Probing Random Media With Singular Waves

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PROBING RANDOM MEDIA WITH SINGULAR WAVES

by

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ABSTRACT

In recent years a resurgence of interest in wave singularities (of which optical vortices are a prominent example), light angular momentum and the relations between them has occurred. Many applications in various areas of linear and non-linear optics have been based on studying effects related to angular momentum and optical vortices. This dissertation examines the use of such wave singularities for studying the light propagation in highly inhomogeneous media and the relationship to angular momentum transfer.

Angular momentum carried by light can be, in many cases, divided in two terms. The first one relates to the polarization of light and can be associated, in the quantum description, to the spin of a photon. The second is determined by the electromagnetic field distribution and, in analogy to atomic physics, is associated with the orbital angular momentum (OAM) of a photon. Under the paraxial approximation appropriate for the case of beam propagation, the two terms do not couple. However, each of them can be modified by the interaction with different media in which the light propagates through processes which involve angular momentum exchange.

The decoupling of spin and orbital parts of light angular momentum can not, in general, be assumed for non paraxial propagation in turbid media, especially when backscattering is concerned. In Chapter 3 of this dissertation, scattering effects on angular momentum of
light are discussed both for the single and multiple scattering processes. It is demonstrated for the first time that scattering from a spherically symmetric scattering potential, couples the spin and the OAM such that the total angular momentum flux density in conserved in every direction.

Remarkably, the conservation of angular momentum occurs also for some classes of multiple scattering trajectories and this phenomenon manifests itself in ubiquitous polarization patterns observed in back-scattering from turbid media. It is newly shown in this dissertation that the polarization patterns a result of OAM carrying optical vortices which have a geometrical origin. These geometrical phase vortices are analyzed using the helicity space approach for optical geometrical phase (Berry phase). This approach, introduced in the context of random media, elucidates several aspects specific to propagation in helicity preserving and non-preserving scattering trajectories.

Another aspect of singular waves interaction with turbid media relates to singularities embedded in the incident waves. Chapter 4 of the dissertation discusses how the phase distribution associated with an optical vortex leads to changes in the spatial correlations of the electromagnetic field. This change can be used to control the properties of the effect of enhanced backscattering in a way which allows inferring the optical properties of the medium. A detailed theoretical and experimental study of this effect is presented here for the first time for both double-pass geometries and diffusive media. It is also demonstrated that this novel experimental technique can be used to determine the optical properties of turbid media and, moreover, it permits to sense the depth of reflective inclusions in opaque media.
When considering a regime of weakly inhomogeneous media, the paraxial approximation is still valid and therefore the spin and OAM do not couple. If, in addition, the medium is optically isotropic then the polarization is not affected. However, when the medium is non-axially symmetric for any specific realization, the OAM does change as a result of interaction with the medium. This effect can be studied using a newly developed method of coherent modes coupling which is presented in Chapter 5. This approach allows studying the power spread across propagating modes which carry different orbital angular momentum. The powerful concept of coherent modes coupling can be applied to fully coherent, fully polarized sources as well to partially coherent, partially polarized ones. An example of this scattering regime is atmospheric turbulence and the propagation through turbulence is thoroughly examined in Chapter 5.

The results included in this dissertation are of fundamental relevance for a variety of applications which involves probing different types of random media. Such applications include remote sensing in atmospheric and maritime environments, optical techniques for biomedical diagnostics, optical characterization procedures in material sciences and others.
To my wife, Orit, and to my children, Tomer, Shakked and Lilach
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LIST OF SYMBOLS

\( E \)  Electric Field vector
\( B \)  Magnetic induction vector
\( A \)  Amplitude of the electric field
\( n \)  Index of refraction
\( k \)  Wave number
\( \lambda \)  Wavelength
\( r \)  Position vector
\( r, \theta, \phi \)  Spherical coordinates
\( \vec{S}(\theta, \phi) \)  Amplitude scattering matrix
\( \Omega \)  Solid angle, Geometric phase
\( P \)  Power
\( \sigma_s, \sigma_a \)  Scattering and absorption cross sections
\( f(\theta, \phi) \)  Scattering phase function
\( x \)  Size parameter
\( g \)  Anisotropy parameter
\( I, Q, U, V \) The Stokes parameters
\( \mu_s, \mu_a \) Scattering and absorption coefficients
\( l_s \) Scattering mean free path
\( l_t \) Transport mean free path
\( D \) Diffusion coefficient
\( q \) Momentum transfer vector
\( P \) Degree of Polarization
\( L \) Radiance or Specific intensity
\( D_n \) Refractive index structure function
\( C_n^2 \) Refractive index structure constant
\( S \) Spin volume density
\( L \) Orbital angular momentum volume density
\( J \) Total angular momentum volume density
\( s \) Spin flux density
\( l \) Orbital angular momentum flux density
\( j \) Total angular momentum flux density
CHAPTER 1

INTRODUCTION

Scattering is a process in which the interaction of electromagnetic radiation with material inhomogeneities leads to changes in the propagation characteristics of the radiation [1]-[3]. These changes may be classified as attenuation of the original intensity and as the redistribution of some of the intensity in different directions. The scattered radiation can not be fully characterized in terms of intensity alone and the coherence, spectral and polarization characteristics of the scattering process all come into play and affect the coherence, spectrum and state of polarization of the scattered radiation. The scattering process is also governed by requirements for the conservation of the fundamental physical quantities - energy, linear and angular momentum. In classical low energy physics, conservation laws are a manifestation of the basic symmetries of space and time [4]. The conservation of energy is the manifestation of the symmetry of time translation, i.e. nothing in a system is changed if we arbitrarily set the time origin. The conservation of linear momentum is the result of symmetry under spatial translation (i.e. space homogeneity) and the conservation of angular momentum is the result of symmetry under rotation (i.e. space isotropy). In a closed interacting system the conservation laws apply to the sum of all the components; the components may exchange
energy, linear momentum and angular momentum. The conservation applies to a specific component only if the interaction itself has a specific symmetry.

The electromagnetic field carry energy, linear and angular momentum. The angular momentum may have two components, one which relates to the polarization (or spin angular momentum -SAM) and one which relates to the spatial distribution of the field and is termed (inspired by atomic physics) orbital angular momentum (OAM). The transfer of energy from an electromagnetic field to a medium may occur due to absorption. Linear and angular momentum are also affected by absorption, but also by scattering. Linear momentum transfer gives rise to radiation pressure while angular momentum transfer leads to changes in the polarization state of the fields as well as to torques applied to particles. In this dissertation we will discuss the mechanisms by which the scattering process couples SAM to OAM and how different effects related to OAM are manifested in multiple scattering.

In many cases, the scattering process is a deleterious effect which interferes with the optical system’s tasks to efficiently collect or transmit optical radiation. Such is the case, for example, when inhomogeneities in transparent optical elements or roughness in reflective ones, cause glare and degrade the resolution and the dynamic range of imaging systems. Another example is related to imaging and beam propagation through the atmosphere which involves scattering by turbulence [7]-[9] and aerosols [10]. In all these cases, the study of the scattering is important in order to mitigate its effects.
In other situations, the intimate relation between scattering processes and the basic properties of the media can be used to study these properties. Methods based on measuring spatial, angular, temporal, polarization, and spectral features of scattered light are widely used in biomedical applications [11], material science [12], remote sensing and others.

The simplest example of scattering is that of an initially plane wave propagating at a specific direction, which impinges on a non-absorbing scattering medium and results in multiple waves propagating at different directions. The component which continues to propagate unaffected is usually referred to as the ballistic component (or the reduced specific intensity) and is attenuated exponentially for a monochromatic wave. In a highly scattering medium, this component may become negligible after a relatively short propagation distance. When the initial wave is a coherent one, the random interference of scattered and unscattered waves creates a random intensity pattern, known as speckle [5],[6]. This pattern will fluctuate due to the random fluctuation in the medium and a long-term averaging will result in a broad intensity pattern. Chapter 2 is a review of scattering theory and contains results which are pertinent for the dissertation.

Scattering processes will usually cause the light to depolarize, i.e. to loose the initial polarization, and, upon spatial or ensemble average, to reach an unpolarized state. When no absorption is included the total energy flux in the scattered intensity pattern is the same as for the incident wave but the loss of directionality and the depolarization are a manifestation of the transfer of linear and angular momentum to the medium. The aspects of angular momentum transfer, and it’s relation to depolarization processes are studied in
Chapter 3 where it is demonstrated that polarization related effects can be explained by applying constraints for angular momentum conservation.

It is of immense interest to explore new ways of exciting scattering processes such that to improve on our capability to sense the medium properties. In this dissertation I will present the use of singular waves, i.e. radiation fields with wave front dislocations, as a source for scattering studies and I will analyze the benefits of this procedure.

Over the last three decades, the study of optical wavefront dislocations has consolidated into a new area of optics called singular optics [13]-[16]. Many different phenomena are related to singular structures which are associated with points of zero amplitude and undetermined phase. Singularities are found as a result of scattering by random media [17] as well as in ordered systems, such as the well known Laguerre-Gauss (LG) modes of the axis symmetric laser resonator [18]. LG modes are an example of the so-called screw phase dislocations or phase vortices which will be discussed later in this report. Beams containing screw dislocations are known to posses angular momentum [19]-[21] and are employed extensively in optical micromanipulation [22]. As will be demonstrated in Chapter 4, an incident wave containing singularities offers unique opportunities for the study of scattering media. The opportunities arise from the fact that, even though scattering is associated with randomization of phase and loss of coherence, there are effects, especially close to the exact forward and backward direction, which depend on to the correlations of the illuminating wave. Those correlations can be manipulated by introducing singularities in a manner that
allows extracting useful information about the scattering medium. A detailed theoretical and experimental study will be presented in Chapter 4.

In the case of a weak inhomogeneous medium, such as atmospheric turbulence it will be shown that a novel method, coherent mode coupling, can be employed in order to explore the power coupling between different modes carrying orbital angular momentum. In chapter 5, this approach will be outlined and its usefulness will be demonstrated for a variety of other applications as well.
CHAPTER 2

REVIEW OF PROPAGATION IN SCATTERING MEDIA

2.1 Propagation in homogeneous media

The interaction of electromagnetic radiation in general and optical radiation in particular with matter can be described at the atomic or molecular level by scattering and absorption processes. If the frequency of the radiation is far from one of the resonances of the medium, the effects of absorption are small. When the medium is dense such that there are many atoms or molecules within one cubic wavelength, the phase of the electromagnetic field does not change appreciably from one scatterer to the other, and the different scattering contributions add coherently. In this case it turns out that the field is not attenuated and the direction (as determined by the wave vector) is not altered and the only effect is to change the phase velocity [23],[24]. Close to resonances, one has to account for absorption as well, but again without any change in direction of propagation. The macroscopic propagation characteristics of optical radiation in a non-magnetic medium, which are determined by
the coherent superposition of multiple scattering contributions, can be described in terms of macroscopic optical properties of the medium i.e., the dielectric susceptibility or equivalently the complex refractive index. The real part of the index is associated with the change in the phase velocity of light and refractive effects and the imaginary part is associated with absorption and the change of the intensity upon propagation. The optical properties are generally frequency dependent, which leads to dispersion effects. The dependence of optical properties on the strength of the field, due to nonlinear effects, which to some degree exists in any material, will be disregarded in this work. The refractive index is not, in general, a scalar property but depends on the polarization of the propagating wave, a dependence called birefringence. For a general non-isotropic material, the birefringence will cause changes in the state of the polarization of the propagating radiation, which are usually deterministic and uniform across the propagating wave’s wavefront.

For a homogenous optical material, i.e. one for which the optical properties do not change as a function of the position in the material, the propagation can be dealt with using basic wave propagation theory as is appropriate for the specific problem. This is also the case when the changes can be characterized in a deterministic way. A macroscopic example for such a case is an optical system, which can be very complicated but still can be described by a collection of elements, each of which has specific and homogeneous optical properties and is usually quite large compared to the wavelength. Another situation is where the features are of the order of a wavelength and examples for this case are gratings, waveguides and photonic crystals; in those cases, light is said to be refracted or diffracted or both.
2.2 Propagation in scattering media

Inhomogeneous bulk media is characterized by a random variation of the optical properties manifested as the polarizability in the microscopic case and complex index of refraction in the macroscopic case. One may consider a background homogeneous medium in which elements with some other refractive index are embedded. These elements may be composed of a material which is different from the background medium or are of the same material but with different properties, such as density or temperature, that directly affect the refractive index. In the most general case, the refractive index can vary within such an element. Another form of randomness can be introduced at interfaces between different media (which may be completely homogenous) as the roughness of a surface. The effect of light interaction with the inhomogeneities is called scattering and its result is to change the field amplitude, phase and polarization at any point in space with regard to the result of propagation in homogeneous medium. Due to the spectral and polarization dependencies of the scattering process the spectrum and state of polarization of the original radiation will be modified by the propagation in the random media.

The result of scattering from multiple inhomogeneities (which from now on will be called scatterers) that can include cascaded scattering events (multiple scattering) is, of course, very complicated and usually has to be dealt with in terms of statistical measures. It should be noted that in practical situations, there are neither ideally homogeneous media nor completely smooth surfaces. It depends on the specific application whether the inhomogeneity
of the medium or the roughness of the surface are large enough to merit attention or can be disregarded.

There is a wide range of possible cases of interest for which scattering is a process to be reckoned with. Those cases can be categorized by the length scales of the problem and the magnitude of variations of the index of refraction. The length scales to be considered are an average wavelength of light which is the basic yard-stick of the problem, the typical size of the scattering element, the mean distance between them (which is related to the density of scatterers) and the overall propagation length. In the following sections I will discuss the effects of the different length scales.

2.2.1 Scatterer size and refractive index contrast

Let us consider a linearly polarized plane wave which impinges on a scatterer [1],[2]. We will assume the propagation direction to be \( z \) and dropping the time dependence the field is described by

\[
E_i(r) = A \exp \{i \, n_1 \, k_0 \, z \} \hat{e}_i, \quad (2.1)
\]

where \( E_i(r) \) is the incident field vector at a position \( r \) in space; \( A \) is the field amplitude, \( n_1 \) is the refractive index of the background medium which we assume to be uniform but not necessarily real; \( k_0 \) is the vacuum wavenumber defined by \( 2\pi/\lambda \) where \( \lambda \) is the wave length of light in vacuum; \( z \) is the propagation distance and \( \hat{e}_i \) is a unit vector indicating the state
of polarization of the incident field. This vector can be in the the basis of linear polarization or the basis of circular polarization. The linear basis is defined in terms of a 2D Cartesian reference frame in a plane transverse to the propagation direction and will be designated as \( \hat{e}_x \) and \( \hat{e}_y \). Alternatively, a circular polarization basis can be defined as \( \hat{e}_L = 1/\sqrt{2} (\hat{e}_x + i \hat{e}_y) \) and \( \hat{e}_R = 1/\sqrt{2} (\hat{e}_x - i \hat{e}_y) \), where the \( L \) and \( R \) subscripts stand for left and right handed polarizations respectively.

The presence of scatterer, which we assume is located at the origin and is defined by an index of refraction \( n_2 (r) \) within some typical size \( a \), modifies the electric field distribution in space. Far from the particle, i.e. at distances large compared to \( a^2/\lambda \), the field distribution is described as a superposition of the incident wave and a scattered component which is represented by an outgoing spherical wave

\[
\mathbf{E} (r) = \mathbf{E}_i (r) + \mathbf{E}_s (r) = A \exp \{ i \ n_1 \ k_0 \ z \} \hat{e}_i + \frac{A}{k_r} \exp \{ i \ n_1 \ k_0 \ r \} \langle \rangle \hat{S} (\theta, \phi) \hat{e}_i, \quad (2.2)
\]

were \( \langle \rangle \hat{S} (\theta, \phi) \) is the dimensionless (and possibly complex) \( 2 \times 2 \) amplitude scattering matrix which relates the amplitude and polarization of the outgoing wave to those of the incident one. This matrix is dependent on the wavelength and the scattering index distribution and it’s form depends on the choice of polarization basis. The power which is removed from the incident beam is obtained by integrating the outgoing wave intensity over a sphere at some large distance \( r \)

\[
P^{(i)}_s = \frac{A^2}{k^2} \int d\Omega \left[ \langle \rangle \hat{S} (\theta, \phi) \hat{e}_i \right] \left[ \langle \rangle \hat{S} (\theta, \phi) \hat{e}_i \right]^*, \quad (2.3)
\]
where $d\Omega$ indicates a solid angle differential element $(\sin \theta \, d\theta \, d\phi)$ and the symbol $\dagger$ denotes the conjugate transpose operation. The efficiency of the scattering process can then be described in terms of an effective scattering cross section which is defined by

\[
\sigma_s^{(i)} = \frac{P_s}{A^2} = \frac{1}{k^2} \int d\Omega \left[ \overleftrightarrow{S}(\theta, \phi) \hat{e}_i \right] \left[ \overleftrightarrow{S}(\theta, \phi) \hat{e}_i \right]^\dagger
\]

\[
= \frac{1}{k^2} \int d\Omega \hat{e}_i \left[ \overleftrightarrow{S}(\theta, \phi) \right]^\dagger \overleftrightarrow{S}(\theta, \phi) \hat{e}_i. \tag{2.4}
\]

For unpolarized light one obtains $\sigma_s = \frac{1}{2} \left( \sigma_s^{(1)} + \sigma_s^{(2)} \right)$.

It is convenient to define the scattering efficiency $Q_s$ by dividing the scattering cross-section by the geometrical cross-section of the scatterer ($\pi a^2$ for a spherical scatterer with a radius equal to $a$). If one is considering unpolarized light, the tensor form reduces to a scalar function. The differential scattering efficiency, also known as phase function is defined as

\[
f(\theta, \phi) = \frac{1}{2k^2} \sum_{i=1,2} \left[ \hat{e}_i \overleftrightarrow{S}(\theta, \phi)^\dagger \overleftrightarrow{S}(\theta, \phi) \hat{e}_i \right] \sigma_s^{(i)}, \tag{2.5}
\]

and the integral of $f(\theta, \phi)$ over all the solid angles is unity. When the scatterers are symmetric about the axis defined by the direction of incidence, there is no dependence on $\phi$ and the dependence on $\theta$ is through $\cos(\theta)$. When the scatterers are very small compared to the wavelength and the field can be considered as uniform within such a scatterer the scattering is called Rayleigh scattering and is characterized by a dipole radiation pattern. For larger spherical scatterers, Mie theory can be applied [1],[2] but analytical results were obtained for other shapes as well [2]. Three example of phase functions obtained for spherical scatterers are presented in Fig. 2.1. The examples are given for different values of the size
parameter which is defined as

\[ x = \frac{2\pi a}{\lambda} \]  

(2.6)

Figure 2.1: Scattering phase functions for scatterers with a refractive index contrast equal to 1.5 and size parameters 1, 2, and 0.5

It is clear that when the size parameter increases, the scattering is less isotropic as the forward scattering becomes more pronounced. The anisotropy is represented by the \( g \) parameter which is equal to \( \langle \cos(\theta) \rangle \), i.e. to the mean of scattering cosine. For a highly forward scattering the anisotropy parameter is close to 1, for an isotropic scattering it is 0 and it can also be negative for the case of backward scattering. For the cases of \( x = 1, 2, 0.5 \) presented above \( g \) is equal to 0.2, 0.63 and 0.05 respectively. As the index contrast decreases the anisotropy reduces. For instance, for \( x = 1 \) if the refractive index of scatterer is set to
1.1 the anisotropy $g$ parameter is 0.17, if the index is increased to 2 then $g$ becomes 0.27.

The overall scattering efficiency can be calculated as a function of $x$ and two examples are shown in Fig. 2.2. The efficiencies were calculated for $n_2$ equal to 1.5 and 1.33. The second result is plotted again as a function of a corrected size parameter $x (1.33 - 1)/(1.5 - 1)$. The results demonstrates that the main features of the scattering efficiencies are a function of $x \Delta n$. One can also observe the well known phenomena of Mie resonances. In the cases shown here resonance appears for a corrected size parameter of about 4 and the peak value is equal to about 4, i.e. the effective scattering cross section is about 4 times the geometric cross section of the scatterer. For very large size parameters the scattering efficiency factor approaches a value of 2. This result can be explained if one recalls that the same effect is obtained in a diffraction by a disk - the relative power of light removed by diffraction is equal to that removed by the geometrical shadow. In the case of a large Mie scatterer we see both the effects of refraction and diffraction.

When absorption has to be accounted for, one can define the total attenuation cross section which is given by the sum of the scattering and absorption cross sections, $\sigma = \sigma_s + \sigma_a$. The particles absorption have to be added, eventually, to the medium absorption, if any exist.

The phase function which deals with the angular distribution of scattered intensity is actually only one element of the 4 by 4 matrix which describes the vectorial scattering effects on the Stokes parameters of the incident light (also referred to as the Stokes vector).
Figure 2.2: Scattering efficiencies plotted versus size parameter. Two cases are shown for $n_2$ equal to 1.5 and 1.33. The second data set is plotted again versus a corrected size parameter.

The Stokes vector is defined in a linear basis as [2]:

$$\begin{bmatrix}
I \\
Q \\
U \\
V
\end{bmatrix} = \begin{bmatrix}
E_x E_x^* + E_y E_y^* \\
E_x E_x^* - E_y E_y^* \\
E_x E_y^* + E_y E_x^* \\
i (E_y E_y^* - E_x E_x^*)
\end{bmatrix}. \quad (2.7)$$

The scattering matrix relates the Stokes vector of the scattered radiation to that of the incident radiation. The 16 elements of the scattering matrix are derived from the 4 elements of the amplitude scattering matrix [2]. The amplitude scattering matrix is complex so it has up to 8 independent parameters and since global phase is arbitrary it is reduced to 7, therefore we can conclude that the different elements of the scattering matrix are not
independent but should have some constraints imposed on the relations between them. If
the particles have some symmetries those constrains are even more severe.

2.2.2 Effects of the mean distance between scatters

The mean distance between scatters is given by reciprocal of the cubic root of the number
density. For very small scatterers there is a possibility of having many scatterers within a
cubic wavelength. We have seen that when this number is very large the combined effect is
to induce a change in the phase velocity without any scattering effect. When this number
is very small, i.e. the mean distance is very large the scattering effects can be considered
as incoherent and the scattering cross-section are simply added. The intermediate case, in
which the number of scatterers is neither very small nor very large is, as usual, the most
complicated one. As the particles get more closely packed one has to account for the spatial
correlations in the spatial locations. The net effect will be to effectively decrease scattering
effects and to increase the transport mean free path that will be discussed in the next section.

It is worthwhile discussing briefly the problem of scattering by a gas. Scattering effects
are evident in the atmosphere (the blue sky, red sunset, polarization of the sky, etc.). Some of
those effects are due to aerosols but most of it is due to the molecules of different species which
constitute our atmosphere. In the standard conditions there are many millions of molecules
is a volume unit defined by a wavelength cubed- so why do we see such a pronounced effect?
It can be shown, in fact, that the scattering effect is due to the thermodynamic fluctuations in the number of scatterers per such a unit volumes [24].

### 2.2.3 Scattering and transport mean free paths

In the case of a dilute scattering medium, one can assume independent scattering events, and easily evaluate a scattering mean free path, \( l_s \), which represents, using the photon terminology, the statistical average of propagation distances between successive scattering events\(^1\). It is given by

\[
l_s = \frac{1}{N\sigma_s}, \tag{2.8}
\]

where \( N \) is the number density of the scatterers (assuming that there is only one species of scatterers). The scattering coefficient, \( \mu_s \), is defined as \( 1/l_s \). The probability of scattering in an infinitesimal length \( ds \) is then given by \( \mu_s \, ds \). Similarly the probability of absorption is given by \( \mu_a \, ds \), where \( \mu_a \) is the absorption coefficient given by \( N\sigma_a \). Solving for the effect of scattering (or absorption) on the irradiance \( I \), one obtains the well know Bear-Lambert law of exponential decay

\[
I(z) = I_0 \exp \left( -\mu_{s,a} z \right) \tag{2.9}
\]

The scattering mean free path embeds both the scattering characteristics of a single element (which depend on the size and the refractive index) and the density of the elements.

\(^1\)In the wave terminology one may refer to this length as a mean coupling distance.
When $l_s$ is larger than the propagation length the scattering regime considered is the so-called single-scattering regime. The probability of having more than a single scattering event, as the light transverse the medium, is very small. At the other extreme, when $l_s$ is much smaller then the propagation length, the case is a highly multiple-scattering one.

A single scattering event and even multiple scattering events might not randomize the direction of a wave (or equivalently its k-vector). When the anisotropy factor $g$ is close to one, the scattering is highly peaked in the forward direction and very small randomization occurs. From radiative transfer theory, that will be discussed next, one finds that k-vector randomization occurs at a length scale given by $l_s/(1 - g)$. Assuming that the absorption is negligible this length is called the transport mean free path. An illustration of the two scale lengths is presented in Fig 2.3. A more complete definition, which includes the effects of absorption will be given in the next section.

A heuristic estimation of this length is obtained when one considers the net displacement of a wave in the original propagation direction after a succession of scattering events. From geometric consideration; this displacement can be written as

$$l_0 + l_1 \cos \theta_1 + l_2 \cos \theta_2 \cos \theta_1 + ... = l_0 + \sum_{i=1}^{\infty} l_i \prod_{n=1}^{i} \cos \theta_n,$$

where $l_i$ is the distance propagated after the $i$ scattering event and $\theta_i$ is the angle between the directions prior and after the $i$ scattering event. Since the scattering events are independent the result of an ensemble average will be $\sum l_s g^i$ which is simply a geometric sum. If we
take an infinite number of scattering events then this sum converges to \( l_t = l_s/(1 - g) \). The physical significance of the limit is that the correlation between the direction of the initial wave and the direction of the scattered wave after the \( i \) scattering event goes like \( g^i \) so it approaches zero as the displacement approaches \( l_t \).

The reduced scattering coefficient, \( \mu'_s \), is defined by \((1 - g)\mu_s\). One can see that it can be calculated as

\[
\mu'_s = N \sigma_s \int d\Omega \ f(\theta) \ (1 - \cos \theta) .
\] (2.10)

In the case of dense packing, when spatial correlations in the position of scatterers have to be accounted for this calculation is modified to \([25],[26]\)

\[
\mu'_s = N \sigma_s \int d\Omega \ f(\theta) \ S(\theta) \ (1 - \cos \theta) .
\] (2.11)
where $S(\theta)$ is the static structure factor, defined as

$$
S(q) = 1 + N \int [C(r) - 1] \exp(i \cdot q \cdot r) \, d^3r,
$$

where $q$ is the elastic momentum transfer vector which is related to the scattering angle by $q = 2k \sin(\theta/2)$, and $C(r)$ is the pair correlation function. Any correlation will decrease the reduced scattering coefficient, increasing the transport mean free path.

It should we noted that when the scatterers population is not composed of a single species but have a polydisperse distribution of sizes the scattering and absorption cross sections are calculated as a weighted mean of the contributions of all the sizes.

