Decoherence In Quantum Dot Charge Qubits: The Role Of Electromagnetic Fluctuations

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Decoherence in Quantum Dot Charge Qubits: the Role of Electromagnetic Fluctuations

by

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A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in the Department of Physics in the College of Sciences at the University of Central Florida Orlando, Florida

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Lateral semiconductor quantum dot structures have been proposed as an effective quantum bit (qubit) for quantum computation. A single excess electron with the freedom to move between two capacitively coupled quantum dots creates a ‘pseudo’-spin system with the same qubit behavior as the more natural two level system of a single electron spin. The excess electron in the double dot system is restricted to one of the two dots, thereby creating two separate and distinct states (usually referred to as $|L\rangle$ and $|R\rangle$). The benefit of these charge qubits lie in the relative ease with which they can be manipulated and created. Experiments have been performed in this area and have shown controllable coherent oscillations and thus efficient single-qubit operations. However, the decoherence rates observed in the experiments is still quite high, making double dot charge qubits not very appealing for large-scale implementations. The following work describes the effect of the electromagnetic (EM) environment of the double quantum dot system on the decoherence of the charge state. Sources of decoherence in similar systems have been extensively investigated before and this paper follows a close theoretical framework to previous work done in the area. The effect of the EM environment can been seen in the calculations discussed below, although it is clear...
that the decoherence seen in experiments cannot be fully explained by the voltage fluctuations as they are investigated here. The limitations of the calculations are discussed and improvements are suggested.
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used in previous figures are used again.
CHAPTER 1

INTRODUCTION

Quantum mechanical algorithms have proven to be more efficient their classical counterparts for a few key computational problems such as factoring and searching [1]. Since people became aware of the possible benefits of quantum computation, several attempts to realize a practical quantum computer have appeared. The experimental effort has been guided by a few rules. In their introductory book on this subject, Nielsen and Chuang state the four basic requirements for a quantum computer as follows [2]:

1. “robust” physical representation of qubits;

2. ability to perform controlled unitary operations;

3. effective preparation of the initial state of the qubit;

4. fidelity in the measurement of the qubit (the output of the computation).

There are many different physical systems that potentially meet these requirements, including ion traps [3], nuclear spins [4], superconductor junctions [5], and
quantum dot structures [6]. This variety reflects the large current effort toward the implementation of a practical quantum computer. Each of these proposals offers its own pros and cons, and at this moment, it is not yet clear which one will prevail.

Among the proposed realizations of quantum computers are quantum dots (QDs). QDs are isolated charge structures which are sometimes referred to as ‘artificial atoms’ or ‘charge islands’. Solid-state heterostructures (usually of GaAs and AlGaAs) are fabricated to trap electrons in a two-dimensional plane at the material interface (the so-called two-dimensional electron gas, 2DEG). Metal electrodes (usually Au) are deposited on the surface of the structure, above the 2DEG, in very specific geometries to create laterally confining potential. The gates and the 2DEG are not in electric contact. When a negative voltage is applied to the gates, the electric field depletes the portion of the 2DEG underneath, creating electron islands of given shapes isolated from the surrounding charges (the charge reservoir) by potential barriers. Several gates may be used and changing their voltages allows for direct control over both the tunneling barrier and the offset of the internal energy levels in the dot. The latter also provides a method of adjusting the number of electrons in the quantum dot. The setup is schematically shown in Fig. 1.1 and 1.2.

While any two-level quantum system can be thought of as a qubit (quantum bit), only a few stand out as being practical. Quantum dots are looked upon fa-
Figure 1.1: A simple schematic of the experimental setup used in the construction of these DQD systems. The voltage gates on the surface create the dots in the 2DEG about 100 nm below. The gates are labeled as they appear later in the model equations.

vorably in the implementation of quantum computation because they potentially meet the requirements listed above and seem to be scalable to feasible quantum computers. Two-qubits operations have been performed successfully using other set-ups [7, 8, 9], but the methods for the construction and manipulation of quantum dot structures are well established and follow traditional semiconductor techniques. QD structures may provide an additional advantage in terms of integrability with conventional electronics. Thus, quantum dots are a likely
candidate for the construction of large-scale quantum computers which would be required for any realistic applications.

The spin of an isolated electron in a quantum dot is an intrinsic two-level system and has been proposed as a qubit [6]. It has several advantages, including the long decoherence times of spins in semiconductors [10]. But the manipulation of the electron spin in a quantum dot requires precise control over the magnetic properties of the materials and the ability to fast-pulse local magnetic fields. The spin qubit in quantum dot structures may be a viable method for quantum computation, but the engineering problems are currently a significant obstacle. An alternative, more accessible, possibility is to use the electron’s charge rather than its spin. Multi-dot structures turn out to be very appropriate for such an
application. For instance, a qubit can be constructed using the two localized states an excess electron can have when hopping between two neighboring quantum dots. A similar system is schematically shown in Fig. 1.4.

Figure 1.3: This picture is an image of a DQD system setup used by A. Chang’s group at Duke University.

The charge qubit created in the double quantum dot (DQD) system acts as a ‘pseudo-’spin system, and can be formally treated in a manner similar to the electron spin degree of freedom. The two states of the DQD system can be defined as $|L\rangle (|10\rangle)$ and $|R\rangle (|01\rangle)$, where the state is defined by the location of the single excess charge in the system (either the left or the right dot). The state of the system can be changed simply by transferring the electron from one dot to the other, and the states can be isolated from the rest of the Hilbert space by not allowing the charge to leave the dot once it has been trapped. Single-qubit manipulations can be achieved by varying the interdot coupling (tunnel barrier) and the relative bias.
(see Fig. 1.3). It is has been shown that coupled qubits (like the quantum dot system) can be used to create a CNOT gate [8, 11], which, along with single qubit gates, can be used to create any multiple qubit gate [2].

Figure 1.4: (i) The two quantum dots are empty (do not contain any excess electrons) and are separated from each other and from the charge reservoirs by tunneling barriers. (ii) The left tunneling barrier is decreased to allow a charge to tunnel into the left dot. The system is now in the $|L\rangle$ state. (iii) The left tunneling barrier is raised and the center tunneling barrier is lowered to allow the electron to tunnel into the right dot. The system is now is the $|R\rangle$ state. (iv) The center barrier is raised and the right barrier is lower to allow the charge to exit the system.

In order to explain more precisely how control of the number of excess electrons in the DQD is achieved, consider a situation where there is an unoccupied energy
level on each dot. \(^1\) Then, the DQD system can only have four possible low-lying states (\(|00\rangle, |10\rangle, |01\rangle, \text{ and } |11\rangle\)). The Coulomb blockade energy of each dot prevents any extra electron from entering or leaving any single dot. By varying the gate voltages as to lower the energy of the 10 and 01 configurations with respect to the \(|00\rangle\) and \(|11\rangle\), a single charge can be trapped in the system (see Fig. 1.4). Hence, the two states left, \(|10\rangle\) and \(|01\rangle\) form the basis of a two-dimensional Hilbert subspace. The energy difference between the two states can be controlled by varying the gate voltages along the line shown in the phase diagram of Fig. 1.5.

\[\begin{array}{c}
(0,0)\\(0,1)\\(1,0)\\(1,1)\\
v_{g1}\\v_{g2}
\end{array}\]

Figure 1.5: A schematic graph of the voltage gate phase space. The solid lines separate regions of distinct occupation numbers. \(v_{g1}\) and \(v_{g2}\) are defined in Fig. 2.1. The blue dotted line represents the path trace by a system alternating between the \(|L\rangle\) and \(|R\rangle\) states.

\(^1\)We will neglect the electron spin thereafter by assuming that a sufficiently strong magnetic field is applied to the system.
A major problem with any physical realization of a qubit is decoherence, and quantum dot charge qubits are no different. If the decoherence times of the charge DQD qubit are faster than the operation times, than the system is useless for coherent quantum computation. Preparing a system with in an initial state which will change before any operations can be performed creates a final state of the system which is unreliable.

