_i^2} F_{\boldsymbol{\alpha}}(\mathbf{x}) dP(\boldsymbol{\alpha}) \leq \left(\int \frac{\partial}{\partial x_i} F_{\boldsymbol{\alpha}}(\mathbf{x}) dP(\boldsymbol{\alpha}) \right)^2.$$

Since $F_{\boldsymbol{\alpha}}(\mathbf{x})$ is $MDRHR$, thus $\tau_{iF_\alpha}(\mathbf{x})$ is decreasing in x_i, $\mathbf{x} \in R_p^+, i = 1(1)p$. This provides us the following inequality

$$F_{\boldsymbol{\alpha}}(\mathbf{x}) \times \frac{\partial^2}{\partial x_i^2} F_{\boldsymbol{\alpha}}(\mathbf{x}) \leq \left(\frac{\partial}{\partial x_i} F_{\boldsymbol{\alpha}}(\mathbf{x}) \right)^2,$$

Simple algebraic rule provides the desired result. ■

Theorem 4.6: If $F_{\boldsymbol{\alpha}}(\mathbf{x})$ is $MDRHRGML$ then so is $F(\mathbf{x})$, provided that the first and the second derivatives of $F_{\boldsymbol{\alpha}}(\mathbf{x})$ respect to the x_i are bounded, for $i = 1(1)p$ and $\forall \boldsymbol{\alpha}$.

Proof: It follows by implementing the same arguments as the proof of Theorem 4.5.

Theorem 4.7: Under the assumption of Theorem 4.4, if $F_{\boldsymbol{\alpha}}(\mathbf{x})$ is *MIMIT* then so is $F(\mathbf{x})$.

Proof: we define the failure function corresponding to the equilibrium distribution of $F_{\boldsymbol{\alpha}}(\mathbf{x})$ as

$$G_{\boldsymbol{\alpha}}(\mathbf{x}) = \frac{\int_0^{x_i} F_{\boldsymbol{\alpha}}(\mathbf{x}_{(u_i)})du_i}{\int_0^{\infty} F_{\boldsymbol{\alpha}}(\mathbf{x}_{(u_i)})du_i}.$$

Now, the failure function corresponding to the equilibrium distribution $F(\mathbf{x})$ is

$$G(\mathbf{x}) = \frac{\int_0^{x_i} F(\mathbf{x}_{(u_i)})du_i}{\int_0^{\infty} F(\mathbf{x}_{(u_i)})du_i}$$

$$= \frac{\int_0^{x_i} \int F_{\boldsymbol{\alpha}}(\mathbf{x}_{(u_i)})dP(\boldsymbol{\alpha})du_i}{\int_0^{\infty} F(\mathbf{x}_{(u_i)})du_i}$$

$$= \frac{\int\int_0^{x_i} F_{\boldsymbol{\alpha}}(\mathbf{x}_{(u_i)})du_i dP(\boldsymbol{\alpha})}{\int_0^{\infty} F(\mathbf{x}_{(u_i)})du_i}, \text{ by Fubini's theorem}$$

$$= \frac{\int G_{\boldsymbol{\alpha}}(\mathbf{x})\int_0^{\infty} F_{\boldsymbol{\alpha}}(\mathbf{x}_{(u_i)})du_i dP(\boldsymbol{\alpha})}{\int_0^{\infty} F(\mathbf{x}_{(u_i)})du_i}$$

$$= \int G_{\boldsymbol{\alpha}}(\mathbf{x})\frac{\int_0^{\infty} F_{\boldsymbol{\alpha}}(\mathbf{x}_{(u_i)})du_i dP(\boldsymbol{\alpha})}{\int_0^{\infty} F(\mathbf{x}_{(u_i)})du_i}$$

This is a mixture of $G_{\boldsymbol{\alpha}}(\mathbf{x})$ with the mixing distribution

$$dW(\boldsymbol{\alpha}) = \frac{\int_0^{\infty} F_{\boldsymbol{\alpha}}(\mathbf{x}_{(u_i)})du_i}{\int_0^{\infty} F(\mathbf{x}_{(u_i)})du_i} dP(\boldsymbol{\alpha}) \geq 0$$

