Characterization Of Critical Network Components Of Coupled Oscillators

2006

Gregory Holifield
University of Central Florida

Find similar works at: https://stars.library.ucf.edu/etd

University of Central Florida Libraries http://library.ucf.edu

Part of the Computer Engineering Commons

STARS Citation

https://stars.library.ucf.edu/etd/1088

This Doctoral Dissertation (Open Access) is brought to you for free and open access by STARS. It has been accepted for inclusion in Electronic Theses and Dissertations by an authorized administrator of STARS. For more information, please contact lee.dotson@ucf.edu.
Characterization of Critical Network Components of Coupled Oscillators

by

GREGORY A. HOLIFIELD
BS University of Southern Mississippi, 1990
MSCpE University of Central Florida, 2003

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Computer Engineering in the School of Electrical Engineering and Computer Science in the College of College of Engineering at the University of Central Florida Orlando, Florida

Fall Term
2006

Major Professors: Avelino Gonzalez and Annie S. Wu
© 2006 Gregory A. Holifield
This dissertation analyzes the fundamental limits for the determination of the network structure of loosely coupled oscillators based on observing the behavior of the network, specifically, node synchronization. The determination of the requisite characteristics and underlying behaviors necessary for the application of a theoretical mechanism for determining the underlying network topology in a network of loosely coupled natural oscillators are the desired outcome. To that end, this effort defines an analytical framework where key components of networks of coupled oscillators are isolated in order to determine the relationships between the various components. The relationship between the number of nodes in a network, the number of connections in the network, the number of connections of a given node, the distribution of the phases of the network, and the resolution of measurement of the components of the network, and system noise is investigated.
To ... Daynee and Timothy
ACKNOWLEDGMENTS

I would like to express my sincere appreciation to my committee. Their mentorship has had a positive impact on my research and future success. I would also like to thank all of my friends and colleagues that endured long conversations and helped me to undergo this effort. I especially want to thank my co-chairs, Dr Avelino Gonzalez and Dr. Annie Wu. Without Dr. Gonzalez’s strong leadership and drive for good research practices, I would still be wondering around in the dark. In a similar manner, Dr. Annie Wu’s perseverance, patience and desire for a deeper understanding of complex material allowed me to search for the answers I needed. I have to thank my sixteen-year-old editor and son, Timothy, for all of his time and effort to help me succeed (I hope to return the favor soon). Lastly, I thank my best friend and wife, Daynee, who I hope will grow old along with me ....
TABLE OF CONTENTS

LIST OF FIGURES .................................................. xiii

LIST OF TABLES .................................................... xvii

1 INTRODUCTION ................................................... 1

1.1 Strategy and Methodology ....................................... 4
1.2 Contributions .................................................... 4
1.3 Overview of Dissertation ....................................... 6

2 BACKGROUND - PRIMER .......................................... 8

2.1 Couple Oscillators and Synchronization ....................... 9
  2.1.1 Huygens Clocks ............................................. 12
  2.1.2 Natural Oscillators ....................................... 15
2.2 Networks ........................................................ 20
  2.2.1 Scale-Free Hierarchical Modular Networks ............... 22
  2.2.2 Social Networks ........................................... 25
2.2.3 Network Motifs .......................................................... 27
2.3 Dynamics ................................................................. 28
  2.3.1 Limit Cycles ......................................................... 29
  2.3.2 Arnold Tongues .................................................... 32
  2.3.3 Phase Transitions and Network Structures and Sync ............ 34
2.4 Summary ........................................................................ 36

3 BACKGROUND - NETWORKS OF COUPLED OSCILLATORS ....... 37
  3.1 Current Work ............................................................. 38
    3.1.1 Winfree’s Work on Coupled Oscillators ....................... 38
    3.1.2 Kuramoto’s Work on Coupled Oscillators ................. 39
    3.1.3 Strogatz’s Work on Coupled Oscillators ..................... 39
  3.2 Models ........................................................................ 40
    3.2.1 Strogatz’s Example to Demonstrate Models ............... 41
    3.2.2 Winfree’s Model ....................................................... 42
    3.2.3 Kuramoto’s Model ..................................................... 44
    3.2.4 Model Summary ....................................................... 46
    3.2.5 Review of Locally (Rings and Lattices) Coupled Oscillators .... 47
    3.2.6 Review of Nonlocally Coupled Oscillators ............... 49
3.3 Summary ................................................................. 50

4 TRADITIONAL APPROACH ............................................ 51

4.1 Review of Current work on Determination of Connectivity .......... 53

  4.1.1 EMA and IPA ....................................................... 54

  4.1.2 Mutual Prediction Approach (MPA) ............................ 56

  4.1.3 Cellular Automata Modeling of Coupled Oscillators .......... 58

4.2 Summary ................................................................. 60

5 ANALYSIS OF NETWORK STRUCTURE DETERMINATION .......... 61

5.1 First Modification of Kuramoto’s Model ............................. 64

  5.1.1 Estimated Model .................................................. 65

  5.1.2 Actual Model ...................................................... 69

  5.1.3 Error $\psi$ .......................................................... 72

  5.1.4 Error $r$ ............................................................. 75

5.2 Second Modification - Additional Connections to a Single Node .... 81

  5.2.1 Estimate Model in Vector Notation ............................. 81

  5.2.2 Actual Model in Vector Notation ............................... 83

  5.2.3 Group Error $\psi$ .................................................. 85

  5.2.4 Group Error $r$ .................................................... 88
5.2.5 Actual Model in Vector Notation .................................. 91

5.3 Matrix Notation ................................................................ 95
  5.3.1 Estimate Model in Matrix Notation .............................. 96
  5.3.2 Actual Model in Matrix Notation ............................... 98
  5.3.3 Relationships ....................................................... 100

5.4 Ambiguity Probability ..................................................... 101
  5.4.1 Doppelganger Probability ...................................... 102

5.5 New Relationships ....................................................... 106

5.6 Summary ...................................................................... 108

6 STOCHASTIC MODEL ...................................................... 109

  6.1 Class Node ............................................................. 110

  6.2 Class Model ........................................................... 113
    6.2.1 Functions ....................................................... 116
    6.2.2 Init() ............................................................ 116
    6.2.3 calcPsi() ..................................................... 118
    6.2.4 calcR() ......................................................... 119
    6.2.5 calcTheta() .................................................. 120
    6.2.6 calcStdDev() ............................................... 121
6.2.7 calcPD() ................................. 122
6.2.8 calcErrorTheta() ........................... 124
6.2.9 calcErrorPsi() ............................ 124
6.2.10 calcErrorR() ............................. 125
6.3 Model Iteration .............................. 126

7 EMPIRICAL ANALYSIS .............................. 128

7.1 Ambiguity and Variations of Phase Range ......................... 129
  7.1.1 First Look - Varying Standard Deviations of Phases ........... 131
  7.1.2 Extended View - Varying Standard Deviations of Phases ....... 132
7.2 Number of Nodes -vs- Ambiguity and Variations of Phase Range .... 136
  7.2.1 First Look - Varying Number Nodes ............................ 136
  7.2.2 Extended View - Varying Number Nodes .......................... 139
7.3 Phase Resolution ................................ 142
  7.3.1 First Look - Varying Phase Resolution ......................... 142
  7.3.2 Extended View - Varying Phase Resolution ..................... 143
7.4 One Additional Connection ..................................... 145
  7.4.1 First Look - One Additional Connection ....................... 145
  7.4.2 Extended View - One Additional Connection .................... 146
7.5 Multiple Additional Connections ................................. 147

7.5.1 First Look - Multiple Additional Connections ............... 148
7.5.2 Extended View - Multiple Additional Connections .......... 149

7.6 Determination with One Additional Connection .................... 150

7.6.1 First Look - One Additional Connection ....................... 151
7.6.2 Extended View - One Additional Connection ................. 153

7.7 Determination with Multiple Additional Connections .............. 155

7.7.1 First Look - Multiple Added Connections ........................ 156
7.7.2 Extended View - Multiple Added Connections ................. 157

8 CONCLUSIONS AND FUTURE EXTENSIONS ......................... 159

8.1 Summary of Research ............................................... 160
8.2 Further Extensions ................................................... 162

A APPENDIX - CELLULAR AUTOMATA ................................. 164

A.1 Wolfram’s Simple Cellular Automata ............................. 166

A.1.1 Example: Wolfram’s Simple Cellular Automata ................ 166
A.1.2 John von Neumann’s Basic Cellular Automata ................. 169
A.1.3 Example Cellular Automata Following Von Neumann .......... 170

A.2 Determination of Unknown Rule Sets and Neighborhood ......... 176
LIST OF FIGURES

2.1 Pendulum Clock .................................................. 11
2.2 Two Pendulum Clocks ........................................... 12
2.3 Two Pendulums Loosely Coupled ............................... 13
2.4 Synchronization Region .......................... 14
2.5 Bridges of Königsberg ........................................... 21
2.6 Demonstration of the phase transition ....................... 23
2.7 Hierarchical Scale-Free Network 1 ....................... 24
2.8 Hierarchical Scale-Free Network 2 ....................... 24
2.9 Hierarchical Scale-Free Network 3 ....................... 24
2.10 Three-node networks .......................................... 28
2.11 Pendulum Clock relationship to Sinusoid ............... 30
2.12 Pendulum Phase Space. ........................................ 31
2.13 Limit Cycle ..................................................... 31
2.14 Stroboscopic measurement of group beginning to synchronize ........ 32
7.1 Basic Node Structure - Ring Topology ........................................ 130
7.2 Multiple 10-Node varying Standard Deviation Populations .............. 132
7.3 Multiple 100-Node varying Standard Deviation Populations .............. 133
7.4 Multiple 100-Node varying Standard Deviation Populations .............. 134
7.5 100-Node Populations with Varying Random Seeds ....................... 135
7.6 $p(\text{Ambiguity})$ and $\sigma_{\theta_i}$ - 100 Node Network ................... 137
7.7 $p(\text{Ambiguity})$ and $\sigma_{\theta_i}$ - 50 Node Network ..................... 137
7.8 $p(\text{Ambiguity})$ and $\sigma_{\theta_i}$ - 10 Node Network ..................... 138
7.9 Phase standard deviation - 10-100 Node Network ......................... 138
7.10 Ambiguity probability - 10-100 Node Network ............................. 139
7.11 First generation over with an ambiguity probability of > 0.60 .......... 140
7.12 First generation over with an ambiguity probability of > 0.60 .......... 140
7.13 Average and Standard Deviation of First generation with trend line .... 141
7.14 Probability of Ambiguity at various resolutions ......................... 143
7.15 Probability of Ambiguity at various resolutions for multiple tests ...... 144
7.16 Modified Node Structure ..................................................... 145
7.17 $p(\text{ambiguity})$ comparison of two different network topologies ....... 146
7.18 $p(\text{ambiguity})$ comparison of two different 100 model experiments .. 147
7.19 Modified Node Structure. .................................................. 148
7.20 p(ambiguity) comparison of two different network topologies ............. 149
7.21 p(ambiguity) comparison of two different 100 model experiments. ........ 150
7.22 Error $\theta_i$ between the estimate and actual networks. .................. 152
7.23 Mean square error difference of various models .............................. 154
7.24 Error $\theta_i$ between the estimate and actual networks. .................. 156
7.25 Mean square error difference of various models .............................. 157
A.1 Wolfram’s Numbering for all possible three-node rule sets ............... 167
A.2 Simple Cellular Automata .................................................... 167
A.3 Generation 2 of Simple Cellular Automata. .................................. 168
A.4 Wolfram Rule 30, Rule Set. .................................................. 168
A.5 Wolfram Rule 30, Example. .................................................. 169
A.6 Simple Cellular Automata. .................................................... 171
A.7 Simple CA with all cells at state $d_0$. ...................................... 173
A.8 Example $\tau$, shown as a majority rule. ...................................... 174
A.9 Example $\lambda$, shown as a majority rule. ...................................... 174
LIST OF TABLES

7.1 Average and Standard Deviation of First generation . . . . . . . . . . . . . . 141
7.2 Average ambiguity probability at the time of an error . . . . . . . . . . . 154
7.3 Average ambiguity probability at the time of an error . . . . . . . . . . . 158
INTRODUCTION

Synchronization is woven into the very fabric of our understanding of the most basic concepts of the universe. The process of synchronization in its own amazes us with qualities we apply to intelligence. From symphonies and figure skaters, to flocks of animals and fundamental particles, the results of synchronization are evident. Beyond amazement, the models of synchronization may well serve as the first opportunity for a new understanding required for efforts involving dynamical systems. Of particular importance here is the synchronization of networked nodes.

Networks dominate our understanding of the universe [Bar03b, BA99]. They exist in the very language in which this work is written, with the relationships between various components of syntax being tied to others in ways that allow us to communicate [Bar03b]. Networks are essential in the more modern communication systems, with phone and computer connectivity crisscrossing the planet [Bar03b, Bar03a]. Networks are the very thread of our social structure, with family, friend and professional relationships providing the primary linkages between us [Wat03]. Even more fundamental, networks exist in the very foundations of interactions in biological systems, with complex food-webs, regulatory gene
networks and complex molecular relationships forming the very building blocks of the natural world [OB02, Dav02, Lee02, Rav02, RPK04, BI99, Har99, Mil02]. The most surprising extent of network influence is the complex relationships in the strange worlds of quantum system, where interactions have to be completely inferred because of the inability to measure components on that scale [Str03].

Less obvious is the role of synchronization of networks in the perception of intelligence, coordination and emergent behavior [Str03, BRV01]. This can be seen by observing complex dance moves performed by a pair or a troupe. Likewise, it can also be seen in the complex interactions of a symphonic orchestra or the team play in sports. At the same time, natural systems like flocks of birds or swarms of insects are seen as having complex intelligent interaction and have become a topic of modeling for artificial machine optimization systems [KR01].

The focus of the research presented here is to discover the limits for determining the network structure of loosely coupled oscillators based on the observation of the behavior of the network, specifically, node synchronization. Moreover, this work defines the requisite characteristics and underlying behaviors necessary for the application of a theoretical mechanism for determining the underlying network topology in a network of loosely coupled natural oscillators. This effort is motivated by the opportunities for using synchronization in the characterization of network structure and growth. The basic components and resulting parameter relationships provide the basis for determining the system requirements for
construction of a measurement system necessary to determine the network structure based on the behavior of the nodes. The more distant motivation is the potential for manipulating synchronization and/or network structure and growth, with further parallels for impacting overall system stasis.

The primary underpinning that enables this research is the premise that the development of synchronization among a group of loosely coupled oscillators is indicative of the nature of the underlying network structure. That is, cells that tend toward synchronization are inter-connected, whether directly or otherwise. By observing the trends toward synchronization, the underlying network structure may be determined and the growth modeled. Through this, the network structure and system stasis could be manipulated.

The genesis of this work is traced to the steady growth of literature driving the understanding of the nature of networks [Str01]. This has included tools that expand the basis of our view of complex networks from random graph theory, Erdős Rényi Graph Theory [Erd97b] [Erd97f] [Erd97e] [Erd97h] [Erd97a] [Erd97d] [Erd97g] [Erd97c], to the still growing fields of Chaos [Str03], Nonlinear Dynamics [Str94] and the Theories of Scale-Free Networks [Bar03a].
1.1 Strategy and Methodology

The general approach to this effort is to define a simplistic analytical framework in which key components of networks of coupled oscillators can be isolated in such a way as to determine the relationships between the various components. That is, the relationship between the number of nodes in a network, $N$, the number of connections in the network, $L$, the number of connections of a given node, $\eta$, the distribution of the phases of the network, $\sigma_\theta$, and the resolution of measurement of the components of the network, $\rho$, and system noise, $\gamma$. The relationship of values of these components will be shown to provide the limits of ability to ascertain the underlying network structure. While there is a brief discussion of the current state of the art on network synchronization in Chapter 3, the ability of a network to synchronize is not specifically addressed in this work, as this is a question that will require some time to conclude for all but the most simplistic networks [Str00] [Str03] [SK04] [Win01] [RPK04]. Therefore, it is assumed that the parameters described are within the necessary limits of synchronization.

1.2 Contributions

The significant results of this work can be characterized as follows:
• **A new approach.** The basis for this work is a network model and an underlying mathematical model that is modified to demonstrate the relationship between the variables in network construction. This framework provides a new fundamental approach to the difficulties of working with complex systems. A parallel idea that explains this issue is described well in Wil McCarthy’s work on underlying science required for nanotechnology [McC03]. McCarthy explains that the interactions at the nano-materials level is too complex to utilize the statistical models provided for each component at the quantum scale and too weird to utilize the simple mechanics-based approaches [McC03]. This provides a perfect example for the new math that Strogatz insists must be found in order to deal with the reality of the complex dynamics which we are now facing [Str94].

• **Empirical Model.** To reinforce the notions proposed in the analytical framework, a stochastic model of a sample system is provided to demonstrate the relationship of the proposed limits.

• **Theoretical and Empirical Evidence.** The culmination of the Theoretical and Empirical Study provides a rigorous set of parameters that can be applied to subject networks to determine the feasibility of ascertaining the underlying topology based on the behavior of the nodes in the system.
1.3 Overview of Dissertation

The next two chapters provide the detailed background material required for later discussions. In Chapter 2, the core concepts of the nature of oscillators, oscillatory behavior, network topologies and brief explanations about common nonlinear dynamics concepts are described. The definitions for the key components are provided along with several common examples to emphasize the major concepts. Chapter 3 expands on these notions with a series of models that have been built as a framework of the current understanding of network behavior of groups of loosely coupled oscillators. The primary emphasis of the works cited is in the determination of the long term qualities of the network based on the topology and characteristics of the individual nodes or oscillators. There is a discussion of the brief body of work related to determining network structures from the behavior of the oscillators. The obvious trait found in this work is the trivial nature of the groups of oscillators described.

In Chapter 4, a traditional approach to determining network structures utilizing cellular automata is demonstrated including a detailed discussion of the foundation of automata theory necessary for application to the problem of determining network structure. After a detailed walk-through of the traditional approach, the shortcomings and complications of this approach are highlighted. This leads directly to the new model and technique derived for the analysis of network structures. A discussion on a return to cellular automata theory and an insertion of the techniques derived from this work are presented in Chapter 8.
The next section of the work describes the primary analysis conducted during this work. In Chapter 5, an extensive view of the new approach and models is described along with the isolation of the related variables into a form that can be used for a detailed discussion of the relationship of the variable both to each other and their role in determining the underlying network topology. This is followed in Chapter 6 by a stochastic computational model that exercises the relationships described in Chapter 5. The results from both the analytical and stochastic models are provided in Chapter 7.

The material in Chapter 8 is provided as thoughts on additional related work that followed the same topic but was limited in the initial analysis. A detailed discussion of these limitations and the extensions that will allow a new approach and an expansion into other areas of research is also provided. Continuing that theme, Chapter 8 provides the outcomes and limitations of the entire body of work as well as possible future extensions.
BACKGROUND - PRIMER

The study of networks of coupled oscillators can be characterized as typical of the current
trends in interdisciplinary scientific research through the professional communities. Fields as
diverse as sociology and theoretical quantum mechanics are brought together to determine
commonalities in order to develop badly needed analytical tools, techniques and understanding [Str94].
The majority of the early work produced on networks of coupled oscillators is
found in research related to biophysics [Win01]. Still, network theory has evolved in recent
years and has begun to mark the study of networks of coupled oscillators with advances away
from the overly simplified models required in early scientific analysis [BK04] [SK04] [Wat99]
[AS05].

The number of works surrounding the theory of networks, synchronization and behavior
of natural coupled oscillators has continued to grow but is relatively new. In the theory of
networks alone, the work in Random Network Theory was not developed until the 1960’s
[Erd97b] [Erd97f] [Erd97e] [Erd97h] [Erd97a] [Erd97d] [Erd97g] [Erd97c]. More recently,
with the shift from the basics of random network theory, a marked growth in the significant
body of popular literature has begun to appear [Str03], [Wat03], [Bar03a]. At the same time,
the literature in a myriad of fields has been newly peppered with scientific works on these topics [BCS97] [Win01] [RPK04] [BI99] [Dav02] [Har99] [Mil02] [Lee02].

The supporting material provided in this chapter is divided into three sections. Section 2.1, Coupled Oscillators and Synchronization, provides the basic definitions used throughout this work and represents the most common explanations in Synchronization Theory. The definitions and explanations include nodes and basic node behavior, oscillation, as well as fundamentals of coupling of nodes. Section 2.2, Networks, reviews the growing body of work on the basics of networks, emphasizing the common themes regardless of the system in question. Lastly, Section 2.3, Dynamics, is an introduction to a set of concepts from Nonlinear Dynamics and Chaos theory that provide the tools for working on the subject systems.

2.1 Coupled Oscillators and Synchronization

The individual elements that are considered in this work are referenced as nodes. Nodes are simply an abstract fundamental self-contained atomic unit and are analogous to cells in Cellular Automata or the similar abstraction of a Finite State Machine. The common feature of the nodes in this discussion is their oscillatory behavior. Specific models of the individual behavior will be discussed in greater detail in Chapter 3. For now, the nature of the oscillators in question is simply regular periodic functions.
Pikovsky, Rosenblum and Kurths provide good working definitions for oscillators of the type discussed throughout this work [PRK01]. The points provided by Pikovsky, Rosenblum and Kurths are that the individual components or nodes are self-sustained oscillators with the following key characteristics [PRK01]:

1. Each node is an active system whereby it has an internal “source of energy” that is used to sustain constant oscillation.

2. Likewise, each node is able to maintain oscillation without external forces.

3. In the absence of external forces, each node tends to the same oscillatory behavior regardless of the start conditions. This oscillatory behavior is known as the natural frequency of that node.

4. When presented with small, transient external forces, the oscillation will be disturbed and then return to its natural oscillatory behavior.

These traits are critical in the concepts discussed in the later chapters. Another way to maintain the perspective is that just as with Cellular Automata, and more directly, Finite State Machines, the nodes are considered as autonomous dynamical systems in the mathematical sense [PRK01].

More basic to the understanding of oscillation are the concepts of frequency and period. As an example, a common pendulum on a clock swings at a given frequency, which can be seen as a number of oscillations in a specified unit of time. The period is the amount of
time between the recurrence of the same point between two cycles. This can be seen directly in the example of a pendulum clock illustrated in Figure 2.1. The cyclic frequency, $f$, is inversely proportional to the period, $T$, $f = 1/T$.

![Diagram of a pendulum clock](image)

**Figure 2.1: Pendulum Clock demonstrating the relationship between angle $\alpha$, change in $\alpha$ over time $\dot{\alpha} = \frac{d\alpha}{dt}$ and the Period, $T$.**

The further extension and the form more often used is the idea of angular frequency, $\omega$. Angular frequency $\omega = 2\pi f$ and will be maintained as the preferred or natural frequency of the node in the absence of interaction with neighbors.

The next concept and component required to embody synchronization is the idea of connection of nodes. Central to the type of connection for this study is *coupling*, and with it, *synchronization*. That is, nodes that are independent and that have their own internal drive and preferred frequency, when exposed to neighbors to which they ”couple”, synchronize by

---

11
adjusting their frequencies in an attempt to match their coupled neighbors. The distinction that is necessary is that the neighbors are not directly connected as shown in Figure 2.2, but rather, change through indirect influence. This indirect influence is best seen in the earliest known scientific discovery of synchronization, Huygens Clocks.

![Figure 2.2: Two Pendulum Clocks shown with direct connection not loose coupling.](image)

### 2.1.1 Huygens Clocks

Huygens Clocks were the first scientifically studied coupled oscillators [Str03, PRK01]. The clocks were found as part of the attempt to improve the accuracy of ocean navigation. At the time, navigation was achieved by relating the position of celestial objects to a measure of the earth’s rotation via chronological time. This meant that the more accurate
the chronological time of celestial measurements, the more accurate the navigation. The Dutch physicist, Chistriaan Huygens, in 1665 made an attempt to improve this accuracy by placing multiple pendulum clocks on board a ship and by using the two and their difference to determine time \[\text{Str03}\]. What Huygens noticed was that the two clocks synchronized the swing of their pendulums. It was eventually realized that this was occurring because of undetectable vibrations passing through the beam in which the two clocks were hung as seen in Figure 2.3 \[\text{PRK01}\]. The beam provides a loose coupling.