### 2.3 Depolarization effects in random media

As described in a preceding section, single-scattering alters the state of polarization of the radiation. The cumulative effect of multiple scattering acts to randomize the polarization of a wave, a process which called *depolarization*. One can then define another scale length which indicated what is a typical distance in which light becomes depolarized. A parameter which allows to quantify the depolarization effect is the *degree of polarization* which is defined as

$$
P = \frac{\sqrt{Q^2 + U^2 + V^2}}{I},
$$

(2.13)
where $I, Q, U, V$ are the Stokes parameters. For a fully polarized wave $P = 1$, and for a fully unpolarized one $P = 0$. If the incident wave on a random medium is fully polarized then the degree of polarization will decrease upon propagation. In many works it was found that the scale lengths are different for an initially linear or circular polarization, and that they are different for different kind of scattering media [27]. In general, when the particles are large and forward scattering is dominant circular polarization will survive for a longer distance. On the other hand when the particles are small, and the scattering is Rayleigh like, linear polarization will survive longer. For small particles the linear depolarization length (for which $P$ reduces to about $1/e$) is about $2.7l_s$ [28] and for large particles it is about $l_t$ [27]. It is important to note that the degree of polarization is a statistical measure of the distribution of polarization states on the Poincaré sphere (It is actually the norm of the center of gravity locus in normalized $Q, U, V$ space). Being a first order moment it does not gives a full description of the states of polarization of the scattered waves. A situation in which $P = 0$ may represent, for example, a case in which we have linearly polarized waves with all the possible orientations, two circular polarizations of equal power, etc. High order statistical moments (with respect to the distribution of states on the Poincaré sphere) contains more information [29]. The connection between depolarization, geometric phase and exchange of angular momentum with the medium will be discussed in Chapter 3.
2.4 Radiative transfer and the diffusion approximation

Due to the inherent randomness of the scattering processes it is usually very difficult to describe the propagation in terms of fields and waves. One then turns to using a description based on the transfer of specific intensity and also to energy conservation considerations. This methodology is included in what is known as radiative transfer equation (RTE) [30],[3]. Even within this framework it is usually very difficult to obtain analytic results and some simplifications were made leading to the so-called diffusion approximation which was found to be very useful in simple geometrical configurations. The terminology used in the context of radiative transfer and the diffusion approximation is based on the photon as an energy carrying basic unit and the random walk it undergoes due to the scattering process. It is important to note though that interference effects can be still significant and those are not always easily described within the RTE framework.

Radiative transfer is dealing with the transport of the quantity called specific intensity or radiance, \( L(\mathbf{r}, \mathbf{s}, t) \). The units of this quantity are \([W \ m^{-2} \ sr^{-1}]\) and it is defined as the power flowing through a differential surface element at position \( \mathbf{r} \), at the direction \( \mathbf{s} \) at time \( t \) (assuming a monochromatic wave, spectral dependence was excluded, but it can be included if necessarily). Accounting for processes of scattering in and out of the direction \( \mathbf{s} \), absorption and possible emission inside the scattering volume one obtains the time-dependent radiative transfer equation:

\[
\left[ \frac{1}{c} \frac{\partial}{\partial t} + \mathbf{s} \cdot \nabla + (\mu_s + \mu_a) \right] L(\mathbf{r}, \mathbf{s}, t) = \mu_s \int_{4\pi} L(\mathbf{r}, \mathbf{s}', t) P(\mathbf{s}, \mathbf{s}') \ d\Omega' + Q(\mathbf{r}, \mathbf{s}, t). \tag{2.14}
\]
The function $P(s, s')$ is the just the phase function or the the probability that light is scattered from any direction $s'$ to the direction $s$ and can be usually written as $P(s \cdot s')$.

The function $Q(r, s, t)$ is a possible source function (fluorescence etc.) while $c$ is the velocity of light inside the medium (or more specifically the velocity of the flow of energy).

It is difficult to obtain analytical results for the radiative transfer equation and in many cases numerical discretization or probabilistic Monte-Carlo (MC) methods are employed. Approximations can be made using the spherical harmonics expansion for the radiance and the most common, and probably useful, is the one that retains only the first order (the so called $P1$ approximation) which leads to the diffusion equation. The $P1$ approximation is equivalent to a Taylor expansion in which the radiance is approximated as an isotropic term and a gradient term. For this approximation to be valid one has to assume that the gradient term is much smaller than the isotropic term, such that neglecting higher terms can be justified. This requirement limits the validity of the diffusion approximation to cases in which the absorption coefficient is much smaller than the reduced scattering coefficient (high absorption will cause a large gradient) and to large separations in space and in time, from the source. The diffusion equation is usually presented in terms of the energy fluence $U(r, t)$ which is the isotropic term of the radiance and is calculated as

$$U(r, t) = \intop_\Omega L(r, s, t) \, d\Omega.$$  \hspace{1cm} (2.15)

Another important quantity is the flux which is given by

$$J(r, t) = \intop_\Omega s \cdot L(r, s, t) \, d\Omega.$$  \hspace{1cm} (2.16)
In diffusion theory the radiance is approximated as

\[ L(\mathbf{r}, s, t) \approx U(\mathbf{r}, t) + \frac{3}{4\pi} J(\mathbf{r}, t) \cdot s \]  

(2.17)

Introducing this expression into the radiative transfer equation and preforming the integrals one obtains the time dependent diffusion equation

\[
\frac{1}{c} \frac{\partial U(\mathbf{r}, t)}{\partial t} - D \nabla^2 U(\mathbf{r}, t) + \mu_a U(\mathbf{r}, t) = q(\mathbf{r}, t),
\]  

(2.18)

where \( q(\mathbf{r}, t) \) is assumed to be an isotropic source term and \( D \) is the diffusion coefficient which is conventionally defined as

\[
D = \frac{1}{3 [\mu_a + \mu_s^t]}.
\]  

(2.19)

It should be noted that this definition is still the subject of an on-going debate, specifically on the subject of its dependence on absorption [31]-[35] (just to cite a few of the papers). It was suggested that the diffusion constant should be modified to [35]

\[
D = \frac{1}{3 [a (\mu_s, \mu_a, g) \mu_a + \mu_s^t]}.
\]  

(2.20)

where \( a (\mu_s, \mu_a, g) \) is a function which is 0.2 in the case of isotropic scattering and changes in the range 0.2-0.6 for anisotropic scattering. The transport mean free path, is given by

\[
l_t = 3D.
\]  

(2.21)
Assuming an infinite scattering medium one can solve for the time depended propagator of the fluence by assuming a source which is a delta function both in time and in space, $\delta (0, 0)$.

The solution can be obtained by standard techniques, such as the Fourier transform method and is given in terms of the photon fluence (which is just the energy fluence divided by the energy carried by each photon $h\nu$, were $h$ is the Plank constant and $\nu$ is the light frequency) - as [36]

$$\Phi (r,t) = \frac{U(r,t)}{h\nu} = c \left( 4\pi c D t \right)^{-3/2} \exp \left( -\frac{r^2}{4Dct} - \mu_0ct \right). \quad (2.22)$$

When one examines this solution it is immediately evident that some precautions should be taken in using the results of the diffusion approximation. For any $t$ larger than zero the spatial distribution is an unbounded Gaussian, implying that some photon moved faster than the speed of light. This Gaussian distribution, which can be inferred from random-walk considerations[37], becomes a better approximation as $t$ becomes larger and larger. Noninvasive experimental measurements of scattered light are done at the boundary between a scattering medium and non scattering medium, as a reflectance experiment or a transmittance experiment therefore boundary effects should be introduced and the reflectance or transmittance calculated. In order to make this calculation, several assumptions are performed. The first one is that a narrow collimated beam (a so called "pencil- beam") is incident normally on a planar surface of the scattering medium which is taken to be infinite in the transverse directions and finite ("slab") or semi infinite in depth (for $z > 0$). The cumulative effects of scattering from the beam are replaced by an equivalent isotropic point
source which is located at a depth of one transport mean free path, $l_t$. This assumption is justified when we are interested at times which are long and locations far from the incident beam. Another assumption is concerned with specifying the boundary conditions. If we the scattering and non-scattering media are index matched such that no Fresnel reflection occurs at the boundary the reasonable physical boundary condition is to set the radiance to zero at the boundary for $s$ which points outside the scattering medium. This constraint causes a discontinuity which is incompatible with the assumptions of the diffusion approximation which assumes a dominant isotropic component and a small directional term. It was found\cite{38} that in order to obtain a proper solution of the diffusion equation which approaches the solution of the radiative transfer equation deep within the scattering media one has to specify a mixed boundary condition in the form

$$\left[ \Phi - z_e l_t \frac{\partial \Phi}{\partial z} \right]_{z=0} = 0. \quad (2.23)$$

This boundary condition is equivalent to specifying a zero fluence condition on an extrapolated boundary which is exterior to the physical boundary and is located at a distance given by $-z_e l_t$ where \cite{38},\cite{39}

$$z_e = \frac{2}{3} \frac{1 + R_{eff}}{1 - R_{eff}}. \quad (2.24)$$

$R_{eff}$ is an effective reflection parameter which is calculated using the Fresnel reflection coefficients. This boundary condition can be imposed by using a negative mirror image of the isotropic source specified at the previous assumption \cite{36},\cite{43}. A graphical representation of this method, familiar from the electrostatic theory, is presented in Fig. 2.4, for the case of
a semi-infinite medium. In the case of a finite medium more sources have to be added to account for the boundary conditions at the exit plane. Formally, an infinite number of pair sources ("dipoles") but in most practical situations the sum is truncated at some finite term.

![Graphical representation ofsource, extrapolated boundary, and the mirror source used to obtain the boundary conditions.](image)

Figure 2.4: A graphical representation of the source, extrapolated boundary, and the mirror source used to obtain the boundary conditions.

Eventually, the solution for the fluence is obtained in the form

\[
\Phi (\rho, z, t) = c (4\pi cDt)^{-3/2} \exp (-\mu_a ct) \times \\
\left\{ \exp \left[ -\frac{\rho^2 + (z - l_t)^2}{4Dct} \right] - \exp \left[ -\frac{\rho^2 + (z + l_t + 2zl_t)^2}{4Dct} \right] \right\}.
\]

(2.25)

Due to cylindrical symmetry the transverse radial coordinate, \(\rho\), and the propagation coordinate, \(z\), were introduced In order to calculate the reflectance one has to integrate the radiance over the back hemisphere. In the diffusion theory, the radiance is given as a sum
of a dominant isotropic fluence term and a much weaker directional flux. At the boundary
the flux term can be calculated from Fick’s law [37] as

$$\mathbf{J} (\rho, t) = -D \nabla \Phi (\rho, z, t)|_{z=0}.$$  \hspace{1cm} (2.26)

Using Eq.2.17 one can obtain the expression for the reflection as

$$R (\rho, t) = \frac{1}{4\pi} \int_{2\pi} d\Omega [1 - R_f (\theta)] \cos (\theta) \times$$

$$\left[ \Phi (\rho, z = 0, t) + 3D \frac{\partial \Phi (\rho, z = 0, t)}{\partial z} \cos (\theta) \right],$$

were \( R_f (\theta) \) is the Fresnel reflection factor for unpolarized light. The last expression can
be written as

$$R (\rho, t) = A_1 \Phi (\rho, 0, t) + A_2 R_f (\rho, t),$$  \hspace{1cm} (2.28)

were

$$R_f (\rho, t) = \frac{1}{2} (4\pi cD)^{-3/2} t^{-5/2} \exp (-\mu_a c t) \times$$

$$\left\{ l_t \exp \left[ -\frac{\rho^2 + l_t^2}{4Dt} \right] + (l_t + 2zcl_t) \exp \left[ -\frac{\rho^2 + (l_t + 2zcl_t)^2}{4Dt} \right] \right\}$$

and \( A_1, A_2 \) are coefficients which depend on the boundary index mismatch. It should be
noted that the integral in Eq. 2.27 was presented for reflectance into the back hemisphere
but the domain of integration should be modified according to the collection geometry.
Similar expression can be calculated for the transmittance [36]. If one is interested in the
distribution of path lengths traversed by light in the media then \( t \) should be replace by \( s/c \). The path lengths distribution can be probed by temporal measurements or by low coherence interferometry [44]. Low coherence interferometry is indicative of the coherence effects which are manifested by light propagating in random media.

The stationary solutions can be obtained by setting the time derivative in the diffusion equation (Eq. 2.18) to zero. The solution can be written as

\[
Rs(\rho) = A_1 \Phi s(\rho,0,t) + A_2 Rsf(\rho,t),
\]

where,

\[
\Phi s(\rho,z) = \frac{1}{4\pi D} \times \\
\left\{ \frac{\exp\left(-\mu_{eff} \left[ (\rho^2 + (z - l_t)^2)\right]^{1/2}\right)}{\left[\rho^2 + (z - l_t)^2\right]^{1/2}} - \frac{\exp\left(-\mu_{eff} \left[ \rho^2 + (z + l_t + 2z_e l_t)^2\right]^{1/2}\right)}{\left[\rho^2 + (z + l_t + 2z_e l_t)^2\right]^{1/2}} \right\},
\]

and

\[
Rsf(\rho,t) = \frac{1}{4\pi} \left\{ l_t \left( \mu_{eff} + \frac{1}{(\rho^2 + l_t^2)\left[\rho^2 + (1 + 2z_b)^2 l_t^2\right]^{1/2}} \right) \frac{\exp\left[-\mu_{eff} \left( \rho^2 + l_t^2\right)^{1/2}\right]}{(\rho^2 + l_t^2)} + \\
l_t \left( 1 + 2z_b \right) \left( \mu_{eff} + \frac{1}{(\rho^2 + (1 + 2z_b)^2 l_t^2)\left[\rho^2 + (1 + 2z_b)^2 l_t^2\right]^{1/2}} \right) \frac{\exp\left[-\mu_{eff} \left( \rho^2 + (1 + 2z_b)^2 l_t^2\right)^{1/2}\right]}{(\rho^2 + (1 + 2z_b)^2 l_t^2)} \right\}.
\]

and

\[
\mu_{eff} = \left[ 3\mu_a (\mu_a + \mu_b^a) \right]^{1/2}.
\]
In the case of negligible absorption the solutions are simplified greatly and become

\[
\Phi_s (\rho, z) = \frac{1}{4\pi D} \left\{ \frac{1}{[\rho^2 + (z - l_t)^2]^{1/2}} - \frac{1}{[\rho^2 + (z + l_t + 2z_l l_t)^2]^{1/2}} \right\}.
\] (2.34)

and, respectively,

\[
R_{sf} (\rho, t) = \frac{1}{4\pi} \left[ l_t \frac{1}{(\rho^2 + l_t^2)^{3/2}} + l_t (1 + 2z_b) \frac{1}{(\rho^2 + (1 + 2z_b) l_t^2)^{3/2}} \right].
\] (2.35)

### 2.5 Atmospheric turbulence

Atmospheric turbulence is an example of a random medium which is inhomogeneous in the sense that the index of refraction is fluctuating due to random variations of temperature and pressure.

Turbulence is characterized by a very small variation of refractive index (order of \(10^{-6}\)) over large length scales compared to the wavelength. A continuous distribution of turbulence eddies ranges in size from a few millimeters to many meters; thus propagation in turbulence is characterized by extremely forward peaked scattering. Since we are usually concerned with propagation over very long paths, the effects accumulate and degrade the performance of imaging systems (such as astronomic telescopes) and laser beam transmitting and receiving devices. In fact, propagation through turbulence is not usually described in terms of scattering, but in terms of wavefront degradation and loss of coherence. Another manifestation
of the forwardly peaked "scattering" in turbulence is the fact that depolarization effects are very small and usually they are neglected.

An approximate expression for the index of refraction of air in the visible part of the EM spectrum is [9]

\[ n = 1 + 79 \times 10^{-6} \frac{P}{T}, \]

(2.36)

where \( T \) is temperature in units of kelvin and \( P \) is pressure in units of millibar. The statistical function used to describe the variations in the index of the refraction is the structure function which is defined as

\[ D_n(\Delta r) = \left\langle [n(\mathbf{r} + \Delta \mathbf{r}) - n(\mathbf{r})]^2 \right\rangle, \]

(2.37)

and is found to be [9]

\[ D_n(\Delta r) = \begin{cases} C_n^2 (\Delta r)^{2/3} & l_0 < \Delta r < L_0 \\ C_n^2 l_0^{-4/3} (\Delta r)^2 & \Delta r < l_0 \end{cases}, \]

(2.38)

where \( l_0 \) and \( L_0 \) are the inner (small) and outer (large) scale lengths of turbulent eddies. \( L_0 \) represents the length scale in which energy is introduced into the turbulent system and in the atmosphere it is assumed to be of the order of the height above the ground. The length scale in which energy dissipates out of the turbulent system is \( l_0 \) and it is of the order of a few millimeters. The scale range of between them is called the inertial range. \( C_n^2 \) is the structure constant of the index of refraction and is the most important parameter in describing turbulence strength. When it is smaller then about \( 10^{-17} \text{ m}^{-2/3} \) the turbulence is referred to as weak. When it is larger then about \( 10^{-13} \text{ m}^{-2/3} \) the turbulence is referred to
as strong with an intermediate range in between. The statistical variations of the index of refraction can be represented as a 3D power spectrum $\Phi_n(\kappa)$ where $\kappa$ is the spatial frequency.

The relation between the 3D power spectrum and the structure function is

$$D_n(\Delta r) = 8\pi \int_0^{\infty} \kappa^2 \Phi_n(\kappa) \left( 1 - \frac{\sin \frac{\kappa \Delta r}{\kappa \Delta r}}{\frac{\kappa \Delta r}{\kappa \Delta r}} \right) d\kappa.$$ (2.39)

Different models have been developed for the power spectrum of the index fluctuations. One of them is the Von Kármán power spectrum is given by

$$\Phi_n(\kappa) = 0.033C_n^2 \exp \left( -\frac{\kappa^2}{\kappa_m^2} \right) \frac{1}{(\kappa^2 + \kappa_0^2)^{11/6}},$$ (2.40)

where $\kappa_m = 5.92/l_0$ and $\kappa_0 = 1/L_0$. Setting $\kappa_0$ to zero (infinite outer scale) leads to the Tatarskii power spectrum (which is valid when beam size is much smaller then the upper scale of the turbulence). Further setting $\kappa_m$ to infinity (no inner scale) we obtain the well known Kolmogorov power spectrum which is valid when the beam width far from both the lower and upper limits.
CHAPTER 3

ANGULAR MOMENTUM IN SINGLE AND MULTIPLE SCATTERING

As discussed in the previous Chapter scattering theory deals mainly with redistribution of the energy of electromagnetic field. Polarization aspects are dealt with as well, but polarization represents only a part of a more basic property of the electromagnetic field, the angular momentum. In this Chapter we discuss how angular momentum is affected by single and multiple scattering and how one can relate specific polarization effects to the transfer of angular momentum to the media.

If we consider the interaction of a plane wave with a spherical, non-absorbing particle, i.e., a scattering process as described in Chapter 2, we can observe that the interaction is axially symmetric (with respect to the direction of propagation) but is not translational symmetric since the electromagnetic (EM) field is different before and after the particle. This implies that the interaction conserves the energy and angular momentum of the EM field but that linear momentum is exchanged. This exchange of linear momentum gives rise to the radiation pressure on the particle [1]. In the following sections we will discuss the linear and
angular momentum of the electromagnetic field, radiation pressure and also angular momentum transfer in single and multiple scattering. It will be demonstrated that even though the total angular momentum of the electromagnetic field is conserved in scattering from a spherical particle, there is a redistribution between the polarization and orbital angular momentum related terms. This redistribution can be observed in multiple scattering where the appearance of specific polarization patterns are the result of phase vortices acquired by the waves. The waves which contribute to those patterns are the ones which do not exchange angular momentum with the medium.

It should be noted that effects which are related to spin-orbit interaction of light were discussed by Zeldovich et al. [45].

3.1 Linear and angular momentum of the EM field

The volume density of linear momentum carried by light in vacuum is given by [46]

\[ p_v = \varepsilon_0 E \times B. \quad (3.1) \]

where \( \varepsilon_0 \) is the permittivity of vacuum. Note that this is not a time averaged quantity. For a plane wave it is easy to define a linear momentum flux density in the form

\[ p = c\varepsilon_0 E \times B = \frac{\Pi}{c}. \quad (3.2) \]
where $c$ the velocity of light and $\mathbf{\Pi}$ is the Poynting vector. The time averaged quantity is given as $I_0/c$ where $I_0 = \varepsilon_0 c E_0^2 / 2$. As it is well known, this corresponds to a quantum description in which each photon in the incident field carries a linear momentum of $\hbar \mathbf{k}$ as well as an energy of $\hbar \omega$ where $\hbar$ (h bar) is the Plank constant divided by $2\pi$ and $\omega$ is angular frequency. In analogy to the definition of angular momentum in mechanics, one can define angular momentum of the EM field as [46],[21]

$$J = \int \mathbf{r} \times \mathbf{p} \, d^3r,$$  \hspace{1cm} (3.3)

where $\mathbf{r}$ is the position vector with respect to some reference origin, $\times$ denotes the cross product between vectors and $d^3r$ is an infinitesimal volume element. One may observe that the cross product represents the angular momentum density. Using Maxwell equations and some vector identities the result is separated into two terms which can be identified as intrinsic (i.e. do not depend on the choice of the origin of the reference system) and extrinsic (do depend on the choice of origin). The intrinsic and extrinsic terms are identified in atomic and particle physics as spin angular momentum (SAM) and orbital angular momentum (OAM). In particular, the two terms can be written in terms of the transverse part of the vector potential (which is gauge invariant) as

$$S = \int \varepsilon_0 (\mathbf{E}^\perp \times \mathbf{A}^\perp) \, d^3r$$  \hspace{1cm} (3.4)
As it turns out the spin term is related to the polarization, a fact that was originally noted by J.H. Poynting \([47]\). The left handed circular polarization (LCP) is associated with a positive helicity and the right handed circular polarization (RCP) is associated with a negative helicity. The measurement of angular momentum carried in the polarization was demonstrated by R.A. Beth using a very precise measurement of the torque applied on a birefringent plate which modifies the state of polarization of light \([48]\). This corresponds to the quantum description of light in which a photon carries an angular momentum which can be \(\pm \hbar\) \([4]\). Linear polarization is a coherent combination of both helicities and carry no angular momentum. The angular momentum flux density of a plane wave is \(\sigma I_0/\omega\) where \(\sigma\) is the helicity.

When one considers the more general case of a paraxial wave for which the electric field can be described as

\[ E = \exp(ikz) F(x, y, z) \]  \hspace{1cm} (3.6)

then the \(z\) component of the angular momentum flux density is \([49]\)

\[ j_z(x, y, z) = \frac{c \varepsilon_0}{2 i \omega} \sum_{k=x,y} F_k^* \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) F_k + \frac{c \varepsilon_0}{2 i \omega} (F_x^* F_y - F_y^* F_x). \]  \hspace{1cm} (3.7)
The second term is zero for linear polarization and reflects the angular momentum carried by circular polarization. The first term relates to the transverse distribution of the field and is the OAM term. If the paraxial beam transverse field distribution can be written as

\[
F(r, \phi) = u(r) \exp(im\phi) \hat{F}
\]  

(3.8)

and recognizing that \(\left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}\right) = \frac{\partial}{\partial \phi}\) we find that

\[
j_z(r, \phi) = \frac{c}{2\omega} \frac{\varepsilon_0}{(m + \sigma)} |u(r)|^2 = (m + \sigma) \frac{I}{\omega}.
\]  

(3.9)

The beams described by Eq.3.8 are called vortex beams and they exhibit an axial singularity since the phase is indeterminate on the beam axis. This phase singularity leads, due to destructive interference, to an intensity null. The integer factor \(m\) is called the topological charge of the singularity and it is an invariant of the propagation, a fact we should expect since it is related directly to the orbital angular momentum and under the paraxial approximation each of the angular momentum terms is conserved independently.

### 3.2 Transfer of linear momentum in scattering and radiation pressure

When one considers elastic scattering and neglects the recoil of the massive scattering particle, the wave vectors of the incident and scattered waves have the same magnitude. The linear
momentum flux density component in the original direction which is carried by the scattered waves in the far field (assuming unpolarized incident light) can be calculated as

\[ p_z = \frac{I_0}{c} k^2 \sigma_s \int d\Omega \ f(\theta) \cos \theta. \] (3.10)

The linear momentum flux density which is removed can be then directly related to the force on particle

\[ F_r = \frac{I_0}{c} k^2 \sigma_s \int d\Omega \ f(\theta) (1 - \cos \theta) = \frac{I_0}{c} k^2 \sigma_s (1 - g). \] (3.11)

Let us now consider the effects of multiple scattering in a backscattering geometry. For simplicity we will consider a half space \((z > 0)\) of scattering spheres which is index match with the half space \((z < 0)\) so that we can ignore boundary reflections effects. A narrow beam is incident on the surface and is backscattered. The total energy flux reflected from the surface is equal to the incident one so if we consider the mean radiance of backscattered light \(L(r, \theta)\) the total scattered power is

\[ P = 2\pi \int d^2r \int_0^{-1} d\cos \theta \ \cos \theta \ L(r, \theta), \] (3.12)

and it should of course be equal to the incident power. The linear momentum flux of the scattered waves (we consider the \(z\) component since the other components vanish due to axial symmetry) is

\[ p_z = \frac{2\pi}{c} \int d^2r \int_0^{-1} d\cos \theta \ (\cos \theta)^2 \ L(r, \theta). \] (3.13)
If we assume the radiance to be a function only of location (i.e. any point on the surface is a lambertian emitter) we get that

\[ p_z = -\frac{2}{3c} P. \quad (3.14) \]

For an ideal mirror the result is \(-P/c\) which indicates a radiation pressure on the mirror of \(2P/c\). In the case of a Lambertian reflector this pressure is \(5P/3c\). The angular distribution for a diffusive reflector is somewhat more complicated and may depend on the the boundary index mismatch and on the anisotropy parameter of the single scattering process [39], so the numerical factors are different, but in any case smaller than that of a mirror. The "softness" of the diffusive reflector is an indication of the transverse momentum transfer to the scattering particles.

### 3.3 Transfer of angular momentum in single scattering

In elastic scattering from a non absorbing spherical particle, two parameters of the electromagnetic field are conserved: the energy and the angular momentum component along the propagation direction. Linear momentum as a whole is also conserved, of course, but some of it is transferred to the particle leading to radiation pressure. The conservation of energy (which is a scalar quantity) leads to a normalization condition for the integrated energy flux density, which is further used in defining the scattering cross section [1, 50]. Similarly, the conservation of angular momentum should be related to the angular momentum flux den-
sity. The continuity conditions for the angular momentum density can be described by three equations (one for each component of the vector) or one equation for a tensor [51].

We are interested in calculating the angular moment flux density of the electromagnetic field which results from scattering of a circularly polarized wave from a non-absorbing spherical particle. It was demonstrated that scattering of circularly polarized wave does not exert torque on the particle and that transfer of angular momentum from the field to the particles is mediated only by absorption [52]. Therefore, the angular momentum of the field is preserved and it should be interesting to know how the flux density of the angular momentum is distributed between the spin or polarization term which will be henceforth designated by $s$ and the orbital angular momentum (OAM), which will be denoted by $l$. The total angular momentum flux density, $j$, is given by the sum of these two terms. The problem of torques applied to particles was treated extensively in the context of particle manipulation or "optical tweezers" [53]. In this work we will limit ourself to cases in which no torque is applied and we will discuss the angular momentum carried by the scattered electromagnetic wave.

