Supporting Secondary Teachers' Proof and Justification of Calculus Concepts Through the Intentional Use of Dynamic Technology

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SUPPORTING SECONDARY TEACHERS’ PROOF AND JUSTIFICATION OF CALCULUS CONCEPTS THROUGH THE INTENTIONAL USE OF DYNAMIC TECHNOLOGY

by

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A dissertation submitted in partial fulfillment of the requirements
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ABSTRACT

Learning calculus concepts plays a huge role in understanding phenomena in STEM-related disciplines. Those concepts tend to be dynamic in nature and the visual exploration and representation of calculus concepts using paper and pencil is limited compared to pedagogically and intentionally using dynamic geometry software. As such, a primary component of this dissertation study involves the integration of dynamic technology. Additionally, previous studies have shown that students have difficulties constructing proofs related to calculus concepts. Despite the existing body of research on students' comprehension of proof and justification, there has not been much focus on teachers' knowledge and perception of proof and justification in connection to the ways that prospective secondary teachers can teach and learn calculus concepts. This study uses a qualitative methodology to investigate the ways in which integrating technology could help both in-service and pre-service secondary teachers gain a deeper understanding of the process of proof. Through a multiple case study approach, research participants were engaged with different mathematical tasks to explore geometric series and subsequently construct and prove conjectures through the integration of dynamic technology. This study showed that dynamic geometry software could help teachers to appreciate the value of visual representation in teaching and learning mathematics. Those technological pieces helped them with exploring different ideas which is crucial in the process of proving. However, a lack of experience both with visual representations and constructing conjectures held participants back from using their full potential. When it comes to mathematical proofs in school mathematics, it should be considered as a process of exploring ideas, making conjectures and checking the validity of those conjectures and not a single notion, and visual representations - specifically
dynamic ones that are created by technology – play a huge role in deepening teachers understanding of the process through their connection with key ideas.
To Mohsen Mohammadpour and children/students that we lost in bloody November
To Reera Esmaeilion and children/students that we lost in downing of PS752
To Farhad Khosravi and children/students that we lost because of poverty
   To all Iranian children/students who have not access to education
       To all Afghan children/students who are left alone
           To all children/students who don’t have a voice
The world would have been a better place with you and your voice
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<td>Association of Mathematics Teacher Educators</td>
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<tr>
<td>CAS</td>
<td>Computer Algebra Systems</td>
</tr>
<tr>
<td>CCSSM</td>
<td>Common Core State Standards for Mathematics</td>
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CHAPTER ONE: INTRODUCTION

Introduction

Zazkis, Weber & Mejia-Ramos (2014) state that even though there are different definitions for mathematical proof in the field of mathematics education, researchers agree that mathematical proofs and proving activities have a central role in communicating mathematical knowledge. According to Stylianides & Stylianides (2006), mathematics educators agree that in the K-12 school mathematics curriculum, proof needs to have a central place. They reason that in mathematics itself, proof has a central role; therefore, K-12 mathematics should maintain that structure (centrality of proof) for students in the primary and secondary grades. Having that central role should not be limited to a particular grade, but it should also not be limited to specific content, and students should have experiences with mathematical proofs in and across all content areas (Stylianides, 2007).

The Principles and Standards for School Mathematics (PSSM; NCTM, 2000) advocates for all students to graduate from secondary school with an understanding of how to construct mathematical proofs. The document takes it a further step and mentions "reasoning and proof" as one of five different process standards. PSSM also states that reasoning and proof cannot be taught and learned in a single course. From prekindergarten to grade 12, every mathematics classroom should include mathematical reasoning and proof in its curriculum (NCTM, 2000). To ensure students grasp a profound knowledge of mathematical proofs, PSSM states that "instructional school programs" should help students achieve some items regarding learning reasoning and proof. Students should recognize reasoning and proof as essential elements of mathematics. Moreover, they should construct mathematical conjectures and analyze them.
Students should also develop and assess mathematical arguments and proofs, and since different kinds of reasoning come to play in constructing different mathematical proofs, students need to have experiences with all of the different modes of reasoning.

Experiencing situations in which students learn mathematical proofs by exploring examples, constructing conjectures, and evaluating those conjectures, as well as studying more advanced proofs (in which they need to work with more than one key idea) as they move up through the grades help students have higher standards and criteria for analyzing and accepting arguments and reasonings (NCTM, 2000). As a result, students’ thinking and reasoning become more sophisticated and precise, and students become more skilled in analyzing complex explanations, and eventually, they will become more confident in accepting or rejecting given statements through analyzing based on their constructed knowledge.

**Students Difficulties with Proof and Justification**

Although there is a great emphasis on the importance of reasoning and proof and the key role they play in school mathematics, there is a considerable body of research showing that students face difficulties when it comes to learning about mathematical proofs (Balacheff, 1988; Bell, 1976; Chazan, 1993; Coe & Ruthven, 1994; Fischbein & Kedem, 1982; Galbraith, 1981; Healy & Hoyles, 2000; Knuth et al., 2002; Porteous, 1990; Schoenfeld, 1986; Senk, 1985; Stylianides et al., 2005; Usiskin, 1987).

A few arguments partly explain why students face obstacles with proofs or when validating the truth of a mathematical statement by relying on certain propositions and definitions using logical reasoning. Alibert (1988), in his study, states that some students consider proving to be an immoderate activity that does not necessarily help to deepen their understanding of mathematical concepts and arguments. Schoenfeld (1994) claims that when we
introduce a statement to students and ask them to prove it, they might think that proof is only about checking the validity of a statement and is not about exploring mathematics and discovering new ideas. This way of thinking about proof and justification is thought to make it less attractive to students.

According to Knuth, Choppin, and Bieda (2009), another reason for students' difficulties with proving activities relates to proof by example, where students decide the validity of a mathematical statement just by checking some examples. They also mention that in the United States, students are much too dependent on examples when it comes to proving a mathematical statement. Even though exploring examples and different cases of a general idea is very helpful in developing reasoning and thinking about proofs, merely checking the validity of a statement by checking a limited number of examples is not sufficient (Stylianides & Stylianides, 2009).

**History of Learning Proof and Justification**

Davis and Hersh (1981) investigated the history of mathematical proofs in school mathematics. They assert that students' school mathematics experiences with proofs had been limited to geometry, and there are no mathematical proving activities in other subjects such as algebra and arithmetic. The reason lies in the roots of proof and the fact that mathematical proof had been developed mostly in geometry in Western mathematics throughout history. Thinking of mathematics as an axiomatic system, Harel and Sowder (2007) argue that approximately 2600 years ago, Greeks had developed the concept of mathematical proof to manifest the validity of geometric propositions deductively and not empirically. This approach led to certain standards and characteristics of mathematical proof for the next 2500 years. Herbst (2002), furthermore, claims that because of the historical origins of mathematical proof in school mathematics, students' exposure to activities such as proving has been limited to proofs in geometry.
Progression of Learning Proof and Justification

Heinze and Reiss (2010) consider students’ motivation and interests in justification and proof. They show that there exists a correlation between affecting factors such as interest in mathematics with students’ performance in activities connected to proof and justification. They recommend that students must be engaged in doing mathematics regularly and consistently. They also claim that the same holds true for doing proof-related activities, and the fact that there is no clear focused progression for proof in school mathematics is an obstacle to achieving that goal (Heinze & Reiss, 2010).

According to Knuth, Choppin, and Bieda (2009), the foundation of proof in secondary mathematics is being constructed in elementary mathematics classes when students learn to make arguments and explain what those arguments mean verbally and justify them and provide reasons why those arguments are valid. Despite the importance of this transition from justification to proof, it happens suddenly. Students need to transition too quickly from working with visual and concrete justifications to more complex and abstract justifications, going from elementary mathematics subjects to secondary ones, with almost no emphasis on working with proof in middle school years. Additionally, during the middle school years, the progress of students’ ability to justify and prove mathematical statements is minimal. However, it is claimed that there is much room for improvement (Knuth et al., 2009).

Proof and Justification in Secondary Schools

Content Limitation of Proof in Secondary Curriculum

Herbst (2002, 2010) explains the history and answers why students’ learning opportunities around proof are mostly limited to high school geometry. In the late 19th century, the Committee of Ten concluded that formal geometry in high school is the best course for
students to learn deductive reasoning and proving. Hanna (1995) states that after the end of the new math era, in which teaching and learning proof was expanded to all the subjects in school mathematics, the importance of proof in the curriculum was diminished, and proof's domain in school mathematics was downgraded and limited to Geometry once again. Wu (1996) and Greeno (1994) claim that in school mathematics, proof can mostly be found in geometry because the nature of the concept of proof is not fully understood and represented in the right way. Comprehending the key role of proof in school mathematics and its effects on empowering students with complicated reasoning skills and therefore educating a better generation of members of society will lead to the idea that proof must be included in all the content areas in school mathematics.

**Transitioning from Secondary to Post-Secondary Mathematics**

According to McClain (2010) and Knuth and colleagues (2009), students gain experiences with justification and proving at the secondary level. This experience starts with exploration, sense-making, justifying, and constructing informal proofs. During the middle school years, students begin to use symbols while engaging in proving tasks. However, that is minimal. By the transition from middle school to high school, students move from informal proofs to formal proofs. In high school, students use more deductive reasoning and inductive reasoning than in middle school.

Stylianou and colleagues (2010) state that students start work with written formal mathematical proofs by entering college. Here, written formal mathematical proof means writing and checking logical inferences to the point that the student reaches fundamental axioms of mathematics. In college, their reasoning and structural thinking get more complex, and "students who major in mathematics-related areas and, hence, study formal mathematics are expected to
Role of Teachers in Facilitating Student Understanding

Content Knowledge for Teaching Proofs and Justification

McClain (2010) mentions that it is our responsibility as teachers and educators to help students understand the nature of proof and get access to the means of proving and being engaged in proving tasks. Engaging students in significant mathematical discussions could build an environment where mathematical arguments enter a refinement process, leading to a more sophisticated and efficient argumentation. McClain (2010) also argues that even though there has been much focus in mathematics education on students' perceptions and understanding of proof and how they learn mathematical proofs, little research focuses on teachers' knowledge about proof or how to teach proof in school mathematics. This study focuses on working with secondary pre-service and in-service teachers and trying to learn how we can help them be prepared to teach mathematical reasoning and proof to their students in the future. Steele (2012) states that teachers should have the knowledge and skills to "identify" if a mathematical argument is a mathematical proof or not (p.162). Not only that, Steele adds that teachers also need to determine what counts as proof and what is a non-proof statement across different representations.

Pedagogical Knowledge for Teaching Proof and Justification

Other than teachers' content knowledge about proof, three other factors could determine pedagogical beliefs toward proof: teachers' beliefs about the nature and role of proof in mathematics, teachers' beliefs about the role of proof in school mathematics, and teachers' beliefs about themselves as mathematical thinkers in the context of proof (Cabassut et al., 2012).
Conner (2007) concludes that how teachers support argumentation is aligned with how they think about the notion of proof, its purpose, and its role in school mathematics. According to Healy and Hoyles (2000) and Knuth (2002b), teachers do not think proof should have a central role in mathematics classrooms. Instead, the idea of proof being central should be limited to advanced mathematics classrooms. However, the same teachers in their studies believed that informal proof and argumentation should be central. Furinghetti and Morselli (2009) realized teachers have two different ways of thinking and using proof in classrooms. One is teaching theorems, in which the role of the proof is convincing others and systemizing facts. The other is teaching via proof. In teaching via proof, teachers use proof to deepen students' understanding. Smith (2006) found that students' perception of proof in lecture-based courses is insufficient. Students in problem-based courses showed a more productive effort in constructing proofs and making sense of ideas, which led to a better understanding of the notion of proof (Smith, 2006).

Connections to Calculus

Importance of Calculus Concepts

There are many reasons why learning calculus concepts are vital. The very first reason is that in many disciplines (economics, physics, engineering, science, etc.), comprehending calculus concepts plays an essential role in understanding the phenomena happening in that field, and one needs to know those concepts to make sense of different phenomena (Bressoud et al., 2016). One important question needs to be answered here. If there are other subjects that are at least as important as calculus, so why do we need to teach calculus concepts to students? Bressoud (1992) has a short yet convincing answer: "Modern scientific thought has been formed from the concepts of calculus and is meaningless outside this context (p. 615)."
According to Ellis and colleagues (2014), more than half of post-secondary students in the United States who take calculus have taken a calculus course in secondary school. Bressoud and colleagues (2013) state that the primary way students enter technical and science fields is through Calculus I. Their study shows a common belief among undergraduate, master's, and two-year college students and students who attend research institutions regarding student success in calculus. According to that common belief, students who are successful in Calculus in college/university tend to have taken and studied a Calculus course in high school (Bressoud et al. 2013).

Calculus concepts are dynamic in nature, and as such, to explore ideas and theorems in Calculus, simple static diagrams (such as the ones that can be constructed by using paper and pencil only) might not be good enough for exploring and discovering dynamic concepts. Here is where technology can aid in making sense of key ideas that will help us teach and learn mathematical reasoning and proof. Interactive technologies such as dynamic geometric software help us in many ways. They allow students to explore more examples, conjectures, and diagrams in a shorter time. In addition, they also help students in having a dynamic visual interactive representation while working with calculus concepts. However, it is crucial to mention that exploring many cases doesn't mean that a mathematical proposition is proven. It helps students improve their visualization abilities and reasoning by interacting with those animation and software, which is crucial in learning, especially concepts connected to algebra and calculus.

**Connections to Geometric Series**

In school mathematics, students see calculus concepts almost from the beginning of their elementary years, when they work with numbers and try to reason arithmetically. In middle school years, when they determine terms of different mathematical sequences and predict how
the sequence will behave, they also work with calculus concepts. During their elementary and middle school years, students use inductive reasoning to find solutions for those kinds of mathematical challenges. In their high school years, students represent mathematical sequences algebraically. They are also asked to think about the summation of all terms in different sequences when they encounter the concept of mathematical series in their algebra and calculus courses.

As it is believed and the standards suggest, there should be a progression for proof and justification in k-12 mathematics (Harel & Fuller, 2010). Mathematical content should be aligned with those progressions, so students can develop their thinking and knowledge of the notion of proof as they move forward with their studies.

Focusing on Teachers

According to Bramlett & Drake (2013), preparing students to understand and eventually develop formal and informal proofs is entirely dependent on the level of teachers' preparation in comprehending and teaching proof. Expecting K-12 teachers to educate students and engage them in proving activities is meaningless if teachers themselves are not provided with opportunities to be involved in the process of proving and understanding its importance (Bramlett & Drake, 2013; Roy et al., 2017).

Therefore, in this study, the focus is on both pre-service and in-service teachers. Also, the mathematical content is chosen in a way that helps both groups of teachers to think about potential lessons that include calculus concepts, in which students use different kinds of reasoning. It also helps teachers understand how students' reasoning and proving skills develop when learning the content (Nolan et al., 2016).
Sequences and series are mathematical concepts that provide those opportunities both for students and teachers. It can be said that students are continuously exposed to different aspects of sequences and series as they move forward in the school system from the very beginning. For example, different counting strategies such as skip counting, and addition and subtraction are connected to arithmetic series. Finding multiples, iterating fractions, and unit fractions, as well as in later years learning about what happens when we multiply or divide fractions by whole numbers vs. dividing by fractions less than one are connected to geometric series. Leveraging connections in elementary and secondary mathematics play an important role in helping students achieve a better understanding of conceptions and avoid misunderstandings (Abbaspour & Safi, 2021). Thus, when working with teachers, and since students have had the experience of working with sequences and series, it is an excellent opportunity to focus more on the progression of proofs and the process of proving itself. It would also be informative to study the development of teachers' abilities to reason and prove (formal and informal) and how those skills develop.

The kind of technology that is going to be used in this study is dynamic geometric software, such as Desmos (www.desmos.com) and GeoGebra (www.geogebra.com). According to Andreasen and Haciomeroglu (2013), dynamic geometric software such as GeoGebra helps students make sense of mathematical problems. Based on Zbiek and Heid's (2012) work, dynamic geometric software provides different strategies and connects verbal, symbolic, algebraic, and graphical representations, which leads to a better conceptual understanding. As Bostic and Pape (2010) state in their work, by using technology and connecting those different representations, teachers can help students' cognitive skills to be developed easier.
One problem with working with sequences and series is that sequences are not limited even though they are countable. It is impossible to show all the terms of a sequence numerically. It is even more difficult when students start to think about sequences and represent them visually. Because showing all the terms either numerically or visually is not a possible task to do, learners need to work with more steps and terms of the sequences. Therefore, paper and pencil are not efficient enough to work with the concept of geometric sequences and series. Here, technology's role becomes important since it provides efficient numerical and visual representations of the calculus concepts. Also, because calculus concepts are dynamic, and for example, in the case of geometric series, depending on the number of sentences the visualization might change, dynamic geometric software such as GeoGebra and Desmos could help us with the problem of visually representing the concepts by providing tools such as sliders.

Rationale for the Study

The importance of the role of proof has been emphasized in the National Council of Teachers of Mathematics (NCTM) (2000) *Principles and Standards for School Mathematics*. Further, according to NCTM’s (2020) *Catalyzing Change in Middle School Mathematics*, teachers should support students’ ability to construct proofs by focusing on informal reasoning to prepare them to work with proofs in high school. In another guidance document, the Association of Mathematics Teacher Educators (AMTE) (2017) *Standards for Preparing Teachers of Mathematics* (SPTM) does not provide much information about how to teach proof in school mathematics. It does not provide any specific standards related to the characteristics of teachers of mathematics connected to proof. In fact, the term proof appears only six times in AMTE’s SPTM. The first time proof is mentioned in the document, the authors
emphasized the importance of argumentation and justification. By stating the importance of connecting reasoning and evidence, they claimed that it “helps set the foundation for mathematical arguments in later grades, including inductive and deductive proof and the analysis, representing, reasoning, revising, and reporting demands of mathematical modeling at the high school level” (p. 77). The second time that proof is mentioned is after stating the importance of teachers’ roles in classrooms and how they can help students to understand the role of teachers, textbooks, and smart classes in establishing facts. “Beginning teachers help students unpack the limitations of these notions while also engaging students in more robust forms of argument and proof” (p. 86). On page 123, it is mentioned in SPTM that it is important for students to develop a perspective in which the importance of algebra is understood. The statement is followed by a claim that this perspective is vital to learning calculus, Introduction to Proofs, and Linear and Abstract Algebra. The next three times that proof is mentioned is about written proof being the final stage of the process of proving, and two very general statements about proof and other subjects’ placement in programs that prepare mathematics teachers. They even go a step further and make suggestions about lowering the proof-related subjects in the content courses of teacher preparation programs.

With all this being said, it is evident that there is much room for improvement in focusing on proof-related teacher preparation content. Assuming that the standards, suggestions, and practices are based on research and considering the lack of recommendations for preparing teachers to learn and teach proof, it is reasonable to conclude that more study is needed on this topic.
Proof’s Connection to Key Ideas and Visual Representations

Raman (2003) mentions a vital difference between professors' and students' points of view toward mathematical proofs. Using questions about some features of functions that students get to know in calculus, Raman mentions that professors use key ideas while students do not. Key ideas are "heuristic ideas which one can map into a formal proof with an appropriate sense of rigor" (p. 323). She also mentions another difference between those two groups regarding proving activities. Professors tend to start thinking about mathematical proofs and constructing them by drawing diagrams, while students are more reluctant to work with visual representations.

Even though, as Tall (1991) states, diagrams could be misleading and lead us to reasoning errors, they could also be beneficial in providing us with mathematical intuition and leading us toward knowing the key ideas that Raman mentioned. If not the heart of mathematical proving and justifying, it could be said that key ideas play a crucial role in using different kinds of reasoning and constructing proofs. When students face a challenge in proving a mathematical statement, it is often the case that they do not know what to do. As outlined in Nelsen's (2002) book, Proof Without Words, diagrams are often linked to key ideas and can help us figure out what needs to be done next or where we should go from a current situation to build proof through the whole process.

Mejia-Ramos and Weber (2019) observed 73 students constructing seven proving tasks in calculus. They reported that even though using diagrams has the potential to help students generate mathematical proofs, many students do not know how to benefit from the use of diagrams. They suggest more research is needed to study the process of "making inferences from diagrams and translating visual arguments into formal proofs (p. 487)". Students need to realize
the importance of diagrams, and that is something that, like proof itself, cannot happen over a
course or one grade or two, but rather over a series of connected efforts across several grades.
Students should be provided with opportunities to think and sketch diagrams as an alternative
representation of a mathematical idea whenever there is a proof-related task. If more scaffolding
is needed, students should access diagrams connected to a certain proof while knowing that
drawing diagrams doesn't mean a proof is completed or finished. Students need opportunities to
translate the visual representation using semiotic language since, after all, one of the functions of
mathematical reasoning and proving is fostering communication and sharing ideas with each
other.

**Purpose and Research Questions**

The purpose of this study is to investigate how by integrating DGS educators can assist
pre-service and in-service secondary mathematics teachers and deepen their understanding with
proofs when the content is connected to calculus concepts. To further explore proof and
justification with pre-service and in-service teachers, this study will be guided by the following
research question:

*In what ways can dynamic technology integration and visual representations of concepts
and ideas support pre-service and in-service secondary teachers’ experiences and beliefs
regarding the process of proving calculus concepts?*

**Significance of Study**

This study aims to contribute to the field by focusing on teachers and teaching and
learning proof by the use of technology. As it is mentioned before the field of mathematics
education can benefit from studies in which the focus is on teachers in teaching and learning
proofs. Inspired by Raman (2003), Stylianides (2009) and NCTM’s (2018) *Catalyzing Change in High School Mathematics*, this study takes a step further to investigate the effect of visual representations through connecting them to key ideas by the use of technology on pre-service and in-service secondary teachers’ understanding of the process of proof.

**Summary**

In the next chapter, the literature review investigates the nature of proof, its definition in mathematics education, and different perspectives in mathematics education. Then, the next section of the literature review provides the research background related to technology integration and the ways in technology contributes to teaching and learning in K-12 settings. The literature review also shares crucial research explaining the importance of working with different representations, especially visual representations. The next part of the second chapter synthesizes the research related to technology, calculus concepts, teacher content knowledge, and mathematical modeling that have influenced this dissertation.

Chapter three will describe the selected research methods, designed tasks, and interventions. Questions related to the study such as sampling methods, participants, data collection methods, data analysis procedures, and more will be answered and clarified in the methodology chapter. Also in chapter three, the theoretical frameworks that have been chosen to design the tasks related to proof and justification and different modes of reasoning (Inductive and Deductive) are shared. Next, the study introduces the chosen cases and justifies why the researcher has selected them for this dissertation while using other ideas to serve the purposes of the study. In chapter four, the collected analyzed data case by case right before the cross-case analysis and comparing the cases. Chapter five is dedicated to the cross-case comparison of three
participants of this study and chapter six will summarize the findings, conclusions and discusses the potentials for future studies related to proof.
CHAPTER TWO: LITERATURE REVIEW

Introduction

This chapter starts with introducing different definitions in the guidance documents that affected researcher’s perspective towards proof and justification. It continues with introducing seminal articles about calculus and technology and the intersection of each pair of them. In the end of this chapter studies that focus on the intersection of all three will be briefly mentioned and the conceptual framework that has been used for this dissertation will be introduced.

Proof and Justification

According to NCTM (2010), justification is part of mathematical reasoning and proof. Reasoning and justification are used to create and evaluate mathematical statements, arguments and proofs. In the Common Core State Standards for Mathematics (CCSSM, 2010), being a practice, justification is defined as constructing valid arguments and analyzing others’ reasonings. Cioe and colleagues (2015) describe justification as “a critical mathematical practice that must play a role in teaching and learning at all grade levels (p.485). They also add that “having students share their reasoning and explain how they know something is true or correct is the process of justification (p.485)”.

Defining proof is a complicated endeavor in itself. Philosophers see it as a strictly syntactic object existing in a formal theory. Pelc (2009) defines proof as “arguments used in mathematical practice in order to justify correctness of theorems (p.86)”. However, this definition of proof doesn't help us answer questions such as how proofs are constructed and how they help mathematics understanding develop (Pelc, 2009). Therefore, that definition is of little value from a pedagogical point of view.
Weber (2014) states that seeing proofs from a mathematician’s standpoint and what they work with as mathematical proof is not useful as well. Mathematicians’ view of proof and what they read and write as proofs is too broad to benefit educators pedagogically. Inspired by Lakoff’s (1987) clustered model, Weber defines proof as a cluster concept. According to Lakoff a cluster model is a model in which "a number of cognitive models combine to form a complex cluster that is psychologically more basic than the models taken individually (p.74)". Weber (2014) claims that, similar to Lakoff’s clustered model, the concept of proof is also a clustered concept. Based on his definitions, proof is:

1. A convincing argument,
2. A deductive argument,
3. A transparent argument where the gaps can be filled by a mathematician,
4. A perspicuous argument that helps to understand why a theorem is true.
5. An argument within a representation system that satisfies communal norms.
6. An argument that is approved by the mathematical community.

Weber (2014) concludes that direct instruction is not the best way to teach and learn. If proof is a cluster concept, and all models of proof are connected, the best way of making sense of proof is through communication and community practice. The communication idea is in line with other definitions of proof. Stylianides's (2007) school definition of proof that is accepted widely considered proof as a mathematical argument that includes three characteristics:

1. The class community uses true and accepted statements that need no more justification.
2. The class community uses valid reasonings that are valid and known conceptually.
3. The class community communicates through those making reasons and arguments that are appropriate and within their conceptual understanding's reach.

NCTM’s *PSSM* (2000) defines proof as “a formal way of expressing particular kinds of reasoning and justification (p.56)” and expects students to be able to understand and construct proofs after finishing secondary school. The standards mention that instructional K-12 programs should help students appreciate proof and justification as the core of mathematical activities, build and investigate conjectures, develop and check the validity of mathematical arguments and have opportunities to work and develop their ability to use different forms and methods of proving and justification. In NCTM’s *Developing Essential Understanding of Proof and Proving*, mathematical proof is a specific kind of argument that connects different deductive, logical statements when supporting or rejecting a mathematical claim (Ellis et al., 2012). According to NCTM’s (2018) *Catalyzing Change in High School Mathematics*, proof is using deductive reasoning to investigate the validity of mathematical conjectures when applying generalization (by using inductive reasoning) to certain instances.
**Table 1. Definitions of proof in the literature**

<table>
<thead>
<tr>
<th>Source</th>
<th>Definition</th>
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<tbody>
<tr>
<td>Pelc (2009)</td>
<td>“Arguments used in mathematical practice in order to justify correctness of theorems (p.86)”</td>
</tr>
<tr>
<td>Weber (2014)</td>
<td>A cluster concept containing convincing, deductive, transparent, perspicuous arguments satisfying communal norms and approved by the mathematical community</td>
</tr>
<tr>
<td>Stylianides (2007)</td>
<td>A series of mathematical arguments that mathematical class communities use as accepted statements, valid reasonings to communicate</td>
</tr>
<tr>
<td>NCTM’s PSSM (2000)</td>
<td>“A formal way of expressing particular kinds of reasoning and justification (p.56)”</td>
</tr>
<tr>
<td>Ellis and colleagues (2012)</td>
<td>A specific kind of argument that connects different deductive, logical statements when supporting or rejecting a mathematical claim</td>
</tr>
<tr>
<td>NCTM’s Catalyzing Change in High School Mathematics (2018)</td>
<td>Using deductive reasoning to investigate the validity of mathematical conjectures when applying generalization (by using inductive reasoning) to certain instances</td>
</tr>
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</table>

Despite the pedagogical benefits of definitions of proof provided by NCTM's (2018) and Stylianides (2007), AMTE's SPTM (2017) does not provide a clear definition of formal proof in school mathematics. However, it recommends that teachers engage students in "more robust forms of argumentation and proof". Also, Sinclair and colleagues (2012) mentions that proof is a process, and to complete a proof, students need to write the proof as to the last stage of this process (the process of proving).