![Figure 2.3: Two Pendulums Loosely Coupled; The common beam transfers information between the two clocks and acts a connection. Adapted from \text{PRK01}.](image)

Pikovsky, Rosenblum and Kurths provide their version of Huygens’s experiment in the following example \[\text{PRK01}\]:

13
Begin with two pendulum clocks in separate locations. The clocks can have their natural frequencies adjusted mechanically by adjusting the length of the pendulum. Each clock is set to its own natural frequency. These natural frequencies are labeled $f_1$ and $f_2$. The difference between the two is $\Delta f$ where $\Delta f = f_1 - f_2$. This is can be seen as the beat frequency. The combined difference changes over time, the two clocks frequencies come together, separate, and come back into tune at $\Delta f$. Continuing the experiment, the clocks are placed on Huygens’s common beam, and the new frequency of each clock is measured. These new frequencies are $F_1$ and $F_2$. The new $\Delta F$ is the difference between $F_1$ and $F_2$, $\Delta F = F_1 - F_2$. An important insight is gained by looking at the plot of $\Delta F$ versus $\Delta f$ as shown in Figure 2.4.

![Figure 2.4: Synchronization Region: Demonstrates the effect of loose coupling between pendulums with the difference in coupled frequency, $\Delta F$ plotted against the difference in natural frequency $\Delta f$. Adapted from [PRK01].](image-url)
From Figure 2.4, the synchronization regions indicate the area in which the coupling strength between the two clocks and their natural frequencies are such that the two clocks are willing to synchronize or adapt their natural frequencies to match another frequency.

This type of behavior can be contrasted by naturally-occurring dynamics such as population changes due to predator-prey interaction, where the life cycles adjust but are forced by the cycle of the other species [PRK01]. This type of dynamic is more similar to the two clocks shown in Figure 2.1. In both cases, the similar oscillation is a result of a direct connection not an indirect influence.

2.1.2 Natural Oscillators

While Hyguens Clocks provide a good foundation for understanding the fundamentals of coupled oscillators, there are many examples that have been documented and studied in varying extremes. Among these are species of fireflies, applause in European audiences, menstrual cycles of women, the Millennium Bridge, Optical Lasers, Josephson Junctions, and others.
2.1.2.1 Fireflies

The central character in Strogatz’s popular account of synchronization is that of fireflies in Thailand and the United States [Str03]. The remarkable accounts had circulated for years and had been recorded in Sir Frances Drake’s log from 1577 [Str03]. In these accounts entire valleys would fill with simultaneously flashing fireflies. There had been many interpretations for the effect including a view that the fireflies must have a conductor in order to act in such a great symphony. Ermentrout and Rinzel studied the range of synchronization of the *Pteroptyx malaccae* firefly in detail, including adjusting the fireflies’ frequency by stimulating them with an artificial light near their natural frequency [PRK01]. Through this experiment, the limits of synchronization could be explored.

2.1.2.2 Applause

An effect that occurs in European audiences is synchronous applause [PRK01] [NRB00]. While audiences in the United States tend to applaud in an asynchronous manner, European audiences tend to clap in unison. The synchrony occurs without direction and is a democratic effect, with individuals choosing their preferred rate and adjusting it to match their neighbors.
2.1.2.3 Menstrual Cycles

While often viewed as controversial, Strogatz notes research by McClintock documenting the relationship of chemical secretion in women in close quarters to changes in their menstrual cycles [Str03]. The study was conducted by observing changes in the women's biological cycles in relation to swabs of sweat that were placed on their lips from samples taken from other women at different stages in the biological cycle from their own. The evidence provided that the women adjusted their cycles in a predictable manner.

2.1.2.4 Millennium Bridge

One of the most amusing accounts of naturally occurring synchronization is the occurrence of a lateral sway of London's Millennium Bridge when it opened June 10, 2000. The bridge was marveled as an open display to engineering construction. However, as soon as it opened, it was noticed that the bridge began to sway laterally. A number of explanations for the lateral sway were suggested. In the end, an analysis determined that the amplified oscillation was generated by the shifting of weight by humans when they walked (think of the extreme lateral exaggeration of speed skaters) [SAM05]. As a sufficient group of individuals randomly synchronized, the bridge shifted slightly. As the bridge shifted, the number of individuals in the group that were synchronized increased. As the number of individuals in
the group increased, the lateral sway increased. This continued until the dynamic range of
the bridge and the people reached equilibrium.

2.1.2.5 Lasers as Oscillators

Strogatz provides an excellent explanation of the effect of laser coherence [Str03]. He gives
a creative metaphor of waking up on an alien planet and finding a stool and watermelon.
Wondering what the purpose of the two items are, you place the watermelon on the stool. As
soon as you do, the watermelon begins to become agitated and falls off the stool, breaking
on the floor, and a single watermelon seed shoots off like a bullet. This is an example of how
normal light is formed by raising the potential and creating the release of the single photon.
In the case of laser light, there are hundreds of watermelons on hundreds of stools, except at
the point a seed fires from the first watermelon, it passes through another watermelon. As
it passes, it goes directly through and continues without interference. The remarkable effect
is that when it reemerges, there is a second seed identical to the first and the second seed
is has an identical trajectory. The seed has in effect been cloned. The quandary is why the
photons choose to synchronize.
2.1.2.6 Josephson Junctions

Josephson Junctions are a strange coincidence of complex interactions at the hairy edge of quantum dynamics that exist in the realm of Einstein-Bose Condensates. The Josephson Effect results as current flows across two semiconductors separated by a dielectric barrier. While the phenomenon is strange, the most remarkable notion is that the governing equations are those that occur in simple pendulum clocks [Jos74]. Due to the simple equations and the promises of advanced technology developments that Josephson Junction are such a popular model for research [Jos74].

2.1.2.7 B-Z Soup

Belousov-Zhabotinsky soup is a remarkable oscillating chemical reaction that provides a great deal of insight to the underlying physical traits of naturally occurring oscillators [Win01]. Boris P. Belousov was a Russian biophysicist attempting to replicate the Krebs cycle via a chemical reaction in the 1950s. Ultimately he found that a solution of citric acid, acidified bromate (BrO3-), and a ceric salt formed the basis for a self-propagating reaction that cycled from yellow to clear.

The process was refined by Anatol M. Zhabotinsky, who replaced citric acid with malonic acid. The results of the process were published during a conference in Prague in 1961, and
were picked up by the world community. The process also became a primary emphasis of Arthur Winfree whose work will be discussed in more detail in Chapter 3.

Belousov-Zhabotinsky soup goes beyond a simple novelty of chemistry. More importantly, the base dynamics are similar to those associated with other naturally occurring, self-sustaining oscillatory interactions of groups of cells, namely cardiac cells [Ali95].

2.2 Networks

Having reviewed a variety of oscillators and the basics of coupling and connectivity, the importance of the ideas surrounding the topology and make-up of various connected groups now emerges. As a basis, the theory of networks is rooted in early graph theory. One of the most historical problems in graph theory is presented in the Bridges of Königsberg. In Albert-László Barabási’s popular book Linked [Bar03a], Barabási conveys the historical account of the problem provided by the Bridges of Königsberg. Simply stated, the question is, can a closed route be created in which one would cross all seven bridges and only cross them exactly once? A layout of the problem is shown in Figure 2.5. Euler proved in 1736 that such a route cannot be constructed [Bar03a]. Euler began his proof by constructing a graph similar to the one shown in Figure 2.5. In the graph, the areas divided by the rivers become nodes (A-D) and the routes between them become edges (a-f). By showing that the
nodes with an odd number of edges must be a start or end point, he could show that there was no way to construct a route with the graph provided [Bar03a].

Figure 2.5: The Seven Bridges of Königsberg is a classic problem in graph theory. Adapted from [Bar03a].

When it comes to the question of determining the behavior of large populations of connected individuals, the theoretical took a very different turn. In the 1960’s, Hungarian mathematicians Paul Erdős and Alfréd Rényi published an extensive theoretical account of how networks are formed [Bar03a] [Erd97b] [Erd97f] [Erd97e] [Erd97h] [Erd97a] [Erd97d] [Erd97g] [Erd97c]. As a prime component of the theory, the underlying assumption is that the networks formed connections at random. To view the utility of this, the typical question could be asked of how a group of people might share some piece of information. To determine this, the connectivity of the group would have to be supposed. By using random graph
theory, we suppose that the probability of a connection is random and we can now model the diffusion of the information throughout the group.

One the most fascinating aspects of the random theory of networks, and one that has strong corollaries to other effects later in this work, is the effect of how groups of items might be connected with the addition of an increasing number of random links. In looking at this intuitively, the number of connections and total number of nodes connected might seem to grow on some form of exponential curve. The surprising fact is that the growth of the number of nodes connected do not have this effect but rather a significantly different one. By looking at the number of nodes connected that form the largest group, which is characteristic of the total nodes connected, what is found is known as a phase transition. This is shown in Figure 2.6. Phase transitions are found throughout nature and will be discussed in more detail in Section 2.3.3.

2.2.1 Scale-Free Hierarchical Modular Networks

While the majority of the last forty years of network theory has been dominated by analysis of random networks, a significant shift in the understanding of networks occurred, partly spurred by an analysis of the Internet \cite{BA99}. The discovery that the Internet was not organized as a random network but had a scale-free hierarchical modular structure paralleled
Figure 2.6: Demonstration of the phase transition nature of connectivity in random graphs. As the number of random individual connections approaches unity, the number of nodes in the largest group of connected nodes rises dramatically. Adapted from [Wat99].

A simple consequence of the study can be seen in the research of Barabási and Albert where the underlying structure of the Internet was described [BA99] [BRV01]. It was believed that the Internet would be structured as a random network due to the random nature in which it was constructed [BA99] [BRV01]. The reality is that the structure was in fact a scale-free modular hierarchical design. A sample of the construction of a hierarchical scale-free network can be seen in Figure 2.7, Figure 2.8 and Figure 2.9.
Figure 2.7: Hierarchical Scale-Free Network developed from one node to four.

Figure 2.8: Growth of a Hierarchical Scale-Free Network developed from one node to four to sixteen.

Figure 2.9: Hierarchical Scale-Free Network developed from one node to four to sixteen to sixty-four.
The growth of this network is surprisingly simple \cite{BA99, BRV01}. In the first step, the growth occurs by making a copy of the node and connecting the nodes to each other and the center node in Figure 2.7. In the second step, copies of the three additional copies of the entire group are constructed, and the peripheral nodes are connected to the center node of the center group and the center nodes of the groups are connected to one another, see Figure 2.8. This process is repeated in Figure 2.9 where sixty four nodes are now connected. The Iterative and hierarchical nature of the attachments define the network. This type of structure seems to be one of the most predominant in nature and is the structure of the Internet \cite{BA99, BRV01}.

\subsection*{2.2.2 Social Networks}

Hierarchical Scale-Free Networks are very prolific and even appear in the common setting of social networks \cite{Wat03}. Duncan Watts’s popular book \textit{Six Degrees: The Science of a Connected Age} provides a good perspective on the relationship between the growth of hierarchical scale-free networks in the previous section to the random network theory of Erdős and Rényi.

To frame the discussion, Watts revisits the famous experiment by the Harvard sociologist and psychologist Stanley Milgram \cite{Wat03}. Milgram conducted his experiment in 1967, when he sent 160 letters at random to people from Boston to Omaha and instructed them to
forward the letter to a person they knew by first name who was most likely to know someone who could get the letter to a single specific individual Milgram had picked in Boston [Wat03].

Milgram’s study showed that the median number of steps the letters took were 5.5 [BA99]. This experiment, as well as popular literary works, developed into the now famous Six Degrees of Separation. The popularity of the belief that all people could be connected to all other people by six intermediate steps reached a high point with the game “The Six Degrees of Kevin Bacon” where the object is to connect any actor to Kevin Bacon through six intermediate acting roles [Wat03]. The game had several variations, perhaps the most ironic is the version in which scientists and mathematicians discover their connectivity to Paul Erdős [Sin02].

More importantly, Watts took the distance between connections and related it to both random networks, scale-free networks and locally connected networks or rings [Wat03] [Wat99] [WS98]. By starting with networks that were locally-connected ring topologies, and adding random connections to distant neighbors Watts was able to show the emergence of “small worlds” or six-degrees of separation; however, the addition of more connections ultimately resulted in random networks [Wat03] [Wat99] [WS98]. By adding a probability of preferential attachment to the randomness, the networks further emerge into hierarchical scale-free networks that are also “small-worlds” [Wat03] [Wat99] [WS98].

From this work, the most exciting outcome of the work in social networks is the experimentation regarding the robustness of various types of networks to failure and the propaga-
tion of information within these networks [Wat03, WS98, Wat99, AJB00]. The total sum of these studies has led to a much clearer ability to visualize the vulnerability and sensitivity of various networks and components of networks to a range of attacks.

2.2.3 Network Motifs

A slightly different approach to the study of networks is the one undertaken by Milo et al. where they analyze the preponderance of various network motifs or configurations of nodes and connections [Mil02]. In studying the various motifs, Milo et al. used an enumerated list of the possible three-node interconnection schemes [Mil02]. These can be seen in Figure 2.10.

One of the key components of the study by Milo et al. is the detailing of an algorithm that is used to determine network motifs. These motifs are defined as interconnection schemes that occur orders of magnitude more often than randomly connected networks with the same parameters. Milo et al. used an algorithm to derive various systems’ motifs and demonstrate that motifs are unique to the particular functions examined, and that there are trends in the relationship between the motifs and the types of functions performed. The functions examined ranged from gene transcription to digital circuits, including a reference to test functions for digital logic.
2.3 Dynamics

The difficulty of studying systems of networks of coupled oscillators is the significant lack of analytical tools available for problems of this type [Str94]. The primary shift that is necessary is one in which the perspective changes from a quantitative to a qualitative view of a system. This shift is encapsulated in the history of the “Three Body Problem”.

Newton solved the problem of the effect of two celestial bodies (his focus was the Sun and the Earth) on each other by providing the differential equations that define the relationship [Str94]. The problem then turned to a similar set of differential equations for defining the relationship between three objects (the Sun, the Earth and the Moon). To Newton’s dismay, the differential equations were not tractable. Eventually, it was determined by Poincaré that it was impossible to provide explicit formulas that define the motion of three general objects by gravitational interaction [Str94]. From the remarkable discovery that explicit formulas could not be defined, Henri Poincaré’s geometrical description of stability of planetary orbit...
emerged. Strogatz provides that the change in perspective was a change from providing the location of the planets at any given time to whether the system was stable and if objects were going to fly off \[Str94\]. This change marked the birth of the study of Dynamics \[Str94\].

Strogatz states that “This [dynamics] is the subject that deals with change, with systems that evolve in time” \[Str94\].

\subsection{Limit Cycles}

One of the primary tools used in dynamics is the display of the geometric relation of motion and change in motion shown by limit cycles demonstrated in a special case of the Poincaré map found by stroboscopic measurement of the frequency of oscillation. While this may sound complicated, it is in fact quite practical. Stroboscopic measurement is common in a wide variety of optics-related applications \[PRK01\] and more commonly in automotive mechanics. It is demonstrated by focusing a strobing light on another rotating or oscillating object. If the strobe of the light and the objects in motion are at the same frequency, the object will appear stationary. If, on the other hand, the two are at different frequencies the object will appear to move. The motion will appear along the path of the object. By further setting the amplitude of the object to a common reference, i.e. an amplitude of 1, then the focus is simply the difference of the phase or stage of the periodic cycle. This is particularly relevant when looking at groups of coupled oscillators.
This can be seen by the pendulum example plotted as position $x$ and momentum $\dot{x}$.

Begin by looking at a relationship of the angle of pendulum to the sinusoidal wave plotted over time in Figure 2.11.

![Diagram of pendulum clock](image)

**Figure 2.11:** Pendulum Clock demonstrating the relationship to a sinusoid by the relationship of the vertical angle $\alpha$, to the change in vertical angle $\alpha$ over time $\dot{\alpha} = \frac{d\alpha}{dt}$ and the Period, $T$.

Next, move the sinusoid to plot the angle, $\alpha$, against the rate of change of $\alpha$ over time $\dot{\alpha} = \frac{d\alpha}{dt}$ and the Period, $T$, as shown in Figure 2.12. In this case, the amplitude of the sinusoidal wave provides the radius of the circle generated in Phase Space in Figure 2.12.

On one hand, if a plot is generated of a number of different pendulums as they synchronize, the attracting cycle in phase space, or the cycle in which all pendulums move toward, defines the stable limit cycle of the group of oscillators will be found (assuming it exist) as
seen in Figure 2.13. If however, the stroboscopic measurement is taken and the phase of each of the pendulums is plotted by observing the relationship to a common frequency, then a Poincaré map can be shown. This can be seen in Figure 2.14 for a group of oscillators moving toward synchronization and Figure 2.15 for a group of synchronized oscillators. The size of the radius of the circle is related to the amplitude of the oscillation function or sine wave. With different amplitudes, there are different rings.
2.3.2 Arnold Tonges

Pikovsky and Rosenblum and Kurths provide a slightly different view, but an important perspective in the theory of synchronization of coupled oscillators, is the region of synchronization \cite{PRK01}. Limiting the discussion to the simplest possible model, in order for one oscillator to synchronize with another oscillator or a group of oscillators, the first oscillator must have a coherence factor, $\varepsilon$, that affects their willingness to adjust their frequency. The
range of natural frequencies, \( \omega \), plotted against the difference of frequency of the other node or nodes, \( \Omega \), and the natural frequency of the reference node as seen in Figure 2.16. The plot is based on a fixed coherence factor, \( \varepsilon \). Likewise a similar plot shown over a range of coherence factors, \( \varepsilon \), phase difference and phases provides the graph in Figure 2.17.

**Figure 2.16:** Synchronization window for a fixed coherence factor. This is shown for a fixed coherence factor, \( \varepsilon \), and plotted over range of natural frequencies, \( \omega \) (shown on the horizontal-axis), against the difference of a driving frequency \( \Omega \) and the natural frequency, \( \omega \) (shown on the vertical-axis). Adapted from [PRK01].

**Figure 2.17:** Synchronization window for a range of coherence factors, known as an Arnold Tongue (adapted from [PRK01]).
Phase transitions are important natural phenomena. The simplest demonstration is in the freezing of water. Until the water reaches 0 °C, no ice formation occurs, but at the point where water reaches 0 °C, ice formation instantly begins. A graph of similar effect seen in Figure 2.18 and is explained in the next paragraph.

![Graph](image)

**Figure 2.18:** Demonstration of the phase transition nature of connectivity in random graphs. Adapted from [Wat99].

Erdős Rényi theory of random networks provides that for a given set of nodes or vertices and various sets of a number of random edges or connections that connect one node to another, that as the number of random connections increases, the percentage of the population of nodes in the largest connected group demonstrates a phase transition [Wat03], as shown in Figure 2.18. At first, while the number of random connections is less than an average
of one per node, the percentage of nodes that are in the largest group is near zero. At a critical point, when the number of random connections reaches an average of one per node then, just as in the case of freezing water, instantly the percentage of the population in the largest connected group rises quickly to 1.

Similarly, Winfree provided that as a group of oscillators becomes more homogeneous, the percentage of the population at the same synchronized rate also demonstrates a phase transition [Str03], as seen in Figure 2.19.

![Figure 2.19: Demonstration of the phase transition nature of synchronization in diverse populations. Adapted from [Str03].](image-url)
2.4 Summary

There are a number of tools and phenomena that have been developed and discovered in the last few years for studying networks of coupled oscillators. While the advances are substantial, the field is still hampered by the mathematical intractability of most of the problems. With that, the characterization of the systems has become the norm. After looking at the basic components and individual tools the next chapter looks at the groups of oscillators and studies of group effects.
BACKGROUND - NETWORKS OF COUPLED OSCILLATORS

Scaling up from the basic components required for the study of networks of coupled oscillators, this chapter turns to the studies of the subject as a whole and to detailed models that have been used to establish the current knowledge. Arthur Winfree’s work is perhaps the most comprehensive, especially in terms of historically documenting the efforts on the subject and maintaining a close relationship with the practical application of the study [Win01]. In the mid 1970’s, Yoshiki Kuramoto began to expand Winfree’s work by placing it on firmer mathematical footing [Str00]. Kuramoto did so at the cost of trivializing the real-world applicability of the models in order to make the math sufficiently tractable. Steven Strogatz sought Winfree out in 1982, during a summer position at Winfree’s lab [Str03]. Strogatz, who eventually received his Doctorate from Harvard and became a faculty member at MIT and Cornell in Applied Mathematics, worked with Winfree on B-Z experiments and towards a mathematical framework for scroll rings, three dimensional propagation of waves through excitable media [Str03] [Win01]. Strogatz’s larger impact has been to make the topic available to a broader community, both in mathematical depth and as application to a wide range of fields.
The next section in this chapter further details the work of each of these men. Then in Section 3.2, the high-points of the models developed by Winfree and expanded by Kuramoto are provided. It also includes a visual example provided by Strogatz to explain the development and meaning of the models. While the models are sufficiently general, the last section briefly describes current extensions to locally and non-locally coupled oscillators.

3.1 Current Work

The current work described in this section show the growth of the understanding of the prediction of the behavior, characterization and simulation of networks of coupled oscillators. The follows chronologically from Winfree to Kuramoto and Strogatz. The work of each of these progresses into the models that are described in the next section.

3.1.1 Winfree’s Work on Coupled Oscillators

Winfree’s work was honored in 2000 with the Norbert Wiener Prize in Applied Mathematics. Winfree’s theoretical detailing of his applied research to an understanding of biological cycles, oscillation and synchronization provides the foundation for the current theories and, more importantly for this work, the underlying mathematical models of synchronization [Win01]. Winfree made tremendous advances in providing a detailed perspective on the
workings in synchronization of biological oscillators through his primary approach, demonstration and simulation of the subject models \cite{Str00}.

### 3.1.2 Kuramoto’s Work on Coupled Oscillators

Kuramoto’s work differs from Winfree’s in two notable ways \cite{Str00}. First, the level of mathematical analysis is far more detailed than any previous work including Winfree’s. A witness to Winfree’s intuition, much of the detailed work begins with Winfree’s ideas and expands them to explain in depth the justification for Winfree’s results. The second difference is a result of the priority of the first. That is, the detailed mathematical analysis required simplification of the models, which tend to deviate much further than Winfree was willing to do to maintain biological relevance.

### 3.1.3 Strogatz’s Work on Coupled Oscillators

Strogatz popularized the work in synchronization of coupled oscillators with his 2003 book *Sync: How Order Emerges from Chaos in the Universe, Nature and Daily Life* \cite{Str03}. This book was followed by a number of articles in *Science* and *Nature* \cite{SAM05, WS98, Str01}. These articles were associated and timed near Barabasi’s network article, also appearing in popular literature \cite{BA99, AJB00}. Strogatz went a long way to explain and expand the
detailed work by both his mentors, Winfree and Kuramoto. Strogatz’s ability to create excellent visual metaphors for various critical effects is very enlightening. Strogatz spends a great deal of effort explaining the synchronization effect of fireflies in Thailand and the Appalachian Mountains. He also explains the intricacies of the models of Winfree and Kuramoto \cite{Str00}.

Strogatz separates from Winfree and draws to Kuramoto primarily in the applied versus theoretical mathematical approach. While Strogatz utilizes the visual advances made by Winfree’s biological analogies, simple mathematical models and stochastic simulations, he also goes farther to tie these to Kuramoto’s restricted models and detailed analysis. Strogatz also brought his own intuition about a new set of mathematical tools that could be applied to this area of research. These are primarily tools from nonlinear dynamics, more directly, stability analysis.