Let us consider an incident plane wave which is monochromatic (angular frequency $\omega$), left circularly polarized, has an amplitude $E_0$ and propagates in the direction $\hat{z}$ (the 'hat' denotes a unit vector). The flux density of the angular momentum of the incident plane wave which is carried by the spin term is equal to $I_0/\omega$ where $I_0 = \varepsilon_0 c E_0^2/2$ ($\varepsilon_0$ being the permittivity of vacuum and $c$ the velocity of light). This corresponds to a quantum description in which each photon in the incident field carries a spin angular momentum of $+\hbar$. The rate at which the angular momentum is removed from the incident field can be
derived from the scattered power and is given by

$$\frac{\partial \mathbf{J}_0}{\partial t} = \sigma_{sc} \frac{I_0}{\omega} \hat{z},$$  

(3.15)

and has a component only in the direction of propagation. This is the source term for the angular momentum of the scattered field and it should be recovered by integrating over all directions the angular momentum flux density of the scattered field.

From general scattering theory it is known that the scattered electric field in the far zone can be written as

$$\mathbf{E}(\theta) = \begin{pmatrix} E_L(\theta) \\ E_R(\theta) \end{pmatrix} = \frac{\exp(ikr)}{r} \begin{pmatrix} S_{LL}(\theta) & S_{RL}(\theta) \\ S_{LR}(\theta) & S_{RR}(\theta) \end{pmatrix} \begin{pmatrix} E_0 \\ 0 \end{pmatrix},$$  

(3.16)

where the matrix $S$ is the scattering matrix in the circular polarization basis, the $L$ and $R$ subscripts designate the left and right circular polarizations. In Eq.(3.16), $k$ is the wave number and $r$ is the distance from the center of the scattering particle which is both the origin of the coordinates frame and the reference point for the angular momentum calculations. The scattering matrix in the circular base can be related to the more usual amplitude scattering matrix given in terms of the parallel and perpendicular electric field components with respect to the scattering plane. For a spherical particle, this scattering matrix is diagonal with elements $S_2(\theta), S_1(\theta)$ and the relation is [1]

$$\begin{pmatrix} S_{LL}(\theta) & S_{RL}(\theta) \\ S_{LR}(\theta) & S_{RR}(\theta) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} S_2(\theta) + S_1(\theta) & S_2(\theta) - S_1(\theta) \\ S_2(\theta) - S_1(\theta) & S_2(\theta) + S_1(\theta) \end{pmatrix}.$$

(3.17)
The notation used in this section is illustrated in Fig. 3.1.

It is important to remember that scattered field components are given in terms of a coordinate frame which is rotated such that the local scattered field is given in terms of a component is a plane which is perpendicular to the scattering direction. However, in the following it is necessary to express all the scattered waves in the same reference frame such that spatial derivatives can be made consistently. The most convenient system of coordinates is the one associated with the incident wave and, in this case, it is found that the rotation introduces a phase term $\exp(i\phi)$ where $\phi$ is the azimuth angle [50°]. We note here that in the case of an incident wave which is right circularly polarized, the phase factor is $\exp(-i\phi)$. 

Figure 3.1: Illustration of the notation used, including the definitions of the angles in the spherical coordinates and the set of orthonormal unit vectors associated with the spherical coordinates.
The scattered field can now be written as

$$E(\theta, \phi) = \frac{\exp (i k r)}{r} E_0 \exp (i \phi) \left\{ S_{LL} (\theta) \, \hat{L} + S_{LR} (\theta) \, \hat{R} \right\}$$  \hspace{1cm} (3.18)$$

and expanding further the left and right circular unit vectors in terms of the locally transverse unit vectors

$$\hat{L} = \frac{1}{\sqrt{2}} (\hat{\theta} + i \hat{\phi})$$

$$\hat{R} = \frac{1}{\sqrt{2}} (\hat{\theta} - i \hat{\phi})$$  \hspace{1cm} (3.19)$$

one finally obtains

$$E(\theta, \phi) = \frac{\exp (i k r)}{r} E_0 \exp (i \phi) \frac{1}{\sqrt{2}} \left\{ [S_{LL} (\theta) + S_{LR} (\theta)] \hat{\theta} + i [S_{LL} (\theta) - S_{LR} (\theta)] \hat{\phi} \right\}$$

$$= \frac{\exp (i k r)}{r} E_0 \exp (i \phi) \left[ S_{\theta} (\theta) \, \hat{\theta} + S_{\phi} (\theta) \, \hat{\phi} \right] .$$  \hspace{1cm} (3.20)$$

where the notation $S_{\theta} (\theta) = \frac{[S_{LL} (\theta) + S_{LR} (\theta)]}{\sqrt{2}}$ and $S_{\phi} (\theta) = i \frac{[S_{LL} (\theta) - S_{LR} (\theta)]}{\sqrt{2}}$

has been used

Having found $E$, we can now proceed to calculate the angular momentum flux density through a radially oriented infinitesimal area in the radiation zone. When the wave is approximated locally as a plane wave the spin term can be found using Eq.(3.7) :
This result is physically reasonable: the spin is simply the difference of intensities of the two radially outgoing orthogonal circular polarization components divided by the angular frequency of radiation. The common phase term $\exp(i\phi)$ does not play a roll in this calculation.

Due to axial symmetry, all the components of $s$ average to zero when an integration over the angles is performed except for the $z$ component which is

$$s_z(\theta) = \frac{\epsilon_0c}{2} \frac{E_0^2}{\omega r^2} \left[ |S_{LL}(\theta)|^2 - |S_{LR}(\theta)|^2 \right] \cos(\theta).$$  \hspace{1cm} (3.22)$$

For a Rayleigh scatterer, one can immediately find that the expression in Eq. (3.22) reduces to

$$s_z(\theta) = \frac{3\epsilon_0c}{16\pi \omega r^2} \sigma_{sc} \cos^2(\theta).$$ \hspace{1cm} (3.23)$$

In order to calculate the total scattered flux, the spin flux density must be integrated over a sphere of radius $r$ to obtain

$$\Phi_z = \frac{1}{4} \frac{\epsilon_0c}{\omega} E_0^2 \sigma_{sc} = \frac{1}{2} \sigma_{sc} \frac{I_0}{\omega}.$$ \hspace{1cm} (3.24)$$
The bar superscript in Eq. 3.24 indicates integration over a sphere of arbitrary radius, in the far field. Notably, one can see that only half of the angular momentum flux removed from the incident wave is contained in the spin term. Of course, this conclusion is similar to the results obtained for the case of a field radiated by a rotating dipole [54].

In order to evaluate the more complex case of a Mie scatterer let us rewrite Eq. 3.22 as

\[ s_z(\theta) = \frac{1}{\omega} [I_L(\theta) - I_R(\theta)] \cos(\theta) = \frac{1}{\omega} V(\theta) \cos(\theta), \] (3.25)

where \( I_L(\theta), I_R(\theta) \) are the scattered intensities with left and right hand circular polarization respectively and \( V(\theta) \) is the fourth Stokes parameter. In our case the incident wave is characterized by a Stokes vector of the form \([1, 0, 0, 1]\) while the scattered wave Stoke vector is \([F_{11}(\theta), F_{21}(\theta), F_{34}(\theta), F_{44}(\theta)]\) where \( \overrightarrow{F}(\theta) \) is the angle dependant \( 4 \times 4 \) scattering matrix [1] which is block diagonal for spherical particles. Since we obtained that for a left hand circular incident wave \( V(\theta) = F_{44}(\theta) \), we can calculate how much of the angular momentum flux density is contained in the spin term, normalized by the scattered angular momentum:

\[ \frac{\omega s_z}{\sigma_{sc} I_0} = \frac{2 \pi \omega \int \sin(\theta) d\theta \ F_{44}(\theta) \cos(\theta)}{\sigma_{sc} I_0}. \] (3.26)

It is worth mentioning that this expression is similar to the asymmetry factor defined by MacKintosh and John [55] (designated there as \( A \)) but note that their definition lacks the
\[ \cos(\theta) \] in the numerator and therefore it reflects the overall helicity flip rather then the \( z \) component of the spin.

Numerical evaluations of Eq.(3.26) based on the Mie theory suggest that a good approximation of this ratio is

\[ \frac{\omega \sigma_z}{\sigma_{sc} I_0} \approx \frac{1 + g^2}{2}, \]  

(3.27)

where \( g = \langle \cos(\theta) \rangle \) is the so-called scattering asymmetry parameter. For highly forward scattering which is helicity preserving, this ratio is close to 1 meaning that the total angular momentum is concentrated in the polarization term. Some examples are presented in Fig. 3.2. The calculations were made for several relative indices of refraction: 1.09 which is comparable to the case of silica spheres in water, 1.18 which represents polystyrene spheres in water and a hypothetical higher contrast material with a relative index of refraction of 1.25. One can observe that the agreement with the dependence suggested in Eq.(3.27) is excellent for \( g \) below 0.1 and above 0.75. A higher order polynomial which will be better in the intermediate range is of course possible. It is also interesting to examine the mean helicity (the Mackintosh_John parameter) of the scattered field which is illustrated in Fig 3.3. One can note that up to \( g = 0.7 \) there is a good linear dependence with a slope of about 1.3.

Let us now turn our attention to the more complex issue of the OAM term which is [49, 54].
Figure 3.2: The ratio of the spin flux to the total angular momentum flux as a function of the anisotropy parameter for several values of the relative index of refraction as indicated.

\[ l(\theta) = \frac{\epsilon_0 c}{i2\omega} \mathbf{E}^* (\hat{r} \times \nabla) \mathbf{E}. \]  

(3.28)

In order to evaluate this expression we employ the spherical coordinates form of the gradient operator and recall that the unit vectors have to be differentiated as well; for example \( \frac{\partial \phi}{\partial \phi} = \cos(\theta) \hat{\phi} \) and \( \frac{\partial \phi}{\partial \phi} = (\hat{z} \cos(\theta) - \hat{r})/\sin(\theta) \). Also note that the common phase term, \( \exp(i\phi) \), is important in this case and should be considered in the differentiation.
Figure 3.3: Mean helicity of the scattered field for the cases presented in Fig. 3.2. Below $g=0.7$ there is a good linear fit with a slope of about 1.3.

Accounting only for the terms that contribute to the $z$ component, we obtain after some algebra that

$$l_z (\theta) = \frac{\epsilon_0 c}{2 \omega r^2} \left\{ |S_{LL} (\theta)|^2 + |S_{LR} (\theta)|^2 - \left[ |S_{LL} (\theta)|^2 - |S_{LR} (\theta)|^2 \right] \cos (\theta) \right\}. \quad (3.29)$$

This result, together with the one expressed in Eq.3.22 indicate that $s_z (\theta)$ and $l_z (\theta)$ sum up to an expression which is proportional to the scattered intensity. For each $\theta$, the total angular momentum flux density is simply the intensity divided by the angular frequency of the radiation. Integrating over a spherical surface in the far field leads indeed to a
manifestation of the conservation of angular momentum flux:

\[ \overline{j}_z = \overline{s}_z + \overline{l}_z = \sigma_{sc} \frac{I_0}{\omega} \hat{z}. \]  

(3.30)

Moreover, we note again that at each \( \theta \), the ratio \( j_z (\theta) / I (\theta) \) is constant. In the quantum description one may say that a photon scattered in any direction carries the same angular momentum as an incident photon, but, in different directions it is distributed differently between the spin and OAM terms. It is now evident that in the forward and backward directions the angular momentum is carried only by the polarization. In other words the helicity is fully preserved in the forward direction and it is fully reversed in the backward direction. For some angles, which for a Rayleigh scatterer is only 90°, on the other hand, the scattered light is linearly polarized and therefore does not carry any spin angular momentum. At these angles the angular momentum is carried only by OAM term.

Fig. 3.4 illustrates the three dimensional distribution of the spin term (relative to the scattered intensity) plotted as a function of scattering direction for several values of the size parameter \( x = 2\pi a / \lambda \), where \( a \) is the radius of the particle and \( \lambda \) is the wavelength. The orbital angular momentum distribution is the complementary one (as it will be given by one minus the spin). It is interesting to note that for small size parameters the changes in the normalized distribution are small. For large particles on the other hand, in which Mie resonances are dominant, one can observe that there are directions in which the spin term is negative- meaning that the normalized orbital term is larger then 1. The transition to such
behavior happens for particles with a size parameter of about $\pi$ (i.e. a particle diameter which is approximately equal to the wavelength).

In order to illustrate the distinction between spin and orbital angular momenta one may use the following "gedanken" experiment. Let us consider a small dielectric sphere which is slightly absorbing, and is placed in the far field of a Rayleigh scatterer. In the exact forward direction the particle will rotate about itself due to the absorption of circularly polarized light. At an angle of 90° with respect to the direction of incidence, on the other hand, the test particle will "orbit" the scattering particle due to the phase gradient in the scattered field. We would like to emphasize that this effect is not due to absorption and that, of course, the test particle will experience a radial force due to radiation pressure.

Finally, we note that the orbital angular momentum of scattered light can also be interpreted as the result of a slight direction dependant shift of the apparent origin of the scattered waves, a shift which introduces an "impact parameter" of the order of $\lambda/2\pi$. The origin of this shift can be identified in the classical electromagnetic theory where it is known that the Poynting vector of radiation from a rotating dipole (equivalent to a Rayleigh scatterer illuminated by a circularly polarized light) spirals in the near field. As a result the far field radiation seems to be emerging not from the center of the dipole but from a shifted position [56] as illustrated qualitatively in Fig. 3.5. In this figure the incident light direction is into the page and the scattered wave is observed in a transverse plane.
Figure 3.4: Three-dimensional angular distributions of the normalized spin term (angular momentum content of the spin term divided by the intensity) for several size parameters. The relative index of refraction was 1.09. Left circularly polarized light is incident along the $z$ axis (the direction is indicated by the red arrow). The color bars represent the spin and a complementary illustration can be obtained for the OAM.
Figure 3.5: Qualitative illustration of the Poynting vector direction for radiation scattered by a Rayleigh scatterer illuminated by a circularly polarized light which is perpendicular to the plane of the figure as indicated. The spiraling of the Poynting vector results in an apparent angular shift of the light at the far field.
The angle $\alpha$ is approximated by $\lambda/2\pi r$ and the angular momentum at the point $r$ can then be calculated to be $k r \alpha = 1$. This means of course that the orbital angular momentum carries all the angular momentum. In our gedanken experiment, the slight tilt of the $k$-vector induces a rotating motion of the test particle placed at $r$. This is the mechanism which couples angular momentum from the electromagnetic field to the medium.

In this section we elucidated the mechanism by which the scattering from a spherical non-absorbing particle is transferring angular momentum carried by an electromagnetic field from the spin term to the orbital angular momentum term. The relative part of the angular momentum contained in the spin term can be calculated exactly and it ranges from 0.5 for a Rayleigh scatterer to 1 in the case of a highly forward scatterer. In any scattering direction the ratio between the total angular momentum flux density carried and the intensity, is constant. Moreover, this ratio has the same value as the corresponding one evaluated for the incident wave. The orbital angular momentum complements the spin angular momentum and in some directions may carry all the angular momentum. The orbital angular momentum carried by the scattered field mediates the transfer of angular momentum to the medium.
3.4 Transfer of angular momentum in multiple scattering and the geometrical phase

As demonstrated in the previous section scattering of light on a single spherical particle, due to symmetry, conserves the angular momentum of the field. However, when we consider a succession of scattering centers it is very unlikely that any particular arrangement will have axial symmetry even though the statistical distribution is isotropic. The scattering and gradient forces which are generated by the field amplitude and phase distribution inside the medium act to generate torques which transfer angular momentum between the field and the medium. One can expect that the angular momentum carried by the wave will dissipate and that a complete depolarization will occur. This is generally the case but there are phenomena which exhibit some order within the disorder. It will be sown that those phenomena of geometrical polarization patterns are a manifestations of a class of scattering paths which do not exchange angular momentum with the medium.

3.4.1 Geometrical phase in optics

In order to proceed with the analysis of angular momentum transfer we need to introduce the concept of geometrical phase and to elucidate the connection between geometrical phase and angular momentum. The concept of geometrical phase (in some places called topological phase) was introduced by Berry in the context of quantum mechanics [57]. Chiao and Wu
applied it to optics but it was a sort of reintroduction since as early as 1938 and 1941 Rytov and Vladimirskii have anticipated it. Let us compare two circularly polarized light waves which start with the same direction and travel along two paths who have the same length, but one is straight and the other coiled, such that they both emerge in the original direction. Both waves will acquire a dynamical phase which is related to the path length but the wave traveling along the coiled path will acquire an extra phase which is equal to the multiplication of minus the helicity by the solid angle spanned by the close curved inscribed in direction space (or k-vector space) by the wave direction. This result is summarized in Eq. 3.31

$$E_{L,R}^f = A E_{L,R}^i \exp(iks) \exp(-i\sigma\Omega), \quad (3.31)$$

where the $i$ and $f$ superscripts denote initial and final amplitudes respectively, The $L$ and $R$ subscripts denote left and right circular polarization, respectively, $A$ is a real amplitude transmission factor, $s$ is the pathlength, $\sigma$ is the helicity and $\Omega$ is the solid angle. It should be noted that the geometric phase, remarkably, does not depend on the wavelength.

The solid angle in direction space is illustrated In Figure 3.6.

The concept of geometrical phase can be applied not only to a continuously curving path, such as a fiber, but also to a series of reflections from mirrors. In that case the different directions are connected by arcs which are parts of great circles of the sphere (a great circle is a circle which lies on the surface of the sphere and its diameter cross the center of the sphere). A mechanical equivalent can be found if we consider a spring and evaluate the
Figure 3.6: (a) The path of light propagating in a fiber which has an helical shape. (b) The K-space curve which substands a solid angle equal to $2\pi (1 - \cos \theta)$ where $\theta$ is the pitch angle of the helix.

extra winding that occurs due to twisting of the spring. In the case of circularly polarized light one may imagine the extra winding of the fields due to the twist in the trajectory. A simple derivation of this phase which relays on solving the Maxwell equations in a rotating frame was presented by Lipson [59]. If the light is linearly polarized this geometrical phase will be manifest as a polarization rotation due to the different phases acquired by the two circular components. This is equivalent to the concept of parallel transport of the linear polarization. It is worth mentioning that other types of geometrical phases arise in optics. The well known Pancharatnam phase is acquired when the state of polarization is light is
changed in a cyclic way such that the area covered on the Poincaré sphere is non-zero [60]. Another type of geometrical phase is associated with the cyclic conversion of LG modes [61].

Geometrical phase was introduced to scattering problems as an alternative explanation of polarization patterns [62],[63]. In this work we expand on that explanation and elucidate its connection with conservation of angular momentum.

Let us first review here the methodology used to calculate the geometric phase.

1. We shall define a configuration space which is a modified k-space, representing the multiplication of the normalized k-vector by the helicity. This space is actually the spin space, which replaces the direction of the k vector with the direction of the spin vector. Each point on the unit sphere in this configuration space represents a left handed circularly polarized wave propagating in some direction or a right handed wave propagating in the opposite direction. The definition used here is similar to the one used by Kitano et al. [64], even though they did not associated it directly with the helicity.

2. An adiabatic change in the direction of propagation leads to a continuous and smooth curve on the unit sphere surface. If the change is cyclic then the curve is closed and the geometrical phase is given by the solid angle subtended by the closed curve with respect to the center of the sphere. If the curve is not closed, the final and initial points need to be connected by an arc which is a part of great circle ("the geodesic rule", see for example Kwon et al. [65]). The closed curve thus obtained is used to calculate
the solid angle. In a case in which the points are antipodal, i.e. points in opposite
directions in configuration space, there is no unique choice of the great circle. In this
case we may choose an arbitrary, but consistent, reference path. The concept may be
applied to a nonadiabatic case as well, in which discrete points on the sphere, which
may be due to reflections or scattering are connected by geodesics in order to create
a continuous (but not smooth) curve. In order to impose a specific reference plane we
can add, as the last point in the series, a virtual direction (for example \([1,0,0]\)).

3. If helicity flip occurs, due to a scattering event for example, the appropriate point is
drawn on the unit sphere and connected with the previous point with a geodesic. An
helicity change is always associated with a change of direction, but it will not necessarily
generate a new point on the sphere. In exact back scattering, a single scattering event
results in helicity flip, but since the direction is flipped as well, the states prior and
after scattering are represented by the same point. In a multiple backscattering process
which resulted in a helicity flip no reference path is required on the spin sphere because
the curve is closed.
3.4.2 Calculations methods for geometrical phase

A series of discreet points on the sphere connected by geodesics defines a spherical polygon. The area inclose by this polygon, which is equal to the solid angle can be obtained by applying the Gauss-Bonnet theorem [66], which on a sphere reduces to the formulae

\[
\Omega = \sum_{i=1}^{n} \alpha_i - (n - 2) \pi,
\]

where \( n \) is the number of vertices and \( \alpha_i \) is the turn angle at each vertex. If we designate the series of vertices as unit vectors \( s_1, s_2, \ldots \) then the turn angle is given by the expression

\[
\alpha_i = \cos \left[ \frac{(s_{i+1} \times s_i) \cdot (s_i \times s_{i-1})}{\| (s_{i+1} \times s_i) \| \| (s_i \times s_{i-1}) \|} \right] = \cos \left[ \frac{(s_{i+1} \cdot s_i)(s_i \cdot s_{i-1}) - (s_{i+1} \cdot s_{i-1})}{\| (s_{i+1} \times s_i) \| \| (s_i \times s_{i-1}) \|} \right],
\]

which is simply the angle between the normals to the planes which contain the intersecting great circles. Another method of calculating the solid angle involves mapping onto a plane. Since the map is a tool for calculating the area we should choose an area preserving projection (which in cartography is known as the cylindrical Lambert area preserving projection). In this projection the spherical angular coordinates \((\phi, \theta)\) on the sphere are projected onto \((\phi, \cos \theta)\) on a planar Cartesian axes. The shape of the closed curves on the sphere surface is distorted, but the area is preserved. This is immediately evident for the sphere as a whole which is projected onto a rectangle that has the area \(2 \times 2\pi\). Some of the other features
of the mapping include: "longitude lines", which are great circles passing through the two poles, are mapped into vertical straight lines; other great circles will not be mapped into straight lines (the exception is the "equator" which is the only latitude line which is a great circle). One should note that the poles are not uniquely mapped and in that case a point is mapped into a line. The position on the line is determined by the direction of the trajectory intersecting the pole.

Applying Green’s theorem to find the area $A$ inclosed by a closed curve $C$ in a plane one obtains

$$A = \frac{1}{2} \oint_C (xdy - ydx).$$ (3.34)

An area calculated in this way is signed. The sign depends on the direction of travel around the curve. Traveling around anti clockwise generates a positive area, while in the clockwise direction the area will be negative. A symmetric figure 8 according to this convention has a zero area! A signed area (or solid angle) will result from applying the Gauss-Bonnet theorem as well.

The last expression can be applied to projected map thus obtaining an expression for the solid angle $\Omega$

$$\Omega = \frac{1}{2} \oint_C (\phi d \cos \theta - \cos \theta d\phi) = -\frac{1}{2} \oint_C (\phi \sin \theta d\theta + \cos \theta d\phi) =$$

$$-\frac{1}{2} \int \left( \phi (t) \sin \theta (t) \frac{d\theta}{dt} + \cos \theta (t) \frac{d\phi}{dt} \right) dt.$$ (3.35)

The r.h.s of the last equation assumes a parametrization $\theta (t), \phi (t)$. For an arbitrary scattering trajectory one needs usually to use a numerical method to calculate the solid angle.
As mentioned before the paths between discreet events need to be completed by drawing a great circle. One can generate such a circle by the following procedure. If two points on the unit sphere (which are not antipodal) are given by 

\[ s_1 = [\sin \theta_1 \cos \phi_1, \sin \theta_1 \sin \phi_1, \cos \theta_1] \]

and \( s_2 = [\sin \theta_2 \cos \phi_2, \sin \theta_2 \sin \phi_2, \cos \theta_2] \), the normal to plane in which they as well as the sphere center lie, is given by \( n = s_1 \times s_2 \). The angle between the two vectors, \( \gamma \) is given by 

\[ \gamma = \arccos (s_1 \cdot s_2). \]

Any number of points on the arc connecting the two points can be generated by rotating vector \( s_1 \) about an axis given by \( n \) by an angle which is the appropriate fraction of \( \gamma \), i.e. some angle \( \beta = f \gamma \) where \( 0 < f < 1 \). This rotation is performed by applying the rotation matrix (constructed according to Rodrigues’ rotation formula [67])

\[
\begin{pmatrix}
\cos \beta + (1 - \cos \beta) (n_x^2 - 1) & -n_z \sin \beta + (1 - \cos \beta)n_x n_y & n_y \sin \beta + (1 - \cos \beta)n_x n_z \\
- n_z \sin \beta + (1 - \cos \beta)n_x n_y & \cos \beta + (1 - \cos \beta) (n_y^2 - 1) & -n_x \sin \beta + (1 - \cos \beta)n_z n_y \\
- n_y \sin \beta + (1 - \cos \beta)n_z n_x & n_z \sin \beta + (1 - \cos \beta)n_y n_x & \cos \beta + (1 - \cos \beta) (n_z^2 - 1)
\end{pmatrix}
\]

In this way one can generate a close path with an arbitrary number of points on the sphere given by \((\phi_i, \cos \theta_i)\) with \( i = 1 \) to \( N \) and \( \phi_1 = \phi_N \) and \( \theta_1 = \theta_N \). The simplest way to perform the integration is then to compute the sum

\[
\Omega = \frac{1}{2} \sum_{i=2}^{N-1} \left[ \phi_i \cos \theta_{i+1} - \phi_{i+1} \cos \theta_i \right].
\]
3.4.3 Geometrical phase and depolarization

Let us discuss the case of transmittance through a turbid slab. We assume that the light is incident normal to the input plane and the detected light is the one that emerges normally to the output plane. A speckle spot which is imaged on the surface may be the result of interference of several propagation paths within the media. The state of polarization of the speckle will be the result of interference effects between the electric fields of the waves carried in the different paths. Let us now assume that the incident wave is circularly polarized. Each of the contributions to the speckle will have different amplitudes, dynamical phases (pathlength dependant), and geometrical phases but a change in the state of polarization can occur only if some of the contributions to the speckle have the opposite helicity, that is a helicity flip has occurred. In a forward scattering medium which is helicity preserving the probability for a helicity flip is very small and a long propagation length is required to observe depolarization. On the other hand if we start with linear polarization, and consider the decomposition in circular modes, each may acquire different dynamical and geometrical phases which leads to rapid depolarization. The helicity flip is less important in this case due to the fact that it has the same probability applies for going both ways. The rate of the depolarization is related to the rate in which the width of the probability distributions of dynamical phases (which are related to the path length) and the geometrical phases (related to the path shape) are spreading. This qualitative explanation is consistent with the known results for depolarization lengths of circular and linear polarization light [27].
3.5 Polarization patterns and the conservation of angular momentum

In this section it will be demonstrated that conservation of angular momentum in multiple scattering leads to an ubiquitous observable phenomenon: the polarization patterns that are observed in backscattering from turbid media. When one considers angular momentum in multiple scattering one has generally to account for the torque which is the result of the force imparted by the scattered fields. This torque transfers orbital angular momentum to the medium, but if we consider light propagating along paths which are confined to a plane which is perpendicular to the boundary (paths which represent an important component of backscattered light) no exchange of the $z$ component of the angular momentum can occur. A component of this multiply scattered light may, however, preserve its helicity, which means that it is flipped with respect to the original direction. A paradox arises since some of the light in the backscattered direction seems to have angular momentum in the opposite direction with respect to angular momentum carried by the incident wave. We will show here that the "missing" angular momentum can be recovered if one considers the geometrical phase acquired by the helicity preserving light which is related to the OAM of light. For the incident beam, $l = 0$, and so it is for the single-backscattered light. We will demonstrate that the multiply scattered light which preserves its helicity acquires phase vortices which carry OAM such that the total angular momentum is conserved. These phase vortices lead, in turn, to observable polarization vortices.
The phase and polarization vortices discussed here are examples of wavefront singularities that were explored extensively in the last two decades as a new and exciting area of optics, singular optics, has emerged [14]-[16]. An optical phase vortex is an example of such singularity in which the phase changes linearly with respect to the azimuth angle around a point, and thus is not defined at that point leading to an amplitude null. Integrating over the phase any closed curve around the singularity yields an integer multiple of $2\pi$ termed the topological charge of the singularity.