A few experimental attempts to implement DQD charge qubits have already appeared in the recent literature [12,13]. The decoherence rates observed in these experiments were relatively low. It is then crucial to try to understand what causes such a poor quantum performance and whether it can be mitigated.

There are many potential sources of decoherence in a DQD structure, most of which are due to the fact that quantum dots are embedded in a dense medium and must be controlled in a macroscopic manner. For instance, vibrations in the heterostructure lattice can lead to decoherence in the system through direct interaction between phonons and the charge state [14,15,16,17], but microscopic calculations suggest that this is probably not the main mechanism. Higher-order tunneling (cotunneling) effects can dephase the system as well, although these effects can be minimized by using sufficiently small gate voltages [18].

Another source of decoherence thought to be important in DQD systems is the background noise of the voltage gates used to define the quantum dots. This is a well-investigated problem in another qubit realization, namely, superconductor
Josephson junctions. At any time, the voltage in the gates experience fluctuations because the charge carriers in the metals always undergo some random thermal motion. This phenomena is referred to as Johnson-Nyquist noise and can lead to decoherence by adding a random-phase component (or detuning) to the qubit basis states.

This thesis is a theoretical treatment of the decoherence resulting from the gate voltage fluctuations which closely follows previous work that has been carried out for superconductor Josephson junctions, and borrows from the formalism recently used in the study of noise and dissipation in electronic transport through DQDs.
CHAPTER 2

DQD SYSTEM MODEL

The DQD system shown in Fig. 1.1 and 1.3 can be represented with the following circuit diagram (see Fig. 2.1).

![Circuit Diagram](image)

Figure 2.1: Above is the circuit model of the double quantum dot system which was used to perform the electrostatics calculations. The variables used in the figure are the same as those used in the equations. The impedance term $Z$ is explained more fully in chapter 3.

Here, the tunneling barriers between the two dots, and between each dot and the reservoirs, are represented by capacitors [20]. The source is coupled to the first dot through the capacitance $C_L$ and has voltage $V_L$; the drain is coupled
to the second dot through the capacitance \( C_R \) and has voltage \( V_R \). The two dots are coupled to each with a tunneling barrier represented by \( C_m \) and to the gate voltages \( V_{g1(2)} \) through capacitances \( C_{g1(2)} \). The gate voltage fluctuations mentioned above are represented here by the frequency-dependent impedances \( Z_{1(2)}(\omega) \). The number of electrons in dot 1(2) is \( N_{1(2)} \) and is controlled by voltage \( V_{1(2)} \), with \( Q_{1(2)} \) being the total charge. The electrostatic energy of the DQD system can be calculated using the above diagram and the following method [20].

The electrostatic energy is defined as the volume integral over the electric field squared [21]

\[
U = \frac{1}{8\pi} \int E^2 dV,
\]

which can be shown to be

\[
U = \frac{\vec{V}\vec{Q}}{2} = \frac{\vec{V}C\vec{V}}{2} = \frac{\vec{Q}C^{-1}\vec{Q}}{2},
\]

where \( \vec{V} \) is the voltage vector, \( \vec{Q} \) is the total charge vector, and using the relationship

\[
\vec{Q} = C\vec{V}
\]

with \( C \) defined as the capacitance matrix. \( C \) and \( \vec{Q} \) can be defined as follows:

\[
C_{jj} = \sum_{k=0}^{N} c_{jk}
\]

\[
Q_j = \sum_{k=0}^{N} q_{jk}
\]

\[
= \sum_{k=0}^{N} c_{jk}(V_j - V_k),
\]
where \( c_{jk} \) is the capacitance between the \( j \)th and \( k \)th conductors and the total charge \( Q_j \) is defined as the sum of the charges on all \( c_{jk} \).

Using Eq. 2.2 and Fig. 2.1, an equation for the total charge \( Q_{1(2)} \) on dot 1(2) can be written as:

\[
Q_1 = C_L(V_1 - V_L) + C_{g1}(V_1 - V_{g1}) + C_m(V_1 - V_2), \tag{2.6}
\]
\[
Q_2 = C_R(V_2 - V_R) + C_{g2}(V_2 - V_{g2}) + C_m(V_2 - V_1). \tag{2.7}
\]

The above equations can rewritten as

\[
Q_1 + C_L V_L + C_{g1} V_{g1} = C_L V_1 + C_{g1} V_1 + C_m(V_1 - V_2)
= V_1(C_L + C_{g1} + C_m) - C_m V_2, \tag{2.8}
\]
\[
Q_2 + C_R V_R + C_{g2} V_{g2} = C_R V_2 + C_{g2} V_2 + C_m(V_2 - V_1)
= V_2(C_R + C_{g2} + C_m) - C_m V_1, \tag{2.9}
\]

which can then be reorganized in the form

\[
\begin{pmatrix}
Q_1 + C_L V_L + C_{g1} V_{g1} \\
Q_2 + C_R V_R + C_{g2} V_{g2}
\end{pmatrix} =
\begin{pmatrix}
C_1 & -C_m \\
-C_m & C_2
\end{pmatrix}
\begin{pmatrix}
V_1 \\
V_2
\end{pmatrix}, \tag{2.10}
\]

where

\[
C_1 = C_L + C_{g1} + C_m, \tag{2.11}
\]
\[
C_2 = C_R + C_{g2} + C_m. \tag{2.12}
\]
Using Eq. 2.2, Eq. 2.10 can be rewritten as

\[
\begin{pmatrix}
V_1 \\
V_2
\end{pmatrix} = \frac{1}{\| C \|} \begin{pmatrix} C_2 & C_m \\ C_m & C_1 \end{pmatrix} \begin{pmatrix} Q_1 + C_L V_L + C_{g1} V_{g1} \\ Q_2 + C_R V_R + C_{g2} V_{g2} \end{pmatrix}
\]

(2.13)

where \( \| C \| \) is the determinant of the capacitance matrix, namely,

\[
\| C \| = C_1 C_2 - C_m^2.
\]

(2.14)

Equation 2.13 can be directly plugged into Eq. 2.2, yielding

\[
U = \frac{1}{2 \| C \|} \left( Q_1 C_2 + Q_2 C_1 + 2 Q_1 Q_2 C_m \\
+ Q_1 C_2 C_L V_L + Q_1 C_m C_R V_R + Q_1 C_2 C_{g1} V_{g1} + Q_1 C_m C_{g2} V_{g2} \\
+ Q_2 C_m C_L V_L + Q_2 C_1 C_R V_R + Q_2 C_m C_{g1} V_{g1} + Q_2 C_1 C_{g2} V_{g2} \right)
\]

(2.15)

(2.16)

To help simplify these equations, the following substitutions can be made:

\[
Q_{1(2)} = -N_{1(2)} |e|,
\]

(2.17)

\[
E_{C1} = \frac{e^2}{2 \| C \|} \frac{C_2}{},
\]

(2.18)

\[
E_{C2} = \frac{e^2}{2 \| C \|} \frac{C_1}{},
\]

(2.19)

\[
E_{Cm} = \frac{e^2}{2 \| C \|} \frac{C_m}{},
\]

(2.20)
where $e$ is the elementary charge. Here it is assumed that there is no bias voltage, hence $V_R = V_L = 0$. Thus, the electrostatic energy $U$ can be written as

$$U = E_{C1}N_1^2 + E_{C2}N_2^2 + 2E_{Cm}N_1N_2$$

$$- |e|N_1(C_2C_{g1}V_{g1} + C_mC_{g2}V_{g2})$$

$$- |e|N_2(C_mC_{g1}V_{g1} + C_1C_{g2}V_{g2})$$

(2.21)

which can finally be written in a more concise form by introducing

$$N_{g1} = \frac{Q_{g1}}{|e|} = C_{g1}V_{g1}$$

$$N_{g2} = \frac{Q_{g2}}{|e|} = C_{g2}V_{g2}$$

(2.22)

and completing the square. The result is

$$U = E_{C1}(N_1 - N_{g1})^2 + E_{C2}(N_2 - N_{g2})^2 + 2E_{Cm}(N_1 - N_{g1})(N_2 - N_{g2}).$$