**The Importance of Proof’s Role**

Proof has been mentioned several times in Florida’s B.E.S.T standards (2020). In the standards, proof is strongly connected to the course of Geometry and students are expected to learn how to construct different kinds of proofs, among these are proof by contradiction and
proof by induction. Students also should learn proofs, such as pictorial proof, paragraph proof, flow chart proof and etc.

Although there are different definitions for mathematical proofs, it is generally agreed upon that proof and proving have a central role in school mathematics (Zazkis, Weber & Mejia-Ramos, 2014). Mathematics educators also agree that in the K-12 school mathematics curriculum, proof needs to have a central place (Stylianides & Stylianides, 2006) because, in mathematics itself, proof has a central role, and K-12 mathematics should maintain that structure for students.

Despite proof's importance and central role, studies have shown that students in different grades face difficulties when it comes to learning about mathematical proofs (Weber, 2001; Healy & Hoyles, 2000; Knuth et al., 2002; Stylianides et al., 2005). Even mathematics majors often struggle when asked to prove a theorem (Zazkis, Weber & Mejia-Ramos, 2014).

Lester (1975), investigated 80 students’ ability to write valid proof as part of their problem-solving developmental skills. The results were impressive. Middle school students (grades 7-9) had the same capability of doing the tasks as high school students (grades 10-12). At the elementary level, lower grades elementary students (grades 1-3) were not as successful as secondary school students in working with elementary proof-related tasks; however, by having extra time, higher grade elementary students (grades 4-6) showed the same level of ability as secondary students. Lester concluded that elementary students could understand certain aspects of mathematical proof. Students in all grades can perform mathematical proving and justify viable arguments, even when they face difficulties.
Although students might follow each step in a formal proof, they might not understand formal proof as a whole. To further the notion of proof, the goals of mathematical proof should be discussed. One goal of the process of proving a mathematical statement is to start with certain assumptions, reason deductively and reach new statements/arguments at each step. Another goal is to explain why and how each step works and makes sense. As the final product of a formal proof, certain statements and theorems that could be used and put together to build an axiomatic systematic mathematical theory (Tall, 1988).

Additional focus is needed on the teaching aspects and instructional strategies concerning proving and justifying mathematical arguments, particularly related to concepts that pose challenges to students.

**Teacher Preparation in Proof and Justification**

Usiskin (1987) believes that the difficulties students face with proving are because the introduction of proof in high schools is not continuous and not supported. Preservice teachers learn about the concepts involved in proving and justifying; however, they need explicit experiences to enrich their teaching knowledge. When considering Shulman's (1986) teacher content knowledge model, it can be said that even though preservice teachers might get to study the subject matter; there could be more opportunities to be exposed to pedagogical content knowledge and ways to teach effectively and engage students with activities related to justification and proof.

Knuth (2002) worked with 17 secondary school mathematics teachers. He asked them about their thoughts and conceptual understanding of mathematical proof. In his findings, teachers often viewed proof as a concept that is appropriate for only a few students. In other terms, only a small group of students are able to learn to prove. Teachers in the study often also
thought that proof is only a subject that needs to be taught and not a way of reasoning to communicate with each other. This point of view causes many equity problems within the classroom and in teaching and learning mathematics. Justification and proof, or, as Stylianides (2008) states, reasoning and proving, is a fundamental mathematical practice that helps students develop a rigorous mathematical understanding. Still, it is also a way for students to communicate with each other with the teacher. To have more equitable learning opportunities, students need to have access and agency - developing a feeling that they can do mathematics and work with mathematical ideas – and justifying and proving helps them with those two factors (Bieda & Staples, 2020). This equitable purpose is also aligned with Weber's (2014) definition of proving as a cluster concept. As stated in NCTM's (2018) Catalyzing Change in High School Mathematics, examining proof as a cycle of inquiry and justification, teachers should use that cycle to attend to the eight mathematics teaching practices and "their connections to equitable teaching".

Proof and Cognition

The logical system plays the most important role in proving tasks, which is the outcome of our brain's multiprocessing complex decision-making system (Tall, 1991). To operate this system emphasizes focusing on the essential information and neglecting the details that are not vital to be processed. Barnard and Tall (1997) define the cognitive unit as a component of cognitive structure that our brain can pay attention to it at one time. A cognitive unit could be a symbol, diagram, concept, definition, theorem, and so on; manipulating those cognitive units is our ability to think mathematically. When it comes to mathematical proof as one of the most vital goals of learning mathematics, we need to be really good at thinking mathematically (Barnard & Tall, 1997). Two factors are important:
1. The ability to compress information to fit into cognitive units.

2. The ability to make connections between cognitive units so that relevant information can be pulled in and out of the focus of attention at will (p.41).

Raman (2003) states that there is an important difference between professors' and students' points of view toward mathematical proofs. Using questions of some features of functions that students get to know in calculus, Raman mentions that professors use key ideas while students do not. Key ideas are “heuristic ideas which one can map into a formal proof with an appropriate sense of rigor” (p.144). Raman also observed that one reason for students facing difficulties when working with proofs is being reluctant to use diagrams.

Mejia-Ramos and Weber (2019) observed 73 students constructing seven proving tasks in calculus. They reported that even though using diagrams has the potential to help students generate mathematical proof, many students don't know how to benefit. They suggest more research is needed to study the process of ”making inferences from diagrams and translating visual arguments into formal proofs” (p.9).

Visualization and Intuition

Visual intuition could either play a constructive role in our understanding of mathematical concepts and theorems or mislead us and direct us to falsified mathematical statements that do not hold true (Tall, 1991). Visual representations should be accompanied by logical reasoning to prevent that kind of misunderstanding. In learning calculus concepts, the visualization could help us build an intuition that eases the process of cognitively learning these concepts; however, compared to other subjects such as algebra and because of the dynamic nature of calculus concepts, visualizations also could be deceptive. Therefore, when working with visual representations, it is important to examine ideas concluded and formally prove
concluded theorems by pairing them with a chain of logical reasons and explanations. Despite the fact that visualization could be misleading, Tall (1991) also believes that eliminating that kind of representation (visuals) does not seem to be a good idea for mathematics students. He states that in the history of mathematics, visualizations have helped mathematicians to access mathematical ideas, and students should have the same opportunity both from a learning and a historical point of view.

**Calculus and Proof**

To improve students' understanding of calculus concepts, it is necessary to have them explore ideas by using visual representations and diagrams. Research has shown that, as with proof (Raman, 2003), students are reluctant to visualize calculus concepts, and similar to proof and in general, students' skill sets to visualize calculus concepts are limited as well (Tall et al., 2008). In order for them to become not only comfortable but also practice that and gain the competency of visualizing, students not only need to work with diagrams, but they need to connect those visualizations to their corresponding mathematical idea (Figure 1).

![Figure 1. The Continuous Cycle of Connecting Visuals and Concepts](image)

To refine ideas, students should turn ideas into diagrams and vice versa. The first visual representation of an idea might not be the best one, so students can refine the picture by
connecting it to the mathematical idea and exploring the connection using deductive reasoning. By refining the image, the student could either develop the idea itself or get a more profound understanding of the concept/idea that helps them connect one idea to another (Tall, 2019).

Speaking of the importance of visualization in empowering students and enriching their understanding, it must be said that technology can improve students' visualization skills in the very first stages of learning mathematical concepts. Technology can provide learners with the comprehension of formal calculus concepts such as ideas such as infinitesimals, change, accumulation, etc. (Bressoud et al., 2016).

**Calculus Concepts**

At the secondary and post-secondary levels, calculus plays an essential role as a course that not only engineers, economists, and mathematicians take but also teachers take as well. Students who take Calculus use it in problem-solving and mathematical modeling activities in different disciplines. While in other parts of the world, students take Calculus in high school, here in the United States, students have the option to take it either at the secondary level or tertiary level (Rasmussen et al., 2014).

The calculus reform movement, which started in the 1990s in the U.S., was primarily focused on designing curriculum in a way that might have helped students to comprehend calculus concepts better. The people responsible for developing the curriculum were mathematicians and not mathematics educators. That affected the research in the field to take a path that pedagogical approaches were not being considered. Therefore, there was no focus on the ways students understood the concepts and studied the theorems. Another result of developing the curriculum by mathematicians was the scarcity of research on teaching practices.
The studies lacked the link and interplay between teaching and research was missing (Rasmussen et al., 2009).

Since 1971, the workforce demand in the U.S. has increased for STEM graduates; however, on average, only about 30% of graduates have pursued a STEM-related major (Rasmussen et al., 2014). Another important finding shows that the more students are engaged in Calculus I, the more they tend to continue a STEM major in the future when compared to students without experience in Calculus 1. One factor that determines and affects students' engagement is pedagogical. When instructors show the class how to solve a problem, some students see that as a personal scaffold for others and not them (Ellis et al., 2014). Another explanation is that less engaged students cannot see the connection between what they are being taught in class and what they are expected to do in assignments and tests; therefore, when solving problems, they feel less prepared than others in the class. A third factor also exists.

Providing extra material in calculus classes to support students' learning and to help them overcome misconceptions positively correlates with students receiving help and therefore continuing to engage more in class (Carnevale et al., 2011). According to Carnevale and colleagues (2011), another pedagogical reason for keeping students motivated to learn calculus is having a whole-class discussion, so students feel involved; by whole-class discussion, they do not mean asking a question from the whole class and asking students to answer it. From the student perspective, a whole class discussion means that the instructor explicitly involves them in-class conversations (Carnevale et al., 2011).

According to Rasmussen and Ellis (2013), a considerable portion of students who do not take Calculus II are the "well prepared" and "hard-working" ones in Calculus I. They also
mention that one reason students switch to majors that do not require them to take more mathematics courses is that Calculus I takes so much time and energy from them, yet they have a negative experience. A primary reason for that is an instructional one. They claim that most of the instructions given to students in Calculus courses are lecture-based, and lectures are not the best way to teach Calculus (Rasmussen & Ellis, 2013).

It is worth mentioning that high school students’ demand to take pre-calculus and calculus has changed drastically over the last few decades (NAEP, 2012). According to the U.S. Department of Education’s report, the number of students who took a Calculus course at high school in 2012 is three times more prominent compared to 1978 (NAEP, 2012). Associations including NCTM and Mathematical Association of America (MAA) agree that the push toward getting to take the Calculus courses in high school has had a negative effect on students and teaching and learning mathematics in general (Bressoud et al., 2017). Despite the educational system making students rush to have Calculus on their high school transcripts, it seems that taking Calculus during high school years has not let them build a strong foundation of mathematical knowledge required for STEM fields. There exist related statistics that affirm those associations’ opinions on this matter. For example, in 2004, almost 17% of students who took Calculus in high school had learning difficulties with mathematics which led them to take an extra course to help them correct their misunderstandings (Ingels et al., 2004).

Considering different disciplines, it can be said that Calculus is a fundamental course, since it has been used in different contexts (Bressoud, 1992). Not only is it a prerequisite for many courses in undergraduate Science, Technology, Engineering, and Mathematics (STEM), so universities require it, but it can also be said that without a strong understanding of calculus
concepts, students will face difficulties in their eventual STEM fields of study. The reason lies in the dynamic nature of calculus concepts that makes it the core of any dynamical modeling widely used in any subject in STEM fields. This argument can be generalized to other areas, such as the humanities, where students need to develop quantitative analytic methods, including probability and statistics. Investigating students' understanding of concepts in Calculus has concealed that students face difficulties if they are asked unfamiliar questions such as the ones they expect to see in assignments, exams, etc. They learn how to answer standard, predictable questions in an expected and reasonable way (Tall, 1990).

**Preparation of Secondary Teachers of Mathematics**

One question that arises immediately is how well-prepared are pre-service teachers when it comes to teaching calculus? Although there is a significant body of research in mathematics education related to students' understanding, little research has been done on how calculus should be taught (Larsen, Marrongelle, Bressoud & Graham, 2017). Furthermore, we know that the instructor's quality and how they teach is a critical factor in encouraging students to learn more about calculus concepts (Bressoud, Carlson, Mesa & Rasmussen, 2013).

**Inequities in Calculus**

A survey administered by Bressoud and Rasmussen's (2015) showed that how calculus is taught in colleges and universities lowers students' confidence in themselves to do mathematics and also lowers their willingness to pursue studies and subjects that need more mathematics. In the same study, students indicated that they actually enjoyed mathematics less after completing calculus courses. The other problem mentioned in the same study was calculus instructors' perceptions of their teaching. While they consider themselves traditional instructors who give
lectures, none were entirely on board with the statement that "calculus students learn best from lectures" (p. 144).

Hagman (2019) describes that the characteristic of successful calculus programs is "diversity, equity, and inclusion practices." In Hagman's opinion, calculus programs should ensure access so that all students are set up for success, particularly those from historically marginalized groups in STEM, such as students of color and female students. Digging deeper into this issue, according to Haciomeroglu, Chicken, and Dixon (2013), students' gender has no significant effect on their cognitive ability and learning outcome in calculus. However, calculus programs are designed in a way that female students enrolled in them are twice as likely to discontinue a field/career that needs them to take another Calculus course (Bressoud et al., 2013).

Studies show that compared to men, women are 1.5 times more likely not to pursue a major in STEM after taking calculus. That causes a gender gap in the workforce in those fields and leads to females being underrepresented in STEM-related professions. Also, compared to men, more women claim that they do not grasp a deep understanding of calculus. The findings suggest that the phenomenon could be explained more by women (compared to men) having self-confidence in their mathematical skills and understanding of the content but needing to gain confidence in their mathematical abilities (Ellis et al., 2016).

**Technology**

Electronic technologies such as calculators and computers are essential in doing mathematics; therefore, they are critical tools to teach and learn mathematics. Electronic technology can support students' thinking in different areas of mathematics. With the use of technology, students and teachers can focus more on reflection, analyzing feedback that
technology provides, reasoning, and problem-solving rather than computational tasks (Dick & Hollerbrands, 2011). In short, technology is an instrument that enriches both students' understanding and teachers' effective teaching skills; it is not a replacement for understanding mathematics. According to Cullen and colleagues (2020), teachers should be provided with opportunities in which they experience using technology efficiently to learn mathematical contents. Experiencing the integration of technology in learning leads teachers to leverage the affordance of it in their future classes.

The Common Core State Standards for Mathematics (CCSSM, 2010) states that technology is a powerful tool to help students understand and solve real-world problems and situations modeled mathematically. The role of technology is emphasized in the PSSM (NCTM, 2000). The technology principle says, "Technology is essential in teaching and learning mathematics; it influences the mathematics taught and enhances students' learning" (p. 11). Students should have the opportunity to explore mathematics concepts with the use of technology. Therefore, integrating technology (as an instrument and not as a goal itself) is an inseparable part of students' learning and understanding of mathematics concepts. According to NCTM's (2018), technology such as Computer Algebra Systems (CAS) can assist learners in exploring patterns, making conjectures about generalizations in all content domains. In that way, technology can help students to strengthen their understanding of mathematical concepts and explore mathematical relationships to explore more and solve problems. The document also insists that while working with technology, students' focus should be on developing their understanding and interpreting results. NCTM (2014) emphasizes the role of technology in meaningful mathematics classrooms "An excellent mathematics program integrates the use of
mathematical tools and technology as essential resources to help students learn and make sense of mathematical ideas, reason mathematically, and communicate their mathematical thinking” (p. 78).

**Affordances and Challenges Related to Equity**

Fey (1989) states two critical benefits to the use of technology in mathematics education. First, teachers and students can focus more on conceptual understanding rather than computations that technological tools can do; secondly, it shifts the curriculum from the traditional methods to more complex mathematics ideas. Pedagogically, technology helps mathematics classes to become more student-centered. By the purposeful use of technology, teachers can become facilitators and task setters/managers rather than someone who gives lectures only. It can also connect teachers to teachers and teachers to students in a more productive way, and students - by making changes in the technology that they use - can become engaged actively in the process of problem-solving (Kaput & Thompson, 1994). However, technology integration does not come without barriers and challenges. Some barriers may include financial cost and student or teacher access, and the teachers should consider when it makes sense pedagogically to integrate technology in teaching and learning mathematics (Heid, 1997).

The integration of technology brings some complexities to the field. Many different factors affect the process of implementing technology in our classrooms, such as social, political, economic, and cultural (Assude et al., 2009). To achieve equitable teaching as a goal, educators should be aware of and intentionally address those factors and complexities to prevent potential problems related to equity.
Technological and Pedagogical Knowledge for Teaching

Those features of using technology to teach and learn mathematics gave place to many questions that needed to be asked. Some of the most critical questions were about the new pedagogical concerns and standards for teachers on integrating technology in teaching mathematics and mathematics classrooms. What should be the technological standards for teachers to get prepared to use technology in their classrooms? What kind of other knowledge should teachers/educators have to design a mathematical course incorporating the use of technology?

Niess (2005) adds four components to teachers' pedagogical content knowledge (PCK):

“1. The concept and meaning of integrating technology to teach a particular subject and its effect on students' learning outcomes.
2. Strategies that can be used in teaching specific subjects by the use of technology.
3. Knowledge of students' sense-making with technology.
4. Knowledge of curriculum changes when integrating technology” (p. 511).

Mishra and Koehler (2006) built a framework based on Shulman's (1986) teacher content knowledge model by adding technological knowledge (TK), technological content knowledge (TCK), technological pedagogical knowledge (TPK), and technological pedagogical content knowledge (TPCK). A model in which the content and pedagogy parts are integrated with technology will be more productive, resulting in better teaching/learning outcomes (Lee & Hollerbrand, 2008; Rakes et al., 2022). The model can be seen in Figure 2.
Niess and Colleagues (2009) introduce a developmental model for TPACK after observing teachers’ progress with integrating technology into their mathematics classrooms. They presented five stages for their developed model: Recognizing, Accepting, Adapting, Exploring, and Advancing. Those items respectively refer to recognizing relationships between technology and mathematical content, agreeing or disagreeing with the use of a specific technology for teaching and learning mathematics, engaging in activities to choose the appropriate technology, actively integrating that appropriate technology with teaching and learning mathematics and evaluating the outcome of the integration of the appropriate technology.

To develop students' skills in using diagrams such as graphs, drawings, and pictures, teachers should know the value of visual representations. While paper and pencil could be helpful for this goal because of the dynamic nature of mathematical sequences and series, they are limited. Technologies such as dynamic geometric software such as GeoGebra and Desmos provide learners with opportunities to go beyond static visualizations. By working with tools
such as sliders, students can vividly observe how a change in one variable affects other variables in an imposed problem.

Even though TPACK provides a seamless framework to design this study and including Technology, more development is needed to be added to Mishra and Koehler's model. Niess and colleagues (2009) introduced a five-stage developmental process regarding teachers' learning to add technology to their classrooms and use it to teach students:

- "Recognizing (knowledge), where teachers are able to use the technology and recognize the alignment of the technology with mathematics content yet do not integrate the technology in teaching and learning of mathematics.
- Accepting (persuasion), where teachers form a favorable or unfavorable attitude toward teaching and learning mathematics with an appropriate technology.
- Adapting (decision), where teachers engage in activities that lead to a choice to adopt or reject teaching and learning mathematics with an appropriate technology.
- Exploring (implementation), where teachers actively integrate teaching and learning of mathematics with an appropriate technology.
- Advancing (confirmation), where teachers evaluate the results of the decision to integrate teaching and learning mathematics with an appropriate technology (p. 12)."

Dynamic Geometric Software
Oldknow (1997) claims that the major event that led the Dynamic Geometric Software (DGS) to enter the educational settings happened in 1985 when Judah Schwartz and Michal Yerushalmy created the Geometric Supposer at MIT. He further mentions that since Geometry had been an important course in the U.S. mathematics school curriculum, the emergence of such software was inevitable. The Geometric Supposer helped students to explore geometrical
entities. As a result, many other DGS were developed such as *Cabri Geometre, Geometer’s Sketchpad, Geometry Inventor,* and *Thales.* The other breakthrough of such technology in education was when Texas Instruments launched its TI-92 graphing calculator using a *Cabri* package to visualize mathematical concepts.

Two of the currently most used DGS are Desmos ([www.desmos.com](http://www.desmos.com)) and GeoGebra ([www.geogebra.com](http://www.geogebra.com)). Desmos is a DGS on an online platform that provides services such as graphing and scientific and matrix calculators to more than 40 million teachers and students annually. Launched in 2011, Desmos is one of the most recent DGS that has been used in mathematics classrooms widely. Being a result of a master's thesis project in 2002, GeoGebra aimed to integrate all the features of the aforementioned software as well as Computer Algebra Systems (CAS). By providing different graphical, algebraic, and spreadsheet views, GeoGebra allows students to work with different representations of mathematical concepts (Hohenwarter & Lavicza, 2010).

Research has shown that DGS such as GeoGebra helps students make sense of mathematical problems (Andreasen & Haciomeroglu, 2013). Software provides different strategies and connects symbolic, algebraic, and graphical representations, which leads to a better conceptual understanding (Zbiek & Heid, 2012). Students' cognitive skills can be developed easier by connecting those representations (Bostic & Pape, 2010).

**Proof and Justification and Calculus**

Several studies have shown that students have difficulties constructing proofs and producing counterexamples in calculus. Ko and Knuth (2009) studied calculus students' abilities and mathematical understandings of proofs within the domain of continuous functions. Finding the majority of the participants have difficulties in solving problems related to proving and
thinking of counterexamples, they suggest more studies should be done on how to teach and learn about proofs. Iannone and Inglis (2010) studied 53 undergraduate students from the U.S. and U.K. They found that within the domain of functions, example-generating tasks of a concept as a pedagogical method did not help students learn to prove mathematical statements.

To know and appreciate calculus concepts, human beings go through a cognitive mental process in which they make sense of functions, derivatives, integration, and connections. Therefore, Tall (2004) provides a framework to describe and explain this mental cognitive process. As it can be seen in Figure 3, His framework consists of three stages/worlds of thinking mathematically.

![Figure 3](image)

*Figure 3. Informed by Tall’s Framework of Cognitive Development of Mathematical Thinking*

The first stage is about our environment and the world around us and its connection to the concepts and properties of mathematical entities we see and make sense of in our minds. This is why this stage is also the conceptual stage, in which we get a visuospatial representation of the mathematical concepts and ideas. To make those concepts and properties thinkable in a way so that we work with them and connect them to procedures, we symbolize them. We can think about concepts, manipulate them to develop procedures and communicate using symbols. This world is where the learner takes actions to develop thoughts through symbols. This stage is named the procedural-perceptual world. Mathematics is an axiomatic system, meaning that we
build up new valid statements and arguments by using logical reasoning, starting with certain axioms and propositions that we all, as the math society, accept and them being true. The last world, which is called the axiomatic/formal world, is built on "formal definitions and proofs". At this stage, the learner develops new meanings and turns them into formal concepts (Tall, 2004).

Tall (2006) connects this framework to humans' natural social experiences and abilities that we gain in our everyday life and communicating with others: Recognizing patterns, repeating them, and using language to talk about them and clarify the patterns and their meanings.

**Technology and Calculus**

The roots of calculus can be traced back to 350 years ago, being one of the recent fields developed compared to other topics such as number theory, algebra, and geometry. Among all different mathematical subjects, calculus is one where the integration of technology has happened sooner and developed further compared to other topics (Tall et al., 2008). Technology has been integrated into calculus in many different forms: to visualize concepts and theorems dynamically, to program conceptually, or as a way to work easier with symbolic and numerical representations. In teaching and learning mathematics, technology itself is not that helpful unless the way to introduce and use it becomes more vivid, more intentional, and at the service of students and teachers. That integration should be executed in a way to not only help students with calculations and visualization but also assist them in developing both their procedural fluency as well as their conceptual understanding (Bressoud et al., 2016).

The calculus reform movement recommends the use of technology in teaching and learning calculus. With the help of technology, students can better understand the applications of calculus because laboratory courses allow them to investigate and explore more, which leads to a
better conceptual understanding (Schoenfeld, 1994). Bringing technology to classrooms might cause some complexities, and one is the problem of working with technology itself. Baraza and Abel (2016) studied 35 calculus students before and after integrating technology into their classrooms. When they asked students about their comfort level of using technology, every student responded that they have either more or the same confidence level in using technology while studying calculus.

Kendal and Stacey (1999) investigated the role of using technology in teaching calculus and students' outcomes. In their study, students had access to technologies such as calculators and Computer Algebra Systems in their learning environments, both at home and school. The course was designed with the help of three teachers who were involved in their studies. One used paper and pencil, the other focused on integrating CAS in the classroom, and the third teacher was more willing to use calculators to visualize concepts and theorems.

Using the calculator to draw diagrams, the teacher used graphical explanations and representations more often than the other two teachers. Although the assessment showed that students shared almost the same mean score on the mathematical tests, their responses to questions differed. Students who had the opportunity to explore and examine the material more visually showed a better and deeper understanding of calculus concepts. In contrast, the other students were better at procedures.

Flores and Park (2016) used GeoGebra to reinvent mathematics education majors’ definition of limit, a basic yet fundamental and challenging concept for students to learn calculus. Through the use of technology, students could deepen their knowledge and also grasp
an excellent comprehension of the formal definition of limit. They all had an informal understanding of limits that, in some cases, were not complete nor correct.

In Hahkioniemi’s (2004) study, students started to build their understanding of calculus concepts (derivative in that case) by using and working with all kinds of perceptual representations. This allowed them to see the concept as an object, which in the long term, enables students to take actions that eventually help to connect concepts to their symbolic representations. However, Hahkioniemi underlines that students still need some guidance from their instructor as depending on the subject (limits in this case), it might be difficult for students to see the relationship between the different representations.

All the studies related to calculus mentioned above equip students with visual representations. Therefore, the reasoning and seeing their connections deepen students' understanding of calculus concepts, which aligns with the calculus reform movement (Haciomeroglu, Aspinwall & Presmeg, 2010).

Proof and Justification and Technology

Senk (1985) reported that after studying 1520 students' ability to prove geometric theorems, only 30% of them reached 75% mastery of proof, and about 30% were not able to write even one proof. Through a five-year project funded by the National Science Foundation, named Proof in Secondary Classrooms (PISC), it is shown that nowadays, compared to 1985, more students are successful in writing proofs, and fewer are reluctant to learn proofs (Cirillo & Hummer, 2019). One of the reasons for this, as mentioned, is integrating technology into mathematics classrooms.

Technology is linked to proof & justification through facilitating visualization. Although Tall (1991) states that suggesting false theorems is a weakness of visualization, he asserts that
mathematicians need to think visually to prove and generalize theorems. We, as educators, need to help our students to grow that skill too.

Although using technology seems to be promising in understanding proofs, there are some concerns regarding using technology in understanding proofs. Could technology undermine students' appreciation of mathematical proof? Chazan (1993) claims that the answer to this question is "no." In his interviews with students, he states that he is convinced that even extensive use of technology helps some questions be raised that students would not ask in a situation without technology. He goes a step further and states that by answering those questions, students will grasp a good understanding of "important and unique" aspects of mathematics, such as proof.

So far, it has been claimed that working with different representations is vital in teaching and learning mathematics. Also, the importance and necessity of integrating technology to help students with their visualization skills and enriching their understanding to achieve that goal have been emphasized. However, it needs to be mentioned that another fundamental practice for students is to see the connections between different representations and relate the perceptual representations to the symbolic ones (Gray & Tall, 2011).