There is a menagerie polarization singularities [14] of which we will discuss the simple example of the polarization vortex. A polarization vortex is a field in which linear polarization states are distributed such that the field vectors rotate about themselves over an angle which is proportional to the azimuth with respect to the vortex axis. Two examples are the azimuthal and radial polarization distributions for which the polarization completes one rotation around the vortex. A polarization vortex can be constructed by a superposition of right circular ($RC$) and left circular ($LC$) polarization waves with opposite topological charges. The superposition of the two circular modes, which are assumed to have the same radial amplitude distribution can be written as

$$
E(r, \phi) = u(r) \left[ \exp(i m_L \phi) \hat{L} + \exp(i m_R \phi + \delta) \hat{R} \right] \\
= \frac{u(r)}{\sqrt{2}} \left[ \exp(i m_L \phi) (\hat{x} + i \hat{y}) + \exp(i m_R \phi + \delta) (\hat{x} - i \hat{y}) \right], \quad (3.38)
$$
where $\delta$ is a possible phase shift between the two modes. If we take the topological charges of the two waves to be $m_L = m$ and $m_R = -m$, for the $RC$ and $LC$ waves respectively, then the field distribution is expressed as

$$E(r, \phi) = \frac{u(r)}{\sqrt{2}} \left[ \exp(im\phi)(\hat{x} + i\hat{y}) + \exp(-im\phi + \delta)(\hat{x} - i\hat{y}) \right]$$

$$= \frac{u(r)\exp(i\delta/2)}{\sqrt{2}} \left\{ \hat{x} \left[ \exp(im\phi - \delta/2) + \exp(-im\phi + \delta/2) \right] + \hat{y} \left[ \exp(im\phi - \delta/2) - \exp(-im\phi + \delta/2) \right] \right\}$$

$$= \sqrt{2}u(r) \left[ \hat{x} \cos(m\phi - \delta/2) - \hat{y} \sin(m\phi - \delta/2) \right].$$

In this case, the polarization topological charge (the number of times the polarization vector rotates around the singularity point) is $m_p = (m_L - m_R)/2 = m$. For example, if $m_L = -m_R = 1$, the phase charges annihilate and the polarization charge is 1 resulting in a rotation of $2\pi$ of the polarization vector. If there is no phase difference the distribution is radial (i.e. the field vectors point parallel to the radial direction), whereas if it is $\pi$, the distribution is azimuthal (i.e. the field vectors are perpendicular to the radial direction). In both cases, analyzing the intensity distribution through vertical or horizontal polarizers, leads to two-fold patterns, rotated with respect to each other by $90^\circ$. If we take $m_L = -m_R = -2$, the polarization topological charge is now 2 and the orthogonal linear polarization patterns will have a fourfold structure, rotated respectively by $45^\circ$ as illustrated in Fig. 3.7.
Figure 3.7: Polarization distribution obtained by superposing a right circularly polarized phase vortex with a topological charge equal to -2 and a left circular phase vortex with a topological charge equal to 2. (a) The intensity image where the vectors indicate the electric field direction. (b) The intensity distribution after a horizontal linear analyzer. (c) The intensity distribution after a vertical linear analyzer.
Figure 3.8: The fourfold pattern obtained by viewing backscattered light through a cross-polarized analyzer. The scattering medium was a suspension of silica particles with a size parameter of 3.5 and a transport mean free path of about 1 mm.

The patterns shown on Fig 3.7 resemble the spatial distributions of the intensity backscattered from turbid media when linearly polarized light is incident on it and it is viewed through linear analyzer [68]—[72] (see also Fig. 3.8).

As we shall see in the following, phase vortices can be associated with these patterns as well. In order to explore the origins of the phase vortices, let us first specify the characteristics of the multiple scattering process we are examining. We will consider media characterized
by an homogenous random distribution of spherical elastic scatterers which are large with respect to the wavelength and therefore scatter preferentially in the forward direction, i.e. have a anisotropy parameter which is close to unity. We recall that scattering at a small angle is known to be, with very high probability, helicity preserving. We will also consider a narrow incident beam which is axially symmetric and does not contain OAM. Finally, the backscattered light is collected over a narrow angle that is centered about the exact backscattering direction. In other words, if the incident light is normal to the surface we impose the constraint that light is collected only if it emerges normal to the surface. This is exactly the radar geometry used in observing the polarization patterns discussed here. The backscattered light has, in this case, a significant component which traveled along planar, nearly semi-circular paths [73]–[75]. This effect can be understood by realizing that, at each scattering event, light is scattered preferentially at small angles and a large portion is being redirected towards the surface along arced trajectories. The physical situation is such that the backscattered light is dominated by adiabatic transport along helicity preserving paths, which allow us to consider the evolvement of optical geometrical phase [58]. A consequence of this constraint is that the waves under consideration can be treated as paraxial.

We can apply the methodology of calculating the geometrical phase in the backscattering geometry. As such, any backscattered path acquires a phase equal with the solid angle determined by the reference and actual paths. It should be noted again that in forward scattering media the planar circular paths are the most probable in backscattering geometry [75], leading to a solid angle which is equal with the area defined by great circles on unit
Figure 3.9: Scattering paths illustrated in (a) real and (b) k-space. In k-space, the path describes the locations of the tip of the k-vectors after each scattering event (they all lay on the surface of a sphere since the scattering considered is elastic). A nearly semi circular path in real space will have a corresponding path in k-space which is nearly a geodesic, staring at the forward pole and ending in the backward one. The geometric phase corresponding to this path is the solid angle subtended between the actual path and the reference path as indicated.
sphere surface as shown in Fig. 3.9. The acquired geometrical phase is \(-\sigma\Omega\), where \(\sigma\) is the helicity (+1 for \(LC\) and -1 for \(RC\)) and \(\Omega\) is the solid angle in k-space [58]. In the case described here, it is evident that \(\Omega\) is simply \(2\varphi\) where \(\varphi\) is the azimuth angle with respect to the reference path projection on the interface. The geometrical phase increases from 0 to \(4\pi\) as \(\varphi\) grows from 0 to \(2\pi\) (this is the known result for a solid angle spanned by two intersecting great circles creating a geometrical shape known as a \(Lune\)). The geometrical phase is robust with respect to small perturbations to the scattering path, so the actual scattering path need not be an exact semi-circle. One can immediately observe that the \(RC\) wave acquires a phase vortex with a topological charge equal to +2 while the \(LC\) one acquires a \(-2\) topological charge.

These geometrical phase vortices are the origin of the four-fold symmetrical polarization patterns in backscattering. The linearly polarized light which is incident on the scattering medium can be decomposed into \(RC\) and \(LC\) modes, which subsequently acquire different phase vortices. The coherent superposition of these modes gives rise to the polarization patterns observed in backscattering. It is important to emphasize that the amplitude distribution in one realization of the scattering media is far from being uniform and that the polarization patterns are obtained only as a result of an ensemble average over different manifestations of the medium. We should also note that in any experimental configuration the incident beam width is finite and the observed backscattering should be considered as a convolution of the backscatter point spread function and the incident intensity distribution. The reduction of contrast away from the center, as seen in Fig. 3.8, is due to the effects
of interference between waves which travel along different paths and therefore acquires different dynamical phases and geometrical phases. The visibility of the patterns is indicative of the fact that the variances of those phases are small close to the insertion point i.e. the subdiffusive regime.

Here we reach the main result of this section which connects the polarization patterns with the geometrical phase vortices and the conservation of the total angular momentum of light. It is well known [19], that optical vortices can be associated with the OAM flux density and that the total angular momentum flux density in the presence of an axial vortex is proportional with the sum of the helicity and the topological charge, i.e. \( s_z \propto \sigma \), \( l_z \propto m \) and \( j_z \propto (\sigma + m) \). The OAM contribution is obtained by applying the operator of the z component of the OAM on the field distribution \( l_z \propto -i \left\langle \overline{E}^* \frac{\partial}{\partial \phi} E \right\rangle \) [49]. The angular brackets denote an ensemble average over the realizations of the medium. If we consider the propagation of circularly polarized modes along planar paths we can write the field distribution corresponding to one realization of the medium (by summing over all the contributions of different paths in that realization) as \( A(r, \phi) \exp(\pm i2\phi) \) where \( A(r, \phi) \) is a complex amplitude factor. Applying the OAM operator we find that

\[
l_z \propto \left\langle -iA^*(r, \phi) \frac{\partial}{\partial \phi} A(r, \phi) \pm 2|A(r, \phi)|^2 \right\rangle . \tag{3.40}
\]

The first term in the expression vanishes after an ensemble average over many realizations of the random, homogenous and isotropic scattering medium. The second term, however, survives and gives rise to a nonvanishing OAM contribution which is proportional to +2 for
$RC$ light and $-2$ for $LC$ light, i.e. the geometrical phase vortex is contributing to the OAM of backscattered light.

Noting this result, it is important to realize that the total angular momentum changes sign due to both single and (planar) multiple scattering. For $RC$ light, for example, $\text{sign} \ (j_z) = -1 \ (\sigma = -1, \ m = 0)$ before scattering and $\text{sign} \ (j_z) = +1$ after scattering for both the single and the multiple scattering we have considered. For single scattered light this is the result of the helicity flip ($\sigma = +1, \ m = 0$). For the multiply scattered light this is due to the fact that $\sigma$ remained -1 but $m$ is now 2, and the sum is again equal to +1. Because the sign of the total angular momentum is relative to the direction of propagation, we conclude that it is conserved. We note here that the role of the geometric phase vortex in backscattering is compatible with the suggestion that the physical origin of geometrical phases of light is related to the conservation of angular momentum [77].

3.6 Probability distributions for the geometrical phase

In the following we will briefly demonstrate the effects of the number of scattering events and of the single scattering anisotropy on the probability distribution of geometrical phase. The scenario considered here is backscattering where the light is incident normally on the surface and the emitted light is observed, again normally to the surface, and at an azimuth angle with respect to the reference frame (defined by the polarization). The first example shown in Fig. 3.10 gives the trajectory in K-space and it’s area preserving mapping for a
double scattering event which is a trivial planar path. In this case the solid angle is given by $2\phi$. The second example illustrated in Fig. 3.11 shows an arbitrary example of a non-planar scattering path involving 6 scattering events. Both trajectories were generated by a conventional Monte Carlo simulation (see for example [78]). The function used to randomize the scattering angle was well known Henyey-Greenstein [79] phase function

$$f_{HG}(\theta) = \frac{1}{4\pi} \frac{1 - g^2}{(1 + g^2 - 2g\cos\theta)^{3/2}}.$$  \hfill (3.41)

The MC simulation was used to qualitatively demonstrate the effects of the number of scattering events and anisotropy parameter on the distribution of geometrical phases. In the MC simulation the last scattering event is constrained to be in the exact backscattering direction which actually imposes a constraint on the preceding event as well since we also specify a radial location. The probability of that scattering event is therefor calculated and applied to correct for the bias which is imposed by the constraints. The results illustrated in following figures are the results of 10000 runs and show approximate probability distributions for geometric phase and path length (dynamical phase) for some specific cases. For all the figures the distance from the incidence point is one transport length and the light is emitted normal to the surface. The azimuthal angle is $90^\circ$ (or $\pi/2$ rad). Figure 3.12 shows the geometrical phase distributions for 3, 5, 7 and 9 scattering events where the anisotropy parameter is equal to 0.96. One can observe that as the number of scattering events increases the geometrical phase distribution widens (even though, for the examples shown it is still
Figure 3.10: Trajectory in K-space (a) and the area preserving mapping (b) for a double scattering event ($g=0.9$). Scattering trajectory is blue and reference curve is magenta. The enclosed solid angle is $2\phi$ in radians ($\phi = 30^\circ$).
Figure 3.11: Trajectory in K-space (a) and the area preserving mapping (b) for a scattering path which involved six scattering events (g=0.9).
maximized at \( \pi \)). The asymmetry of the distribution is the result of wrapping the distribution which is defined on a range of \( 4\pi \) on to \( 2\pi \) range which is the physically observable one. The tail of distribution between \( 2\pi \) and \( 4\pi \) is wrapped to such that it lifts the distribution wing closer to the origin. Figure 3.15 shows the pathlength distribution for the case described here. Figures 3.14 and 3.13 show the geometrical phase and pathlength distributions for the case an anisotropy parameter which is equal to 0.5. It is clear, that for this case of a more isotropic scattering, the geometrical phase distribution are wider. We can conclude from the examples presented here that for a small number of scattering events, especially for a highly forward scattering media, planar trajectories are a significant part of the entire population of scattering trajectories population. One can observe that the obtained distributions (of which only a few example are brought here) are comparable to the probability distribution of a sum of a constant phasor and a random phasor [6]. When the constant phasor is large (here the planar trajectories contribution) compared to the random phasor (the 3D trajectories contribution) the distribution can be approximated as normal.
Figure 3.12: Geometrical phase distribution for several numbers of scattering events. The anisotropy parameter was equal to 0.96. The emission point was one transport mean free path away from the incidence point. The asymmetry of the distribution is explained in the text.
Figure 3.13: Pathlength distribution for several numbers of scattering events. The anisotropy parameter was equal to 0.96. The emission point was one transport mean free path away from the incidence point.
Figure 3.14: Geometrical phase distribution for several numbers of scattering events. The anisotropy parameter was equal to 0.5. The emission point was one transport mean free path away from the incidence point. The asymmetry of the distribution is explained in the text.
Figure 3.15: Pathlength distribution for several numbers of scattering events. The anisotropy parameter was equal to 0.5. The emission point was one transport mean free path away from the incidence point.
3.7 Spatially resolved Mueller matrices

Up until now we discussed polarization patterns which arise when the medium is illuminated by linear polarization and is then observed through linear analyzers. Let us consider the more general case of polarization patterns generated by backscattering from a turbid media, polarization patterns which can be described in terms of spatially resolved Mueller matrix. We will assume an incident narrow collimated beam (the legendary "pencil beam") and also assume that the collected light is the one that is backscattered normal to the surface. The single-scattered light is then confined to the location of the illumination beam and, in most cases, an effort will be made to block it out. Multiple scattering give rise to a much broader distribution which can be sensed thorough polarization optics (typically a polarizer and a quarter wave plate) in order to produce spatially resolved Mueller matrix elements. It is important to note that different coordinate systems are associated with the in-going and out-going light. The first one to which we shall refer as the observational frame (as it is usually the one which an experimentalist will refer to) is a right handed frame with the \( Z \) axis pointing into the media (along the direction of propagation of the incident beam) and the second is a right handed one with the \( Z \) axis pointing out (along the direction of propagation of the scattered light). The transverse \( XY \) frame for the second case is obtained from the first by reflecting the \( Y \) axis with respect to the \( XZ \) plane. As a result the vertical and horizontal polarizations are the same in both frames, but the diagonal as well as the
circular ones are exchanged. As we shall see later this is important for proper interpretation of the Mueller matrix elements.

If we now assume that a significant part of the multiply scattered light travels along helicity preserving nearly planar paths we can associate a phase vortex with each of the circular polarization modes in the incident light, as discussed in Section 3.4.4.

Any state of the incident polarization can be represented in the circular basis in the usual way

$$E_i = E_L \hat{L} + E_R \exp(i\delta) \hat{R} \quad (3.42)$$

where $$\hat{L} = 1/\sqrt{2} (\hat{x} + i \hat{y})$$, $$\hat{R} = 1/\sqrt{2} (\hat{x} - i \hat{y})$$ and $$\delta$$ is the relative phase between the two circular modes.

We will now assume that the two circular modes undergo a helicity preserving scattering process. In that case the two modes acquire an appropriate phase vortex and the scattered light field (for one statistical realization of the media) is described by

$$E_s = E_L A_L (r, \phi) \exp[i\psi_L (r, \phi)] \exp(-i 2 \phi) \hat{L} +$$

$$E_R \exp(i\delta) A_R (r, \phi) \exp[i\psi_R (r, \phi)] \exp(i 2 \phi) \hat{R}. \quad (3.43)$$

then the resulting Stokes parameters will be given by
\[ I = \langle [E_L A_L (r, \phi)]^2 + [E_R A_R (r, \phi)]^2 \rangle, \]
\[ Q = \langle 2 \text{Re} [E_L E_R A_L (r, \phi) A_R (r, \phi) \exp [i \psi_L (r, \phi) - i \psi_R (r, \phi) - \delta] \exp (-i 4 \phi)] \rangle, \]
\[ U = -\langle 2 \text{Im} [E_L E_R A_L (r, \phi) A_R (r, \phi) \exp [i \psi_L (r, \phi) - i \psi_R (r, \phi) - \delta] \exp (-i 4 \phi)] \rangle, \]
\[ V = \langle [E_L A_L (r, \phi)]^2 - [E_R A_R (r, \phi)]^2 \rangle. \]

The circular modes are scattered along the same paths so we can take \( \psi_L (r, \phi) = \psi_R (r, \phi) \)
and using also \( A_L (r, \phi) = A_R (r, \phi) = A (r) \). In addition we can also consider a statistically
isotropic scattering process to finally obtain
\[
I = (E_L^2 + E_R^2) \langle A^2 (r) \rangle, \quad (3.45)
\]
\[ Q = \langle 2 \text{Re} [E_L E_R A^2 (r) \exp (-i 4 \phi)] \rangle = 2E_L E_R \langle A^2 (r) \rangle \cos (4 \phi + \delta) \]
\[ U = -\langle 2 \text{Im} [E_L E_R A^2 (r) \exp (-i 4 \phi)] \rangle = 2E_L E_R \langle A^2 (r) \rangle \sin (4 \phi + \delta) \]
\[ V = (E_L^2 - E_R^2) \langle A^2 (r) \rangle. \]

We will not concern ourself here with the exact form of \( \langle A^2 (r) \rangle \). We will also comment
that, as \( r \) gets larger, the geometric phase distribution becomes broader, the phases of
the two components becomes uncorrelated and the contrast of the patterns diminishes and
disappears.

Applying this methodology for six incident fields described by six different Stokes vectors,
\([1, \pm 1, 0, 0], [1, 0, \pm 1, 0], [1, 0, 0, \pm 1] \), we can derive the corresponding Mueller matrix. Let
us consider, for example, the first pair of input Stokes vectors. In the circular polarization basis, the incident fields are $[E_L, E_R] = [1/\sqrt{2}, \pm i/\sqrt{2}]$ and the scattered light Stokes vectors can be calculated using Eq. 3.45. Using these two input Stokes vectors the first column of the Mueller matrix can be evaluated as $1/2[I_+ + I_-, Q_+ + Q_-, U_+ + U_-, V_+ + V_-]^T$ and the second column by $1/2[I_+ - I_-, Q_+ - Q_-, U_+ - U_-, V_+ - V_-]^T$ where the superscript $T$ designates the transpose operation. The notation $I_+, Q_+, U_+, V_+$ means the intensity measured when the incident light has the Stokes parameters given by $[1, +1, 0, 0]$, $I_-, Q_-, U_-, V_-$ means the intensity measured when the incident light has the Stokes parameters given by $[1, -1, 0, 0]$. The remaining three columns are calculated using the other four incident Stokes vectors. For the case discussed above the obtained matrix has the form shown in Fig. 3.16. We stress that we disregarded the unpolarized component which of course acts to change the magnitude of the matrix elements.

This figure is drawn with the convention that the in going and outgoing Stokes vectors are defined with respect to the same frame of reference (the observational one). This it is the reason why the element $M_{44}$ is negative even though we have asserted that the scattering process is helicity preserving. In the observational frame the outgoing wave has the opposite helicity which give rise to the minus sign. If the convention is that the Stokes vectors are defined each with the corresponding frame the lower half of the matrix should multiplied by -1.

Let us now consider that some helicity mixing is taking place and this is quantified by some factor $a$ (where $0.5 \leq a \leq 1$). The rule applied is then simple. If some circular
Figure 3.16: The Mueller matrix elements of exact backscattered light which preserved its helicity.
polarization contribution emerging has the same helicity as the incident wave (which may be the result of going through an even number of helicity flips, all at same plane) then it acquired a phase vortex. If the wave has gone thorough an odd number of helicity flips it does not acquire a phase vortex, a conclusion which is, again, a manifestation of the angular momentum conservation.

This rule can be demonstrated to be the result of considering the geometrical phase acquired by the light waves. If we consider the helicity configuration space then the wave which has flipped its helicity will follow a path which is different then the one which did not. If there is no additional flip the path on the unit sphere is returning to the original pole and, for a planar scattering path, the accumulated phase turns out to be zero.

The expressions for the outgoing field can be generalized by using the following amplitude coupling matrix written in the circular polarization basis:

\[
\begin{pmatrix}
E_L^{(s)}(r, \phi) \\
E_R^{(s)}(r, \phi)
\end{pmatrix} = N \sqrt{\langle A^2(r) \rangle} \begin{pmatrix}
 a(r) \exp(-i \ 2 \ \phi) & [1 - a(r)] \\
[1 - a(r)] & a(r) \exp(i \ 2 \ \phi)
\end{pmatrix} \begin{pmatrix}
E_L^{(i)} \\
E_R^{(i)}
\end{pmatrix}.
\] (3.46)

In Eq. (3.46) \( N \) is a power normalization factor, \( \langle A^2(r) \rangle \) is the mean intensity distribution and \( a \) is the helicity preserving parameter which may be a function of distance from the incidence point. The first case we discussed is obtained by assuming that \( a \) is constant and equal to 1. If we take \( a \) to be 0.5 (i.e. complete helicity mixing which is appropriate for Rayleigh scatterers) we obtain the Mueller matrix shown in Fig. 3.17
Figure 3.17: The Mueller matrix elements of exact backscattered light for which the helicity is mixed ($\alpha = 0.5$).
Several important features are worth discussing. In comparison with Fig. 3.16 the M44 element disappeared because of the helicity mixing assumption and twofold symmetric patterns appeared in elements M12, M13, M21 and M31. Also the elements in the central $2 \times 2$ block changed. M22 and M33 have now a positive 4-fold pattern while M23 and M32 are weaker than in the helicity preserving case. It is interesting to examine for this case the co- and cross polarized patterns which are obtained when one illuminates the medium with linearly polarized light and observed the backscattered light through linear polarizers.

![Figure 3.18](image)

Figure 3.18: Backscattered patterns obtained from a medium of helicity mixing scatterers when the illumination is horizontally polarized. (a) co-polarized pattern, (b) cross-polarized pattern.
This is illustrated in Fig. 3.18 and we make the important observation that not only the symmetry is different (two fold vs. four fold) compared to the helicity preserving case but also the intensity of the cross polarized pattern is much lower (the total power is of course the same, but it is distributed over a different number of "leaves").

Let’s examine now an intermediate case. For example, The Mueller matrix for the case $\alpha = 0.75$ is shown in Fig. 3.19.

Figure 3.19: The Mueller matrix elements of exact backscattered light for an intermediate case of helicity mixing ($\alpha = 0.75$).
We are now in the position to discuss the meaning of the helicity mixing parameter. For a scattering particle with a very high anisotropy parameter the backscattered light is dominated by the helicity preserving channel. This means that scattering paths which are the result of multiple events in which the light is scattered at very small angles and thus the helicity is maintained with a very high probability. The transfer of power between the different helicities can be described by coupled rate equations. Of course, after an infinite number of scattering events the power in each helicity is equal but the rate of convergence is determined by the helicity preserving property of a single scattering event. When the single scattering event is very highly helicity preserving the helicity will be maintained over trajectory composed from a significant number of scattering events. For a Rayleigh scatterer the mixing is already complete after one scattering event and remains so for any number of scattering events (recall from Fig. 3.3 that the mean helicity of the scattered waves for the Rayleigh case is zero).

We note helicity mixing processes are usually discussed in terms of power transfer between the orthogonal polarizations. If we start with some initial intensity at the original helicity, $I_1 = I_0$ then the relative power coupled to the orthogonal polarization, $I_2/I_0$ will be designated as $b$. The amplitude coupling parameter can then be obtained as a solution to a quadratic equation and is

$$a = 1 - \frac{b - \sqrt{b - b^2}}{2b - 1}. \quad (3.47)$$

For $b = 0.0$, $a$ is equal to 1 (helicity preserving) and in the limit $b \to 0.5$, $a$ approaches 0.5 as well (helicity mixing).
In the cases discussed above we assumed that the helicity mixing is uniform over the entire back surface, an assumption which is approximately correct only for the extreme cases discussed here. In general, $a$ can be a function of the radial distance from the point of the light injection.

3.7.1 The effects of off-axis light

In the results presented in Figs 3.16, 3.17 and 3.19 we notice that the elements in the last row and last column (except $M_{44}$) are identically zero. It is interesting to note that this observation is at the origin of numerous discrepancies between different experiments as well as different MC simulations, that were preformed, supposedly, for similar scattering conditions. Here we elucidate this by proving that the magnitudes of those elements are actually determined by a parameter which is rarely reported: the acceptance angle of the collecting optics.

Up until now we have considered light which is scattered in the exact backscattering direction but in any practical situation the light is collected, of course, over some finite angular aperture. To allow for this effect we have to consider some small perturbation to the geometrical phase vortices described above. A perturbation can be conveniently introduced by considering a helicity mixing parameter which is complex and has the form $a \exp(i\beta)$ where $\beta$ is a small phase factor which is related to the field of view. This phase factor depends on the angular distribution of scattered light but for small fields of view one can
obtain that the mean perturbation to the geometrical phase is given by the half-angle FOV multiplied by a factor of the order of unity. The origin of the perturbation is illustrated in Fig. 3.20. Light is injected to the medium on the axis of the collection optics which collected emitted light over some acceptance angle. The emitted light angular distribution is biased such that the radiance in a direction away from the axis is somewhat larger. For each point on the surface this is manifested as skewed distribution on the helicity sphere which introduces a small phase bias, decreasing the geometrical phase. As an example, Fig. 3.21
presents the results for a helicity maintaining parameter of $\exp(-i\ 0.1)$ which corresponds to a FOV half angle of about $5^\circ$.

Figure 3.21: The Mueller matrix elements for a case of backscattered light which preserved its helicity and a collection optics with a finite angle of acceptance ($a = \exp(i\ 0.1)$).

Similar results for the case $a = 0.85\exp(-i\ 0.1)$ are shown in Fig. 3.22.