(2.23)

Up until this point, the DQD system has been assumed to be classical. However, the behavior of the system needs to be explained quantum mechanically and a Hamiltonian needs to be defined for the DQD system. The first step in that direction is to introduce a simple change of variable to the equations, namely,

$$n \equiv N_1 - N_2,$$

$$n_{g} \equiv N_{g1} - N_{g2}.$$  

(2.24)

(2.25)

The counting variables $N_1, N_2, N_{g1},$ and $N_{g2}$ can be rewritten by recognizing that, by assumption, there can be only one excess electron in the system ($N_1 + N_2 = 1$).
Thus,

\[
N_1 = \frac{1}{2} + \frac{n}{2},
\]

(2.26)

\[
N_2 = \frac{1}{2} - \frac{n}{2},
\]

(2.27)

\[
N_{g1} = \frac{1}{2} + \frac{n_g}{2},
\]

(2.28)

\[
N_{g2} = \frac{1}{2} - \frac{n_g}{2}.
\]

(2.29)

Plugging Eq. 2.29 into Eq. 2.23 yields

\[
U = \frac{E_{C1}}{4} (n - n_g)^2 + \frac{E_{C2}}{4} (n_g - n)^2 + E_{C_m} (n - n_g)(n_g - n)
\]

(2.30)

Here it is important to comment on the behavior of the \( n \) variable. If the DQD system has a single electron in the left dot and no electron in the right dot, the state is \( |L\rangle \) (or \( |10\rangle \)) and \( n = +1 \); if the system is in the opposite configuration, the state is \( |R\rangle \) (or \( |01\rangle \)) and \( n = -1 \). If the electrons is shared equally between the two states (on the boundary line between (1,0) and (0,1) in Fig. 1.5), \( N_1 = \frac{1}{2}, N_2 = \frac{1}{2}, \) and \( n = 0 \). The value of \( n \) is restricted to \(-1 < n < +1\) and provides information on the exact state of the system. Changing \( n \) continuously from \(-1\) to \(+1\) traces the dashed line in Fig. 1.5. Expanding Eq. 2.30 and eliminating the higher order terms (because the interest lies in the behavior of \( n \) and not on \( n^{2,3,...} \) ) gives

\[
U = \frac{nn_g}{2} \left(2E_{Cm} - E_{C1} - E_{C2}\right) + \frac{n^2 + n_g^2}{4} \left(E_{C1} + E_{C2} - 2E_{Cm}\right)
\]

\[
= \left(2E_{Cm} - E_{C1} - E_{C2}\right) \frac{nn_g}{2}
\]

\[= \varepsilon n,
\]

(2.31)
where $\varepsilon = \frac{n}{2}(2E_{Cm} - E_{C1} - E_{C2})$. The system is completely described by a $2 \times 2$ density matrix $\rho$ ($2 \times 2$ because there are only two available states). In Eq. 2.31, a value of $n = 1$ puts the DQD system in $|L\rangle$ while $n = -1$ corresponds to $|R\rangle$. Hence, $n$ may be reverted to an operator in the qubit subspace. Comparing this behavior with the density matrix shows $n$ acts as the Pauli matrix $\sigma_z$. Using this correspondence allows us to write a Hamiltonian for the system in the form

$$H_S = \varepsilon \sigma_z. \quad (2.32)$$

It is clear that the system does not just exist in a single state, but is free to transit between the two states in its Hilbert space. Therefore, the system Hamiltonian must include a term to account for changing the state of the DQD system. This new term appears in the off-diagonal elements of the Hamiltonian. Another way to arrive at that is to rewrite the state $|L\rangle$ as the column vector $(1, 0)$ and noticing the state can be changed to $|R\rangle$ (or $(0, 1)$) simply by multiply by the Pauli matrix $\sigma_x$. The physical mechanism behind this term is tunneling through the potential barrier separating the two dots (as seen in Fig. 1.1). The total system Hamiltonian reads

$$H_S = \varepsilon \sigma_z + \nu \sigma_x \quad (2.33)$$

where $\nu$ is the interdot tunneling amplitude. This final Hamiltonian is not the most general form one can write for a two-dimensional quantum systems (it is missing the Pauli operator $\sigma_y$ for that matter), but it can be shown that it already contains sufficient elements to allow for the most general single-qubit operations.
In Chapter 1, the DQD system was initially treated classically and then a key variable (the charge imbalance) was identified as quantum mechanical. A similar approach will be applied here to introduce a quantum mechanical Hamiltonian for fluctuations in gate voltage. This is a standard approach in the literature of quantum dissipative systems. The novelty here is the application to the DQD system.

The first step is to model the system comprised by the gates and the surrounding electromagnetic environment by a simple effective circuit. The gates are considered to be an infinite transmission line with distributed inductive and capacitive elements as shown Fig. 3.1. The frequency dependence of the gate’s effective impedance, $Z(\omega)$, is defined by the particular choice of inductances and capacitances of the line elements.

The electrostatic energy of a capacitor was defined in Eq. 2.2. The response of the LC circuit shown in Fig. 3.1 is analogous to a system of system harmonic
Figure 3.1: The impedance \( Z \) seen in Fig. 2.1 is modeled as an infinite series of resistors and capacitors.

oscillators (SHO). In order to understand that, recall that the kinetic energy term,

\[
E_K = \frac{p^2}{2m}.
\]  

Looking at Eq. 2.2, the charge \( Q \) in the LC circuit acts as the momentum \( p \) in the mechanical system while the capacitance \( C \) acts as the mass \( m \). Likewise, the potential energy term of the SHO,

\[
E_P = \frac{k\Delta q^2}{2}
\]

also has a simple analogy. The restoring force constant \( k = m\omega_n^2 \) is related to the inverse inductance of the LC circuit \( 1/L \) (here \( \omega_n \) is the natural frequency of oscillation). The variable \( q \) is the canonical conjugate variable of the momentum and a similar case has to made for the electrical case. We define the variable \[22\]

\[
\phi = \frac{e}{\hbar} \int_{-\infty}^{t} V(t')dt'
\]

where \( V \) is the voltage in the circuit. The generalized flux \( \frac{\hbar}{e}\phi \) is the canonical conjugate variable to charge. Thus, the Hamiltonian for the SHO mechanical
system reads

\[ H = \frac{p^2}{2m} + \frac{k(\Delta q)^2}{2} \]  

and by following the analogy to its logical end, the Hamiltonian for the transmission line (for a single inductor and capacitor) can be written down as the following

\[ H = \frac{Q^2}{2C} + \left(\frac{\hbar}{e}\right)^2 \frac{(\Delta \phi)^2}{2L} \]  

(3.5)

Now, the LC circuit model of the transmission is an infinite series of inductors and capacitors, not simply a single inductor and capacitor in series. Moreover, the real system consists of the two quantum dots themselves grounded to the 2DEG along with the transmission line.

Figure 3.2: Both impedances in Fig 2.1 are modeled by Fig. 3.1, and the quantum dots themselves are modeled as capacitors. Note that the gates (modeled as transmission lines) are capacitively coupled to the 2DEG which is grounded.
Figure 3.2 shows a circuit model which is closer to reality. Here the quantum dots are represented as capacitors and the transmission line is again modeled as an infinite series of inductors and capacitors. Using this new schematic, a more accurate Hamiltonian can be written down as

\[
H = \sum_{i=1}^{2} \left\{ \frac{Q_{0,i}^2}{2C_g} + \sum_{l=1}^{\infty} \left[ \frac{Q_{l,i}^2}{2C_{ti}} + \left( \frac{\hbar}{e} \right)^2 \left( \frac{\phi_{l+1,i} - \phi_{l,i}}{2L_i} \right)^2 \right] \right\},
\]

(3.6)

where the \( i \) summation index runs over the two gates. The first capacitive energy term in Eq. 3.6 comes from the two quantum dots modeled as capacitors (\( C_g \) is the gate capacitance of the DQD circuit and \( Q_0 \) is the charge on the dot). The infinite series in Eq. 3.6 is simply Eq. 3.5 applied to the infinite number of inductors and capacitors in series used to model the transmission line. The \( Q_l \) and \( C_l \) terms are the charge and capacitance associated with the transmission line individual elements.