Technology serves four important roles in the process of teaching and learning mathematics: The first one is promoting different steps in the process of proof; meaning exploration, constructing conjectures, and finally checking the validity of conjectures by either proving or disproving them. The other important role of it is connecting different representations, and the last two are supporting reasoning when investigating cases and examples and serving as a tutee (Cullen et al, 2020).
Conceptual and Analytical Frameworks

The main components of the study that interplay with each other are preservice secondary mathematicians, the intentional use of dynamic interactive technology such as GeoGebra, and the experiences that pre-service and in-service teachers gain by exploring proving and justifying calculus concepts and theorems. One of the indirect goals of this study is to help students make sense of calculus concepts. To achieve that goal and as it is mentioned through this chapter, more studies need to be done to investigate teachers’ beliefs and perceptions of proof and justification, role of technology and the interplays between them. Therefore, it could be said that this study is specifically interested in the interplay between pre-service and in-service secondary teachers and their experience with proof of justification, how they think of proof of justification when the subject matter is calculus concepts and how having those experiences influences their perceptions. As shown in Figure 4, additional interplays that will be analyzed are technology’s effects on both participants and their experiences with proof and justification. Teachers are the main focus of this study, because the aforementioned goal cannot be achieved without teacher preparation programs and providing pedagogical support so that they can support their future students’ reasoning, sense making, and appreciation of proof and justification as well as technology and visually representing mathematical key ideas. This study fuses different mathematical teaching and learning aspects to investigate the research question, exploring secondary mathematics teachers' understanding of mathematical proof and reasoning. The different aspects that are mentioned are justification and proof, teacher content knowledge, and technology and dynamic visual representations. The literature review supports each aspect that this study focuses on. For example, Knuth (2002) found that one reason that might hinder teachers to teach proofs could lie behind their beliefs towards proof. According to Cirillo and
Hummer (2019), we know that one reason that students are more successful in writing proof compared to students in 1985 is the use of technology, which is in line with Dick and Hollerbrand’s (2011) findings that state the strategical use of technology improves teaching and learning mathematics. This study tries to dig deeper and investigates different parts of this dynamic setting (Figure 4) of teaching proof.

![Conceptual Framework](image)

*Figure 4. Conceptual Framework*

As mentioned before, the mathematical subject chosen for this study is calculus concepts. Specifically, geometric sequences and series are chosen for specific and intentional reasons. First, mathematical sequences and series are calculus concepts that are dynamic in nature. Secondly, students have experiences working with sequences and series – explicitly or indirectly throughout various grades beginning at the elementary grades and proceeding to secondary grades. Third, geometric series connect with concepts that students need to explore, make
Students are exposed to these dynamic concepts from the beginning of their K-12 education in mathematical classrooms. Even though students do not formally learn the names of different mathematical series, they do learn a basic arithmetic sequences at the elementary level when they start counting by ones. Students explore other arithmetic sequences when they skip count by other numbers starting from a certain number. Later on, students work with geometric sequences when they start from a number and multiply it repeatedly by a specific number. Students get introduced to a more formal form of sequences and series at the secondary level when they take Algebra courses and are requested to work with different numerical patterns and analyze how such patterns will continue.

Students have different kinds and levels of experiences with the mentioned concepts in their K-12 school years. They use both inductive and deductive reasoning and need to explore, conjecture, generalize and prove. Therefore, it can be assumed that the selected content is rich enough for educators to design a series of tasks investigating teachers' beliefs, content knowledge, and pedagogical knowledge.

Considering the definition of mathematical reasoning and proof selected for the purposes of this study, the selected content is directly related to mathematical activities and thinking including:

1. Identifying patterns
2. Making conjectures
3. Checking if the conjectures are true or false and generalizing
4. Proving arguments and checking the validity of the provided generalizations
Stylianides's (2008) analytic framework of proving and reasoning is well suited to this study's core theoretical framework. It has been the main framework to choose and design problems, tasks, and interventions.

The provided reasons and the alignment between the mathematical component of Stylianides's framework and sequences and series support the decision to select this as a theoretical framework. Since the aim of this study is to work with pre-service and in-service secondary mathematics teachers, the pedagogical component of the selected framework benefits the researcher in investigating participants' thoughts about teaching mathematical proof through providing them with productive challenges (Abbaspour & Safi, 2021).

One important issue related to reasoning and proof is the distinction between inductive reasoning and deductive reasoning and the ways in which preservice teachers think about these different kinds of reasoning. It is also critical to know their point of view about the relationship between those kinds of reasoning and how learners could benefit from each type of reasoning to comprehend the notion of proof better.

The framework itself gives room to connect different modes of reasoning/tasks to prove itself. Stylianides (2005) has studied the curriculum and analyzes the provided opportunities embedded in the curriculum that is connected to reasoning and proving. In his dissertation, Stylianides (2005) divides reasoning and proving tasks into two categories. In one category, learners start with widely accepted truths and proven arguments using inductive reasoning. Patterns and conjectures fall under the other group. This kind of reasoning begins by observing and analyzing a few limited cases, observing and determining the patterns, trying to make
conjectures about the patterns, and as the last stage, further the process by making generalizations mathematically (Roy et al., 2015).

To grasp a deeper understanding of proof, teacher educators need to work with teachers to develop their comprehension of inductive and deductive reasoning and investigate connections between them. They should also make a transition within the deductive mode from providing students with the solutions only or giving them the solutions with some additional guidance (Stylianides, 2015). Merely giving the solutions doesn't allow learners to understand the nature of proof and therefore build a sense of appreciation for this key element of mathematics.

Although inductive reasoning and deductive reasoning are different, they do share aspects. In both modes of reasoning, learners work with key ideas. As mentioned in the other chapters of this study, exploring key ideas is one significant difference between mathematicians and students when it comes to proof (Raman, 2006). However, it is unrealistic to expect one to construct all the key ideas themselves. Students face difficulties when they participate in activities related to proving mathematical statements. One reason that they face obstacles at different stages of reasoning and proving is that they don't know what to do when they encounter an obstacle; for instance when they don’t find a key idea. Key ideas show students what needs to be done to go logically from one point to the other in the process of proving. It is essential to know what key ideas will be needed and why. Therefore, approaches to introducing key ideas are important because handing the students the key ideas is no different from giving them the solution and depriving them of understanding the value of proving. Students need teachers to have an appreciation of exploring key ideas using different representations. In that way, pedagogically, they can help students understand them by working with other representations.
CHAPTER THREE: RESEARCH DESIGN AND METHODOLOGY

This study plans to use qualitative methods to examine and investigate how using technology and providing dynamic visual representations of calculus concepts could potentially enrich secondary teachers' perceptions of mathematical proof and justification.

To design this research, a collective (multiple) case study approach was used (Creswell, 2008; Yin, 1994; Merriam, 1998; Stake, 2006) to investigate how by providing visual representations of key ideas and through integrating technology, educators can address teachers' understanding content related to calculus concepts and pedagogical knowledge regarding proof and justification. This study addresses the following research question at the center of this study was:

- In what ways can dynamic technology integration and visual representations of concepts and ideas support pre-service and in-service secondary teachers’ experiences and beliefs regarding the process of proving calculus concepts?

In 1986, Shulman addressed the missing paradigm, which refers to a blind spot in most research in teaching. The blind spot is the of research on the relationship between teachers’ understanding of mathematical content and the education they provide for students. To address this, Shulman suggested that there was a need to distinguish among three different kinds of knowledge: subject matter content knowledge, pedagogical content knowledge, and curricular knowledge. Beyond the subject matter knowledge, pedagogical content knowledge includes subject matter knowledge for teaching (i.e., understanding what makes a particular topic easy or difficult to understand). One year later, Shulman (1987) introduced his pedagogical reasoning and action model, consisting of a cycle with six crucial stages: comprehension, transformation, instruction, evaluation, reflection, and new comprehension. Inspired by Shulman, Ball, Phelps,
and Thames (2008), refined the popular concept of pedagogical content knowledge to a broader concept named mathematical knowledge for teaching. In their framework, knowledge of content and teaching (KCT) combines teaching mathematics and knowing mathematics (Figure 5).

![Mathematical Knowledge for Teaching (adapted from Ball, Phelps & Thames, 2008)](image)

*Figure 5. Mathematical Knowledge for Teaching (adapted from Ball, Phelps & Thames, 2008)*

Ball, Phelps and Thames's (2008) framework for Mathematical Knowledge for Teaching (MKT) eventually led the field to construct a framework for proof and justification as well. Based on Knuth's (2002a, b) and Steele's (2006) work, Steele and Rogers's (2012) created a framework for Mathematical Knowledge for Teaching Proof (MKT-P). Even though they claim that their framework doesn't address teachers' complete domains of knowledge to teach proof, it is a good starting point to frame teachers' knowledge about proof and justification (Figure 5).
The main four components of MKT-P are defining proof, identifying proofs and non-proofs, creating mathematical proofs, and understanding the role of proof in mathematics. Despite the existence of different definitions for proof, Steele and Rogers (2012) work with a definition of proof driven from several research articles:
"a proof is a mathematical argument that is general for a class of mathematical ideas and establishes the truth of a mathematical statement based on mathematical facts that are accepted or have been previously proven (p. 161)."

The framework suits this study because both the elements mentioned for proof knowledge and the represented criteria for evaluating components are consistent with the study's main theoretical framework, which is Stylianides's framework.

**Proposed Research Design**

This study aims to investigate the effect of using DGS as an intervention to build and make sense of proofs related to calculus concepts and theorems associated with it through dynamically visualizing key ideas. This study requires gaining insight into secondary teachers’ thinking including conceptions of calculus concepts and theorems. It is also important to understand not only pre-service and in-service secondary teachers’ perception regarding the construction of proofs but also the ways in which understanding proof and justification may translate to teaching and learning of proof at the secondary school level.

According to Creswell (2014), since the topic of the study has not been studied before and the goal and focus have been narrowed down in a new and authentic way; the study merits a qualitative approach and design. Therefore, a qualitative research approach was used to further shed light on this research endeavor. Specifically, this study employed a qualitative research design (Yin, 1994). A multiple case study was selected for this research study (Merriam, 1998; Creswell, 2013). The reason for that selection was the fact that a multiple case study allows researchers to achieve an in-depth multi-dimension comprehension of the studied subject (Stake 2006).

Prior to using data collection methods, and after defining the purpose of the study and constructing the research question, the researcher reviewed the literature review regarding the
three main topics: proof and justification, technology, and calculus as well as any intersection among them. To select the cases from the pool of participants a maximum variation method was used (Creswell, 2013; Safi, 2009; Abassian, 2018). To record each case in details and to decrease the chance of missing information multiple sources of information were collected from the participants, which allowed the researcher to study each case entirely and thoroughly (Yin, 1994).

In analyzing the data, the researcher first focused on each case separately to find any emerging themes, the researcher also interpreted those themes and findings case by case. As the next step he conducted a cross-case analysis, in which the themes that were found was analyzed in a cross-case analysis (Creswell, 2013).

**Sampling and Participants**

The population selected for this study was undergraduate and graduate students enrolled in secondary mathematics education programs at a metropolitan research university in the southeastern United States. The sample chosen from the population included those who had finished core mathematics content courses, including the Calculus sequence and at least a logic and proof course.

Students who were enrolled in the *Teaching Algebra in Secondary Schools* course were selected as this study's research population. The 19 students in the class varied regarding their major, mathematics background, and being a graduate or undergraduate student. The students in the class who gave consent to be involved in the study were divided into two groups, graduate students and undergraduate students. Of the undergraduate students, those who had taken the Calculus course sequence as well as a course in logic and proof were selected as potential participants. The group of graduate students were divided into two groups as well. One with
students who also had at least one year of teaching experience and the other one included students who had not. One graduate student was selected from both groups (graduate students with teaching experience and graduate students without teaching experience) to be part of the study to comprise the different cases from the graduates. Eight students (five graduates and three undergraduates) of the total 16 students met the criteria and were purposefully chosen. Therefore, it could be said that this study has used a purposeful sampling method to select its participants (Creswell, 2013).

Figure 6. Participant Selection Criteria
A maximum variation approach was used in selecting the cases. Using a purposeful sample allows the researcher to sample cases or individuals that vary from each other characteristically (Palinkas et al. 2015). This study's method to select participants was convenient, too, because all the participants were willing to participate in the study and were asked to provide consent. The study aims to focus on three individuals to serve as case study participants. Eight students were selected to ensure access and opportunity to gather data for multiple case study participants. So, if some who chose to participate would not be able to continue their participation during the time window of the study's data collection, the likelihood of having a case study with three participants was greater. According to Burkholder and colleagues (2020) “in a multiple-case study, having three to four distinct cases for comparison is probably the most cases that one can realistically handle (p. 248)”. Therefore, the researcher in this study aimed to have three participants from the pool of eight (five graduate students and three undergraduate students) who gave consent to participate.

**Researcher Positionality**

In this study, the researcher’s role is merely being an observer and facilitator of the course in which the study takes place. The researcher was not the instructor of the course nor the teaching assistant of the course. The researcher interviewed the participants, recorded the data, and facilitated some of the small group conversations and classroom conversations. The instructor did not grade any course assignments or have any control or say in students’ course grades, assignments, or progression in their programs.

At the time of the study, the researcher had eight years of teaching experience, mostly in secondary schools. He also had been a graduate teaching assistant and instructor of record for three and a half years for several content courses and one methods course at the institution that
this study was conducted. Among the eight participants of the study, only one of them previously had a course with the researcher. However, that one person was not selected to be one of the final three selected participants. The researcher is also biased towards the positivity of integrating technology to deepen teachers’ understanding of dynamic concepts such as calculus concepts. However, in this study, the researcher didn’t have any biases towards participants’ perceptions of the process of proof other than what he had studied for the literature review.

Data Sources

Before conducting the research, the researcher obtained the course instructor’s permission to conduct this study in his classroom. In the next step, the researcher requested the Institutional Review Board (IRB) to review the study’s structure and protocols. The IRB granted the researcher permission to conduct the study, and upon receiving the permission, the researcher started the study by going on-site, inside the classroom.

The plan for this study to collect data considered using different methods of data collection. At the beginning of the study, participants received consent forms and were asked to participate in the study. The timeline of the study for the following interventions is presented in Figure 8.

- Participants completed questionnaires, answering questions related to educational background and course work they have taken.
- Audio and video recording took place during the multi-week intervention
- Artifacts of student thinking were collected
- Interviews took place before and after the intervention to investigate students’ thinking of concepts and proofs
The data collection included one pre-interview (Pre-I) and three post-interviews (Po-I). It also included students' recorded videos while working on class assignments. The researcher also gathered students' artifacts that were submitted through the online system of the course. Students needed to work on three assignments (As) for the study, and they were required to think aloud and record themselves while working on them. After submitting each assignment, one of the post-interviews took place that was related to that certain assignment's goal. The researcher also audio-recorded and video-recorded the whole class (Cl), especially focusing on group discussions and whole-class discussions. Each assignment was designed in a way that participants needed to solve a task and submit it to get access to the next task. The interviews were semi-structured, allowing for the number and selection of questions to be modified based on the students' responses to class activities, the assignment, and their reflection.

In chapters one and two, the importance of teachers' perceptions and beliefs towards proof was briefly discussed. Researchers have shown that there is a strong relationship between teachers supporting argumentation in mathematics classrooms and how they view proof, the purpose of proof, and its role in school mathematics (Conner, 2007; Knuth 2002). They have also shown that many teachers do not believe in proof's central role in mathematics classrooms, and they think that proof and proving activities should be limited to a certain group of students in mathematics classrooms since not all students are able to understand proofs and construct proofs. In their opinion, a general mathematics classroom is not necessarily a class in which proof should be introduced to students, and proving activities should remain in advanced mathematics classrooms where those students who can understand and construct proofs are present (Hoyles, 2000; Knuth, 2002b). Because of those findings and prior to having the pre-service and in-
service secondary teachers work on the assignments related to proof, and in the pre-interview, the researcher aimed to understand participants' beliefs and understanding of proof and proving activities, the role of proof in mathematics classrooms, if they thought that all students could learn proof. In the pre-interview, the researcher also tried to investigate participants' experience with proving tasks. The pre-interview took place in weeks four, five, and six of the class based on participants' convenience, immediately after obtaining students' consent to participate in the study. The pre-interview lasted between 60 to 90 minutes for each participant. Some of the questions related to proof that was asked from the participants in the pre-interview are shown in Figure 7. All the mentioned interviews were recorded.

**Table 3. Participants’ participation in different tasks during the study**

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Inspired by Stylianides’s analytical framework of proving and reasoning, the researcher considered mathematical proof as a process that includes identifying patterns, constructing conjectures and providing non-proof arguments, as well as constructing proofs. Using that framework led the researcher to ask about participants’ beliefs about those entities and also investigate their knowledge and experience about those items in the pre-interview.

Since the framework mentioned above is the central framework of this study, the design of the assignments that the participants and other students in the class needed to work on were aligned with the framework. After the pre-interview, all students were asked to work on the first assignment. They had one week to work on the tasks and submit their work online. The assignments were designed based on Stylianides’s (2008) analytical framework for reasoning and proof. There were three tasks, and each one was related to only one component of the process of proof. According to the analytical framework for reasoning and proof, the first one was about identifying patterns, the second one was related to constructing conjectures, and the last assignment included activities related to investigating conjectures and trying to prove them. Participants were required to work on a problem using paper and pencil and submit their work
online. They were also asked to work with DGS and audio and video record themselves on Zoom when working with the DGS on their computers. The researcher had asked them to think aloud while making the recordings.

The other data source collected in this study was audio and video recording of the three classes after each assignment was completed. The discussions in those classes were about the tasks that students worked on in those assignments. Those discussions took place in weeks six, eight, and 11 during the semester. The researcher observed the class and focused on two target groups specifically.

**Data Collection Procedures**

This study used multiple data sources, such as observation, video recordings, computer screen recordings, interviews, teachers' artifacts such as submitted assignments, reflections, and interviews. The researcher of the study, in this case, is an observer and facilitator of the Teaching Algebra in Secondary Schools course that participants are enrolled in and were selected from. The designed tasks and assignments that were used in this study were all part of the course's required assignments, tasks and classroom activities. However, if a student elected not to participate in the study, that student's work would not be included in the data collection. They were also told by the researcher that it was their right to opt out of the study at any time during the study. It was explained to them that the study is part of a graduate students’ dissertation preparation, and there are no financial source or incentives involved. Also, students were informed that either participating or not participating in the study would not affect their grades in learning outcomes.

The data collected and the schedule in which collecting data took place can be seen in Table 2 and Figure 8. The data collection process of the study started and ended with
participants’ interviews about their perception of the notion of proof, its importance, kinds of reasoning, the role of diagrams, key ideas, technology, differences between concepts such as reasoning, argumentation, justification and proof and how they think is best to teach those concepts to students. For this purpose, and inspired by Knuth's works (2002), a protocol was developed for the interviews. It included questions to prompt participants' thinking and helped the researcher gaining additional insights into pre-service and in-service teachers’ conceptions related to proof and justification. Interview questions for the use of technology and in regards to the TPACK framework were inspired by Mudzimir's (2012) study on developing secondary PSTs technological pedagogical content knowledge.

During the class and for the assignments, participants were reminded about the think-aloud method. The think-aloud method has its disadvantages since every thought process should be represented verbally. According to Charters (2003), because some thinking happens faster than we can talk about it, there is simply no chance to represent it using words. Also, there are always some unconscious processes going on in our minds that we are not even aware of to verbalize (Sugirin, 1999). As a research tool, the think-aloud technique is one of the most successful tools to gather data on the human thinking process and analyze that data (Olsen et al. 1984). Ericsson and Simon (1980) also argue that this method is very reliable when the human thinking process is the subject of the study. To record participants' thinking while working on tasks, they were asked to work on specific assignments during the course of three weeks. In those assignments, they worked with DGS and tasks that were designed for the course and the study. While working on the tasks, they were on Zoom and were asked to think aloud while working with technology, and record their work. This allowed the researcher to virtually be with the
participants and reminded them to think aloud. At the same time, this design assisted with the data collection process since participants' thinking could be recorded along with what they did when using the technological instrument.

Following students’ individual work on assignment including the recording of the individual think-aloud, the researcher facilitated small group discussions and then whole-class discussions around the individual activities that participants explored. Students' interactions were video recorded in class, and the researcher collected their artifacts and any notes they have created. Artifacts included their interaction with the DGS. Therefore, and to account for the thoughts that students might have but were not be able to verbalize, their computer screen was recorded. By doing that and aligning their think-aloud data with what they did on their computers, the researcher sought to provide a more precise and accurate account. The timetable of the study can be found in Figure 8. It must be said that the pre-interview took place after posting the first assignment. Even though that the first task was about exploring two geometric series and visualizing them and the researcher purposefully hadn’t mentioned any words such as proof, justification, conjecture and etc. it must be mentioned that it could have affected participants responses. The same holds true for the other interviews and assignments. It must be taken to account that despite not knowing the time the selected cases had first opened the
assignments, they all submitted each of their works after each set of interviews.

<table>
<thead>
<tr>
<th>Week of</th>
<th>Mode</th>
<th>Course Topics</th>
<th>Undergraduate PST</th>
<th>Graduates</th>
<th>Pre-Interview</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Jan 10</td>
<td>F2F</td>
<td>(Sky)</td>
<td>(Dream)</td>
<td>(Mamba)</td>
</tr>
<tr>
<td>2</td>
<td>Jan 17</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Jan 24</td>
<td>F2F</td>
<td>Introduction of the study + Consent Form</td>
<td>CF 1/26</td>
<td>CF 1/27</td>
</tr>
<tr>
<td>4</td>
<td>Jan 31</td>
<td>F2F</td>
<td>Consent Form</td>
<td>CF 2/5</td>
<td>Pre-Int 2/4</td>
</tr>
<tr>
<td>5</td>
<td>Feb 7</td>
<td>MM</td>
<td>Assignment 1 Geogebra Tech Activity 1</td>
<td>Pre-Int 2/9 Pre-Int 2/11</td>
<td>CF 2/7</td>
</tr>
<tr>
<td>6</td>
<td>Feb 14</td>
<td>F2F</td>
<td>Classroom Discussion 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>Feb 21</td>
<td>MM</td>
<td>Assignment 2 Geogebra Tech 2</td>
<td>Int 1 2/23 Int 1 2/25</td>
<td>Int 1 2/22</td>
</tr>
<tr>
<td>8</td>
<td>Feb 28</td>
<td>F2F</td>
<td>Classroom Discussion 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>March 7</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>Mar 21</td>
<td>F2F</td>
<td>Classroom Discussion 3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>Mar 28</td>
<td></td>
<td></td>
<td>Int 3 3/31 Int 3 4/1</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>Apr 4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Figure 8. Data Collection Time-Table*

**Trustworthiness**

Showing that this study is credible is not enough for claiming that it can be trusted. Therefore, enough information is provided so if any other researcher wants to replicate the study, they know how to design the study and get consistent results. The tasks are introduced at the end of this chapter. It needs to be mentioned that he design of the study including the tasks, deadlines, interviews and the schedule of the study was planned and conducted by the agreement of the instructor of the course. Providing information on details of data collection methods and data analyzing methods is an important step towards ensuring the trustworthiness of the study. In
this way, readers can follow them to find the origin of every decision that the researcher made. By doing this, the possibility of results being affected by the researcher's bias will be minimized as well (Burkholder et al. 2020).

This allowed for triangulation of the data, helping the study to not only identify the complexity and authenticity of cases (Yin, 1994), but also to ensure the internal validity of the study (Cresswell, 2013). By providing information about his positionality and bracketing - refraining from judgement (Moustakas, 1994) - the researcher has tried to be as neutral as possible in the process of collecting and analyzing data through mentioning and being aware of his biases. The researcher also has tried to be as clear as possible about the procedures and interventions of the study by providing the tasks, the interview questions, so the study could be repeated by other researchers to check the consistency of the results (Burkholder et al., 2020).

As mentioned before, for the credibility of the study, the researcher used the triangulation in conducting the study through using multiple data sources such as audio and video recording, interviews and teacher artifacts. Different kinds of data were collected to support the truth of the claimed existing patterns and categories. This allowed the researcher to investigate how well the collected data from various methods of data collecting (for example, the data collected by interviewing the participants and the data produced by observation) supported the study’s claims (Creswell, 2013).

**Data Analysis Procedure**

A qualitative analysis of interviews and video and audio recordings during the intervention was conducted in order to construct a theme of students' understanding of formal proofs involving calculus concepts using technology. All of the transcribed data was reviewed independently case by case by the researcher to identify common themes and patterns. For each
case the researcher also noted if any statements/arguments were repeated; the researcher added those to the collected themes. In the next phase the researcher completed a cross-case analysis. For this purpose, the researcher also noted the repeated patterns and constructed provided themes (Marshall and Rossman, 2014).

This study uses Cobb and Yackel’s (1996) interpretive framework for the qualitative analysis of collected data. The reason for that decision comes from the emergent perspective in which learning is a phenomenon that has both individual and social dimensions. According to the emergent perspective and associated learning theory, the psychological perspective of learning and social learning is intertwined, and they depend on each other. So, the social and psychological perspectives cannot be isolated and studied without the other. In this study, both tasks were designed in a way that both dimensions were accounted for. Participants worked on certain assignments and DGS activities individually and then interacted with their peers in the class with a facilitated discussion. Both the individual and social dimensions together, and their interactions built participants’ knowledge (Cobb, 2000; Cobb and Yackel, 1996; Roy et al., 2014). According to Cobb and Yackel (1996), individual learning and social learning happen at the same time and none of them has priority over the other one.
As it can be seen in Table 4, both social and psychological perspectives are divided into three subgroups. The social perspective includes social norms, sociomathematical norms, and classroom mathematical practices and the psychological perspective consists of the individual’s beliefs about his or her role, others’ roles, and the general nature of mathematical activity; mathematical beliefs and values; and mathematical conceptions and activity. In this study, the focus is on participants’ understandings and beliefs about mathematical justification and proof. Therefore, and even though the setting of the study is designed considering social aspects of learning, in this study, mathematical beliefs and values as well as mathematical conceptions and activity components from the framework were used to analyze the data solely. This study also uses Dick and colleagues’ (2020) conceptual framework for teacher noticing of students’ work in a technology-mediated environment to analyze the data.

After transcribing the raw data, common themes were found in the pre-interview for each case. The themes were about participants’ beliefs and perceptions of proof, justification, reasoning, technology, calculus and also their experiences with each of them both as a K-12 and as a college student. In the process of coding each mentioned key word was highlighted and the participants’
responses were collected. In the next phase the researcher focused on each concept only, read the responses carefully, interpreted them and recorded the interpretation. Same process was conducted for each notion in each case.

In each post assignment interview, and within each case, the researcher derived the common themes and repetitive instances that resulted in the emergence of themes. Also, and for each post assignment interview the researcher extracted sub-themes that might support the main emerging themes. For the assignments and think-aloud submissions the same approach was used by the primary investigator.

The coding structure that was used to analyze the data in this study was inspired by Bailey and colleagues’ (2021) coding structure. Similar to their rubric the designed coding structure of this study, has three different levels “lacking”, “limited” and “robust”. By integrating Raman’s (2003) work, each level was assigned to show participants’ understanding of the problem, ability to visualize them, and solve them through connecting the visuals to the key idea involved in solving the tasks.