Now, by using Theorem 4.3, $F_{\boldsymbol{\alpha}}(\mathbf{x})$ is $MIMIT \Leftrightarrow G_{\boldsymbol{\alpha}}(\mathbf{x})$ is $MDRHR$. In view of the assumption, and by using Theorem 4.5, $G(\mathbf{x})$ is $MDRHR$, which means $F(\mathbf{x})$ is $MIMIT$. ■

Theorem 4.8: Under the assumption of Theorem 4.4, if $F_{\boldsymbol{\alpha}}(\mathbf{x})$ is $MIMITGML$ then so is $F(\mathbf{x})$.

Proof: It follows by providing the same fashion in the proof of Theorem 4.7. ■

4.5 New Inequalities in Multivariate Case

4.5.1 Extension the Univariate Concepts to Multivariate Cases

Recalling that the random variable X has decreasing reversed hazard rate distribution if F is log-concave or, equivalently if $-\log F(x)$ is a convex function, and distribution F is increasing mean inactivity time if $-\log\int_0^t F(u)du$ is a convex function. We consider the following extensions of these concepts of univariate $DRHR$ aging to the multivariate (bivariate) case.

Definition 4.6: Distribution function F belongs to the class of $MDRHR$ if

$F^2(2^{-1}(\mathbf{x}+\mathbf{y})) \geq F(\mathbf{x})F(\mathbf{y})$, where $\mathbf{x}=(x_1,x_2,\ldots,x_n)'$ and $\mathbf{y}=(y_1,y_2,\ldots,y_n)'$

Definition 4.7: Distribution function F belongs to the class of $MIMIT$ if

$G^2(2^{-1}(\mathbf{x}+\mathbf{y})) \geq G(\mathbf{x})G(\mathbf{y})$, where $G(x,y)=\int_0^x\int_0^y F(w_1,w_2)dw_1dw_2$

Remark 4.3: F belongs to $BIMIT$ if and only if

$$\int_0^{\frac{x_1+y_1}{2}}\int_0^{\frac{x_2+y_2}{2}} F^2(w_1,w_2)dw_1dw_2 \geq \left(\int_0^{x_1}\int_0^{x_2} F(w_1,w_2)dw_1dw_2\right)\left(\int_0^{y_1}\int_0^{y_2} F(u_1,u_2)du_1du_2\right)$$

Proof: Utilizing Schwarz's Inequality, Definition 4.7 can be rewritten in the form of

$$F \in BIMIT \Leftrightarrow \int_0^{\frac{x_1+y_1}{2}} \int_0^{\frac{x_2+y_2}{2}} F^2(w_1,w_2)dw_1dw_2 \geq \left(\int_0^{x_1}\int_0^{x_2} F(w_1,w_2)dw_1dw_2\right)\left(\int_0^{y_1}\int_0^{y_2} F(u_1,u_2)du_1du_2\right)$$

$$G^2(x,y) = \left(\int_0^x \int_0^y F(w_1,w_2)dw_1dw_2\right)^2 = \left(\int_0^x \int_0^y F(w_1,w_2)dw_1dw_2\right)\left(\int_0^x \int_0^y F(u_1,u_2)du_1du_2\right)$$

$$= \int_0^x \int_0^y \left[\int_0^x \int_0^y F(w_1,w_2)dw_1dw_2\right] F(u_1,u_2)du_1du_2$$