### 3.2 Models

The models discussed in this section represent the current best understanding of the prediction of the behavior, characterization and simulation of networks of coupled oscillators. The work is provided by Strogatz, Winfree and Kuramoto. These models will make up the primary approach used in this dissertation.
3.2.1 Strogatz’s Example to Demonstrate Models

Strogatz provides a very good analogy of groups of oscillators seen as a group of individuals in a running club [Str03]. The individuals run around a circular track. The distribution of the number of fast versus slow runners can be varied. This distribution is considered the homogeneity of the population of oscillators. The individuals can speed up and slow down at will, still have a preferred pace, and are motivated by calls from their friends in the club. The influence of a given runner, and the attention of a given runner to another, could be adjusted by a parameter that is provided. An illustration of three such clubs and their respective dispersion on the track is shown in Figure 3.1

![Running Club Analogy](image)

Figure 3.1: Running Club Analogy with three different clubs each dispersed around the track. Adapted from [Str03].
3.2.2 Winfree’s Model

Looking specifically at Winfree’s model, it consist of three components, (1) a node’s preferred pace, (2) a node’s sensitivity to influence by connected nodes, and (3) the total influence of other nodes [Str03]. This is provided by Strogatz in the expression of Winfree’s model by Equation 3.1 [Str00]. The key points are that the change in phase $\dot{\theta}_i$ is due to $X$, a phase-dependent influence function, and $Z$, a sensitivity function. Additionally, $\omega$ represents the $i^{th}$ node’s natural frequency, and $\theta$ is each node’s current phase as found by stroboscopic measurement. Additionally $j$ and $i$ are specific increments of the total population, $N$, of individuals.

$$\dot{\theta}_i = \omega_i + \left( \sum_{j=i}^{N} X(\theta_j) \right) Z(\theta_i) \quad (3.1)$$

Using Strogatz’s analogy, $\dot{\theta}_i$ is the change in pace of a given runner. Likewise, $\omega$ is the preferred pace of the runner in the absence of other runners. The function $X$ is the influence of one runner on another. The phase dependence can be thought of as how close the two runners are on the track when one calls out to the other. The function $Z$ is the willingness of a runner to change pace.

Looking back at the information from Chapter 2, if the function for $X$ and $Z$ are fixed, and various populations are explored, then the percentage of the population synchronized is
a function of the homogeneity of the population as shown in Figure 3.2 [Str03]. The effect is that of the phase transition as described in Section 2.3.3 and shown in Figure 3.3 [Str03].

Figure 3.2: Running Club distribution with three different clubs with various degrees of homogeneity Adapted from [Str03].

Figure 3.3: Demonstration of the phase transition nature of synchronization in a diverse population of running clubs. Adapted from [Str03].
3.2.3 Kuramoto’s Model

Kuramoto made a significant breakthrough by adapting Winfree’s original model using the assumption that all nodes were connected and that the connection had an equal influence [Win01]. From this, the equation for the changes in phase for each element becomes:

\[ \dot{\theta}_i = \omega_i + Kr \sin(\psi - \theta_i) \]  \hfill (3.2)

As with Winfree, the change in phase, \( \dot{\theta}_i \), of the \( i \)th node remains a function of the natural frequency of the node \( \omega_i \). The change is that now \( X \), the phase dependent influence function, and \( Z \), a sensitivity function, are defined as \( K \), the sensitivity function, normally set to \( K = 1 \), additionally, \( r \) is the coherence of phases of the entire group (defined below) and \( \sin(\psi - \theta_i) \) is the difference of the phase of the individual \( \theta_i \) and the average phase of the group \( \psi \). The sine function acts as a way to normalize the difference in phase space.

Expanding more, the order parameter is provided by \( re^{i\psi} \). This shows the state of the entire population as a function of a single complex value. Plotted on the complex phase space, the coherence of the group is expressed in the value \( r \) plotted as the radius and \( \psi \), the average phase, shown as the angle on the complex plane.

\[ re^{i\psi} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j} \]  \hfill (3.3)
The perspective of Winfree can be seen in Figure 3.4 with the view of various groups of runners with fixed influence functions. This perspective is further extended by Kuramoto in Figure 3.5 where the coherence of the group can now be characterized in a single complex value as is prevalent in the varying length of the radius of the complex value. The angle of the arrow is provided by the average phase of the group $\psi$. The length of the vector is determined by the value $r$, which conveys how tightly the group is correlated in phase.

![Figure 3.4: Winfree’s perspective of the group of runners in phase space.](image)

![Figure 3.5: Kuramotos’s extension of the group of runners in phase space.](image)

Looking again at Strogatz’s analogy, as the group of runners begin to form coherent groups, this is directly expressed in Kuramoto’s coherence factor. From Figure 3.6, the group on the left is strongly coherent and therefore has a coherence value $r \approx 1$, the center
group is not as coherent and has a value $r < 1$, and the group on the right is very incoherent and has a value $r \approx 0$.

![Figure 3.6: Example of Coherence Measurement.](image)

### 3.2.4 Model Summary

To provide a sample of the outcome of these two models, Strogatz noted the following about the relationship of the models to the structure of the network:

When the system was self-synchronizing, Winfree found that no oscillator was indispensable. There was no boss. Any oscillator could be removed and the process would still work. Furthermore, the pack did not necessarily run at the speed of its fastest member. Depending on the choice of influence and sensitivity functions, the group could run at a pace nearer to the average speed of those in the pack, or it could go faster or slower than any of its members. It was all wonderfully counterintuitive. Group synchronization was not hierarchical, but it wasn’t always purely democratic either \cite{Str03}.
The simplicity of the model and its universal nature allows for application regardless of the specific underlying function. For the best foundation, oscillators, as they are being discussed, meet the criteria provided by Pikovsky, Rosenblum and Kurths, [PRK01], and shown in Section 2.1, have the following key characteristics:

1. Each node is an active system whereby it has an internal "source of energy" that is used to sustain constant oscillation.

2. Likewise, each node is able to maintain oscillation without external forces.

3. In the absence of external forces, each node tends to same oscillatory behavior regardless of the start conditions.

4. When presented with small transient external forces, the oscillation will be disturbed and return to its natural oscillatory behavior.

Given these characteristics, the models provide a broadly applicable qualitative perspective that was founded by Poincaré [Str94].

3.2.5 Review of Locally (Rings and Lattices) Coupled Oscillators

Rings and lattices provide another good modeling approach for a variety of reasons. The equations are well behaved in that the symmetry provided allows for a uniformed approach for entire populations. Additionally, the connection scheme has broad applicability in chemistry.
and physics, with behaviors in many specialties acting as ring or lattices. For this reason, cellular automata have been a popular modeling tool.

Fairly early in his research, Winfree provided a basic model for interaction for ring topologies [Win01]. His base equation is provided in Equation 3.4. In this equation, the change in phase, \( \phi_i \), of the \( i^{th} \) node in respect to time is a function, \( f \), of the phase of the \( i^{th} \) node, \( \phi_i \), plus the difference of phase of the \( i^{th} \) node and its neighbors phases, \( \phi_{i-1} \), and \( \phi_{i+1} \).

He did choose a scalar constant to show the coupling and sensitivity but later complained about the constant, explaining that because \( (\phi_i - \phi) \) is somewhat nonsensical on a circle, the linear multiplier of \( k \) is later replaced with the "odd periodic function" \( \sin \) [Win01].

\[
\frac{d\phi_i}{dt} = f(\phi_i) + k(\phi_{i-1} - \phi_i) + k(\phi_{i+1} - \phi_i)
\]  

(3.4)

Winfree then chose to expand the base equation to reflect a more familiar form of the diffusion equation shown in Equation 3.5, where \( D \) is the diffusion coefficient and \( h \) is spacing of the nodes. This expansion allows \( k \) to be provided by \( \frac{D}{h^2} \).

\[
\frac{d\phi_i}{dt} = f(\phi_i) + D\left(\frac{\phi_{i-1} - \phi_i}{h}\right) - \left(\frac{\phi_{i+1} - \phi_i}{h}\right)
\]  

(3.5)

A rigorous treatment of ring topologies is also provided by Pikovsky, Rosenblum and Kurths [PRK01]. They provide a base model that is very close to Kuramoto’s global connectivity model [Str00] and a graphical presentation of the transition to synchronization.
Their presentation shows the relationship of a range of coupling strengths, $K$ (in Kuramoto terms [Str00]) or $\varepsilon$ (as indicated in Pikovsky, Rosenblum and Kurths [PRK01]), to a set of locally coupled oscillators. The bifurcation plot shows the tendency of the nodes to form clusters of synchronized nodes (in local connections) until all nodes are synchronized. Similar results demonstrating the same phenomenon are shown in Bressloff, Coombes and de Souza [BCS97].

### 3.2.6 Review of Nonlocally Coupled Oscillators

In the last few years, there has been a new body of work on nonlocally coupled oscillators. That is, oscillators that are not globally coupled (all to all) and not restricted to nearest neighbor local connections. The most interesting of these is Abrams’ and Strogatz’s [AS05] review of Kuramoto’s introduction of Chimera States [BK04, SK04]. These works explain an unexpected phenomenon where both synchronized and incoherent states coexist in populations of identical nonlocally coupled oscillators. All analysis suggests that populations of identically coupled oscillators should exist in two states (1) synchronization (2) or waves – a solitary wave in one dimension, spirals in two dimensions and scroll rings in three. In nonlocally coupled identical oscillators however this third state of synchronized and incoherent structures has been found with certain experimental parameters [AS05]. While this research is still very young, the question of the existence of Chimera States may well be explained in hyperdimensional space and the approach may be interesting.
3.3 Summary

The current models provide a good foundation for developing an approach to determining the network structure based on the behavior of the nodes over time. While the work by Winfree, Kuramoto, and Strogatz has continued to grow in strength, complexity, and applicability, the work is still limited to specific network types and not scalable through a range of networks. The broader applicability is one of the goals of this work. The next chapter presents more traditional approaches to the problem of determining network structure from the behavior of the network. The approaches are in two forms, the primary discussion is the determination of cellular automata rules from automata behavior. The secondary discussion is the more popular approach using signal processing algorithms.
TRADITIONAL APPROACH

The work in the previous chapters builds toward the novel approach to be presented in this dissertation. Prior to presenting the new approach, a review of traditional approaches to determining network structure based on the behavior of underlying behavior is provided. To frame the state of the previous approaches to determining network structures, Strogatz provides a perspective and the direct motivation for this work in the following:

The connectivity of networks is typically unknown, except for a few small systems of neurons. Without knowing who is interacting with whom, it’s impossible to test the models quantitatively. In a tree full of fireflies, for example, you would have to figure out which bugs can see which, measure all their intrinsic flash rates one by one, and finally measure each insect’s sensitivity and influence functions. No one has even tried this experiment for two fireflies, let alone a whole congregation of them. [Str03]

Strogatz makes a number of assumptions about the underlying information requirements necessary to predict the behavior of the system. While these assumptions may be fundamentally true from a theoretical mathematics perspective, it is not necessarily the case with a
practical implementation perspective. While other limitations may exist, the full knowledge of the components listed by Strogatz (intrinsic flash rates, sensitivity and influence functions and line of sight) are not the practical limit. This argument will be provided in more detail in Chapter 5.

The direct approaches to determining the coupling of a group of oscillators can be categorized into two loose groups. One of the most notable collections provided is a series of works by Rosenblum *et al* [CRF03] [RCB02] [RP01] [RPK04]. The material includes the techniques derived by Rosenblum *et al* for the detection of coupling direction through signal processing and information theory approaches. Their review also provides insight to the methods applied by Otnes and Enochson [OE72], Pompe [Pom93], Voss and Kurths [VK97], Schiff *et al* [SSC96], Schmitz [Sch00], as well as, mutual information theory by Quiroga, Arnhold and Grassberger [QAG00] and Drepper [Dre00]. As an alternative to the methods discussed, there is a body of work on both modeling coupled oscillators via cellular automata and theoretical and practical work on determining Cellular Automata rule sets from data.

The work reviewed by Rosenblum *et al* is the emphasis of the first section of this chapter. A detailed background and theoretical foundation for cellular automata approaches is provided in the following section along with a more detailed review of previous work on the basis for determining cellular automata rule sets from data. The final section walks through a typical approach to determining the network structure using cellular automata and demonstrates the challenges provided by this approach.
4.1 Review of Current work on Determination of Connectivity

Rosenblum et al has provided a series of studies of qualification and quantification of connectivity of two nodes [RP01] [RCB02], [CRF03], [RPK04]. One of the major highlights is a summary of approaches to determining the causal relationship in cardiorespiratory interaction [RCB02]. Since their studies specifically emphasize causal relationships, they conclude that typical signal processing techniques like cross-spectra [OE72], mutual information [Pom93], maximal correlation [VK97], cannot be applied because of the underlying assumption of symmetry required for each [RCB02]. The applicable approaches are subdivided into three categories; information theory approach based on entropy measures, mutual predictability [SSC96] [Sch00] [Dre00] QAG00, and an original approach by Rosenblum et al was introduced in [RP01].

The base model that they apply for analysis is shown in Equation 4.1.

\[
\dot{\phi}_{1,2} = \omega_{1,2} + \varepsilon_{1,2} f_{1,2}(\sin(\psi_i(t) - \theta_i(t)))
\] (4.1)

Equation 4.1 is similar to Kuramoto’s base equation provided in the previous chapter and shown again in Equation 4.2.

\[
\dot{\theta}_i(t) = \omega_i + Kr_i(t) \sin(\psi_i(t) - \theta_i(t))
\] (4.2)
The rate of change of the phases over time, $\dot{\varphi}_{1,2}$, is directly related to Kuramoto’s $\dot{\theta}_i(t)$. The natural frequencies in both equations are provided by variable $\omega$. The variable $\varepsilon_{1,2}$ is the functional equivalent of Kuramoto’s $Kr_i(t)$, which provides the coupling strength of the oscillator. In the case of Rosenblum et al, the assumption is that there is only one pair of nodes. Therefore, the coupling strengths are independent and can take on any value. They are assumed to be very small, specifically provided as much less than the natural frequencies $\varepsilon_{1,2} \ll \omega_{1,2}$.

The articles provide a series of algorithms for the detection and quantification of coupling direction. They begin with the conversion of the data to phase estimates. The techniques vary from Hilbert transforms to Wavelet analysis and the equivalence of stroboscopic measurements. Once the data are converted to phases, there are three approaches mentioned in the literature: (1) The Evolution Map Approach comparison of synchronization indices (2) The Instantaneous Period Approach (3) The Mutual Prediction Approach [RP01].

### 4.1.1 EMA and IPA

In the Evolution Map Approach (EMA), the dependency of the trajectory, or change, of the average of phases is compared to the trajectory of each of the individual phases [RCB02]. The map perspective of the trajectories is shown in Figure 4.1. The trajectories of each phase can be seen going from the open circles towards the closed circles. The trajectory shown
by the open and closed boxes demonstrated the combination of the two. Rosenblum *et al* generate the trajectory through mean square error fitting of the average through the use of a finite Fourier series. The integration of the finite Fourier series for both individual phase trajectories is then used to determine directionality indexes that are normalized and averaged. The result is a index $d^{1,2}$, where $-1 < d^{1,2} < 1$ that provides the directionality. Key to this approach is the assumption that the nodes are connected in some form and that they are limited to only two nodes. However, the premise of projecting a trajectory based on a model of the system and determining the associated error is critical to the approach and will be seen again and utilized in the approach recommended in this study. An additional assumption made is that the influence is continuous and that the fit of the trajectories is a continuous difference.

![Figure 4.1: Evolution Map Approach, adapted from [RCB02].](image)

Figure 4.1: Evolution Map Approach, adapted from [RCB02].
In the Instantaneous Period Approach (IPA), the key component is the calculation of the time for each phase $\phi_{1,2}(t_k)$ to increase by $2\pi$ [RCB02]. This provided the instantaneous periods also known as the Poincaré return times, which are related to the Poincaré limit cycle discussed in Section 2.3.1. Rosenblum et al are able to utilize the Poincaré return times in the same series of analysis arrived at through the Evolution Map Approach but with a remapped characterization of the asymmetry [RCB02]. It is unclear as to why exactly the shift occurs and what the meaning of it is from the limited analysis provided. The results and the underlying assumptions remain the same, with the limitations and focus on continuous influence and projection of trajectories based models and feedback combined with mean square error estimates for the results of the coupling.

4.1.2 Mutual Prediction Approach (MPA)

In the mutual prediction approach, the question is asked whether the prediction of one system can be improved with information from the other system [RCB02]. This approach can be demonstrated by taking a series of points and predicting additional points using some function.

Take a system consisting of two oscillators connected in some manner. A difference can be calculated between the measured value and the estimate provided by the chosen estimating function. The first error considered is univariate and is calculated as $E_1(t_k) =$
$|\phi_1(t_k) - \phi_1'(t_k + \tau)|$. In this case, the error for node 1, $E_1$, at time, $t_k$, is equal to the absolute difference in phase $\phi_1(t_k)$, at time, $t_k$, minus the estimate of the phase, $\phi_1'$, at time, $t_k$, plus a selected time increment $\tau$.

Continuing with the same system, the univariate error, $E_1$, is next compared to a bivariate error using the same prediction function with two inputs or one which uses two variable, $E_{1,2}$. If $E_{1,2} < E_1$, then the phase of the second node, $\phi_2$, has influence, otherwise it does not. The root square error over all times $k$ is what Rosenblum et al refer to as the predictability improvement, $I_{1,2}$ [RCB02]. As in the earlier approaches, a directionality index is found by taking the average of the difference of the two predictability improvement variables, $I_{1,2}$ and $I_{2,1}$.

Again, the important things to remember are (1) the assumption of some form of connectivity in the final directionality index, (2) multiple models in which predictions are made and a difference of the predictability of each model is compared to minimize the error. These important factors are similar to techniques applied in modern Kalman Filters used in signal processing [Sim01], a point to which we will return in the new approach described in Chapter 6.
4.1.3 Cellular Automata Modeling of Coupled Oscillators

After reviewing the foundations of cellular automata, an example of work involving the simulation of Cardiac Arrhythmia should be provided. Bardou, Auger, Seigneuric and Chassé provided a computational experiment titled *Cellular Automata Models and Cardiac Arrhythmias; An approach to sudden cardiac death prevention* [BAS99]. The experiment provides cellular automata models for simulating the wave propagation in the myocardium. Looking back, this simulation is the computational equivalent of the Belousov-Zhabotinsky experiments described in Section 2.1.2.7.

In the model by Bardou *et al*, a grid of 2500 cells is provided with the state of each cell being determined by the a state diagram that provides a series of discrete states of the cell in the absence of influence. This state diagram is combined with an influence rule set that allows a change in the discrete state based on influence of other cells with a neighborhood radius. The influence function is calculated based on the Huygens’ principle for determining the position of a wave front based on the position of a secondary wave generated from each point on the wave [BAS99]. This combination of the influence function and the state diagram results in an alternative state diagram that unfortunately is not described directly by Bardou *et al*.

The outcome of Bardou’s model is representative of the distinctive wave propagation models that match those found in the Belousov-Zhabotinsky experiments and are representative of the models of the myocardium [BAS99].
Approaches like the one detailed in Appendix A are also provided by Adamatzky [Ada94]. In his book *Identification of Cellular Automata*, Adamatzky provides an algorithm specifically to perform the task outlined. The first step in the algorithm provides the first limit in the approach. Adamatzky begins

An algorithm for the identification of \( U \) consists of the following principal steps:

1. Choose and initial radius of neighborhood \( u(x) : r = 1(k = 3^d) \) [Ada94].

While this technique provides a degree of insight and potential utility, the limitations are fairly obvious.

1. The size of the neighborhoods has to be assumed. While the lower bound of the neighborhood can be determined to be greater than some value by the existence of contradictory rules, determination of a the true lower bound is limited by the ambiguities in the total collection of elements.

2. There are \( N \)-dimensional different possible arrangements of neighborhoods.

3. The state space creates rule sets that require larger and larger data sets in order to determine lower bound of possible neighborhood size.
4.2 Summary

The previous approaches can be described as one of two categories. First signal processing techniques have been applied to simple two node networks. While these have shown success, they are limited to two nodes systems, and they require a degree of complexity due to the directionality assumption required by each. Additionally, the continuous nature of the algorithms has further complicated the study and make mathematical tractability more difficult. Secondly, the simple finite state machine nature of cellular automata makes them very nice as a modeling paradigm for determining unknown rule sets; however, the complexity of the formalization of Cellular Automata, the complexity of converting continuos functions into finite states, and the requirement to assume the underlying topology make cellular automata a poor approach.
The most relevant questions in this study can be summarized as follows:

*Under what conditions is it possible to determine that an amount of change of a loosely coupled oscillator could most likely be associated with only a particular individual node?*

This question provides the limiting conditions for determining the underlying network structure.

The primary factor associated with this question is the probability that two nodes have the same phase. This factor is driven by two variables, the range of possible phases and the number of distinguishable phases within that range. The range of possible phases changes over time is a function of the network topology. The network topology is driven by the total number nodes in the network, the number of connections of each node, and the total number of connections within the entire network. Also associated with the determination of the network topology is the secondary factor associated with the referenced question – the relative contribution of each node based on the topology of a given node.
This chapter provides the basic analysis required for modeling and discovery of the limits of the determination of network structures of coupled oscillators due to their underlying behavior. This chapter supports the hypothesis that given a network of nodes with phases that are significantly different that will synchronize, the changes in phases over time will provide underlying network structure.

A network model and an underlying mathematical model are constructed to demonstrate the relationship between the variables in network construction. Sources of error in the determination of the network error are described in detail. The analysis in this chapter is implemented in stochastic models in Chapter 6.

The contributions provided in this chapter are as follows:

1. The error of the correlation coefficient for a group of nodes between an actual model and an ideal estimated model is attributable to a specified set of parameters.

2. The error of the average phase of a group of nodes between an actual model and an ideal estimated model is attributable to a specified set of parameters.

3. The error estimates of the correlation coefficient and the average phase a group of nodes between an actual model and an ideal estimated model is attributable to a specified set of parameters within a specified vector.
4. The error estimates of the correlation coefficient and the average phase a group of nodes between an actual model and an ideal estimated model is attributable to a specified set of parameters within a specified matrix.

5. The probability that two nodes will be indistinguishable is provided by a given probability function and is a function of the number of possible measurable phases and the number of nodes in the system.

6. The number of possible measurable phases is provided by the measurement resolution and distribution of phases as calculated by the standard deviation of phases.

7. The measurement resolution of the system is given by a measurement factor and the system noise floor.

The first step in the analysis provided in this chapter is to define an idealized model where the deterministic behavior of the model could be studied.

The model chosen was a simple ring topology in which all nodes are connected to their two nearest neighbors (left and right). An example of this topology is shown in Figure 5.1. From a practical perspective, the chosen model was believed to be the most tractable, and provided the greatest dynamic range as it was modified. From an application perspective, a model that heuristically matches the most likely match would provide the lowest model noise. The issue of model noise will be addressed at the end of the chapter.
The selected starting configuration is a known, relatively simple connectivity scheme of \( n \) locally connected coupled oscillators and all connections are considered bimodal. The governing equation used to determine the state of each node at time \( t \) is a new simple variation of Kuramoto’s model for global connectivity shown in its pure form, Equation 5.1.

![Figure 5.1: Basic Node Structure.](image)

### 5.1 First Modification of Kuramoto’s Model

Kuramoto’s basic model has been used to provide a framework for several analytical works [Str00] [Win01] [PRK01]. The mathematical framework provides that the change in phase of the \( i^{th} \) node is based on the naturally desired frequency of the \( i^{th} \) node, \( \omega_i \), plus a set of additional coefficients that relate the \( i^{th} \) node to its connected neighbors. The set of coefficients contains \( K \) which is known as the coupling coefficient which represents the \( i^{th} \) nodes desire to couple with other nodes. The second coefficient \( a \) is the factor provided
by Kuramoto that relates the \(i^{th}\) node’s desire to couple with how closely the nodes are to each other in phase. The remaining coefficient represents the phase difference of the group of coupled oscillators to its current individual phase.

\[
\dot{\theta}_i(t) = \omega_i + K r_i(t) \sin(\psi_i(t) - \theta_i(t))
\]

(5.1)

The next step is to establish a the full set of equations and determine the system state based on the simplified model, which will be considered an estimate.

### 5.1.1 Estimated Model

By assuming that all nodes in the collection are connected in the same manner, that is, to their left and right neighbors, an estimate of the behavior of the group is provided by the governing equations. Because an actual connectivity scheme might be different, we utilize the \(\tilde{\cdot}\) to represent estimated values based on the underlying assumption.