As it could be seen in Table 5, there were two categories defined in the coding structure; one for the tasks in which the participants were required to work on the problems only using paper and pencil, and another one that they worked with DGS. It could be said that the latter was more inclusive compared to the first one and it included the first category in some ways. The researcher had considered three items in coding the interpreted data: Visualizing the task by the use of paper and pencil, making sense of the DGS and connecting the ideas behind each solution to the DGS they worked with.
Table 5. Coding structure of participants’ adapted from Bailey et al. (2021), and Raman (2003)

<table>
<thead>
<tr>
<th>Solving problems</th>
<th>Robust</th>
<th>Solving the task and providing visual representations</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Exploration, making</td>
<td>Limited</td>
<td>Solving the task but not providing visual representation</td>
</tr>
<tr>
<td>conjectures, Proving)</td>
<td>Lacking</td>
<td>Not solving the task</td>
</tr>
<tr>
<td>Visual Representation</td>
<td>Robust</td>
<td>Drawing visual representations, making sense of the DGS, connecting the DGS to the concepts</td>
</tr>
<tr>
<td>with technology</td>
<td>Limited</td>
<td>Not drawing visual representations, making sense of the DGS, connecting the DGS to the concepts</td>
</tr>
<tr>
<td></td>
<td>Lacking</td>
<td>Not drawing visual representations, Not being able to make sense of the DGS, Unable to connect the DGS to the concepts</td>
</tr>
</tbody>
</table>

Tasks

Three assignments are designed for this study, and each assignment has a different topic and numbers of tasks. The details about the tasks and topics of the assignments could be found in table 4.
Table 6. Topics and number of tasks in each assignment

<table>
<thead>
<tr>
<th>Assignment</th>
<th>Topic</th>
<th>Number of Tasks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st Submission</td>
<td>Exploring Patterns</td>
<td>5</td>
</tr>
<tr>
<td>2nd Submission</td>
<td>Making Conjectures</td>
<td>2</td>
</tr>
<tr>
<td>3rd Submission</td>
<td>Proving Conjectures</td>
<td>2</td>
</tr>
</tbody>
</table>

Assignment I, Exploration

Connecting concepts to real world situations is a key factor in teaching and learning mathematics (Abbaspour et al., 2021). Therefore, and as the opening of the tasks a real world problem was chosen. The tasks in this study start with presenting a mathematical modeling problem. The problem was selected from NCTM's (2000) *Principles and Standards for School Mathematics* (p.303) since it directly connects to geometric sequences and series and includes a real-world situation. The problem is as follows:

"A student strained her knee in an intramural volleyball game, and her doctor prescribed an anti-inflammatory drug to reduce the swelling. She is to take two 220-milligram tablets every 8 hours for ten days. If her kidneys filtered 60% of this drug from her body every 8 hours, how much of the drug was in her system after ten days? How much of the drug would have been in her system if she had continued to take the drug for a year? (p.303)"

Using a medical context, the problem required students to construct a model in which they needed to work with geometric patterns and geometrics series as the connected content (Figure 9):
The purpose of selecting a modeling problem comes from Alibert's (1988) and Schoenfeld's (1994) studies. The tasks are designed in a way to show that mathematical proving is not an irrelevant activity, but that it is a purposeful activity designed to help learners deepen their understanding of mathematics. Modeling tasks are also chosen carefully to show that proof is connected to exploring and discovering patterns and ideas. Therefore, intentionally, a real-world mathematical modeling problem was selected that is connected to mathematical sequences and series. In that way, the mathematical challenge is connected to problem-solving as well.

For the second task in assignment one, students were given a special case of geometric series with first term and common ratio being equal to $\frac{1}{2}$. Students were asked to use only paper and pencil and try to visualize the series and by using that visual, find the sum of that series (Figure 10).

Figure 9. Assignment I – Task I Prompt Questions

Use paper and pencil and provide a visualization of the following Geometric series:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$$

If you come up with a visual idea that is not precise enough in your opinion, please add some explanations about how it relates to the series. Does your visual help you to figure out the sum of the terms above? How?

Figure 10. Assignment I – Task II
In the third task of the first assignment students were asked to work with a DGS that visualized the geometric series given in the previous question. The DGS provided a dynamic visual by having a slider. Students were able to change the shape and see the construction of the visual step by step and after adding each term of the geometric series one by one (Figure 11).

![Figure 11. Assignment I – Task III GeoGebra Activity (adopted from https://www.geogebra.org/m/xajygxfc)](image)

There was a list of prompts posted with the link to the DGS that students could get help from to have a better understanding of what they needed to talk about (Figure 12).
You are required to submit a video ~10 to ~20 minutes. The following might help you with the think-aloud process:

- Before checking the formula play with the sliders: What does each slider do?
- Where could we see the geometric series in the shape?
- Where could we see the sum of the geometric series in the shape?
- How are the series and the sum of them connected to each other in the activity?
- How does the activity help you in proving the conjecture?
- If anything needs to be added to the proof please talk about it and write it as well.
- Was your conjecture valid? If not what changes did it need?

**Figure 12. Assignment I – Task III Prompt Questions**

Task 4 of assignment one was similar to the second task. Students were given a geometric series with the first term and common ratio of \( \frac{1}{4} \) and they were asked to draw a visual representation and use the visual to try to determine an approximation for the sum of the series (Figure 13).

Use paper and pencil and provide a visualization of the following Geometric series:

\[
\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots
\]

If you come up with a visual idea that is not precise enough in your opinion, please add some explanations about how it relates to the series. Does your visual help you to figure out the sum of the terms above? How?

**Figure 13. Assignment I – Task IV**

In the last task of the first assignment, students were provided with another DGS that visualized the second example of the geometric series. Again, by using a slider the DGS dynamically visualized a geometric series with the first term and common ration of \( \frac{1}{4} \). In this DGS, a triangle was used, a triangle with area 1 was used and divided into four equal-sized smaller triangles and always the same iteration happened to the smaller triangles on top after each step (Figure 14).
Students were asked to determine the connection between the dynamic visual representation and the geometric series that could be found in task 4. The prompt questions that were given to students can be found in Figure 15.

You are required to submit a video ~10 to ~20 minutes. The following questions might help you with the think-aloud process:

- How is the visual related to the series?
- What happens when you move the slider?
- Is there a number that the sum of the terms gets close to?
- How does this applet help you to understand the series better?

Figure 15. Assignment I – Task V Prompt Questions

Assignment II, Conjecture

In the second assignment, the class worked on two questions individually. In the first task they were asked to construct their own conjecture for the sum of the geometric series that share the same number as their common ratios and first terms. They were also asked to investigate two
more geometric series and visualize them, one with the common ratio and first term of \( \frac{1}{3} \) and the other one with the common ratio and first term \( \frac{1}{5} \). The first question is shown in Figure 16.

Think about the following series:

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \\
\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots \\
\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots \\
\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \ldots 
\]

a) How would you determine the sum of each series?

b) Is there a case that works for all Natural numbers (\( \mathbb{N} \))?

\[
\frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \ldots 
\]

c) What type of conjectures can you make for the sum of any Geometric series?

d) Please provide a visual representation of your conjecture/general case using only paper and pencil.

**Figure 16. Assignment II – Task I**

The next task provided a DGS in which a visual form of the general case of the geometric series was constructed dynamically. Unlike the first two DGS that students worked with in assignment 1, this DGS had two sliders and was more sophisticated. The students needed to interact with the DGS, connect it to the general form of the geometric series and use it to calculate the sum of the series approximately (Figure 17).
With any DGS there were some prompt questions presented to help students with their think aloud process. Those questions could be found in Figure 18.

*Figure 18. Assignment II – Task II Prompt Questions*

Assignment III, Proof

In the third and last assignment, students were required to check the validity of their conjecture, prove it if it was correct or disprove it, in the case that their conjecture was not working in general. They were also required to change their conjecture and modify it in the latter case, then try to prove the modified version (Figure 19).
In our last activity, you made some conjectures about the sum of the Geometric series that works all the time. Still, we don’t know if the conjectures are true or not, and we need to check their validity.

Please try to prove your conjecture using only paper and pencil. You can also make changes in your conjecture if necessary and prove the new conjecture if needed.

**Figure 19. Assignment III – Task I**

The last task of the proof assignment was the last one in the whole sequence of tasks. Students needed to work with a DGS to make sense of the dynamic visual representation. They were required to connect the DGS to the geometric series, find out the idea behind the proof and make sense of the introduced proof in the DGS. The key idea that was embedded in the DGS’s proof and visualization was geometric with strong algebraic connections. The DGS had used the idea of similar triangles to construct proportions. As it can be seen Figure 16, the symbolic representation was also accessible in the DGS (Figure 20).

![Diagram of similar triangles with labels](https://www.geogebra.org/m/Fx8DpKdx)

**Figure 20. Assignment III – Task II GeoGebra Activity (adopted from https://www.geogebra.org/m/Fx8DpKdx)**
CHAPTER FOUR: RESULTS

This chapter is divided into three sections. Each section is dedicated to one participant/case, respectively Unicorn, Sky and Magic. Each section starts with a brief introduction of each participant’s education level, background knowledge and teaching experience. Then the section proceeds with the pre-interview stage in which the researcher provides the analyzed data about the participant’s beliefs, perception, and knowledge of subjects such as proof, justification, dynamic technology, and visual representation more. After the pre-interview, each section provides information about each assignment, participant’s work and the post interview.

Case I: Unicorn

Pre-Interview

The pre-interview was conducted before the start of the class in which students worked on exploring patterns and geometric series, making conjectures about the sum of the geometric series and proving the validity of those conjectures. Technology software and visually representing the ideas had been integrated in classroom instructions, discussions, and assignments. In the pre-interview, the researcher asked Unicorn about her background, educational experiences, teaching experiences, and also beliefs about justification and proof and the role of technology in teaching and learning mathematical concepts as well as visually representing ideas.

Unicorn is a white female graduate student who had no prior school teaching experiences. Her math teaching experience at the time of the study had included only proctoring AP classes. Being recently graduated with an undergraduate degree in a business-related field, this was Unicorn’s first semester in Mathematics Education’s master’s program. She also had not studied
NCTM’s *Principles and Standards for School Mathematics*. Unicorn’s mom was a mathematics teacher who taught Calculus to high school students and Algebra to senior students who wanted to enter college. Unicorn stated that she loved mathematics and that she had always been in the honors courses. When being asked about her experience with algebra, Unicorn uttered that the structure of algebra and finding the answer to algebraic questions were the characteristics that made her like the subject:

I think… the part that I like the most is that Algebra gave you an answer and I like having that answer… that’s why I like math too much, there's an answer… I like the step by step, the organization that math had, especially Algebra.

Despite emphasizing on finding the whys, Unicorn asserted that one of the reasons that she liked Algebra was its step-by-step procedures in solving a problem. Unicorn’s opinion about calculus was similar to her opinion about algebra; mentioning that she really liked calculus, she described concepts connected to calculus concepts as a subject that deepened her understand of mathematical concepts:

It stretched my brain. I don’t think I ever really had understood the whys behind math until I got to Calculus.

She pointed out concepts such as rate of change as her favorite subjects in Calculus. She also talked about some connected concepts in science such as velocity and acceleration and how visual representations and graphs helped her to understand those concepts better. Unicorn asserted that solving calculus problems required understanding algebraic concepts and being fluent with skills related to those concepts. Asking a question about the difference between algebra and calculus, she replied that one of the main differences was about the simplicity or complexity of the steps one needs to take to solve a problem:
In Algebra, it’s like… okay, if you plugged in the numbers correctly, you get the right answer. It is very one-dimensional, whereas Calculus, you have to look at kind of more than one piece.

According to Unicorn it was during Covid era that she found out she didn’t like to study business anymore and wanted to change her major and study a math related field. Therefore, she decided to pursue a master’s degree in teacher education with a focus on mathematics education. Unicorn’s experience with math in college had mostly been related to work with a lot of graphs and formulas:

Just plug and type situations and that’s not really what I like to do, I like to understand why it’s happening… They just try to cover so much content as fast as possible. A lot of excel-based data. Once you plug it in, then you just can copy and paste and the formulas and the stuff like that… Math side I like teaching it, I like understanding and letting other people understand the whys.

When the researcher asked Unicorn about the difference between school mathematics and college/university mathematics, she mentioned that the way mathematics was taught in university was very procedural while in K-12 school they focused more on the reasons, and it had been more conceptual. Unicorn asserted that focusing on conceptual knowledge is what she really likes when studying a mathematical concept:

We are going to use Excel in daily life so it’s how I am going to use Excel, I don’t necessarily need to know the why behind it, I just need to know that this is the function that you are using and plug in the numbers. So that’s what the university taught me… In high school we learned a lot more of the whys. Well, why is calculus doing what you know, what’s the whys behind it, and I think that was really important in understanding how to do math growing up…

The interviewer explored Unicorn’s beliefs and knowledge of different mathematical entities. Trying to investigate her understanding of justification, Unicorn was asked about what she thought of mathematical justification and its role in school mathematics. While noting the
importance of justification in teaching and learning mathematics, Unicorn asserted her belief by making an example:

I think it’s important that you can justify your math. I found that X equals seven, well, why does X equals seven?... But justifying is important, I think, especially in teaching. Because now, the kids are understanding that this number represents something. It’s not just a number that you’ve found.

The next question in the interview involved her definition and beliefs towards mathematical proofs. Unicorn’s initial response to the question was that “Proof is showing how you get the answer”. Unicorn seemed to not be sure about her way of defining proof; therefore, she was asked about the difference between justification and proof. Despite providing definitions and since the definitions were similar to each other, Unicorn used justification and proof interchangeably. There was no clear distinction between them in Unicorn’s description when she talked about their difference:

I think proof is more numbers, more number based and then the justification is okay well how did you use those numbers and why did you use those numbers, why did you use those steps to get the answer… and you’re proving why you got those numbers.

In the second half of the interview, Unicorn was asked about her opinion on what a mathematical proof is again, and Unicorn mentioned that proving means to talk about whys and reasons behind getting an answer to a numerical question. The interviewer asked Unicorn about a contradiction in the definitions:

Researcher: You use the word “because”.

Unicorn: Uhum.

Researcher: So, we are talking about the whys?

Unicorn: Right.
Researcher: But in the beginning you had a different definition. You told me that proof means that we need to know the hows… Has something changed in the past 20-25 minutes?

Unicorn: I think it is the same thing. The hows and the whys are closely related… This is how I did it, and this is why I did it. If you answer both questions, you are proving yourself.

In order to gain more insight into Unicorn’s perspective and beliefs about proof and its importance, she was asked her opinion about what constitutes proof and when we can say a proof is complete. Unicorn replied to this question by saying “When nobody can ask why again”. Unicorn showed a good understanding of when we can say a proof is constructed and valid. According to what has been showed in the literature review, the concept of proof is also a concept that might be constructed personally or in groups, but it can be checked by the society as a whole which is similar to a lens that has been used in this study. Another evidence that led the researcher to this conclusion was Unicorn’s response to a question related to mathematical arguments. As a response she expressed her idea about at least two people interacting with each other in describing argumentation and proof. She needed to make more clarification about her thinking but there was that social element present in her sayings. Unicorn’s definition of mathematical argument was looser than justification and proof, and according to her answers what argumentation, justification and proving have in common is that they all are about answering the “why” questions. Unicorn also believed that all students are able to learn proofs and mathematical proofs are not some concepts that only a certain group of students could learn. Unicorn was asked about the age that is appropriate for students to start learning about mathematical proofs. She thought that during middle school years, students are at the best age to start learning proofs.
Unicorn was asked about her knowledge of different kinds of reasoning such as inductive reasoning and deductive reasoning. Unicorn’s initial thought on inductive reasoning was:

Find like that the answer that doesn't make sense or two, for me, I always did I always thought of it as like when you're doing multiple choice questions which ones don't make sense. So that That way, you can eliminate it...

Then when she faced a question about deductive reasoning and the differences between deductive and inductive reasonings Unicorn responded that she didn’t know. She also changed her answer and expressed that the definition that she provided for inductive was in her opinion deductive reasoning and inductive reasoning is to think of the correct answers and we use deductive reasoning to eliminate the wrong answers.

When asked about contents in which she had seen proof more, Unicorn replied, Calculus. She also added that back in school as a K-12 student she worked more with proofs in Geometry classes. “I think we did a big unit in Geometry” she replied and mentioned that the proofs that she experienced working with in Geometry required lots of memorizations. This was an interesting point, because despite the belief that mathematics in school was more conceptually taught and learned compared to being procedural in college and university, the activities related to proving seemed to rely less on answering the whys and focusing more on memorizing “eight or nine different proofs that you had to use”. However, later, Unicorn indicated that even though students needed to memorize those proofs they also needed to know the whys behind the solutions of each proof.

Unicorn was asked her opinion on different representations and the role of technology such as DGS in the process of teaching and learning. Unicorn described herself as a “symbol person” who also likes visual representations which are not her first option in solving a
mathematics problem. She also connected the visual representation to proving by saying that different representations can assist us by showing us that we are on the right path in proving a statement or verifying if reasoning is correct or not. The researcher realized that Unicorn made a distinction between mathematical representation and visual representations in a way that seemed to her visual representations were not mathematics. Unicorn believed that most students start solving problems and proving statements by using numbers and variables and not diagrams since they are taught in a way that focuses more on those kinds of representations contrasted to visual representations. Unicorn admitted that students need to have opportunities to work with visual representations but at the same time she was worried that as a teacher she was not prepared and didn’t know how to help students with visual representations:

I think it definitely needs to be practiced, but I don't know. I think it's hard, because I know for me like if I was told I have to teach visually I wouldn't necessarily even know how… I would definitely need help.

Mentioning that all assignments had to be submitted online, Unicorn’s experience with technology as a student was limited to having iPads and teachers projecting PowerPoint documents and slide by using a shared Apple TV screen. She expressed that she hadn’t used DGS such as GeoGebra or Desmos as a student “I haven’t worked with it a ton. I have worked with it a little bit, but it wasn’t a huge thing when I was in school… I didn’t use it… I am just starting to use it now”. Unicorn also stated that despite preferring to work with paper and pencil, she thought that GeoGebra and Desmos were beneficial in teaching and learning mathematical concepts and by providing visual representations, and that dynamic geometric software could help us with providing more reasonings. She even went further and indicated that DGS could help you prove that an answer is correct or you can even prove a statement visually.
Near the end of the pre-interview Unicorn was asked about generalization and its connection to proofs in mathematics. She showed a good understanding of what generalization is by saying “… if we have a statement and we want to make it more general, we are going to take out maybe the most important pieces of knowledge… make it more widespread to fit a whole bunch more categories”. She was asked a question regarding the connection between generalization and mathematical proofs and Unicorn’s response was surprising because it was contrary to the framework that the researcher had chosen to work with in this study (Stylianides, 2008). Unicorn thought that generalization is “the opposite of proof. I think generalization is, okay, here is the overall umbrella, but to prove something you have to find the specific”. While affirming the connection between exploring mathematical cases of a problem and making conjectures with proofs, Unicorn didn’t remember any activities in school mathematics in which she was asked to construct a mathematical conjecture. However, she again emphasized that understanding proofs means “giving the justification of why you’re doing it” and a proof is constructed when somebody can justify a statement completely.

Assignment I, Exploration

Unicorn’s work showed that she was able to successfully create a mathematical model to depict the scenario by using geometric series. By analyzing her work, the researcher realized that Unicorn also understood that the sum of the geometric series converges to a point, meaning that it gets close but will never be the exact amount. Unicorn had used statements like “having around” or “this is approaching a limit of” that revealed her understanding of limits and convergence (Figure 21).
Figure 21. Unicorn’s Response to the Modelling Problem

For task two, Unicorn visualized the problem by using a box with an area of $\frac{1}{2}$ and halving that box to get another box with an area of $\frac{1}{4}$ and she kept halving the boxes to create all the terms that exists in the given series (Figure 22). She was also able to figure out their relationship to each other and put them together in a way that constructed a bigger box that was almost a square. She figured out that the series will get close to the sum of the area that she was able to create visually which was one. At the same time, she expressed in her submitted work that the sum of the series will never be exactly one. Her work also showed that she had a
background knowledge of what geometric series look like in general. She knew that each geometric series could be shown uniquely by having its first term (a) and common ration (r).

Figure 22. Unicorn’s Response to the First Geometric Series Exploration

The idea of the visualization in GeoGebra was the same that Unicorn came up with by using only paper and pencil. Students were also required to think aloud while interacting with the technology piece and record themselves. By analyzing Unicorn’s recorded video the researcher found out that Unicorn didn’t consider visually representing the series to be mathematical. She claimed that she first looked up how geometric series looked like in the general form symbolically, this was interesting since Unicorn’s response to the first question showed her ability to model the problem numerically which meant that she constructed the geometric series
without looking up and searching for them in the accessible resources. She also mentioned that after that she started thinking about the relationship between terms and how each term in the series was half of its previous term. She also talked about what she remembered from before and stated that having $n$ number of terms in the geometric series meant that the first term had been multiplied $n - 1$ times by $\frac{1}{2}$. Unicorn was able to make sense of the DGS and mentioned the similarity between how she visualized the series and what was on the DGS. Her recorded video again showed that she understood the shape never became a full square so the sum of the series would never be exactly equal to one. According to Unicorn’s recorded work, Unicorn believed that symbolic and numerical representations are mathematical and visual representations are not. That was consistent with her beliefs that were revealed in the pre-interview. She also thought that students first needed to understand a mathematical concept “mathematically” and then visually. Visuals and technology that help visualizing ideas for Unicorn played a role of affirming understanding.

As it can be seen in Figure 23, Unicorn started working with numbers and mentioned that creating a visual for this geometric series was more difficult. She had a visual on her submitted work but the connection between the visual and the shaded area with the sum of the series that she claimed to be close to $\frac{1}{3}$ was not clear enough. However, by using a unit square again and dividing it into four smaller equal-sized squares, Unicorn visualized $\frac{1}{4}$ and did the same to one of the smaller squares to create $\frac{1}{16}$. She was able to create all the terms in the given geometric series by using the same pattern. As mentioned, she was not quite satisfied with her visual and wrote using a triangle would have been “more realistic”. As it can be seen in the next part of this
assignment, Unicorn’s idea of using a triangle was a valid idea for visually representing the geometric series in task 4. The researcher also realized that Unicorn had found a stronger connection between the geometric terms and exponents of a number compared to the first case.

Figure 23. Unicorn’s Response to the Second Geometric Series Exploration

In her recording, Unicorn stated that it was more difficult for her to connect the DGS to the geometric series in this case. Unicorn also mentioned that she first worked on the problem
“mathematically” which in her words meant working with numbers and symbols. She was able to connect the dynamic visual representation to the geometric series and she argued that after each step the numbers get smaller and smaller and that is why the sum of the geometric series was approaching a definite number. Unicorn reasoned that if the common ratio was a bigger number the sum of the geometric series would not have been approaching a certain number. The researcher believed that Unicorn was able to make sense of the condition of the common ratio being less than one in order for the geometric series to approach a certain number. Unicorn tied her reasoning to the visual by mentioning that after each step the new constructed shapes become smaller and smaller. In the end she also expressed that she was not sure how she got $\frac{1}{3}$ but that was the right answer to the problem.

One interesting misunderstanding that Unicorn mentioned was her thinking of the terms in the second geometric series as half of the terms in the first geometric series. She expressed that at first, she thought $\frac{1}{4^n}$ is half of $\frac{1}{2^n}$ since $\frac{1}{4}$ is half of $\frac{1}{2}$ and so the sum of the geometric series in the second case should had been half of the sum of the geometric series for the first one, but using the visual she was able to see that was not the case and some terms were missing.

One week after turning in the first assignment, students had their first class to go over the problems and discussed their solutions and understanding of the first assignment by sharing their work and having small group and whole class discussions. Unicorn was in a group with four other graduate students, and group members were fixed for the whole period of the study being conducted. The setting of the class and also the order of assignments, classes and interviews were the same for all the activities. There were three phases and in each phase students worked on an assignment, then went to class and then were interviewed about their individual experience.
with the assignment and also how the group and whole class discussions affected their understanding.

In interview 1, Unicorn was asked about her experience with visual representations. She mentioned that as a K-12 student she always learned a mathematical concept or idea "mathematically" (symbolically or numerically) and then they worked with visual representations occasionally as another way to explain what they had learned. She then added that based on what she had observed in the class, some students first learn a concept visually and then connect their initial knowledge to numbers and symbols in their learning process. When asked about her experience at school and college, Unicorn stated that in school they had more opportunities to work with visual representations than in college to make sense of what they needed to learn. She mentioned that in middle school they had more opportunities to work with visuals compared to high school. According to her, the reason was that in middle school they were more focused on leaning conceptually, so they were provided with more representations such as visuals to understand the content. Unicorn’s opinion was that the opportunities students get to work with visual representations were enough, what could have helped them was having more visually represented ideas than the ones existed in mathematics textbooks. She added that in college, the course they used the visuals the most was Calculus. She said “… as I got into the higher-level math classes, the visual representation kind of disappeared”. She also added that in schools, the extent of using visual representations was dependent on the teachers’ ability and willingness. Unicorn also believed one challenge to working with visuals was that real world situations were hard to visualize.
Unicorn had not provided any reasons for the answers she got for the sum of the geometric series in tasks 2 and 4. Hence, in interview 1, the researcher asked her about how she was able to solve the problems. Unicorn replied that she remembered the sum of the series from Calculus II, but she was not able to exactly explain and justify her work. She told the researcher that she was trying to figure it out. The researcher understood that even though the participant came up with visuals and made sense of the DGSs, she was not satisfied since she claimed that she didn’t solve them “mathematically” (Symbolically/numerically). When being asked about the way she felt after being required to visualize the series, Unicorn responded:

I was very stuck… because that’s not how my brain works… like where is the formula? Let me find the answer, like, I know I can find the answer, whereas visually now it’s like okay that’s a second step for me. Someone almost has to find the visual example for me to understand how to do it, and I think that’s the hard part. I visually couldn’t understand how to do that next step, or visually do it. I could mathematically solve it, but visually I couldn’t solve it.

As it can be seen in her last sentence, Unicorn again made a distinction between solving a problem mathematically versus visually. That led to the researcher asking a direct question and wanting Unicorn to elaborate. Unicorn’s expression showed that there was a shift happening in her mind about the use of visual representations in teaching and learning mathematics:

I think they work hand in hand. The math and the visual work together, and it could be vice versa. I think that’s something that I’m learning in this class… is the visual piece can be your first step of mathematically solving the equation.

Unicorn believed that one reason that she was not comfortable to start visualizing concepts and ideas was the lack of opportunities to do so in school. When the researcher asked her about her experience using the DGS and comparing it with the use of paper and pencil, she pointed out that the DGS was more exact and it allowed her to “zoom and go farther and farther”, so she could explore more steps in a shorter time. Even though Unicorn had started to understand the
importance of visual representations and mentioned that a few minutes before, symbolic and numeric representations were more “mathematical” for her than visuals:

**Researcher:** This is the second one (showing her submitted work for the next geometric series), you were asked to visualize the series. What were your thoughts?

**Unicorn:** Again, I actually did the math first.

**Researcher:** By “the math”, you mean numbers and symbols?

**Unicorn:** Numbers and symbols, yep, I did the symbols and numbers first.

Unicorn indicated that the visual she provided didn’t help her to figure out the sum of the series for the second geometric series. She had thought about using a triangle, but she was not able to build it on her own. When asked about the reasoning behind the answer she came up with in her submission, Unicorn said that she used a graphing calculator to plot a line that represented the sum of the series (simply by adding some of the first terms in the series) and then she looked at the number it was getting close to. It was interesting for the researcher to see that despite emphasizing on symbols and numbers, Unicorn still used a different visual to find the answer to the sum of the geometric series. The participant mentioned that the second DGS was more difficult to understand, and it was not until she hit the solution button and saw the triangle divided into three colored equal-sized parts that she realized $\frac{1}{3}$ could be right answer to the problem.

In the final part of interview 1, Unicorn was asked to talk about her experience in the class and how it influenced her thinking. According to Unicorn, the class had two main functions. Firstly, by having discussions in her small group, she understood the idea behind the DGS and the geometric series in a more profound way. She indicated a person in her group and said, “without her explaining it and then the group explaining it, I wouldn’t have understood it”.

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The class also changed her perspective about how different people learn differently, and how visual representations helped students to start thinking about mathematical concepts and ideas initially.

Despite being in the class and having classroom and group discussions, Unicorn was not able to relate $\frac{1}{3}$ to her drawing for the second case. The researcher showed another student’s visual representation that was depicted in the class and Unicorn were able to connect it to her answer by using it. The other student’s work was very similar to Unicorn’s work and had more details that was helping illustrate understandings (Figure 24). That work could be seen in Figure 10. Unicorn thought that the difference between the other student’s work and hers was that she only focused on showing the terms and leaving the rest of the square not shaded, while the other student tried to go further and see if the pattern gets repeated somewhere else in the square, and it was connected to $\frac{1}{3}$.