The canonical quantization of Eq. 3.6 follows by replacing the charge \( Q \) and the phase \( \phi \) by the continuous operators \( \hat{Q} \) and \( \hat{\phi} \), respectively, and by applying the commutation relation [22] \([\hat{\phi}_m, \hat{Q}_n] = i\hbar \delta_{mn} \) (again, in analogy to the relation of \([\hat{q}_m, \hat{p}_n] = i\hbar \delta_{mn} \) for the SHO system).

The \( Q_n \) and \( \phi_n \) enter in the Hamiltonian as positive definite quadratic forms with only short-range (nearest-neighbor) couplings and can therefore be simultaneously diagonalized. Following Ref. [19], the Hamiltonian is rewritten in terms of a new set of operators \( \hat{Q}(x) \) and \( \hat{\phi}(x) \), where \( \hat{Q}(x) \) and \( \hat{\phi}(x) \) are the inverse Fourier
transform of $\hat{Q}_n$ and $\hat{\phi}_n$, respectively. The relations between these two sets are

$$\hat{Q}_{li} = \sqrt{2} \int_0^1 dx \cos \left[ \left(l + \frac{1}{2}\right) \pi x \right] Q_i(x), \quad (3.7)$$

$$\hat{\phi}_{li} = \sqrt{2} \int_0^1 dx \cos \left[ \left(l + \frac{1}{2}\right) \pi x \right] \phi_i(x), \quad (3.8)$$

where $\hat{Q}_{li}$ and $\hat{\phi}_{li}$ obey the commutation relation $[\hat{Q}_i(x), \hat{Q}_j(y)] = i\hbar \delta(x-y)\delta_{ij}$. It is convenient to assume that $C_{gi} = C_{ti}$. The Hamiltonian in Eq. 3.6 can then be rewritten as

$$H = \sum_{i=1}^{2} \left\{ \sum_{l=0}^{\infty} \left[ \frac{\hat{Q}_{li}^2}{2C_{ti}} + \left(\frac{\hbar}{\epsilon}\right)^2 \frac{(\hat{\phi}_{l+1,i} - \hat{\phi}_{li})^2}{2L_i} \right] \right\}, \quad (3.9)$$

This form can be diagonalized through Eqs. 3.8 to yield

$$H = \sum_{i=1}^{2} \int_0^1 dx \left\{ \frac{\hat{Q}_{li}^2}{2C_{ti}} + \left(\frac{\hbar}{\epsilon}\right)^2 \frac{2}{L_i} \sin^2 \left(\frac{\pi x}{2}\right) \hat{\phi}_{li}^2(x) \right\}. \quad (3.10)$$

The Hamiltonian in Eq. 3.10 represents a sum of independent quantum harmonic oscillators. These eigenmodes can be represented in terms of bosonic creation and annihilation operators $\hat{a}_x$ and $\hat{a}_s$ which obey the usual relation $[\hat{a}_x, \hat{a}_s^\dagger] = \delta(x-y)$. The connection between $\hat{Q}(x)$ and $\hat{\phi}(x)$ and the creation and annihilation operators is

$$\hat{Q}(x) = \sqrt{\frac{\lambda_{xi}}{2}} (\hat{a}_{xi} + \hat{a}_{xi}^\dagger), \quad (3.11)$$

$$\hat{\phi}(x) = \frac{e}{i\hbar \sqrt{2\lambda_{xi}}} (\hat{a}_{xi} - \hat{a}_{xi}^\dagger), \quad (3.12)$$

where

$$\lambda_{xi} = \hbar \sqrt{\frac{C_{ti}}{L_i}} \sin \left(\frac{\pi x}{2}\right). \quad (3.13)$$

1This assumption does not hold in general and may need to be removed in a more accurate calculation.
Substituting Eq. 3.12 into Eq. 3.10 finally reveals the Hamiltonian for the boson bath

\[ H_B = \sum_{i=1}^{2} \int_{0}^{1} dx \hbar \omega_{xi} \left( \hat{a}_{xi}^{\dagger} \hat{a}_{xi} + \frac{1}{2} \right), \tag{3.14} \]

where, following Ref. [19]

\[ \omega_{xi} = 2\omega_{ci} \sin \left( \frac{\pi x}{2} \right), \tag{3.15} \]

with

\[ \omega_{ci} = \sqrt{\frac{1}{L_{i}C_{ii}}}. \tag{3.16} \]

As we will see later, \( \omega_{ci} \) plays the role of the high-frequency cutoff.
CHAPTER 4

SYSTEM-BATH INTERACTION

The high carrier density in the metal electrodes allows one to treat the electromagnetic noise in the gates as classical and thus temperature driven. The frequency-dependent voltage fluctuations in the gates are assumed to follow the Johnson-Nyquist expression \[23\]:

\[
\langle \delta V^2 \rangle = \hbar \sqrt{C_i/L_i} \omega \coth(\hbar \beta \omega/2),
\]

with \(i = 1, 2\) and \(\beta = 1/k_B T\). These voltage fluctuations manifest themselves explicitly in the total Hamiltonian of the system and are responsible for the coupling between qubit and bath. To see that recall that

\[
\varepsilon = \frac{n_g}{2} \left(2E_{Cm} - E_{C1} - E_{C2}\right)
\]

and, from Eqs. 2.25 and 2.22,

\[
n_g = \frac{1}{|e|} \left(C_{g1} V_{g1} - C_{g2} V_{g2}\right).
\]
As a result, fluctuations in $V_{g1}$ and $V_{g2}$ will cause fluctuations in $n_g$, $n_g = n_g^{(0)} + \delta n_g$, leading to an additional term to the Hamiltonian:

$$H_S = \epsilon \sigma_z n_g \longrightarrow + \epsilon \sigma_z n_g^{(0)} + \frac{\epsilon \sigma_z}{|e|}(C_{g1} \delta V_{g1} - C_{g2} \delta V_{g2}). \quad (4.4)$$

For quantized bath modes, the voltage fluctuations $\delta V_g$ can be related to the boson creation and annihilation operators through Eq. 2.3 (recall that $\hat{Q}_{0,i}$ is the charge in the $i$-th quantum dot),

$$\delta V_{gi} = \frac{\hat{Q}_{0,i}}{C_{gi}} = \frac{1}{C_{gi}} \int_0^1 dx \cos \left(\frac{\pi x}{2}\right) \sqrt{\hbar \omega_{xi}} C_{ti} (\hat{a}_{xi} + \hat{a}_{xi}^\dagger). \quad (4.5)$$

which matches a similar equation in Ref. [19]. Hence, the interaction between the system and the bath can be written down as

$$H_{SB} = \frac{\epsilon}{|e|} \sigma_z (C_{g1} \delta V_{g1} - C_{g2} \delta V_{g2}) \quad (4.6)$$

$$= \frac{\epsilon}{|e|} \sigma_z \sum_{i=1}^2 \eta_i C_{gi} \delta V_{gi} \quad (4.7)$$

$$= \frac{\epsilon}{|e|} \sigma_z \sum_{i=1}^2 \eta_i \int_0^1 dx \cos \left(\frac{\pi x}{2}\right) \sqrt{\hbar \omega_{xi}} C_{ti} (\hat{a}_{xi} + \hat{a}_{xi}^\dagger), \quad (4.8)$$

with $\eta_1 = +1$ and $\eta_2 = -1$.