(by Schwarz's inequality) $\leq \int_0^x \int_0^y F^2(w_1,w_2)dw_1dw_2$ ■

Theorem 4.9: If $G^2(2^{-1}(\mathbf{x}+\mathbf{y})) \geq G(\mathbf{x})G(\mathbf{y})$, then for all integers $\mathbf{r} > \mathbf{3}$,

$$E(\prod_{i=1}^{p} 2^{-r_i+3}\tilde{X}_i^{-r_i+3}) \geq E(\prod_{i=1}^{p}(r_i-1)^{-1}(X_{1i}+X_{2i})^{-r_i+3})$$

where $\tilde{X}_i = \max(X_{1i}, X_{2i})$, $\mathbf{X_1} = (X_{11}, X_{12}, \ldots, X_{1p})$, and $\mathbf{X}_2 = (X_{21}, X_{22}, \ldots, X_{2p})$. $\mathbf{X}_1$ and $\mathbf{X}_2$ are two copies of random vector $\mathbf{X} = (X_1, X_2, \ldots, X_p)$.

Proof:

$$LHS = \int_0^\infty \dots \int_0^\infty \prod_{i=1}^{p} (x_i + y_i)^{-r_i} \Lambda(2^{-1}(\mathbf{x}+\mathbf{y})) d\mathbf{x} d\mathbf{y}$$

$$= \int_0^\infty \dots \int_0^\infty \prod_{i=1}^{p} u_i^{-r_i+1} \Lambda(2^{-1}(2^{-1}\mathbf{u})) d\mathbf{u}$$

$$= \int_0^\infty \dots \int_0^\infty \prod_{i=1}^{p} u_i^{-r_i+1} \int_{\mathbf{0}}^{\frac{u_1}{2}} \dots \int_{\mathbf{0}}^{\frac{u_p}{2}} F^2(\mathbf{w}) d\mathbf{w} d\mathbf{u}$$

$$= \int_0^\infty \dots \int_0^\infty F^2(\mathbf{w}) \int_{2w_1}^\infty \dots \int_{2w_p}^\infty \prod_{i=1}^{p} u_i^{-r_i+1} d\mathbf{u} d\mathbf{w}$$

$$= \prod_{i=1}^{p} (r_i - 2)^{-1} E\left(\int_{\tilde{X}_1}^\infty \dots \int_{\tilde{X}_p}^\infty \prod_{i=1}^{p} (2w_i)^{-r_i+2} d\mathbf{w} \right)$$

$$= E(\prod_{i=1}^{p} 2^{-r_i+3} (r_i - 2)^{-1} (r_i - 3)^{-1} \tilde{X}_i^{-r_i+3})$$

$$RHS = \int_0^\infty \dots \int_0^\infty \prod_{i=1}^{p} (x_i + y_i)^{-r_i} G(\mathbf{x}) G(\mathbf{y})) d\mathbf{x} d\mathbf{y}$$

$$= \left\{ \int_0^\infty \dots \int_0^\infty \prod_{i=1}^{p} (x_i + y_i)^{-r_i} (\int_{\mathbf{0}}^{x_1} \dots \int_{\mathbf{0}}^{x_2} F(\mathbf{w}) d\mathbf{w}) \right\} G(\mathbf{y}) d\mathbf{x} d\mathbf{y}$$

$$= \left\{ \int_0^\infty \dots \int_0^\infty F(\mathbf{w}) \int_{w_1}^\infty \dots \int_{w_p}^\infty \prod_{i=1}^{p} (x_i + y_i)^{-r_i} d\mathbf{x} d\mathbf{w} \right\} G(\mathbf{y}) d\mathbf{y}$$

$$= \left\{ \int_0^\infty \dots \int_0^\infty F(\mathbf{w}) \prod_{i=1}^{p} (r_i - 1)^{-1} (w_i + y_i)^{-r_i + 1} d\mathbf{w} \right\} G(\mathbf{y}) d\mathbf{y}$$

$$= \prod_{i=1}^{p} (r_i - 1)^{-1} (r_i - 2)^{-1} E \left\{ \int_0^\infty \dots \int_0^\infty \prod_{i=1}^{p} (X_{1i} + y_i)^{-r_i + 2} G(\mathbf{y}) d\mathbf{y} \right\}$$