Figure 24. The Other Student’s Visual Representation of the Second Sequence
Being able to connect the other student’s visual representation to hers, Unicorn changed her opinion about the usefulness of what she had submitted for this assignment:

*Researcher:* Have your opinion changed about your visual after seeing this?

*Unicorn:* I think it definitely… I don’t know if it necessarily changed, but it definitely showed me that I was on the right track to understanding it.

*Researcher:* Did you think initially that your visual was a good representation?

*Unicorn:* No, I did not think it was a good representation, but once I saw that… I knew that I was close.

*Assignment II, Conjecture*

Because students individually and as a whole class were able to imagine the terms of geometric series as exponential fractions (such as $\frac{1}{2^n}$), the researcher gave them the general form of the geometric series by using variable in the exponential forms. In her submitted work, Unicorn didn’t demonstrate any visual representations. Not only did she not provide any visuals, but she also used the general form and the formula of the sum of the geometric series that she looked up in other resources such as course materials, books and internet. What she had done was completely similar to her perception of how mathematics was being taught in colleges and universities. She substituted the numbers in the formula she had found and didn’t provide any justifications why they worked (Figure 25).
Despite the fact that Unicorn relied more on what she memorized and despite the fact that she used a statement extracted from external resources and used it without explaining how and why it was working, there was some parts of her submission showed her ability to construct conjectures. Unicorn connected what she had found online with the series that she was given in
the assignment. She identified that by using the formula for the sum of the geometric series and replacing the first term of the geometric series $a$ and the common ratio $r$ with $\frac{1}{n}$ she would get the formula $\frac{1}{n}$ for the general case:

$$\frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \cdots$$

Later in her work she went further and mentioned the denominator (supposed to be $x$ in her work) of the first fraction she was seeing in the posted assignment, and she wrote that the sum could be calculated by plug in the denominator as $x$ in $\frac{1}{x-1}$.

This evidence spoke to the fact that Unicorn probably could have constructed a conjecture for the general case but she less confident in her ability to do that (Figure 26).

Remembering what she had studied in Calculus II, Unicorn was more confident in trusting her memory and hence she found the formula by inspecting it on online resources on Internet.

Figure 26. Evidence of Unicorn’s Ability to Construct Conjectures

The next question provided a DGS in which a visual form of the general case was constructed dynamically. The students needed to interact with the GeoGebra, connect it to the general form of the geometric series and use it to calculate the sum of the series approximately.

Based on the submitted recorded video of Unicorn’s work, Unicorn made sense of the dynamic visual representations. She understood what each slider represented and was able to
interpret what would have happened to the geometric series after changing each slider. The connection between the DGS and her submitted work was clear to her and she mentioned that analyzing the DGS helped her to grasp a deeper understanding of the general case and the answer to the sum of the series in the general form.

Because Unicorn had not demonstrated how she came up with the formula $\frac{a}{1-r}$ and why it worked, the researcher started interview two, by investigating her line of thought and how she reached the answer she submitted. Unicorn’s expression affirmed the researcher’s theory that Unicorn relied more on her memory while working on the conjecture problem. When asked how she came up with her conjecture, she replied that she knew from previous knowledge, went back to her notes and looked up the formula online.

When asked about the lack of visual representations in her work, she replied that it was hard for her to visualize the geometric series. Unicorn stated that she was confused about how somebody would have come up with the visual at first place “… I still, I was kind of confused, of how they even found, like, like for the first one… Where it’s like a fourth, and then they use a triangle… How did they figure out that triangle is the best way to describe?” She added that after discussing in her class group she was able to understand the visuals for the other examples. Those examples can be seen in the other cases of this study. The turning point happened in the middle of the interview where Unicorn expressed that she would start teaching the geometric series to her future students by visually representing the ideas:

I think I would start with the visual and be like okay so what do we think… how do we solve that… You can either give them the equation at that point and be like Okay, this is the equation that most people use let's see how they compare, or you can try and work through it, I would need more practice with how to work through…
Unicorn claimed that it took a few seconds for her to figure out how the DGS represented the geometric series dynamically in general. Her claim was consistent with the researcher’s analysis of the submitted recorded video. When being asked how Unicorn would teach her future students who will be more comfortable with using symbols and numbers, she replied:

I think it's important as a teacher to give both. I know that's definitely something where I always am like let's just get to the math, but I think as a teacher, you always have to, try to do both mathematics, or the numerical values and then also a visual representation… I think, also with series and how numbers are formed, I think giving both options give every different type of learner a different way of finding that example or finding out how. It would be so I think that I would start with the visual. And then maybe teach the numerical side of it and then go back to the visual.

The researcher again investigated Unicorn’s thinking on why she didn’t describe visual representations as being mathematical. Unicorn believed the reason was that she had always thought of numbers and equations built by variables when she worked on a problem. She thought that her emphasis was on numbers and variables because she was taught mathematics in a way that it stressed too much on using representations other than visuals. In addition, she talked about proofs in school mathematics and how students were required to prove everything symbolically and then they might be given or asked about a visual representation to certify their work. She also realized that on the other hand there were students (like some of her classmates) who would have started with the visual representations and then construct their proofs. The researcher noticed that even though Unicorn had started to think about the necessity and benefits of working with visual representations, her experiences as a student and the huge emphasis on symbols were hindering her from becoming comfortable with visualizing mathematical ideas:

I think mathematics, in my mind is numerical. There’s no other way… I was taught and there is no other, like I was taught numerically… when I think mathematics, I think numbers, paper and pencil.
Assignment III, Proof

Unicorn’s submitted work had little to do with proving her conjecture. She had repeated what she had written in her response to task 1 of assignment II. Her work showed that the formula she conveyed was working for a special numerical case (Figure 27). Again, Unicorn had used both formulas. She had found out one of the formulas by looking at her notes from Calculus II, and the conjecture that she had made and had connected to the denominator of the first term of the geometric series (In this case it was equal to the common ratio as well) was constructed through working with DGS. In her submitted work, she had tried to show the validity of the formula that she had remembered by making an example that was working. The researcher concluded that she needed to be reminded about the connection between generalizations and proofs. The researcher also connected her work to her initial perception of the notion of proof and Unicorn’s thinking that proof was the opposite of generalization. By noting that and accepting generalization as a part of the process of proving a statement she might have noticed that showing one case did not mean the validity in the general form and it was impossible to check if her conjecture and also memory retrieved formula would have worked for all the natural numbers, because it was impossible to check infinite different cases numerically.
activity #3-
my conjecture was →

~ for any geometric series with
initial value (a) and rate (r) equal
to each other and both less than
and a sum of the series can be
found.

If ~ a and r = each other and
~ |r| ≤ 1

Then the sum = \( \frac{1}{x-1} \) where x = the denominator
x-1 we started with.

e.g.,
\[
\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots \quad a = \frac{1}{4} \quad r = \frac{1}{4} \quad \text{sum} = \frac{\frac{1}{4}}{\frac{4}{1}} = \frac{1}{3}
\]

mathematically
\[
\sum \frac{a \cdot r^{n-1}}{1-r} = \frac{a}{1-r} \quad \text{when} \ |r|<1
\]
\[
a = \frac{1}{5} \quad \frac{1}{5} = \frac{\frac{1}{5}}{\frac{4}{5}} = \frac{1}{4}
\]
\[
\frac{1}{25} + \frac{1}{125} + \cdots = \frac{\frac{1}{25}}{\frac{1}{125}} \quad \text{because} \ \frac{1}{25} = \frac{1}{25}
\]
In the final task of assignment 3, Unicorn needed to work with a DGS to make sense of the dynamic visual representation. She was asked to connect the DGS to the geometric series, find out the idea behind the proof and make sense of the introduced proof in the DGS. The key idea that was embedded in the DGS was geometric with a strong algebraic connection. The DGS had used the idea of similar triangles to construct proportions. As it can be seen Figure 456, the symbolic representation was also accessible in the DGS.

In her video recorded submission, as her response to the DGS, Unicorn showed that she had realized the role of each slider. One role was changing the amount of common ratio and the other changing the number of the present terms in the geometric series. But according to her thoughts, this DGS had been confusing for her. Even though she saw the connection between the visual representation and the geometric series, her initial thought made her to focus on the areas of the triangles and not their similarities. In her video, she tried to connect \( \frac{1-r^6}{1-r} \) to the series and after a few not very successful attempts she stopped trying. The researcher observed that while working with the DGS, Unicorn’s eyes were more focused on the symbolic representation that was accessible on the bottom of the DGS. She rarely focused more than a few moments on the visual representation. After focusing on the visual part of the DGS for a few seconds, her eyes were jumping to the formula and lingered there quite a bit and the same situation repeated many times. Despite having a superficial understanding of the DGS and the components of it, Unicorn was able to answer the first two items of the prompt questions for this task successfully.

Interview 3 took place after the class in which students shared their work on assignment 3 with each other. In the class an algebraic proof for the conjecture was studied. The primary investigator played the role of a facilitator and, with the help of students themselves by asking
direct questions, assisted them to make sense of the last DGS and proof step by step. When asked about the challenges Unicorn faced in the last assignment. She mentioned that at first, it was not challenging and it was in the class that she got “frustrated” because the facilitator and students kept asking why questions:

I think it wasn’t challenging at first. It was like Oh, this is what I’m going to do, but then, when we started the class and you guys kept asking, well, why why why, it got more difficult and had to search different ways.

She mentioned that this time making sense of the DGS was hard for her. She reasoned that since in all previous DGS, area of shapes was used but in this one focusing on the area was not helpful. So, she eventually didn’t find out the similar triangles while working with the DGS individually. She believed that working on her own with the DGS was not beneficial this time. However, after the class discussions and breaking down each part of the DGS and studying them one by one, all made sense for Unicorn and she not only made sense of the similar triangles, but she also understood how the proof worked.

Repeating that she was not a visual learner, and she was not taught mathematics visually, Unicorn claimed that she didn’t put a lot of trust in visual proofs. At the same time, she admitted the importance of visual representations because of what she had observed in the class and from her classmates. To indicate the importance of being in a class and having group and classroom discussions she said:

They (teachers) are good at explaining how this applet works for what we’re learning, and I think that’s really cool, and I think it like, if you look at the bigger picture it's okay, these teachers are helping other teachers. A lot of your information doesn't come by working by yourself, it actually comes from learning and talking to other teachers as well.

According to what she said, learning is a social activity eventually. To make an example she made an instance of the last GeoGebra activity, in which she was not able to make sense of it
in the individual part. But after attending the class she felt that she had understood the DGS and proof better. Another reason that Unicorn mentioned as a factor that undermines students’ ability to visualize ideas was the disconnection between Geometry and Algebra in school mathematics as described:

There are ways of proving we're talking about in an Algebra class, and you think algebra, you think numerical values… but you can take it more to a geometry standpoint. More of shapes and stuff like that, and I think that's the hard part, especially like I know in my curriculum it goes Algebra and then Geometry like Algebra one, Geometry, Algebra two... I know my… my Algebra one kids would never understand that.

When asked about the three given assignments as a whole and how they affected Unicorn’s understanding of geometric series and proofs, she said that her perception didn’t change that much. She went further and asserted that she became frustrated since they were “doing the same problem over and over again”. It seemed that her point of view was more content focused and the difference between exploration tasks, conjecture related tasks, and proving tasks was not that clear to Unicorn. On the other side, she claimed that working with different visuals and connecting them to other representations was a productive challenge for her.

According to her, to prove a statement it was better to have more representations to make sense of the proof, she still believed that proof was closely related to answering the why questions. Her opinion about the role of technology in proving activities was that technology opens more new doors to group discussions related proofs and proving activities. In the end, her belief about the use of visual representations developed more and more to a point that Unicorn mentioned she would expect her students to come up with more than one representation, and among those there should be a visual. She took it further when she said that she expected her
future students to make sense of DGS and visual representations as well as they make sense of numerical and symbolical representations.

**Summary**

Unicorn showed that she had a good understanding of what mathematical proof is in general. At first, she connected proof to explanation and justification to support a mathematical claim through reasoning, but despite that and despite using proof and justification interchangeably, in the end she connected proof to providing reasons when somebody faces “why” questions. However, she showed that there is a lot of room for improvement to understand different kinds of reasoning such as inductive and deductive and their connections to mathematical proofs.

Unicorn’s case is a good example of how much experiences we get as a K-12 student can affect and determine our beliefs and perspectives towards different mathematical entities and notions. As a student she was more exposed to certain kinds of representations such as numerical and symbolic to an extent that at first, she stated visual representations are not mathematical. It was during the interventions that she started to admit the importance of visual representations. She even indicated that visual representation were the ones that she would like to start her classes as a teach in the future, but she required help and stated that she didn’t know what to do and how to do it. Unicorn also mentioned that she was not able to remember many instances that she was required to construct conjectures as a K-12 student, she thought that lack of experience resulted in her not being confident enough to come up with a conjecture in the second assignment and relying more on her memory.

When it came to key ideas, and connecting those to visual representations, Unicorn showed her ability to find and make those connections despite not being able to construct the
visuals herself. Unicorn believed that the first to DGS she worked with were helpful in understanding the geometric series better and the third one was not helpful. Even though she stated that compared to the first two DGS the third one was not as helpful, she was making sense of the proof of the sum of the geometric series by working with the DGS connected to the proof. But overall, in her opinion the DGS played a positive role in visualizing dynamic concepts and therefore understanding them better.

Case II: Sky

Pre-Interview

Sky is a white female undergraduate student in her junior year, studying secondary mathematics education. Her mother was a substitute teacher who was also the source of inspiration for Sky to become a teacher. Sky mentioned that growing up she attended public schools and her first experience of teaching mathematics and working with students dated back to her freshman or sophomore during her high school years when she started tutoring students. She tutored students in various mathematics classes ranging from middle school to high school. Sky loved teaching and described her teaching experience as being “fun”:

It was a lot of fun, because at first they don't necessarily understand the concepts… because that's why I'm there. And then, once we go over it's like you have seen it clicking in their head and that's just a really great experience to have. It's just having kids actually understand what they're learning.

Sky asserted that mathematics was her favorite subject at school and the only time that she didn’t enjoy mathematics was when she had an English teacher who taught her and her classmates Geometry. That experience helped her to notice the importance of the pedagogical aspect of teaching mathematics. Since the teacher taught mathematics procedurally, Sky didn’t like that specific mathematics class. Furthermore, she helped her classmates in understanding the
concepts they were studying in that class to overcome difficulties in learning mathematics procedurally.

Sky had taken an Algebra course while she was in the eighth grade. Her experience with algebra was something unique. Since she was moving with her family in the aforementioned year, she attended an online class, in which she struggled with:

Doing Algebra online was a little rough to start, but thankfully because my mom did have a math degree, she was able to help me a lot with that… The teacher wasn't always that present when it came to, like teaching the lesson. She just kind of gave you like a PowerPoint or a few slides with all the Information and then just some textbook questions. I did pretty well, I know I struggled at the beginning of the class just adjusting to like online learning and stuff, but, overall, overall, it was it wasn't that engaging either it was like one time giving the Info, and now you know it.

In college, Sky had completed the Calculus series and was enrolled in the Logic and Proof course during the semester of conducting this study. Her favorite course was the AP Calculus course that she had taken in high school. Sky argued that her teacher was the reason that she liked AP Calculus so much. The teacher was the teacher of the year, and he also had tried to make connections between the life and real events and the calculus concepts they were going to learn, and it was engaging for students. When asked about the difference between algebra and calculus, Sky argued that algebraic concepts are the basis of calculus concepts that are more abstract like differentiation and integral. Speaking about her experience, she added that the way Calculus was taught was more conceptual compared to Algebra.

Next, the researcher asked Sky about mathematical justification and what she thought about it. According to her response, the participant described her definition of mathematical justification in the following manner:
Justification is being able to backup your information, with like reasoning, whether it's words or like explaining your work that like on the side of how you did each and every step and how and why it works.

She was also asked about proof and what it means to prove a mathematical statement. There was some ambiguity in her response. While she seemed to know the relative relationship between justification and proof by saying “It’s a little bit of justification like when you do it”, her response also showed that there was lot of space for expanding her verbal definition of mathematical proof:

It is relatively the same thing, except I from what I've experienced it's more of just explaining. I think a proof is more of the explanation aspect and justification is the why behind the explanation, so like why you would do something…

The researcher decided to try and dig deeper in order to explore further in relation to justification and proof. When he asked her about the importance of proof in school mathematics, the participant stated that proof is “really important”. She claimed that students not only need to know how theorems and statements work, but they need to understand why they work. In her opinion proving those theorems/statements was the key to grasp the understanding of the whys behind. That statement was completely in contrast to Sky’s first response, and by pointing that out the researcher wanted the participant to clarify what she thought. After thinking for a few seconds Sky responded by saying:

I think that proofs and justification go hand in hand. Because You can't really prove anything without justifying your reasoning, why you’re proving it to be this sort of way.

Sky made her definition more precise by connecting proof to reasoning and also mentioned that justification was a part of proof but not the whole proof. She also thought that proof was related to applications. Sky argued that without knowing why some theorems work and without understanding the reasoning behind mathematical procedures students were not able to connect
what they do to real world applications. Sky also indicated that she had realized a difference between the proofs that she had seen in Algebra versus the ones she saw in Geometry. In her opinion algebraic proofs were connected more to finding variables and studying patterns while in Geometry, proving activities were more about trying to find out the world around, wondering why certain constructions act the way they are constructed. Mentioning that she had seen proving activities mostly in Geometry, and then in Calculus, she asserted that the forms of proofs in calculus was different from the proofs she dealt with in her Geometry courses:

   I want to say where you write out why you're doing things… it doesn't necessarily have to be in the structured order that geometry was in...

   When the researcher asked her about the difference between the proofs that she had seen in school mathematics compared to proofs she had worked with in college, Sky responded that proofs became more difficult as she transitioned from high school to college. Furthermore, she said that as a student and in the Geometry class she was given the statement, all the other givens were represented to them by the teacher and she was asked to prove the given statement using the theorems and facts, which were introduced in the class:

   it's a statement, but it, it becomes a lot simpler to understand because they give you all the givens and then they're like okay use these specific tools use these specific.

That was in contrast with her experience in upper secondary mathematics classrooms. According to her claim, students were given a statement and they were asked to either prove it or disprove it by coming up with counter examples. It was the student’s responsibility to check the validity of the given statements at the college level. Also, students needed to find suitable statements or theorems that could be used to prove a certain statement:

   …like you have to go through each option to see which does work and which doesn't work and that's just kind of how I see like this class being much… So you try different
ideas and see if it works or not. Sometimes it's a little more specified, like with quizzes, it's like, it's going to be based on what we just learned sort of thing versus like tests, where it's like a more generalized; like you've this amount of knowledge and it could be any of these things, so you have to understand what it is.

Sky mentioned another difference between the proofs she had worked with in different grades. She indicated that at college level, there was a great emphasis on how students needed to write proofs. She said that the instructor “is very very strict on how she wants it to be written”.

Sky thought that depending on the content, proof could be more or less connected to exploring mathematical ideas. She also stated that proof is connected to generalizations as well. However, she added that proofs are a part of generalizing an idea, and not the other way. According to her, logic is the foundation of mathematical proof and without logic students cannot understand how to construct a proof. She said that logic and proof go “hand in hand” and for a proof to be valid it must be “logically sound”.

Based on the evidence provided, it was evident that Sky was in favor of using visual representations. The conclusion came from her mentioning that using a visual representation could help students understand not only the concepts better, but it also help them with explanation of the steps in procedures and justification of them as well. Sky’s response came before the researcher even mentioned the term “visual representation”. She made a geometry example to clarify what she was saying. In her example cutting out triangles and putting them on top of each other to see if they match had helped students to better understand the concepts, relationships and differences of congruent and similar triangles. To further investigate the meaning of visual representations the researcher asked Sky what she thought of when she spoke about those kinds of representations. In her mind, any kind of hands-on activities like working with physical manipulatives, watching videos and using DGS like the ones she had seen in her
courses in her undergraduate program such as GeoGebra and Desmos, were all different kinds of visual representations.

Sky believed that visually representing ideas helped students to explore mathematical ideas, and through trial and error student could grasp a sense of validity of a mathematical statement. Eventually, according to Sky, working with different representations and specifically visual representations, helps students to form their logic, “build” and “develop” their understanding of proofs “a bit more”, and “reinforce” their comprehension of different ideas. Sky also claimed that she was in favor of using visual representations in her classrooms in future, however she added that some students don’t learn concepts visually, therefore, other representations should be provided for those students. Despite being on board with the idea of using visual representations in mathematics classrooms, Sky stated that her experience at school was limited to seeing some visual representations of mathematical ideas on PowerPoints and textbooks. She claimed that as students, they mostly worked with numerical and symbolic representations when solving problems. According to her, she was exposed more to physical and visual representations when she started her undergraduate degree in education and took mathematics education courses at her institution.

At this point, the researcher asked Sky about technology that assist students in visualizing ideas and the difference between paper and pencil and DGS such as GeoGebra and Desmos. As stated by Sky, using both (DGS and paper and pencil) provides students with opportunities to explore ideas. She uttered that using paper and pencil only is “repetitive” and by doing that, students could not expand their understanding.
Sky believed that all students were capable of learning proofs. She reasoned that since kids at a young age start to ask “why” questions. She concluded because of that, kids are able to not only learn but also explain why certain mathematical ideas work. So “they should be able to come up with a proof to why”.

Sky was also asked about the age that she thought is best for students to learn mathematical proofs. According to Sky’s response it all depended on logic, and she thought that students learn logic from an early age.

Researcher: When do you think is the best age to learn proofs?

Sky: I think at elementary school, it would be valuable to them.

Researcher: Why?

Sky: Because logic is something that you can use, not just in math. You can use logic in science stuff… you can use it to understand possibly the construction of possibly an essay if you are taught logic at a young age. Then you can see all the applications and see how it works.

Researcher: Got it.

Sky: Maybe if logic is introduced to them earlier than they could possibly learn about proofs… I know the logic and proof go hand in hand, but just like the introduction of like truth versus false at a young age… and then Introducing them to the concept of like why something is true or why something is false…

Because Sky got to the importance of logic in learning mathematical proof she was asked a series of questions to explore Sky’s opinion about different kinds of reasoning. Sky didn’t seem to be confident enough (like her other answers) in what she was recalling while answering the questions; moreover it seemed that Sky thought of inductive proof when she heard the term inductive reasoning.
Inductive reasoning was like having the steps of proving… I did a little bit of this in my intro to discrete class. It wasn’t a ton, but I remember having to basically explain why one way works, because another way also works like it was the sum of like $N$, and then a formula and then $K$ plus one in the formula. So, it is basically expanding the knowledge that not only does this one way work but multiple ways work because of this one.

She also mentioned that in her eyes, deductive reasoning was the “inverse” of inductive reasoning. She believed that deductive reasoning was understanding “why something works, even if something else doesn’t”. Sky added that in proving activities students use deductive reasoning more than inductive reasoning.

When asked about her experiences with conjectures, Sky replied that she didn’t remember any specific cases that she was required to come up with a conjecture. Despite her experience she thought that conjectures were connected to proving activities. Overall, Sky thought that explorations, constructing conjectures and generalizing ideas are all related to mathematical proofs, however she was not sure about the order. Despite not knowing the order, Sky stated that conjectures were part of explorations. She added that there might be other components related to proofs, but she was not sure about them. The participant indicated that in her opinion students should come up with their conjectures and it was better to ask students to explore and construct statements rather than providing them with statements and wanting them to prove them.

**Assignment I, Exploration**

Sky’s work showed that instead of connecting the modeling task to geometric series, in her model she solved the real world problem using linear equations (Figure 28). By analyzing her work, the researcher was able to identify the reason behind Sky’s answer. She had not connected each filtration period to its previous periods. Sky had treated each 8 hours independently. That was why she had reached an answer different from the correct one. The other less important
point was Sky was not considering the two dosages that the patient was taking each eight hours. That led Sky to use number 220 milligrams instead of 440 milligrams of drug.
a) If her kidney filtered 60% of this drug from her body every 8 hours?

\[
(1 - 0.60)(2.20) = \text{amount left after 8 hours}
\]

\[
10 \left[ \frac{(24:8)(0.4)}{2.20} \right] = T
\]

b) How much of the drug was in her system after ten days?

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<th>days</th>
<th>0</th>
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<td>Total</td>
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<td>2200</td>
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</tr>
</tbody>
</table>

\[ d \left[ \frac{(24:8)(0.4)}{2.20} \right] = T \]

\[ d = \text{days} \]

\[ \text{If (a) is applied} \]

\[ d \left[ \frac{(24:8)(1-0.6)(2.20)}{2.20} \right] = \text{total} \]

\[
7.65 \left[ \frac{(24:8)}{2.20} \right] = T
\]

\[
3.65 \left[ \frac{(24:8)(1-0.6)(1.5)}{2.20} \right] = \text{year dosage}
\]

\[ \text{Yes, the equation for the drug trial remained the same throughout the questions. Even as the days increase, the values still increased daily at the same rate.} \]

Figure 28. Sky’s Response to the Modelling Task
Sky used a triangle to show the geometric series in the second task. In her work the area of the triangle was one. Also, it seemed that she used an isosceles triangle to represent the series. She had divided the triangle into two seemingly congruent and therefore equal-sized smaller triangles, and the area of each new smaller triangle was \( \frac{1}{2} \). As it is shown in Figure 29, and in the next step, Sky took the triangle on the right and divided it in half, and she kept doing the same division over and over again. Despite the fact that Sky didn’t connect the different steps to each other in the modeling problem, here she was able to connect each term of the series to the previous term. She also was able to create all the terms in the given geometric series and cover almost all the area of the triangle. That was how Sky reasoned that the sum of the given geometric series starting with \( \frac{1}{2} \) and with the common ratio of \( \frac{1}{2} \) will approach one. Despite finding the sum of the series, Sky had not mentioned if the sum of the series would be exactly one or was going to be approximately one.
Figure 29. Sky’s Response to the First Geometric Series Exploration

In her audio and video recorded work, Sky stated that she had seen the given visual representation in the first DGS before. When working with the DGS she was able to connect the series to the dynamic visual representation. She also showed that she was able to connect terms in the series to their previous ones and found the final answer based on the area of the unit square and how by color coding and moving the slider more and more area of the square got covered. Eventually, Sky connected the DGS to her visual representation that she had constructed using paper and pencil. She also talked about the three important key ideas of exploring the given series visually:

1. Starting with a shape that the area of it was one.
2. Halving the shape into two equal-sized pieces

3. Keeping one piece and doing the halving over and over again

Figure 30 shows that the participant successfully visualized the second geometric series with first term and common ratio both being equal to $\frac{1}{4}$. Despite her first visual work that she used a triangle to visually represent the first geometric, for this one she used a square to represent the geometric series. Sky started with a unit square and divided it into four smaller equal-sized squares. Each of the new smaller squares were $\frac{1}{4}$. At each step, she took the top left square and continued the same process to create the consecutive terms of the geometric series. Despite showing the second geometric series visually, there was no sign of the sum of the geometric series in Sky’s work.

Figure 30. Sky’s Response to the Second Geometric Series Exploration
After exploring the given problem by using the slider for a few seconds, Sky described the second GeoGebra activity as “interesting”. Sky was able to connect the visual representation to the given geometric series. In her recording the researcher realized that the convergence point of the sum of the series was not clear for Sky until the middle of her recording and before the point she clicked on the solution button. After seeing the visual representation being color coded, Sky explained how she saw the same pattern repeated two other times one time in orange and another time in green. She concluded that three of the same patterns cover the whole area of the initial triangle, she could divide the area of the first one by three and therefore the sum of the geometric series would be \( \frac{1}{3} \). She added that the DGS being color coded helped her to realize the sum of the series since it was difficult for her when the other patterns were white. She argued that she didn’t realize the sequence got repeated on the sides of the triangle, when it was not colored.

One week after turning in the first assignment students had their first class to review the problems and discuss their solutions and understanding of the first assignment by sharing their work and having small group and whole class discussions. Sky sat with two other undergraduate students in a group.