The interaction Hamiltonian has a bilinear form and can factorized into two independent parts acting on the qubit and bath respectively:

$$H_{SB} = K \Phi, \quad (4.9)$$

where $K$ acts the two-dimensional qubit space,

$$K = \frac{\sigma_z}{2}. \quad (4.10)$$
and Φ acts only on the electromagnetic bath modes and is defined as

\[ \Phi = \frac{2e}{|e|} \sum_{i=1}^{2} \eta_i \int_0^1 dx \cos \left( \frac{\pi x}{2} \right) \sqrt{\hbar \omega_{xi}} C_i (\hat{a}_{xi} + \hat{a}_{xi}^\dagger). \]  

(4.11)
CHAPTER 5

DENSITY MATRIX EQUATIONS

The complete Hamiltonian of the total system (qubit-bath) has now been described as

\[ H = H_S + H_B + H_{SB}, \]  

(5.1)

where \( H_S \) is the system Hamiltonian described in Chapter 1, \( H_B \) is the bath Hamiltonian described in Chapter 3, and \( H_{SB} \) is the system-bath interaction Hamiltonian described in Chapter 4. The total Hamiltonian can be used to solve for the density matrix which completely describes the dynamics of the system.

Note that we are not interested in following the dynamics of the bath modes. In fact, despite being describe as a sum of linear (noninteracting) harmonic oscillators, the bath quickly thermalizes with the remaining environment. Moreover, the bath degrees of freedom are not directly measurable. In order to quantify the decoherence in the qubit, we must be able to describe its dynamics taking into account the bath in an effective manner. Thus, a scheme must be employed to
absorb the effect of the bath modes into an effective equation of motion that only addresses variables in the qubit two-dimensional space.

For a completely isolated qubit, the quantum dynamics is fully contained in a $2 \times 2$ density matrix that obeys the standard Liouville equation

$$\dot{\rho}_S(t) = -\frac{i}{\hbar} [H_S(t), \rho_S(t)]. \quad (5.2)$$

When the qubit is coupled to the environment, a more appropriate starting point is to consider the density matrix of the full system, qubit-bath,

$$\dot{\rho}(t) = -\frac{i}{\hbar} [H(t), \rho(t)], \quad (5.3)$$

where $H(t)$ represents the total Hamiltonian. One then defines the reduced density matrix of the qubit through

$$\rho_S(t) = \text{Tr}_B[\rho(t)] = \sum_{n_B} \langle n_B | \rho(t) | n_B \rangle, \quad (5.4)$$

i.e. by tracing out the bath degrees of freedom. Here, $\{|n_B\rangle\}$ are eigenstates of the time-independent bath Hamiltonian $H_B$. The standard procedure for obtaining an equation of motion for the reduced density matrix is to formally solve Eq. 5.3 and then apply the trace shown in Eq. 5.4. The technical details can be found in Ref. [24] and in advanced text books such as Ref. [25]. Under the Markov and Born approximations, one finds the so-called Redfield equation [26]

$$\dot{\rho}_S(t) = -\frac{i}{\hbar} [H(t), \rho_S(t)] + \frac{1}{\hbar^2} \left\{ [\Lambda(t) \rho_S(t), K] + [K, \rho_S(t) \Lambda^\dagger(t)] \right\}, \quad (5.5)$$
where the extra terms to the right of the Louivillian describe the relaxation due to the coupling to the bath. The auxiliary matrix $\Lambda(t)$ is defined as

$$\Lambda(t) = \int_0^{\infty} dt' B(t') e^{-it' H_S(t)/\hbar} K e^{it' H_S(t)/\hbar},$$

(5.6)

where $K$ is the qubit dependent part of $H_{SB}$ (see Chapter 4) and $B(t)$ is the bath correlation function, which is defined as

$$B(t) = \text{Tr}_B [\Phi(t) \Phi(0) f(H_B)].$$

(5.7)

Here, $\Phi(t)$ is the bath operator as defined in Chapter 4 and $f$ represents the thermal occupation factor: $f(H_B) = e^{-\beta H_B}/\text{Tr}_B [e^{-\beta H_B}]$. All time-dependent operators shown above are defined in the Heisenberg picture.

For convenience, in the following calculations, $\Phi$ (originally defined in Eq. 4.11) is rewritten as

$$\Phi = \int_0^1 dx \sum_{i=1}^2 g_{x_i} (\hat{a}_{x_i} + \hat{a}_{x_i}^\dagger),$$

(5.8)

where we introduce the auxiliary constant

$$g_{x_i} = \frac{2e}{|e|} \eta_i \cos \left( \frac{\pi x_i}{2} \right) \sqrt{\hbar \omega_{x_i} C_{ii}}.$$  

(5.9)

Defining $E(\{n_{x_i}\})$ and $|\{n_{x_i}\}\rangle$ as the energy eigenvalues and eigenvectors of $H_B$ for a set of occupation numbers $\{n_{x_i}\}$ allows the bath correlation function to be written down as

$$B(t) = \sum_{\{n_{x_i}\}} \langle \{n_{x_i}\} | \Phi(t) \Phi(0) f(H_B) | \{n_{x_i}\} \rangle,$$

(5.10)
with \( Z = \text{Tr}_B [e^{-\beta \hat{H}_B}] \). Taking the thermodynamic average and plugging in \( \Phi(t) \),

\[
B(t) = \frac{1}{Z} \sum_{|n_{xi}|} e^{-\beta E(|n_{xi}|)} \left\langle |n_{xi}| \left| e^{i \hat{H}_B t / \hbar} \Phi \right. \left| n_{xi} \right\rangle \right. \tag{5.11}
\]

\[
= \frac{1}{Z} \sum_{|n_{xi}|} e^{-\beta E(|n_{xi}|)} e^{iE(|n_{xi}|)t / \hbar} \left\langle |n_{xi}| \left| \Phi e^{-i \hat{H}_B t / \hbar} \right. \left| n_{xi} \right\rangle \right. . \tag{5.12}
\]

Inserting the identity matrix \( I \equiv \sum_{|m_{xi}|} |m_{xi} \rangle \langle m_{xi}| \) yields

\[
B(t) = \frac{1}{Z} \sum_{|n_{xi}|} \sum_{|m_{xi}|} e^{-\beta E(|n_{xi}|)} e^{i \mathcal{E}(|n_{xi}|) - E(|m_{xi}|)t / \hbar} \left\langle |n_{xi}| \left| \Phi |m_{xi} \rangle \right. \left. \langle m_{xi}| \Phi |n_{xi} \rangle \right. \tag{5.13}
\]

Notice that \( \Phi \) is a function of the creation and annihilation operators \( \hat{a} \) and \( \hat{a}^\dagger \), hence \( \langle |n_{xi}| \Phi |m_{xi} \rangle \) is only nonzero when \( |n_{xi}| \) and \( |m_{xi}| \) differ by just one quanta (in any of the normal modes). Therefore, the bath correlation function can be written as

\[
B(t) = \frac{1}{Z} \sum_{|n_{xi}|} \left[ e^{-\beta E(|n_{xi}|)} \int_0^1 dx \sum_{i=1}^2 (g_{xi})^2 e^{-i \omega_{xi} t} \left( n_{xi} + 1 \right) \right] \\
+ \frac{1}{Z} \sum_{|m_{xi}|} \left[ e^{-\beta E(|m_{xi}|)} \int_0^1 dx \sum_{i=1}^2 (g_{xi})^2 e^{i \omega_{xi} t - \beta \hbar \omega_{xi}} \left( m_{xi} + 1 \right) \right] . \tag{5.14}
\]