To graphically portray the effect of a set of nodes to the values presented in Kuramoto’s equation, take for example the collection of three different groups of oscillators with various phases illustrated in Figure 5.1.1. Using this example, Figure 5.1.1 provides the corresponding perspective for Kuramoto’s equation. The angle of the arrow is provided by the average
phase of the group $\psi$. The length of the vector is determined by the value $a$, which conveys how tightly the group is correlated in phase.

![Figure 5.2: Example of Coherence Measurement.](image)

The first definition beyond the base equation is that of the estimated neighborhood $\tilde{\eta}$.

**Define:** $\tilde{\eta}|\tilde{\eta}$ = the estimated neighborhood size including the reference node.

For a symmetrical neighborhood the following additional definition is provided.

**Define** $\tilde{\eta}|\tilde{\eta}$ mod $2 = 1$

The estimate of the average phase for the $i^{th}$ node, $\tilde{\psi}_i$, is provided by simply averaging all of the phases of the nodes connected in a neighborhood of the $i^{th}$ node.

**Define:** $\tilde{\psi}_i|\tilde{\psi}_i$ = sum of all phases in an estimated neighborhood divided by the estimated neighborhood size $\tilde{\eta}$.

This is definition is shown in the example provided by Equation 5.2.
Example $\tilde{\eta}|\tilde{\eta} = 3$

\[
\psi_i(t) = \frac{\theta_i(t) + \theta_{i-1}(t) + \theta_{i+1}(t)}{\tilde{\eta}}
\]  

(5.2)

Applying the example, the estimate of the average phase is given by Equation 5.3.

\[
\psi_i(t) = \frac{\theta_i(t) + \theta_{i-1}(t) + \theta_{i+1}(t)}{3}
\]  

(5.3)

In order to generate a more compact form of the equations, the index $j$ is defined.

Define: $j|j = \text{symmetrical index representing the distance to the furtherest node from the center node in a symmetrical neighborhood}$

Define: $j|j = \frac{\tilde{\eta}-1}{2}$

Thus, the more compact form of the estimated average phase is given in Equation 5.4

Example From the example $j|j = \frac{3-1}{2}, j = 1$.

\[
\psi_i(t) = \frac{1}{\tilde{\eta}} \sum_{k=(i-j) \mod N}^{(i+j) \mod N} \theta_k(t)
\]  

(5.4)

The second variable of concern for the analysis is the value $r$ given in Kuramoto’s correlation equation. As mentioned above, This value shows the relationship of a group of nodes
to one another or "closeness" in phase of the group. Kuramoto’s correlation coefficient is
given by Equation 5.5.

\[ \tilde{r} e^{i \tilde{\psi}(t)} = \frac{1}{\tilde{\eta}} \sum_{k=(i-j) \mod N}^{(i+j) \mod N} e^{i \theta_k(t)} \]  

(5.5)

Solving for \( \tilde{r} \) allows the isolation of the correlation of the phases within the group. This
is given in equation 5.6. As a side note, \( e^i \), as expected, is the base of natural logarithm
raised to the square root of negative one , \( \sqrt{-1} \)

\[ \tilde{r} = \frac{\sum_{k=(i-j) \mod N}^{(i+j) \mod N} e^{i \theta_k(t)}}{e^{i \tilde{\psi}(t)} \tilde{\eta}} \]  

(5.6)

For clarity and perspective, the expanded version of the summation for the estimated
neighborhood is provided in equation 5.7.

\[ \tilde{r} = \frac{(e^{i \theta_{i-1}(t)} + e^{i \theta_i(t)} + e^{i \theta_{i+1}(t)})}{e^{i \tilde{\psi}(t)} \tilde{\eta}} \]  

(5.7)

With establishment of the state based on the simplified model, now considered an esti-
mate, the model is modified to determine a deviation from the estimate.

68
5.1.2 Actual Model

The next step in the analysis is to determine the effect of a deviation of the underlying connectivity model. This determination is done by providing a single additional bimodal connection between two nodes in the group. This scheme is shown in Figure 5.3.

![Figure 5.3: Modified Node Structure.](image)

As before, we define the variable for the neighborhood size. To do so an additional variable for representing the number of additional neighbors from the estimate is given by $\eta_{\mu i}$.

**Define:** $\eta_{\mu i}|\eta_{\mu i} = \text{additional neighbors than the estimated.}$

With this new variable we can now define the actual neighborhood size as follows:
Define: $\hat{\eta}|\hat{\eta} = \text{actual neighborhood size including the reference node for both symmetrical and asymmetrical neighborhoods. It is equivalent to the estimated } \tilde{\eta}\text{ and the additional } \eta_{\mu i}$

Define: $\hat{\eta}|\hat{\eta} = \tilde{\eta} + \eta_{\mu i}$

Likewise, a new variable for the phase of an additional node is defined as $\theta_{\mu i}(t)$.

Define: $\theta_{\mu i}(t)|\theta_{\mu i}(t) = \text{the phase of an additional node from the estimate.}$

With the new neighborhood size and additional node established, the actual average phase can be established and is shown in an given by Equation 5.8

$$\hat{\psi}_i(t) = \frac{\theta_i(t) + \theta_{i-1}(t) + \theta_{i+1}(t) + \theta_{\mu i}(t)}{\hat{\eta}}$$

(5.8)

Example

From the example network in Figure 5.3. The node with the additional connection is now provided as in equation 5.9.

$$\hat{\psi}_i(t) = \frac{\theta_i(t) + \theta_{i-1}(t) + \theta_{i+1}(t) + \theta_{\mu i}(t)}{4}$$

(5.9)
Keeping \( j = 1 \) from the estimated \( \hat{\eta}^{-1} \) and converting to the more compact form the equation for the actual phase is given by (5.10)

\[
\hat{\psi}_i(t) = \frac{1}{\hat{\eta}} \left\{ \sum_{k=(i-j) \mod N}^{(i+j) \mod N} \theta_k(t) + \theta_{\mu i}(t) \right\}
\] (5.10)

Again, expanding to Kuramoto’s correlation coefficient gives Equation (5.11)

\[
\hat{r} e^{i\hat{\psi}(t)} = \frac{1}{\hat{\eta}} \sum_{k=(i-j) \mod N}^{(i+j) \mod N} e^{i\theta_k(t)}
\] (5.11)

As before, \( r \) is solved for giving the correlation value in isolation, as shown in Equation (5.12).

\[
\hat{r} = \frac{\sum_{k=(i-j) \mod N}^{(i+j) \mod N} e^{i\theta_k(t)} + \theta_{\mu i}(t)}{e^{i\hat{\psi}(t)\hat{\eta}}}
\] (5.12)

Once again for perspective, the expanded version of \( r \) is provided in Equation (5.13)

\[
\hat{r} = \frac{e^{i\theta_{i-1}(t)} + e^{i\theta_i(t)} + e^{i\theta_{i+1}(t)} + e^{i\theta_{\mu i}(t)}}{e^{i\hat{\psi}(t)\hat{\eta}}}
\] (5.13)
In addition, if in the trivial but desired case where the estimated configuration is the same as the actual configuration then the governing equation is the same as above with \( \theta_{\mu i} = 0 \) and \( \eta_{\mu i} = 0 \)

With both sets of governing equations those for an estimated configuration and those for an actual configuration that is different that the estimate, a degree of error generated from the difference can now be explored. This analysis is given in the next section with two error values. The first error studied is error due to differences in the estimated versus actual average phase of a group of oscillators. This is shown as Error \( \psi \). The second error is the correlation of the nodes of a group of nodes to each other or how closely grouped in phase they become. This second error is shown as Error \( a \) from Kuramoto’s \( a \) coefficient.

### 5.1.3 Error \( \psi \)

The first error we examine is error due to differences in the estimated versus actual average phase of a group of oscillators. This type of error is provided as Error \( \psi \).

**Define** Error \( \psi \) for the \( i^{th} \) node at time \( t \), \( \varepsilon_{\psi_i(t)} \) is the difference of the actual \( \hat{\psi}(t) \) from the estimated \( \tilde{\psi}(t) \). This is further provided in Equation 5.14

\[
\varepsilon_{\psi_i(t)} = \hat{\psi}_i(t) - \tilde{\psi}_i(t) \quad (5.14)
\]
Let’s begin by examining the example with the case provided in Equation 5.14. Equation 5.14 can then be expanded by inserting Equations 5.4 and 5.8. This provides for the full perspective in Equation 5.15.

\[ \varepsilon_{\psi_i(t)} = \frac{\theta_i(t) + \theta_{i-1}(t) + \theta_{i+1}(t) + \theta_{\mu_i}(t)}{\hat{\eta}} - \frac{\theta_i(t) + \theta_{i-1}(t) + \theta_{i+1}(t)}{\hat{\eta}} \]  
(5.15)

Next, the fractions are expanded to facilitate combining the two equations as shown in Equation 5.16

\[
\varepsilon_{\psi_i(t)} = \frac{\hat{\eta}\left\{\theta_i(t) + \theta_{i-1}(t) + \theta_{i+1}(t) + \theta_{\mu_i}(t)\right\}}{\hat{\eta}\hat{\eta}} - \frac{\hat{\eta}\left\{\theta_i(t) + \theta_{i-1}(t) + \theta_{i+1}(t)\right\}}{\hat{\eta}\hat{\eta}} \]
(5.16)

With the expanded version the two equations can be combined giving the new form as seen in Equation 5.17.

\[
\varepsilon_{\psi_i(t)} = \frac{\hat{\eta}\theta_i(t) + \hat{\eta}\theta_{i-1}(t) + \hat{\eta}\theta_{i+1}(t) + \hat{\eta}\theta_{\mu_i}(t)}{\hat{\eta}\hat{\eta}} - \frac{\hat{\eta}\theta_i(t) + \hat{\eta}\theta_{i-1}(t) + \hat{\eta}\theta_{i+1}(t)}{\hat{\eta}\hat{\eta}} \]
(5.17)

The two equations can be fully combined as in Equation 5.18
\[
\epsilon_{\psi_i(t)} = \frac{\tilde{\eta}_i(t) + \tilde{\eta}_{i-1}(t) + \tilde{\eta}_{i+1}(t) + \tilde{\eta}_{\mu i}(t) - \tilde{\eta}_i(t) - \tilde{\eta}_{i-1}(t) - \tilde{\eta}_{i+1}(t)}{\tilde{\eta}(\tilde{\eta} + \eta_{\mu i})}
\] (5.18)

Once more expanding the equation by expanding \( \tilde{\eta} \) where \( \tilde{\eta} = \tilde{\eta} + \eta_{\mu i} \) see Equation 5.19.

\[
\epsilon_{\psi_i(t)} = \frac{\tilde{\eta}_i(t) + \tilde{\eta}_{i-1}(t) + \tilde{\eta}_{i+1}(t) + \tilde{\eta}_{\mu i}(t)}{\tilde{\eta}(\tilde{\eta} + \eta_{\mu i})} + 
\frac{-(\tilde{\eta} + \eta_{\mu i})\theta_i(t) - (\tilde{\eta} + \eta_{\mu i})\theta_{i-1}(t) - (\tilde{\eta} + \eta_{\mu i})\theta_{i+1}(t)}{\tilde{\eta}(\tilde{\eta} + \eta_{\mu i})}
\] (5.19)

The expanded \( \tilde{\eta} + \eta_{\mu i} \) is now combined directly with the respective phases in Equation 5.20.

\[
\epsilon_{\psi_i(t)} = \frac{\tilde{\eta}_i(t) + \tilde{\eta}_{i-1}(t) + \tilde{\eta}_{i+1}(t) + \tilde{\eta}_{\mu i}(t)}{\tilde{\eta}(\tilde{\eta} + \eta_{\mu i})} + 
\frac{-(\tilde{\eta} + \eta_{\mu i})\theta_i(t) - (\tilde{\eta} + \eta_{\mu i})\theta_{i-1}(t) - (\tilde{\eta} + \eta_{\mu i})\theta_{i+1}(t)}{\tilde{\eta}(\tilde{\eta} + \eta_{\mu i})}
\] (5.20)

At this point, the two components are combined to provide a single component in Equation 5.21.
The final form of the Error $\psi$ for the $i^{th}$ node at time $t$ is reduced to relationship shown in Equation 5.22.

$$
\varepsilon_{\psi_i(t)} = \frac{\tilde{\eta} \theta_{\mu}(t) - \eta_{\mu i} \tilde{\theta}_i(t) - \eta_{\mu i} \theta_{i-1}(t) - \eta_{\mu i} \theta_{i+1}(t)}{\tilde{\eta}(\tilde{\eta} + \eta_{\mu i})}
$$

(5.21)

With the Error $\psi$ completed, the attention is now focused on the second variable in the correlation function the Error $r$.

### 5.1.4 Error $r$

The second error is the correlation of the nodes of a group of nodes to each other or how closely grouped in phase they become. This second error is shown as Error $a$ from Kuramoto’s $a$ coefficient. As before, the Error $r$ is defined as the Error in the corresponding correlation of the nodes between the estimated network configuration for a group of nodes and the actual configuration for a group of nodes. This is provided in the following definition;
Define Error $r_i$ for the $i^{th}$ node at time $t$, $\varepsilon_{r_i}(t)$ is the difference of the actual $\hat{r}$ from the estimated $\tilde{r}$ as shown in Equation note-eq-28.

$$\varepsilon_{r_i}(t) = \hat{r}_i(t) - \tilde{r}_i(t) \quad (5.23)$$

Expanding Equation 5.23 from Equations 5.12 and 5.7 provides an example in the full form of $\varepsilon_{r_i}(t)$, as shown in Equation 5.24.

$$\varepsilon_{r_i}(t) = \frac{(e^{i\theta_{i-1}(t)} + e^{i\theta_i(t)} + e^{i\theta_{i+1}(t)} + e^{i\theta_{\mu}(t)})}{e^{i\psi(t)}} \hat{\eta} - \frac{(e^{i\theta_{i-1}(t)} + e^{i\theta_i(t)} + e^{i\theta_{i+1}(t)})}{e^{i\psi(t)}} \tilde{\eta} \quad (5.24)$$

The $(e^{i\theta_{\mu}(t)})$ is separated by expanding the form to $e^{i\psi(t)}(e^{i\theta_{\mu}(t)})$ in order to allow reduction of the complexity between the two corresponding components the estimate and the actual. See Equation 5.25.
\[
\varepsilon_{r_i(t)} = \frac{(e^{i\theta_{i-1}(t)} + e^{i\theta_{i}(t)} + e^{i\theta_{i+1}(t)})}{e^{i\psi(t)}\eta} - \frac{(e^{i\theta_{i-1}(t)} + e^{i\theta_{i}(t)} + e^{i\theta_{i+1}(t)})}{e^{i\psi(t)}\eta}
\]

The two components are now adjusted by the respective \(\eta\) to allow them to be combined. This shown in Equation 5.26.

\[
\varepsilon_{r_i(t)} = \frac{\tilde{\eta}e^{i\tilde{\psi}(t)}(e^{i\theta_{i-1}(t)} + e^{i\theta_{i}(t)} + e^{i\theta_{i+1}(t)})}{e^{i\tilde{\psi}(t)}e^{i\tilde{\psi}(t)}\tilde{\eta}^2} - \frac{\tilde{\eta}e^{i\tilde{\psi}(t)}(e^{i\theta_{i-1}(t)} + e^{i\theta_{i}(t)} + e^{i\theta_{i+1}(t)})}{e^{i\tilde{\psi}(t)}e^{i\tilde{\psi}(t)}\tilde{\eta}^2}
\]

To reduce the complexity, a new variable \(\nu\) is defined as the phases of the nodes of the example estimate, which are now applied to both the estimate and the isolated portion of the actual component.

Define \(\nu|\nu = (e^{i\theta_{i-1}(t)} + e^{i\theta_{i}(t)} + e^{i\theta_{i+1}(t)})\)

The insertion of \(\nu\) is shown in Equation 5.27.
\[ \varepsilon_{r_i}(t) = \frac{\tilde{\eta} e^{i\tilde{\psi}(t)}(\nu) + \tilde{\eta} e^{i\tilde{\psi}(t)}(e^{i\theta_\mu(t)})}{e^{i\tilde{\psi}(t)}e^{i\tilde{\psi}(t)}} - \frac{\tilde{\eta} e^{i\tilde{\psi}(t)}(\nu)}{e^{i\tilde{\psi}(t)}e^{i\tilde{\psi}(t)}} \tag{5.27} \]

The two components are now combined in Equation 5.28.

\[ \varepsilon_{r_i}(t) = \frac{\tilde{\eta} e^{i\tilde{\psi}(t)}(\nu) + \tilde{\eta} e^{i\tilde{\psi}(t)}(e^{i\theta_\mu(t)}) - \hat{\eta} e^{i\tilde{\psi}(t)}(\nu)}{e^{i\tilde{\psi}(t)}e^{i\tilde{\psi}(t)}} \tag{5.28} \]

Since, \( \hat{\psi}(t) = \frac{\tilde{\psi}(t)+\theta_\mu(t)}{\eta} \) the following definition can be provided.

Define \( e^{i\hat{\psi}(t)}|e^{i\triangledown(t)} = e^{i\frac{\tilde{\psi}(t)+\theta_\mu(t)}{\eta}} \)

Inserting the expanded form from the definition gives Equation 5.29

\[ \varepsilon_{r_i}(t) = \frac{\tilde{\eta} e^{i\tilde{\psi}(t)}(\nu) + \tilde{\eta} e^{i\tilde{\psi}(t)}(e^{i\theta_\mu(t)}) - \hat{\eta} e^{i\tilde{\psi}(t)}(\nu)}{e^{i\tilde{\psi}(t)}e^{i\tilde{\psi}(t)}} \tag{5.29} \]

Further expanding \( \hat{\eta} \) found in \( \frac{\tilde{\psi}(t)+\theta_\mu(t)}{\eta} \) by \( \hat{\eta} = \tilde{\eta} + \eta_\mu \) gives the new form Equation 5.30

\[ \varepsilon_{r_i}(t) = \frac{\tilde{\eta} e^{i\tilde{\psi}(t)}(\nu) + \tilde{\eta} e^{i\tilde{\psi}(t)}(e^{i\theta_\mu(t)}) - \hat{\eta} e^{i\tilde{\psi}(t)}(\nu)}{e^{i\tilde{\psi}(t)}e^{i\tilde{\psi}(t)}} \tag{5.30} \]

Likewise the same expansion for \( \hat{\eta} \) in \( \hat{\eta} e^{i\tilde{\psi}(t)}(\nu) \) is provided in Equation 5.31.
\[ \varepsilon_{r_i}(t) = -\frac{\bar{\eta}e^{i\bar{\psi}(t)}(\nu) + \bar{\eta}e^{i\bar{\psi}(t)}(e^{i\theta_\mu(t)}) - (\bar{\eta} + \eta_{\mu})e^{\frac{i\bar{\psi}(t) + \theta_\mu(t)}{\eta + \eta_{\mu}}}(\nu)}{e^{i\bar{\psi}(t)}e^{i\bar{\psi}(t)}(\bar{\eta} + \eta_{\mu})} \] (5.31)

Once again the same expansion is given for \( \bar{\eta} \) is provided in the denominator which gives the form in Equation 5.32.

\[ \varepsilon_{r_i}(t) = \frac{\bar{\eta}e^{i\bar{\psi}(t)}(\nu) + \bar{\eta}e^{i\bar{\psi}(t)}(e^{i\theta_\mu(t)}) - (\bar{\eta} + \eta_{\mu})e^{\frac{i\bar{\psi}(t) + \theta_\mu(t)}{\eta + \eta_{\mu}}}(\nu)}{e^{i\bar{\psi}(t)}e^{i\bar{\psi}(t)}(\bar{\eta} + \eta_{\mu})} \] (5.32)

Finally, by replacing \( \nu \), the full error is shown in Equation 5.33

\[ \varepsilon_{r_i}(t) = \frac{\bar{\eta}e^{i\bar{\psi}(t)}(e^{i\theta_{i-1}(t)} + e^{i\theta_{i}(t)} + e^{i\theta_{i+1}(t)}) + \bar{\eta}e^{i\bar{\psi}(t)}(e^{i\theta_\mu(t)}) - (\bar{\eta} + \eta_{\mu})e^{\frac{i\bar{\psi}(t) + \theta_\mu(t)}{\eta + \eta_{\mu}}}(e^{i\theta_{i-1}(t)} + e^{i\theta_{i}(t)} + e^{i\theta_{i+1}(t)})}{e^{i\bar{\psi}(t)}e^{i\bar{\psi}(t)}(\bar{\eta} + \eta_{\mu})} \] (5.33)

Even though Equation 5.33 appears complicated, it is in fact a set of six complex number number interacting through a sum and product to form a complex number. If we convert
the exponential from of each of the six different variable above to a single complex variable
the final outcome becomes clearer.

Define: \( z_1 \) \( z_1 = e^{i\theta_1(t)} \)

Define: \( z_2 \) \( z_2 = e^{i\theta_2(t)} \)

Define: \( z_3 \) \( z_3 = e^{i\theta_3(t)} \)

Define: \( z_4 \) \( z_4 = e^{i\theta_4(t)} \)

Define: \( z_5 \) \( z_5 = e^{i\theta_5(t)} \)

Define: \( z_6 \) \( z_6 = e^{i\tilde{\psi}(t)} \)

With these new definitions Equation 5.34 can be rewritten in the simpler form shown in

Equation 5.34

\[
\varepsilon_{ri(t)} = \frac{\tilde{\eta} z_6 (z_1 + z_2 + z_3 + z_4) - (\tilde{\eta} + \eta_{\mu i}) z_5 (z_1 + z_2 + z_3)}{z_5 z_6 \tilde{\eta} (\tilde{\eta} + \eta_{\mu i})} \tag{5.34}
\]
5.2 Second Modification - Additional Connections to a Single Node

With a set of base equations established for a collection of nodes in the simple configuration, the next variation studied is the effect of multiple connections on one node. An example of the described network configuration is shown in Figure 5.4.

![Modified Node Structure](image)

Figure 5.4: Modified Node Structure.

5.2.1 Estimate Model in Vector Notation

In order to more efficiently represent this new configuration, vector forms of the phases of the nodes and of each nodes neighbors are provided.