Since Sky analyzed the first question in a different way, the researcher wanted to investigate her thinking and how she had tried to figure out the mathematical model behind the real-world problem. Sky mentioned that it was during class time and while sharing ideas and solutions with other member groups that she had found some different interpretations that made her think about her solution again. First of all, she had realized that she had not read the question thoroughly, and that led her to not consider the patient taking two pills each eight hours. The
researcher asked her about the table she had provided in her submitted work and asked her if she saw any connections between the first problem and the geometric series in the next questions of assignment one. The participant asserted that she had not understood the question well enough and that was the reason that in her model and after filtering each those the remaining amount of drug seemed to remain in the body forever. She mentioned that after the class discussions and reading the question thoroughly she was able to see the geometric series behind the mathematical model in the solution of the problem one.

When asked about the next question, she responded that visualizing the first geometric series was not challenging for her.

I did it in a triangle shape because, I don't know, I just felt like it was really easy to cut in half and see how each half was split a bit easier than… I would say, even the square, you have to specify that it's one unit in total.

The instructor asked her about the symmetric unit triangle, and how she was able to figure out that the area of the triangle had to be one before even drawing it. Sky responded that she had seen the series before and had already known the answer to the sum of the series in the first place. The researcher was also interested in Sky’s selection of a triangle to depict the first geometric series and asked Sky about it.

*Researcher:* How did you figure out that the area of the triangle would get closer to one?

*Sky:* Because I had seen the series before and I had worked it all, well… if you look at the picture, in general, like it halves what is in it, takes half, and splits it in half, so you see how these triangles are forming infinitely and are potentially filling the one triangle.

*Researcher:* So, did you know the answer to the sum of the series before starting to draw this visual?

*Sky:* Yes.
Sky: And why did you use a triangle?

Sky: When I go about it and draw it like the different fractions in like a circle form, and then filling it like how other students were doing it… I realized that you can view it as a circle and split it and split it… I feel like with this one (triangle) it was a lot simpler.

The researcher asked Sky about her experience with the first DGS. Sky stated that she was able to see the series that way, but it was easier for her to visualize the series using a triangle. When asked about the reason behind it, Sky answered “the triangle already has a half in its area formula”. She thought that it was more convenient for her since the common ratio of the first geometric series was \( \frac{1}{2} \) and in the formula to calculate the area of a triangle there was a \( \frac{1}{2} \) as well, and she thought that the mentioned coefficient was connected to dividing the shape into halves in some ways.

Sky said that the DGS and using a square was beneficial. She explained that from a different point of view, it was easier to start with a square since “do I split it in half length wise, do I split it in half height wise? It doesn’t really matter, because it is still going to give me the same result”. She compared it with her use of triangles and mentioned in working with triangles she needed to pay attention to how she was going to halve it.

When Sky was asked about the difference between using a paper and pencil and technology pieces like GeoGebra, she said that she preferred drawing the series herself since in that way she would understood why she was drawing what she was drawing as well. Sky thought that with students it was better to have them visualize the series on their own before working with DGS when wanting to reinforce a concept, while for purposes such as starting a conversation, using DGS would be a great way.
At this point the researcher wanted to explore Sky’s thinking about sum of the series and if it was going to be exactly one or very close to one. Sky’s response showed that she knew the sum of the series was approaching one but was never going to be one exactly:

it's kind of how the fractions are going, because you're approaching the limit of zero but you're never actually reaching it… I think that, in terms of an infinite series you kind of have to approximate or like estimate…

After realizing that Sky’s idea was the sum of the series would approach the convergence point, she was asked about her visual representation of the second geometric series. Sky stated that she used a square since she was inspired by the first DGS, in which she had worked with a square. The researcher also wanted to know what Sky thought of the sum of the series, Sky’s answer showed that despite not writing that in her submitted work, she had the right idea about the sum of the second geometric series:

For this one, I wanted to focus on how each section was split into fourths. So, each time I would take like a different color and a fourth. This is kind of like how the triangle is (In the second DGS). I really liked how one of the kids (students in the class) explained it. If you rotate it, it kinda just looks like an infinite path. That’s something you learn in your art class, where it’s like the focal point of your perspective point or like as I call it the horizon line, where things eventually end up. So that shows very well that this is like split into a third.

Hearing terms like horizon, perspective and focal point made the researcher to dig a little bit deeper in Sky’s background and it was then that Sky mentioned that before switching into education, she was an architecture major in her first semester, and the reason that she labeled the activities with the term “interesting” was that she was able to connect the architecture ideas to the mathematical ideas in the activities.

I got interested in it in high school when we worked on like a perspective project, and most people just did it on like random objects, and I’m like… I want to make buildings like go on infinitely. So, this is pretty similar to that.
When comparing the visual representations of the two geometric series, Sky mentioned in her response that the second geometric series would be more challenging for students, because they needed to work with a fourth and not a half, and drawing $\frac{1}{4}$ was “messier”. Sky had shown that she knew the answer to the second geometric series. However, since she had not written any numbers in her submitted work about the sum of the series and the number it was approaching, she was asked if she was able to figure out the answer before, during or after drawing the visual representation.

*Researcher:* So, when you when you first saw this series were you able to estimate what number gets close to.

*Sky:* Umm… I think so. I did see that on either side there were like an equal part to what we shaded it in… Like there were two equal parts to what we shaded. So, the third part would be one third.

*Researcher:* Right, but what about before doing that, before providing the visual? If you were only asked to work with the numbers?

*Sky:* If I was only asked work with the numbers, I feel like I’d have a lot harder time because just working with those numbers, it doesn’t seem like the most clear thought.

*Researcher:* So, would you be able to estimate the sum of the series without the visual?

*Sky:* Eventually, yes. But not right off the bat.

In the last part of the first post assignment interview, Sky was asked about her experience with the second DGS and how it affected her understanding of the second geometric series. According to Sky, she liked the second DGS more than the first one; the reason she provided for her claim was that for the second one, she was able to zoom more and more, and it gave her a sense of doing the same process over and over again (in this case dividing each equilateral white triangle
on top into 4 equal-sized new smaller triangles and make the middle one blue). “… really helped, because you saw them go on infinitely” she replied, to add to her explanation.

She also mentioned that being able to click on the “solution” button and shading the other two sections helped her to see the patterns a lot easier. In that way, she was able to “break the sections apart… and see the actual series… and how the colors orient… your thinking”. Sky also added that unlike the first problem and DGS, for this activity she liked the DGS more than the visual she had drawn using paper and pencil. She thought that the DGS allowed her to explore the case more, especially when considering that a fourth was more difficult to visualize than a half.

When asked about the class and how the class affected her learning, Sky indicated that she learned more than what she was able to learn on her own and it widened her horizon about the problems they worked on and the concepts they dealt with in the assignments:

I learned like a significant amount more, because I’m not just like learning how to explain my process of how I view things, but I’m getting to see… like the two people at my table never had the same exact answer as me… It’s not like solidified, like, hey you can do it this way, but also expanding my mind to like, looking at things, cuz I would have never put it on a number line and she just like thought about that so easily and, like her brain is like. But it was really cool, because then the other girl had, like all her circles and which would you like divide. My brain thought about this way their brain thought about it that way, and just like being able to understand like how other people are viewing things is helping me view things in different ways.

It was interesting for the researcher to see the social impact of sharing ideas in the classrooms and how it led to a development of feelings in which the importance of not only different kind of representations but different representations in the category of visual representations were realized. It seemed that technology played a large role in understanding different concepts, but in cases like this, it was not necessarily the best tool to start thinking
about problems. Also assigning individual assignments without having group and class
discussions will not do justice to the process of teaching and learning. What was added to Sky’s
knowledge during class and the change in her perspective towards visual representations and
how students solve problems in different ways was very valuable. When asked what she would
have missed if there was no classroom discussion, Sky replied:

I feel like I would have gotten probably less out of it because I mean if you consider no
group discussions and no class discussions, I feel like I would have just like skimmed
over this, sort of a bit more, but like having it… reinforces the idea of like visual
representation.

Assignment II: Conjecture
Because the students individually and as a whole class were able to imagine the terms of
geometric series as exponential fractions (they also had shown that in their submitted works),
they were given the general form of the geometric series by using variable in the exponential
forms. Sky had not provided the numerical answers to the sum of the series given in the first
question, instead she had introduced the general formula and the procedures to follow to get the
answers. Her answer gave existence to more new questions. Questions such as, How did Sky get
the formula \( \frac{a}{1-r} \)? Was the formula working? And if so, why? The researcher got more curious
since Sky used \( a \) and \( r \), to show the first term and the common ratio, while the general formula
represented in the assignment had used only one variable \( n \) to show the general form of the
geometric series. Therefore, the researcher decided to further investigate Sky’s thinking about
the connection between her answer and the represented form and most importantly the process
that Sky had used to come up with her constructed conjecture.
a) How would you determine the sum of each series?
I would first find the rate of change between any 2 consecutive numbers. Then find the first term in the series. From there I would use the general formula:
\[ \text{sum} = \frac{a}{1-r} \]
From there I would find the sums of these series.

b) Is there a case that works for all natural numbers \( \mathbb{N} \)?
Yes, this formula above works for this as well.

c) What type of conjecture can you make for the sum of any geometric series?
For any geometric series that looks like
\[ \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \cdots, \]
I can conclude that the sum will never be greater than the value of 1. As long as \( n \geq 1 \), it is less than 1. You can conclude the sum is infinite.

d) Please provide a visual representation of your conjecture/guess for the general case using only paper and pencil.

\[ n = 2 \]
\[ \begin{array}{c}
\text{Square: } 1 \\
\text{Circle: } 1 \\
\text{Triangle: } 1
\end{array} \]

\[ n = 3 \]

\[ n = 4 \]

\[ n = 1 \]

Figure 31. Sky Inserting Numbers in the General Formula
Sky stated that she had relied on her memory to answer the question about the conjecture. She said that she knew the answer from “previous experiences”, and she remembered the formula. According to her, she didn’t follow the order of the questions. She first had tried to sketch the visual representations “…I drew my shapes and stuff first just to like, visualize it, and then from that I made the conjecture…” She continued her thought process and how she had realized that if \( n = 1 \), then the sum of the geometric series would have “an infinite value”, and by increasing \( n \) she said the sum of the series would be finite and became smaller and smaller. Then she started to explain her formula and what each variable meant and tried to find the answers to the given examples by substituting in the numbers. When asked how she constructed the formula and came up with the conjecture, Sky responded that she tried to remember the formula. It seemed Sky needed more help to get comfortable with constructing the conjecture rather than relying on memory. Also, the connection between Sky’s formula and the general representation of the geometric series in the assignment was not clear enough. Therefore, the researcher went over the numerical examples and tried to see how and what Sky would develop as the conjecture. The following is the conversation between the researcher and Sky after talking about the visuals and the numerical answer to the geometric series.

**Researcher:** What are doing in your mind?

**Sky:** I guess… I’m seeing a pattern within each step…

**Researcher:** …what happens?

**Sky:** So, from what I see… what I see pattern wise… it’s always going to be one over the value of \( n \), minus one.

**Researcher:** So, why not using this conjecture? Seems to be more natural to get.
Sky: I thought the conjecture was more about like the words side of things, like how we would explain, like the end result, and just like the general data of something… I just don’t know what I thought.

Sky continued to explain that she had not worked with geometric series for a long time and when asked about her thought process, she was not sure. “I haven’t done sequences and series in a bit… when it came to conjecture I just kind of like, I don’t know”. She said that she had thought that she needed to explain how the formula worked or what the pictures represented rather than what could had been concluded from the visual representations. When asked about constructing a conjecture and how challenging it was for her, she replied that she was comfortable with the task. However, she was not “exactly sure” what to do, and that led her to rely more on her memory rather than trying to construct a conjecture for the general case. The researcher thought that could be because a lack of experience with activities in which students needed to build statements after exploring ideas instead of being given statements that they know are valid. After more investigation, Sky stated that she rushed towards the generalization activity and didn’t try to connect the exploration part to the conjecture section.

Sky had some difficulties at first when she worked with the third DGS. She described the DGS “confusing”. After a few minutes of playing with the sliders and tracking the changes in the visual representations, Sky concluded that the first slider was connected to the number of the sides and the common ratio of the geometric series. Figuring out the second slider’s function took longer for Sky. Eventually she was able to make sense of it and connect it to the progression of the sum of the series. The only part of the DGS that Sky had some problems with was the color coding of different sections.
When asked about her experience with the DGS, Sky stated that while it was interesting, the GeoGebra activity was not “entirely clear”. She said that she had not understood the different aspects of the technology until another student in the class showed his visual representations. “I think I comprehended it better once he had explained his method, because I was a little bit unclear about the sliders and stuff”. She also mentioned that the coloring of the GeoGebra piece was confusing since in her eyes the blue and the purple were almost exactly the same. Sky mentioned that the visualization ideas behind the DGS and using regular polygons was not clear nor close to what she had on her mind.

“I don't think it was that beneficial. I feel like for me, I would have understood it better if I’d drawn it out myself. like I didn't necessarily think about it in terms of like the six sided Polygon versus anything else. I just kind of view it in terms of like any sort of shape and when the sides kind of grew I was a little confused because the parts don't look entirely equal to me.

When Sky was asked for more explanation, she responded that the section in the middle (the smaller similar polygon) was difficult for her to figure it out and how it was connected to the series. She added that in her mind and when visualizing the series, it was more comfortable for her if all the different parts were “perfectly equal”, so she could “stack up” them. She thought that the DGS was not beneficial like the DGS she had worked with before.

Sky was asked about her class experience and Sky mentioned that she felt bad about herself since she was not able to explain her reasoning behind her conjecture and how she constructed it. Sky stated that the reason behind that was the fact that she had relied too much on her past knowledge and memory. The participant added that seeing other students’ work in the class and especially the visual representations of other students in the class made her to see different aspects of teaching the series using visual representations. When asked about the class
and how it affected her understanding, Sky replied that the class discussions gave her another way of looking at the sequences and series in a visual way, especially considering that she had not felt that she grasped a good comprehension of the second DGS. She added that another benefit of the classroom discussions was that those discussions had given her more confidence in explaining sequences and series. Sky also stated that if she had not attended the class, she probably would have relied only on her last experience like what she had shown in her submitted work, which in her opinion was not the most beneficial way for students to learn. As the last question in this interview Sky was asked again about the reasons that led her to be more dependent on her memory rather than her reasoning skills, and her answer was valuable for the researcher:

> I think… if I don't necessarily remember the concept very well, if it wasn't really taught to me very well, I tend to rely on what I know… Because my teachers didn't necessarily have us explore the concepts or topics. If I explored them in the past from teachers that have encouraged us to like look at it in different ways, I think, then I'd more comfortable.

*Assignment III, Proof*

In her work, Sky used the conjecture she came up with in the second interview with the researcher. She used the $\frac{1}{n-1}$ instead of the formula $\frac{a}{1-r}$ that she brought in her second submission from her memory. Sky showed different numerical examples to show the validity of the formula. She also used the same visual representation that she had worked with in the DGS she worked with. The only exception was for the case in which $n = 3$ that she provided a new visual representation using a triangle.
In the last task of assignment 2, which was the last one in the whole sequence of tasks, Sky needed to work with a DGS to make sense of the dynamic visual representation. She was asked to connect the DGS to the geometric series, find out the idea behind the proof and make sense of the introduced proof in the DGS. The key idea that was embedded in the DGS’s proof and visualization was geometric with strong algebraic connections. The DGS had used the idea of similar triangles to construct proportions, and the symbolic representation was also accessible in the DGS.

The recorded video showed that Sky almost figured out the functions of each slider, she stated that the first slider added more sections and related that to the progression of the geometric series. Her opinion about the second slider needed more clarification, she mentioned that the second one was changing the triangle, but she didn’t provide more information about how and in
which ways the second triangle would be changed. She was also able to see the connection of the symbolic representation of the geometric series to the length of the side \( AD \) in the shape.

She also thought more about the area of the triangles \( AEF \) and \( ADG \) rather than the similarity between them. Sticking to the area of triangles, Sky did not connect the sum of the series to \( \frac{1-r^6}{1-r} \). Also, after clicking on the formula and observing the proportional relationship between the sides of the two triangles, the participant said that she could not see how they are related/connected. Therefore, and naturally, Sky didn’t provide any connections between her conjecture made in interview 2 and what the DGS was providing her with.

Interview 3 took place after the class in which students shared their work on assignment 3 with each other. In the class, an algebraic proof for the conjecture were studied. The primary investigator played the role of a facilitator and with the help of students themselves by asking guided questions assisted them to make sense of the last DGS step by step and also the proof in the same DGS. When asked about the challenges Sky faced in the last assignment, she mentioned that it was not easy for her to prove her conjecture despite believing that there should be a visual proof. She added that in some cases “that’s the only way to understand anything, is just like a visual proof”:

I was kind of stuck with this one idea and I just kind of reiterated it and showed it, not only in a pattern as in pictures, but also in a table above.

She mentioned that this time making sense of the DGS was hard for her. She reasoned that since in all previous DGS, area of shapes was used while in this one focusing on the area was not helpful. So, she eventually didn’t find out the similar triangles while working with the DGS individually. She believed that working on her own with the DGS was not beneficial this time. However, after the class discussions and breaking down each part of the DGS and studying them
one by one, all made sense for Sky and not only she made sense of the similar triangles, but she also understood how the proof worked. Sky stated that it was during their group discussions that they concluded, in the last DGS, they preferred to be given the key idea. She indicated that she thinks it is more beneficial for them to explore the key idea rather than coming with the key ideas by themselves. She also believed that students have an easier access to some key ideas but that is not the case for all key ideas. She reasoned that some key ideas are more rudimentary, and some need more knowledge to have, and those properties determine how much an idea is accessible:

I feel like it's easier to view those because it's like the basis of mathematics because it's like what you initially learn counting wise is reinforced in. Because even if you're like lacking in the skills of like multiplication or division, you can usually rely on just like basic grouping or counting to help you in those circumstances. Well, in those you have to rely on so many other aspects of like mathematics, that you have to have remembered.

When asked about the role of visual representations in learning mathematical proofs, Sky responded that “visual representations help in some circumstances and other circumstances they confused” her until she had a conversations about them “in a group or with other people”:

I feel like having a visual thing to manipulate and to view, we become a bit more comfortable understanding like why things work in that sort of sense. I don't know I at least that's how I am when it comes to like seeing things visually it helps my brain to manipulate other things, to work towards it.

She also described her experience with proof and proving activities as being disconnected from exploring, she said “we never really explored anything, it was just like okay prove is this is true or false”. Sky was asked about what had changed in her view towards mathematical proofs, and she responded that:

I feel like mathematical proof should be more, it should be less of just like proving whether things are true or false but rather just kind of gaining and comprehending the concept and understanding why it works.
Summary

At first, Sky didn’t mention a clear distinction between justification and proof. She almost took those notions as equals at the beginning of the study. Step by step she developed her definition of what she understood of proof, justification and their relationship. Sky believed that justification was only a part of proof; in her opinion, proof was also related to exploring ideas and making general arguments. Sky showed that she needed more resources to build her knowledge about different modes of reasoning such as inductive and deductive reasoning.

Despite stating that visual representations are difficult for some students to work with, Sky was in favor with visually representing ideas, especially by using DGS. In her opinion DGS provided more opportunities in exploring different ideas and since exploring ideas was connected to generalization and proofs, DGS would help students with proving activities. In her opinion, the last DGS didn’t help her to understand the key ideas and the proof of the sum of the geometric series, but as it was mentioned many times in her interviews, Sky thought that working with DGS assisted her in understanding the concepts (in this case calculus concepts) better.

Sky was very comfortable with exploring the geometric series and visualizing them, however when it came to making conjectures, Sky was not comfortable; Sky relied more on her memory and looked up general formulas from the resources she had instead of generalizing ideas and constructing conjectures. In the interviews she showed that she was able to construct conjectures, and stated that one reason that she was not comfortable with making conjectures was the fact that as a K-12 student she was not exposed to activities in which she was required to make conjectures. The same was asserted by Sky about realizing connections between visual representation and key ideas. According to her she was able to see some connections easier
compared to other connections, simply because as a K-12 student she got more experience with some of them than others.

Case III: Magic

Pre-Interview

Having recently received his bachelor’s in secondary education and mathematics, Magic is currently a graduate student earning his master’s degree in mathematics education. He is also a full-time high school mathematics teacher focusing on grades six to twelve. Taking Algebra I in seventh grade and geometry in eighth grade, Magic described himself as a fast learner in learning mathematical content. He stated that he learned high school level mathematics courses in middle school and when he was in high school, he started taking college level mathematics courses such as college Algebra, and pre-Calculus, despite being able to skip those because of his test scores. After finishing high school, Magic went to university and took Calculus courses, where he describes himself as a “bad student”:

I went to university after I graduated high school, so I started with Calculus I. To be honest, I was a bad student and failed Calculus I…

However, eventually, Magic was able to finish all his Calculus courses successfully. He explained that he was “pretty good at math”, but he didn’t know what he wanted to do and what path to take. According to Magic, he didn’t have any real passion towards any specific mathematical topic or subject. It was a conversation with his sister that helped Magic to realize he liked helping other students:

Back in high school, I used to help, tutor younger students for SAT… It was fun and enjoyable to help, teach other students… I was interested in being a teacher and so she pushed me to change my direction from math to math education, and so I changed my major.
During his undergraduate years, Magic’s teaching experience was limited to the two internships that he experienced during his studies. He had spent a whole academic year working with a high school teacher helping student in an Algebra II class.

When the researcher asked Magic about his opinion on algebra and calculus, he responded that those two courses are not different. In his opinion, calculus has some new concepts such as differentiation and integration compared to Algebra; still in solving calculus problems, students need to use their arithmetic and algebraic skills.

Magic claimed that he was not able to explain justification well. He first described justification as proving, but later he changed it into “giving meaning to a specific standpoint”. But he also added that justification should be followed up by “proving how it work… And evidence, by doing examples…”.

In the next step, Magic was asked about proof and how he would define it. Magic indicated that proof is “an idea or property that can be used to explain how something works”. Magic also stated that to prove something students don’t necessarily need to use concrete examples, but they can use abstract ideas and variables. Since the definitions that Magic provided for justification and for proof needed more clarification, the researcher decided to investigate Magic’s experiences with proof as a student. Magic told the investigator that the only time his teachers mentioned proof explicitly was in a Geometry course and he didn’t remember other courses in which he heard the term “proof” other than Trigonometry. Magic described his experience as follows:

In K through 12 they don’t necessarily explain it (proof) all that too well. They just say this is what it is, and it may not be the… you know, it may not be the proof itself, but how the teacher applies it… we just replicate what they write on the board.
According to Magic, not only his experience with proof was limited to certain subjects, but the teachers taught proof in those instances procedurally, so understanding the concept itself was not achievable for students.

The researcher also asked Magic about his experience with proof in post-secondary settings. He said that in post-secondary the proofs that he worked with were more abstract and he used variables to prove statements. What he described as his experience was close to the definition he provided. It seemed that his post-secondary experience with proof had a greater impact of his understanding of mathematical proof. He also mentioned another difference, namely in post-secondary the definitions were explained and written in detail. He compared it to what he had seen in schools. According to him in schools, the definitions and proofs were given by teachers, and they focused more on the numerical applications of those proven statements.

When asked about the role of proof in school mathematics, Magic stated that proving a statement means that statement would always work:

> It shows that whatever you’re trying to prove will work, all the time, within its… I guess boundaries or definitions, and so it’s something that can be true… and you can always use that strategy or technique… it can serve a purpose to be another teaching strategy or another tool that a student can use to solve a particular type of problem.

Magic’s recent statement showed a deeper understanding of proof through connecting proof to generalization without mentioning the term itself. Also, when he was asked about the importance of teaching and learning proofs in school mathematics, Magic replied that students will eventually see proofs in higher level mathematics courses, but there won’t be enough time to explain why proof is being used. Magic believed that with enough time and effort all students are able to learn proof. In Magic’s opinion, we can start teaching proof to students at an early age like elementary years. He mentioned that since elementary years is an era in which students get
to explore mathematical ideas and concepts, those years could be beneficial for students to commence learning proofs and their connection to explorations.

Before bringing up the role of representation, Magic talked about its importance in the process of proof. Magic thought that using different representations would help students have a better understanding when they explore concepts and prove statements. The reason that he preferred students to start learning the notion of proof at an early age was that during those years students tend to work more with physical representations and as they grow and advance in their studies, the use of physical representations become less and less to a degree that when they enter university, they mostly work with variables. Magic stated that working with more representations could provide students with a deeper understanding of ideas they explore and eventually with proofs. He also believed that among different representations, visual representations are the most convenient representations to start with when teaching students, since visual representations are a “more basic and simple form that everyone can agree to…”. Magic preferred to start with visual representations and then connect them to other forms of representations such as symbolic and verbal. At this point, Magic refined his definition of proof, he described proof to be “a guess, a generalization” with given constraints. He also stated that logic is “following rules”.

The researcher realized that Magic was constantly developing his idea of mathematical proof. In different parts of the interview, he connected different stages of the process of proving to its definition. Exploration, conjecture (guess) and generalization. Therefore, Magic was asked about different modes of reasoning. Magic indicated that he didn’t remember what inductive and deductive reasoning was, because it was a long time ago that he had heard them. After thinking for a few seconds, he provided the procedure of proof by induction as the definition of inductive
reasoning. He also mentioned that deductive reasoning “would be more abstract”. However, and when asked about the form of reasoning students work more with, he showed a clearer image of inductive reasoning:

I believe (in K-12) we see more inductive reasoning and going through trial and error and doing more examples…

The researcher asked Magic about his approach in teaching proofs, and he responded that it depended on students’ content knowledge background. He stated that in an ideal scenario he would have his students to work in small groups and learn from each other through working together. However, because of COVID and students being absent constantly he was not able to that in his classes. Supposing that students all have at least a decent amount of content knowledge, Magic told the researcher that while teaching proof he would ask students about the ideas they might have to start proving a statement. He also added that by using trial and error, students would get a better sense of their own idea and find out if the idea would work for proving a certain statement or not. Based on Magic’s approach if a student’s idea doesn’t work, he would give the student some “subtle hints” so that student would come up with other ideas.

According to Magic:

I say a fair amount of time that the student will try something out and they’ll be able to do it… It will be less of me guiding them and more of them doing the activity themselves…

The researcher realized that Magic was constantly developing his idea of mathematical proof. In different parts of the interview, the participant connected different stages of the process of proving to its definition, exploration, conjecture (guess) and generalization. Magic stated that as a student he was not provided with enough opportunities in which he constructed a conjecture:
I don’t recall making any conjectures really or none that are memorable enough for me to use.

Magic also emphasized the importance of conjecture in the process of proving:

If a student says one thing, well now, they have to prove that what they said is true. So, others will have to challenge them to disprove it or see if there’s any faults in their reasoning that ties into proofs. A conjecture will be an early phase of a proof in a sense.

Magic also defined argumentation as proving a point on a given topic and checking the validity of reasoning. He pointed out that in the process of proving and while working with ideas, the most important and most difficult part is connecting different representations since each representation has a level of difficulty to be understood. He further explained the fact that students learning in different ways adds to the complexity of connecting representations. He argued that some students are in favor of using visual representations while others might be more comfortable with symbolic representations and therefore it is difficult to start with either one.

The last part of the pre-interview was about the role of technology in teaching and learning proof. Magic answered that DGS such as GeoGebra gives us the ability to show variables and ideas in “real time”. It also allows us to test a lot of examples “at a very extreme rate” so it benefits us timewise. According to Magic, using technology helps students and teachers in exploring ideas, however he stated that visually representing math ideas by the use of technology might be misleading.

