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Multicolor Ramsey and List Ramsey Numbers for Double Stars

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MULTICOLOR RAMSEY AND LIST RAMSEY NUMBERS FOR DOUBLE STARS

by

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ABSTRACT

The core idea of Ramsey theory is that complete disorder is impossible. Given a large structure, no matter how complex it is, we can always find a smaller substructure that has some sort of order. For graphs G and H we write $G \longrightarrow (H;k)$ if every k-edge-coloring of G contains a monochromatic copy of H in color i for some $i \in \{1, 2, \ldots, k\}$. Similarly, we write $G \not\longrightarrow (H;k)$ if there exists a k-edge-coloring of G with no monochromatic H. Such a coloring c is a *critical k-coloring*. The k-color Ramsey number of the graph H, denoted $r(H;k)$, is the smallest integer N such that $K_N \longrightarrow (H;k)$, where K_N is the complete graph on N vertices. Despite active research for decades, very little is known about Ramsey numbers of graphs. This is especially true for $r(H;k)$ when $k \geq 3$, also known as the multicolor Ramsey number of H. Let S_n denote the star on $n+1$ vertices, the graph with one vertex of degree n (the center of S_n), and n vertices of degree 1. The *double star* $S(n,m)$, where $n \geq m \geq 1$, is the graph consisting of the disjoint union of two stars S_n and S_m together with an edge joining their centers. In this thesis, we study the multicolor Ramsey number of double stars. We obtain upper and lower bounds for $r(S(n,m);k)$ when $k \geq 3$ and prove that $r(S(n,m);k) = nk + m + 2$ when $k \geq 3$ is odd and n is sufficiently large. We also investigate a generalization of the Ramsey number known as the list Ramsey number. Let $L : E(K_n) \to \binom{\mathbb{N}}{k}$ be an assignment of k-element subsets of $\mathbb{N}$ to the edges of K_n. A coloring $c : E(K_n) \to \mathbb{N}$ is said to be an *L-coloring* if $c(e) \in L(e)$ for all $e \in E(K_n)$. The k-color *list Ramsey number* $r_\ell(H;k)$ of a graph H is defined as the smallest n such that there is some $L : E(K_n) \to \binom{\mathbb{N}}{k}$ for which every L-coloring of K_n contains a monochromatic copy of H. In this thesis, we study $r_\ell(S(1,1);p)$ and $r_\ell(S_n;p)$ where p is an odd prime number.

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TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1: INTRODUCTION

We begin this thesis by introducing basic definitions and notation. Let $\mathbb{N}$ be the set of natural numbers. For any $n \in \mathbb{N}$, define $[n] \coloneqq \{1, 2, \ldots, n\}$. For a finite set V, we use $|V|$ to denote number of elements in V and $[V]^2$ to be the set of 2-element subsets of V. A graph is a pair $G = (V, E)$ of sets where $E \subseteq [V]^2$. The vertex set of a graph G is denoted $V(G)$, its edge set $E(G)$. We say that $|V(G)|$ is the *order* of G and $|E(G)|$ is the *size* of G. We frequently write the order and size of G as $|G|$ and $e(G)$ respectively. For convenience, we write the edge $\{u, v\} \in E(G)$ as uv. We say u, v are *adjacent* if $uv \in E(G)$. If $e = uv \in E(G)$, then u and v are *incident* with e. The two vertices incident with an edge are the *ends* of that edge. A graph G is *complete* if its vertices are pairwise adjacent. The *complement* of a graph G, denoted $\overline{G}$, is the graph with vertex set $V(\overline{G}) = V(G)$ and edge set $E(\overline{G}) = \{uv : uv \notin E(G)\}$. A graph H is said to be a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For $U \subseteq V(G)$, $G[U]$ is the graph with vertex set U and edge set $\{uv \in E(G) : u, v \in U\}$. The *neighborhood* of a vertex $v \in V(G)$ is the set $N_G(v) = \{u : uv \in E(G)\}$, and the *degree* of v in G is $d_G(v) = |N(v)|$. When the context is clear we omit the subscript in $N_G(v)$ and $d_G(v)$. The *minimum degree*, denoted $\delta(G)$, is $\min\{d(v) : v \in V(G)\}$. Similarly, the *maximum degree*, denoted $\Delta(G)$, is $\max\{d(v) : v \in V(G)\}$. For subsets A, B of $V(G)$ an (A, B)-edge is an edge with one end in A and one end in B.

Now we define some useful graphs. The *join* of two graphs G and H, written $G + H$, is the graph obtained from disjoint copies of G and H by adding an edge between each pair of vertices u, v where $u \in V(G)$ and $v \in V(H)$. Given a graph G and positive integer n, the graph nG is the disjoint union of n copies of G. The *complete graph* on n vertices is a K_n and $\overline{K_n}$ is an *independent set*. The complete bipartite graph $K_{n,m}$ is $\overline{K_n} + \overline{K_m}$. A *path* is a nonempty graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$, edge set $E = \{v_i v_{i+1} : i \in [n-1]\}$ and is denoted P_n. We often write a path as a sequence of vertices $v_1v_2\dots v_{n-1}v_n$, where v_iv_{i+1} are adjacent for all $i \in [n-1]$. A *matching* of size n is a nK_2. The graph S_n is a *star* on $n+1$ vertices and is a graph with exactly one vertex with degree n and n vertices of degree 1. The *center* of a S_n is the vertex of degree n and a *leaf* is a vertex of degree 1. The *double star* $S(n,m)$, where $n \geq m \geq 1$, is the graph consisting of the disjoint union of two stars S_n and S_m together with an edge joining their centers. A *subdivided star* S_n^t where $n \geq t \geq 1$ is the graph obtained from S_n by subdividing t distinct edges exactly once.

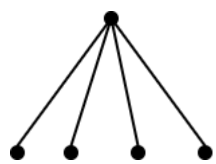

Figure 1.1: S_4

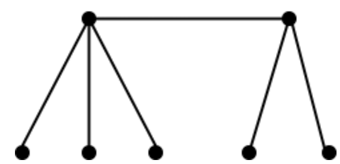

Figure 1.2: $S(3,2)$

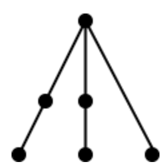

Figure 1.3: S_3^2

We end this part of the introduction with definitions concerning k-edge colorings and Ramsey numbers. A k-edge-coloring of a graph G is a function $\tau : E(G) \to [k]$ that assigns a number to each edge in G. A k-edge-coloring is *proper* if adjacent edges receive different colors. Given a graph G, the smallest positive integer k such that there exists a proper k-edge-color of G is the chromatic index of G, written $\chi'(G)$. A subgraph H of a k-edge-colored graph G is *monochromatic* if $\tau(e) = i$ for all $e \in E(H)$ where $i \in [k]$. For graphs $G, H_1, H_2, \dots, H_k$ we write $G \longrightarrow (H_1, H_2, \dots, H_k)$ if every k-edge-coloring of G contains a monochromatic copy of H_i in color i for some $i \in [k]$. Similarly, we write $G \not\longrightarrow (H_1, H_2, \dots, H_k)$ if there exists a k-edge-coloring of G with no monochromatic H_i in color i for all $i \in [k]$. Such a coloring c is a *critical k-coloring*. When $H = H_1 = H_2 = \ldots = H_k$ we write $G \longrightarrow (H; k)$ and $G \not\longrightarrow (H; k)$ respectively. The *k-color Ramsey number* of the graphs $H_1, H_2, \ldots H_k$ is the smallest integer N such that $K_N \longrightarrow (H_1, H_2, \ldots, H_k)$ and is denoted by $r(H_1, H_2, \ldots, H_k)$. When $H = H_1 = H_2 = \ldots = H_k$ we write $r(H; k)$ instead of $r(H_1, H_2, \ldots, H_k)$.