The actual geometrical perturbation phase dependents both on the angular distribution of scattered light and on the polarization properties of the collecting optics. It seems that for a medium composed of a suspension of spherical particles the information that may be
Figure 3.22: The Mueller matrix elements of backscattered light for an intermediate case of helicity mixing and a collection optics with a finite angle of acceptance \( (a = 0.85 \exp(i \cdot 0.1)) \)
carried in those elements is in any case redundant. When one considers off-axis effects single scattered light contributes to the patterns as well.
3.7.2 Contrast distribution

It should be noted that the spread of the distribution of geometric phases, described in a previous section, is responsible for the reduction in contrast of the polarization patterns. This is demonstrated by using the well known result[6]:

\[
\langle \exp (i\Omega) \rangle = \exp (i \langle \Omega \rangle) \exp \left( -\frac{1}{2} \sigma_\Omega^2 \right),
\]

where \( \langle \Omega \rangle \) the mean value of the geometric phase, \( \sigma_\Omega \) is the standard deviation and the probability distribution of the phase is approximated as a normal distribution. As the variance of the geometric phase increases the azimuth dependant terms decrease leading to a contrast reduction. As one look further from the incidence point the emerging trajectory involves, in average, more scattering events, and the geometric phase variance increases, leading to a contrast reduction. Regretfully, there is no established theoretical result describing the mean number of scattering events for the subdiffusive regime, and therefore it is difficult to obtain an analytical result for the dependence of the phase variance on the scattering medium parameters. As demonstrated before Monte Carlo simulations can be used. In both simulations and experiments one observes that the typical radius of the patterns is of the order of the transport mean free path.
3.8 Conclusions

In this chapter, we were able to demonstrate that all the details of measured spatially resolved Mueller matrices for backscattering from random media can be explained in terms of geometrical phase vortices. We newly introduced a treatment of backscattering phenomena in terms of circularly polarized modes. It was shown that multiply scattered light which propagates along planar trajectories and preserves the original helicity is also acquiring a geometrical phase vortex while the light which flipped it’s helicity does not. This is a manifestation of the conservation of angular momentum as demonstrated in Section 3.5. It was found that the relative contribution of helicity preserving and non-preserving components is determined by the size of the particles.

The formalism developed in this chapter was used to provide a consistent description of polarization phenomena in backscattering from turbid media and it explained the source of several ambiguities in recent experiments.

Finally we note that similar considerations can be applied in the transmission scenario, but here the roles are reversed, the helicity preserving light does not acquire a phase vortex while the one the flipped do.

In the next Chapter we will discuss modification to a backscattering phenomenon due to vortices which are embedded in the incident waves.
CHAPTER 4

ENHANCED COHERENT BACKSCATTERING OF
SINGULAR WAVES

4.1 ENHANCED COHERENT BACKSCATTERING

In the previous chapter we discussed the appearance of geometric phase vortices which are
due to the backscattering geometry. In the following we will discuss effects in backscattering
which are related to phase vortices embedded in the incident waves. Specifically we will
demonstrate that spatial correlation in vortex fields can change the enhancement in the
coherent enhanced backscattering effect in a way that is predictable and related to the
optical properties of the medium.

Coherent enhancement of backscattering from random media was first observed by Ishi-
maru and Kuga [80] and has attracted significant interest over the last two decades since the
effect was recognized as a manifestation of the so-called weak localization of optical waves
[81],[82],[83]. The enhanced backscattering (EBS) cone was shown to be the result of the
constructive interference of waves propagating along multiple scattering paths and the conju-
gate waves following time-reversed paths (as illustrated in Fig. 4.1). The effect was studied theoretically and investigated experimentally in many scattering media including highly diffusive materials [84] and double-pass configurations [85] (a combination of a phase screen and a mirror in which the waves pass twice through the phase screen). The phenomenon was found to occur in the reflection from a rough surfaces [86] and was applied to explaining the sharp surge in brightness of planetary bodies while in opposition (near zero phase angles) [87]. A review of the much of the early research done in the subject can be found in [88].

For extended media and fully coherent illumination, the maximum enhancement factor is twice the mean background level due to constructive interference. Effects modifying the shape of the peak can be related to the scattering media. For instance, it has been shown that destroying the time reversibility by the application of magnetic field [89], or removing long path contributions by increased absorption or limited thickness [90] lowers the peak magnitude. The EBS cone shape is also modified by the extent of correlations in the scattering medium [91]. In addition the peak’s magnitude and shape are affected by the angular properties, spatial coherence, and temporal coherence of the illumination beam [92],[93],[94]. Using singular waves as the illumination source for EBS experiments offer an attractive way to modify the field spatial correlations in a way that allows one to infer the optical characteristics of the scattering media. This chapter contains detailed theoretical explanation of the effect and experimental results [95],[96],[97].
4.2 Coherent effects in forward and backward scattering

Let us examine the diagram in Fig. 4.2 which describes the possible single and double scattering paths in the forward and backward directions for the case of two identical scatterers in a finite slab. If we consider single forward scattering then we see that the paths 1-a-3 and 2-b-4 will have the same phase irrespective of the position of the scatterers and therefore will
Figure 4.2: Possible single and double scattering paths for a simple case of two scatterers.

interfere constructively. On the other hand the paths 1-a-1 and 2-b-2, i.e. single scattering in the backward direction, will have different phases that will change as the scatterers move randomly and the interference will wash-out. The situation is reversed for the case of multiple scattering paths. If we consider the paths 1-a-b-4 and 2-b-a-3 we obtain that the phases are different and changing randomly. For the backward case, on the other hand, the paths 1-a-b-2 and 2-b-a-1 have the same phase and will interfere constructively, irrespective of the scatterers positions. These are the time reversed paths which contribute to the enhanced back scattering. It should be noted that when the scatterers are not identical the phase difference will not, generally, vanish in the forward direction.
4.2.1 Physical optics description of EBS

In order to explore quantitatively the effects of a phase vortex on the EBS cone, let us first examine the effects of a general non-uniform field distribution incident on the surface. Assuming that both the first and the last scattering events occur at the surface, the contribution of a point on the surface \( r_m \) to the scattered field at some other point \( r_n \) can be written as the sum

\[
E_s(r_n) = E_i(r_m) \sum_s A_s(r_n, r_m) = E_i(r_m) \cdot C(r_n, r_m),
\]

where \( E_i \) and \( E_s \) stand for the incident and scattered fields respectively, \( r_n \) and \( r_m \) are two arbitrary points on the surface and \( A_s(r_n, r_m) \) is the complex scattering amplitude associated with the propagation along a scattering path \( s \). The sum is performed over all possible paths and \( C(r_n, r_m) \) represents the sum of all possible scattering amplitudes. We also consider that scattering paths are uncorrected and therefore the only contribution that survives the ensemble averaging over the realizations of the random medium is the time reversible term. We also assume that the field is not normally incident but rather has a non-zero transverse incident wavevector \( k_i \perp \). The contribution of the time reversed paths to the far-field distribution of the field in terms of the transverse component of the scattered wave vector \( k_f \perp \) is
\[ E_f (\mathbf{k_s}) = [E_i (\mathbf{r}_n) + E_i (\mathbf{r}_m) e^{i \mathbf{k}_\perp \cdot (\mathbf{r}_m - \mathbf{r}_n)}] C(\mathbf{r}_n, \mathbf{r}_m) \]

\[ = [E_i (\mathbf{r}_n) + E_i (\mathbf{r}_m) e^{i (\mathbf{k}_f + \mathbf{k}_\perp) \cdot (\mathbf{r}_m - \mathbf{r}_n)}] C(\mathbf{r}_n, \mathbf{r}_m). \quad (4.2) \]

In terms of the momentum transfer vector, \( \mathbf{q} = (\mathbf{k}_f + \mathbf{k}_\perp) = \hat{\mathbf{e}} k (\sin \theta_i - \sin \theta_f) \), this can be further written as

\[ E_f (\mathbf{q}) = [E_i (\mathbf{r}_n) + E_i (\mathbf{r}_m) e^{i \mathbf{q} \cdot (\mathbf{r}_m - \mathbf{r}_n)}] C(\mathbf{r}_n, \mathbf{r}_m), \quad (4.3) \]

where \( k \) is the wavenumber, \( \theta_i, \theta_f \) are the incident and scattering angles respectively and \( \hat{\mathbf{e}} \) is a unit vector laying in plane defined by the surface of the medium. The direction dependent component of the intensity distribution can be obtained by multiplying the field in Eq. 4.3 with its complex conjugate. Dropping some multiplicative factors one obtains

\[ I_f (\mathbf{q})_{n,m} \propto \text{Re} \left\{ [E_i (\mathbf{r}_n) E_i^* (\mathbf{r}_m) e^{i \mathbf{q} \cdot (\mathbf{r}_m - \mathbf{r}_n)}] P(\mathbf{r}_n - \mathbf{r}_m) \right\} \]

\[ \propto \text{Re} \left\{ [E_i (\mathbf{r} + \Delta \mathbf{r}) E_i^* (\mathbf{r}) e^{i \mathbf{q} \cdot \Delta \mathbf{r}}] P(\Delta \mathbf{r}) \right\}. \quad (4.4) \]

In Eq. 4.4 we have invoked the homogeneity of the medium and have replaced \( |C(\mathbf{r}_n, \mathbf{r}_m)|^2 \) by \( P(\mathbf{r}_n - \mathbf{r}_m) \) which can be interpreted as the probability of a photon incident the surface at an infinitesimal area around \( \mathbf{r}_m \) to emerge at an infinitesimal area around \( \mathbf{r}_n \). In the case of an isotropic medium this will be eventually reduced to \( P(|\Delta \mathbf{r}| = |\mathbf{r}_n - \mathbf{r}_m|) \). In the second row of Eq. 4.4 we have also performed the change of variables \( \mathbf{r}_m \rightarrow \mathbf{r} \) and \( \mathbf{r}_n \rightarrow \mathbf{r} + \Delta \mathbf{r} \).
Integrating over both $r$ and $\Delta r$ and then normalizing by the background intensity one obtains the following formula describing the shape for the normalized EBS cone

$$
\hat{I}_f(q) = \frac{\text{Re} \left\{ \int d^2 \Delta r P(\Delta r) \ e^{i \cdot q \cdot \Delta r} \int d^2 r \ E_i(r + \Delta r) E_i^*(r) \right\}}{\int d^2 r \ E_i(r) E_i^*(r)}. \quad (4.5)
$$

If the incident field is uniform then the inner integral in the nominator can be factored and it cancels out with the integral in the denominator. As it is well known, the EBS cone shape will then become proportional to the Fourier cosine transform of $P(\Delta r)$[98]. In the case of a partially coherent field an ensemble average over the term $E_i(r + \Delta r) E_i^*(r)$ will produce the field correlation function which will act to modify the EBS cone shape [93].

In a case of a deterministic but non-uniform field the term in the curly brackets can be regarded as the Fourier transform of $P(\Delta r)$ multiplied by the field spatial correlation function which will act as a filter in the $\Delta r$ domain, attenuating the contributions of large separations. Let us assume for now that he correlation function has only a radial dependence which will be designated by $\Gamma(\Delta r)$. Using again the basic properties of the Fourier transform, the expression in Eq. 4.5 becomes

$$
\hat{I}_f(q) \propto \text{Re} \left\{ \tilde{P}(q) \otimes \tilde{\Gamma}(q) \right\}, \quad (4.6)
$$

where the notation $\tilde{P}(q)$ denotes the 2D Fourier transform of $P(\Delta r)$ and $\otimes$ denotes the convolution operation. $\tilde{\Gamma}(q)$ is the angular power spectrum of the illumination field. It is evident that the effect of the field’s non uniformity is to smooth out the shape of the EBS
cone. The details of the convolution depend on the characteristics of the complex amplitude
distribution of the illumination field as well as on the transverse scattering probability.

4.2.2 EBS from diffusive media

The shape of the EBS cone in the case of reflection from a diffusive semi-infinite media was
evaluated taking into account the random walk of photon scattering in random media [98].
The evaluation was originally performed assuming the stationary solution of the diffusion
equation in the no absorption case and taking into account the fluence contribution (Eq.
2.34) as the transverse scattering probability distribution. The obtained expression was

\[
\tilde{I}_f(q) = \frac{3}{8\pi} \left\{ 1 + \frac{2z_e}{l_t} + \frac{1}{(1 + q l_t)^2} \left[ 1 + \frac{1 - \exp(-2q z_e)}{q l_t} \right] \right\}. 
\]

It should be noted that the expression was derived for the isotropic scattering case \((g = 0)\)
and \(l_s\) was replaced by \(l_t\) based on heuristic arguments. Over the years this approximate
expression was found to represent quite well experimental results and was modified to take
into account absorption, boundary conditions, finite slab size and more [111]-[113],[89]. It
was found that the FWHM of the EBS cone can be approximated quite well by [114]

\[
\theta_{fwhm} \simeq \frac{0.7}{k l_t} (1 - R_{eff}). \tag{4.7}
\]
4.2.3 Polarization aspects of EBS

Previous sections of this Chapter dealt with EBS from the scalar point of view, i.e. neglecting the vector nature of the EM radiation. This is, of course, only an approximation. The effects of polarization, or rather depolarization, associated with propagation in random media, manifest themselves in EBS as well. In order to explore the polarization effects it is beneficial to consider again geometrical phase effects in enhance backscattering. If on a non-planar 3D trajectory in the scattering media the geometric phase accumulated is \( \Omega \), then on the reversed path the geometric phase is \( 4\pi - \Omega \). One can observe that the geometrical phase, unlike the dynamic phase, is not in general reversible! As was originally pointed out by Freund[117] if the geometrical phase is an odd multiple of \( \pi/2 \) the beams will interfere destructively. When one considers illumination by linear polarization this case corresponds to rotation of the field vector by \( \pi/2 \) in opposite directions- leading to a \( \pi \) phase difference between the two waves. On the other hand, it can be shown from geometrical reasoning, that for a planar trajectory the geometric phase for the revered trajectory is the same (up to a \( 2\pi \) phase difference which is unmeasurable). Therefore we can conclude that planar trajectories contribute preferentially to the EBS effect. According to this interpretation, the fact that the enhancement factor measured in many cases is very close to the theoretical value of two is an indication that in exact backscattering planar trajectories are overwhelmingly dominant.

One can therefore consider the polarization patterns of backscattered light discussed in Chapter 3. Those patterns are not axially symmetric and therefore it is not appropriate
to apply Hankel transform. For instance, in the case of a medium of Rayleigh scatterers illuminated by linear polarized light, we saw that the backscattered patterns can be described by an overlay of an isotropic unpolarized distribution and a polarized pattern (see Fig. 3.18). The polarized pattern has a two fold "bow tie" like shape in the co-polarized channel and a less intense four fold pattern in the cross-polarized channel. When one considers the Fourier transform of those patterns (in order to qualitatively describe the contributions to the enhanced backscattering) it is clear that the peak in the cross polarized channel is weaker and more symmetric, while in the co-polarized channel it is anisotropic: it is narrower in a direction perpendicular to the polarization as indeed it is found in many measurements (see for example [28],[88]). For large particles, the co- and cross-linear polarized patterns are more similar (in symmetry and magnitude) and therefore the EBS cones are also similar.

### 4.3 SINGULAR BEAMS

The electric field of an electromagnetic wave at some transverse plane can be described as the vectorial complex quantity $\mathbf{A}(x,y) \exp[i\phi(x,y)]$, involving amplitude and phase. At the locations were the amplitude goes to zero the phase is undefined as it can assume any value. These points are called wave dislocations and there study has emerged into a new area of optics called singular optics [13]-[16]. As a phase singularity is associated with an amplitude (or intensity) null, such a null will emerge upon propagation, due to destructive interference
effects, if a singularity is somehow imposed on a field. In Chapter 3 it was also demonstrated that backscattered light in turbid media acquires geometrical phase singularities.

4.3.1 Optical vortices-general description

Optical singularities were found to emerge naturally in speckle fields which are generated by scattering in random media [17]. Specifically those singularity of a type called screw phase dislocations or vortex fields. These singularities are characterized by an azimuthally varying phase which upon integration of over any close curve around the singularity yields an non-zero integer multiple of $2\pi$ which is known as the topological charge (non-integer topological charges are possible as well, yielding the so-called mixed screw-edge dislocations). Beam containing vortices arise also from well ordered systems, the best example being the Lagguerre - Gaussian (LG) modes of the axis symmetric laser resonators [18]. Let us consider radially symmetric vortex fields which have the general form

$$E(r) = u_m(r) e^{im \phi} \hat{E},$$

(4.8)

where $m$ is the topological charge of the vortex and $r$ and $\phi$ are the cylindrical coordinates. The amplitude distribution $u_m(r)$ will be generally dependent on the charge $m$ and will exhibit a null at $r = 0$ due to interference effects upon propagation. $\hat{E}$ is a polarization unit vector. If the phase vortex is imposed on the source, the phase vortex distribution might exist without a null (the so-called point singularity). For simplicity, we have suppressed the
Figure 4.3: Helical wavefront which characterize an optical vortex.

spectral dependence and ignored any phase distribution other then the vortex. The exact form of the amplitude function may depend on the nature of the source, the topological charge, the optical system and propagation distance. The phase front of such wave exhibits a helical structure. The beam, of course does not propagate as a helix, but the k-vector rotates about the direction of propagation and therefore does not have any component in beam direction. An example of a helical wave front which characterize the optical vortex is shown in Fig. 4.3.

The rotation of the k-vector implies the existence of orbital angular momentum and indeed these type of beams were found to carry orbital angular momentum which is directly related to the topological charge \([19]-[21]\) as was discussed in Chapter 3. The intensity null
feature was found to be very useful in the area of optical micromanipulation [22], super-resolution fluorescence microscopy [99] and more.

### 4.3.2 Field distributions for propagating vortex beams

A variety of optical methods were explored for generating singular beams [21]. A computer generated hologram can be used to generate a LG mode as a diffraction order when illuminated by a fundamental Gaussian beam (TEM00) [100]. Another method is based on using optical mode convertors, based on combination of cylindrical lenses, to convert Hermite-Gauss (HG) modes to LG modes [101]. The computer generated method was implemented using a Spatial Light Modulator (SLM) [102] which also allows for the dynamic generation and manipulation of multiple singularities [103]. A method based on the direct imprint of a phase vortex was implemented using a spiral phase plate [104] as well as using a SLM [105].

Laguerre-Gauss modes are eigen modes of the Helmholtz equation and can be considered as a natural basis for vortex beams. A LG mode is described by

\[
E_{pm}(r, \phi, z) \propto \left(1 + \frac{z^2}{z_R^2}\right)^{-1/2} \left[\frac{\sqrt{2r}}{w(z)}\right]^m L_p^m \left[\frac{2r^2}{w^2(z)}\right] \exp \left[-\frac{r^2}{w^2(z)}\right] \\
\times \exp \left[-\frac{i k r^2}{2R(z)}\right] \exp (-im\phi) \\
\times \exp \left[i(2p + m + 1)\tan^{-1}\left(\frac{z}{z_R}\right)\right],
\]

(4.9)
where \( z_0 = \frac{k w_0^2}{2} \), \( w(z) = w_0 \sqrt{1 + \left( \frac{z}{z_0} \right)^2} \), \( R(z) = z \left( 1 + \left( \frac{z}{z_0} \right)^2 \right) \) and \( L_m^p(x) \) are the associated Laguerre polynomials [106].

The direct imprint methods does not produce a pure LG mode but a mixture of such modes in the far-field [104]. Analytical solutions for the field distribution, at any range, were calculated [107] and will be discussed shortly. It should be noted that the analysis of optical fields with singularities and optimization of the methods to generate them is a matter of on going current research. We are considering the field which is the result of near field propagation from a point vortex, that is a phase vortex which is imprinted on Gaussian mode

\[
E(r) \propto \exp\left( -\frac{r^2}{w_0^2} \right) e^{i m \phi},
\]

where \( w_0 \) is the waist of the beam. An expression for the field in the general case can be derived and rewriting Eq.4.8 as[107]

\[
E(r,z) = A_t(r,z) e^{i \Phi(r,z)} e^{i m \phi},
\]

it was obtained that

\[
A_t(r,z) = \frac{\sqrt{\pi}}{2} \left( \frac{r}{w_0} \right) \left( \frac{z_0}{z} \right)^{5/4} \left( \frac{z_0}{R(z)} \right)^{3/4} \exp \left[ -\frac{r^2}{w_0^2(z)} \right] \left[ I_{l-\frac{1}{2}}(\gamma) - I_{l+\frac{1}{2}}(\gamma) \right],
\]

\[
\Phi(r,z) = -l\frac{\pi}{2} + \frac{\pi}{4} + \left( \frac{r}{w_0} \right)^2 \frac{z_0}{z} \left[ 1 - \frac{w_0^2}{2w^2(z)} \right] - \frac{3}{2} \tan^{-1} \left( \frac{z}{z_0} \right),
\]

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where $\gamma = (1/2) \left[ r/w(z) \right]^2 (1 + iz_0/z)$. The amplitude function is $A_m(r,z)$ complex so the phase is not completely described by the function $\Phi(r,z)$. In the near field ($z \ll z_0$) we can approximate $\gamma \approx (i/2) \left[ r/w(z) \right]^2 (z_0/z)$ and find that

$$A_m(r,z) = \frac{\sqrt{\pi} i^{-(l-1)/2}}{2} \frac{r}{w_0} \left( \frac{z_0}{z} \right)^{1/2} \exp \left[ -\frac{r^2}{w_0^2} \right] \left\{ -\frac{r^2}{w_0^2} \right\} \left\{ J_{l-1} \left[ \frac{1}{2} \left( \frac{r}{w(z)} \right)^2 \left( \frac{z_0}{z} \right) \right] \right\} -$$

$$i J_{l+1} \left[ \left( \frac{1}{2} \right) \left( \frac{r}{w(z)} \right)^2 \left( \frac{z_0}{z} \right) \right] \right\} ;$$

$$\Phi(r,z) = -\frac{i \pi}{2} + \frac{\pi}{4} + \frac{1}{2} \left( \frac{r}{w_0} \right)^2 \frac{z_0}{z} - \frac{3}{2} \tan^{-1} \left( \frac{z}{z_0} \right) , \quad (4.13)$$

Those expressions can be reduced to a separable form in which the amplitude is real and in the phase function we omitted the global terms

$$A_m(r,z) = \frac{\sqrt{\pi}}{2} \left( \frac{r}{w_0} \right) \left( \frac{z_0}{z} \right)^{1/2} \exp \left[ -\frac{r^2}{w_0^2} \right] \left\{ J_{m+1} \left[ \frac{1}{2} \left( \frac{r}{w(z)} \right)^2 \left( \frac{z_0}{z} \right) \right] \right\}^2 +$$

$$\left\{ J_{m+1} \left[ \left( \frac{1}{2} \right) \left( \frac{r}{w(z)} \right)^2 \left( \frac{z_0}{z} \right) \right] \right\}^{1/2} ;$$

$$\Phi(r,z) = \frac{1}{2} \left( \frac{r}{w_0} \right)^2 \frac{z_0}{z} - \tan^{-1} \left\{ \frac{J_{m+1} \left[ \left( \frac{1}{2} \right) \left( \frac{r}{w(z)} \right)^2 \left( \frac{z_0}{z} \right) \right] }{J_{m+1} \left[ \left( \frac{1}{2} \right) \left( \frac{r}{w(z)} \right)^2 \left( \frac{z_0}{z} \right) \right] } \right\} . \quad (4.14)$$

An example of the results for the intensity and the phase for a topological charge equal to one and six and for $z/z_0 \approx 0.035$ are shown in Fig. 4.4. As can be seen, most of
the power removed from the inner core is concentrated into the first ring and just some minor modulations are evident beyond that (for comparison, the original Gaussian beam is also plotted). For higher topological charges, the core is larger and the first ring is more intense. Looking at the relative power content up to $r = w_0$, we find that it remains about $0.86 \pm 0.04$ up to a topological charge of 10. As for the phase, one can observe some radial phase dependence (which becomes more pronounced for higher charges) but most of the phase change occurs at the core region which has small intensity and therefore the effects should be minimal.

When one considers an experimental implementation of generating a singular beam using a SLM, some practical considerations have to be addressed. These include finite resolution, phase quantization, phase mismatch and noise (inhomogeneity of the SLM array) etc. In order to analyze these effects, a propagation code was developed. This code is based on generating the appropriate field distribution, imposing degrading conditions, and then propagating the beam using FFT methods (the field is transformed, multiplied by the scalar free space transfer function [108], and transformed back). Fig. 4.5 demonstrates two simulated intensity patterns for the case of a wavelength equal to 488 nm and waist equal to 4 mm. A topological charge equal to one was imprinted and the beam propagated for 0.5 m. The ideal situation is seen at Fig. 4.5a and it can be seen that the numerical results correspond well with the analytical solution. In Fig 4.5b a phase discontinuity of $0.2\pi$ was applied and as well as a normally distributed phase noise with a standard deviation of $0.02\pi$. The propagation code was used to study the effects of phase perturbations on the EBS results and
Figure 4.4: Near field intensity and phase distributions for a source having topological charges $m = 1$ (solid lines) and $m = 6$ (dashed lines) point vortices. The intensity of the basic Gaussian beam is plotted for comparison. The beam width, $w_0$, is 1.5 mm and the relative distance, $z/z_0$, is 0.035.
Figure 4.5: Two simulated intensity patterns for the case of a wavelength equal to 488 nm and waist equal to 4 mm. A topological charge equal to one was imprinted and the beam propagated for 0.5 m. (a) corresponds to an ideal situation and the numerical results correspond well with the analytical solution. (b) A phase discontinuity of $0.2\pi$ was applied and a normally distributed phase noise with a standard deviation of $0.02\pi$ was added.

they were found to be relatively small for the conditions of the experiments described later in this chapter.

In the case of propagation to the far field, or in the Fourier plane of a lens, the field distribution is described by

$$
\tilde{I}_m (q) \propto q^2 \exp \left[ - \frac{q^2 w_0^2}{4} \right] \left| I_{m-1} \left( \frac{q^2 w_0^2}{8} \right) - I_{m+1} \left( \frac{q^2 w_0^2}{8} \right) \right|^2,
$$

where $q$ is the appropriate angular frequency coordinate.
4.3.3 Implementing a phase spiral on a spatial light modulator

The SLM that is used in the experimental setup is a Hamamatsu X8267 PAL-SLM. PAL stands for Parallel aligned Liquid Crystal [109]. In such a device the liquid crystal molecules are aligned perpendicular, when no voltage is applied, to the optical axis with no twisting. An applied voltage will cause the molecules to rotate and align more horizontally along the axis, changing the effective refractive index seen by light with a linear polarization which was initially aligned along the molecules. The changing index will produce a phase shift. Light with a perpendicular linear polarization will practically be unaffected. A schematic description of the principle of operation is shown in Fig. 4.6.

The particular device in use is optically addressed i.e., the voltage is applied by illuminating a layer of amorphous silicon which is coated as a dielectric mirror. The "write light" which originates from a laser diode and is projected thorough a LCD which modulated spatially the intensity falling on the silicon mirror. The voltage across the PAL is proportional to the charge accumulated in the silicon layer, which is directly proportional to the light intensity. The read-out light (which is properly polarized) and is reflected from the silicon mirror. Traversing the PAL layer it acquires a phase which is proportional to the write light intensity. The spatial modulation is high resolution and the optical writing eliminates the diffraction effects of electrodes structure and reduces (but not eliminate) pixilation effects. The general structure is depicted in Fig. 4.7.
Figure 4.6: Operation Principle of a Parallel Aligned Liquid crystal Spatial Light Modulator (PAL-SLM). The molecules are aligned, with no voltage applied, case (a) perpendicular to the optical axis. The two perpendicular polarization experience different refractive indices. As voltage is applied, case (b), the molecules rotate to become more aligned with the axis. The vertical polarization experience a change of index and a phase shift while the horizontal is unaffected.
Figure 4.7: The general structure of the optically addressed PAL-SLM.
Figure 4.8: An example of a phase mask for generating a singular beam with a topological charge equal to 5.