$$= \prod_{i=1}^{p} (r_i - 1)^{-1} (r_i - 2)^{-1} E \left\{ \int_0^\infty \dots \int_0^\infty \prod_{i=1}^{p} (X_{1i} + y_i)^{-r_i + 2} \int_{\mathbf{0}}^{y_1} \dots \int_{\mathbf{0}}^{y_2} F(\mathbf{w}) d\mathbf{w} d\mathbf{y} \right\}$$

$$= \prod_{i=1}^{p} (r_i - 1)^{-1} (r_i - 2)^{-1} E \left\{ \int_0^\infty \dots \int_0^\infty F(\mathbf{w}) \int_{w_1}^\infty \dots \int_{w_p}^\infty \prod_{i=1}^{p} (X_{1i} + y_i)^{-r_i + 2} d\mathbf{y} d\mathbf{w} \right\}$$

$$= \prod_{i=1}^{p} (r_i - 1)^{-1} (r_i - 2)^{-1} E \int_0^\infty \dots \int_0^\infty F(\mathbf{w}) \prod_{i=1}^{p} (w_i + X_{1i})^{-r_i + 2} d\mathbf{w}$$

$$= \prod_{i=1}^{p} (r_i - 1)^{-1} (r_i - 2)^{-1} E \int_{X_{21}}^\infty \dots \int_{X_{2p}}^\infty \prod_{i=1}^{p} (w_i + X_{1i})^{-r_i + 2} d\mathbf{w}$$

$$= \prod_{i=1}^{p} (r_i - 1)^{-1} (r_i - 2)^{-1} (r_i - 3)^{-1} E (\prod_{i=1}^{p} (X_{1i} + X_{2i})^{-r_i + 3})$$

CHAPTER FIVE: CONCLUSION AND FUTURE WORK

5.0 Chapter Plan

In this concluding chapter we present a summary of the main ideas of this work, a brief discussion about new classes in life testing and a description of possible extensions and future directions for the aforementioned notions and classes.

5.1 Summary of Thesis

Introducing the new notions of guaranteed minimum length life enabled us to address a novel solution to the one of the most challenging survival data sets, interval-censored data. We implemented U- statistics theory, negative moment and Laplace transform inequalities to establish some new nonparametric test procedures against the alternatives. Extending these univariate notions and classes of aging to multivariate cases in the view of different approaches completed the first part of our study.

5.2 Future Work

The notion of inactivity time of a unit with a guaranteed minimum length life under shock models is a novel way of building some realistic univariate aging classes. Such models are more applicable in the study of reliability and life testing of a unit when its lifetime has become shortened or prolonged due to tempered events (shocks). Future research involves extending the aforementioned notions and classes in univariate and multivariate cases. In this study, we will consider a unit which is observed at pre-scheduled visits that is receiving shocks from one or more sources. Each of these shocks decreases or prolongs the life of unit. Based on this notion, we will define the new functions such as reversed hazard rate of a unit with a guaranteed minimum length life under shocks and mean inactivity time of a unit with a guaranteed minimum length life under shocks (RHRGML-sh, MITGML-sh). Also, we will study the probabilistic properties of these new univariate aging classes such DRHRGML-sh and IMITGML-sh. As a part of this research, we will study stochastic comparisons between probability distributions and some preservation results under typical reliability operations such as convolution and mixture. To estimate the magnitude of departure from the null hypothesis, we use the obtained inequalities from the previous step and utilize U-statistics theory. It will enable us to establish new nonparametric test procedures against alternatives. Another novel method of establishing nonparametric test procedures is using negative moment and Laplace transform methods. These tests should be simple to devise, calculate and study. It is of interest to see whether these new nonparametric test procedures have higher relative efficiency than the ones obtained from the previous methods. To this end, we plan to calculate the relative efficiency and power of the proposed nonparametric test procedures through simulation work.

The specific research aims of this future study will be, as follows:

A1. Develop the concept of inactivity time of a unit with a guaranteed minimum length life under shock models in different cases:

Case 1: Threshold value, say x, is fixed and shocks are governed by a homogeneous Poisson process with constant intensity λ.