The main focus of this thesis is an area of mathematics called Ramsey theory. Ramsey theory is a subfield of combinatorics that is primarily concerned with finding or estimating the smallest size of a collection of objects that guarantees the existence of a specific ordered pattern. In this thesis we focus on graph Ramsey theory which grew from the following theorem known as Ramsey's theorem.

Theorem 1.0.1 (Ramsey [15])**.** *For a given positive integer k, and any positive integers $n_1, n_2, \ldots, n_k$, there exists a number $r(n_1, n_2, \ldots, n_k)$ such that if the edges of a complete graph G on $r(n_1, n_2, \ldots, n_k)$ vertices are colored with k distinct colors, then for some $i \in [k]$, G has K_{n_i} as a subgraph, with all edges of K_{n_i} colored by color i. That is, $r(n_1, n_2, \ldots, n_k)$ is the minimum number of vertices r such that $K_r \longrightarrow (K_{n_1}, K_{n_2}, \ldots, K_{n_k})$.*

Ramsey theory is a notoriously difficult field of mathematics. Despite being very active, Ramsey numbers are known for very few classes of graphs. To illustrate the idea behind Ramsey theory, we look at a simple example.

Example 1. $r(S(2,1); 2) = 6$.

Proof. In order to prove that $r(S(2,1); 2) = 6$, we must show that $6 \leq r(S(2,1); 2)$ and $r(S(2,1); 2) \leq 6$. First we show $6 \leq r(S(2,1); 2)$. To do this we show $K_5 \not\longrightarrow (S(2,1); 2)$. We can partition K_5 into edge-disjoint C_5 and $\overline{C_5}$. Note $\overline{C_5}$ is also a C_5. Color the edges of one C_5 using blue and the other red. Under this coloring K_5 does not contain a monochromatic

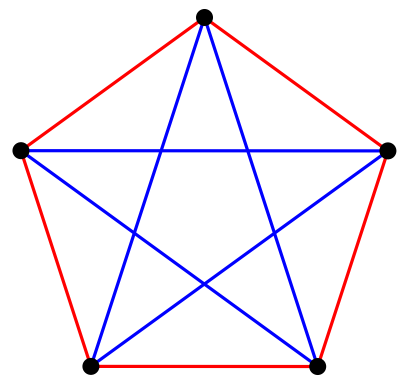

Figure 1.4: A 2-edge-coloring of K_5 with no monochromatic $S(2,1)$.

copy of $S(2,1)$, as desired. This coloring is shown in the figure above. Now we show that $r(S(2,1);2) \leq 6$. We must show that $K_5 \longrightarrow (S(2,1);2)$. Let G be a 2-edge-colored K_6 and label the vertices of G as $v_1, v_2, \ldots, v_6$. By the pigeonhole principle, v_6 is incident with at least three edges of the same color, say red. We may assume $v_6 v_i$ is red for all $i \in [3]$. If there exists $i \in [3]$ and $j \in \{4,5\}$ such that $v_i v_j$ is red, then we have a red $S(2,1)$. We may assume that for all $i \in [3]$ and $j \in \{4,5\}$, $v_i v_j$ is blue. Then we have a blue $S(2,1)$ where v_4 is the center of a blue S_2 with leaves $\{v_1, v_2\}$ and v_3 is the center of a blue S_1 with $\{v_5\}$ as a leaf. This completes the proof. $\square$

While this example is simple, very little is known about Ramsey numbers. The prominent mathematician Paul Erdős once said if aliens demanded that we find $r(K_6;2)$ in a year we would have no choice but to launch a preemptive attack, suggesting the difficulty of computing Ramsey numbers.

In Chapter 2 we introduce Ramsey theory as a field of study and discuss previous work related to this thesis. We discuss the history, classical problems, and its interaction with other fields of mathematics.

In Chapter 3 we introduce the list Ramsey number, a recent variation on the Ramsey number. We discuss the previous work on list Ramsey numbers, and open questions.

In Chapter 4 we begin presenting our original research. The focus in this section is our results on Ramsey numbers of double stars and subdivided stars.

In Chapter 5 we prove original results for the list Ramsey numbers of stars, double stars, and subdivided stars.

In Chapter 6 we discuss future work.

CHAPTER 2: RAMSEY THEORY AND KNOWN RESULTS

Background

While our focus is on graph Ramsey theory, problems in Ramsey theory deal with many types of mathematical structures. Ramsey's original theorem said, in essence, that complete disorder is impossible; every large enough collection of objects contains an ordered pattern. One of the earliest results in Ramsey theory is Van Der Waerden's Theorem in 1927, which studied patterns in subsets of the positive integers. A *k-term arithmetic progression* is a sequence of positive integers of the form $a + id$, where $a, d \in \mathbb{N}$ and $i \in \{0, 1, \ldots, k-1\}$.

Theorem 2.0.1 (Van Der Waerden)**.** *Every finite coloring of the positive integers contains arbitrarily long monochromatic arithmetic progressions.*

This theorem was later generalized by Szemeredi and played a vital role in the proof of what is now known as the Green-Tao theorem. Other interesting applications of Ramsey theory are Schur's theorem and the Ramsey-theoretic proof that Fermat's last theorem is false over finite fields. For more information on applications of Ramsey theory, ranging from Ergodic theory to automated theorem proving see [16].

The legendary mathematician Paul Erdős is responsible for the rapid development of Ramsey theory, and combinatorics during the twentieth century. Erdős' first exposure to Ramsey theory was due to Esther Klein who proposed the question: "Is it true that for all n, there is a least integer K(n) so that any set of K(n) points in the plane in general position must alwyas contain the vertices of a convex n-gon?" [12]. Erdős and Szekeres answered this question in the paper "A combinatorial problem in geometry "[6]. In this paper they proved the following theorem.

Theorem 2.0.2 (Erdős and Szekeres[6]). *Let s, l be positive integers. Then*

$$r(K_s, K_l) \leq \binom{k+l}{2}.$$

This bound was best for over 50 years. Trying to improve the bounds for the Ramsey numbers of cliques, $r(K_s, K_l)$, has received considerable attention and lead to the development of many new mathematical tools. For example, the development of the probabilistic method and random graphs by Paul Erdős was largely influenced by his desire to bound $r(K_s, K_t)$. These tools have come to revolutionize combinatorics, theoretical computer science and numerous other fields.

Known results

In this section, we discuss results in Ramsey that are useful to this thesis. In 1973, Burr and Roberts in [3] completely determined $r(S_{n_1}, \ldots, S_{n_k})$ for all positive integers $n_1, \ldots n_k$.

Theorem 2.0.3 (Burr and Roberts [3]). *Let $n_1, \ldots n_k$ be positive integers and t be the number of these that are even. Then*

$$r(S_{n_1}, \ldots, S_{n_k}) = \sum_{i=1}^{k} n_i - k + \varepsilon_t,$$

where $\varepsilon_t = 1$ if t is even and $\varepsilon_t = 2$ otherwise.

By letting $n_1 = \ldots = n_k$ we obtain a useful corollary.

Corollary 2.0.4 (Burr and Roberts [3]). *Let n and k be positive integers. Then*

$$r(S_n; k) = \begin{cases} (n-1)k+1 & \text{if } n \text{ and } k \text{ are even,} \\ (n-1)k+2 & \text{otherwise.} \end{cases}$$

Stars are one of two classes of graphs whose Ramsey number is completely determined. The other class of graphs whose k-color Ramsey number is known for all $k \geq 2$ are matchings. The k-color Ramsey number for matchings was determined by Cockayne and Lorimer in [4].

Theorem 2.0.5 (Cockayne and Lorimer [4]). *For all positive integers n and k,*

$$r(nK_2; k) = nk + n - k + 1.$$

The Ramsey number for P_4 is well-studied and is known for all k not congruent to 0 modulo 3 due to Irving in [11]. Note $P_4 = S(1, 1)$.