The PAL-SLM allows the projection of phase mask which has a nominal resolution of 768 by 768 pixels over a 20mm by 20 mm active area (effective resolution is 19 lp/mm). A phase change of over $2\pi$ can be obtained for the prescribed wavelength with a modulation dynamic range of 200:1. The device is computer controlled in a simple manner as just another display. Phase masks can be set up as an image in the central part of a 1024x768 display. A Matlab program was written to generate appropriate phase masks of the form $\exp\{i \mod [m \phi, 2\pi]\}$. An example is shown in Fig. 4.8.
Figure 4.9: A phase mask for a "vortex lens"

The program can be also used to generate a "vortex lens" which combines the effects applying a singularity and focusing [110]. An example is presented in Fig. 4.9.

Examples of actual intensity distributions will be presented in the following chapter. The Matlab program can be controlled to generate a sequence of phase masks in an automatically controlled experiment.
4.4 EBS OF SINGULAR BEAMS

4.4.1 Theory

In the case of an illumination field which contains an on-axis singularity it is straightforward to show, following Eq. 4.6 that

$$e_m(q)/Z_0 \propto \left| \int_0^\infty dr \ r \ u_m(r) J_m(qr) \right|^2 \; ,$$  \hspace{1cm} (4.16)

where $J_m$ is the Bessel function of the first kind of order $m$. We can easily recognize the result as being the square of the Hankel transform of order $m$ of the function $u_m(r)$. Of course, this result can also be applied to cases in which the topological charge is zero, such as finite beams of circular cross section, Gaussian beams with finite phase curvatures and others. A general result can be inferred from this expression about the behavior at small $q$ by expanding the Bessel function in the Hankel transform into a power series and retaining the lowest order, i.e.

$$\tilde{T}_m(q) \propto \left| \int dr \ r \ u_m(r) (rq)^m \right|^2 = q^{2m} \left| \int dr \ r^{m+1} u_m(r) \right|^2 \; ,$$  \hspace{1cm} (4.17)

and, as expected $\tilde{T}_m(0) = 0$. One can explore the effects of phase distribution versus the effects of the amplitude distribution of a vortex by comparing the result in Eq. 4.6 to the one obtained by a Hankel transform of order 0 of a amplitude distribution related to a topological charge equal to $m$.

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\[ \tilde{\Gamma}_{m,0}(q) \propto \left| \int_0^\infty dr\, r\, u_m(r)\, J_0(qr) \right|^2. \quad (4.18) \]

In this case the on-axis result is non-zero which implies that the major effect of the phase vortex should be attributed to the phase distribution. Several cases of non zero topological charge can be examined for the field distribution. If we assume an amplitude distribution compatible with the so-called large-core singularity \([115]\), which can be recognized as a LG mode with \(p = m\) (see Eq. 4.9).

\[ u_m(r) \sim r^m \, e^{-\left(\frac{r}{w_0}\right)^2}, \quad (4.19) \]

where \(w_0\) is the width of the field distribution, we find that

\[ \tilde{\Gamma}_m(q) \sim q^{2m} \, e^{-\frac{w_0^2 q^2}{2}}. \quad (4.20) \]

This result is a manifestation of the self reciprocity of functions of the form \(r^m \exp(-r^2/2)\) under a Hankel transform of order \(m\)[116]. The final result will, of course, depend also on \(P(\Delta r)\). We will assume the scattering probability distribution to be a simple single-scale Gaussian of the form

\[ P(\Delta r) \propto e^{-\left(\frac{\Delta r}{r_s}\right)^2}, \quad (4.21) \]

where \(r_s\) represents a characteristic transverse scattering length. This assumption is not over simplifying since it is compatible with the scattering distribution in double pass configura-
tions [85]. In a diffusive medium the multi-scale transverse scattering distribution can be regarded as a superposition of single scale distributions - as will be discussed later. We also note that in the majority of practical cases $r_s << w_0$. In the case of $m = 0$, the expression for the EBS cone shape is simply the convolution of two Gaussian functions which leads to the result

$$
\hat{I}_f(q, m = 0) = \left(1 + \frac{r_s^2}{2 w_0^2}\right)^{-1} e^{-\left(1 + \frac{r_s^2}{2 w_0^2}\right)^{-1} \frac{q^2}{4}}.
$$

In order to evaluate the convolution in Eq. 4.6 for $m > 0$, we employ the explicit form of the convolution integral in Cartesian coordinates

$$
\hat{I}_f(q_x, q_y, m) \propto \iint_{-\infty}^{\infty} (q_x^2 + q_y^2)^m \exp\left[-\frac{w_0^2 (q_x^2 + q_y^2)}{2}\right] \exp\left[-\frac{r_s^2 \left((q_x - q_x')^2 + (q_y - q_y')^2\right)}{4}\right] dq_x' dq_y'
$$

$$
= \exp\left[-\frac{r_s^2 (q_x^2 + q_y^2)}{4}\right] \iint_{-\infty}^{\infty} (q_x^2 + q_y^2)^m \exp\left[-\left(\frac{w_0^2}{2} + \frac{r_s^2}{4}\right) (q_x^2 + q_y^2)\right] \times
$$

$$
\exp\left[\frac{r_s^2 (q_x q_x' + q_y q_y')}{2}\right] dq_x' dq_y'.
$$

Transforming back to polar coordinates we can write

$$
\hat{I}_f(q, \theta, m) \propto \exp\left[-\frac{r_s^2 q^2}{4}\right] \int_{0}^{\infty} dq' q'^{(2m+1)} \exp\left[-\left(\frac{w_0^2}{2} + \frac{r_s^2}{4}\right) q'^2\right] \int_{0}^{2\pi} d\theta' \exp\left[\frac{r_s^2 q' q \cos(\theta' - \theta)}{2}\right].
$$

Identifying the integral over the angular coordinate as the zero order modified Bessel function of the first kind, $I_0$, the integral can then be written as

$$
\hat{I}_f(q, m) \propto \exp\left[-\frac{r_s^2 q^2}{4}\right] \int_{0}^{\infty} dq' q'^{(2m+1)} \exp\left[-\left(\frac{w_0^2}{2} + \frac{r_s^2}{4}\right) q'^2\right] I_0\left(\frac{r_s^2 q' q}{2}\right).
$$
The integration yields

\[ \hat{I}_f(q, m) = \frac{w_0^2}{2} m^{-1} \exp \left[ -\frac{r_s^2 q^2}{4} \right] \left( 1 + \frac{r_s^2}{2 w_0^2} \right)^{-1} \times \left\{ \frac{r_s^4 q^2}{16} (m - 1) \right\} + \]

\[ 1 \text{F}_1 \left[ m; 1; \frac{r_s^4 q^2}{8 w_0^2} \left( 1 + \frac{r_s^2}{2 w_0^2} \right)^{-1} \right] \left[ \frac{r_s^4 q^2}{16} + \frac{m w_0^2}{2} \left( 1 + \frac{r_s^2}{2 w_0^2} \right) \right], \quad (4.26) \]

where \( 1 \text{F}_1 \) is the confluent hypergeometric function of the first kind. The cone shapes for several values of \( w_0/r \) are presented in Fig. 4.10. It is evident that when the intensity width is comparable to the transverse scattering length, a depression is created in the case of high topological charges. This effect can be regarded as a manifestation of anti-localization of light in backscattering [117] as the intensity is decreased in the exact backscattering direction rather than enhanced. This is a feature which is unique to the illumination by a singular beam as compared to other coherent or partially coherent beams which always demonstrate the larger enhancement in the exact backscattering direction.

The trend of the shape changes with increasing topological charge for large values of \( w_0/r_s \) suggests that a simpler expression could be obtained for practical use. Setting \( q \) to 0 and keeping the lowest orders in \( r_s/w_0 \), we can approximate the confluent hypergeometric functions to unity. The result for the peak value is then

\[ \hat{I}_f(0, m) = \left( 1 + \frac{r_s^2}{2 w_0^2} \right)^{-1} \simeq \left( 1 - (m + 1) \frac{r_s^2}{2 w_0^2} \right), \quad (4.27) \]
Figure 4.10: The normalized EBS cone shape as calculated from Eq. 4.26 for \( w_0 / r_s = 1.5, 4, 10, 20 \). The horizontal axis is normalized by half the FWHM of the cone corresponding to the case of a plane wave illumination. The EBS magnitude reduces and the peak broadens when the topological charge is increased from 0 to 10.
showing an approximately linear decrease of the peak value with increasing topological charge.

A field which has the form given in Eq. 4.19 has attractive features because of the shape invariance upon propagation but it is not easy to implement experimentally. Another case is when the field amplitude is a Gaussian without an intensity null - the so called point singularity. This case might be implemented by illuminating a spiral phase mask (projected, for example, by an spatial light modulator) with a laser beam and imaging it onto the surface of the scattering medium. The Hankel transform for this field was evaluated [118] and in our notation the angular power spectrum can be written as

\[ \tilde{I}_m(q) \propto q^2 \exp \left[ -\frac{q^2w_0^2}{4} \right] \left( I_{m-1} \left( \frac{q^2w_0^2}{8} \right) - I_{m+1} \left( \frac{q^2w_0^2}{8} \right) \right)^2, \]  

(4.28)

where \( I_n \) is the modified Bessel function of the first kind of order \( n \). For small \( q \) we can keep the lowest orders of the power series expansion of the Bessel function and obtain that the small \( q \) dependence is \( q^{2m} \) as well. Calculating the convolution in Fourier domain is a bit more involved in this case and as an alternative, we will use an argument for the approximately linear dependence of the peak value on the topological charge. We will assume that the angular power spectrum can be written in the form

\[ \tilde{I}_m(q) \propto \frac{1}{m!} \left( \frac{w^2q^2}{2} \right)^m e^{-\frac{w_0^2q^2}{2}} \times \sum_{j=0}^{\infty} a_j(m) \left( \frac{w^2q^2}{2} \right)^j. \]  

(4.29)
Obviously, $a_j(0) = \delta_{j0}$ where $\delta$ is the Kronecker delta and, as stated above, $a_0(m) > 0$.

The normalization requirement can be written as

$$\frac{1}{m!} \sum_{j=0}^{\infty} a_j(m) \Gamma(m + j/2 + 1) = 1$$  \hspace{1cm} (4.30)

As a result of the convolution one obtains,

$$\tilde{I}_f(0, m) = \left(1 + \frac{r_s^2}{2 w_0^2}\right)^{-\langle m+1 \rangle} \frac{1}{m!} \sum_{j=0}^{\infty} a_j(m) \Gamma\left(m + \frac{j}{2} + 1\right) \left(1 + \frac{r_s^2}{2 w_0^2}\right)^{-j/2}$$ \hspace{1cm} (4.31)

Keeping again the lowest orders in $r_s/w_0$ and using the normalization condition one finds that we should expect a linear dependence of the peak on the topological charge.

### 4.4.2 Experimental setup and results for a double-pass system

In order to test the theoretical and numerical predictions the experimental setup depicted in Fig.4.11 has been used. The double-pass configuration is a good model for a single scale scattering media and allows for a convenient scaling of the transverse scattering length by a longitudinal movement of the mirror behind the rotating phase-screen. We assume that the angular scattering probability of the phase screen is given by a Gaussian function

$$\exp\left(-\frac{\alpha^2}{\sigma_{\alpha}^2}\right)$$

where $\alpha$ is the scattering angle (with respect to the incident beam axis) and $\sigma_{\alpha}$ is a measure of the angular scattering width. Due to the double pass geometry, the transverse scattering length is given by $r_s = 2 d \sigma_{\alpha}$, where $d$ is the distance between the phase screen and the mirror.
Figure 4.11: Experimental setup: LA - Argon ion laser -488 nm (Melles Griot). P- Polarizer, BE- pin hole and beam expander, M1,M2 - mirrors, BS- beam splitters, SLM- Spatial light modulator (Hamamatsu X8267), A - optional aperture, RPS- rotating phase screen. L - lens, CCD- digital CCD camera (Pulnix 1040), BA- Beam analyzer (Spiricon HWA).
The random phase screen used was a textured plastic sheet. The beam propagated for about 50 cm before illuminating the phase screen which means that this is a near field situation. We chose the illumination scheme in this way because high order singularities are sensitive to perturbations and will "brake up" into clusters of first order singularities upon propagation[17]. The perturbations may be the result of an imperfect phase mask imposed by the SLM. However, for a short propagation distance the singularity will still be stable. Some examples of the obtained intensity distributions are presented in Fig. 4.12. The intensity distributions are compatible with the results of Eq.4.14. The polarizers in the setup were aligned to observer the co-polarized backscattering. As the phase screen was rotated, 40 frames were acquired and averaged to remove the speckle noise and the EBS cone shape was analyzed. The generation of the phase masks on the SLM, the image acquisition and the data analysis were preformed automatically using a combination of Matlab and Labview programs. In order to first characterize the setup we compared the results obtained for $m = 0$ with the analytical results (which compare with known results in the literature[85]). A comparison can be made between the experimental measurement and the predicted $1/e$ half-width, $\Delta \theta \approx 2/ (k r_s)$. In the case of our setup where the mirror-phase screen separation is about 3.5 mm, the characteristic transverse scattering length was about 0.22 mm. The beam width was about 1.5 mm which is appreciably larger then the scattering length. The fact that the peak value is not two is most probably due to the SLM residual beam non-uniformity which diminish the beam quality.
Figure 4.12: Intensity distributions for several topological charges. The images were acquired at a distance of 0.5 m (near field). Some structure is visible due to SLM non-uniformity. Intensity gradients are due to residual phase discontinuity.

Let us now examine the results when phase vortices are applied. The results for a separation of 3.5 mm are presented in Fig. 4.13. The decrease and rounding of the peak and the increase in the width are evident. In Fig. 4.14 we show the linear dependence of the normalized enhancement factor on the topological charge for several phase screen - mirror separations. The dependence is indeed linear and the slope is increasing with the distance as expected from Eq. 4.27. For a separation of 3.5 mm, the slope indicates a scattering
Figure 4.13: The effect of applying phase vortices with topological charges ranging from 0 (no vortex) to 6 on the EBS cone shape. The distance between the phase screen and the mirror was 3.5 mm.
Figure 4.14: The linear dependence of the normalized enhancement factor on the topological charge for several distances $d$ between the phase screen and the mirror. The slope of these dependencies is increasing with the distance $d$ as predicted from Eq. 4.27.
length which is larger by a factor of about two then the one obtained from the cone width. This slight discrepancy may be due to the actual scattering probability distribution which might not be Gaussian and to the beam quality. In the specific case of a double-pass system either the distance of the mirror from the phase screen or the phase screen characteristic angular spread could be inferred if the other is known (as the scattering length is given by \( r_s = 2 d \sigma_\alpha \)).

4.4.3 EBS of singular beams in diffusive media

In the case of volume scattering media, within the diffusion approximation, the transverse scattering probability density function can be described as a superposition of multiple single scale contributions. Let us consider the transverse scattering probability of a semi-infinite diffusive media for a specific scattering path length \( s \) which can be written in the form (assuming absorption is negligible) [43]

\[
P(\Delta r, s) = A(l_t) \ Z_B(l_t, s) \ \exp \left( -\frac{3\Delta r^2}{4 \ l_t \ s} \right),
\]

where \( A(l_t) \) is a normalization factor and \( l_t \) is the transport mean free path. In Eq. 4.32, \( Z_B(l_t, s) \) is a function which depends on the refractive index mismatch at the boundary.
where $z_e$ is known as the extrapolation length ratio which is of the order of unity and is determined by the boundary index mismatch \cite{38,39}, $z_0$ is the mean penetration ratio at which it is assumed that the light is isotropically scattered and is taken to be equal to unity. The constants $C_1$ and $C_2$ depend on the index mismatch and on the geometry of the collection system.

Considering the expression in Eq. 4.32 as a function of $\Delta r$, the contribution of a class of paths with a specific length $s$, can be recognized as a Gaussian probability distribution function with a characteristic transverse scattering length scale $r_s = 2 \sqrt{l_t s / 3}$ and with a weighting factor which is given by $A(l_t) \cdot Z_B(l_t, s)$. The overall effect of the phase vortex on the enhancement factor at the exact backscattering direction, $I_f(0, m)$, is obtained by considering the effect of the vortex beam on each Gaussian component (associated with a specific class of scattering path lengths and a certain transverse scattering length scale) by applying Eq. 4.27 and then summing the effects on an intensity basis with the appropriate weighing factor. As a result one obtains the following integral form:

$$
\tilde{I}_f(0, m) \propto \int_{s_{\text{min}}}^{\infty} \left(1 + \frac{2 l_t s}{3 w_0^2}\right)^{-(m+1)} A(l_t) \ Z_B(l_t, s) \ ds.
$$

The choice of the lower limit of integration, $s_{\text{min}}$, merits a brief discussion. It is well known that the diffusion approximation is not valid for path lengths or separation distances which...
are smaller then a few $l$, [40],[41]. In particular the diffusion results predict a significant cumulative probability at path lengths shorter then $\Delta r$, which has no physical meaning. Over the years, several suggestions were made to correct the diffusion expressions which introduce an effective lower limit [55],[42] and compensate for the over-estimation of short path contributions. Here we take a simpler approach and introduce a sharp lower limit to the integral. The sensitivity of the results to this choice will be examined later when the theoretical results are compared to the experimental ones.

### 4.4.4 Experimental setup and results for diffusive media

In order to test the theoretical and numerical predictions fro diffusive media an experimental setup similar to that depicted in Fig.4.11 has been used but the diffuser-mirror combination were replaced by volume scattering samples. The beam, which has a waist of 4 mm, propagated for about 50 cm before illuminating the samples and generated a field distribution which was described in detail in a previous section. The sample was in one case a Spectralon plate (Labsphere, North Sutton, NH) which was mounted on a rotating stage in order to preform ensemble averaging, and in the other case a solution of polystyrene microspheres (Polysciences, Warrington, PA) in a quartz cuvette. The polarizers were aligned such that the detection was in the co-polarized channel. The sample was tilted slightly to remove a much as possible the specular reflection. Images taken by the CCD were averaged over time.
and the peak enhancement was analyzed as a function of the topological charge applied to the beam.

The mean free path of the Spectralon plate can be evaluated from the FWHM of the EBS cone in the absence of any vortex by using the approximation [114]

$$l_t \simeq \frac{0.7}{k \theta_{fwhm}} (1 - R_{eff}),$$  

(4.35)

where $R_{eff}$ is the effective reflectance of the interface that should be calculated by averaging over the Fresnel reflection coefficients weighted by the diffusive angular radiance [38]. Neglecting any reflectance effects (i.e. assuming $R_{eff} = 0$) leads to an estimate of $28 \pm 1 \mu m$ based on the theoretical expression of the line shape [98] (See Fig. 4.15) which is an overestimate. Spectralon is made by compressing a Teflon powder and the effective index at the interface is determined by the distributions of particles and pores and by the micro roughness of the surface. Assuming a bulk refractive index in the range $1.3 - 1.35$ one can further assume an effective index of about 1.2 (this is a reasonable assumption considering results for the effective index of glass frits, compared to the index of the bulk glass [39]). One then obtains an effective reflectance of about 0.29 which leads finally to a value of $20 \pm 1 \mu m$ for the mean transport length.

Experimentally it is usually very difficult to obtain the maximal theoretical enhancement factor of two due to many reasons which include a background of single-scattering events, the finite size of the sample, beam quality and more. In the case of the Spectralon sample the maximal enhancement obtained was about 1.7. In order to explore the effects of the
phase vortex the peak magnitude normalized under the assumption of the existence of a background component which is not involved in the EBS effect (again, due to single scattering events, illumination filed characteristics etc.). Under this assumption the measured peak enhancement, can be written as [119]

$$I_f(0, m) = \frac{1 + 2x(m)}{1 + x(m)} - 1,$$

(4.36)

where $x(m)$ is the ratio of the coherent component to the non-coherent background for a given topological charge. This ratio can be directly derived for each topological charge and normalized to its value at the no-vortex case. After this re-normalization the enhancement factor is compared to theoretical results based on Eq. 4.34, where $Ze$ was calculated to be 0.94, $C_1$ - 0.166 and $C_2$ - 0.4. The comparison is presented in Fig. 4.16. The theoretical results are presented for a minimal path lengths of 5, 10 and 15 transport mean free paths (but those curves are almost overlapping). The experimental results corresponds to a transport free mean path of about 15 $\mu$m which is somewhat less then our previous estimation. The discrepancy at higher topological charges may be attributed to the fact that due to the finite resolution of the SLM, a significant non-diffracted component emerges which illuminates the sample as well.

In the second sample the microspheres diameter was 0.588 $\mu$m, and the solid fraction was 2.6% leading to an expected mean free path, according to Mie scattering theory [1], of 83 $\mu$m. This result was verified by a measurement of the EBS cone width in the case of no vortex. The peak enhancement in this case was 1.3. After a process of normalization the experimental results, which were obtained in the presence of vortices, are presented in
Fig. 4.17 and compared to model predictions. The experimental results, compare quite well with a transport mean free path of 83 $\mu m$. The choice of a minimal path length is more significant in this case with a longer mean free path. The best fit is obtained with a minimal path length of 10 mean free paths. The parameters employed in this case were (taking into account the quartz cuvette) $Z_e$ -1.7, $C_1$ - 0.2 and $C_2$ - 0.6. The discrepancy at large topological charges is, again, probably due to the presence of an undiffracted component in the illumination beam.
Figure 4.15: Measured cross section of the EBS cone in the case of Gaussian beam illuminating a Spectralon sample, fitted to the theoretical expression for a mean free path of 28 \( \mu m \).
Figure 4.16: Normalized enhancement factor for the Spectralon sample. Experimental results are compared to theoretical model predictions with minimal path lengths equal to 5, 10 and 15 transport mean free paths. The theoretical curves almost overlap. The results corresponds to a mean free path of 15 $\mu m$. 
Figure 4.17: Normalized enhancement factor for the Polystirene solution. Experimental results are compared to theoretical model predictions with minimal path lengths equal to 5, 10 and 15 transport mean free paths. The results corresponds to a mean free path of 83 $\mu m$. 

\[ \text{Normalized Enhancement} \]

\[ \text{Topolgical charge} \]

\[ \text{Exp. Results} \]
4.4.5 Depth sensitivity and detection of a reflective inclusion

In the previous section we studied the case of an homogenous semi infinite medium. It is interesting to explore how the introduction of an inclusion modifies the results obtained above. As it was discussed in previous sections, the applications of singularities removes the contribution of large separations on the surface and therefore long scattering paths to the coherent enhancement. Long scattering paths probe more efficiently deeper layers inside the medium [73]-[75] and therefore a singular beam offers a preferential probing of the shallow layers. It will be now explored how the functional dependence of the peak magnitude on the topological charge is varied in the case of an embedded inclusion.

In order to probe the depth sensitivity of the singular EBS technique the following experiment was performed. A reflective needle, 0.5 mm in diameter was inserted vertically in the suspension of polystyrene particles described in the previous section. The needle was positioned along the diameter of the illumination beam (such that it crosses its center) and was located at several depths with regard to the inner wall of the cuvette. These distances were 0.6 mm (in which the needle was barely visible), 1.2 mm (in which the needle was not visible at all), 1.8 mm and 3 mm. It should be noted that these distances correspond to 7, 14, 21 and 36 transport mean free paths. A reference measurement was taken for the case of no needle. The experimental results are summarized in Fig. 4.18 and are overlaid with linear fits.
Figure 4.18: Enhancement factor for several positions of the needle and for the case without a needle.
When examining the results, the first observation one can make is that the enhancement is increased by the presence of the reflective inclusion. This effect [88] which can be attributed to the addition of new reversible scattering paths which can contribute to the EBS cone. These trajectories involve single scattering events which previously could not contribute to EBS and reflection from the inclusion surface. An example of such a path is shown in Fig. 4.19. It is interesting to note that the effect persists for depths which are many tens of the transport mean free path which was 0.083 mm in this experiment.

Figure 4.19: An example of a reversible path involving one scattering event (from the small sphere representing a particle) and a reflection (from the large sphere representing an inclusion).
As the needle is placed deeper into the medium, the effect diminishes. This is evident in the magnitude but also in the slope of the linear dependencies. The slopes associated with the linear fits to the experimental data are presented in Fig. 4.20. It should be noted that the last point (10 mm) actually represents the case of no needle.

![Graph showing the dependence of slope on depth](image)

Figure 4.20: Dependence of slope of linear fits on the depth of needle. The last point (10 mm) represents the case without needle.

The fact that the slope is smaller for shallower depth is an indication of the increased relative contributions of short paths. As the needle is set deeper this relative increase diminishes and the response is similar to that of an inclusion free bulk. It is interesting to
note that the deviation of the slope is still significant for a depth of 3 mm in which the enhancement curves themselves almost overlap.

4.5 Conclusions

In conclusion, it was demonstrated theoretically that phase vortices affect the phenomenon of enhanced coherent backscattering by modifying the spatial correlations of the illumination field. It was shown both theoretically and experimentally that this effect can be used to determine the scattering properties of both double-path configurations and turbid media. In the case of turbid media, the effect can be explained within the diffusion approximation when one correctly accounts for the over-estimation of small path length contributions. The effects of boundary conditions are considered in the context of the diffusion approximation, but polarization effects were not considered at this time. The influence of absorption should be examined in future work as well.

The overestimation of short path lengths was corrected by setting a lower limit to the path lengths distribution. This limit was found to be of the order of 10 transport mean free paths. The experimental results can be used to determine the value of the scattering mean free path.

It is worth mentioning that EBS measurements are usually quite demanding since the effects of single scattering, spurious reflection, beam quality and more have to be eliminated or
accounted for. A more robust technique can be developed based on the approach developed in this work. This new technique is "self-referenced" in the sense that the measurements for sequentially increasing topological charges are normalized to the case of no-vortex illumination. As a result, all other effects which contribute to the reduction of the enhancement cone are effectively canceled. Another notable feature of this approach is that the information about the scattering properties, i.e. the transport mean free path, is obtained in a robust geometry with no moving parts and without using angular resolved measurements. The evaluation procedure is based only changes introduced by the SLM which is electronically controlled and on on-axis far field measurement.
CHAPTER 5

COHERENT MODES COUPLING APPROACH TO PROPAGATION IN TURBULENCE

5.1 introduction

In the previous chapters we discussed effects which relates to scattering in strongly inhomogeneous media. In a weakly inhomogeneous medium, which we also assume to be optically isotropic, the paraxial approximation still holds and the spin and OAM of propagating light beams are considered to be uncoupled. The polarization is not modified by propagation, but the OAM content of the beam may change due to the fact that for different realizations the distribution of the scattering potential is not axially symmetric. This redistribution can be studied by following the power coupling between different modes, which carry different OAM.

An important case of propagation in a weakly inhomogeneous medium is the laser beam propagation in atmospheric turbulence. Understanding how laser beams propagate along atmospheric paths is significant for many applications, including remote sensing and optical
communications. Propagation of laser beams, treated as emanating from completely coherent sources, was studied extensively and several comprehensive reviews and books were published on the subject [8],[9]. Recently, attention has also been given to the more general case of propagation of partially coherent beams (PCB) generated by partially coherent sources [122],[123],[124],[125],[126].