Assignment I, Exploration

In his work, Magic showed that he was able to determine the pattern and model the real-world problem in task one mathematically (Figure 33). Based on his work, Magic had been able to work with geometric series and he also found out the point of convergence of the geometric series he came up with to solve the problem.
In his response to the next task, Magic visualized the first given geometric series. His idea to visualize the series was the same idea behind the first DGS in task two. Despite not mentioning the size of the sides in his visual, it is obvious from his work that he started with a unit square and on each step, he halved the square to create the terms in the geometric series and showed that eventually it will cover the area of the square. Based on his submitted work, it was not clear if Magic had thought the answer to the geometric series would become exactly one or would converge to one (Figure 34).
Figure 34. Magic’s Visual Representation of the First Geometric Series

Because the idea he used to visualize the first geometric series by paper and pencil was the same in the first DGS, Magic faced no difficulties in working with the first technology. According to his recorded video, Magic was able to interpret the activity, make sense of it and connect it to the symbolic representation of the geometric series. Magic was in favor of the dynamic properties of the DGS and described his experience as being “great”.

For the next task, Magic again started with a square to visualize the geometric series. He divided the square into two equal parts and divided one of those two equal-sized sections into two smaller equal-sized rectangles to create a piece with the area of $\frac{1}{4}$. He then moved to the next piece and divided it into four pieces. So, he was able to make a piece with an area of $\frac{1}{16}$. Despite showing the pattern behind creating all the sentences in the series, it was not clear in his work how the sum of the series converged to $\frac{1}{3}$. Magic was asked about the process it took him to
get to that answer. According to him and in the last question, Magic faced some difficulties at first. He was not able to clearly recognize what was happening in the dynamic visual representation. It was not until he clicked the coloring option in the DGS that made him see the pattern of the geometric series. After seeing the shape being color-coded, Magic indicated that on each level he was seeing a trapezoid and each trapezoid was divided into three equal sized triangles, and this pattern was being repeated over and over again. So, the answer to the sum of the series would be $\frac{1}{3}$. In the end Magic stated that he enjoyed the DGS more than his visual representation since seeing the connection between the visual representation and the symbolic representation was easier (Figure 35).

![Figure 35. Magic’s Visual Representation of the Second Geometric Series](image)
The interview started with Magic’s response to the first question. Magic stated that it took him some time to figure out the mathematical model behind the problem. However, it was not difficult for him to realize the geometric pattern. Magic also stated that he was familiar with the idea behind visualizing the first geometric series. Magic was then asked about the difference between visualizing the series by the use of paper and pencil and technology. Magic thought that using technology had some advantages over using paper and pencil:

If we just print a picture… most of the work is essentially done, all the shapes are filled in and all the fractions are filled in… the applet shows each iteration as they come into place. If we just put the picture you can’t zoom in because it’s just a static image, whereas, since this is more dynamic, you can actually zoom in and see more. More than what human eye can see as it keeps getting smaller and smaller.

The participant told the researcher that visualizing the second geometric series was a frustrating experience. He stated that he had seen the idea for the first one before, but he had no clue about visualizing the second one. One reason that Magic mentioned was the difficulty to create visualization for pieces that were going to be smaller and smaller. The reason that he started with a square was because the common ratio and first term reminded him of breaking chocolate pieces into smaller pieces and the shape of chocolate pieces were mostly rectangles or squares.

As mentioned before, Magic’s drawing wasn’t showing why he reached $\frac{1}{3}$ as the answer. Magic stated that he knew the result should be less than $\frac{1}{2}$ since the terms in the second geometric series was going to be less than the terms in the first one. The participant added:

Since it’s a geometric series, it has to go towards some common type of fraction, so I took kind of a leap of faith or some type of intuition and assumed that it’s heading towards $\frac{1}{3}$… It’s more intuition and guessing.
He further described his visual as not convincing which in his mind led him to be “frustrated”, “uncertain” and “confused”. Magic claimed that both visualizing the geometric series with the first term and common ratio of $\frac{1}{4}$ and working with the related DGS was challenging. He again felt frustration at first, because he was not able to see the convergence point which was $\frac{1}{3}$.

This one made me feel confused at the moment. Because I didn’t really necessarily find like how did they do the $\frac{2}{3}$ portion or you know…how does it go towards my conjecture… then I clicked the solution… the pattern makes sense at the moment. Now I can actually see, well, there’s three color-coded patterns.

Magic compared the dynamic visualization of the second geometric series with the one he provided, saying that his was not convincing while the one in the DGS was. He also argued that it was possible that if he had more time he would have come up with the same visual representation. The researcher asked him about the difference of working with the DGS and using the slider versus only glancing at a screenshot of the final image when the solution button was pressed. Magic told the researcher that it would be more difficult. He reasoned that looking at a static image would not help learners to see the iteration and how it was happening:

If I’m given all these triangles immediately, I wouldn’t really realize that it’s one fourth of the previous one… because you are just given all the information immediately, and it’ll be harder to make sense of it, in my opinion.

The subject of the conversation between the interviewer and the interviewee in the last part of the first interview was the classroom discussions (Magic was in a group with four other graduate students) and their effect on Magic’s understanding. For Magic the most important part was experiencing different viewpoints. Mentioning the difference in working with various representations. According to Magic some students prefer to work with symbols and numbers
while others are more comfortable working with tables and figures. Claiming that he belonged to the former group, Magic stated that seeing others using a square to visually represent the second geometric series was valuable and made him explore other ideas and visuals along with his groupmates. Not only the classroom discussions helped Magic to see other ideas, but he was able to connect different visual solutions to each other, which in his mind it was interesting. He had realized that rotating Sky’s visual will provide us with a pattern similar to what we saw in the second DGS.

Assignment II, Conjecture

For the second assignment, Magic constructed a conjecture that was claimed to work for the geometric series in general. He had claimed that the sum of the series

$$\frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \ldots$$

would be $$\frac{1}{n-1}$$, where n could be any number other than one. In his work, Magic visualized five different geometric series. For $$n = 2$$ and $$n = 3$$, he had used different ideas to visually represent them. While for $$n$$ equal to four, five and six, the participant had started with a regular polygon with an area equal to one. He also had built a similar regular polygon inside the first one with an area equal to the area of first one divided by $$n$$. Magic had kept repeating the same pattern inside his regular polygons to create similar, smaller regular polygons with the areas of $$\frac{1}{n}$$, $$\frac{1}{n^2}$$, $$\frac{1}{n^3}$$ and so on. Therefore, and for example, for the first iteration he had a smaller polygon with area of $$\frac{1}{n}$$. The area between the smaller similar polygon and the original polygon was $$\frac{n-1}{n}$$. Since all his shapes were regular polygons, Magic was able to connect the center of the shapes to the vertices. By doing so $$n-1$$ trapezoid appeared, with area of each of them being $$\frac{1}{n}$$. By repeating the same
he was able to create all the terms in the geometric series (Figure 36).

Magic was not only able to visually construct a conjecture, but he also visualized the series in a way that was generalizable to all cases. His visual had been more complete than the one the class was provided in the DGS. Therefore, while working with the DGS it was not difficult for Magic to realize the function of each slider and the connection between different parts of the visual with the geometric series themselves. He also shared his conjecture and visuals with the students in the class as a whole class discussion. While most of the class was confused about the DGS and making conjectures, after Magic’s explanation the class was in a better place understanding the concept, the visual, the conjecture and the connections between them. Students who participated in the study stated that
Figure 36. Magic’s Conjecture of the General Case
Magic’s explanation helped them to make sense of the idea behind the dynamic visual representations and its connection to the content.

In the second post assignment interview, Magic indicated that it took him about one hour to one hour and a half to build the visual representation, which he submitted. He reasoned that in the first assignment he was not satisfied with his visual, and it didn’t convince him, so he tried to come up with a clearer and more convincing image that could strengthen his reasoning. Magic also claimed that he knew there was a formula for the general case, but he didn’t want to look it up and tried to build the conjecture on his own by getting help form his visualizations for different examples:

I honestly forgot how to find it so I wrote that out, so I limited myself or restricting myself to certain conditions. I’m not going to use the formula, or at least not outright to I would try to do a visual but the visual has to be convincing enough, at least from my standards, and so I had to go into the actual geometric series itself.

Magic had tried different shapes before choosing to use regular polygons. Starting with triangles, he had understood that triangles were hard to be divided into any arbitrary number of equal-sized smaller pieces. Therefore, he had tried to work with other shapes, given different numbers for $n$.

A triangle can’t be partitioned into five well, it can be, but it doesn't look good enough, at least not for my standards… after many trials… I had to think of another condition to place upon myself, or at least my conjecture, and that is to… whenever I divide I have to get the same shape again, or at least I have to get the same category of shapes so if I get a quadrilateral, it’ll be another quadrilateral. Whenever I divide, the next iteration has to have the same shape again.

The participant also justified why he chose to connect all the corners of the regular polygons to have the same number of areas as the value of $n$ itself. He also argued that since for
every example what he had come up as the final answer to the sum of the series had been one over the number of sides minus one, he had built his conjecture that way.

Since in his work, Magic first had introduced his conjecture and then the visual the researcher asked him about the order of constructing the conjecture and creating the visuals. Magic responded that he first tried to come up with the visuals and it was based on the visuals that he created his conjecture.

Magic stated that working with the DGS was not challenging for him and it didn’t take much effort. He mentioned that he only needed to find the role of the sliders and since he had already built the visuals on his own by drawing, he had enough experience to understand the DGS and connect it to the other forms of representations. He also said that he noticed students’ confusion with the technological piece. According to him, because the DGS gave too much information at the beginning so processing what was going on in the DGS became difficult for other students. Magic believed that students had difficulties with noticing the progression in the dynamic visual representation. He told the researcher that it was easier for him to see those connection since he had developed each piece of the visual by himself. Therefore, he had a good understanding of the details, which were important to notice, especially since there were two sliders in that DGS. Another downside of the technology that Magic mentioned was the color-coding of the visual. Since the used colors were similar, they had led to some sort of confusion in determining different pieces of the visual representation in the DGS. Magic also concluded that not having a visual representation, makes it harder to construct conjectures and understanding how the series work.
When he was asked about the class activities and how it affected his perspective, Magic mentioned it widened his horizon about the process of learning:

If I hadn’t attended this class, I would have missed out on what it’s like to be in the shoes of a learner again. Because once we reach certain stages… by my own example, once I became a teacher, you know, you kind of forget what it’s like to be a student again… and you forget how frustrating it can be to figure things out by yourself.

He also added that in everyday practice as a teacher, there are important things such as different representations that he overlooks. The participant thought that especially not having enough experience with those representations as a student intensifies the lack of using them in classrooms. According to him the classroom experience opened his eyes that other methods exist, and sometimes those methods are easier to understand and make more sense.

Being a teacher, you have to meet certain criteria every week, or certain standards… the most efficient or fastest method of teaching, so that we can meet these deadlines. While we forget that there’s other representations and foundations that are important. So, especially in this class there's so many visuals or any or other methods, besides using symbolism, the symbolism, is the most primary method. At least in my experience for high schools and I'm teaching it especially that's, the main thing that we use. So, we forget that there’s visual included. For we never experienced visuals ourselves.

Assignment III, Proof

In the last assignment, Magic explained why his conjecture was working. He used both visual and symbolic representations to support the validity of his conjecture. In his submitted work, Magic explained the mechanism behind his visual representation, how he produced it and how it could be constructed for any given natural number, \( n \), automatically (Figures 37 & 38).
Given the geometric series
\[ \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \cdots = \frac{1}{n-1}, \]
such that \( n \) is any integer greater than 1.

A regular polygon with the sides equal to \((n-1)\) can be used to represent what the sum of the geometric series approaches to, with the exception of \( n=2 \) and \( n=3 \), since a polygon must have at least 3 sides (in this case \( n \geq 4 \)).

To do so, the regular polygon can be partitioned into a configuration with an inner regular polygon of the same shape, with line segments connecting the vertices of the original polygon and the vertices of the inner polygon. This constructs \( n \) pieces with \( n-1 \) trapezoids and 1 inner polygon.

Let us also assume that the pieces are all equal in area of \( \frac{1}{n} \). This process can be repeated infinitely to partition one of the pieces of \( \frac{1}{n} \) by \( n \) pieces to get \( \frac{1}{n} \cdot n = \frac{1}{n^2} \); \( \frac{1}{n^2} \cdot n = \frac{1}{n^3} \), and so on. By shading one of the trapezoidal pieces and all of the smaller adjacent trapezoids to it, the area of the shaded region approaches a triangle that is the area.

Figure 37. Magic’s Verbal Explanation/Justification of His Visual Proof
Figure 38. Magic’s Visual Supporting the Proof of His Conjecture
In the post interview, Magic stated that he thought he had proven his conjecture by using drawings. He claimed that his proof was convincing for himself. However, it was during the class discussion that he had realized for his proof to be credible he needed to convince others as well.

The classroom discussion in which they thought about the definition of mathematical proofs and their properties led Magic to see the process of proof and determining validity of a conjecture as a social phenomenon. He showed that he used a new lens after the class discussion to understand proof. He knew that his proof was convincing for him but was thinking if his proof made sense for others or not.

It's not just what I think it's convincing, rather, is it convincing enough for my peers and others who are also doing this? Is it convincing enough for them to see as well? so in terms of where actual proof I would need more you know other people to see whether they agree or not.

Magic described his experience with the DGS as being “lost” and “confusing”. He thought that part of that experience was due to the fact that there was a gap between the classes because of spring break. In his first glance he had thought that there were some communication errors on the assignment, since the introduced series in the DGS was not similar to the one, they had represented in their assignment number two and therefore in the class discussions. Despite that, Magic tried to make sense of the DGS. At his first glance what he noticed was the area of the yellow triangles. Later on, he had thought that could not have been the case since the geometric series was visualized on the bottom side of the rectangle. It was interesting to hear Magic say he didn’t think about the proportional relation of side lengths of similar triangles until he clicked on the solution button while working with the DGS. Comparing the key ideas (area of the similar triangles vs the sides lengths of the same triangles being proportional), it seemed that
the former idea was more accessible to the participant while the latter was not as accessible as the first one. It seemed that the ideas have a level of priority to choose from when encountering a problem-solving activity.

Listening to the input from other students made it easier for me to understand that we were looking at the ratios of the inside length of the triangle within the rectangle… these proportions do make sense… now looking back at it… I get why we would use similar triangles.

The researcher also asked Magic about visualizing fractions and division and what came to Magic’s mind after thinking about his conjecture and drawing $\frac{1}{n-1}$. Magic described his thought process as thinking about partitioning and segmentation and initial piece. He stated that he imagined a whole piece and tried to segment it into smaller equal-sized individual pieces. The researcher posed a question about the differences between visualizing addition, subtraction and multiplication versus drawing an image for division and why it was difficult for him to visualize division by the use of the concept of a line’s slope. Magic reasoned that in school mathematics students spend more time on visualizing operations like addition and subtraction compared to division, and even within division there is little focus on the connections between the slope of a line and related concepts such as division and fractions.

Magic also mentioned that it was not until having the last classroom discussion that he realized the connection between the series represented in the DGS, which was:

$$1 + r^2 + r^3 + \cdots$$

To the form that was used in the second assignment:

$$\frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \cdots$$
Magic again stated that the classroom discussions helped him to understand the relationship between the different symbolic representations of the geometric series and also the sides of the triangles being proportional.

In the end, Magic described his experience in this study being “great” and “fun”. He even had shared his assignments with his students and received positive feedback from them.

and I also showed them to some of my students at school and they had like eye opening moments that oh that's very creative, so it brought a lot of creativity…

He also added that seeing other students in class sharing their work of the exploration task brought him “a lot of creativity” which he “would never thought of”. He described the conjecture assignment as being “very valuable” since it “forced a lot of productive struggle”.

So, I was very proud of myself and it was I had a very good sense of accomplishment and I made a lot of connections with the different types of series and then eventually to the generalized one, so I thought I made a breakthrough, at least for myself.

Magic was asked what would have happened if they had only the last assignment. Magic responded that they he would probably work with the symbolic representation and not the visual representation, and he would not comprehend geometric series the way he was understanding them after being involved in all three assignments. He indicated that teachers need to include more visual representation in mathematics classrooms, “because you can make a lot more connections to the abstract representations”. He argued that working only with symbolic representation is difficult and after middle school there are fewer opportunities for students to work with visuals. He concluded that using technologies like GeoGebra could help both teachers and students in bringing those into the classroom.
Summary

Magic’s first response to defining proof was connecting it to explaining procedures. Later on, Magic developed his idea of proof and connected it to reasoning and justification. During developing his idea of mathematical proof, Magic connected proof to exploring ideas, making conjectures and generalizations. Magic described logics as the rules that need to be followed in proving a statement, and he had not a clear definition of different modes of reasoning.

According to Magic visually representing ideas helped him to have a more profound understanding of the concepts. As a teacher also Magic preferred to start his classes by using visual representations, since he believed that visual representations are easier to understand and they assist students in connecting the other representations. At the same time Magic believed that DGS helps students and teachers with visually representing dynamic ideas such as what he had seen in this study. However, Magic thought that to benefit from DGS, students should be able to have the opportunity to construct their visuals by the use of paper and pencil prior to having access to DGS. In his opinion trying to visualize concepts before using DGS, results in understanding the concepts better through seeing the connections between the dynamic visuals in DGS and the mathematical ideas clearer.

Despite not remembering many cases in which he was asked to make a conjecture, Magic did make a conjecture and showed its validity through proving it. He stated that it was difficult for him to come up with a conjecture and visually representing it. Magic indicated that he put a lot of time and effort on the second assignment because he had experienced his students facing difficulties with conjectures, and he wanted to experience the same.
CHAPTER FIVE: CROSS CASE SYNTHESIS

In this chapter, the cross-case analysis will be provided. It will start with comparing Unicorn, Sky and Magic’s opinion and beliefs about proof and justification. In the next step, the three cases are compared based on their responses to each assignment. By using the coding structure introduced in chapter 3, this chapter tries to show a clear image of the effect of interventions on each participants and how they were different and similar at each step of the process of proof, working with DGS and visualizing the mathematical ideas.

General Beliefs and Perceptions

The participants Unicorn, Sky and Magic all had their similarities and differences in academic backgrounds, life experiences and beliefs about justification, proof, visual representations, the role of technology, and so on.

All participants who gave consent to be part of the study believed that all students are able to learn mathematical proof. However, their opinion about the appropriate age that students should start learning proof was different. Sky believed that elementary students could learn proof, and Magic thought that teaching and learning proofs should start during those years and it would be late to postpone learning proofs. However, Unicorn stated that not elementary years, but during middle school years, students are prepared to learn proofs.

When they were asked about courses in which they were required to work with proofs as students themselves, all mentioned Geometry as the major course in which they were engaged with proving activities. Unicorn and Sky mentioned Calculus as another course in which they were required to work with proofs and Magic said Trigonometry.

All participants had a difficult time explaining different modes of reasoning, meaning inductive and deductive reasoning. Not being sure about her definitions, Unicorn defined
inductive reasoning as a kind of reasoning that students use when trying to find the right answer and deductive reasoning a kind of reasoning that is being used when somebody wants to eliminate an answer in a multiple choice problem. Sky provided a definition of proof by induction for inductive reasoning and defined deductive reasoning as the inverse of proof by induction. Magic at first claimed that he didn’t remember inductive and deductive reasonings; however, eventually he had a similar opinion to Sky on inductive and deductive reasoning.

Table 7. Participants’ perception of learning proof

<table>
<thead>
<tr>
<th>Participant</th>
<th>All students are able to learn proof</th>
<th>Best grades to start learning proofs</th>
<th>Mathematics courses you have seen proofs more</th>
<th>Knowing different modes of reasoning</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unicorn</td>
<td>Yes</td>
<td>Middle School</td>
<td>Geometry Calculus</td>
<td>No</td>
</tr>
<tr>
<td>Sky</td>
<td>Yes</td>
<td>Elementary School</td>
<td>Geometry Calculus</td>
<td>No</td>
</tr>
<tr>
<td>Magic</td>
<td>Yes</td>
<td>Elementary School</td>
<td>Geometry Trigonometry</td>
<td>No</td>
</tr>
</tbody>
</table>

All participants agreed that visual representations were useful to teach and learn mathematics. Despite agreeing on that matter, they had different perspectives on the level of using visual representations. Unicorn agreed that using visual representations as a side resource could help students’ understanding of different concepts. But she also didn’t see visual representations as “mathematical” at first; even by the end of the study she was skeptical about identifying visual representations and mathematical entities. Sky thought that the visual representations were mathematical and would help students’ understanding in learning mathematical concepts. But she admitted that some students were not comfortable using those kinds of representations and it would not be her first choice to start a discussion about a problem,
instead it would be a great supportive way to help students’ comprehension. On the other hand, Magic claimed that he is more comfortable using other representations such as symbolic and numerical and he asserted that as a teacher he preferred to start with visual representations first.

As K-12 students, their experience with using technology was very limited. It was limited to using PowerPoint slides in classrooms and seeing some pictures and videos on the slides. They had not used dynamic geometric software as students until they started their post-secondary programs. At one point, Unicorn stated that the lack of experience caused her to not know what to do and how to start using dynamic geometry programs in her future classes.

During the interviews, all participants were changing their answers to defining justification and proof. In one case, the definition was developed bit by bit, while in other cases the definition became different to how proof is defined in the mathematics education. Unicorn’s first definition of proof was focused on the answers to the “how” questions in the process of solving problems and later she changed it to the answers to the “why” questions. According to her a proof is constructed when nobody could ask more “why” questions. She also told the researcher that proof was more connected to numerical problems and representations. In her last response she added that the process of proving was the opposite of the process of generalization. Unicorn was using proof and justification interchangeably.

At first Sky claimed that proof and justification were the same concept. She asserted that proof was about parts being connected to explanations in a problem-solving activity and justification was more about the “whys” behind those steps. She had some doubts about the relationship between justification and proof. At one point, she told the researcher that justification and proof are the same notion. Another time she stated that justification and proof
go “hand in hand”. Despite that there was a lot of room for improvement in defining each, Sky depicted a clear image of the relationship between them by saying that in proving a statement, justification needed to be present while in justification, there is not necessarily a need for mathematical proof.

Magic described justification as same as proving, and in the middle of the pre-interview he defined it as “giving meaning to a specific standpoint”. He thought that justification needed to be followed by proving. On his mind, proof was “an idea or property that can be used to explain how something works”. Magic also argued that exploration, constructing conjectures and generalization all were connected closely to proof.

All participants indicated that they hardly remembered constructing any kind of conjectures during their school years. Also, based on their progression through four interviews, all participants looked at proof with social lens at the end of the study.

Table 8. Participants’ perception of representing and defining proof

<table>
<thead>
<tr>
<th>Participant</th>
<th>Visual representations</th>
<th>Using of proof vs using justification</th>
<th>Experience of using technology (DGS) in K-12 as a student</th>
<th>Experience of constructing conjectures in K-12 as a student</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unicorn</td>
<td>Side resource/not mathematical</td>
<td>Interchangeably</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Sky</td>
<td>Helps to understand/not for all students</td>
<td>In the beginning Interchangeably/proof is a subset of justification</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Magic</td>
<td>Preferred to start teaching by using visuals</td>
<td>At first Interchangeably/justification needed to be followed by proving</td>
<td>No</td>
<td>No</td>
</tr>
</tbody>
</table>
Assignment I, Exploration

Unicorn had successfully modeled the real-world problem. However, visualizing the geometric series was challenging for her. She stated that a lack of experience in working with visuals was the reason behind it being so challenging and taking so much time and effort from her. She used a square to visualize both geometric series and was able to correctly determine the sum of the series for the first geometric series. But for the second geometric series and despite visualizing it, she was not able to use the visualization to support her answer.

Even though, in her model, Sky hadn’t connected the problem to the geometric series, she visualized both geometric series. She had used a triangle to visualize the first geometric series and then a square to visualize the second one. She was also able to use her drawings to find the sum of the given geometric series and support her answers.

Magic had found the geometric patterns behind the posed problem and successfully had modeled the problem mathematically. He also used squares to visualize the given geometric series. His first drawing had helped him to find the answer to the sum of the geometric series. But in his second drawing and partitioning the square into smaller rectangles, he was not able to use his visual to support the answer he got for the sum of the second geometric series. His answer to that question came from conjecture and comparing the two geometric series.

All participants thought that the DGS had helped them eventually in understanding the geometric series better. Unicorn, Sky and Magic were able to make sense of the first DGS, they successfully connected the dynamic visual representation to the geometric series. Also, for all the participants, the second DGS was more challenging compared to the first one. One reason was
that they either had come up with the same idea for the first DGS or had seen it somewhere before.

The participants also agreed on the effects of the classroom discussion on their understanding in different ways. The first classroom discussion and group discussion had helped Unicorn to understand the DGS in a more profound way. It also helped her realize how students learn in different ways. Sky also stated that it was during classroom discussion and group discussions that she understood the importance of visually representing mathematical ideas and how different visual representations enriched their understanding in different ways. Magic also described the effect of group discussions as eye opening. It was an exploration of students’ learning patterns and connecting different representations that helped them in understanding the material.

*Table 9. Participants’ responses to assignment I (exploration)*

<table>
<thead>
<tr>
<th>Participant</th>
<th>Modelling tasks</th>
<th>Exploration task/drawing visual I</th>
<th>Making sense of technology I</th>
<th>Exploration task/drawing visual II</th>
<th>Making sense of technology II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unicorn</td>
<td>Robust</td>
<td>Robust</td>
<td>Robust</td>
<td>Limited</td>
<td>Robust</td>
</tr>
<tr>
<td>Sky</td>
<td>Limited</td>
<td>Robust</td>
<td>Robust</td>
<td>Robust</td>
<td>Robust</td>
</tr>
<tr>
<td>Magic</td>
<td>Robust</td>
<td>Robust</td>
<td>Robust</td>
<td>Limited</td>
<td>Robust</td>
</tr>
</tbody>
</table>

*Assignment II, Conjecture*

The conjecture construction was one of most challenging parts for participants. Unicorn had relied more on her memory rather than constructing a conjecture on her own. She also had not come up with a visual representation to support her conjecture. Unicorn stated that the lack of opportunities during school years in working with visual representations prevented her from
visualizing a general case. She also thought the same reason lies behind the fact that she was not comfortable with creating a conjecture and therefore she looked up the formula for the sum of the geometric series in other resources. However, she was able to understand the DGS, find the functions of the sliders and connect them to the geometric series. According to her the classroom discussions and Magic’s explanation of his work (both visualization and conjecture) helped Unicorn to grasp a better understanding of the DGS as well as the geometric series themselves.

Sky had the same issue with constructing conjectures as well. She also tried to remember what she had known from before when she was asked to build a conjecture. That led to Sky feeling bad about herself and what helped her was being in class, seeing some other students facing the same problem made her feel better. She also benefited from the whole class discussion and stated that it helped her understand the visual representation of the general form better. Although it was in the class that she filled the knowledge gaps of the DGS, Sky didn’t believe that the third DGS was helpful. In her interview, she also showed that she was able to make the conjecture and like Unicorn a lack of experience made her be not comfortable to trust her own abilities in constructing a conjecture.

Just as the other two participants, Magic’s experience with conjecture was limited, however having a teaching experience at school made him curious to know how a learner could visualize the general case and come up with a conjecture. That led him to be confident in trying to answer the questions in the second assignment. He spent about an hour and a half to visualize the general case and used his visual representation to make a conjecture. In his visualization, he used the same key idea that was embedded in the DGS. That helped him to easily understand different aspects of the DGS, connect it to the geometric series and grasp a good understanding
of the whole concept. He also believed that coming up with a visual representation on his own and then working with the DGS was beneficial. He reasoned that without using paper and pencil and directly working with the technological piece would have deprived him from opportunities in which developed his mental image of the concepts.