Theorem 2.0.6 (Irving [11]).

$$r(P_4; k) = \begin{cases} 2k+2 & \text{if } k \equiv 1 \pmod 3, \\ 2k+1 & \text{if } k \equiv 2 \pmod 3, \\ 2k \text{ or } 2k+1 & \text{if } k \equiv 0 \pmod 3. \end{cases}$$

Now we look at bounds for the Ramsey numbers of more general families of graphs. Erdős and Graham proved bounds for the Ramsey numbers of trees and forests in [5].

Theorem 2.0.7 (Erdős and Graham [5]). *Let T be a tree on n vertices. Then,*

$$(k-1)\frac{n-1}{2} < r(T; k) \leq 2kn + 1.$$

In [2] Burr notes it is likely that $r(T;k) \sim kn$ as $k \to \infty$. The situation for forests seems more complicated.

Theorem 2.0.8 (Erdős and Graham [5])**.** *If F is a forest on n edges, then*

$$\tfrac{k\sqrt{n}-1}{2} < r(F;k) < 4kn.$$

Moreover, if $k \leq n^2$, then

$$A\sqrt{kn} < r(F;k),$$

where A is a positive universal constant.

Interestingly, when the number of colors is small compared to the number of vertices there is the possibility that $r(F;k)$ grows in $\sqrt{k}$. This is confirmed for the disjoint union of stars, mS_m.

Theorem 2.0.9 (Erdős and Graham [5])**.** *There is a constant A_1 such that, if $k \leq m$, then*

$$r(mS_m;k) < A_1\sqrt{k}m^2.$$

Also, if $k \geq 3m^2$, then

$$r(mS_m;k) \leq 3km.$$

This phenomena where the Ramsey number behaves differently depending on the relationship between the parameters is very common. It occurs for double stars below and in a number of other results in this paper. See [2] for more information on the above results and multicolor Ramsey numbers.

In this paper we are interested in Ramsey numbers for double stars, $r(S(n,m);k)$. For $k=2$, there are two main results.

Theorem 2.0.10 (Grossman, Harary and Klawe [10]).

$$r(S(n,m);2)=\begin{cases}\max(2n+1,n+2m+2) \ \textit{if } n \textit{ is odd and } m\le 2,\\ \max(2n+2,n+2m+2) \ \textit{if } n \textit{ is even or } m\ge 3, \textit{ and } n\le\sqrt{2}m \textit{ or } n\ge 3m.\end{cases}$$

In [10] the authors also conjecture the following:

Conjecture 2.0.11 (Grossman, Harary and Klawe [10]).

$$r(S(n,m);2)=\begin{cases}\max(2n+1,n+2m+2) \ \textit{if } n \textit{ is odd and } m\le 2,\\ \max(2n+2,n+2m+2) \ \textit{otherwise.}\end{cases}$$

In 2016, Norin, Sun and Zhao in [14] disproved the conjecture of Grossman, Harary, and Klawe and extended their result.

Theorem 2.0.12 (Norin, Sun and Zhao [14]).

$$r(S(n,m);2)\ge\begin{cases}\frac{5}{6}m+\frac{5}{3}n+o(m) \ \textit{for all } n\ge m\ge 1,\\ \frac{21}{23}m+\frac{189}{115}n+o(m) \ \textit{for all } n\ge 2m.\end{cases}$$

Furthermore, for $1\le m\le n\le 1.699(m+1)$, *we have*

$$r(S(n,m);2)=\max(2n+2,n+2m+2).$$

This disproves Conjecture 2.0.11 for $\frac{7}{4}m+o(m)\le n\le\frac{105}{41}m-o(m)$.

In general, computing Ramsey numbers is very difficult, for $k = 2$ little is known and even less is known for $k \geq 3$. Due to the difficulty, the field has been concerned with determining the asymptotic behavior of the Ramsey function. There are also many variants of the classical Ramsey number. These variants may consider parameters other than the size of the graph to guarantee the existence of an ordered substructure. Some example of well-studied variants of the classical Ramsey number are the size Ramsey number, induced Ramsey number, and degree Ramsey number. The next chapter of this paper is devoted to a new variant of the classical Ramsey number defined by Alon, Bucić, Kalvari, Kuperwasser, and Szabó [1].

CHAPTER 3: LIST RAMSEY NUMBERS

Recently, Alon, Bucić, Kalvari, Kuperwasser, and Szabó [1] developed a generalization of the Ramsey number known as the list Ramsey number. This generalization is motivated by analogous generalizations in the field of graph coloring, which is central to graph theory. Let $L : E(K_n) \to \binom{\mathbb{N}}{k}$ be an assignment of k-element subsets of $\mathbb{N}$ to the edges of K_n. A coloring $c : E(K_n) \to \mathbb{N}$ is said to be an *L-coloring* if $c(e) \in L(e)$ for all $e \in E(K_n)$. A graph G is *k-edge-choosable* if for every $L : E(G) \to \binom{\mathbb{N}}{k}$, there exists a proper L-coloring of $E(G)$. The list chromatic index of G, $\chi'_\ell(G)$, is the smallest k such that G is k-edge-choosable.

The k-color *list Ramsey number* $r_\ell(H; k)$ of a graph H is defined as the smallest n such that there is some $L : E(K_n) \to \binom{\mathbb{N}}{k}$ for which every L-coloring of K_n contains a monochromatic copy of H.

By choosing L to assign each edge the same list, we see that $r_\ell(H; k) \leq r(H; k)$. The two primary papers on list Ramsey numbers are [1] and [7]. In [1] the authors introduce the notion of list Ramsey numbers and prove several results. One of the main questions is to find graphs for which the two Ramsey numbers agree.

Question 3.0.1. *Find graphs H such that for all $k \geq 2$, $r_\ell(H; k) = r(H; k)$.*

Little is known for Problem 3.0.1. The authors of [1] proved the following results. First they proved a lemma that relates the list Ramsey number of stars to the list chromatic index of graphs. This lemma is crucial to their lower bound constructions.

Lemma 3.0.2 (Alon, Bucić, Kalvari, Kuperwasser, and Szabó [1])**.** *Let $G_1, \ldots, G_s$ be graphs that partition the edge set of K_n. If $\chi'_\ell(G_i) \leq k$ for all i and each vertex of K_n belongs to at most $t-1$ of G_i's, then*

$$r_\ell(S_t;k) > n.$$

Theorem 3.0.3 (Alon, Bucić, Kalvari, Kuperwasser, and Szabó [1])**.** *For any $n,k \in \mathbb{N}$, except possibly finite integers n for each odd k, we have $r_\ell(S_n;k) = r(S_n;k)$. More precisely,*

(a) For every $n,k \in \mathbb{N}$, we have $(n-1)k+1 \le r_\ell(S_n;k)$. In particular, $r_\ell(S_n;k) = (n-1)k+1 = r(S_n;k)$ whenever n and k are both even.

(b) For every $k \in \mathbb{N}$ there exists $w(k) \in \mathbb{N}$ such that the following holds. For every k and $n \ge w(k)$ that are not both even, we have $r_\ell(S_n;k) = (n-1)k+2 = r(S_n;k)$.

They conjecture the two Ramsey numbers are always equal for stars.

Conjecture 3.0.4. *For any $r,k \in \mathbb{N}$*

$$r_\ell(S_n;k) = r(S_n;k).$$

Interestingly, authors of [1] proved that unlike for stars, the ordinary Ramsey number $r(tK_2;k) = tk+t-k+1$ is significantly larger than the list Ramsey number $r_\ell(tK_2;k)$ for most values of the parameters. This shows how the relationship between the list Ramsey number and Ramsey number can differ drastically depending on the class of graphs being studied.