Here it is important to notice that

\[
\frac{1}{Z} \sum_{|n_{xi}|} n_{xi} e^{-\beta E(|n_{xi}|)} = N_B(\hbar \omega_{xi}) = \frac{1}{e^{\beta \hbar \omega_{xi}} - 1} , \tag{5.15}
\]

where \( N_B \) is the Bose-Einstein distribution function. Using Eq. 5.15 and replacing the index of the second sum from \( |m_{xi}| \) to \( |n_{xi}| \) gives

\[
B(t) = \int_0^1 dx \sum_{i=1}^2 \left[ N_B(\hbar \omega_{xi}) + 1 \right] (g_{xi})^2 \left[ e^{-i \omega_{xi} t} + e^{i \omega_{xi} (t - \beta \hbar)} \right] . \tag{5.16}
\]

Here the following simple relation is used,

\[
1 + N_B(\hbar \omega_{xi}) = \frac{e^{\beta \hbar \omega_{xi}}}{e^{\beta \hbar \omega_{xi}} - 1} . \tag{5.17}
\]
to yield

\[ B(t) = \int_0^1 dx \sum_{i=1}^2 (g_{xi})^2 \left\{ e^{i\omega_{xi}t} N_B(\hbar \omega_{xi}) + e^{-i\omega_{xi}t} [1 + N_B(\hbar \omega_{xi})] \right\}. \]  

(5.18)

Finally, a convenient form for the bath correlation function can be introduced,

\[ B(t) = \int_0^\infty d\omega \nu(\omega) \left\{ e^{i\omega t} N_B(\hbar \omega) + e^{-i\omega t} [1 + N_B(\hbar \omega)] \right\}, \]  

(5.19)

where the spectral function \( \nu(\omega) \) has been defined as

\[ \nu(\omega) = \int_0^1 dx \sum_{i=1}^2 |g_{xi}|^2 \delta(\omega - \omega_{xi}) \]  

(5.20)

\[ = \int_0^1 dx \sum_{i=1}^2 \left( \frac{2\epsilon}{|\epsilon|} \right)^2 \cos^2 \left( \frac{\pi x}{2} \right) \omega_{xi} C_{ti} \delta(\omega - \omega_{xi}). \]  

(5.21)

The auxiliary matrix \( \Lambda \) can now be solved for (see Eqs. 5.6 and 2.33),

\[ \Lambda(t) = \int_0^\infty dt' B(t') e^{-it'H_S(t)/\hbar} K e^{it'H_S(t)/\hbar} \]  

(5.22)

\[ = \frac{1}{2} \int_0^\infty dt' B(t') e^{-it'(\epsilon_n \sigma_z + \epsilon_m \sigma_x)} \sigma_z e^{it'(\epsilon_n \sigma_z + \epsilon_m \sigma_x)}, \]  

(5.23)

where \( H_S \) and \( K \) have been substituted by Eq. 2.33 and Eq. 4.10, respectively.

Now, assuming that the system is operating at the degeneracy point, namely, \( n_g = 0 \), we find

\[ \Lambda(t) = \frac{1}{2} \int_0^\infty dt' e^{-iv_m t' \sigma_x} \sigma_z e^{iv_m t' \sigma_x} \]  

(5.24)

Notice that

\[ e^{-iv_m t' \sigma_x} \sigma_z e^{iv_m t' \sigma_x} = [\cos (v_m t') \sigma_0 - i \sin (v_m t') \sigma_x] \sigma_z \]  

\[ \times [\cos (v_m t') \sigma_0 + i \sin (v_m t') \sigma_x] \]  

\[ = \sigma_z \cos (2v_m t') - \sigma_y \sin (2v_m t'). \]  

(5.25)
Therefore,
\[
\Lambda(t) = \frac{1}{2} \int_0^\infty dt' \left[ \sigma_z \cos(2\nu_m t') - \sigma_y \sin(2\nu_m t') \right],
\]
(5.26)
which can also be written in the form
\[
\Lambda(t) = \frac{1}{2} \left( \gamma_1 + i\gamma_3 \right) \sigma_z - \frac{1}{2} \left( \gamma_2 + i\gamma_4 \right) \sigma_y,
\]
(5.27)
where
\[
\begin{pmatrix}
\gamma_1 + i\gamma_3 \\
\gamma_2 + i\gamma_4
\end{pmatrix} = \int_0^\infty dt' B(t') \begin{pmatrix}
\cos(2\nu_m t') \\
\sin(2\nu_m t')
\end{pmatrix}
\]
(5.28)

Now, recall that the density matrix \(\rho_S(t)\) is Hermitian and therefore can be written down as
\[
\rho_S = \begin{pmatrix}
\rho_{11} & \rho_{12} \\
\rho_{21} & \rho_{22}
\end{pmatrix} = \begin{pmatrix}
\rho_{11} & \rho_{12} \\
\rho_{12} & \rho_{22}
\end{pmatrix},
\]
(5.29)
with \(\rho_{11} + \rho_{22} = 1\) at any time. Let us assume that \(\rho_{11}(0) = 1\) and \(\rho_{22}(0) = 0\) initially, i.e., that the system has is in initial state \(|L\rangle = |10\rangle\) (the excess electron of the charge qubit is in the left quantum dot). The above assumptions and Eq. 5.29 show that there are only three real independent components to \(\rho(t)\). For time-independent \(\nu_m\), the Redfield equation (Eq. 5.5) can be solved exactly for \(t > 0\), since all the coefficients in Eq. 5.5 are constant. Following Ref. [15], the solution for those independent components are
\[
\rho_{11}(t) = \frac{1}{2} + \frac{1}{2} e^{-\gamma_1 t/2} \left[ \cos(\omega t) + \frac{\gamma_1}{2\omega} \sin(\omega t) \right],
\]
(5.30)
\[
\text{Re}[\rho_{12}(t)] = -\frac{1}{2} \left( 1 - e^{-\gamma_1 t} \right) \tanh(\beta \hbar \nu_m),
\]
(5.31)
and

\[
\text{Im}[\rho_{12}(t)] = \frac{2v_m + \gamma_2}{2\omega} e^{-\gamma_1 t/2} + \sin(\omega t),
\] (5.32)

where

\[
\omega^2 = 4v_m\left(v_m + \frac{\gamma_2}{2}\right) - \frac{\gamma_1^2}{4},
\] (5.33)

\[
\gamma_1 = \frac{\pi}{2} v(2v_m) \coth(\beta\hbar v_m),
\] (5.34)

and

\[
\gamma_2 = -\int_0^\infty \frac{dy}{y^2 - 1} v(2v_m y) \coth(\beta\hbar v_m y),
\] (5.35)

where the latter expression involves a principal value integral.

These equations are well-known in the quantum dissipation literature and are characteristic of the so-called weakly-coupling spin-boson model [27, 25]. They are believed to hold when the relaxation times are larger than the oscillation period and other intrinsic time scales of the combined qubit-bath system. This is directly related to the Born and Markov approximations used in the derivation of the Redfield equations.

Here we have been able to make a connection between the frequency and relaxation rates of the qubit to specific microscopic parameters related to thermal fluctuations in the electromagnetic environment of the DQD qubit. This is a novel result that will allow the decoherence rates for DQD charge qubits to be estimated under realistic conditions.
The quality (Q) factor is a simple ratio meant to measure the “quality” of a resonant system. Any oscillatory system (like the DQD qubit) responds to external driving. When the frequency of the driving forces are close to the system’s natural frequencies, the system shows the highest amplitudes in its steady-state response. The Q-factor measures how sharp these resonances are. The higher the Q-factor the sharper the resonances are in frequency. Another way to interpret the Q-factor is to look at the transient response: In this case, one makes a sudden change in an external parameter of the system and observes its damped oscillatory response. Then, the Q-factor measures the number of oscillations within the relaxation time of the system and little damping is indicated by a high Q-factor.