In this work we will approach the subject by using the method of coherent modes decomposition of the cross-spectral density function of PCB. This approach was used before [126] in treating the spreading of PCB, and, using some approximations for the power spectrum of the index of refraction fluctuations, analytical results were obtained for the spreading of each mode. The spreading of the total beam can then be obtained by a superposition of the individual modes. The approach suggested in this work examines the perturbational effects of turbulence on power coupling between free space propagating modes, a coupling that does not occur in free space. This model might be useful for numerical simulations of beam propagation, beam quality assessments, intensity distribution optimization and others. It will be demonstrated that in turbulence the modal decomposition of the cross-spectral density can be represented in terms of free space modes which exchange power and get correlated upon propagation. In this work we will concentrate on the effects of power coupling. The change in the decomposition is obtained by using mode coupling coefficients which are calculated numerically using the structure function of the wavefronts passing through turbulence. We will demonstrate this procedure by applying it to approximate calculations of intensity distribution which are compared to results obtained by other methods.
A partially coherent electromagnetic beam can be described using the cross-spectral density matrix defined by \([127]\)

\[
\overline{W} = W_{ij}(r_1, r_2, \omega) = \langle E_i^*(r_1, \omega) E_j(r_2, \omega) \rangle, \tag{5.1}
\]

where \(i\) and \(j\) stand for the Cartesian coordinates (the definition can be in fact used with any two orthogonal polarization vectors, such as the circular basis, or any two orthogonal elliptic polarization vectors), \(E_{i,j}\) is the fluctuating electric field component, \(r_1\) and \(r_2\) are position vectors of two points in space, \(\omega\) is the angular frequency of the optical wave, the asterisk denotes complex conjugate and the angular brackets denote statistical ensemble averaging. Assuming ergodicity this ensemble average is equivalent to a long time (with respect to correlation time scales) averaging. The cross-spectral density matrix, offers a comprehensive description of the spectral, coherence and polarization properties of optical fields. If the application is limited to a scalar beam, one can simplify the formalism to that of the cross-spectral density function which is given by

\[
W(r_1, r_2, \omega) = \langle E^*(r_1, \omega) E(r_2, \omega) \rangle. \tag{5.2}
\]

It was shown that under very general conditions, the cross-spectral density function describing a partially coherent optical field in the source plane located at \(z = 0\) (which from
now on will be referred to as the sources cross-spectral density) can be represented as a coherent modes decomposition given by the following Mercer’s expansion \cite{128,129,130}

\[ W(\mathbf{\rho}_1', \mathbf{\rho}_2', 0, \omega) = \sum \lambda_l(\omega) \phi^*_l(\mathbf{\rho}_1', \omega) \phi_l(\mathbf{\rho}_2', \omega) , \]

(5.3)

\( \mathbf{\rho}_1' \) and \( \mathbf{\rho}_2' \) being the position vectors of two points in the source plane (see Fig. 5.1 which explains the notation used). The source modes \( \phi_l(\mathbf{\rho}', \omega) \) are an infinite set of statistically uncorrelated and spatially orthonormal functions which obey the following integral equation\cite{128}.

\[ \int \int W(\mathbf{\rho}_1', \mathbf{\rho}_2', 0, \omega) \phi_l(\mathbf{\rho}_1', \omega) \ d^2\mathbf{\rho}_1' = \lambda_l(\omega) \phi_l(\mathbf{\rho}_2', \omega) . \]

(5.4)

The associated eigenvalues, \( \lambda_l(\omega) \), constitute an infinite set of real, non-negative coefficients that are generally dependent on frequency. The spectral density distribution (or intensity) can then be calculated as

\[ S(\mathbf{\rho}', \omega) = W(\mathbf{\rho}', \mathbf{\rho}', 0, \omega) = \sum \lambda_l(\omega) \phi^*_l(\mathbf{\rho}', \omega) \phi_l(\mathbf{\rho}', \omega) . \]

(5.5)

Noting this result, one can interpret the expansion coefficients \( \lambda_l(\omega) \) as the power carried in each mode.

It has been shown that the propagation of the cross-spectral density function in free space can be described by [130]
Figure 5.1: Illustration of the notation used in the propagation description.

\[
W(\rho_1, \rho_2, z, \omega) = \int \int W(\rho'_1, \rho'_2, 0, \omega) \ K_0(\rho_1 - \rho'_1, \rho_2 - \rho'_2, z, \omega) \ d^2\rho'_1 d^2\rho'_2, \quad (5.6)
\]

where \( K_0(\rho_1 - \rho'_1, \rho_2 - \rho'_2, z, \omega) \) is the free space propagator defined by

\[
K_0(\rho_1 - \rho'_1, \rho_2 - \rho'_2, z, \omega) = G_0^*(\rho_1 - \rho'_1, z, \omega) \ G_0(\rho_2 - \rho'_2, z, \omega), \quad (5.7)
\]
in terms of Green’s function, $G_0 (\rho - \rho')$, for the propagation from a point $(\rho', 0)$ in the source plane to a point $(\rho, z)$ in the half space $z > 0$. Under the paraxial approximation, the Green’s function is

$$G_0 (\rho - \rho', z, \omega) = -\frac{ik}{2\pi z} \exp \left\{ i k |\rho - \rho'|^2 \right\}, \quad (5.8)$$

where $k = \omega/c$ is the wave number.

It has been shown [130], that the cross-spectral density at any point in a plane $z > 0$ can be written as a Mercer’s expansion of propagating coherent modes in the form

$$W (\rho_1, \rho_2, z, \omega) = \sum_l \lambda_l (\omega) \psi_l^* (\rho_1, z, \omega) \psi_l (\rho_2, z, \omega), \quad (5.9)$$

where the expansion coefficients maintain their values in the source plane. The propagating modes, $\psi_l (\rho, \omega)$, are uncorrelated and constitute an orthonormal set under the paraxial approximation [131]. The orthonormality condition is met, in fact, in any case for which the contribution of evanescent waves can be neglected. If the source modes are a complete set (for example Hermite-Gauss or Laguerre-Gauss modes) then the propagating modes will also constitute a complete set for any plane $z > 0$. This is a consequence of the fact that the propagation of an homogenous field (i.e. a field that does not contain evanescent waves) between two parallel planes can be represented by a unitary transformation [132]. In the following we will also assume that the propagating modes are, indeed, a complete set which can be used to describe any beam-like field distribution.
Using the orthonormality conditions of the coherent modes we can readily obtain that
the propagator can also be expanded in terms of coherent modes:

\[
K_0 (\mathbf{p}_1 - \mathbf{p}'_1, \mathbf{p}_2 - \mathbf{p}'_2, z, \omega) = \sum_{n,m} \phi_n (\mathbf{p}'_1, \omega) \phi^*_m (\mathbf{p}'_2, \omega) \psi_n (\mathbf{p}_1, z, \omega) \psi^*_m (\mathbf{p}_2, z, \omega).
\]

This result can be verified by substituting it the r.h.s. of Eq. 5.6, using the Mercer’s expansion for the source cross-spectral density and applying the orthonormality conditions. This result will be used in the following section.

5.3 Propagation in Atmospheric Turbulence

In atmospheric turbulence the propagation of the cross-spectral density, \(W_{AT} (\mathbf{p}_1, \mathbf{p}_2, z, \omega)\), is described by the integral equation,

\[
W_{AT} (\mathbf{p}_1, \mathbf{p}_2, z, \omega) = \int \int W (\mathbf{p}'_1, \mathbf{p}'_2, 0, \omega) K_{AT} (\mathbf{p}_1, \mathbf{p}'_1, \mathbf{p}_2, \mathbf{p}'_2, z, \omega) \ d^2 \mathbf{p}'_1 \ d^2 \mathbf{p}'_2.
\]

where \(K_{AT} (\mathbf{p}_1 - \mathbf{p}'_1, \mathbf{p}_2 - \mathbf{p}'_2, z, \omega)\) is the appropriate propagator associated with atmospheric turbulence. Recently, it has been shown [122],[123], that this takes the form (with some change of notation):
\[ K_{AT}(\mathbf{\rho}_1, \mathbf{\rho}_1', \mathbf{\rho}_2, \mathbf{\rho}_2', z, \omega) = \langle G^n_{AT}(\mathbf{\rho}_1-\mathbf{\rho}_1', z, \omega) G^n_{AT}(\mathbf{\rho}_2-\mathbf{\rho}_2', z, \omega) \rangle \]

\[ = G^n_0(\mathbf{\rho}_1-\mathbf{\rho}_1', z, \omega) G_0(\mathbf{\rho}_2-\mathbf{\rho}_2', z, \omega) \langle \exp [\psi^*(\mathbf{\rho}_1-\mathbf{\rho}_1', z, \omega) + \psi(\mathbf{\rho}_2-\mathbf{\rho}_2', z, \omega)] \rangle \]  

(5.12)

\[ = K_0(\mathbf{\rho}_1-\mathbf{\rho}_1', \mathbf{\rho}_2-\mathbf{\rho}_2', z, \omega) \exp \left[ -\frac{1}{2} D_{sp}(\mathbf{\rho}_1-\mathbf{\rho}_2, \mathbf{\rho}_1'-\mathbf{\rho}_2', z, \omega) \right], \]

where \( G_{AT}(\mathbf{\rho}-\mathbf{\rho}', z, \omega) \) is the Green’s function for propagation in one realization of the turbulence:

\[ G_{AT}(\mathbf{\rho}-\mathbf{\rho}', z, \omega) = G_0(\mathbf{\rho}-\mathbf{\rho}', z, \omega) \exp \{ \Psi(\mathbf{\rho}-\mathbf{\rho}', z, \omega) \}. \]  

(5.13)

In Eq. (5.13), \( \Psi(\mathbf{\rho}-\mathbf{\rho}', z, \omega) \) is the phase acquired due to atmospheric turbulence upon propagation from point \( \mathbf{\rho}' \) in the source plane to point \( \mathbf{\rho} \) in the observation plane and the angular brackets denote ensemble averaging. The phase term is generally complex and reflects both the log-amplitude variation and pure phase. The final result in Eq.(5.12) is obtained under the assumption that the phase is a Gaussian distributed random variable \([8],[9]\). \( D_{sp}(\mathbf{\rho}_1-\mathbf{\rho}_2, \mathbf{\rho}_1'-\mathbf{\rho}_2', z, \omega) \) is the \textit{two-point spherical wave structure function} which is defined by \([9]\)

\[ D_{sp}(\mathbf{\rho}_1-\mathbf{\rho}_2, \mathbf{\rho}_1'-\mathbf{\rho}_2', z, \omega) = 8\pi^2 k^2 z \int_0^\infty \int_0^\infty \Phi_n(\kappa) \left\{ 1 - J_0 \left[ (1 - \xi) (\mathbf{\rho}_1-\mathbf{\rho}_2) + \xi (\mathbf{\rho}_1'-\mathbf{\rho}_2') \right] \kappa \right\} d\kappa d\xi. \]

(5.14)
Here $\Phi_n(\kappa)$ is the 3-D power spectrum of the index of refraction fluctuations, $\kappa$ is the spatial frequency and $J_0$ is the zero order Bessel function of the first kind. Since we are considering here propagation along horizontal paths the power spectrum of the index of refraction fluctuations does not change with the distance. The two-point spherical wave structure function is a measure of the phase variation between two paths starting on two points in the source plane separated by vector $\rho'_1 - \rho'_2$ and ending on two respective points in the observation plane separated by vector $\rho_1 - \rho_2$. We rewrite Eq. (5.12) as a sum of the free-space propagator and turbulence induced perturbation:

$$K_{AT}(\rho_1 - \rho'_1, \rho_2 - \rho'_2, z, \omega) = K_0(\rho_1 - \rho'_1, \rho_2 - \rho'_2, z, \omega)$$

$$- K_0(\rho_1 - \rho'_1, \rho_2 - \rho'_2, z, \omega) \left\{ 1 - \exp \left[ -\frac{1}{2} D_{sp}(\rho_1 - \rho_2, \rho'_1 - \rho'_2, z, \omega) \right] \right\}.$$

Finding the propagating modes in atmospheric turbulence is a difficult task because of the complicated integrals involved. Instead, we will use a perturbation approach based on the free-space propagating modes, which, as discussed before, may be used in many practical cases as a complete basis for the description of a paraxial field distribution. We will assume this approach to be valid in the regime of weak turbulence.

Let us now consider the contribution of the $n^{th}$ mode of one realization of the source to the field at an observation plane located at a distance $z$ after propagation through atmospheric turbulence. This contribution will be designated as $\chi_n(\rho, z, \omega)$. This function of $\rho$ can be expanded in terms of the orthonormal set of free space propagating field modes in the form
\[ \chi_n (\rho, z, \omega) = \sum_l b_{nl} (z, \omega) \psi_l (\rho, z, \omega), \quad (5.16) \]

where \( b_{nl} (z, \omega) \) are random variables which depend on the statistical properties of the turbulence. Applying the ensemble averaging over different realizations of the turbulence, the cross-spectral density of the field distribution can now be evaluated as:

\[
W_n (\rho_1, \rho_2, z, \omega) = \langle \chi^*_n (\rho_1, z, \omega) \chi_n (\rho_2, z, \omega) \rangle = \\
= \sum_{l,m} \langle b^*_{nl} (z, \omega) b_{nm} (z, \omega) \psi_l^* (\rho_1, z, \omega) \psi_m (\rho_2, z, \omega) \rangle \\
= \sum_{l,m} \langle b^*_{nl} (z, \omega) b_{nm} (z, \omega) \rangle \psi_l^* (\rho_1, z, \omega) \psi_m (\rho_2, z, \omega). \quad (5.17)
\]

In Eq. (5.17) the cross spectral density is divided into sums of "diagonal" terms (involving modes of the same index) and "mixed" terms (involving modes of different indices). This argument can, of course, be repeated for each source mode. Since the source modes are uncorrelated the cross spectral density of the total field in the plane \( z \) will be the incoherent superposition of the contributions arising from each of the source modes, i.e.
\[
W_{AT}(\rho_1, \rho_2, z, \omega) = \sum_n \sum_l \langle |b_{nl}(z, \omega)|^2 \rangle \psi_l^*(\rho_1, z, \omega) \psi_l(\rho_2, z, \omega) + \sum_n \sum_{l \neq m} \langle b_{nl}^*(z, \omega)b_{nm}(z, \omega) \rangle \psi_l^*(\rho_1, z, \omega) \psi_m(\rho_2, z, \omega).
\] (5.18)

The first of two double sums represent the power content of the propagating modes and the second the correlations between the modes. It is clear that in the turbulence the propagating modes cannot be considered as uncorrelated. However, since in this work we are interested in only the power coupling, in the following the effects of modes correlations will be disregarded. In order to derive the power redistribution between the modes we will set \(\rho_1 = \rho_2\) and integrate over the plane \(z\). Due to modes orthonormality, the mixed terms vanish and the result is:

\[
\iint W_{AT}(\rho, \rho, z, \omega) \, d^2\rho = \sum_n \sum_l \langle |b_{nl}(z, \omega)|^2 \rangle .
\] (5.19)

The left hand side of Eq. (5.19) is the total power of the beam at distance \(z\), and, because of energy conservation, it is equal to the total power of the source which is the sum of the source modes expansion coefficients. The physical significance of \(\langle |b_{nl}(z, \omega)|^2 \rangle\) in Eq.(5.19) can be understood to be the power which originated from the \(n^{th}\) source mode and contributed to the power of the \(l^{th}\) propagating mode.

Having established that the coefficients \(\langle |b_{nl}(z, \omega)|^2 \rangle\) represent the redistribution of power, they can also be written in terms of the original power of any mode and the balance of power coupled in and out of that mode. Thus the first double sum of the cross spectral density
in Eq.(5.18) (the "diagonal" one) can be written in the form (the order of summation was interchanged as well)

\[
W_{AT}^{(d)}(\rho_1, \rho_2, z, \omega) = \sum_l \left[ \lambda_l(\omega) + \sum_n \lambda_n(\omega) d_{nl}(z, \omega) \right] \psi_l^*(\rho_1, z, \omega) \psi_l(\rho_2, z, \omega),
\]  

(5.20)

where \(d_{nl}(z, \omega)\) are modes power coupling coefficients which are function of both the propagation distance \(z\) and the frequency \(\omega\). The superscript \((d)\) signifies that we consider only the sum of the diagonal elements.

On substituting from the right hand side of Eq. (5.20) in to the left hand side of Eq. (4.27) and on using Eq. (5.15) and (4.21) we obtain

\[
\sum_l \left[ \lambda_l(\omega) + \sum_n \lambda_n(\omega) d_{nl}(z, \omega) \right] \psi_l^*(\rho_1, z, \omega) \psi_l(\rho_2, z, \omega) = \sum_l \lambda_l(\omega) \psi_l^*(\rho_1, z, \omega) \psi_l(\rho_2, z, \omega) - \\
\int \int \left[ \sum_l \lambda_l(\omega) \phi_l^*(\rho_1', \omega) \phi_l(\rho_2', \omega) \sum_{n,m} \phi_n(\rho_1', \omega) \phi_m^*(\rho_2', \omega) \psi_n^*(\rho_1, z, \omega) \psi_m(\rho_2, z, \omega) \times \right. \\
\left. \left\{ 1 - \exp \left[ -\frac{1}{2} D_{sp}(\rho_1 - \rho_2, \rho_1' - \rho_2', z, \omega) \right] \right\} \right] \ d^2\rho'_1 d^2\rho'_2.
\]  

(5.21)

The sum on the right hand side of Eq.(5.21) may include mixed combinations of propagating modes but in the following derivation they will be disregarded since we are seeking the power coupling coefficients. The derivation of the modes coupling coefficients from Eq. (5.21) is still a daunting task even when a numerical method is employed. Therefore, before proceeding with the calculations we will make some simplifications in order to derive an
approximate expression for the coupling coefficients. We will assume that the source is narrow and the propagation distance is long enough so that the phase variance for two propagation paths is dominated by the separation of the end points in the observation plane. Under this assumption the integration over the source plane will not change significantly the final result if we let \( \rho'_1 - \rho'_2 \approx 0 \). This approximation may be considered as a the zero order approximation for the source plane separation. The result of this assumptions is that the two-point spherical wave structure function is replaced by the spherical wave structure function \( D_\Phi (\rho_1, \rho_2, \omega) \) [9].

As noted in Section 3, we will assume that the sources are narrow and that the propagation distance is long enough so that the phase variance is dominated by the separation in the observation plane. Under this assumption the integration over the source plane will not affect significantly the final result if we if we let \( \rho'_1 - \rho'_2 \approx 0 \). The result of this assumption is that the two-point spherical wave structure function is replaced by the one point spherical wave structure function, \( D_\Phi (\rho_1, \rho_2, \omega) \). Preforming the substitution in Eq.(5.21) we obtain:
\[
\sum_l \left[ \lambda_l (\omega) + \sum_n \lambda_n (\omega) d_{nl} (z, \omega) \right] \psi_l^* (\mathbf{p}_1, z, \omega) \psi_l (\mathbf{p}_2, z, \omega) = \\
= \sum_l \lambda_l (\omega) \psi_l^* (\mathbf{p}_1, z, \omega) \psi_l (\mathbf{p}_2, z, \omega) - \\
\iint \sum_l \lambda_l (\omega) \phi_l^* (\mathbf{p}_1', \omega) \phi_l (\mathbf{p}_2', \omega) \sum_{n,m} \phi_n (\mathbf{p}_1', \omega) \phi_m^* (\mathbf{p}_2', \omega) \psi_n^* (\mathbf{p}_1, z, \omega) \psi_m (\mathbf{p}_2, z, \omega) \times \\
\left\{ 1 - \exp \left[ -\frac{1}{2} D_\Phi (\mathbf{p}_1, \mathbf{p}_2, z, \omega) \right] \right\} d^2 \mathbf{p}_1' d^2 \mathbf{p}_2' \\
(5.22)
\]

Multiplying both sides by \( \psi_j^* (\mathbf{p}_1, z, \omega) \psi_j (\mathbf{p}_2, z, \omega) \) and performing the appropriate integration we obtain using the orthonormality relations,

\[
\sum_l \sum_n \lambda_n (\omega) d_{nl} (z, \omega) \psi_l^* (\mathbf{p}_1, z, \omega) \psi_l (\mathbf{p}_2, z, \omega) = \\
- \sum_l \lambda_l (\omega) \psi_l^* (\mathbf{p}_1, z, \omega) \psi_l (\mathbf{p}_2, z, \omega) \left\{ 1 - \exp \left[ -\frac{1}{2} D_\Phi (\mathbf{p}_1, \mathbf{p}_2, z, \omega) \right] \right\} .
\]
\[
\sum_n \lambda_n (\omega) d_{nj} (z, \omega) = \sum_l \lambda_l (\omega) \int \psi^*_l (\rho_1, z, \omega) \psi_l (\rho_2, z, \omega) \psi^*_j (\rho_1, z, \omega) \psi_j (\rho_2, z, \omega)
\]

\[(5.24)\]

\[
\times \left\{ 1 - \exp \left[ -\frac{1}{2} D_\Phi (\rho_1, \rho_2, \omega) \right] \right\} d^2 \rho_1 d^2 \rho_2
\]

It should be noted due to the approximation used no mixed terms appear at all. We now make use of the facts that \(n\) and \(l\) are dummy indices that are used for summation and that the result should be valid for any initial set of source coefficients to conclude that (for convenience we replaced the \(j\) subscript by \(l\)) The expression obtained for the mode coupling coefficients (which can be regarded as elements of a matrix) is

\[
d_{nl} (z, \omega) = -\int \psi^*_n (\rho_1, z, \omega) \psi_n (\rho_2, z, \omega) \psi^*_l (\rho_1, z, \omega) \psi_l (\rho_2, z, \omega)
\]

\[(5.25)\]

\[
\times \left\{ 1 - \exp \left[ -\frac{1}{2} D_\Phi (\rho_1, \rho_2, \omega) \right] \right\} d^2 \rho_1 d^2 \rho_2.
\]

This expression has the form of an “overlap”- like integral which involves propagating modes and a “perturbation” interaction induced by the random medium. Another consequence of using the approximate structure function is that no mixed terms emerge so we may consider these terms as the result of higher order approximations.
It should be noted that the mode coupling matrix $d_{nl}$ is Hermitian and after some manipulation it can be shown that the modified mode expansion coefficients are real and non-negative as required. An important property of the matrix arises from energy conservation considerations. Neglecting any backscattering effects or absorption and recalling that the expansion coefficients were interpreted in terms of power content for each mode, we have to demand that

$$\sum_l \left[ \lambda_l (\omega) + \sum_n \lambda_n (\omega) d_{nl} (z, \omega) \right] = \sum_l \lambda_l (\omega) \Rightarrow \sum_l \left[ \sum_n \lambda_n (\omega) d_{nl} (z, \omega) \right] = 0. \quad (5.26)$$

Changing the order of summations we obtain

$$\sum_n \left[ \sum_l \lambda_n (\omega) d_{nl} (z, \omega) \right] = 0 \Rightarrow \sum_n \lambda_n (\omega) \sum_l d_{nl} (z, \omega) = 0. \quad (5.27)$$

Since the set of coefficients can be chosen arbitrarily we must have

$$\sum_l d_{nl} (z, \omega) = 0. \quad (5.28)$$

The diagonal elements can be interpreted as the power “flowing out” of a certain mode and the other elements of a row represent the power coupled to any other mode. The sum of any row and any column must therefore be zero – with only the diagonal elements being negative and all the other positive. This condition implies that the rows are not linearly independent and that we can obtain an equivalent matrix with a null row (simply by adding
all the other rows to any row). It can be shown that for this matrix the dimension of the kernel (the set of vectors with a zero eigenvalue) is one.

Using matrix and vector notation, the propagation effects on the vector which represents the mode power distribution are described by the expression

$$\vec{\lambda}(z, \omega) = \left[ I + \vec{d}(z, \omega) \right] \vec{\lambda}(0, \omega),$$  

(5.29)

with $I$ designating the identity matrix. We will call the sum in the square brackets the propagation matrix.

It should be stressed that the $z$ dependence of the expansion coefficients is the result of using the free space modes basis to describe non free space propagation.

Since the dimension of the kernel of the $\vec{d}$ matrix is one, only one "modes vector" has a zero eigenvalue and therefore is not affected by propagation. It is easily seen that this is the vector with all the elements equal, i.e. the vector representing a fully incoherent source. It can be stated that only the radiation from a completely incoherent source will be unaffected by propagation through atmospheric turbulence. Since the radiation from a completely incoherent source is hardly a beam, we might also add that no beam will be unaffected by atmospheric turbulence. The matrix has other eigenvectors associated with eigenvalues different from 0 and it can be shown that the sum of those vector’s elements must be zero which implies they can not represent an actual intensity distribution.
Another question arises whether the inevitable effect of turbulence might be a positive one. For example, one may ask whether some initial modes distribution exists that leads to a final distribution with only one element, for example, the lowest order one. Solving for this initial distribution is done by inverting the propagation matrix and multiplying the desired modes distribution. Since the inverse of the matrix in the square brackets of Eq. (5.29) has negative elements at any row and any column, the result of this inverse problem will have negative elements which can not be realized.

5.4 Hermite-Gauss modes coupling for Gaussian Schell - model beams

As a specific example we will consider the propagation of Gaussian Schell - model (GSM) beams in the atmosphere. Gaussian Schell - model (GSM) sources are described by cross-spectral density of the form [130],

\[ W(\rho_1', \rho_2', \omega) = A(\omega) \exp\left(-\frac{|\rho_1'|^2 + |\rho_2'|^2}{4\sigma_s^2(\omega)}\right) \exp\left(-\frac{|\rho_1' - \rho_2'|^2}{2\sigma_\mu^2(\omega)}\right), \] (5.30)

Here \( A(\omega) \) is the spectral density (or intensity) on axis, \( \sigma_s \) is a length scale of the intensity distribution and \( \sigma_\mu \) is a length scale of the correlations.