Theorem 3.0.5 (Alon, Bucić, Kalvari, Kuperwasser, and Szabó [1])**.** *Let $r,k \in \mathbb{N}$. If $2(k+1) \le \log r$, then*

$$2r \le r_\ell(rK_2;k) \le 2r + 42r^{\frac{k}{k+1}}.$$

If $2(k+1) > \log r > 0$, then

$$\frac{rk}{4\log rk} \leq r_\ell(rK_2; k) \leq \frac{34rk}{\log rk}.$$

Theorem 3.0.6 (Alon, Bucić, Kalvari, Kuperwasser, and Szabó [1])**.** *For any fixed $k \geq 2$ and r tending to infinity, we have $r_\ell(rK_2; k) = 2r + o(r)$. In particular,*

$$\frac{r(rK_2;k)}{r_\ell(rK_2;k)} = (k+1)/2 + o(1).$$

For any fixed $r \geq 1$ and k tending to infinity, we have $r_\ell(rK_2; k) = \Theta(k/\log(k))$. In particular,

$$\frac{r(rK_2;k)}{r_\ell(rK_2;k)} = \Theta(\log(k)).$$

The list chromatic index of graphs plays a key role in determining the values of list Ramsey numbers of graphs. In particular, the result of Galvin [9], that $\chi'_\ell(G) = \chi'(G) = \Delta(G)$ if G is a bipartite graph, in the proofs in [1]. Also, Lemma 3.0.2 is key in constructing the lower bound for $r_\ell(S_n; k)$. It is likely that lower bound constructions for other graphs will similarly depend on list chromatic indexes.

In [7] Fox, He, Luo and Xu investigate the growth rate of list Ramsey numbers. In [1] the authors ask if $r_\ell(H; k)$ grows exponentially in k for a fixed graph H with $\pi(H) > 0$. Fox, He, Luo and Xu answer this in the positive. For a set V define $[V]^r$ to be the set of r-element subsets of V where $2 \leq r \leq |V|$. An r-graph is a pair $G = (V, E)$ of sets where $E \subseteq [V]^r$. For an r-graph H, the *Turán density* is defined as $\pi(H) = \lim_{n\to\infty} ex(n, H)\binom{n}{r}^{-1}$, where $ex(n, H)$ is the largest number of edges in an n-vertex graph that does not contain H as a subgraph.

Theorem 3.0.7 (Fox, He, Luo and Xu [7], 2021)**.** *If H is an r-graph which is not r-partite then, for any $k \geq 1$*

$$r_\ell(H;k) \geq c_r(1-\pi(H))^{-k/(r-1)},$$

where $c_r = ((r-2)!/e)^{1/(r-1)}$.

This along with theorem 5 and 6 in [1] implies that $r_\ell(H;k)$ grows exponentially if and only if H is not r-partite. Otherwise, $r_\ell(H;k)$ grows polynomially. The special case when $H = K_3$ has special significance. Determining if $r(K_3;k)$ grows exponentially or superexponentially is one of the oldest and most famous problems in Ramsey theory.

Theorem 3.0.8 (Fox, He, Luo and Xu [7], 2021)**.** *For all* $k \geq 1$, $r_\ell(K_3;k) \geq \frac{1}{e} \cdot 2^k$.

Together with the upper bound obtained in [1], Theorem 3.0.8 implies that

$$\frac{1}{e} \cdot 2^k \leq r_\ell(K_3;k) \leq (4+o(1))^k.$$

It remains unknown whether $r_\ell(K_3;k) < r(K_3;k)$, as the current best lower bound for $r(K_3;k) > 3.199^k$ is not large enough.

CHAPTER 4: MULTICOLOR RAMSEY NUMBERS OF DOUBLE STARS AND SUBDIVIDED STARS

In this chapter we prove results for the Ramsey numbers of double stars and subdivided stars. Our research was motivated by the conjecture of Alon, Bucić, Kalvari, Kuperwasser, and Szabó, that $r_\ell(S_n;k) = r(S_n;k)$. We prove non-trivial upper and lower bounds for $r(S(n,m);k)$. Using these bounds, we explicitly determine the value of $r(S(n,m);k)$ for n sufficiently large. We believe the techniques used will be helpful for determining multicolor Ramsey numbers for other cases of $S(n,m)$ and subdivided stars, S_n^t.

Double stars

We start with the Ramsey number of $S(n,1)$ for $k=2$ and $k=3$.

Proposition 4.0.1. *For all $n \geq 2$*

$$r(S(n,1);2) = r(S_{n+1};2) = \begin{cases} 2n+2 \text{ if } r \text{ is even} \\ 2n+1 \text{ if } r \text{ is odd} \end{cases}$$

.

Proof. Since $S_{n+1} \subseteq S(n,1)$, it suffices to show $r(S(n,1);2) \leq r(S_{n+1};2)$. Let $n \geq 2$ and $N = r(S_{n+1};2)$. Let G be a 2-edge-colored complete graph on N vertices. By definition of $r(S_{n+1};2)$, every 2-edge-coloring of K_N contains a monochromatic copy of S_{n+1}. Let S denote the monochromatic S_{n+1} subgraph of G. Let v be the center and $L = \{v_1, v_2 \ldots, v_{n+1}\}$ be the leaves of S. Assume n is odd and S is monochromatic in color 1. Then $|G \setminus S| = n \geq 2$. If any edge between the leaves of G to $G \setminus S$ is colored using color 1, we have a monochromatic $S(n, 1)$. We may assume that all the edges between L and $G \setminus S$ are colored using 2. Let $u, w \in G \setminus V(S)$. We have a monochromatic S_{n+1}, say S', in color 2 with center u and leaves $\{v_1, \ldots, v_{n+1}\}$. We obtain a monochromatic $S(n, 1)$ in color 2 by adding the vertex w and edge $v_1 w$ to S'. For $n \geq 2$ even this same proof works since, $|G \setminus S| = n + 1 \geq 2$. Therefore, $r(S(n, 1); 2) = r(S_{n+1}; 2)$. □

Now we compute the exact value of $r(S(r, 1); 3)$.

Proposition 4.0.2. *For $r \geq 2$,*

$$r(S(r, 1); 3) = 3r + 3.$$

Proof. First we show that $r(S(r, 1); 3) > 3r+2$. Let $n = 3r+2$ and G_1, G_2, G_3 be a partition of $V(K_n)$ with $|G_1| = r + 2, |G_2| = |G_3| = r$. Color all the edges within G_i using color i. Each G_i is a monochromatic clique of size $|G_i|$ in color i. Fix two vertices $u, w \in G_1$. Color all the edges between $\{u, w\}$ and G_2 using color 2 and the all edges between $\{u, w\}$ and G_3 using color 3. Now color all the edges between $G_1 \setminus \{u, w\}$ and G_2 using color 3 and the edges between $G_1 \setminus \{u, w\}$ and G_3 using color 2. Color the edges between G_2 and G_3 using color 1. This has no monochromatic $S(r, 1)$, and so $r(S(r, 1); 3) \geq 3r + 3$.

It remains to show that $r(S(r, 1); 3) \leq 3r + 3$. Let $n = 3r + 3$ and G a 3-edge-colored K_n. Let H denote a monochromatic copy of S_{r+1} in G. We may assume H is monochromatic in color 1. Let $x, v \in H$ be the center and a leaf in H respectively. If v is incident with an edge in color 1 with end in $G \setminus H$, then we have a monochromatic $S(r, 1)$. So we may assume that the edges between $H \setminus x$ and $G \setminus H$ are in color 2 or 3. Note, $|G \setminus H| = 2r + 1$ and so each $v \in H \setminus x$ is the center of a monochromatic S_{r+1} with leaves in $G \setminus H$, say in color 2. Let

$X \subseteq G \setminus H$ be the leaves of this monochromatic S_{r+1}. Then if there is any $(X, G \setminus H)$-edge in color 2 we are done. Thus, they form a monochromatic $K_{r+1,r}$ in color 3. Since $r \geq 2$, this gives us a monochromatic $S(r, 1)$. $\square$

To prove our next result, we need what is known as Petersen's two factor theorem. A graph G is *d-regular* if $d(v) = d$ for all $v \in V(G)$. Furthermore, a subgraph H of G is *spanning* if $V(H) = V(G)$.