The column vector $\Theta(t)$ is the ordered set from $(1 - N)$ of all phases at time $t$. This is shown in Equation 5.35.
The row vector $\tilde{\mu}_i(t)$ is the estimated neighborhood for the $i^{th}$ node at time $t$. The values are provided as 1 or 0 depending on whether or not there is connection between the $i^{th}$ node and the respective neighbors. The row vector is shown in Equation 5.36.

$$\tilde{\mu}_i(t) = \begin{bmatrix} \tilde{b}_{i1} & \tilde{b}_{i2} & \tilde{b}_{i3} & \ldots & \tilde{b}_{iN} \end{bmatrix}$$ (5.36)

As stated $\tilde{b}_{ij}$ is defined as follows:

Define $\tilde{b}_{ij} = \begin{cases} 1 & \text{if connected} \\ 0 & \text{if not connected} \end{cases}$

From Equation 5.36 and Equation 5.35 the average phase for the $i^{th}$ node can now be found by the vector multiplication form seen in Equation 5.37.

$$\tilde{\psi}_i(t) = \frac{1}{\eta} \tilde{\mu}_i(t) \Theta(t)$$ (5.37)
From Equation 5.37 it is shown that the row vector \( \tilde{\mu}_i(t) \), providing the connectivity of the \( i^{th} \) node to its neighbors is multiplied by the individual phase of each of the nodes. The details can be seen in Equation 5.38. This is further multiplied by the inverse of the number of neighbors, \( \frac{1}{\eta} \), to provide the average phase of the \( i^{th} \) node.

\[
\tilde{\psi}_i(t) = \frac{1}{\eta} \left[ \tilde{b}_{i1} \quad \tilde{b}_{i2} \quad \tilde{b}_{i3} \quad \ldots \quad \tilde{b}_{iN} \right] \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \vdots \\ \theta_N \end{bmatrix} = \\
\frac{1}{\eta} \left( \tilde{b}_{i1} \theta_1 + \tilde{b}_{i2} \theta_2 + \tilde{b}_{i3} \theta_3 + \ldots + \tilde{b}_{iN} \theta_N \right)
\] (5.38)

### 5.2.2 Actual Model in Vector Notation

As with the estimate above, the row vector \( \hat{\mu}_i(t) \) is the actual neighborhood for the \( i^{th} \) at time \( t \). The values are provided as either 1 or 0 depending on whether or not there is connection between the \( i^{th} \) node and the respective neighbors. This is shown in Equation 5.39.
\[ \hat{\mu}_i(t) = \begin{bmatrix} \hat{b}_{i1} & \hat{b}_{i2} & \hat{b}_{i3} & \ldots & \hat{b}_{iN} \end{bmatrix} \] (5.39)

As stated \( \tilde{b}_{ij} \) is defined as follows:

\[
\begin{align*}
\hat{b}_{ij} | \tilde{b}_{ij} &= \begin{cases} 
1 & \text{if connected} \\
0 & \text{if not connected}
\end{cases}
\end{align*}
\]

Since the individual phases of each node are the same in the estimate and the actual, the actual average phase of the \( i^{th} \) node at time \( t \), \( \hat{\psi}_i \), is provided by the corresponding vector form seen in Equation 5.40.

\[ \hat{\psi}_i(t) = \frac{1}{\hat{\eta}} \hat{\mu}_i(t) \hat{\Theta}(t) \] (5.40)

Just as with the related estimates, Equation 5.40 shows that the row vector \( \hat{\mu}_i(t) \), providing the connectivity of the \( i^{th} \) node to its neighbors, is multiplied by the column vector \( \hat{\Theta}(t) \), containing the individual phase of each of the nodes. The details can be seen in Equation 5.41. This is further multiplied by the inverse number of neighbors, \( \frac{1}{\hat{\eta}} \), to provide the actual average phase of the \( i^{th} \) node.
\[
\hat{\psi}_i(t) = \frac{1}{\hat{\eta}} \left[ \hat{b}_{i1} \hat{b}_{i2} \hat{b}_{i3} \ldots \hat{b}_{iN} \right] \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \vdots \\ \theta_N \end{bmatrix} = \frac{1}{\hat{\eta}} \left( \hat{b}_{i1}\theta_1 + \hat{b}_{i2}\theta_2 + \hat{b}_{i3}\theta_3 + \ldots + \hat{b}_{iN}\theta_N \right)
\]

Even with the new forms, the error of average phase of a group of nodes is still simply based on the difference of the actual versus the estimated average phase as given in Equation 5.41.

\textbf{5.2.3 Group Error } \psi

Just as in the case of the first modification of the ring in the previous section, Error in the new configuration is defined as the difference between the actual and the estimate. This is further expressed in Equation 5.42.

\textbf{Define} Error } \psi for the } i^{th} \text{ node at time } t, \varepsilon_{\psi_i(t)} \text{ is the difference of the actual } \hat{\psi} \text{ from the estimated } \tilde{\psi}
\[ \varepsilon_{\psi_i(t)} = \hat{\psi}_i(t) - \tilde{\psi}_i(t) \] (5.42)

Expanding both the estimate and the actual phase averages provides the perspective in Equation 5.43

\[ \varepsilon_{\psi_i(t)} = \frac{1}{\eta} \hat{\mu}_i(t) \Theta(t) - \frac{1}{\tilde{\eta}} \tilde{\mu}_i(t) \Theta(t) \] (5.43)

Since the current phase vector at time, t, \( \Theta(t) \) is the same for both the estimate and the actual averages, it can be factored from both as in Equation 5.44. Additionally, since the sum of two row vectors is also row vector, the square brackets are also used to denote the resulting row vector.

\[ \varepsilon_{\psi_i(t)} = \left[ \frac{1}{\eta} \hat{\mu}_i(t) - \frac{1}{\tilde{\eta}} \tilde{\mu}_i(t) \right] \Theta(t) \] (5.44)

Equation 5.45 provides the replacement of \( \hat{\mu}_i \), the actual neighborhood size with the equivalent in terms of the estimate and the additional neighbors \( \tilde{\eta} + \eta_{\mu i} \).

\[ \varepsilon_{\psi_i(t)} = \left[ \frac{1}{\tilde{\eta} + \eta_{\mu i}} \hat{\mu}_i(t) - \frac{1}{\tilde{\eta}} \tilde{\mu}_i(t) \right] \Theta(t) \] (5.45)
Equation 5.46 is the preparation of the two scalar multipliers, each fractions for combination with each other.

\[
\varepsilon_{\psi_i(t)} = \left[ \frac{\tilde{\eta}}{\tilde{\eta}^2 + \tilde{\eta}\eta_{\mu i}} \hat{\mu}_i(t) - \frac{\tilde{\eta} + \eta_{\mu i}}{\tilde{\eta}^2 + \tilde{\eta}\eta_{\mu i}} \tilde{\mu}_i(t) \right] \Theta(t) \tag{5.46}
\]

The replacement of the notation for the row vectors \( \hat{\mu}_i(t) \) and \( \tilde{\mu}_i(t) \) by their respective values is provided in Equation 5.47.

\[
\varepsilon_{\psi_i(t)} = \left[ \frac{\tilde{\eta}}{\tilde{\eta}^2 + \tilde{\eta}\eta_{\mu i}} \left[ \hat{b}_{i1} \ldots \hat{b}_{iN} \right] - \frac{\tilde{\eta} + \eta_{\mu i}}{\tilde{\eta}^2 + \tilde{\eta}\eta_{\mu i}} \left[ \tilde{b}_{i1} \ldots \tilde{b}_{iN} \right] \right] \Theta(t) \tag{5.47}
\]

The result of the combination of the two row vectors is seen in Equation 5.48. Only the first and last components are provided in the resulting row vector.

\[
\varepsilon_{\psi_i(t)} = \left[ \left[ \frac{\tilde{\eta}}{\tilde{\eta}^2 + \tilde{\eta}\eta_{\mu i}} \hat{b}_{i1} \ldots \frac{\tilde{\eta}}{\tilde{\eta}^2 + \tilde{\eta}\eta_{\mu i}} \hat{b}_{iN} \right] - \left[ \frac{\tilde{\eta} + \eta_{\mu i}}{\tilde{\eta}^2 + \tilde{\eta}\eta_{\mu i}} \tilde{b}_{i1} \ldots \frac{\tilde{\eta} + \eta_{\mu i}}{\tilde{\eta}^2 + \tilde{\eta}\eta_{\mu i}} \tilde{b}_{iN} \right] \right] \Theta(t) \tag{5.48}
\]
The final combination of the two row vectors result in the new row vector with the values

\[ b_{iN} = \left( \frac{\tilde{\eta}}{\eta^2 + \eta \eta_m} \right) \hat{b}_{ij} - \left( \frac{\tilde{\eta} + \eta \eta_m}{\eta^2 + \eta \eta_m} \right) \tilde{b}_{ij} \] as sown in Equation 5.49.

\[ \varepsilon_{\psi_i(t)} = \left[ \left( \frac{\tilde{\eta}}{\eta^2 + \eta \eta_m} \right) \hat{b}_{i1} - \left( \frac{\tilde{\eta} + \eta \eta_m}{\eta^2 + \eta \eta_m} \right) \tilde{b}_{i1} \ldots \left( \frac{\tilde{\eta}}{\eta^2 + \eta \eta_m} \right) \hat{b}_{iN} - \left( \frac{\tilde{\eta} + \eta \eta_m}{\eta^2 + \eta \eta_m} \right) \tilde{b}_{iN} \right] \Theta(t) \quad (5.49) \]

Equation 5.50 shows the possible values of \( b_{iN} \) depending on the corresponding values, \( \tilde{b}_{ij} \) and \( \hat{b}_{ij} \) in the corresponding connectivity row vectors.

\[ \left( \frac{\tilde{\eta}}{\eta^2 + \eta \eta_m} \right) \hat{b}_{ij} - \left( \frac{\tilde{\eta} + \eta \eta_m}{\eta^2 + \eta \eta_m} \right) \tilde{b}_{ij} = \begin{cases} 
\frac{\tilde{\eta}}{\eta^2 + \eta \eta_m} & \text{if } \tilde{b}_{ij} = 0 \text{ and } \hat{b}_{ij} = 1 \\
\frac{-\eta \eta_m}{\eta^2 + \eta \eta_m} & \text{if } \tilde{b}_{ij} = 1 \text{ and } \hat{b}_{ij} = 1 \\
\frac{-\tilde{\eta} - \eta \eta_m}{\eta^2 + \eta \eta_m} & \text{if } \tilde{b}_{ij} = 1 \text{ and } \hat{b}_{ij} = 0 \\
0 & \text{if } \tilde{b}_{ij} = 0 \text{ and } \hat{b}_{ij} = 0 
\end{cases} \quad (5.50) \]

### 5.2.4 Group Error \( r \)

The group error for \( r \) in the new configuration is best explained by the use of both the single variable definitions of the complex variables combined with the row and column vector forms used in the last section.
Beginning with the simple definitions

Define: $z_1|z_1 = e^{i\theta_1(t)}$

Define: $z_2|z_2 = e^{i\theta_2(t)}$

Define: $z_3|z_3 = e^{i\theta_3(t)}$

Define: $\vdots$

Define: $z_N|z_N = e^{i\theta_n(t)}$

Define: $z_{\psi}|z_{\psi} = e^{i\tilde{\psi}(t)}$

With the complex values defined a column vector of these can once again be generated as shown in Equation 5.51.

$$\Theta_2 = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_N \end{bmatrix} \quad (5.51)$$
The row vector $\tilde{\mu}_i(t)$ remains the same and represents the estimated neighborhood for the \( i^{th} \) at time \( t \). The values 1 or 0 depending on whether or not there is connection between the \( i^{th} \) node and the respective neighbors. The row vector is shown in Equation 5.52.

\[
\tilde{\mu}_i(t) = \begin{bmatrix} \tilde{b}_{i1} & \tilde{b}_{i2} & \tilde{b}_{i3} & \ldots & \tilde{b}_{iN} \end{bmatrix}
\]  

(5.52)

As stated above $\tilde{b}_{ij}$ is defined as follows:

Define

\[
\tilde{b}_{ij} = \begin{cases} 
1 & \text{if connected} \\
0 & \text{if not connected}
\end{cases}
\]

With these forms complete a new vector form can be expressed for the estimated correlation $\tilde{r}$ can be established, as shown in Equation 5.53. In this equation, the inverse of the estimated number of neighbors $\frac{1}{\tilde{\eta}}$, is a scalar multiplier times the estimated average phase of the \( i^{th} \) node in complex form is, $z_{\tilde{\psi}}$ which is also a scalar multiplier times the row vector $\tilde{\mu}_i(t)$ which represents the estimated neighbors of the \( i^{th} \) node. All of this is multiplied by the column vector $\Theta_{\tilde{z}}$ which represents the complex representation of the phases of each node. The result is the scalar estimate of correlation $\tilde{r}$.

\[
\tilde{r} = \frac{1}{\tilde{\eta}} z_{\tilde{\psi}} \tilde{\mu}_i(t) \Theta_{\tilde{z}}
\]  

(5.53)

Expanded into the full form Equation 5.54 shows the vector forms of $\tilde{r}$
\[ \hat{r} = \frac{1}{\eta} z_{\psi} \left[ \tilde{b}_{i_1} \tilde{b}_{i_2} \tilde{b}_{i_3} \ldots \tilde{b}_{i_N} \right] \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_N \end{bmatrix} = \frac{1}{\eta} z_{\psi} \left( \tilde{b}_{i_1} z_1 + \tilde{b}_{i_2} z_2 + \tilde{b}_{i_3} z_3 + \ldots + \tilde{b}_{i_N} z_N \right) \] (5.54)

### 5.2.5 Actual Model in Vector Notation

The same process allows the transformation of the actual correlation value, \( \hat{r} \) to be determined. To begin, the complex form of the average phase of the \( i^{th} \) node is defined.

**Define:** \( z_{\psi} | z_{\psi} = e^{i\psi(t)} \)

With the complex values defined a column vector of these can once again be generated as shown in Equation 5.55.
\[
\Theta_z = \begin{bmatrix}
  z_1 \\
  z_2 \\
  z_3 \\
  \vdots \\
  z_N 
\end{bmatrix} \tag{5.55}
\]

The row vector \( \hat{\mu}_i(t) \) remains the same as previously defined and represents the actual neighborhood for the \( i^{th} \) node at time \( t \). The values 1 or 0 depend on whether or not there is connection between the \( i^{th} \) node and the respective neighbors. The row vector is shown in Equation 5.56.

\[
\hat{\mu}_i(t) = \begin{bmatrix}
  \bar{b}_{i1} & \hat{b}_{i2} & \hat{b}_{i3} & \ldots & \hat{b}_{iN} 
\end{bmatrix} \tag{5.56}
\]

As stated above \( \hat{b}_{ij} \) is defined as follows:

\[
\begin{align*}
\hat{b}_{ij} | \bar{b}_{ij} &= \begin{cases} 
1 & \text{if connected} \\
0 & \text{if not connected} 
\end{cases}
\end{align*}
\]

The vector form can be expressed for the actual correlation \( \hat{r} \), as shown in Equation 5.57. In this equation, the inverse of the actual number of neighbors \( \frac{1}{\eta} \), is a scalar multiplier times the actual average phase of the \( i^{th} \) node in complex form is, \( z_\psi \) which is also a scalar.
multiplier times the row vector \( \hat{\mu}_i(t) \) which represents the actual neighbors of the \( i^{th} \) node. All of this multiplied by the column vector \( \Theta_z \) which represents the complex representation of the phases of each node. The results is the scalar actual correlation \( \hat{r} \).

\[
\hat{r} = \frac{1}{\hat{\eta}_\psi} z_\psi \hat{\mu}_i(t) \Theta_z \tag{5.57}
\]

Expanded into the full form Equation 5.58 shows the vector forms of \( \hat{r} \)

\[
\hat{r} = \frac{1}{\hat{\eta}_\psi} z_\psi \begin{bmatrix}
\hat{b}_{i1} & \hat{b}_{i2} & \hat{b}_{i3} & \ldots & \hat{b}_{iN}
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2 \\
z_3 \\
\vdots \\
z_N
\end{bmatrix}
= \frac{1}{\hat{\eta}_\psi} z_\psi \left( \hat{b}_{i1} z_1 + \hat{b}_{i2} z_2 + \hat{b}_{i3} z_3 + \ldots + \hat{b}_{iN} z_N \right) \tag{5.58}
\]

The error \( \varepsilon_{ri(t)} \) can now be determined by subtracting the actual correlation, \( \hat{r} \) from the estimated correlation \( \tilde{r} \) as shown in Equation 5.59.
\[ \varepsilon_{r_i(t)} = \hat{r}_i(t) - \tilde{r}_i(t) \] (5.59)

Expanding Equation 5.59 from Equations 5.57 and 5.53 provides Equation 5.60.

\[ \varepsilon_{r_i(t)} = \frac{1}{\hat{\eta}} z_{\hat{\psi}} \hat{\mu}_i(t) \Theta_z - \frac{1}{\tilde{\eta}} z_{\tilde{\psi}} \tilde{\mu}_i(t) \Theta_z \] (5.60)

The \( \Theta_z \) can be factored out and the sum of the two row vectors with their respective scalar multipliers is equal to a row vector as annotated by the square brackets.

\[ \varepsilon_{r_i(t)} = \left[ \frac{1}{\hat{\eta}} z_{\hat{\psi}} \hat{\mu}_i(t) - \frac{1}{\tilde{\eta}} z_{\tilde{\psi}} \tilde{\mu}_i(t) \right] \Theta_z \] (5.61)

The scalar multiplier, \( \frac{1}{\hat{\eta}} \), can be expanded to \( \frac{1}{\hat{\eta} + \eta_{\mu}} \).

\[ \varepsilon_{r_i(t)} = \left[ \frac{1}{\hat{\eta} + \eta_{\mu}} z_{\hat{\psi}} \hat{\mu}_i(t) - \frac{1}{\tilde{\eta}} z_{\tilde{\psi}} \tilde{\mu}_i(t) \right] \Theta_z \] (5.62)

Expanding the row vectors \( \hat{\mu}_i(t) \) and \( \tilde{\mu}_i(t) \) provides the form in Equation 5.63.

\[ \varepsilon_{r_i(t)} = \left[ \frac{1}{\hat{\eta} + \eta_{\mu}} z_{\hat{\psi}} \begin{bmatrix} \hat{b}_{i1} & \ldots & \hat{b}_{iN} \end{bmatrix} - \frac{1}{\tilde{\eta}} z_{\tilde{\psi}} \begin{bmatrix} \tilde{b}_{i1} & \ldots & \tilde{b}_{iN} \end{bmatrix} \right] \Theta_z \] (5.63)
The two row vectors are now combined to form Equation 5.64.

\[ \varepsilon r_i(t) = \left[ \frac{1}{\eta + \eta_\mu} z_\psi \hat{b}_{i1} - \frac{1}{\eta} z_\psi \tilde{b}_{i1} \ldots \frac{1}{\eta + \eta_\mu} z_\psi \hat{b}_{iN} - \frac{1}{\eta} z_\psi \tilde{b}_{iN} \right] \Theta z \quad (5.64) \]

Equation 5.65 shows the possible values of \( b_{iN} \) depending on the corresponding values, \( \tilde{b}_{ij} \) and \( \hat{b}_{ij} \) in the corresponding connectivity row vectors.

\[
\begin{align*}
\frac{1}{\eta + \eta_\mu} z_\psi \hat{b}_{iN} - \frac{1}{\eta} z_\psi \tilde{b}_{iN} &= \begin{cases} 
\frac{1}{\eta} z_\psi \hat{b}_{iN} & \text{if } \tilde{b}_{ij} = 0 \text{ and } \hat{b}_{ij} = 1 \\
\frac{(\eta + \eta_\mu) z_\psi b_{iN} - \eta z_\psi \hat{b}_{iN}}{\eta (\eta + \eta_\mu)} & \text{if } \tilde{b}_{ij} = 1 \text{ and } \hat{b}_{ij} = 1 \\
\frac{1}{\eta + \eta_\mu} z_\psi \hat{b}_{iN} & \text{if } \tilde{b}_{ij} = 1 \text{ and } \hat{b}_{ij} = 0 \\
0 & \text{if } \tilde{b}_{ij} = 0 \text{ and } \hat{b}_{ij} = 0
\end{cases} 
\end{align*}
\]

5.3 Matrix Notation

The extension to matrix notation is extremely painless after completing the row and column vector notation due to the \([1 \times N]\) and \([N \times 1]\) nature of the vector notations. A such only small examples of the notation are provide in the next two sections. The essence of the change is that the row vectors are formed into a \([N \times N]\) matrix all other portions remain unchanged. The resulting error values are then provided in a \([1 \times N]\) column vector. This change becomes evident in Equation 5.70
5.3.1 Estimate Model in Matrix Notation

In an identical form of the vector notation everything begins with the column vector $\Theta(t)$ which is the ordered set from $(1 - N)$ of all phases at time $t$. This is shown in Equation 5.66.

$$
\Theta(t) = \begin{pmatrix}
\theta_1 \\
\theta_2 \\
\theta_3 \\
\vdots \\
\theta_N
\end{pmatrix}
$$

As in each instance before, the row vector $\tilde{\mu}_i(t)$ is the estimated neighborhood for the $i^{th}$ at time $t$. The values or 1 or 0 depending on whether or not there is connection between the $i^{th}$ node and the respective neighbors, as seen in Equation 5.67.

$$
\tilde{\mu}_i(t) = \begin{pmatrix}
\tilde{b}_{i1} \\
\tilde{b}_{i2} \\
\tilde{b}_{i3} \\
\vdots \\
\tilde{b}_{iN}
\end{pmatrix}
$$

The key change is when the ordered set of all row vectors $\tilde{\mu}_i(t)$ at time $t$ is placed in an the $N \times N$ matrix $\tilde{\mu}(t)$ in the estimated neighborhood for the $i^{th}$ at time $t$. The values
or 1 or 0 depending on whether or not there is connection between the $i^{th}$ node and the respective neighbors. The new form is shown in Equation 5.68.

\[
\tilde{\mu}_i(t) = \begin{pmatrix}
\tilde{b}_{11} & \tilde{b}_{12} & \tilde{b}_{13} & \ldots & \tilde{b}_{1N} \\
\tilde{b}_{21} & \tilde{b}_{22} & \tilde{b}_{23} & \ldots & \tilde{b}_{2N} \\
\tilde{b}_{31} & \tilde{b}_{32} & \tilde{b}_{33} & \ldots & \tilde{b}_{3N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\tilde{b}_{N1} & \tilde{b}_{N2} & \tilde{b}_{N3} & \ldots & \tilde{b}_{NN}
\end{pmatrix}
\] (5.68)

The estimate of the average phase takes the same form as earlier, but is provided again in Equation 5.69.

\[
\tilde{\psi}_i(t) = \frac{1}{\tilde{\eta}} \tilde{\mu}_i(t) \Theta(t)
\] (5.69)

The extended results are shown in Equation 5.70. The key here is that the final form is a $1 \times N$ column vector as mentioned above.
\[ \tilde{\psi}_i(t) = \frac{1}{\eta} \begin{pmatrix} \tilde{b}_{11} & \tilde{b}_{12} & \tilde{b}_{13} & \ldots & \tilde{b}_{1N} \\ \tilde{b}_{21} & \tilde{b}_{22} & \tilde{b}_{23} & \ldots & \tilde{b}_{2N} \\ \tilde{b}_{31} & \tilde{b}_{32} & \tilde{b}_{33} & \ldots & \tilde{b}_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{b}_{N1} & \tilde{b}_{N2} & \tilde{b}_{N3} & \ldots & \tilde{b}_{NN} \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \vdots \\ \theta_N \end{pmatrix} = \begin{pmatrix} \frac{1}{\eta} \left( \tilde{b}_{11}\theta_1 + \tilde{b}_{12}\theta_2 + \tilde{b}_{13}\theta_3 + \ldots + \tilde{b}_{1N}\theta_N \right) \\ \frac{1}{\eta} \left( \tilde{b}_{21}\theta_1 + \tilde{b}_{22}\theta_2 + \tilde{b}_{23}\theta_3 + \ldots + \tilde{b}_{2N}\theta_N \right) \\ \frac{1}{\eta} \left( \tilde{b}_{31}\theta_1 + \tilde{b}_{32}\theta_2 + \tilde{b}_{33}\theta_3 + \ldots + \tilde{b}_{3N}\theta_N \right) \\ \vdots \\ \frac{1}{\eta} \left( \tilde{b}_{N1}\theta_1 + \tilde{b}_{N2}\theta_2 + \tilde{b}_{N3}\theta_3 + \ldots + \tilde{b}_{NN}\theta_N \right) \end{pmatrix} \] (5.70)

### 5.3.2 Actual Model in Matrix Notation

The row vector \( \hat{\mu}_i(t) \) is the actual neighborhood for the \( i^{th} \) at time \( t \). The values or 1 or 0 depending on whether or not there is connection between the \( i^{th} \) node and the respective neighbors, Equation 5.71.
\[ \hat{\mu}_i(t) = \begin{pmatrix} \hat{b}_{11} & \hat{b}_{12} & \hat{b}_{13} & \ldots & \hat{b}_{1N} \\ \hat{b}_{21} & \hat{b}_{22} & \hat{b}_{23} & \ldots & \hat{b}_{2N} \\ \hat{b}_{31} & \hat{b}_{32} & \hat{b}_{33} & \ldots & \hat{b}_{3N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \hat{b}_{N1} & \hat{b}_{N2} & \hat{b}_{N3} & \ldots & \hat{b}_{NN} \end{pmatrix} \] (5.71)

The actual average phase is shown in Equation 5.72.

\[ \hat{\psi}_i(t) = \frac{1}{\bar{\eta}} \hat{\mu}_i(t) \Theta(t) \] (5.72)

The resulting extended perspective and the column vector form is provided in Equation 5.73.
\[
\psi_1(t) = \frac{1}{\eta} \begin{pmatrix}
\hat{b}_{11} & \hat{b}_{12} & \hat{b}_{13} & \ldots & \hat{b}_{1N} \\
\hat{b}_{21} & \hat{b}_{22} & \hat{b}_{23} & \ldots & \hat{b}_{2N} \\
\hat{b}_{31} & \hat{b}_{32} & \hat{b}_{33} & \ldots & \hat{b}_{3N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\hat{b}_{N1} & \hat{b}_{N2} & \hat{b}_{N3} & \ldots & \hat{b}_{NN}
\end{pmatrix}
\begin{pmatrix}
\theta_1 \\
\theta_2 \\
\theta_3 \\
\vdots \\
\theta_N
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{\eta} \left( \hat{b}_{11}\theta_1 + \hat{b}_{12}\theta_2 + \hat{b}_{13}\theta_3 + \ldots + \hat{b}_{1N}\theta_N \right) \\
\frac{1}{\eta} \left( \hat{b}_{21}\theta_1 + \hat{b}_{22}\theta_2 + \hat{b}_{23}\theta_3 + \ldots + \hat{b}_{2N}\theta_N \right) \\
\frac{1}{\eta} \left( \hat{b}_{31}\theta_1 + \hat{b}_{32}\theta_2 + \hat{b}_{33}\theta_3 + \ldots + \hat{b}_{3N}\theta_N \right) \\
\vdots \\
\frac{1}{\eta} \left( \hat{b}_{N1}\theta_1 + \hat{b}_{N2}\theta_2 + \hat{b}_{N3}\theta_3 + \ldots + \hat{b}_{NN}\theta_N \right)
\end{pmatrix}
\] (5.73)

5.3.3 Relationships

With the establishment of the governing equations for the related error, the relationship of varying network structures can be seen.