For example, in a damped mechanical spring system, \( Q = \frac{fM}{R} \) where \( M \) is mass, \( R \) is the mechanical resistance coefficient, and \( f = \frac{1}{2} \sqrt{k/M} \) is the natural oscillation frequency, with \( k \) being the spring constant. Damping a system will lower the Q factor.
The Q-factor of coherent oscillations for the DQD qubit will be defined as

\[ Q = f \cdot T_2 \]  

(6.1)

where \( f \) is the oscillation frequency and \( T_2 \) is the customary phase relaxation time. The energy and phase relaxation times, \( T_1 \) and \( T_2 \) respectively, are obtained by following the oscillations of reduced density matrix in the energy eigenbasis when the qubit is initially placed in a state that is not an energy eigenstate. For that purpose, we first rotating from the ‘left-right’ basis \( \{|L\rangle, |R\rangle\} \) to the energy eigenbasis \( \{|-, +\rangle\} \). This rotation yields

\[ \rho_{--}(t) = \frac{1}{2} - \text{Re} \left[ \rho_{12}(t) \right] \]  

(6.2)

and

\[ \rho_{-+}(t) = -\frac{1}{2} + \rho_{11}(t) + i \text{Im} \left[ \rho_{12}(t) \right], \]  

(6.3)

where \( \text{Re} \left[ \rho_{12}(t) \right], \rho_{11}(t), \) and \( \text{Im} \left[ \rho_{12}(t) \right] \) were defined in Chapter 5. The diagonal matrix elements \( \rho_{--}(t) \) are damped by the \( e^{-\gamma_1 t} \) term, which is related to the energy relaxation of the system. This decay term yields the decoherence time \( T_1 = \frac{\gamma_1}{\gamma_1} \).

Similarly, the off-diagonal matrix element \( \rho_{-+} \) has the decay term \( e^{-\gamma_1 t/2} \), which is related to the bath-induced phase relaxation. This second decay term yields the phase relaxation or decoherence time \( T_2 = 2\gamma_1^{-1} \). Again, these general expressions for the relaxation times in terms of \( \gamma_1 \) are well-known in the literature \[27,25\].

The Q-factor can be written as

\[ Q = f \cdot T_2 = \frac{\omega}{2\pi} \cdot \frac{2}{\gamma_1} = \frac{\omega}{\pi \gamma_1}. \]  

(6.4)
Since the Q-factor depends on both $\gamma_1$ and $\omega$ and both depend on the spectral density, it is necessary to spend time investigating some properties of $\nu(\omega)$.

The spectral function was defined in Chapter 5 as

$$
\nu(\omega) = \int_0^1 dx \sum_{i=1}^2 \left( \frac{2\epsilon}{|e|} \right)^2 \cos^2 \left( \frac{\pi x}{2} \right) \omega_{xi} C_{ii} \delta(\omega - \omega_{xi}).
$$

Plugging in the definition of $\omega_{xi}$ (see Eq. 3.15), this function becomes

$$
\nu(\omega) = \int_0^1 dx \sum_{i=1}^2 \left( \frac{2\epsilon}{|e|} \right)^2 \cos^2 \left( \frac{\pi x}{2} \right) 2\omega_{ci} \sin \left( \frac{\pi x}{2} \right) C_{ii} \delta\left(\omega - 2\omega_{ci} \sin(\pi x/2)\right).
$$

Notice that the integral can be eliminated from the expression of the function with a simple change of variable. This is so because the integral part of the spectral function can be solved with the substitution of $z \equiv \sin(\pi x/2)$ as follows:

$$
\nu(\omega) = \frac{1}{\pi \omega_{ci}} \int_0^1 dz \int_0^1 \frac{dz}{\sqrt{1 - z^2}} \delta \left( z - \frac{\omega}{2\omega_{ci}} \right)
$$

$$
= \frac{1}{\pi \omega_{ci}} \left( \frac{\omega}{2\omega_{ci}} \right) \sqrt{1 - \left( \frac{\omega}{2\omega_{ci}} \right)^2} \theta \left( 1 - \frac{\omega}{2\omega_{ci}} \right),
$$

where $\theta$ is the Heaviside step function and we implicitly assumed $\omega > 0$. Using this result, the spectral function can be reduced to

$$
\nu(\omega) = \sum_{i=1}^2 \frac{C_{ii}}{\pi} \left( \frac{4\epsilon}{|e|} \right)^2 \left( \frac{\omega}{2\omega_{ci}} \right) \sqrt{1 - \left( \frac{\omega}{2\omega_{ci}} \right)^2} \theta(2\omega_{ci} - \omega)
$$

or

$$
\nu(\omega) = \sum_{i=1}^2 A_i \left( \frac{\omega}{2\omega_{ci}} \right) \sqrt{1 - \left( \frac{\omega}{2\omega_{ci}} \right)^2} \theta(2\omega_{ci} - \omega),
$$
where the constant $\mathcal{A}_i$ is defined as

$$\mathcal{A}_i \equiv \frac{C_{\text{ii}}}{\pi} \left( \frac{4\epsilon}{|e|} \right)^2. \quad (6.10)$$

An interesting feature of this result is that there is a high-frequency cutoff (the largest between $\omega_{c1}$ and $\omega_{c2}$) for the spectral function. The immediate consequence of that is the absence of any relaxation for driving frequencies (i.e. tunneling rates) beyond the cutoff. This comes about because of the gates were modelled as low-pass transmission lines, which is a realistic assumption. We will discuss the nature of this high-frequency cutoff below.

Notice also that as $\omega \to 0$, the spectral function $\nu(\omega)$ varies linearly with $\omega$, a behavior known in the literature as “ohmic” [27][25] (and in contrast to subohmic for the sublinear case and superohmic when the power in frequency exceeds one). As a result, when the oscillation frequency goes down and becomes much smaller than temperature ($v_m \ll k_B T/\hbar$), one expects $\gamma_1$ to approach a constant value,

$$\gamma_1 \to \frac{\pi}{2\beta\hbar} \sum_{i=1}^{2} \frac{\mathcal{A}_i}{\omega_{ci}}. \quad (6.11)$$

However, one needs to take this result with caution. It is clear from Eq. 5.33 that for sufficiently small values of $v_m$ the oscillation frequency $\omega$ vanishes, indicating a superdamped regime. While this is perfectly possible in a real system, Eqs. 5.33 to 5.35 were derived under the assumption that certain time scales of the system remain smaller than the relaxation times. Therefore, it is not obvious in the weak-coupling regime that these equations remain accurate when $\omega \to 0$. 


We have attempted at quantifying the Q-factor for realistic DQD systems using the microscopic parameters available in the literature \[12\] \[13\]. For the sake of simplicity, we assumed that the gates have identical characteristics, thus \(\omega_{c1} = \omega_{c2} = \omega_c\), \(C_{t1} = C_{t2} = C_t\), and \(C_{g1} = C_{g2} = C_g\). In Fig. 6.1 we show the Q-factor as a function of the tunneling rate \(v_m\) for a wide range of frequencies.

Figure 6.1: Q is plotted versus \(v_m\) as defined in Eq. 6.4. The constants are defined as follows \(\epsilon = 1.0\) meV [20], \(C_{t1} = C_{t2} = 1.0\) aF [13], \(\omega_{c1} = \omega_{c2} = 10^{11}\) rad\(\cdot\)s, and \(T = 40\) mK [15]. Notice the asymptotic behavior of Q as it approaches the cutoff frequency \((f_c = \omega_c/2\pi = 0.16\) THz). Inset is the low frequency range of the same graph.
The Q-factor approaches infinite as the tunneling frequency approaches the cutoff frequency $\omega_c$. The inset of Fig. 6.1 shows the low frequency behavior of $Q$. As mentioned above, there is a point in the low frequency regime where $\omega$ (hence, $Q$) goes to zero.

In Fig. 6.2 we present the same graph as in Fig. 6.1 but over a more relevant range of frequencies.

![Graph showing Q versus $v_m$ for the GHz range. The constants are the same as for Fig. 6.1, except that here three separate values for the cutoff frequency are plotted.](image)

Figure 6.2: $Q$ versus $v_m$ for the GHz range. The constants are the same as for Fig. 6.1, except that here three separate values for the cutoff frequency are plotted.