Theorem 4.0.3 (Petersen)**.** *For every positive integer k, every $2k$-regular graph can be decomposed into k 2-regular spanning subgraphs.*

We use Petersen's two factor theorem to prove a lower bound for $r(S(2, 1); k)$ when k is odd. Observe that this agrees with the lower bound in Theorem 4.0.5, but is a very different construction.

Proposition 4.0.4. *For all odd $k \geq 3$,*

$$r(S(2, 1); k) \geq 2k + 3.$$

Proof. Let j be a nonnegative integer, $k = 2j + 1$, and $n = 2k + 2$. Let $G = K_n$ and $G_1, G_2, \ldots, G_{j+1}$ be a partition of G into disjoint graphs on 4 vertices. For each $i \in [j + 1]$, color all the edges of G_i using color $2j + 1$. Now consider the graph $H = G \setminus \bigcup_{i=1}^{j+1} E(G_i)$ and note that it is $4j$-regular. By Petersen's 2-factor-theorem we can partition H into $2j$ edge-disjoint 2-factors, $H_1, H_2, \ldots, H_{2j}$. For $i \in [2j]$ color all the edges of H_i using color i. For $i \in [2j]$, H_i is 2-regular, so it cannot contain a $S(2, 1)$. The color class for color $2j + 1$ is $\bigcup_{i=1}^{j+1} G_i$. This graph is disconnected and each component has 4 vertices. Thus, it cannot contain $S(2, 1)$ as a subgraph. Hence, this is a k-coloring of $E(K_n)$ with no monochromatic $S(2, 1)$, as desired. $\square$

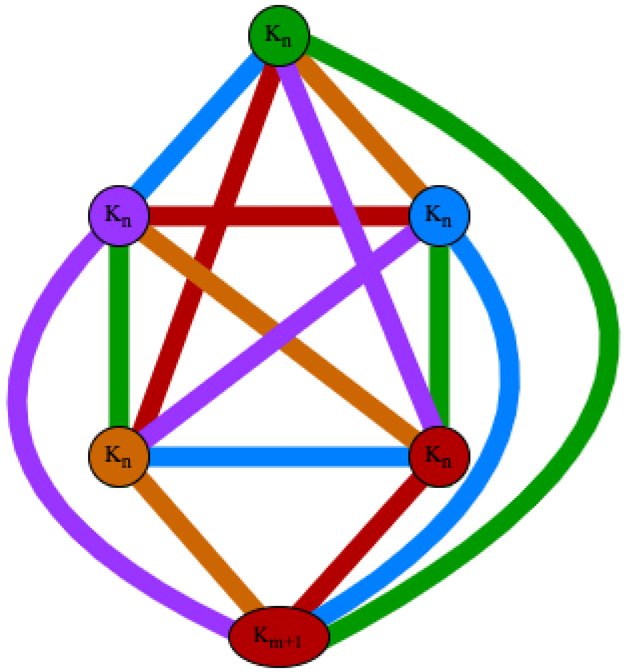

Figure 4.1: $K_{5n+m+1} \not\to (S(n,m);k)$

Now we prove our main results.

Theorem 4.0.5. *For $n \geq m \geq 1$ and odd $k \geq 3$,*

$$r(S(n,m);k) \geq nk + m + 2.$$

Proof. We show that $r(S(n,m);k) > nk+m+1$. First we present a critical k-coloring of K_{nk} and then extend that to a critical k-coloring of K_{nk+m+1}. Let $G = K_{nk}$ and $V_1, V_2, \ldots, V_k$ be a partition of $V(G)$ where each part contains exactly n vertices. Now let $H = K_k$ be obtained from G by contracting each part V_i into a single vertex v_i. Then the largest independent set of edges in H is at most $(k-1)/2$ and $\chi'(H) = k$. This implies that in a proper coloring of $E(H)$ each color class consists of exactly $(k-1)/2$ edges. Let $c : E(H) \to [k]$ be a proper coloring. Let $d : V(H) \to [k]$ be a coloring of $V(H)$ such that $d(v_i)$ is colored using the coloring missing from its edges. Note that d is a bijection. For each pair of distinct $i, j \in [k]$ color all the edges between V_i and V_j using color $c(v_iv_j)$. For each $i \in [k]$ color all the edges with both ends in V_i using $d(v_i)$. Under this coloring of $E(G)$ each color class consists of $(k-1)/2$ disjoint $K_{n,n}$ and one K_n. To extend this to a critical k-coloring of K_{nk+m+1} add $m+1$ vertices, $S = \{u_1, u_2, \dots, u_{m+1}\}$ and color $E(G[S])$ using $d(v_1)$. For each $i \in [k]$ color all the edges between S and V_i using $d(v_i)$. Figure 4.1 illustrates a critical k-coloring of K_{nk+m+1} when $k = 5$. Observe that the color class of $d(v_1)$ consists of a K_{n+m+1} and $(k-1)/2$ disjoint copies of $K_{n,n}$ and all other color classes consist of a $(m+1)K_1 + K_n$ and $(k-1)/2$ disjoint copies of $K_{n,n}$. Thus, there is no monochromatic copy of $S(n, 1)$. □

We now show that the lower bound given by Theorem 4.0.5 is sharp for all $k \geq 3$ odd and n sufficiently large. In order to do this, we need Lemma 4.0.6. Its proof follows from a double counting argument and can be found in [13, Proposition 1.7].

Lemma 4.0.6. *Let $\mathcal{F}$ be a family of subsets of some set X. For each $x \in X$, we define $p(x)$ to be the number of members of $\mathcal{F}$ containing x. Then*

$$\sum_{x \in X} p(x) = \sum_{F \in \mathcal{F}} |F|.$$

Observe that Lemma 4.0.6 is a generalization of the handshaking lemma; for any graph G, $\sum_{v \in V(G)} d(v) = 2\,|E(G)|$.

Theorem 4.0.7. *Let $k \geq 2$ and $n \geq m \geq 1$ be integers. If $(n+1) \cdot \lceil \frac{n+1}{k-1} \rceil > m((k-1)n+m)$, then*

$$r(S(n, m); k) \leq kn + m + 2.$$

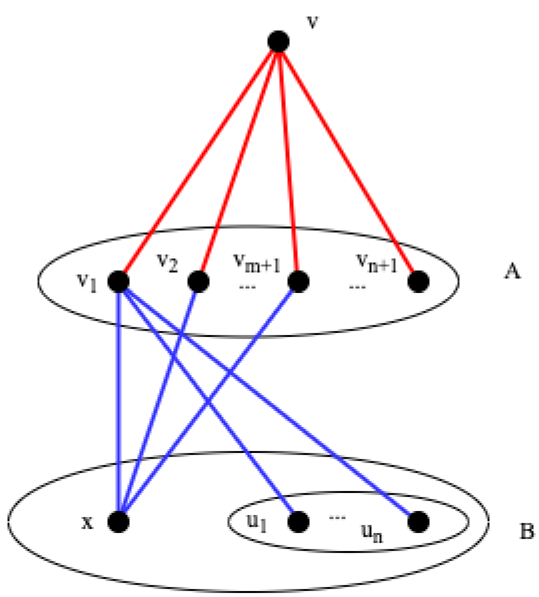

Figure 4.2: How the existence of an x with $p(x) \geq m+1$ implies a monochromatic $S(n, m)$.