As the standard deviation of the phases of the nodes approaches zero, the error of \( \psi \) tends to zero.
\( \sigma_\theta \to 0 | E_\psi(t) \to 0 \)

As the standard deviation of the phases of the nodes approaches infinity, the error of \( \psi \) tends to \( 2\pi \).

\( \sigma_\theta \to +\infty | E_\psi(t) \to 2\pi \)

As the number of nodes tends to \( +\infty \), the error in terms of \( \psi \) approaches 0.

\( N \to +\infty | E_\psi(t) \to 0 \)

As the difference of the estimated versus the actual number of neighbors approaches \( +\infty \), the error in terms of \( \psi \) approaches \( 2\pi \).

\( \hat{\eta} - \tilde{\eta} \to +\infty | E_\psi(t) \to 2\pi \)

### 5.4 Ambiguity Probability

The key issue that arises in the analysis is when do two nodes or a node and a group of nodes appear exactly the same. That is, when does the value of a node and a group of nodes have the same value and/or values over time. These parameters are found in the pairing
probability found in what is commonly referred to as the "Birthday Paradox". The paradox can be stated as follows: "How many people must attend a party before the probability of two of them having been born on the same day of the year is greater than 0.50?"

The answer is generally supposed to be $\frac{1}{2}365$ or half the number of days in a year. However, the actual answer is 27. This can be seen in the following probability equation in the next Section in Equation 5.78.

$$p(n : d) = \begin{cases} 
1 - \prod_{k=1}^{n-1} \left(1 - \frac{k}{d}\right) & \text{if } n < d \\
1 & \text{if } n > d 
\end{cases} \quad (5.78)$$

The Birthday Paradox is formally defined by the Doppelganger Probability shown in Equation 5.78

In the case of groups of coupled oscillators appearing to have the same phase the specific probability is shown in Equation 5.79 Here the probability of two nodes appearing identical is given by $n = \text{number of samples or total number of nodes}$ and $d = \text{total number of possible phases}$. The determination of the total number of possible phases will be provided in more detail below.
\[
p(\text{samples : # of phases}) = \begin{cases} 
1 - \prod_{k=1}^{\text{samples}-1} \left( 1 - \frac{k}{\# \text{ of phases}} \right) & \text{if } n < d \\
1 & \text{if } n > d 
\end{cases}
\tag{5.79}
\]

There are two factors that are used to determine the total number of phases available. The factors are the statistical separation of the phases, which is provided by the Standard Deviation, and the measurement resolution of the system.

### 5.4.1.1 Standard Deviation

The first component is the standard deviation. The standard deviation is a common statistical tool, and from one perspective, it provides the range in which a given portion of the population will fall. The basic formula for calculating the standard deviation is given in Equation 5.80

\[
\sigma = \sqrt{\frac{1}{N-1} \sum_{k=1}^{N} (x_k - \bar{x})^2}
\tag{5.80}
\]

Applying Equation 5.80 directly and assuming a normal distribution of a group of nodes of coupled oscillators, \(\sigma\) is the distance from the average in which 68\% of the population will
exist. An example is shown in Figure 5.4.1.1. The example is calculated by allowing the number of nodes in the population to be represented by $N$ in Equation 5.81. The average phase of the entire population is provided as $\bar{x}$. Each phase, represented by $x_k$ is then subtracted from the average, $\bar{x}$, and the differences of the average is summed. The square root of the sum is then taken. For the purposes of this study, two standard deviations will be used. As such, $\sigma$ will simply be doubled. Doubling the value will provide that 95% of the population to be within the referenced range.

$$\sigma = \sqrt{\frac{1}{\text{number of nodes} - 1} \sum_{j=1}^{\text{number of nodes}} (\phi_j - \bar{\psi})^2}$$  \hspace{1cm} (5.81)

![Standard Deviation](image)

Figure 5.5: Standard Deviation.
5.4.1.2 Resolution and Noise

The second component required to determine ambiguity is the resolution determining the individual frequency of each node. There are several effects that can result in a lower frequency resolution in the referenced system. These effects include the result of noise in the system providing a degree of ambiguity or a floor in which the difference between any node or grouping of nodes cannot be differentiated. Frequency resolution, $w$, can be seen as an inverse of the sampling rate, $f_s$. Sampling theory provides that the minimum sampling rate, or Nyquist Rate, required to determine a given frequency, $f$, is twice the frequency $f_s = 2f$. Since the resolution of an entire range of frequencies is required, Nyquist Theory also provides that the minimum sampling frequency is based on the highest frequency. As an example, if the given bandwidth, $B$, of a set of nodes is from 1-10 cycles per sec, or 1-10 Hertz (Hz), then the Nyquist Rate is twice the highest frequency, $f_s = 2 \times 10 = 20\text{Hz}$. The frequency resolution, $w$, is therefore provided at $w = 1/20 = 0.05$. Since the range of frequencies is from 1 - 10, the total number of possible frequencies, $d$, is a product of the size of range of the number frequencies $d = \langle B \rangle \times w$.

To better represent the range of values of frequencies rather than absolute range as in the previous example two standard deviations are used. If there were $N$ number of nodes in a system at resolution of $w$ and the nodes were grouped statistically in a range of values with a standard deviation of $\sigma_\theta$ at time $t$, then the total number of possible phases $d$ at time, $t$, is given by $d(t) = 2\sigma_\theta \times w$. 

105
5.5 New Relationships

The new relationships can now be summarized as follows:

The system resolution, \( w \), is equal to the inverse of the frequency sampled \( f_s \), in Equation 5.82.

\[
w = 1 / f_s \tag{5.82}
\]

The standard deviation of the phases is provided by Equation 5.83.

\[
\sigma = \sqrt{\frac{1}{\text{number of nodes} - 1} \sum_{j=1}^{\text{number of nodes}} (\phi_j - \bar{\psi})^2} \tag{5.83}
\]

The total number of possible phases at any given time \( t \) is provided by Equation 5.84.

\[
d(t) = 2\sigma_\theta \times w\pi \tag{5.84}
\]

The probability that two nodes will appear the same is given by Equation 5.85.

\[
p(\text{samples : # of phases}) = \begin{cases} 
1 - \prod_{k=1}^{\text{samples}-1} \left( 1 - \frac{k}{\# \text{ of phases}} \right) & \text{if } n < d \\
1 & \text{if } n > d
\end{cases} \tag{5.85}
\]
Assuming a standard deviation of phases greater than zero, $\sigma_\theta > 0$, as the resolution tends to $+\infty$, the probability of ambiguity tends to 1. This is shown in Equation 5.86

$$w \to +\infty | p(\text{samples : # of phases}) \to 1$$ (5.86)

As the resolution tends to 0, the probability of ambiguity tends to 0. This is provided by Equation eq-rel2r1.

$$w \to 0 | p(\text{samples : # of phases}) \to 0$$ (5.87)

Assuming a resolution greater than unity, $w > 1$, as standard deviation of phases, $\sigma_\theta$ tends to zero, the probability of ambiguity tends to unity. This is provided by Equation eq-rel2r2.

$$\sigma_\theta \to 0 | p(\text{samples : # of phases}) \to 1$$ (5.88)

These results will be reviewed in detail in Chapter 7. To provide the review, a stochastic model is provided with the details of the framework established in Chapter 6.
5.6 Summary

This analysis in this chapter provides all of the functionality required for the implementation of a system of coupled oscillators such that the relationships between the network components in terms of standard deviation of phases, number of nodes and connectivity can be explored in terms of the limits of determination of the network structure from the behavior of the network. A stochastic model constructed from this analysis is provided in Chapter 6. Tests utilizing the models built from the analysis is provided in Chapter 7.
The intent of the stochastic modeling provided in this chapter is to verify and further the understanding gained in the analysis of Chapter 5. The mathematical models previously described are provided in computational models that include a number of random variables. The typical software implementation limitations are described in detail. The modified Kuramoto’s model developed for this dissertation and described in Section 5.1 is the primary model used throughout this chapter. This material is divided into sections by the software classes utilized in the implementation. Separate sections provide further details on each of the functions used to calculate various components of the model. The final section shows the entire process with multiple models tied together.

The implementation begins with the basic model detailed in Section 5.1 and provided again in Equation 6.1. The model in Equation 6.1 shows that the rate of change of the phase of the \( i^{th} \) node at time, \( t \), is given by the nodes natural frequency, \( \omega_i \), added to a combination of a coupling coefficient, \( K \), a correlation factor, \( r_i(t) \), and the difference of the current phase of the node, \( \theta_i \), and the average of the nodes phase and its neighbors \( \psi_i(t) \).
\[ \dot{\theta}_i(t) = \omega_i + K r_i(t) \sin(\psi_i(t) - \theta_i(t)) \quad (6.1) \]

### 6.1 Class Node

To represent the model, a **Class Node** is created. The class represents the node and its associated traits. This representation includes the following:

- The current phase is **Theta**, representing \( \theta_i \).
- A row vector given as **Mu** represents the neighbors of the node, \( \mu \).
- The number of neighbors is **eta**, which in the analysis is used as \( \eta \).
- The number of additional neighbors **eta_mu**, which is used in the analysis as \( \eta_{\mu} \).
- The average of the phases of the node and its neighbors is **Psi**, representing \( \psi_i \).
- The natural frequency of the node is **Omega**, representing \( \omega_i \).
- The correlation factor is **r**.

In addition, there are a series of complex number representations that are provided in **Class Node**:

- The current phase, **comp_Theta** representing the complex value of the polar form of \( \theta_i \).
• The average of the phases of the node and its neighbors \texttt{comp\_Psi}, representing the complex value of the polar form of \psi_i.

• The correlation value of the node and its neighbors \texttt{comp\_r}, representing the complex value of the polar form of r_i.

The series of error values related to each node are provided in \texttt{Class Node} as follows:

• The error of the correlation coefficient, r, provided as a difference of a reference model as compared to another configuration is given by \texttt{E\_r}.

• The error of average of the phases of the node and its neighbors, \psi, from this model as compared to another configuration is given by \texttt{E\_psi}.

• The error of the phase of the node, \theta, from this model as compared to another configuration is given by \texttt{E\_theta}.

The entire class is represented as follows:
Class Node{
    Theta,
    Mu,
    eta,
    eta_mu,
    Omega,
    Psi,
    r,
    comp_Theta,
    comp_Psi,
    comp_r,
    E_Theta,
    E_Psi,
    E_r,
}
6.2 Class Model

Because multiple models are compared the Class Model is constructed to encompass all of the parameters, components and functions required to execute the full model. This includes the following parameters:

- The total number of nodes in the model is provided by the parameter `number_Nodes`.
- The common number of neighbors is provided by `number_Neighbors`.
- The level of precision of the measurement of the nodes is provided in `m_precision`.
- The actual level of precision of the nodes in the model is provided by `a_precision`.

In addition to the parameters of the model, a number of components are required. These include the following:

- A collection of all nodes in the model is provided by `nodes[]`.
- The standard deviation of the current phases of all nodes is stored in `sigma`.
- The probability of two nodes being indistinguishable is given by `p_dopple`.

In addition to the parameters and components of the model, several functions are required. Details of the internals of each function is provided in Section 6.2.1. The functions include the following:
An initialization function, `init()`, is used to set each of the parameters to the appropriate values and establish random variables.

Once the initial values are assigned, a value for the average phase can be calculated using the function `calcPsi()`.

The correlation value of each node is calculated by `calcR()`.

The calculation of the average phase of a node and its neighbors is provided by `calcTheta()`.

The standard deviation of the phases of the entire collection of nodes is provided by `calcStdDev()`.

The probability that two nodes are indistinguishable is given in `calcPD()`.

The error of the current phase as compared to another model is calculated in `calcErrorTheta()`.

The error specific to the difference in average phase of a node and neighbors compared to another models representation of that node and its neighbors is found by `calcErrorPsi()`.

The error based on the difference in correlation of a node and its neighbors compared to another models representation of that node and its neighbors is determined in `calcErrorR()`.

The total class is provided as follows:
Class Model{
    number_Nodes,
    number_Neighbors,
    m_precision,
    a_precision,
    nodes[],
    averagePhase,
    sigma,
    p_dopple,
    init(),
    calcPsi(),
    calcR(),
    calcTheta(),
    calcStdDev(),
    calcPD(),
    calcErrorTheta(),
    calcErrorPsi(),
    calcErrorR()
}
6.2.1 Functions

6.2.2 Init()

Upon the call of the initialization function, init(), each node is assigned a natural frequency, Omega. This frequency is a random number within a specified range, normally [0, π]. The value is then scaled to the precision provided in m_perCISION for those nodes that are part of estimated models and a_perCISION for those nodes that are assigned as part of an model representing an actual system.

The phase of each node, Theta, at initialization is set equal to the natural frequency, Omega. Each node is also assigned a number of normal and additional neighbors eta and eta_mu. The row vector Mu is assigned a boolean value indicating the connection of each neighbor.

The pseudo-code for the function is as follows:
init(){
    for( numNodes ){
        Omega = random phase $[0, \pi]$ 
        if (estimate)
            Omega = Omega scaled by $m_{\text{precision}}$
        if (actual)
            Omega = Omega scaled by $a_{\text{precision}}$
        Theta = Omega 
        eta = Normal Neighbors 
        eta\_mu = additional neighbors
        for( numNodes ){
            if (neighbor)
                Mu = 1
            if (not neighbor)
                Mu = 0
        }
    }
}
6.2.3 \textit{calcPsi()}

With the initiation of all of the nodes, an average phase can be calculated. This is provided in the function \textit{calcPsi()}. Equation 6.2, which is used in this function is the same as the equation in Section 5.1.1 and Section 5.1.2.

\[
\psi_i(t) = \frac{1}{\eta} \left\{ \sum_{k=(i-j) \mod N}^{(i+j) \mod N} \theta_k(t) \right\} + \theta_{\mu i}(t) \quad (6.2)
\]

In the case of the computational model, whether the values of the average phase, \( \Psi \), represents an estimate or an actual is determined by the structure of the model as the base equations are the same in either case. The average phase, \( \Psi \), is simply the sum of the phases of the reference node and all of the phases of the nodes neighbors. The sum is divided by the total number of neighbors, \( \eta + \eta_{\mu} \).

\begin{verbatim}
calcPsi(){
    for( numNodes ){
        Psi = (Mu . Theta)/(eta + eta_mu)
    }
}
\end{verbatim}
6.2.4 \( \text{calcR()} \)

With the calculation of the average phases, the correlation of the nodes can be determined by the function \( \text{calcR()} \). This function is derived from Kuramoto’s correlation equations. The equation used is provided in Equation 6.3

\[
 r = \left\{ \frac{\sum_{k=(i-j) \mod N} e^{i\theta_k(t)}}{e^{i\psi(t)}\eta} \right\} + e^{i\theta_{\mu}(t)}
\]

(6.3)

The polar form of \( \Psi \) and \( \Theta \) are \( \text{comp\_Psi} \) and \( \text{comp\_Theta} \) respectively. These values are used to represent \( e^{i\psi(t)} \) and \( e^{i\theta(t)} \). With that, the sum of the phases, \( \text{comp\_Theta} \) of the node and its neighbors is multiplied by the average phase \( \text{comp\_Psi} \) which is then divided by the total number of neighbors \( (\eta + \eta_{\mu}) \).

```
calcR(){
    for( numNodes ){
        comp_r = (Mu \cdot \text{comp\_Theta})/(\text{comp\_Psi} (\eta + \eta_{\mu}))
        r = absolute magnitude (comp_r)
    }
}
```

The final value of value of \( r \) is given by taking the absolute magnitude of the complex number \( \text{comp\_r} \).
6.2.5 *calcTheta()*

With both *Psi* and *r* calculated the emphasis finally turns to calculating the rate of change for each node as provided by the function *calcTheta()*). The equation used is the primary reference equation from both this chapter and the analysis. It is once again provided as Equation 6.4

\[
\dot{\theta}_i(t) = \omega_i + K r_i(t) \sin(\psi_i(t) - \theta_i(t))
\]  

(6.4)

The values for the natural frequency of the node is *Omega*, representing *\( \omega_i \)*. The value *K* is set equal to unity, which is maintained from Kuramoto’s analysis \[Str00\]. The correlation factor is *r*, which is *\( r_i(t) \)* is determined by the function *calcR()*). The value *Psi*, which represents *\( \psi_i(t) \)*, is determined by the function *calcPsi()*). The output of the function is the value for *Theta* which represents *\( \theta_i(t) \)*

```c
calcTheta(){
    for( numNodes ){
        Theta = Omega + r sin(Psi - Theta)
    }
}
```
6.2.6 calcStdDev()

The standard deviation of the phases is a critical component in the determination of the probability of two nodes being indistinguishable from each other. The standard deviation is determined from the equation first provided in Section 5.4.1.1. The relevant form is shown again in Equation 6.5.

$$\sigma = \sqrt{\frac{1}{\text{number of nodes} - 1} \sum_{j=1}^{\text{number of nodes}} (\phi_j - \bar{\psi})^2}$$

(6.5)

In the implementation of the function calcStdDev(), the value of sigma is determined by the square root of product of (1) the inverse of the total number of nodes minus unity, 1/(numNodes -1), and (2) the sum of the squares of difference of the current phases Theta and averagePhase.
6.2.7 calcPD()

The probability of two nodes being indistinguishable is determined in the function calcPD(). The function is based on the equation provided previously in Section 5.4.1 and repeated in Equation 6.6. The output of the function is the value of \( p_{\text{dopple}} \).

\[
p(samples: \# \text{ of phases}) = \begin{cases} 
1 - \prod_{k=1}^{samples-1} \left( 1 - \frac{k}{\# \text{ of phases}} \right) & \text{if } n < d \\
1 & \text{if } n > d 
\end{cases}
\]  

(6.6)
The equation provides for that the probability that two nodes are indistinguishable is given by unity minus the incremented product of (1) the quotient of the difference of unity minus the increment from one to total number of nodes minus unity and (2) total number of possible phases. The number of nodes minus unity is provided by $\text{numNodes} - 1$. The total number of possible phases is found by the product of three standard deviations of phases of nodes, $\sigma$, and the model appropriate precision parameters $m_{\text{Precision}}$ and $a_{\text{Precision}}$ depending on whether the referenced model is an estimate or an actual model respectively.

```c
calcPD()
{
    p_doppel = 1
    for( i = 1 to numNodes - 1 ){
        if (model = estimate)
            p_doppel = p_doppel \cdot (1 - i) / (3 \sigma \cdot m_{\text{Precision}})
        if (model = actual)
            p_doppel = p_doppel \cdot (1 - i) / (3 \sigma \cdot a_{\text{Precision}})
    } 
    p_doppel = 1 - p_doppel 
}
```
6.2.8 calcErrorTheta()

The error of $\Theta$, $E_\Theta$, represents the difference of calculated values of phases, $\Theta$, of two different models as shown in Equation 6.7. Likewise the function calcErrorTheta() is simply the difference of the two calculated $\Theta$ values as derived from the respective calcTheta() functions above.

$$\varepsilon_{\theta_i(t)} = \hat{\theta}_i(t) - \tilde{\theta}_i(t) \quad (6.7)$$

```
calcErrorTheta()
{
    for( numNodes ){
        E_Theta = Theta_(model 1) - Theta_(model 2)
    }
}
```

6.2.9 calcErrorPsi()

The error of $\Psi$, $E_\Psi$, represents portion of the error of theta that is the result of the difference of calculated values of average phases, $\Psi$, of two different models. This error is shown in Equation 6.8. The equation and the detailed analysis applied to it is found
in Section 5.1.3. The function `calcErrorPsi()` is the difference of the two calculated Psi values as derived from the respective `calcPsi()` functions above.

\[
\varepsilon_{\psi_i(t)} = \hat{\psi}_i(t) - \tilde{\psi}_i(t)
\]  

(6.8)

```c

calcErrorPsi()
{
    for( numNodes ){
        E_Psi = Psi(model 1) - Psi(model 2)
    }
}
```

6.2.10  `calcErrorR()`

The error of \( r, E_R \), represents the portion of the error that is a result of the difference of the calculated values of correlation values, \( r \), of two different models. This error is shown in Equation 6.9 and was originally referenced and detailed in Section 5.1.4. The equation and the detailed analysis applied to it is found in Section 5.1.3. The function `calcErrorPsi()` is the difference of the two calculated Psi values as derived from the respective `calcPsi()` functions above.
\[ \varepsilon_{r_i(t)} = \hat{r}_i(t) - \tilde{r}_i(t) \] (6.9)

calcErrorR() {
    for( numNodes ){
        E_r = r_(model 1) - r_(model 2)
    }
}

6.3 Model Iteration

With the components and functions of the models defined, tying the model in a complete iteration for testing is the next task. The model flow is straightforward and follows the sequence of the definitions provided. Everything begins with the parameters defined. Initiation is therefore provided by the function init(). The iteration continues by calculating the values for \( \Psi, r \), and returning to \( \Theta \). The standard deviation, \( \sigma \), and probability of ambiguity of two nodes, \( p_{dopple} \) can then be established. The calculation of the associated error values for \( \Theta, \Psi, \) and \( r \) provided the last step.
Iterate Models()
{
    for( ALL Models ){
        init()
        calcPsi()
        calcR()
        calcTheta()
        calcStdDev()
        calcPD()
    }
    calcErrorTheta()
    calcErrorPsi()
    calcErrorR()
}

The computational framework provides good control and variability for the experimenta-
tion. By providing the direct control over the level of precision in the computational model,
the random drift normally associated round off error with models of natural systems is easily
managed. Round off error still exists, but it is translated into a system noise floor that can
be defined.
EMPIRICAL ANALYSIS

This chapter provides an empirical demonstration of the effect of the various systems variables on the determination of the underlying network structure. This is summarized in the question posed at the beginning of Chapter 5 as follows:

*Under what conditions is it possible to determine that an amount of change of a loosely coupled oscillator could most likely be associated with only a particular individual node?*

As stated, this question drives the exploration of the limiting conditions for determining the underlying network structure.

The most important factor in that exploration is the probability that two nodes have the same phase. As described in Section 5.4, this factor is driven by two variables, the range of possible phases and the number of distinguishable phases within that range. The dynamics of the systems creates a situation where the range of possible phases changes over time and this dynamic is a function of the network topology. The network topology is directly defined by the total number nodes in the network, the number of connections of each node,
and the total number of connections within the entire network. The secondary factor that can be applied to the study of the determination of the network topology is – the relative contribution of each node based on the topology of a given node.