In Ref. [12] the authors mention that pulses turning on the tunnel coupling between dots were applied with a sampling rate of 100 MHz. The amplitude of
the tunneling rate for these pulses was 2.3 GHz and this value has been marked in Fig. 6.2, along with the behavior of Q for several different cutoff frequencies. It is obvious from Figs. 6.1 and 6.2 that the cutoff frequency $\omega_c$ greatly affects the behavior of Q.

Accurately measuring the cutoff frequency requires the knowledge of inductances and capacitances of the circuit elements defining the effective transmission line that models the gates (see Eq. 3.16):

$$\omega_c = \frac{1}{\sqrt{LC}}$$  \hspace{1cm} (6.12)

At this point we encounter the first major shortcoming of our calculation. Recall that in Chapter 3 it was assumed that the capacitance of the dot and gate was equal to that of elements of the transmission lines ($C_g = C_t$). Taking this further would allow us to estimate $C$ in Eq. 3.16 through $C_{gi}$, which can be inferred from the electrostatic coupling energy scales in a plot of conductance versus gate voltages for the DQD \[20]. Using this argument, for current setups one finds that $C \sim 10^{-18}$ F (atto farads). However, the geometry of these setups is such that the gate electrodes are much more exposed to the 2DEG than to the quantum dots. As a result, it is very plausible that $C_g$ and $C_t$ are quite different, perhaps by more than one order of magnitude. In Fig. 6.3 we show the sensitivity of the Q factor to the capacitance $C_t$. 

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Figure 6.3: $Q$ versus $C_g$ for the aF range. The same three cutoff frequencies plotted in Fig. 6.2 are used again here, and $\epsilon = 1.0$ meV, $T = 40$ mK, and the tunneling frequency $v_m = 2.3$ GHz\textsuperscript{[12]}. It is assumed that $C_{g1} = C_{g2}$.

Estimate of the impedances is even more daunting, since experiments do not provide any information on the inductive characteristics of the DQD system. For instance, assuming a cutoff frequency in the THz range and capacitances of atto farads, we arrive at impedances of about 10 mH, which seems a rather large value.

Therefore, unfortunately, without detailed finite-element simulations of the electromagnetic fields in the setup at a wide range of frequencies, the transmission line parameters cannot be seriously input into the expression for the spectral
function. We thus take the opposite view and try to use a comparison between our analytical results and the known Q-factors observed in the experiments to try to arrive at a reasonable, realistic value for the cutoff frequency.

Figure 6.3 shows the Q factor decreases as the capacitance increases, as expected. It also shows an increasing $Q$ with increasing values of the cutoff frequency. Even here the strong of influence of $\omega_c$ is seen.

![Graph showing Q versus $\omega_c$ for the 0.1 THz range. The constants are $\epsilon = 1.0$ meV, $T = 40$ mK, $C_{g1} = C_{g2} = 1.0$ aF and the tunneling frequency $v_m = 2.3$ GHz. Here, it is assumed $\omega_{c1} = \omega_{c2}$.](image)

Figure 6.4: $Q$ versus $\omega_c$ for the 0.1 THz range. The constants are $\epsilon = 1.0$ meV, $T = 40$ mK, $C_{g1} = C_{g2} = 1.0$ aF and the tunneling frequency $v_m = 2.3$ GHz. Here, it is assumed $\omega_{c1} = \omega_{c2}$. 
Figure 6.4 shows the Q factor displaying asymptotic behavior as it approaches the operation frequency of 2.3 GHz \((2.3 \cdot 2\pi = 0.14 \text{ [rad·THz]})\). This behavior makes sense, and agrees with Fig. 6.1. As the cutoff frequency approaches the tunneling frequency of the system (or vice versa) the Q factor increases rapidly towards infinity.

Looking back at the equations defining Q reveals a sensitivity to the temperature of the system as well. We show such relationship in Fig. 6.5.

Figure 6.5: Q versus \(T\) for the mK range. The constants are \(\epsilon = 1.0\) meV, \(C_{g1} = C_{g2} = 1.0\) aF, \(v_m = 2.3\) GHz, and the same three cutoff frequencies used in previous figures are used again.
The shape of Fig. 6.5 is similar to that obtained for the case of decoherence due to phonons [15]. Again, notice the strong influence of the cutoff frequency.
CHAPTER 7

CONCLUSION

The values for the quality factor found here are very high. Figure 6.2 shows $Q$-factor values ranging between 175 and 800, depending on the cutoff frequency $\omega_c$, for the 2.3 GHz operation frequency mentioned in Ref. [12]. Using 1 THz for the cutoff frequency, the $Q$-factor is approximately 260, which corresponds to a decoherence time $T_2$ of approximately 113 ns. The decoherence times seen by Hayashi and co-workers in their experiment are of the order 0.5 to 1 ns, not hundreds of nanoseconds. Petta and co-workers also observed decoherence times in the nanosecond range.

The simplest conclusion is that the values used for the cutoff frequency were incorrect. In Chapter 6, the role of the cutoff frequency was discussed in some detail. It was argued that the values of the inductance and capacitance of the transmission line elements are not easily estimated. The assumption of $C_g = C_l$, which was used here and is common in the literature [19], does not seem appropriate either. Therefore, at this point it was reasonable to take the inverse
approach and try to reverse engineer a value for this enigmatic cutoff frequency. However, this also presented some difficulties.

Figure 6.4 shows the sensitivity of the $Q$-factor to $\omega_c$ at the 2.3 GHz operation frequency. Just as in Fig. 6.1, the $Q$-factor diverges as $\omega_c \to 2\pi f_{op}$ (where $f_{op}$ is the tunneling frequency at which the charge qubit is operating). The $Q$-factor also increases as the cutoff frequency is increased. As $\omega_c \to \infty$, $\nu(\omega)$ and $\gamma_1$ tend to zero (see. Eq. 5.34 and 6.9). Hence, $Q \to \infty$ (see Eq. 6.4). These two separate behaviors act to create a minimum in Fig. 6.4 with a value of approximately 140. This $Q$-factor corresponds to a $T_2$ of approximately 60 ns, which is still an order of magnitude from the experimental values. These results seem to indicate that there is no cutoff frequency for which the theoretical $Q$-factor values will match the experimental results at 2.3 GHz. A value of the $Q$-factor low enough to match the experimental results can be seen in Fig. 6.1, but the tunneling frequencies are very low and it is precisely in this region where the theory is thought to break down by way of its assumptions.

Far from stating that charge fluctuations do not contribute to the decoherence seen in charge qubits, these results seem to indicate that the theory itself needs improvement. Most likely, the contribution of fluctuations in the voltage gates has not been accurately represented with these results. There are two possible reasons for this discrepancy. Either the assumptions and estimates made for
several materials and device parameters need to be revised, or the model itself lacks some critical ingredients.

Perhaps an indication that the important aspects of the coupling between electrons and electromagnetic fluctuations in the setup have not yet been taken into account fully is the following. The drag on charge motion in the gates due to the relatively low conductance in 2DEG (a charge-image effect) is known to be significant in superconducting qubits \([28]\). This has not been taken into account in our model. Another aspect which is missing and may turn out to be relevant is the capacitive coupling between the gate electrodes.

This work can be expanded on with a more careful treatment of the microscopic parameters of the DQD system. The frequency-dependent impedances of the gates and the interaction between gates and the 2DEG can be computationally simulated to produce much more accurate values. The Born-Markov approximations are assumed to become invalid in the low frequency range, but no attempt was made to correct or re-derive the equations for the reduced density matrix in this new regime. Tools for extending the Markov approximation exist, although at a relatively high computational cost \([29]\). This work would be facilitated by more experimental data plots in the vein of Chapter 6. When decoherence times and rates are measured as a function of the tunneling frequency, it is possible to infer some of the time scales that enter into calculations involving electromagnetic fluctuations as well as other types of decoherence mechanisms. Such information
would be invaluable in comparison to plots similar to Fig. 6.1, Fig. 6.2, and similar plots in the literature [15].
LIST OF REFERENCES