Proof. Let G be a complete, k-edge-colored K_{kn+m+2} using colors $[k]$. Then G contains a monochromatic S_{n+1}, say in color k. We use H to represent this monochromatic S_{n+1} in color k. Let A be the leaves of H and $B = G \setminus H$. If any $v \in A$ is incident with m edges with ends in B colored using k we are done. This implies each $v \in A$ is incident to at least $(k-1)n + m - (m-1) = (k-1)n + 1$ edges colored using colors $[k-1]$ with ends in B. By the pigeonhole principle, each $v \in A$ is the center of a monochromatic S_{n+1}, with leaves in B, in some color in $[k-1]$. Furthermore, there are at least $(n+1)/(k-1)$ monochromatic stars $H_1, H_2, \ldots H_{\lceil (n+1)/(k-1) \rceil}$, say in color $k-1$, with centers in A and leaves in B. For each $i \in [\lceil (n+1)/(k-1) \rceil]$, let L_i be the leaves of H_i. Without loss of generality, suppose v_i is the center of H_i for each $i \in [\lceil (n+1)/(k-1) \rceil]$. Now we view $\mathcal{F} = \{L_1, L_2, \ldots, L_{\lceil (n+1)/(k-1) \rceil}\}$ as a family of subsets of B. For $x \in B$ let $p(x)$ be the number of members of $\mathcal{F}$ containing x. Suppose some $x \in B$ has $p(x) \geq m+1$. We may assume x lies in $L_1, L_2, \ldots, L_{m+1}$. Then we obtain a monochromatic $S(n, m)$ where v_1 is the center of an S_n and x is the center of an S_m. This is illustrated by Figure 4.2.This implies that $p(x) \leq m$ for all $x \in B$. By choice of n we have,

$$\begin{aligned}\sum_{x \in B} p(x) &= m((k-1)n + m) \\ &< (n+1) \cdot \lceil (n+1)/(k-1) \rceil \\ &= \sum_{L \in \mathcal{F}} |L|,\end{aligned}$$

a contradiction. □

Combining Theorem 4.0.5 and Theorem 4.0.7 results in the following corollary.

Corollary 4.0.8. *Let $n \geq m \geq 1$ be integers. If $k \geq 3$ is odd and $(n+1) \cdot \left\lceil \frac{n+1}{k-1} \right\rceil > m((k-1)n + m)$, then*

$$r(S(n,m);k) = kn + m + 2.$$

Thus, we have computed the exact value for $r(S(n,m);k)$ provided that n is large enough compared to m and k. Now using a different technique, we obtain a similar result for $S(n,1)$.

Lemma 4.0.9. *Let $n, k \geq 2$ be positive integers. Then every k-edge-coloring of $K_{k(k-1)+1,k(n+1)}$ contains a monochromatic $S(n,1)$.*

Proof. First we show that every 2-edge-coloring of $K_{3,2(n+1)}$ contains a monochromatic copy of $S(n,1)$. Let G be a 2-edge-colored $K_{3,2(n+1)}$ and A and B be the partite sets of size 3 and $2(n+1)$ respectively. Clearly, G must contain a monochromatic S_{n+1}. Let v be the center of this monochromatic star and $L = \{v_1, v_2, \ldots, v_{n+1}\}$ be the leaves. We may assume this star is colored using color 1. If there is a $(A \setminus \{v\}), L$-edge colored using 1, we are done. So we may assume all of the edges between $A \setminus \{v\}$ and L must be colored using color 2. Since $|A \setminus \{v\}| \geq 2$, this gives us a monochromatic $S(n,1)$. Now suppose the statement in this lemma holds for some $k-1 \geq 2$ and let G be a k edge colored of $K_{k(k-1)+1,k(n+1)}$. Let A and B be the partite sets of size $k(k-1)+1$ and $k(n+1)$ respectively. Clearly, G contains $k-1$ monochromatic copies of S_{n+1} in the same color, say k. Let U be the set containing the leaves of these $k-1$ monochromatic copies of S_{n+1} and V be the set containing their centers. If there exists a $(A \setminus V, U)$-edge colored using k, then we are done. So, all the edges between $A \setminus V$ and U must be colored using the colors $[k-1]$. Observe that

$$\begin{aligned} |A \setminus V| &= k(k-1)+1-(k-1) \\ &= (k-1)(k-1)+1 \\ &\geq (k-1)(k-2)+1, \end{aligned}$$

and $|U| = (k-1)(n+1)$. By our inductive assumption, every $k-1$ edge coloring of $K_{(k-1)(k-2)+1,(k-1)(n+1)}$ contains a monochromatic $S(n,1)$. Thus, there exists a monochromatic copy of $S(n,1)$ between $A \setminus V$ and U. $\square$

Theorem 4.0.10. *Let $k \geq 4$ and $n \geq (k-1)(k-3)+1$. Then $r(S(n,1);k) \leq nk+3$.*

Proof. Let $k \geq 4$ and $n \geq (k-1)(k-3)+1$. Let G be a k edge colored K_{nk+3}. We show G contains a monochromatic copy of $S(n,1)$. Since $r(S_{n+1};k) \leq nk+2$, G must contain a monochromatic S_{n+1}. Let v_0 be the center and $L_k = \{v_1, v_2, \ldots, v_{n+1}\}$ be the leaves of a monochromatic S_{n+1} in G. Without loss of generality, suppose this S_{n+1} is colored using k. Let $H = G \setminus \{v_0, v_1, \ldots v_{n+1}\}$. If there are any edges colored using k between H and L_k we are done. So we may assume all (H, L_k)-edges are colored using the colors $[k-1]$. Observe that $|H| = n(k-1)+1$. This implies that each $u \in S$ is the center of a monochromatic S_{n+1}. By choosing $n \geq (k-1)(k-3)+1$ we have that there are $k-2$ monochromatic copies of S_{n+1} in the same color, say $k-1$. Let L_{k-1} be the leaves and X

contain the centers of these $k-2$ monochromatic copies of S_{n+1} in color $k-1$ between H and L_k. If they are not disjoint, then we are done. Furthermore, if there are any edges in colors k or $k-1$ between L_{k-1} and $L_k \setminus X$ we are done. Thus, all the edges between L_{k-1} and $L_k \setminus X$ must be colored using $[k-2]$. Observe, $|L_{k-1}| = (k-2)(n+1)$ and $|L_k \setminus X| = n+1-(k-2) \geq (k-1)(k-3)+1-(k-3) = (k-2)(k-3)+1$. By the above lemma, we have a monochromatic $S(n,1)$, as desired. □

Combining Theorem 4.0.10 and Theorem 4.0.5 we obtain the following corollary.

Corollary 4.0.11. *Let $k \geq 5$ be odd and $n \geq (k-1)(k-3)+1$. Then $r(S(n,1);k) = nk+3$.*

Note that for for even k and n we have a gap of 1 between the upper and lower bounds for $r(S(n,1);k)$ when n is sufficiently large.

Corollary 4.0.12. *Let $k \geq 4$ and $n \geq (k-1)(k-3)+1$ be even. Then*

$$nk+2 \leq r(S(n,1);k) \leq nk+3.$$

Proof. If n is even, then $n+1$ is odd. Observe, $nk+2 \leq r(S_{n+1};k) \leq r(S(n,1);k) \leq nk+3$. □

Remark 4.0.13. *If we show that $nk+2 < r(S(n,1);k)$ for n,k even then we prove the above theorem but for n,k even.*

In the next section we look at the Ramsey number for a new class of graphs that we call subdivided stars.

Subdivided stars

By Proposition 4.0.1, it follows that $r(S_n^1;2) = r(S_n;2)$ for all $n \geq 3$.