Beginning at the end, the first look into the performance of the network of coupled oscillators is the rate of change of probability of ambiguity based on the variations in the range of possible phases or standard deviation of phases.

### 7.1 Ambiguity and Variations of Phase Range

The standard deviation of the phases is the selected measure to determine the range of values used in the probability of two nodes being indistinguishable. By looking at a range of standard deviations based on fixed models, a good perspective on the effect of standard deviation can be achieved. In order to establish the given conditions, the physical and mathematical model proposed in Chapter 5 is utilized.

The specified model is a simple ring topology and the modified version of Kuramoto’s model. A sample of the ring topology is provided in Figure 7.1. The key feature is that nodes are loosely coupled with only the immediate neighbors or those on the left and right. The reference mathematical model is the modified Kuramoto’s equation provided in Equation 7.1. The essential tie to the ring topology is with the value of $\psi_i(t)$, the average of the
phases of the neighbors and the reference node. In the case of the ring topology, $\psi_i(t)$ is calculated using Equation 7.2.

\[
\dot{\theta}_i(t) = \omega_i + K r_i(t) \sin(\psi_i(t) - \theta_i(t))
\] (7.1)

\[
\psi_i(t) = \frac{1}{\eta} \sum_{k=(i-j) \mod N}^{(i+j) \mod N} \theta_k(t)
\] (7.2)

The remaining essential equations are first, the standard deviation as calculated from Equation 7.3 and second, the probability equation for determination of ambiguity as seen in Equation 7.4. For an extended discussion of the standard deviation see Section 5.4.1.1. Likewise, further details on the ambiguity probability is found in Section 5.4.1.
\[ \sigma = \sqrt{\frac{1}{\text{number of nodes} - 1} \sum_{j=1}^{\text{number of nodes}} (\phi_j - \bar{\psi})^2} \quad (7.3) \]

\[ p(\text{samples} : \# \text{ of phases}) = \begin{cases} 1 - \prod_{k=1}^{\text{samples} - 1} \left(1 - \frac{k}{\# \text{ of phases}}\right) & \text{if } n < d \\ 1 & \text{if } n > d \end{cases} \quad (7.4) \]

7.1.1 First Look - Varying Standard Deviations of Phases

To establish the effect of the standard deviation on the ambiguity probability of a group of coupled oscillators, the first three experiments were conducted with populations of ten nodes. The nodes are randomly seeded with natural frequencies distributed over three different closed intervals, \([0, \pi]\), \([0, \frac{1}{2}\pi]\), and \([0, \frac{1}{4}\pi]\). The results are three varying initial standard deviations that each evolve over time. The first look at the described evolution of three different populations is shown in Figure 7.2. The figure demonstrates the nice symmetric relationship between the change in standard deviation to the change in the probability of ambiguity in the population. Restated, as the standard deviation is tending toward zero, the probability that two nodes are indistinguishable approaches unity. The variations in the starting standard deviation also show a clear tendency toward evolution that is relatively...
shorter. The relationship appears almost linear. This appearance is deceiving, but provides the key to one of the best insights of this work. To view this insight, a look at a larger population is provided.

Figure 7.2: Comparison of multiple 10 Node varying Standard Deviation Populations (same random seed).

7.1.2 Extended View - Varying Standard Deviations of Phases

Looking at a ring network with an increased population size of 100 nodes to eliminate possibility that the small sample size is creating anomalies, a 100 node network is seeded in the same manner. Once again, the natural frequencies are distributed over three different closed intervals, \([0, \pi]\), \([0, \frac{1}{2}\pi]\), and \([0, \frac{1}{4}\pi]\). The results, shown in Figure 7.3, clearly demonstrated the same symmetrical relationship between the change in standard deviation and the change in ambiguity probability. Even better, the same apparent linear shift is evident.
between the various starting intervals. This apparent linear shift might lead to a belief that knowing the starting standard deviation might provide the ability to not only know if the determination of the network relationships is possible, but also to project the rate of change over time. Again, this belief is incorrect and can be seen in a second random seed over the same three closed intervals.

![Figure 7.3: Comparison of multiple 100 Node varying Standard Deviation Populations (same random seed).](image)

With a new seed, the three 100-node populations are once again seeded with random natural frequencies over the closed intervals of $[0, \pi]$, $[0, \frac{1}{2}\pi]$, and $[0, \frac{1}{4}\pi]$. The results are shown Figure 7.4. The same linear correlation between the various interval is seen. Looking closely, while the patterns parallel, the shape of the new pattern is clearly different as shown in Figure 7.3.
To emphasize the difference a figure with three populations using the same starting interval but different seeds is provided in Figure 7.5. Shown together, the differences are very evident. This new perspective might lead to the conclusion that an issue might exist in the seeding of the populations. What is true is that the seed provides the same sequence of random numbers. These random numbers are then scaled each time over the closed interval. Therefore, the evolution occurs with nodes with the same relative separation only scaled differently. This evolution leads to a linear difference between the intervals but with scaled differences in the starting populations. The difference between random seeds might better be described as providing a different self-similarity index.

Consider a different perspective. At first all of the nodes are cycling at their natural frequencies. Upon coupling, the nodes tend to change toward a common phase with their
neighbors. The local population becomes much more homogenous. As time progresses, the changes created by local differences are minimal compared to the changes being generated by differences over a larger number of neighbors. Therefore, populations with a lower self-similarity would tend to change more quickly, while populations with a high self-similarity would change more slowly.

While this is an exciting point for future study, for the current topic, the key relationship is not the rate of change, but the symmetric relationship of the ambiguity probability and the standard deviation at any specific time. This later relationship was predicted in the analysis in Section 5.4 and remains true. Later experiments will test this relationship further.
7.2 Number of Nodes -vs- Ambiguity and Variations of Phase Range

With the effect of the standard deviation of the phases to the ambiguity probability isolated by the equations and the stochastic examples, the next relationship explored is that between the number of nodes and the ambiguity. In this case, the number of nodes will be varied while the starting interval of the population, neighborhood structure and resolution is held steady.

To best isolate the effect of the number of nodes, the starting interval is maintained by seeding the population as before with nodes receiving natural frequencies distributed over the closed interval \([0, \pi]\). The neighborhood structure provided is still the ring topology shown in Figure 7.1. The resolution is set to \(1 \times 10^5\), meaning that there are \(1 \times 10^5\) number of detection frequencies over the interval which is also maintained by round off functions at the given resolution.

7.2.1 First Look - Varying Number Nodes

A ten node network using parameters is first explored. The results are shown in Figure 7.6. In the figure, the change of standard deviation and ambiguity probability over time is shown. As mentioned above, the nice symmetrical relationship between the two values is evident. The rate of change becomes more interesting when compared with two other networks of 50 nodes and 100 nodes respectively shown in Figure 7.7 and Figure 7.8.
Figure 7.6: Ambiguity probability and phase standard deviation - 10 Node Network.

Figure 7.7: Ambiguity probability and phase standard deviation - 50 Node Network.

As might be intuitive, the rate of change is slower for larger networks. This corresponds to the flow of information over time and given number of information exchange points. That is, with more points required to spread the information over and a single pipe to pass the
information, the amount of time required is larger. A combined graph showing the rate of change of the standard deviation of the phase for all three networks is shown in Figure 7.9. A similar perspective for probability of ambiguity is found in Figure 7.10.

Figure 7.8: Ambiguity probability and phase standard deviation - 10 Node Network.

Figure 7.9: Phase standard deviation - 10-100 Node Network.
7.2.2 Extended View - Varying Number Nodes

In a more extensive review, 100 different experimental models for each of four different network sizes, 10, 20, 50, and 100 nodes each were constructed. As before, the starting interval is a random number seeding with nodes having natural frequencies distributed over the interval \([0, \pi]\). All networks are constructed with a ring topology. The resolution established as \(1 \times 10^5\). The results are shown in Figure 7.11.

The extended perspective found in Figure 7.11, demonstrates that the rate of flow of information across the similarly constructed networks is a function of the number of nodes, with higher generations required for larger networks. One of the more interesting aspects is, along with the average generation before reaching a probability of 0.60 scaling logarithmically, the standard deviation of the average number of generations also demonstrates a logarithmic
Figure 7.11: First generation with an ambiguity probability of $> 0.60$ for 100 model evaluations of 10-100 node networks - logarithmic scaled y-axis.

Figure 7.12: First generation with an ambiguity probability of $> 0.60$ for 100 model evaluations of 10-100 node networks - normally scaled y-axis.

scaling. Figure 7.12 emphasizes this by removing the logarithmic scaling on the y-axis. The internal details are provided in Table 7.1. The power function and power law trend line of this effect is further shown in Figure 7.13.
Table 7.1: Average and Standard Deviation of First generation over with an ambiguity probability of $>0.60$.

<table>
<thead>
<tr>
<th>Node</th>
<th>Average</th>
<th>Std Dev</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>2394.87</td>
<td>505.08</td>
</tr>
<tr>
<td>50</td>
<td>929.56</td>
<td>118.97</td>
</tr>
<tr>
<td>20</td>
<td>216.46</td>
<td>20.68</td>
</tr>
<tr>
<td>10</td>
<td>66.12</td>
<td>4.93</td>
</tr>
</tbody>
</table>

Figure 7.13: Average and Standard Deviation of First generation over an ambiguity probability of $>0.60$. The trend line demonstrates the log-log nature of the standard deviation and average first generations.

Reviewing the implications, as the number of nodes increases, the average first generation to exceed a given ambiguity probability also increases. The error associated with this average also increases, both exponentially. Not altogether surprisingly, this analysis also reflects the distribution of self-similarity which should be further explored.
7.3 Phase Resolution

The precision and accuracy at which the phases can be measured both externally and internally provides a degree of system noise and directly effects the phase resolution. This variability then further effects the number of available frequencies. The number of available frequencies in turn effects the ambiguity and therefore, the ability to determine the network structure based on the behavior of the nodes in the network. In this set of experiments, the phase resolution is varied over a series of networks. The networks are all provided with 100 nodes. At initiation, the distribution of the natural frequencies is maintained at the closed interval $[0, \pi]$. All networks are still constructed on a ring topology.

7.3.1 First Look - Varying Phase Resolution

In the first demonstration, a network of 100 nodes connected on a ring with natural phases seeded in the range $[0, \pi]$, have the phase resolution varied at $1 \times 10^6$, $1 \times 10^5$, and $1 \times 10^4$. The resulting probability of ambiguity is provided in Figure 7.14.

The data show the expected direct relationship between the variation in resolution and probability of ambiguity. The linear scaling which appears as an exponential difference due to the exponential difference in each of the provided resolutions is clear in Figure 7.14. To view this further, a second series of experiments utilizing a larger set of nodes is explored.
7.3.2 **Extended View - Varying Phase Resolution**

In the extended experiments, 3 sets of 100 different random seeds initiated the natural phases of each network still maintaining the \([0, \pi]\) closed interval. Each set of experiments was conducted with 100 node ring networks. The phase resolution was varied with resolutions provided at \(1 \times 10^6\), \(1 \times 10^5\), and \(1 \times 10^4\).

The results provide a good view of the scaled effect of resolution as shown in Figure 7.15. Once again, the variation in self-similarity appears to generate the extensive range of first generations to exceed the 0.60 probability of ambiguity. The variation does not reflect the exponential scaling and power law plots seen in the previous experiments of variations in the number of nodes. This lack of scaling would also tend to continue to support the self-similarity hypothesis provided earlier. For the current effort, the more relevant result
is the concurrence with the probability relationships suggested in Section 7.15 and given in Section 5.4.

Figure 7.15: Probability of Ambiguity at various resolutions for multiple tests.
7.4 One Additional Connection

With the all of the primitives thus far built on ring topologies, the next set of demonstrations is intended to show the more interesting topology provided by an single additional connection. A sample of the network topology described is shown in Figure 7.16. Networks of sizes 10 and 100 nodes are provided for perspective. The resolution of all of the networks explored is maintained throughout all of the experiments at $1 \times 10^5$. The seeded phase interval is maintained at $[0, \pi]$.

![Figure 7.16: Modified Node Structure.](image)

7.4.1 First Look - One Additional Connection

In the first experiment, the network is constructed with 100 nodes in a ring topology seeded in the $[0, \pi]$ interval, with a phase resolution of $1 \times 10^5$. A duplicate network is
constructed with an additional node connected as shown in the smaller network seen in Figure 7.16. The resulting probability of ambiguity is shown in Figure 7.17.

Figure 7.17: \( p(\text{ambiguity}) \) comparison of two different network topologies - a ring and a ring with a node that connects to one additional node.

The increased slope in the curve representing the network with the additional connection reflects the anticipated change due to the reduction of the distance required for information to flow in the network. This trend is evident in networks of various sizes as shown next.

7.4.2 Extended View - One Additional Connection

In the extended iterations, the 100 different networks are constructed for both 10 and 100 nodes. The nodes are provided in a ring topology and are seeded in the \([0, \pi]\) interval, with a phase resolution of \(1 \times 10^5\). A pair of sets of 100 duplicate networks is also constructed, each with an additional node connected as described above and as shown in the smaller network.
seen in Figure 7.16. The comparison of the two networks in terms of generation in which each model has a $> 0.60$ ambiguity probability is shown in Figure 7.18.

![Figure 7.18: p(ambiguity) comparison of two different 100 model experiments.](image)

The important trend is the nature of the additional node to lower the generation in which the ambiguity probability is greater than 0.60. It is also nofigure that in greater than 0.97 percentage of the experiments all resulted in lower generations. In those cases in which the generation was not lower, the percentage of generations in which the additional node networks exceeded by was greater than 0.01.

### 7.5 Multiple Additional Connections

Looking beyond the first extension, the next group of experiments demonstrates the more extensive power of multiple additional connections. The new network topology is shown in
Figure 7.19 and provides a single node with connections to every other node. As a reminder, all connections in this study are considered bimodal, so information flows in both directions. The new networks are provided in sizes of 10 and 100 nodes for alternative perspectives. The resolution continues to be maintained at $1 \times 10^5$. The seeded phase interval is still $[0, \pi]$.

7.5.1 First Look - Multiple Additional Connections

In the first experiment, the familiar network model is assembled with 100 nodes in a ring topology seeded in the $[0, \pi]$ interval, with a phase resolution of $1 \times 10^5$. A duplicate network is constructed with the new topology with one node selected to be connected to all others as shown in Figure 7.19. The resulting probability of ambiguity is shown in Figure 7.20.

The dramatic increase in the curve of the network with the multiple additional connections is obviously due to the decrease in the distance information has to flow in the network.
This trend is evident in networks of various sizes but has a surprising difference as shown next.

### 7.5.2 Extended View - Multiple Additional Connections

In the larger experiments, two sets of 100 different networks are constructed for both 10 and 100 node networks. The nodes in the first set are provided in the typical ring topology and are seeded in the $[0, \pi]$ interval, with a phase resolution of $1 \times 10^5$. The second pair of 10 and 100 node networks are exact replicas, but an additional node is connected as described above and as shown in the smaller network seen in Figure 7.19. The comparison of the two networks in terms of generation in which each model has a greater than 0.60 ambiguity probability is shown in Figure 7.18.
Figure 7.21: p(ambiguity) comparison of two different 100 model experiments.

The most important trend is the nature of the additional multi-connected node to lower the generation at which the ambiguity probability is $>0.60$. In none of the networks does the additional connected network ever come close to falling in a later generation. What is most fascinating about this case is the view that the networks with the lower number of nodes are effected much less than the larger network. This view can be seen by looking closely at Figure 7.18. The center pair of lines represents the smaller networks. The outer pair of lines is the 100 node networks. Not only is the larger network effected more, but the overall performance is even greater.

7.6 Determination with One Additional Connection

All of the previous experiments have provided the foundation for this series. These experiments provide the final look at the probability of determining the network based on the
behavior of the network. In this series, a reference network is simulated to provide an “actual” network of couple oscillators. The reference networks are constructed of between 10 and 100 nodes. The topology of the “actual” network is a ring topology with a single additional connection. Networks simulating all possible combinations are provided in conjunction. A computational convenience is taken in that, only networks with a single reference node are explored for the one additional node connected to all others. It is possible that another pairing somewhere else in the network provides the reference quantities; however, the new arrangement would have to correct for that pairing and all others.

7.6.1 First Look - One Additional Connection

In the first experiment an “actual” 10-node ring network with one additional connection is constructed. The network is seeded in the closed interval of \([0, \pi]\) with an internal resolution of \(1 \times 10^6\). A corresponding network with only a ring topology is constructed with an external resolution of \(1 \times 10^5\). The phase inputs to this network are the phases from the “actual” network representing a measured input being provided to an alternative model network. The resulting error in terms of phase of the \(i^{th}\) node, \(\varepsilon_{\theta_i}\), is plotted in Figure 7.22.

The perspective in Figure 7.22 provides a nice view of three different phenomena.
The first is the obvious way the two nodes that are significantly different between the two models standout. This view is shown in the two lines with the negative slope in Figure 7.22.

The next is the corresponding noise floor provided by the difference in the resolution between the actual and measured systems. This perspective is not the most common place for noise as the measured system could provide a higher resolution within the physical limits of the measuring device. At the same time, this perspective shows the related noise effect. The same would be true in reverse but would not be measurable. Rather it would provide oversampling of the “real” system.

The last occurrence is the nice relationship shown between the change in declining error and the change in predictability shown by the dark line with the positive slope in Figure 7.22. This perspective demonstrates the predicted ambiguity probability. As a concern, the probability is slightly high due to the difference in the predicted and actual
resolution. This perspective brings the point that an inverse in resolution with the actual being lower than the sample providing lower than actual probabilities of determination.

All of these perspectives show that the error provided in detailed analysis fits the models as anticipated.

### 7.6.2 Extended View - One Additional Connection

The expanded experiments here are conducted using the reference “real” system which consist of a network of 10, 20, 50, and 100 nodes. The nodes are arranged in a ring with the addition of one extra connection between two nodes. All nodes are seeded in the range \([0, \pi]\) with an internal resolution of \(1 \times 10^6\). For each reference model, alternative models of the same size and starting parameters are used with the exception of the phase resolution which is provided at the lower resolution of \(1 \times 10^5\). The number of alternative models for each reference model is equal to the number of nodes in the reference model. That is, the reference model of 10 nodes has 10 each 10 node alternative models compared to it. Each comparative model has a different connection scheme with a different node selected to be coupled with the node 1. An example of the comparison of models can be seen in Figure 7.23.

In the comparison shown in Figure 7.23, the mean square error of each of the models is shown over time. As expected from the detailed analysis, the lowest error model is the
Figure 7.23: Mean square error difference of various estimates compared to actual networks.

one that best matches the reference model. This is shown as the bottom dotted line. As
time progresses and the reference model begins to completely synchronize all models become more accurate. The dark line with a positive slope shows the probability of ambiguity that is predicted. As all of the models, converge the probability reaches its peak.

Table 7.2: Average ambiguity probability at the time of an error in determining the correct model.

<table>
<thead>
<tr>
<th>Node</th>
<th>Average Generation</th>
<th>Average p(doppel)</th>
<th>MSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>100 Node</td>
<td>534.0</td>
<td>0.778</td>
<td>$\pm$ 3.70</td>
</tr>
<tr>
<td>50 Node</td>
<td>115.6</td>
<td>0.511</td>
<td>$\pm$ 10.04</td>
</tr>
<tr>
<td>20 Node</td>
<td>120.04</td>
<td>0.513</td>
<td>$\pm$ 8.40</td>
</tr>
<tr>
<td>10 Node</td>
<td>38.88</td>
<td>0.466</td>
<td>$\pm$10.46</td>
</tr>
</tbody>
</table>
To further validate these ideas, the results of series of 100 experiments on each of the different size models is shown in Table 7.2. Here each of the experiments is initiated as described above. A different random number provided the seed for each experiment. The average ambiguity probability along with mean square error of that probability estimate and average generation in which the model was chosen as the best is shown. The results follow the expected from Section 5.4, with the some concern for the high average estimate in the largest networks. This set of high values should be explored further, but still provided useful insight on the feasibility of the estimate.

7.7 Determination with Multiple Additional Connections

The last series of experiments is provided to demonstrate feasibility determining the network structure in a network with connection shown in Figure 7.19 as compare to the predicted probability of ambiguity provided for the determination of network structure. The networks are constructed as in the previous section before with a reference network of 10, 20, 50 and 100 nodes. The nodes are arranged in a ring topology with one node having additional connections to all other nodes. The nodes are seeded in the \([0, \pi]\) interval with the reference model having the ability to resolve phases to \(1 \times 10^6\). Alternative models are then compared to the reference models.
7.7.1 *First Look - Multiple Added Connections*

In the first experiment a reference model of 10 nodes is constructed as described above. The network has 10 nodes with the 1<sup>st</sup> node having all other nodes connected to it. An estimating network consisting of only a ring topology of 10 nodes is compared. The estimating network is the provided with same phases at the slightly lower resolution of $1 \times 10^5$. The phases are updated between the estimate and the reference model after each time step. The resulting error in terms of estimates of phase, $\theta_i$, is shown in Figure 7.24.

![Figure 7.24: Error $\theta_i$ between the estimate and actual networks.](image)

The relative close nature of the change of all nodes except the two that are connected the same in both the actual model and the predicted model is clear. This relationship can be seen in the eight lines with the negative slopes and the two sets of dotted lines along the bottom in Figure 7.22. The bottom lines correspond to the noise floor provided by the difference in the resolution between the actual and measured systems. The relationship
between the change in declining error and the change in predicability shown by the dark line with the positive slope in Figure 7.22 is again as predicted in Section 5.4.

### 7.7.2 Extended View - Multiple Added Connections

To look further, a series of comparative models is provided against each of the various size reference model in an extensive series of tests. First, the 10 node reference model is compared to 8 alternative 10 node models. The comparative models are constructed as variants of the reference configuration with the connection scheme of a specified node selected to not receive the additional connection with the node 1. An example of the comparison of models can be seen in Figure 7.25

![Figure 7.25: Mean square error difference of various estimates compared to actual networks.](image-url)
From Figure 7.25, the model that most reflects the reference configuration is obvious by the low mean square error rate shown as the dotted line across the bottom of the Figure. At the same time, as the models converge, the probability of ambiguity increases as shown by the dark black line with the positive slope in Figure 7.25. The convergence of the models and the probability correspond nicely as predicted.

To validate this finding, a more extensive series of experiments each with its own random seed is provided. In these experiments, 100 separate iterations of the networks of sizes 10, 20, 50 and 100 are conducted. The comparison of the average generation and average ambiguity probability is provided in Table 7.3.

Table 7.3: Average ambiguity probability at the time of an error in determining the correct model.

<table>
<thead>
<tr>
<th></th>
<th>Average Generation</th>
<th>Average $p($dopple$)$</th>
<th>MSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>100 Node</td>
<td>21.03</td>
<td>0.889</td>
<td>± 4.33</td>
</tr>
<tr>
<td>50 Node</td>
<td>18.38</td>
<td>0.873</td>
<td>± 4.61</td>
</tr>
<tr>
<td>20 Node</td>
<td>21.64</td>
<td>0.872</td>
<td>± 5.92</td>
</tr>
<tr>
<td>10 Node</td>
<td>19.47</td>
<td>0.803</td>
<td>± 6.35</td>
</tr>
</tbody>
</table>

While the results are significant in that the values are tightly correlated. The average ambiguity probability is higher than expected. As such further experimentation, specifically in to the statistical distribution of the nodes should be further explored.
CONCLUSIONS AND FUTURE EXTENSIONS

The nonlinear nature of coupled oscillators provides a serious challenge to the existing methods for explaining, as well as, quantifying and qualifying the existing and future behavior of systems containing them. While these restrictions do exist and new techniques are badly needed, studies such as this one, provide key insights into the behavior and allow for quantification of the predictability at specified times. This work shows that the ability to predict whether a given network structure can be determined at least by enumeration of possible network combinations.

While this study was not intended to provide a specific technique for the determination of network topology by the behavior, the implications of testing most-likely models against the subject systems provides good promise as an approach. The key breakthrough came by picturing the problem as a common tracking problem as in tracking the projected path of object given its history. One of the most popular approaches to this is Kalman Filtering. For an introduction to Kalman Filters see Simon [Sim01].