From the cross-spectral density function (5.30) the spectral density
\[ S(\rho', \omega) = W(\rho', \rho', 0, \omega) = A(\omega) \exp \left( -\frac{|\rho'|^2}{2\sigma^2_\omega(\omega)} \right) \] (5.31)

and the spectral degree of coherence

\[ \mu(\rho'_1, \rho'_2, \omega) = \frac{W(\rho'_1, \rho'_2, \omega)}{\sqrt{S(\rho'_1, \omega)S(\rho'_2, \omega)}} = \exp \left( -\frac{|\rho'_1 - \rho'_2|^2}{2\sigma^2_\mu(\omega)} \right) \] (5.32)

can be evaluated. It is well known that the GSM sources can be expanded by using Hermite-Gauss coherent modes [130]. The expansion is:

\[ W(\rho'_1, \rho'_2, \omega) = \sum_m \sum_n \beta_{mn}(\omega) \phi^*_m(\rho'_1) \phi_n(\rho'_2), \] (5.33)

with the eigenmodes and coefficients given by the expression:

\[ \phi_{mn}(\rho') \equiv \phi_{mn}(x', y') = \frac{1}{w_0 \sqrt{\pi 2^{m+n-1} m! n!}} H_m \left( \frac{\sqrt{2}}{w_0} x' \right) H_n \left( \frac{\sqrt{2}}{w_0} y' \right) \exp \left( -\frac{x'^2 + y'^2}{w_0^2} \right), \]

\[ \beta_{mn}(\omega) = A(\omega) \left( \frac{\pi}{a + b + c} \right) \left( \frac{b}{a + b + c} \right)^{m+n}, \] (5.34)

\[ a = \frac{1}{4\sigma^2_\omega(\omega)}, \quad b = \frac{1}{2\sigma^2_\mu(\omega)}, \quad c = \sqrt{a^2 + 2ab}, \quad w_0 = \frac{1}{\sqrt{c}} \]

The corresponding expressions for the propagating modes are [134] :
\[
\psi_{mn}(\rho, z) \equiv \psi_{mn}(x, y, z) = \frac{1}{w_0\Delta(z)\sqrt{\pi}2^{m+n-1}m!n!} H_m\left(\frac{\sqrt{2}}{w_0\Delta(z)} x\right) H_n\left(\frac{\sqrt{2}}{w_0\Delta(z)} y\right) \times \\
\exp\left[-\frac{x^2 + y^2}{w_0^2\Delta(z)^2}\right] \exp\left[-ik\frac{x^2 + y^2}{2R(z)}\right] \exp[-i(m + n + 1) \Phi(z)]
\] (5.35)

Where \(\Delta(z)\) is the expansion parameter of the beam, the radius of curvature of the phase front, the Rayleigh range and the Gouy phase shift are given by the formulas:

\[
\Delta(z) = \sqrt{1 + \left(\frac{z}{Z_R}\right)^2},
\] (5.36)

\[
R(z) = z \left[1 + \left(\frac{Z_R}{z}\right)^2\right],
\] (5.37)

\[
Z_R = \frac{k w_0^2}{2},
\] (5.38)

and

\[
\Phi(z) = \tan^{-1}\left(\frac{2z}{kw_0^2}\right)
\] (5.39)

Fig. 5.2 illustrates an example for the modes expansion coefficients distribution for a GSM beam which has a global coherence equal to 0.3 (global coherence is defined as the ratio of the coherence length scale to the intensity length scale). A more coherent beam will have distribution which is more concentrated near the 00 mode while a less coherent beam will have a flatter distribution.
Generally, the modal expansion of the propagating cross spectral density function in free space has the form

\[
W(\rho_1, \rho_2, z, \omega) = \sum_{mn} \beta_{mn}(\omega) \psi_{mn}^*(\rho_1, z, \omega) \psi_{mn}(\rho_2, z, \omega),
\]

(5.40)

If the GSM beam propagates in turbulence then the expansion can be written, according to Eq. (5.20), as
\[ W_{AT}^{(d)} (\rho_1, \rho_2, z, \omega) = \sum_{mn} \left[ \beta_{mn} (\omega) + \sum_{kl} \beta_{kl} (\omega) d_{(mn)(kl)} (z, \omega) \right] \psi^*_{mn} (\rho_1, z, \omega) \psi_{mn} (\rho_2, z, \omega). \] (5.41)

Using Eq. (5.25) the mode coupling coefficients can be calculated to be

\[ d_{(mn)(kl)} (z, \omega) = -\frac{1}{[w_0 \Delta (z)]^2 \pi^{2m+n+k+l-2} m! n! k! l!} \int \int H_k \left( \frac{\sqrt{2}}{w_0 \Delta (z)} x_1 \right) H_l \left( \frac{\sqrt{2}}{w_0 \Delta (z)} y_1 \right) \times \]

\[ \times H_m \left( \frac{\sqrt{2}}{w_0 \Delta (z)} x_2 \right) H_n \left( \frac{\sqrt{2}}{w_0 \Delta (z)} y_2 \right) \exp \left( -\frac{x_1^2 + y_1^2}{w_0^2 \Delta^2 (z)} \right) \exp \left( -\frac{x_2^2 + y_2^2}{w_0^2 \Delta^2 (z)} \right) \left\{ 1 - \exp \left[ -\frac{1}{2} D_\Phi (\rho_1, \rho_2, z, \omega) \right] \right\} \] \[ \times dx_1 dy_1 dx_2 dy_2. \] (5.42)

The calculation of the coupling coefficients requires that the explicit expression for the structure function \( D_\Phi (\rho_1, \rho_2, \omega) \) is used. In the present example we assume that the 3-D power spectrum of the index of refraction fluctuations has the Tatarskii form [9]

\[ \Phi_n (\kappa) = 0.033 \ C_n^2 \exp \left( -\kappa^2 / \kappa_n^2 \right), \] (5.43)

where \( 0 \leq \kappa < \infty \), \( \kappa_m = 5.92/l_0 \), \( l_0 \) is the inner scale of turbulence which is of the order of few millimeters. \( C_n^2 \) is the turbulence structure constant which is usually in the range of \( 10^{-16} \) for weak turbulence to \( 10^{-12} \) for strong turbulence.

The spherical wave structure function derived from this power spectrum is then given by [9]:
\[ D_{\Phi}(\rho_1, \rho_2, z, \omega) = D_{\Phi}(\rho = |\rho_2 - \rho_1|, z, \omega) = \frac{1.093 C_n^2 k^2 z l_0^{-1/3} \rho^2}{\left(1 + \frac{\rho^2}{l_0^2}\right)^{1/6}}. \] (5.44)

It is also important to note that in our approach the matrix elements are calculated without any dependence on the initial distribution of modes. The only important parameters are the wavelength, the waist, the inner scale and the turbulence structure constant. When these parameters are constant then after calculating the coupling coefficients we can apply any initial modes distribution and obtain the propagated modes distribution. Note that since we are dealing with long-term temporal averages these mode combinations can represent some coherent beams as well (i.e. some specific single initial mode but not a coherent combination of modes).

It should be noted that the mode coupling approach can be applied to LG modes as well, which are eigenmodes of the angular momentum operator. In fact a somewhat similar approach was very recently used to calculate coupling between LG modes induced by the turbulence [135].

### 5.5 Numerical Results

In order to properly take into account the inner scale effects we performed numerical computation of the coupling coefficients using the explicit form of the structure function from...
Eq. 5.44. Since we have a Gaussian weighting function in Eq.(5.42) it is convenient to use a version of Hermite-Gauss quadrature method [136].

The number of modes used was up to \((n,m) = (9,9)\) and for each case we examined the total power still contained in this set of modes. All the calculations were performed for the wavenumber \(k = 10^7 m^{-1}\), corresponding to a wavelength of approximately 628\(nm\), and \(Cn^2 = 10^{-14} m^{-2/3}\).

In order to define the range of validity of the expression used in our calculations let us first consider the Rytov parameter which is conventionally used as a measure of turbulence strength. This parameter is given by [9]

\[
\sigma^2_1 = 1.23C_n^2 k^{7/6} z^{11/6},
\]  

(5.45)

and, if it has takes on values smaller than unity, then the turbulence is considered to be weak. Using the values of the parameters given above the result is that turbulence can be considered as weak up to a range of about 1000 \(m\) (it should be noted that the Rytov should be considered for plane waves but the result is not modified significantly if one considers the more stringent conditions which apply to Gaussian beams[9]).

Let us now consider the case in which both the inner scale and the beam waist are equal to 1 \(cm\). The coupling coefficients are calculated for a set of ranges from 250 \(m\) to 2000 \(m\) with a 250 \(m\) step and then they can be applied to any set of initial coefficients.
At first we examined the case of an initially fully coherent beam in which the initial power is contained in the \((0, 0)\) mode. In this case, the redistribution of power between some of the modes is illustrated in Fig. 5.3. Some interesting features emerge immediately. The fundamental mode is coupled most strongly to its “nearest neighbors”, as can be also inferred from the characteristics of the overlap integrals of the Hermite-Gauss functions. The initial mode \((0, 0)\) decays monotonically but all the other modes rise from an initially zero power state, reach a maximum at some range and then they all start to decay. Note that this decay starts at longer distances for higher-order modes. The overall tendency is towards power equalization which will occur at very long ranges where the beam loses becomes practically incoherent.

We have also checked how much power is “lost” into modes of higher order than those taken into account in our numerical calculation. Summing over the expansion coefficients we obtain the power fraction still accounted for in the calculation and this result is presented in Fig. 5.4. One can see that in the weak turbulence regime examined our procedure still accounts for 95% of the power emanating from the source. The modified modes power content can be used to obtain an estimate of intensity distribution by a superposition of the modes. The full calculation of the intensity distribution requires a knowledge of the off-diagonal elements of the cross-spectral density as well as the diagonal ones. The use of an approximate intensity distribution based only on the diagonal elements is equivalent to assuming that the propagating modes are completely uncorrelated. This leads to an upper bound estimate on the beam spread since any correlations reduce the beam spread[137]. The
Figure 5.3: The change in the power content of the different modes upon propagation in atmospheric turbulence of a beam produced by a TEM00 mode source ($k = 10^7 \text{m}^{-1}$, $C_n^2 = 10^{-14}\text{m}^{-2/3}$, $l_0 = 1 \text{ cm}$, $w_0 = 1 \text{ cm}$).

The approximate intensity distribution for a range of 1000 $m$ is presented in Fig. 5.5, where the result of the modes calculation is compared with the free space spreading and the analytical formula based on the Gaussian approximation presented in Ref. 2, sec. 6.3.2. One can see that the intensity distributions resulting from the mode calculation and the Gaussian approximation are quite similar but the modes calculation results manifest higher wings due to the contribution of higher modes. This result can be compared to exact calculations of intensity distribution which are based on the Kolmogorov power spectrum [9]. The results are presented in Fig.5.6 and indeed one can observe that exact calculations deviate from the Gaussian approximation in a way which is very similar to the modes results.
Figure 5.4: Total power contained up to the mode TEM99 of the beam for the case of a TEM00 source \( (k = 10^7 \, m^{-1}, C_n^2 = 10^{-14} \, m^{-2/3}, l_0 = 1 \, cm, w_0 = 1 \, cm) \).

The modes coupling coefficients can be used to obtain an estimate of the beam width defined as

\[
\bar{\rho}(z) = \sqrt{\frac{\iiint \rho^2 I(\rho, z) \, d^2\rho}{\iiint I(\rho, z) \, d^2\rho}}. \tag{5.46}
\]

In Eq.(5.46) \( I(\rho, z) \) is the intensity distribution which is equivalent to the spectral density. Using the orthonormality properties of the modes it can be shown that \([138],[126]\)

\[
\bar{\rho}(z) = \left[ \frac{1}{2} \sum_{nm} \beta_{mn}(\omega, z) (m + n + 1) w_0^2 \Delta^2(z) \right]^{1/2}, \tag{5.47}
\]
Figure 5.5: Normalized intensity distribution (with respect to free space propagation) for a TEM00 beam after 1000 meters of propagation ($k = 10^7 \text{ m}^{-1}, C_n^2 = 10^{-14} \text{ m}^{-2/3}, l_0 = 1 \text{ cm}, w_0 = 1 \text{ cm}$).

$\Delta(z)$ is the expansion parameter of the beam defined in Appendix B. For the case presented in Fig. 5.5 the ratio of the beam widths calculated using the modes and the analytical formula is about 1.15.

Next we analyze another case in which the initial mode distribution is a $(1, 0) + (0, 1)$ combination, i.e. the incoherent doughnut mode. In Fig. 5.7 we plotted the power distributed in several modes as this beam propagates through the same conditions of atmospheric turbulence. The $(1, 0)$ mode was not plotted but it has of course the same behavior as $(0, 1)$ mode. The intensity distribution in the range of 1000 meter is plotted in Fig. 5.8. This result is qualitatively comparable with previous results [139]. The on-axis intensity in this
Figure 5.6: Comparison of intensity distributions for free space propagation (dashed line), "Gaussian" approximation and exact theory based on Kolmogorov power spectrum.

case rise from its initial zero value for several hundreds of meters and than declines. The rate in which the intensity null is "filling up" is related to the turbulence parameters and as such can be used, in an experimental setup to evaluate them.

The last case illustrated here is a Gaussian Schell- model beam with $\sigma_s=0.00845 \ m$ and $\sigma_\mu =0.006374 \ m$ (i.e. a global coherence parameter equal to 0.75) so that $w_0$ is again approximately 1 cm. The interchange between different modes is presented in Fig. 5.9. The power accounted for is very similar to the case of for the coherent beam. The calculated beam spreading is presented in Fig. 5.10. We can see that for propagation distances within
Figure 5.7: The change in the power content of the different modes upon propagation of a "doughnut" beam ($k = 10^7$ m$^{-1}$, $C_n^2 = 10^{-14}$ m$^{-2/3}$, $l_0 = 1$ cm, $w_0 = 1$ cm).

In several recent works it was shown that the degree of polarization of partially coherent, partially polarized beams changes upon propagation in free space[148],[140]. It should be
Figure 5.8: "Doughnut" beam mean intensity distribution after propagation of 1000 m in atmospheric turbulence \((k = 10^7 \text{ m}^{-1}, C_n^2 = 10^{-14} \text{ m}^{-2/3}, l_0 = 1 \text{ cm}, w_0 = 1 \text{ cm})\).

noted that in the simple case of two overlapped, uncorrelated beams with orthogonal polarizations, this change is caused by the different spreading characteristics of those beams induced by the different correlations in the source plane. More recently, relying on a new unifying theory of coherence and polarization\cite{141,142}, the problem of polarization changes upon propagation in turbulence was approached\cite{143,144}. It was shown that a complex interplay due to effects of source correlations and to atmospheric turbulence can induce quite dramatic changes of the degree of polarization. As an example, a unpolarized beam could become almost completely polarized on-axis after some propagation distance. After a longer propagation range the beam might return to its initial unpolarized state, again, on axis. It is important to consider the way in which the polarization state changes over the entire beam...
Figure 5.9: The change in the power content of the different modes upon propagation of a GSM beam \( (k = 10^7 \ m^{-1}, C_n^2 = 10^{-14} \ m^{-2/3}, l_0 = 1 \ cm, w_0 = 1 \ cm) \).

cross section. For applications which involve active polarimetric remote sensing, it may also be beneficial to design sources that can produce a specific polarization distribution, taking into account propagation and atmospheric effects. A typical application is the case of a non-depolarizing target located in a natural depolarizing background. To locate the target one may scan the scene with a polarized laser beam and detect the reflected light with two imagers which are sensitive to orthogonal polarizations. Taking the difference of the two images, the higher signal is expected to indicate the target location. The resolution of such detection system is determined mostly by the scanning beam width. The same detection scheme could be implemented with an illumination beam which is not uniformly polarized but has a polarization distribution which will work, as will be demonstrated, to enhance
Figure 5.10: Comparison of beam spread calculation for free space propagation and in turbulence using the results from modes expansion coefficients for a GSM beam

\((k = 10^7 \text{ m}^{-1}, C_n^2 = 10^{-14} \text{ m}^{-2/3}, l_0 = 1 \text{ cm}, w_0 = 1 \text{ cm})\).

The resolution. In this report we will consider both the direct and inverse problem of the propagation of polarization distribution using the approach of coherent mode coupling.

The non-negativity of the modes expansion coefficients, which was discussed in the previous section, is a physical constraint when one considers the intensity distribution but it is not the case when one considers the distributions of other Stokes parameters. We consider the Stokes parameters, as a transverse distribution which is a function of the transverse position vector \(\vec{r}\) at some observation plane situated at a distance \(z\) from the source. The Stokes vector distribution is then given by \(\vec{S}(\vec{r}) = \{I(\vec{r}), Q(\vec{r}), U(\vec{r}), V(\vec{r})\}\). Note
that the transverse distribution of the various Stokes parameters can be entirely different and have different spreading characteristics.

Let us consider the simple case of a beam which can be described as two uncorrelated overlapped partially coherent beams of orthogonal polarizations, for example, horizontal and vertical linear polarizations. The modal decomposition and experimental synthesis of such beams were discussed in the past [145]–[147]. We will assume that the two beams can be decomposed onto the same set of Hermit-Gaussian modes (HG), possibly with different coefficients, and consider the difference of intensities of the two uncorrelated beams, \( I_0 (\vec{r}) - I_{90} (\vec{r}) \), which is the second element of the Stokes vector of the combined beam. It is clear that the mode decomposition of the \( Q \) element of the beam, given by \( \vec{\lambda}_Q (z, \omega) = \vec{\lambda}_0 (z, \omega) - \vec{\lambda}_{90} (z, \omega) \), may contain negative coefficients. In that case it is possible to obtain a source with a \( \vec{\lambda}_Q (0, \omega) \) which is an eigenvector of propagation of a certain distance in turbulence. A source composed in that manner will produce a beam for which the polarization distribution will be replicated, up to a scaling factor, at a specific range for which the eigenvectors were calculated.

Perhaps a more interesting case is one in which a certain source polarization distribution propagates and, at certain range, has a narrow Gaussian distribution which approximates the free space spreading of a Gaussian beam. In order to solve this inverse problem one have to multiply this desired distribution vector (which has only the first element non zero) by the inverse of the propagation matrix and obtain the required source modes distribution. In other words, one has to solve for \( \vec{\lambda}_Q (0, \omega) = [\vec{I} + \vec{d} (z, \omega)]^{-1} \vec{\lambda}_Q (z, \omega) \) where \( \vec{\lambda}_Q (z, \omega) \)
is equal to $[1, 0, 0, ...]^T$. Considering a specific case in which both the beam waist and the turbulence inner range are equal to 1 cm, the turbulence structure constant is equal to $10^{-14} m^{-2/3}$, the wavenumber is $10^7 m^{-1}$ and the horizontal propagation distance is 1000m, we can calculate the coupling matrix for the Hermite-Gauss modes (assuming a Tatarskii index of refraction fluctuations power spectrum) and obtain that the dominant first coefficients (for $HG_{00}, HG_{01}, HG_{10}$ and $HG_{11}$) are 1.599, -0.223, -0.223 and -0.012 (the sum of all the elements is 1 according to our normalization). This can be implemented with a horizontally polarized $HG_{00}$ mode with a 1.599 relative power and a vertically polarized beam made with $HG_{01}$ and $HG_{10}$ uncorrelated modes (both with 0.223 relative power) and a vertically polarized $HG_{11}$ mode with relative power equal to 0.012.

The results for the $Q$ distribution for this specific case are presented in Fig. 5.11. As a reference, we also show the normalized turbulence degraded intensity of a Gaussian beam (the horizontally polarized component of our case) and the free space distribution of the Gaussian beam. It is evident that the $Q$ distribution is narrower and actually approximates well the free space width. The Full Width at Half Maximum (FWHM) is smaller by 12%.

It is interesting to examine the gain in distribution width which is even greater for a range of 2000 meters. In Fig. 5.12, one can see that the wings of the $Q$ distribution are higher than those of the free space Gaussian distribution, but the central lobe overlaps and the FWHM gain is about 15%. It is important to emphasize that there is no power exchange between the orthogonal polarizations and that the polarization distribution is controlled by the different spreading characteristics of the two orthogonal beams.
Figure 5.11: A comparison of the Q polarization distribution from the polarized source (solid line), a Gaussian beam in turbulence (dashed line) and a Gaussian beam in free space (dotted line). All the distributions were calculated for a range of 1000 meters and are normalized to the respective axial value.

Since the HG$_{11}$ mode coefficient is very small we can still obtain a good approximation to the last result by using as a source a combination of two lasers, one emitting a HG$_{00}$ beam polarized horizontally and the other emitting a “doughnut” beam (incoherent superposition of HG$_{01}$ and HG$_{10}$ modes) with an orthogonal polarization. This combination can be viewed as a sharpening spatial filter which is still effective in long-term averaged turbulence. Such a source may be applied for improving the resolution of an active polarization remote sensing system in which the important information is the polarization contrast (the difference between the returns at two orthogonal polarizations).
Figure 5.12: A comparison of the Q parameter distribution from the polarized source (solid line), a Gaussian beam in turbulence (dashed line) and a Gaussian beam in free space (dotted line). All the distributions where calculated for a range of 2000 meters and are normalized to the respective axial value.

To examine the practical aspects of such a source let us consider the impact on of signal to noise which is degraded as the spatial resolution is improved. Let us assume an imaging device, employing two identical imagers with orthogonal polarizes in front of them, producing a polarization image which is the result of subtracting the two intensity images. We will compare a source which is linearly polarized with a source described above which has a polarization distribution. We will consider the power of the single laser source to be equal to the power of the combined lasers. If we assume that the imagers are dark current limited
then the mean signal to noise ratio (SNR), upon reflection from a non depolarizing target, will be given by for a single laser source $SNR_{D1} \propto A\eta \tau P / \sqrt{N_d}$, where $A$ is an attenuation factor related to target reflectance, atmospheric attenuation and collection efficiency, $\eta$ is the imager quantum efficiency, $\tau$ is the integration time, $P$ is the source power and $N_d$ is the mean number of electrons generated by dark current in one integration interval in the pixels which contain the collected reflected light. In the case of the of the two lasers source the expression will be modified to $SNR_{D2} \propto A\eta \tau (P_1 - P_2) / \sqrt{N_d}$ where $P_{1,2}$ are the the powers emitted by the two lasers (as mentioned before, $P = P_1 + P_2$). The ratio of the SNR simply reduces to $(P_1 - P_2) / (P_1 + P_2)$ and implies a reduction , for our example, by 44%. In the case of photon noise limited operation the expressions for the SNR will become $SNR_{P1} \propto \sqrt{A\eta \tau P}$, and $SNR_{P2} \propto \sqrt{A\eta \tau} \sqrt{(P_1 - P_2) / (P_1 + P_2)}$, and the ratio in this case, evaluated again in the center of the beam, will be given by $\sqrt{(P_1 - P_2) / (P_1 + P_2)}$. In the example considered, the signal to noise is reduced by 25%. One can conclude from this rough estimate that in cases where the SNR is not a limiting factor an improvement in spatial resolution may be obtained by using the compound illumination beam.
5.7 Conclusions

In this chapter we introduce the approach of coherent mode coupling and demonstrated a number of applications of this approach. It can be used for fully coherent, fully polarized beams, including OAM carrying modes, as well as partially coherent, partially polarized beams. As an example, we considered the propagation characteristics of a certain class of partially polarized beams using the coherent modes coupling approach for the Stokes parameters. It was demonstrated that this approach can be used to specify sources which will produce a rather narrow polarization distribution even after propagation in atmospheric turbulence. These sources may find applications in active polarimetric remote sensing.
CHAPTER 6

CONCLUSIONS AND SUMMARY OF ORIGINAL CONTRIBUTIONS

For quite some time, wave front singularities associated with random electromagnetic waves were studied extensively in the context of paraxial propagation in weakly inhomogeneous media. In this dissertation, new aspects of propagation of singular waves and of angular momentum transfer in strongly inhomogeneous media were examined with an emphasis on backscattering geometries, which are highly non-paraxial.

Singularities of electromagnetic fields are known to be associated with the orbital angular momentum (OAM) of light. Therefore we can expect that several aspects of light scattering phenomena are connected with the transport characteristics of the angular momentum of light. For instance, the flux density of the total angular momentum of light can be separated in two terms, a polarization or spin angular momentum related term and an OAM term. These two terms do not couple in cases in which the paraxial approximation applies, but there are strong coupling effects at non paraxial situations. In Chapter 3, this coupling was calculated explicitly for the first time in the case of single scattering from a spherically
symmetric potential. Remarkably, it was shown that the flux density of the total angular momentum is a conserved quantity along every direction of scattering.

This dissertation has also discussed the subtle aspects of the conservation of the total angular momentum of light which also manifests in multiple scattering. It was newly found that the polarization patterns which are a ubiquitous phenomena in backscattering from turbid media are in fact the signature of scattering trajectories that conserve the angular momentum. The coupling between spin and OAM in these trajectories leads to 'geometrical phase vortices' which carry OAM, and which in turn generate polarization vortices. The term 'geometrical' refers to the origin of the vortex which is related to the optical geometric phase or Berry phase. This formalism is applied here using the helicity configuration space approach, which permitted to account for both helicity preserving and non preserving scattered waves. The results of this general approach are in excellent agreement with a numerous experimental results.

When studying scattering phenomena, it is important to realize that propagation-induced polarization effects can not be separated from the study of OAM because scattering events couple (i) spin to OAM and (ii) OAM to the medium through an applied torque on the scattering particles. Consequently, one might expect that interaction of singular waves with random media can have unique characteristics. Indeed, as shown in Chapters 4 and 5 of this dissertation, backscattering trajectories lead to phase vortices, while phase vortices embedded in the incident wave modify effects associated with backscattering.
A particular aspect of this work is the novel use of singular waves as illumination sources for the study of random media. The spatial correlations embedded in such waves were found to control the magnitude of the enhanced coherent backscattering effect. This effect, which is the result of constructive interference of waves propagating along time reversible trajectories, relates both to the correlation properties of the random medium and of the illuminating electromagnetic wave. Through a thorough theoretical and experimental study, it was demonstrated here that optical vortices incorporated in the illumination beams can be used as a robust experimental technique for measuring the optical properties of both double-pass scattering systems and diffusive media. The diffusion approximation to radiative transfer equation was used to derive expressions which were found to be in very good agreement with experimental results. It was also demonstrated that the inherent depth sensitivity of this technique can be used to infer information about inclusions embedded in the scattering media.

In a weakly inhomogeneous medium, the paraxial approximation still applies and therefore the spin and OAM decouples. If the medium is optically isotropic, the spin, and therefore the polarization, are both not affected. The OAM, on the other hand, may be affected due to axially nonsymmetric distributions of the scattering potential in different realizations of the medium. Using a novel approach based on coherent modes coupling, several effects of propagation through such media were studied. When applied to the specific case of atmospheric turbulence, this newly developed procedure was found useful for a wide range of applications including quantifying the spreading of beams containing higher order modes.
and the propagation of partially coherent and partially polarized beams. Detailed aspects of this approach for describing the properties of beams propagation in atmospheric turbulence were elucidated in this dissertation. This approach was also applied to demonstrate that a polarized source can be spatially tuned to produce a beam in which polarization features spread upon propagation slower than the intensity distribution.

The results included in this dissertation relates to a number of phenomena encountered in optical scattering. The general procedures and the proof of concept demonstrations should be of interest for a wide range of applications involving probing of random media. These applications, among others, include remote sensing in atmospheric and maritime environments, biomedical optics and various optical diagnostics in material sciences.
APPENDIX

PUBLICATIONS AND PRESENTATIONS
Publications


7. C. Schwartz and A. Dogariu, "Angular momentum conservation in single scattering", to be submitted

8. C. Schwartz and A. Dogariu, "Polarization patterns and symmetries as a manifestation of helicity preserving characteristics of scattering media", temporary name, to be submitted.
Presentations


4. COSI/SRS OSA Topical meeting, June 2005: "Probing random media with singular beams" - Chaim Schwartz and Aristide Dogariu (Paper JMA3)

5. Frontiers in Optics – OSA annual meeting, October 2005: Backscattered Polarization Patterns as a Manifestation of Phase Vortices - Chaim Schwartz and Aristide Dogariu (paper FWT6).


7. SPIE - Defense and Security Symposium, Orlando, April 2006: "Polarization patterns and symmetries as a manifestation of helicity preserving characteristics of scattering media" - Chaim Schwartz and Aristide Dogariu
LIST OF REFERENCES


[133] E. Wolf, “Correlation-induced changes in the degree of polarization, the degree of coherence and the spectrum of random electromagnetic beams on propagation”, Optics Letters 28 1078-1080 (2003)