Proposition 4.0.14. *For $n \geq 5$, $r(S_n^2; 2) = r(S_n; 2)$*

Proof. It suffices to show $r(S_n^2; 2) \leq r(S_n; 2)$. Let $n \geq 5$ be odd and G be a 2-edge-colored K_{2n} using colors red and blue. Given two sets A, B we say they are red (blue) complete if all (A, B)-edges are colored using red (blue). Then G contains a monochromatic copy of S_n^1, say S. Let $V(S) = \{v_0, v_1, \ldots, v_n, u_1\}$ such that $S[\{v_0, v_1, \ldots, v_n\}]$ forms a S_n with center v_0, and $v_1u_1 \in E(S)$. We may assume S is colored using red. Now let $B = G \setminus S$ and $A = \{v_2, v_3, \ldots, v_n\}$. If there are any red edges between A and B, then we have an S_n^2. So we may assume that all (A, B) edges are blue. Suppose $b_0 \in B$ has blue neighbor $u \in G \setminus B \cup A$. Then $b_0, c, v_2, \ldots, v_n$ forms a blue S_n. Since $n \geq 5$, $|B \setminus b| \geq 2$, we can form a blue S_n^2 with $b_0, c, v_2, \ldots, v_n, b_1, b_2$ where $b_1, b_2 \in B$. Now we may assume that each $b \in B$ has all red edges outside of B. Now we can form a red S_n^2 with $v_0, v_1, \ldots, v_n, b_0, b_1, u_1$ where v_0 is the center of a S_n with leaves $b_0, v_1, \ldots, v_{n-1}$ and pendant vertices u_1 and b_1 (u_1b_0, v_1b_1 are red). The proof where $n \geq 5$ even is the same. $\square$

Proposition 4.0.15. *Let $n \geq 10 - \varepsilon$ where $\varepsilon = 1$ if n is odd and $\varepsilon = 0$ otherwise. Then $r(S_n^3; 2) = r(S_n; 2)$.*

Proof. It suffices to show $r(S_n^3; 2) \leq r(S_n; 2)$. Let $n \geq 9$ be odd and G be a 2-edge-colored K_{2n} using colors red and blue. We use $d_r(x)$ and $d_b(x)$ to denote the number of red and blue edges incident to x respectively. Then G contains a monochromatic subgraph $H = S_n^2$. We may assume H is colored using red. Let v be the vertex of degree n in H and $v_1, v_2, \ldots, v_n$ be the neighbors of v in H where $N_H(v_i) = \{v, u_i\}$ for all $i \in [2]$. Let $A = \{v_3, \ldots, v_n\}$ and $B = G \setminus H$. Then $|A| = n - 2$ and $|B| = 2n - (n + 3) = n - 3$. If there is any red (A, B)-edge we are done. So, all (A, B)-edges are blue.

Claim 4.0.15.1. *For each $x \in A \cup B$, $d_b(x) < n$.*

Proof. Let $x \in A \cup B$ satisfy $d_b(x) \geq n$. If $x \in B$, then we can form a blue S_n by taking $n-2$ leaves from A and at most 2 leaves from B. There are still at least $|B| - 3 \geq 3$ other vertices in B. Since every (A, B)-edge is colored blue, we can choose any 3 of the $|B|-3 \geq 3$ remaining vertices in B and this along with x and its n blue neighbors form a monochromatic S_n^3. The proof for when $x \in A$ is the same. □

Thus, each $x \in A \cup B$ has $d_b(x) \leq n-1$ and $d_r(x) \geq n$.

Claim 4.0.15.2. *$G[B]$ is a monochromatic complete graph in color red.*

Proof. Assume $G[B]$ is not a monochromatic complete graph. Then there is a blue edge, xy with $x, y \in B$. If either x or y has blue-degree of at least n we are done. This implies that they have red edges to the rest of the graph. Now we can form a monochromatic red $H' = S_n^3$ where v is the vertex of degree n with neighbors $x, v_1, v_2, \ldots, v_{n-1}$ in H' and $N_{H'}(v_1) = \{v, u_1\}$, $N_{H'}(v_2) = \{v, y\}$, and $N_{H'}(x) = \{v, u_1\}$. □

This implies that $G[B]$ is a red monochromatic K_{n-3} and v is blue complete to B. Now each $x \in B$ has $d_b(x) \leq n-1$. Thus, each $x \in B$ must be red complete to the rest of the graph. If there is a red edge between A and $\{u_1, u_2\}$, then we have a red S_n^3 we the same S_n as H, but with an extra subdivided edge with end in B. Thus, all edges between A and $\{u_1, u_2\}$ are blue. We may assume each $x \in A$ has $d_b(x) \leq n-1$, and so $G[A]$ is a red K_{n-3} and all $(A, \{v_1, v_2\})$-edges are red. Now we obtain a red S_n^3 with center v_1, whose neighbors are all of B, $\{v_3, v_4, v_5\}$, and v_3v_6, v_4v_7, v_5v_8 are the subdivided edges.

Now let $n \geq 9$ be even. Let G be a 2-edge-colored K_{2n-1} using colors red and blue. Then G contains a monochromatic subgraph $H = S_n^2$. We may assume H is colored using red. Let v be the vertex of degree n in H and $v_1, v_2, \ldots, v_n$ be the neighbors of v in H where $N_H(v_i) = \{v, u_i\}$ for all $i \in [2]$. Let $A = \{v_3, \ldots, v_n\}$ and $B = G \setminus H$. Then $|A| = n-2$

and $|B| = 2n - 1 - (n + 3) = n - 4$. If there is any red (A, B)-edge we are done. So, all (A, B)-edges are blue. Now observe that the proofs above claims only rely on the fact that $|B| - 3 \geq 3$. Since $n \geq 10$, $|B| - 3 = n - 7 \geq 3$.

Now we can apply the above claims to get that for each $x \in A \cup B$ has $d_b(x) \leq n - 1$ and $d_r(x) \geq n$ and $G[B]$ is a monochromatic K_{n-4} or else we have a monochromatic S_n^3. If there is any red edge between v and B, then we obtain a monochromatic S_n^3. So, v is blue-complete to B. Now each $x \in B$ has $d_b(x) \geq n-1$, and so B is red-complete to $\{v_1, v_2, u_1, u_2\}$. Suppose there is a red edge between v and $\{u_1, u_2\}$, say vu_1. Now we have a monochromatic S_n^3 with center v neighbors $u_1, v_1, v_2, \ldots, v_{n-1}$, and u_1b_1, v_1b_2, v_2u_2 are red for some distinct $b_1, b_2 \in B$. Now we have v is blue complete to $\{u_1, u_2\}$. If there is any red $(\{u_1, u_2\}, A)$-edge, say u_1v_3 we have a monochromatic S_n^3 with center v has neighbors $v_1, v_2, \ldots, v_n$, and v_1b_1, v_2u_2, v_3u_1 are red for some $b_1 \in B$. Thus, $\{u_2, u_2\}$ and B are blue complete. Observe, that $\{u_1, u_2\}$ are interchangeable with the vertices in B, and so we can assume $d_b(u_1), d_b(u_2) \leq n - 1$. This implies $\{u_1, u_2\}$ and $\{v_1, v_2\}$ are red complete, and $G[B \cup \{u_1, u_2\}]$ is a red K_{n-2}. If there is a red $(\{v_1, v_2\}, A)$-edge, say v_1v_3 we have a monochromatic S_n, H' with center v_1 has neighbors v, u_1, u_2, v_3, and all the vertices in B. Recall, $d_b(v_3) \leq n - 1$, and so v_3 has some red neighbor in A, say v_4. So we obtain a monochromatic S_n^3 by adding v_3v_4, vv_5, u_2v_2 to H'. So, $\{v_1, v_2\}$ is blue complete to A. Now each $x \in A$ has $d_b(x) \geq n$. By Claim 4.0.15.1, this gives us a blue S_n^3 and completes the proof. □

The lower bound below motivates our conjecture that $r(S_n^t; 2)$ will have two regimes of behavior depending on whether $n \leq 2t$ or $n > 2t$.