Case 2: Threshold value x is a random variable with its own distribution function. Here, shocks are governed by a homogeneous Poisson process with constant intensity λ.

Case 3: Threshold value x is a random variable and units receive shocks from multiple sources.

A2. Develop new moment inequalities for inactivity time of a unit with a guaranteed minimum life length under shock models. These inequalities will be obtained by using the negative moment and Laplace transformation methods.

A3. Establish new nonparametric test procedures for testing against specific alternatives.

A4. Perform simulation studies to investigate the efficiency and power of the aforementioned tests.

5.3 Notions under Shock Models: Preliminary Results

5.3.1 Hazard Rate Function of a Unit Subject to Shock Models

Let us consider a random variable Y with distribution function F as life variable of a unit which is subjected to k-shocks governed by a Poisson process with a constant intensity λ. Also, let the random variable X_i denote the amount of damage caused by i^{th} shock which has common distribution G. Therefore, the total damage is $\sum_{i=1}^{k} X_i$ with a convolution distribution $G^{(k)}$. This unit will fail if the total damage at $[0,t]$ exceeds of the fixed threshold value x.

The conditional probability that this unit will fail at or before $t+\delta_t$, given that this unit has age t is defined as:

$$P(Y-\sum_{i=1}^{k} X_i \le t+\delta_t \mid Y-\sum_{i=1}^{k} X_i \ge t)$$
$$=\frac{\int_0^x [F(t+\delta_t+w)-F(t+w)]dG^k(w)}{\int_0^x \overline{F}(t+w)dG^k(w)}$$

Now we can define the hazard rate function of unit under shock models, HR_{Sh}, as:

$$\lim_{\delta_t \to 0}\frac{1}{\delta_t}\times\frac{\int_0^x [F(t+\delta_t+w)-F(t+w)]dG^k(w)}{\int_0^x \overline{F}(t+w)dG^k(w)}$$
$$=\frac{\int_0^x f(t+w)]dG^k(w)}{\int_0^x \overline{F}(t+w)dG^k(w)}$$

5.3.2 Reversed Hazard Rate Function of a Unit Subject to Shock Models

Considering the same condition as before, simply define the reversed hazard rate function of a unit under the shock models, RHR_{Sh}, as follow;

$$\frac{\int_0^x f(t+w)]dG^k(w)}{\int_0^x F(t+w)dG^k(w)}$$

5.3.3 Mean Remaining Life of a Unit Subject to Shock Models

Consider the random variable $Y_{t,k_t,x}$ as the remaining life time of a unite at age t, which is subjected to the k shocks, k_t, governed by a Poisson process. we define $Y_{t,k_t,x}$ as

$$Y_{t,k_t,x} = \left\{ Y - \sum_{i=1}^{k_t} X_i - t \mid Y - \sum_{i=1}^{k_t} X_i \geq t, \sum_{i=1}^{k_t} X_i \leq x \right\}$$

Assuming that the variables Y, $W = \sum_{i=1}^{k} X_i$ are independent.

Definition 5.1: The mean residual life of a unit under shock models, MRL_{Sh} is defined as

$$\begin{aligned} \gamma^x(t,k_t) &= E(Y_{t,k_t,x}) \\ &= E\left\{ Y - \sum_{i=1}^{k_t} X_i - t \mid Y - \sum_{i=1}^{k_t} X_i \geq t, \sum_{i=1}^{k_t} X_i \leq x \right\} \\ &= \frac{\int_t^{\infty} \overline{v}(z)dz}{\overline{v}(t)} \end{aligned}$$

where $\overline{v}(z) = \int_0^x \overline{F}(t+w)dG^{k_t}(w)$.