To arrive at the point of understanding that this might be seen as a a tracking problem, a generalized model and mathematical framework had to discovered. The well worn road
provided by Winfree, Kuramoto and Strogatz provided a perfect foundation. By simplifying the models, as provided here, the range of prediction could be managed. Still, the fact that the relationships between the various components of networks of coupled oscillators are based on differential and nonlinear equations makes predictability subject to extreme limitations and results primarily in qualification of the system based on stability analysis.

8.1 Summary of Research

The problem being addressed in this research is the limits of determining the network structure of loosely coupled oscillators based on the observation of the behavior of the network, specifically, node synchronization.

The hypothesis can be re-summarized by the following statement:

Given a network of nodes with phases that are significantly different and that will synchronize, the changes in phases over time will provide underlying network structure.

My specific contributions were as follows:

1. A network model and an underlying mathematical model are constructed to demonstrate the relationship between the variables in network construction.

2. An empirical model that reinforces the notions proposed in the analytical framework is provided via a stochastic model of a sample system, and this model demonstrates the relationship of the proposed limits.
The analysis from Chapter 5 shows the following:

1. The error of the correlation coefficient for a group of nodes between an actual model and an ideal estimated model is attributable to a specified set of parameters.

2. The error of the average phase of a group of nodes between an actual model and an ideal estimated model is attributable to a specified set of parameters.

3. The error estimates of the correlation coefficient and the average phase a group of nodes between an actual model and an ideal estimated model is attributable to a specified set of parameters within a specified vector.

4. The error estimates of the correlation coefficient and the average phase a group of nodes between an actual model and an ideal estimated model is attributable to a specified set of parameters within a specified matrix.

5. The probability that two nodes will be indistinguishable is provided by a given probability function and is a function of the number of possible measurable phases and the number of nodes in the system.

6. The number of possible measurable phases is provided by the measurement resolution and distribution of nodes as calculated by the standard deviation.

7. The measurement resolution of the system is given by the measurement capabilities and the system noise floor.
The empirical analysis from Chapter 7 shows that the relationships between the number of nodes, number of connections, measurement resolution and standard deviation of the system predicted in Chapter 5 do occur as predicted.

Both analysis support the hypothesis and give a set of attributes that must occur in order to determine the network structure from the behavior of a synchronizing network of couple oscillators.

8.2 Further Extensions

While this study did not attempt to provide a prediction of the future state of the system from its parameters. This limitation severely limits the utility of the findings and begs for future study. Perhaps the best future extension will involve the calculation of the Lyapunov Exponents for various components of the network dynamics. This might provide the characteristic rate of change from a given state given the conditions of rest of the system.

More critically, the predictions provided do have a number of limitations. Most importantly, the limitations of varying degrees of influence and sensitivity generating pseudo-Doppelgangers, i.e. one firefly, looking like another due to sensitivity, influence and and perspective was not approached. It was assumed as in previous studies that all influence and sensitivity was limited to the fixed rates. The exception is the change in influence provided by the correlation coefficient, $r$, found in Kuramoto’s equation. The extensions of multi-
ple nodes acting as one should be reviewed. The effect may well be trivial considering the balancing act required for multiple nodes to achieve that degree of mimicry.

Probably the simplest extension with the best payoff is toward minimal dynamic prediction of network structures to allow for signal integration utilized in almost all normal signal processing methods to allow for extension beyond what are most likely unrealistic restrictions on precision and accuracy of signal processing systems. At the same time, the use of this study may well lead to a between characterization of natural systems ability to resolve others as they synchronize.

The most difficult but arguably most important extension is toward dynamic network structures. Natural systems undoubtedly shift relationships over time. The fact that this study did not depend on signal processing gains allows for predict as provided. This should be a concern for reference in future studies that will most likely have additional heuristics applied.
A

APPENDIX - CELLULAR AUTOMATA
Turning to a more computationally-based approach, the simplicity and vast theoretical foundation of Cellular Automata provides an excellent toolset for detailing the relationship between synchronization and network structure. This section begins with the fundamentals of cellular automata. The first rule sets are basic binary rule sets. Later, the definition of rule sets for oscillatory function is defined. The details provided here are based on an initial approach for determining unknown rule sets from node behavior.

The best known Cellular Automata are perhaps those detailed by Stephen Wolfram [Wol01]. Because of their popularity, Section A.1 provides an example of a simple CA explaining Wolfram’s notation.

Even with the extent of growth of published material on Cellular Automata over the last several decades, it is helpful to step back to their origins. John von Neumann conceived and put Cellular Automata through their paces in extensive *gedanken* experiments involving self-reproduction and propagation of information [Neu01]. Von Neuman had worked on earlier theoretical machines until Stanislaw Ulam suggested cellular space as an alternative [Neu01]. Section A.1.2 provides the highlights of Von Neumann’s structure for basic Cellular Automata.
A.1 Wolfram’s Simple Cellular Automata

Stephen Wolfram documented thousands of pages of details on the basis, phenomenon and utility of simple cellular automata [Wol01]. In 1986, Wolfram provided a widely used naming convention for Linear Binary Cellular Automata [Wol01]. These simple automata with binary states and neighborhoods are best explained using examples of the two nearest neighbors. By enumerating all the possible variations of a rule set for a given number of cells, a rule set could then be defined as the integer representation of the binary encoding of the results of the rules as shown in Figure A.1.

Wolfram goes into great detail showing how the even the simplest 255 rules can create extensive behaviors ranging from trivial to complex and chaotic.

A.1.1 Example: Wolfram’s Simple Cellular Automata

It is helpful to take a cursory look at how a simple set of rules iterated can create complex behavior. We begin with the notion that a cell is one of a group of cells arranged in a line. Each cell considers the cells to the left and right as it neighbors. The cells located on the end treat the cells on the opposite end as its neighbor, as shown in Figure A.2.

The cells have one of two states, 1 or 0. It could just as easily be black or white, A or B; any binary combination will suffice. At the first generation or time increment, $t = 0$, each
Figure A.1: Wolfram’s Numbering for all possible three-node rule sets. Each three node group is used to define the center cell in the next generation. The enumerated set of all possible rules is defined by a rules that result in all possible binary numbers 0 through $2^{2^3}$

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
\]

Wolfram Rule 0

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
\end{array}
\]

Wolfram Rule 1

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
\end{array}
\]

Wolfram Rule 2

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
\end{array}
\]

Wolfram Rule 232

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
\end{array}
\]

Wolfram Rule 256

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\end{array}
\]

Figure A.2: Simple Cellular Automata constructed as a ring where the ends connect creating a continuous sequence.

In the case of the example binary cellular automata, the seed is either 1 or 0. Figure A.2 shows time, $t = 0$, for 12 cells. At time, $t = 1$, each cell looks at its neighbors values and its own value and determines the new value for that cell based on the matching rule. As shown in Figure A.3, cell number 5 takes its value from cell number 167.
generation 0, which is 0, and the values of its left and right neighbors cell’s number 4 and 6 at generation 0, which have the values 1 and 0 respectively, and forms the set, 100. It is then compared to the rule set provided in Figure A.4 and provides the new value, 1, for cell number 5 in generation 1.

![Figure A.3: Generation 2 of Simple Cellular Automata.](image)

![Wolfram Rule 30](image)

**Figure A.4: Wolfram Rule 30, Rule Set.**

Using a larger number of cells and looking at multiple generations, it is hard not to recognize the complexity in the shapes that are interwoven throughout the diagram in Figure A.5. Recalling that all of this behavior is generated from eight simple binary rules, it is difficult to comprehend where the complexity is hidden that generates this intricate behavior.

The difficulty becomes that there is a tendency to concentrate on the apparent complexity of the Wolfram’s diagrams and ignore the basic concept of cellular automata. As such, a look at the origins of cellular automata as a means to experiment with complex ideas follows.
A.1.2 John von Neumann’s Basic Cellular Automata

John von Neumann’s basic Cellular Automata consists of (1) a cellular space, which is an infinite \( n \)-dimensional Euclidean space, (2) a cell which is an abstract “machine” that exists at a finite location in the cellular space (3) a neighborhood which is a finite list of cells that are the neighbors of a given cell (4) discrete increments of time \( t = 0, 1, 2, 3, \ldots \) (5) a finite list of states for each cell, and (6) a rule governing the state transition called the Transition Function.

Von Neumann’s model for Finite Cellular Automaton is a device or system that consists of a 6-tuple \( (D, I, O, \tau, \lambda, d_0) \) where,
1. $D$ is a finite set of internal states that exist at any discrete moment of time $t = 0, 1, 2, 3, \ldots$

2. The cell receives $I$, a finite set of input states, and \ldots

3. \ldots transmits $O$, a finite set of output states.

4. At time zero the internal state of the automaton is in state $d_0$ where $d_0 \in D$.

5. The transition from an internal state to another internal state is determined by a rule set $\tau$, known as the transition function, which uses the automaton’s internal state and the input state.

6. The automaton’s output is determined by the rule set $\lambda$, the output function, which like $\tau$ uses the cells internal state and its input state [Neu01].

\subsection{A.1.3 Example Cellular Automata Following Von Neumann}

To maintain continuity with the Wolfram Example, a simple linear binary Cellular Automata using ten cells $C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9, C_{10}$ is provided. The geometrical arrangement is shown in Fig A.6. It is noted that von Neumann typically used a 2-dimensional lattice, and that he utilized external inputs and outputs. These two items are not utilized in order to maintain continuity with Wolfram.
A.1.3.1 Definitions

Utilizing the representation provided in Figure A.6, the following can be defined:

1. **Cellular space** is a one-dimensional Euclidean space.

2. Each *cell* is an abstract *device or system* located in cellular space at integer $x$ where 

   $$(0 < x < 11)$$

3. Each cell’s *neighbors*, the set $N$, are the two nearest cells, the one to the left and the one to the right. Von Neumann would have more typically considered the cells on the ends to receive or provide external interactions as opposed to the ring connection shown in Figure A.6. To continue to maintain a closed system, for this example we define $N$ by:

   (a) $N_j$ - are the neighbors of the $j^{th}$ cell.

   (b) $\mu$ is the total number of cells.

   (c) $N_j = (C_k, C_l)$ where

       i. $k = (\mu + j - 1) \mod \mu$

       ii. $l = j + 1$
iii. where mod is the modulo operation defined by the remainder of integer division, (i.e.\(13\mod 10 = 3\), where \(13/10 = 1\) with a remainder of 3).

4. The finite set of states is simply binary \((0,1)\).

5. Time is an increment \(t = 0,1,2,3,\ldots\)

6. Additionally, we can establish the rules governing the state transition called the Transition Function. The Transition Function is defined by \(\tau\) and \(\lambda\) in the next section.

A.1.3.2 Extending the Example with Definitions in terms of Von Neumann’s 6-tuple for a Finite Cellular Automata

Each cell can then be defined in terms of von Neumann’s 6-tuple \((D,d_0,O,I,\tau,\lambda)\)

1. \(D\) is the set of all possible internal states of a cell for the example binary states \(D \equiv (0,1)\). Also \(d_t\) is the internal state of the Cell at time \(t\), and \(d_t \in D\).

2. The \(d_0\) is defined as a randomly selected initial state of each cell at time \(t = 0\). The \(d_0\) for the all cells is shown in Figure A.7.

3. The \(O\) is the set of outputs which is defined as the binary states \(O \equiv (0,1)\). Also, \(o_t\) is the output at time \(t\), and \(o_t \in O\).
4. $I$ is the set of inputs, which consist of the $O$ from neighbors cells. For example, in the case of cell $C_5$ whose neighbors are $N_5 = (C_4, C_6)$:

(a) Since, $O$ is the set of possible binary states $O \equiv (0, 1)$, then

(b) $I$ is the set of possible binary states $(0, 1)$ of the outputs which is the set $I \equiv (00, 01, 10, 11)$.

5. We can define the change in internal state Transition Rule Set, $\tau$, as a simple majority rule, Wolfram Rule 232 (see Figure A.1). Here $\tau$ is the set of transitions for all possible input and internal states shown in Figure A.8. That is, $\tau$ looks at the cells current state and the state of the inputs and determines the next internal state by the majority of either 1's or 0's. For example, for a given cell, if the current internal state at time, $t$, is $d_t = 0$ and the current inputs at time $t$ is $i_t = 01$ then the majority is 0. Therefore the next internal state is $d_{t+1} = 0$.

6. We can define the change in output state at any time $t$, $o_t$, by the Transition Rule, $\lambda$, also as a simple a majority rule, Wolfram Rule 232 (see Figure A.1). Here the $\lambda$ looks
at the set of transitions for all possible input and internal states shown in Figure A.9. That is, \( \lambda \) looks at the cell’s current state and the state of the inputs and determines the next output state, \( o_t \), by the majority of either 1′s or 0′s. For example, for a given cell, if the current internal state at time \( t \), \( d_t = 0 \) and the current inputs at time \( t \) is \( i_t = 01 \) then the majority is 0. Therefore the next output state is \( o_{t+1} = 0 \).

7. In many systems, special quiescent states are defined and provided for each of the output and input states. These states are often used as the initial states. Additionally,
special transition rules are provided for these states. In the example used, the $d_0$ is directly provided. Since, $d_t$ and $o_t$ are governed by the same transition rule, majority rule for the internal and input states, $\lambda \equiv \tau$. As such, apart from the undefined initial state, $o_t \equiv d_t$. To simplify matters, $o_0$ or the initial state of $o_t$ at time $t = 0$ is defined as $o_0 = d_0$. This notation leaves the input states at time $t = 0$ undefined. For now, since the primary use of the input states, $i_t$, is to provide a transition rule for a time, $t = x$, and all other variables for $t = 0$ are defined, we will leave $i_0$ undefined.

A.1.3.3 Extending the Example With a Deeper Look at Time $t = 1, t = 2$

For each time increment, $t$, we can define the exact state of all the elements in each cell. These form the 3-tuple $(i_t, d_t, o_t)$.

**Definitions**

1. The $i_t$ at $t = x$ is defined as the state $o_{x-1}$ of each of the neighbors, $N$.

2. The $d_t$ at $t = x$ is provided by $\tau(d_{x-1}, i_x)$

3. The $o_t$ at $t = x$ is provided by $\lambda(d_{x-1}, i_x)$

**Example, $C_5$ and $t = 1$**

Looking at Time, $t = 1$, and using cell $C_5$ as a detailed example:
1. Since the $i_t$ at $t = x$ is defined as the state $o_{x-1}$ of each of the neighbors, $N$, for cell $C_5$, the set of neighbors, $N_5 = (C_4, C_6)$, as defined above (see Subsection A.1.3). Then, at time, $t = 1$, $i_1$ is the $o_0$ for each of the neighbor cells $(C_4, C_6)$. Since $o_0 \equiv d_0$ as provided above (see subsection A.1.3.2), then $i_1 = (o_0(C_4), o_0(C_5))$ which can be seen by looking at Figure A.7 that defines the cells at state $d_0$. Thus, $i_1 = (1, 1)$ for Cell $C_5$

2. The $d_t$ at $t = 1$ is provided by $\tau(d_0, i_1)$. For cell $C_5$, $i_1 = (1, 1)$, and by Figure A.7, $d_0 = 1$. Then, combining $d_0$ and $i_1$, $\tau(1, (1, 1))$ and referencing Figure A.8, then $d_1 = 1$.

3. Similarly, $o_t$ at $t = 1$ is provided by $\lambda(d_0, i_1)$. For cell $C_5$, $i_1 = (1, 1)$, and by Figure A.7, $d_0 = 1$. Combining $d_0$ and $i_1$, $\tau(1, (1, 1))$ and referencing Figure A.9, $o_1 = 1$.

**A.2 Determination of Unknown Rule Sets and Neighborhood**

The purpose of this detailed underpinning is to provide a basis for understanding the process of determining both unknown rule sets and neighborhood sizes from observation of the behavior of a set of cellular automata over time. For simplicity a simple binary rule set is explored. The transition to a more complex rule set involving oscillatory functions is not provided due to the unnecessary complexity. There are three critical notes that need to be discussed prior to attempting to determine unknown rule sets and neighborhood size:

1. The simplest tool for testing the determination of unknown rule sets and neighborhood size is contradictory rule sets. No Cellular Automata rule sets can contradict. In other
words, one cannot have two rules for the exact same transition. If such a rule did exist, a rule for handling the ambiguity would have to exist and the rule for handling the ambiguity would then itself represent a new dimension and a new total rule set.

2. The determination of rule sets larger than the size of the automata is restricted.

   (a) Since the rule set \( \tau \) is defined as the deterministic transition from an internal state to another internal state and is provided by a function of the internal state, \( D \) and the input state, \( I \) such that \( \tau : D \times I \rightarrow D \).

   (b) Since the input state, \( I \), is determined by the neighborhood, \( N \), and the set \( N \), consists of the \( h \) cells where \( h \in \text{total number of cells} \)

   (c) no contradictions are allowed

   (d) Then, \( \tau \) is bound by the total number of cells

3. The establishment of a lower bound of \( N \) for valid rule sets is possible.

   (a) Since no contradictions are allowed,

   (b) Since the rule set \( \tau \) is defined as the deterministic transition from an internal state to another internal state and is provided by a function of the internal state, \( D \) and the input state, \( I \) such that \( \tau : D \times I \rightarrow D \).

   (c) Since the input state \( I \) is determined by the neighborhood \( N \), and the set \( N \) consist of the \( h \) cells where \( h \in \text{total number of cells} \)
(d) If $D \times I \not\rightarrow D$ for $I$ with $N$ where $h = x$ and $x < \text{total number of cells}$, then $h > x$.

The determination of the rule set and neighborhood size for a given Finite Cellular Automata is possible, allowing for ambiguities, by observing the transition of the from one time step to the next in a state diagram.

The determination of the lower bound of a Finite Cellular Automata neighborhood size is provided by contractions in the rule set of a given neighborhood size.

The limitation of the determination is due to ambiguities resulting from either a lack of significant states to determine the entire set of rules or to determine if contradictions would occur.

\section*{A.2.1 Definitions}

1. \textit{Time} is an increment $t = 0, 1, 2, 3, \ldots$

2. \textit{cellular space} is a one-dimensional Euclidean space.

3. Each \textit{cell} is as an abstract \textit{device or system} located at discrete points in cellular space.

4. Each cell’s \textit{neighbors}, the set $N$, are the $h$ cells, determined by a neighborhood rule.

By Von Neumann’s Finite Cellular Automaton the remaining components consist of the 6-tuple $(D, O, I, d_0, \tau, \lambda)$

5. $D$ is a finite set of internal states
6. $O$ is the finite set of outputs.

7. $I$ is the set of inputs, which consist of the $O$ from neighbors $N$.

8. At time zero the internal state of the automaton is in state $d_0$ where $d_0 \in D$.

9. The transition from an internal state to another internal state is determined by a rule set $\tau$, known as the transition function, which uses the automaton’s internal state and the input state $\tau : D \times I \rightarrow D$.

10. The automaton’s output is determined by the rule set $\lambda$, the output function, which like $\tau$ uses the cells internal state and its input state (Burks VonNeumann 1966).

A.2.2 Discussion of the Determination of $\tau$ and Neighborhood, $N$

The most important aspect of cellular automata for this research is how the determination of an unknown rule set $\tau$ might occur. The following example is provided.

Given a linear finite cellular automata and given states $t_1$ and $t_2$ where $t_1$ precedes $t_2$ as follows:

$t_1 = 00001010011100101110111$

$t_2 = 10001010001000100100010$

There are two possible starting points for determination of the unknown rule set; (1) take the largest possible subset in the previous state leading to the largest subset in the
next state (2) take the smallest subset in the previous state leading to the smallest subset in the next state. The possible outcomes are described in the next section with the starting at the largest possible subsets shown as high-high to low-low and starting with the smallest possible subsets shown as low-low to high-high.

A.2.3 High-High to Low-Low

If we take the largest sets, then \( t_1 \to t_2 \) would indicate the entire set \( t_1 \) is the largest possible subset and it generates \( t_2 \) provided by the rule

\[
000001010011100101110111 \to
\]

\[
100001010001000100100010
\]

yielded the entire set \( t_2 \).

We could then increment by one cell and develop a second rule that was

\[
000010100111001011101110 \to
\]

\[
000010100010001001000101
\]

This could continue until there were possible rules for each of the sequences.

While this is valid, it also results on the least likelihood of affirming or contradicting a given ruleset, due to the probability of two sets being equal being least likely for the longest length strings.
A.2.4 Low-Low to High-High

The other choice is to begin with the smallest possible subsets in both the current and previous states. The rules are established by looking at each possible set of $n_1$ characters in $t_1$ and associating them with a set of $n_2$ characters in $t_2$. Looking at each possible combinations and determining if an ambiguity will exist.

if ..

$n_1 = 1$ and $n_2 = 1$

Then ...

The first rule, $rule_1$ would be $0 \rightarrow 1$

by

$t_1 = 0 0 0 0 0 1 0 1 0 0 1 1 1 0 0 1 0 1 1 1 0 1 1 1$

$t_2 = 1 0 0 0 0 1 0 1 0 0 0 1 0 0 1 0 0 1 0 0 0 1 0$

The second rule, $rule_2$, would be $0 \rightarrow 0$

by

$t_1 = 0 0 0 0 1 0 1 0 0 1 1 1 0 0 1 0 1 1 1 0 1 1 1$

$t_2 = 1 0 0 0 1 0 1 0 0 0 1 0 0 1 0 0 1 0 0 0 1 0$.

Because $rule_1$ and $rule_2$ provide a contradiction, the rule sets involving $n_1 = 1$ and $n_2 = 1$ are not possible.
Next look at $n_1 = r_1 + l_1$ and $n_2 = 1$

Then ...

The first rule, rule$_1$ would be $000 \rightarrow 0$

by

\[
t_1 = \underline{0} \underline{0} \underline{0} 0 1 0 1 0 0 1 1 1 0 0 1 0 1 1 1 0 1 1 1
\]

\[
t_2 = 1 0 0 0 0 1 0 1 0 0 0 1 0 0 0 1 0 0 1 0 0 0 1 0
\]

The second rule, rule$_2$, would be $000 \rightarrow 0$

by

\[
t_1 = 0 0 0 0 0 1 0 1 0 0 1 1 1 0 0 1 0 1 1 1 0 1 1 1
\]

\[
t_2 = 1 0 0 0 1 0 1 0 0 0 1 0 0 1 0 0 1 0 0 0 1 0
\]

so far no contradiction....

The third rule, rule$_3$, would be $000 \rightarrow 0$

by

\[
t_1 = 0 0 0 0 1 0 1 0 0 1 1 1 0 0 1 0 1 1 1 0 1 1 1
\]

\[
t_2 = 1 0 0 0 1 0 1 0 0 0 1 0 0 1 0 0 1 0 0 0 1 0
\]

rule$_4$, would be $001 \rightarrow 0$

by

\[
t_1 = 0 0 0 0 1 0 1 0 0 1 1 1 0 0 1 0 1 1 1 0 1 1 1
\]

182
\[ t_2 = 100001010001000100100010. \]

rule\(_5\), would be 010 → 1

by

\[ t_1 = 00000001001000111001011110111 \]
\[ t_2 = 100001010001000100100010. \]

rule\(_6\), would be 101 → 0

by

\[ t_1 = 0000001010011100101110111 \]
\[ t_2 = 100010010001000100100010. \]

rule\(_7\), would be 010 → 1

reconfirming rule 5...

by

\[ t_1 = 00000100100011100101110111 \]
\[ t_2 = 100001010001000100100010. \]

rules\(_8\), would be 100 → 0

by

\[ t_1 = 0000010100011100101110111 \]
\[ t_2 = 100001010001000100100010. \]
The end result is a complete set of rules that does not contradict and provides the smallest possible neighborhoods. While this method is relatively straightforward, it also has significant limitations. The most important of which is that a uniformed neighborhood has to be assumed. Perhaps equally important is that even if the uniformed neighborhood size and configuration is acceptable, the fact that remains that while rulesets can be determined to be greater than a given size based on contradictions, they can never be determined to be complete. There could always exist a rule set greater than the size of the number of cells previously determined and perhaps the entire set.
LIST OF REFERENCES