Table 10. Participants’ responses to assignment II (conjecture)

<table>
<thead>
<tr>
<th>Participant</th>
<th>Conjecture tasks/visually representing the task</th>
<th>Making sense of the DGS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unicorn</td>
<td>Limited</td>
<td>Robust</td>
</tr>
<tr>
<td>Sky</td>
<td>Limited</td>
<td>Limited</td>
</tr>
<tr>
<td>Magic</td>
<td>Robust</td>
<td>Robust</td>
</tr>
</tbody>
</table>

Assignment III, Proof

Magic had proved his conjecture visually and supported his proof using both symbolic and visual representations. However, the DGS challenged him, and despite having a productive struggle he was not able to complete the proof symbolically on his own. He made connections between the DGS and the geometric series. He focused more on the areas of the similar triangles instead of their proportional sides. It was during class discussions that Magic felt that he had completely understood the DGS and the proof it was introducing.

Unicorn used only inductive reasoning to prove her conjecture. She showed a few cases that her formula was working and concluded that it would work in all cases. She also didn’t provide any kinds of visual representations for her work. However, while working with the DGS, Unicorn found out the key idea behind the proof and connected the DGS to the geometric series successfully. She stated that it was after the class discussion that she had realized how important
visual representations are in teaching and learning mathematical concepts as well as proofs. The latter opinion was far different from what she had expressed at the beginning of this study.

Sky also had a difficult time to prove her conjecture. Similar to Unicorn, Sky showed that her conjecture was working for a few examples and then she concluded that it would work all the time. While working with the DGS, like Magic, she spent a lot of time on the areas of the similar triangles, which didn’t help her in understanding the visual proof.

Table 11. Participants’ responses to assignment III (proof)

<table>
<thead>
<tr>
<th>Participant</th>
<th>Proof tasks/visually representing the task</th>
<th>Making sense of the DGS</th>
<th>Kind of reasoning used to prove the conjecture</th>
<th>Key idea I: Slope and fraction</th>
<th>Key idea II: Similar triangles and proportional sides</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unicorn</td>
<td>Limited</td>
<td>Robust</td>
<td>Inductive</td>
<td>Understood after class discussion</td>
<td>Understood after class discussion</td>
</tr>
<tr>
<td>Sky</td>
<td>Limited</td>
<td>Limited</td>
<td>Inductive</td>
<td>Understood after class discussion</td>
<td>Understood after class discussion</td>
</tr>
<tr>
<td>Magic</td>
<td>Robust</td>
<td>Robust</td>
<td>Deductive</td>
<td>Understood after class discussion</td>
<td>Understood while working with GeoGebra</td>
</tr>
</tbody>
</table>

When asked about the reasons behind sticking to one key idea and not using other ones, all participants stated that the reason was as students they worked with some ideas more than others, and they felt more comfortable trying to understand the proof by thinking about the idea that looked more familiar. According to them the lack of visualizing fractions as a slope of a line
also came from a lack of focus on connecting those concepts and visuals when they were students.

Table 12. Participants’ opinion about technology and classroom discussions

<table>
<thead>
<tr>
<th>Participant</th>
<th>Effect of Technology (Tasks I, II and III respectively)</th>
<th>Effect of class/group discussions</th>
</tr>
</thead>
</table>
| Unicorn     | helpful, helpful, not helpful                             | Deepened participants’ understanding of the DGS.  
Realizing the importance of visual representations in teaching and learning proof.  
Thinking of proof as an activity/interaction between at least two persons. |
| Sky         | helpful, helpful, not helpful                             | Understood the importance of visual representations.  
Feeling more comfortable when observing others face the same challenges in some assignments.  
Understanding the technology in a profound way. |
| Magic       | helpful, helpful, helpful                                 | Exploring students’ leaning patterns through connecting different representations.  
Understanding what it is like to be as a learner.  
Widening his horizon about the process of learning.  
Thinking of proof as a social activity; realizing the importance of convincing others after convincing himself. |
CHAPTER SIX: SUMMARY, DISCUSSION, AND RECOMMENDATIONS

This study used a qualitative method to investigate the effect of technology integration in deepening secondary pre-service and in-service teachers’ understanding of mathematical proof. The chosen content for this study was geometric series, a calculus concept. Multiple case study was the selected qualitative method to conduct this study. The question that this study has tried to answer is as follows:

*In what ways can dynamic technology integration and visual representations of concepts and ideas support pre-service and in-service secondary teachers’ experiences and beliefs regarding the process of proving calculus concepts?*

From the eight participants who gave consent to be in the study, three were chosen for this case study:

- **Sky**: A female undergraduate student who completed the Calculus series and took the Logic and Proof course while participating in this study. Her teaching experience was limited to tutoring a few students.

- **Unicorn**: A female graduate student who recently graduated with a major in business and started a graduate mathematics education master's degree. At the time of the study, she had not had any kind of classroom teaching experience.

- **Magic**: A male graduate student who recently got his secondary education degree and had started his master's degree in mathematics education. Magic had one year of experience as an intern in a high school mathematics class during his bachelor's. At the time this study was being conducted, Magic was teaching as a full-time teacher at a high school.

This study used data collection methods such as interviews, observations, teacher artifacts, and video-recorded submissions of the participants. Four interviews were conducted...
with each student who had given consent to participate in the study. There were three assignments, and in each assignment, there was at least one task in which participants were asked to visualize and at least one DGS that students needed to work with, make sense of and record themselves while thinking aloud and working. In total, this study roughly gathered more than 32 hours of interviews, eight hours of classroom discussions, and eight hours of video-recorded submission.

The researcher did not examine proof as an isolated notion in this study. Instead, proof was seen as a process. The process had three stages, exploration, conjecture, and proof itself. That is why three different assignments were embedded in the study, each connected to one component of the process.

*Modes of Reasoning*

According to NCTM’s (2018), students use inductive reasoning to make generalizations. They also use deductive reasoning to investigate the validity of mathematical statements. Inspired by Stylianides’ (2008) framework, the first two stages of the process required participants to use inductive reasoning, and the latter required them to use deductive reasoning. When asked about the different kinds of reasoning and the difference between them, all participants were not sure of the definitions. Inductive reasoning was more accessible to them since all remembered proof by induction, which uses inductive reasoning. However, they were not sure about deductive reasoning. Knowing different modes of reasoning would help teachers to make a clearer distinction between exploration and conjecture and the other stage, which is proof itself. Not knowing deductive reasoning could lead to instances in which a person might think that a statement is proven by showing that it works for a limited number of cases (Unicorn’s case).
Table 13. Study Recommendations on Modes of Reasoning

<table>
<thead>
<tr>
<th>Category</th>
<th>Recommendations</th>
</tr>
</thead>
<tbody>
<tr>
<td>K-12</td>
<td>K-12 teacher education programs should include deductive and inductive reasoning in their mathematics education syllabus and course instruction. It not only helps teachers with teaching mathematical proof as a process, but it also empowers them with a critical lens to understand students’ reasoning</td>
</tr>
<tr>
<td>Future Research</td>
<td>Studying high school students’ knowledge about different modes of reasoning and how it affects middle school and high school students’ perception of mathematical proof.</td>
</tr>
</tbody>
</table>

**Visual Representations**

An efficient way to use technology in mathematics classrooms is to connect different representations while teaching and learning mathematics (Doer & Zangor, 2000). Creating visual representations and connecting them to other forms of representations is where technology could play a huge and important role in both teaching and learning mathematics (Lee & Hollerbrands, 2008). Exploring and constructing conjectures are the first steps of the process of proving. This is when learners become more familiar with the concepts themselves. Verbal, numeric, and symbolic representations are necessary to work with when we explore a mathematical concept. However, at the first stages, visual representations could be beneficial as well. Moreover, by the use of visual representations connecting different representations of a mathematical idea becomes easier, which eventually leads to a better and more profound understanding. Working
with visual representations also helps the participants to make sense of the numerical and symbolic representations through working with visual representations.

Using appropriate visual representations improve students’ intuition towards new concepts and problems they face while studying mathematics. According to the interviews in this study, students rarely use visual representations while working with proofs. Even in cases that they do use visuals, the focus is more on writing the proofs, using symbols and numbers, and in some cases in classrooms visual representations become devalued. That is why it was difficult for the in-service and pre-service secondary teachers in this study to visualize geometric series. Unicorn didn’t think of visual representations being mathematical. She had a hard time coming up with visuals and even though at the beginning she stated that she didn’t consider visual representations a mathematical answer to problems, when she wanted to calculate the sum of the series, she used a graphing calculator. Step by step and assignment after assignment, she gradually changed her mind about using visual representations in her classroom. But again, she stated that because of the lack of experience in working with visual representations she didn’t know how to use them in her future mathematics classes as a teacher.

The mindset of the participants was changing constantly and developing through the course of this study. At the end they claimed that they will integrate visual representations, but a participant like Unicorn asserted that even though she might not want her students to construct visual representations, but she will want them to make sense of those representations and connect them to other forms of representations.
Table 14. Study’s Recommendations on Visual Representations

<table>
<thead>
<tr>
<th>Category</th>
<th>Recommendations</th>
</tr>
</thead>
<tbody>
<tr>
<td>K-12</td>
<td>Students need to be provided with more opportunities in which they are required to visualize the concepts and explore them by the use of technology on their own; especially in middle school and high school.</td>
</tr>
<tr>
<td>Future Research</td>
<td>The field is in need of studies that focus on the role of visuals in demonstrating mathematical ideas and proving activities. Specifically using technology to dynamically visualize dynamic concepts such as geometric series that has been discussed in this study. It is also important to study teachers’ perceptions of visualization and its role in teaching and learning proof.</td>
</tr>
</tbody>
</table>

Key Ideas

Concluded from Raman (2003), a necessary step to promote proof in school mathematics is promoting the use of key ideas and also visual representations. It is important to not only promote using key ideas, but also connect them to their visual representations when encountering a proof task. In the last activity, one of the key ideas that were used in the assignment to prove the general formula of the sum of the geometric series was using similar triangles and their sides being proportional. However, students were more focused on the areas when they saw the DGS, and it was not until the class discussions that students thought about the sides being proportional. It seems the amount of emphasis on different ideas in different subjects has a correlation with noticing the most used ones and overlooking the ones that students have experienced less.

Another interesting observation was visualizing key ideas. All participants’ conjectures had a fraction, and when they were asked about visualizing their conjecture, they thought about starting with a whole and segmenting it. None of the participants thought about the slope of a line and its connection to their proposed fraction. When they were asked about their opinion, they responded that there were a lot of opportunities to work with manipulative and drawings,
starting with whole pieces, and the connection between the slope of a line and division/fractions was not explained enough when they learned those concepts in middle school. So, when they progress and advance in their studies, they might not consider some key ideas which are crucial in proving activities. It seems that we focus more on the key ideas and connections between different representations and pay less attention to some others when we work with students.

Table 15. Study’s Recommendations on Key Ideas

<table>
<thead>
<tr>
<th>Category</th>
<th>Recommendations</th>
</tr>
</thead>
<tbody>
<tr>
<td>K-12</td>
<td>More emphasis is required (in secondary school mathematics) on making connections between different key ideas by the use of technology; especially between concepts in elementary and secondary (e.g., division, fraction and slope of the line, differentiation).</td>
</tr>
<tr>
<td>Future Research</td>
<td>Exploring the ways teacher education programs could deepen in-service and pre-service teachers’ knowledge of connections between different key ideas within and between mathematical subjects through integrating technology.</td>
</tr>
</tbody>
</table>

**Technology**

The findings and recommendations of this study to use technology to explore different mathematical conceptions, constructing conjectures and checking the validity of those conjectures by proving or disproving them are aligned with what the literature suggests (Ball & Stacey, 2005; Laborde, 2001; Wilson, 2008). Ball and Stacey (2005), recommend to use technology in designing courses in a way that students see how to integrate technology in their mathematics classrooms.

Because calculus concepts are dynamic in nature, the use of paper and pencil, which provide us with static images of the concepts, could not be as beneficial as using dynamic geometric software like GeoGebra to visualize them. However, it must be noticed that this doesn’t mean that technological pieces are the best tools to start learning about a subject. First of
all, they must be refined; not all DGS can help students with their understanding. Pre-constructed DGS could give away too much information too soon to the students and deprive them of having opportunities to explore different aspects of a given subject by themselves.

Dynamic geometry software such as GeoGebra and Desmos have been developed recently. That is why most of the current in-service and pre-service teachers have had limited experience with them in teacher preparation programs. Moreover, how to use technology in proving activities is less examined. Those software help learners to:

- Explore more cases in less time
- Explore examples that are difficult to examine using paper and pencil
- Explore dynamic concepts such as concepts we encounter in calculus and require a dynamic image and not a static one
- Improve our ability to represent mathematical ideas and concepts visually

Because of what has been mentioned above, using software such as GeoGebra could enrich learners’ experience in exploring different mathematical subjects related to proof. That itself helps with constructing conjectures cause the users could examine more cases and ideas. Therefore, it is natural to conclude that those kinds of technology could play a huge role in teaching and learning proof.

Table 16. Study’s Recommendations on Technology

<table>
<thead>
<tr>
<th>Category</th>
<th>Recommendations</th>
</tr>
</thead>
<tbody>
<tr>
<td>K-12</td>
<td>Using dynamic geometric software in different stages of the process of proof, exploration, making conjectures and proof.</td>
</tr>
<tr>
<td>Future Research</td>
<td>Creating resources and dynamic visual representations for proof-related activities in secondary mathematics courses other than geometry.</td>
</tr>
</tbody>
</table>
Conjecture

One of the most important roles of technology in the cycle of proof is using it to generate hypotheses and conjectures after exploring connected ideas and concepts. Researchers have emphasized on the role of technology as a checking tool to evaluate conjectures (Cullen et al. 2020; Cuoco & Goldenberg, 1996). We cannot expect anybody to learn mathematical proofs without involving them in activities such as exploring and constructing conjectures. Constructing conjectures was one of the most challenging tasks that participants faced. Two of the participants relied on their memory and had not had enough confidence in themselves to construct conjectures, and the only participant who tried to construct a conjecture for the general case stated that the reason he did that was due to experiencing the same challenge while he worked with his students. In general, it seems that teachers are not comfortable when they are asked to build a conjecture based on their explorations. One major reason for that phenomenon is that in school mathematics, students are not involved in tasks in which they need to make a conjecture. Their experience lacks building conjectures on their own. All the participants in this study stated that they didn’t remember any specific moments in their K-12 years in which they were asked to make a conjecture. The way proof was taught to them was procedural. They were always provided with statements that they needed to prove, and sometimes they were provided with the proofs as well and needed only to know how and why the proof was working. They had not constructed their knowledge about proof, which we can see led to the fact that they are not comfortable with an important step in the process of proving. It still holds them back from trying as teachers and creates negative feelings about themselves and their mathematical skills and knowledge.
Teachers’ backgrounds as students play an important role in how they shape their beliefs towards proof, technology, and visual representations. In this study, and in contrast with what we know from the literature review, participants had a positive point of view about students’ abilities in learning proofs. All participants agreed that all students could learn proof, and proof is not a notion that only a special group of students can learn. According to the findings and results of this study, high school could be late for students to start learning about proofs. The participants of this study believed that students should start learning proofs during their middle school and even elementary years.

*Table 17. Study’s Recommendations on Making Conjectures*

<table>
<thead>
<tr>
<th>Category</th>
<th>Recommendations</th>
</tr>
</thead>
<tbody>
<tr>
<td>K-12</td>
<td>Providing students with opportunities in which they construct a conjecture after exploring ideas.</td>
</tr>
<tr>
<td>Future Research</td>
<td>Examining the opportunities mathematics textbooks provide for students to engage with making conjectures.</td>
</tr>
</tbody>
</table>

Another interesting point was that during their K-12 school years, participants only remembered geometry as the dominant course in which they had worked with proofs. That being said, the other courses that they remembered they saw proofs were Calculus courses that not all students take at high school. The lack of experience with proving activities in school mathematics courses other than geometry indoctrinates that proof is not an inclusive entity in mathematics. That could create the impression that proof is unimportant in school mathematics, undermining its place. This will worsen learning proofs, especially for those students who don’t have a good experience with Geometry.
Proof as a Social Activity

Participants had good knowledge about what a mathematical proof is despite being vague about the difference between justification and proof. All had a decent, acceptable definition of proof on their minds. What was interesting to see was them questioning their idea of proof as an individual activity. At first, their comprehension of proof was not a social phenomenon. After being involved in group and class discussions, they started thinking about other aspects of proof. At the end of the study, Unicorn defined proof as an interaction between two people, and Magic posed the question if it is enough for his proof to convince himself. What about convincing others?

It was during the interactions between students that the participants thought of proving tasks as a group activity. The same holds true when they realized other parts connected to proof, such as using technology to explore ideas and cases and making conjectures, are better understood during group and class discussions.

As it has been said in the literature review, there needs to be more research focusing on teachers and the how they could teach proof in an efficient way. Proof should not be viewed as an isolated activity. Contrarily, and in school, mathematical proofs should always be accompanied by exploration and conjectures. Students need more opportunities in which they construct their own conjecture and therefore construct their knowledge. Because the more students explore, the more they will be prepared for constructing conjectures, technology plays a huge role. Technology provides them with more opportunities, more discussions, and a better understanding because it gives them more information on the subject/concept they learn.

There should be studies that measure the number of activities in which students get engaged in constructing conjectures. Investigating how teachers could build a culture in their
classrooms that encourage students to be comfortable with conjectures is another valuable idea for a proof-related study.

Teachers need to experience working with mathematical proofs in their preparation programs. They need to see proof as a process of exploring, making conjectures, and proving and not only proving. They need to have opportunities to learn about different modes of reasoning, how they are different and where they use each of them in the process. There should be productive challenges in which teachers see the importance of key ideas and visually represent them. Proof should be included in all of school mathematics courses and not only in Geometry. One of the most important parts is the responsibility of organizations such as AMTE. Standards for preparing teachers of mathematics should be clearer, inclusive, more straightforward, and explicit when they discuss the skills and knowledge teachers should achieve to be ready to teach proof.

**Implications**

One important obstacle in teaching and learning proof is teachers’ perceptions and beliefs of proof. Seeing proof as a single notion and not a process will result in not understanding it and also not appreciating its importance. In school mathematics, proof should be seen as a process that starts with exploring ideas, continues to make conjectures and finally checking the validity of those conjectures and proving/disproving them. In that way teachers could help students to connect mathematical proofs to other areas and practices in mathematics and appreciate its importance. Unfortunately, in school mathematics students mostly explore ideas and are asked to prove certain statements. There are few opportunities in school mathematics in which students are required to build their own conjecture. If we suppose that conjecture’s role in the process of proving is acting as a bridge between proving and exploring, not having that experience means
that we teach a notion without really understanding it and connecting it to other mathematical activities. That is why students think proof has little to do with mathematical explorations.

Another important finding of this study is the importance of visualizing concepts and connecting it to key ideas. The lack of experience in working with visual representations could set a mental barrier in appreciating the role and the importance of visual representations. Moreover when it comes to DGS, since most teachers have not experienced working with them as a K-12 student they are either reluctant to use them or they need help to know how to integrate visual representations and connect them to key ideas in a proving process. Despite the importance of integrating technology, it is vital to know how and when to use DGS. According to this study the best way of using DGS to visualize a mathematical concept is using paper and pencil first and then work with DGS to make a better understanding of a concept. Jumping right into DGS might provide learners with too much information too soon and deprive them from having meaningful productive challenges that assist them with the process of proof.

To be added to the findings above, it must be said that in K-12 educational system, proof is limited to the course of Geometry only. Students and teachers don’t encounter proof in other areas in mathematics and that undermines the role of proof in school mathematics. Proof-related activities should be expanded to other subjects in school mathematics as well.

To be able to teach proof in a more efficient way, teacher preparation programs should expose them more to proving activities in different subjects of mathematics and show them the importance of visualizing ideas and using DGS for that purpose, especially when working with dynamic concepts such as calculus concepts. Visualizing ideas should not be limited to
elementary years, and it should be present and practiced regularly in secondary years and in subjects other than Geometry such as Algebra.

One important item to notice is understanding the difference between different kinds of reasoning and their relationship with proof. To avoid mistakes such as showing the validity of a mathematical statement with showing that it works for a few cases, teachers should know the distinction between inductive and deductive reasoning and how each one is connected to a certain step in the process of proof. The lack of that knowledge prevents teacher to do their best when it comes to teaching and learning proof.

Last but not least, proof is a way that teachers and students communicate with each other. All that is mentioned above such as integrating DGS, visualizing ideas, providing opportunities with conjectures should be included in teacher preparation programs in a way that teachers learn through working together.

Based on what has been said above and according to the findings of the study, some recommendations are being suggested both for teaching and learning in K-12 settings and needed future research in the field of mathematics to improve teaching and learning mathematical proofs. Those recommendations could be found in the following tables (Tables 18 & 19).
Table 18. Recommendations for future research

<table>
<thead>
<tr>
<th>Topic</th>
<th>Recommendations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modes of Reasoning</td>
<td>Studying high school students’ knowledge about different modes of reasoning and how it affects middle school and high school students’ perception of mathematical proof.</td>
</tr>
<tr>
<td>Visual Representations</td>
<td>The field is in need of studies that focus on the role of visuals in demonstrating mathematical ideas and proving activities. Specifically using technology to dynamically visualize dynamic concepts such as geometric series that has been discussed in this study. It is also important to study teachers’ perceptions of visualization and its role in teaching and learning proof.</td>
</tr>
<tr>
<td>Key Ideas</td>
<td>Exploring the ways teacher education programs could deepen in-service and pre-service teachers’ knowledge of connections between different key ideas within and between mathematical subjects through integrating technology.</td>
</tr>
<tr>
<td>Technology</td>
<td>Creating resources and dynamic visual representations for proof-related activities in secondary mathematics courses other than geometry.</td>
</tr>
<tr>
<td>Conjecture</td>
<td>Examining the opportunities mathematics textbooks provide for students to engage with making conjectures.</td>
</tr>
</tbody>
</table>
Table 19. Recommendations for teaching and learning in K-12 settings

<table>
<thead>
<tr>
<th>Topic</th>
<th>Recommendations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modes of Reasoning</td>
<td>K-12 teacher education programs should include deductive and inductive reasoning in their mathematics education syllabus. It not only helps teachers with teaching mathematical proof as a process, but it also empowers them with a critical lens to understand students’ reasoning.</td>
</tr>
<tr>
<td>Visual Representations</td>
<td>Students need to be provided with more opportunities in which they are required to visualize the concepts and explore them by the use of technology on their own; especially in middle school and high school.</td>
</tr>
<tr>
<td>Key Ideas</td>
<td>More emphasis is required (in secondary school mathematics) on making connections between different key ideas by the use of technology; especially between concepts in elementary and secondary (e.g., division, fraction and slope of the line, differentiation).</td>
</tr>
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<td>Providing students with opportunities in which they construct a conjecture after exploring ideas.</td>
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</table>
APPENDIX A: INSTITUTIONAL REVIEW BOARD FORMS
UCF IRB Approval Letter

Institutional Review Board
PWA00000031
IRB00001138, IRB00012110
Office of Research
12201 Research Parkway
Orlando, FL 32826-5246

EXEMPTION DETERMINATION

January 11, 2022

Dear Shahab Abbaspour Tazehkand:

On 1/11/2022, the IRB determined the following submission to be human subjects research that is exempt from regulation:

<table>
<thead>
<tr>
<th>Type of Review:</th>
<th>Initial Study, Category 2(i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Title:</td>
<td>SUPPORTING SECONDARY PRE-SERVICE TEACHERS’ PROOF &amp; JUSTIFICATION OF CALCULUS CONCEPTS THROUGH THE INTENTIONAL USE OF DYNAMIC TECHNOLOGY</td>
</tr>
<tr>
<td>Investigator:</td>
<td>Shahab Abbaspour Tazehkand</td>
</tr>
<tr>
<td>IRB ID:</td>
<td>STUDY0000640</td>
</tr>
<tr>
<td>Funding:</td>
<td>None</td>
</tr>
<tr>
<td>Grant ID:</td>
<td>None</td>
</tr>
</tbody>
</table>

Documents Reviewed:
- Shahab Abbaspour IRB - HRP 251 Form, Category: Faculty Research Approval;
- Interview Prompt Questions - Shahab Abbaspour.docx, Category: Interview / Focus Questions;
- MAE 4339 Schedule, Category: Other;
- MAE 4339 Syllabus, Category: Other;
- Shahab Abbaspour IRB GeoGebra Activities, Category: Other;
- Shahab Abbaspour IRB HRP 254, Category: Consent Form;
- Shahab Abbaspour IRB HRP 255, Category: IRB Protocol;
- Shahab Abbaspour IRB-411 Form, Category: Other

This determination applies only to the activities described in the IRB submission and does not apply should any changes be made. If changes are made, and there are questions about whether these changes affect the exempt status of the human research, please submit a modification request to the IRB. Guidance on submitting Modifications and Administrative Check-in are detailed in the Investigator Manual (HRP-103), which can be found by navigating to the IRB Library within the IRB system. When you have completed your research, please submit a Study Closure request so that IRB records will be accurate.

If you have any questions, please contact the UCF IRB at 407-823-2901 or irb@ucf.edu. Please include your project title and IRB number in all correspondence with this office.

Sincerely,

[Signature]

Katja Kluge
Designated Reviewer
Informed Consent Form

EXPLANATION OF RESEARCH

Title of Project: SUPPORTING SECONDARY PRE-SERVICE TEACHERS’ PROOF & JUSTIFICATION OF CALCULUS CONCEPTS THROUGH THE INTENTIONAL USE OF DYNAMIC TECHNOLOGY

Principal Investigator: Shahabeddin Abbaspour Tazehkand
Faculty Supervisor: Dr. Farshid Safi

You are being invited to take part in a research study. Whether you take part is up to you.

The purpose of this research is to study how visual representations can help math educators in better understand calculus concepts. It also aims to investigate pre-service teachers' beliefs toward mathematical proofs and the role that technology can play in both teaching and learning them.

You will be asked to have interviews with the PI about the activities they do for the course such as group discussions, assignments, and technological pieces that they use to explore the course material.

Three one-to-one interviews will take place with the PI, it is estimated for the first interview to last for about 75 minutes and the second and third interviews for about 30 minutes each and one post interview that lasts for 30 minutes too. You will be committed to participate in all three interviews.

You will be audio recorded during this study for the interviews and audio and video recorded while working with technological pieces either when you work on your assignments or have group discussions. If you do not want to be recorded, you will not be able to be in the study. Discuss this with the researcher or a research team member. The recording will be kept in a locked, safe place. The recording will be erased or destroyed after 5 years.

Your participation in this study is voluntary. You are free to withdraw your consent and discontinue participation in this study at any time without prejudice or penalty. Your decision to participate or not participate in this study will in no way affect your academic performance, including continued enrollment, grades, employment or your relationship with the individuals who may have an interest in this study. The PI is the only person who has access to the collected data. The data will be stored for 5 years and then it will be erased and deleted.

You must be 18 years of age or older and be a secondary education major to take part in this research study.

Study contact for questions about the study or to report a problem: If you have questions, concerns, or complaints Shahabeddin Abbaspour Tazehkand, Graduate Student, School of Teacher Education, University of Central Florida, or Dr. Farshid Safi, Faculty Supervisor, Department of Mathematics Education, School of Teacher Education, College of Community, Innovation and Education, University of Central Florida at by email at

IRB contact about your rights in this study or to report a complaint: If you have questions about your rights as a research participant, or have concerns about the conduct of this study, please contact Institutional Review Board (IRB), University of Central Florida, Office of Research, 12201 Research Parkway, Suite 501, Orlando, FL 32826-3246 or by telephone at (407) 823-2901, or email irb@ucf.edu.

Your Name:

Your Initial and Signature:
REFERENCES


Ellis, J., Fosdick, B. K., & Rasmussen, C. (2016). Women 1.5 times more likely to leave STEM pipeline after calculus compared to men: Lack of mathematical confidence a potential culprit. *PloS one, 11*(7), e0157447.


Hersh, R. (2010). What I would like my students to already know about proof. In Teaching and learning proof across the grades (pp. 17-20). Routledge.


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