5.3.4 New Univariate Classes of Life Distributions from Shock Models

Definition 5.2: A distribution function F is said to be a decreasing (increasing) mean residual life of a unit under shock models type one, $DMRL_{Sh-1}$ (or $IMRL_{Sh-1}$) if

$$\gamma^x(t_2,k_{t_2}) \leq (\geq) \gamma^x(t_1,k_{t_1}) \text{ for all } t_1 \leq t_2,\ k_{t_1} = k_{t_2}$$

The interpretation of above definition is similar to the mean remaining life of a component subjected to shocks decreases as it ages.

Let us denote the class of $DMRL_{Sh-1}$ ($IMRL_{Sh-1}$) distributions by D_{sh-1} ($\bar{D}_{sh-1}$).

Definition 5.3: A distribution function F is said to be a decreasing mean residual life of a unit under shock models type two, $DMRL_{Sh-2}$ (or $IMRL_{Sh-2}$) if

$$\gamma^x(t_2, k_{t_2}) \le (\ge) \gamma^x(t_1, k_{t_1}) \text{ for all } (t_1, k_{t_1}) \le (t_2, k_{t_2})$$

i.e, given a unit at age t subjected to k_t shocks, the mean remaining life can be reduced by getting old and by more shocks.

Here, D_{sh-2} ($\bar{D}_{sh-2}$) denotes the class of $DMRL_{Sh-2}$ ($IMRL_{Sh-2}$) distribution.

Remark 5.1: It is easy to see $D_{sh-2} \subseteq D_{sh-1}$.

Remark 5.2: The following are alternative versions of the aforementioned classes

1. A distribution function F is $DMRL_{Sh-1}$ *iff* $\gamma^x(t, k_t)$ is decreasing function in t.
2. A distribution function F is $DMRL_{Sh-2}$ *iff* $\gamma^x(t, k_t)$ is a decreasing function in (t, k_t).

Now, let us consider the class of $p.d.f$'s which are common to D_{sh-1} and $\bar{D}_{sh-1}$, i.e.,

$$\varepsilon_l = D_{sh-l} \cap \bar{D}_{sh-l} \qquad \text{for } l = 1, 2$$

Theorem 5.1: A continuously differentiable distribution function F is $DMRL_{Sh-1}$ and $IMRL_{Sh-1}$ *iff* F is exponentially distributed.

Proof: By definition of class of ε_1, we have

$$\begin{aligned} E\left\{Y - \sum_{i=1}^{k_t} X_i - t \mid Y - \sum_{i=1}^{k_t} X_i \ge t, \sum_{i=1}^{k_t} X_i \le x\right\} &= E\left\{Y - \sum_{i=1}^{k_0} X_i \mid Y - \sum_{i=1}^{k_0} X_i \ge 0, \sum_{i=1}^{k_0} X_i \le x\right\} \\ &= E(Y) = \mu_y \quad \text{(Constant)} \end{aligned}$$

$$\Rightarrow \gamma^x(t,k_t) = \gamma^x(0,k_0)$$

It follows that Y is exponentially distributed.

Theorem 5.2: $\gamma^x(t,k_t)$ is TP_2.

Proof: we need to show

$$\begin{vmatrix} \gamma^x(t_1,k_{t_1}) & \gamma^x(t_2,k_{t_1}) \\ \gamma^x(t_1,k_{t_2}) & \gamma^x(t_2,k_{t_2}) \end{vmatrix} \geq 0 \quad \text{or equivalently we shall show}$$

$$\frac{\gamma^x(t_1,k_{t_1})}{\gamma^x(t_2,k_{t_1})} \geq \frac{\gamma^x(t_1,k_{t_2})}{\gamma^x(t_2,k_{t_2})}.$$

For the numerators we have

$$\begin{aligned}
&\overline{F}(z+w)dG^{k_{t1}}(w) \geq \overline{F}(z+w)dG^{k_{t2}}(w) \\
&or \\
&\int_{t_1}^{\infty}\int_0^x \overline{F}(z+w)dG^{k_{t1}}(w)dz \geq \int_{t_1}^{\infty}\int_0^x \overline{F}(z+w)dG^{k_{t2}}(w)dz
\end{aligned}$$

The same inequality is held for the denominators. This finishes the proof.

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