Lemma 4.0.16. *For $n \leq 2t$,*

$$r(S_n^t; 2) \geq n + 2t + 1.$$

Proof. Let G be a complete graph on $n+2t$ vertices and $A, B \subseteq V(G)$ with $|A| = n+t$ and $|B| = t$. Color all edges in $G[A]$ and $G[B]$ using color blue. Now color all (A, B)-edges using red. Since $|A|, |B| < n+t+1$, $G[A]$ and $G[B]$ do not contain a monochromatic S_n^t. Since $|B| < t+1$, there cannot be a monochromatic S_n^t between A and B. Thus, G contains no monochromatic S_n^t. This proves the result. □

Conjecture 4.0.17.

$$r(S_n^t; 2) = \begin{cases} \max(2n-1, n+2t+1) \ \textit{if } n \textit{ is even,} \\ \max(2n, n+2t+1) \ \textit{if } n \textit{ is odd.} \end{cases}$$

We can apply 4.0.5 to obtain a lower bound for $r(S_n^t; k)$ where k is odd.

Remark 4.0.18. *For $n \geq 3, t \geq 1, k \geq 2$, $r(S_n^t; k) \geq r(S(n-1,1); k)$. In particular, $r(S_n^t; k) \geq (n-1)k+3$ for k odd.*

CHAPTER 5: LIST RAMSEY NUMBERS OF DOUBLE STARS AND SUBDIVIDED STARS

In this section we prove results for the list Ramsey numbers for double stars. We begin by stating some results that will be used throughout this section. These results are necessary for our lower bound constructions.

Theorem 5.0.1 (Galvin [9])**.** *Every bipartite graph G satisfies $\chi'_\ell(G) = \chi'(G) = \Delta(G)$.*

Theorem 5.0.2 (Schauz [17])**.** *$\chi'_\ell(K_{p+1}) = p$ for every odd prime p.*

Theorem 5.0.3 (Häggkvist, Janssen [8])**.** *$\chi'_\ell(K_{2n+1}) = 2n + 1$ for every $n \geq 1$.*

First we prove three results for $S(1, 1)$.

Proposition 5.0.4. *$r_\ell(S(1, 1); 2) = r(S(1, 1); 2) = 5$.*

Proof. We show $r_\ell(S(1, 1); 2) > 4$. Let $L : E(K_4) \to \binom{\mathbb{N}}{2}$ be an assignment of lists to the edges of K_4. We show that there exists an L-coloring $c : E(K_4) \to \mathbb{N}$ such that K_4 contains no monochromatic $S(1, 1)$ under the coloring c. Partition K_4 into K_1 and K_3. Suppose $\bigcap_{e\in E(K_3)} L(e) \neq \emptyset$. Then color $E(K_3)$ monochromatically by $i \in \bigcap_{e\in E(K_3)} L(e)$. Now color the edges between K_1 and K_3 any color except for i. This graph contains no $S(1, 1)$. Similarly, if the lists of the edges incident with K_1 have a common color, say i, color them using i and the remaining edges any color but i. This graph contains no $S(1, 1)$. Now we may assume $\bigcap_{e\in E(K_3)} L(e) = \emptyset$ and $\bigcap_{e\in E(K_4)\setminus E(K_3)} L(e) = \emptyset$. This implies that we can color all the edges in $E(K_3)$ differently and all the edges between K_1 and K_3 differently. In this case, a fixed color is used on at most two edges, so there cannot be a monochromatic $S(1, 1)$. This shows $r_\ell(S(1, 1); 2) = r(S(1, 1); 2) = 5$. □

Proposition 5.0.5. $r_\ell(S(1,1);3) = r(S(1,1);3) = 6$.

Proof. It suffices to show that $r_\ell(S(1,1);3) > 5$. We show that there exists an L-coloring $c : E(K_5) \to \mathbb{N}$ such that (K_5, c) contains no monochromatic $S(1,1)$. Let $L : E(K_5) \to \binom{\mathbb{N}}{3}$ be an assignment of lists to the edges of K_5. If L is constant, then we are done. So we may assume that there exists a vertex, say v_5, in K_5 such that $\left|\bigcup_{i\in[4]} L(v_iv_5)\right| \geq 4$. Now color the edges incident with v_5 differently. Since $\chi'_\ell(K_4) = 3$ we can color the K_4 induced by $\{v_1, v_2, v_3, v_4\}$ such that it has no $K_{1,2}$. Under this coloring K_5 cannot have a monochromatic copy of $S(1,1)$. □

Lemma 5.0.6. $r_\ell(S(1,1);k) \geq k+3$ *for every odd prime* k.

Proof. Let $G := K_{k+2}$. Let $L : E(G) \to \binom{\mathbb{N}}{k}$ be an assignment of lists to the edges of G. If L is constant, then we are done by 2.0.6. We may assume that there exists a vertex, say u, in G such that $\left|\bigcup_{v\in N(u)} L(uv)\right| \geq k+1$. Now color the edges incident with u differently. Since $\chi'_\ell(K_{k+1}) = k$ by 5.0.2, we can color the edges of $G \setminus u$ from L such that it has no monochromatic $K_{1,2}$. It follows that G has no monochromatic copy of $S(1,1)$ under such a coloring. Thus $r_\ell(S(1,1);k) \geq k+3$. □

Now we look at some results for double stars.

Proposition 5.0.7. *For all* $k \geq 2$ *and* $n \geq m \geq 1$,

$$r(S(n,m);k) \geq r(S_{n+1};k) \text{ and } r_\ell(S(n,m);k) \geq r_\ell(S_{n+1};k).$$

Proof. Both inequalities hold because $S_{n+1} \subseteq S(n,m)$. □

Proposition 5.0.8. *Let* $r \geq 3$. *Then* $r_\ell(S(r,1);2) = r(S(r,1);2)$.

Proof. Let $r \geq 3$ and $n = r_\ell(S_{r+1}; 2)$. Then for any $L : E(K_{n-1}) \to \binom{\mathbb{N}}{2}$, there exists an L-coloring $c : E(K_{n-1}) \to \mathbb{N}$ such that K_{n-1} contains no monochromatic S_{r+1} subgraph. Clearly, K_{n-1} contains no monochromatic $S(r,1)$ subgraph, and so $r_\ell(S(r,1); 2) \geq r_\ell(S_{r+1}; 2)$. By Theorem 3.0.3, $r_\ell(S_{r+1}; 2) = r(S_{r+1}; 2)$. Applying Proposition 5.0.7 we have $r_\ell(S(r,1); 2) \geq r(S_{r+1}; 2) = r(S_{r+1}; 2)$. We know that $r_\ell(S(r,1); 2) \leq r(S(r,1); 2)$, so the result follows. $\square$

Proposition 5.0.9. *For all $n \geq 3m$ such that n is even or n is odd and $m \leq 2$,*

$$r_\ell(S(n,m); 2) = r(S(n,m); 2).$$

Proof. If n is even and $n \geq 3m$, then $2n + 1 \geq 3m$, then $2n + 2 > 2 + 2m + 2$. Hence, $r(S(n,m); 2) = 2n + 2$. Note $2n + 2 = r_\ell(S_{n+1}; 2) \geq r_\ell(S(n,m); 2)$. Thus, $r_\ell(S(n,m); 2) = r(S(n,m); 2)$. Now suppose n is odd and $m \leq 2$. If $2n+1 \geq n+2m+2$, then $r_\ell(S(n,m); 2) \leq r(S(n,m); 2) = 2n + 1$. Since, $2n + 1 = r_\ell(S_{n+1}; 2) \leq r_\ell(S(n,m); 2)$ we are done. $\square$

Now we add to the evidence that $r_\ell(S_n; k) = r(S_n; k)$. Our proof relies on the fact that the list edge coloring conjecture is true for complete graphs of prime degree.

Theorem 5.0.10. *For all $t \geq 3$ and every odd prime p,*

$$r_\ell(S_t; p) = r(S_t; p) = (t-1)p + 2.$$

Proof. By 2.0.4 and the fact that $r_\ell(S_t; k) \leq r(S_t; k)$, we have $r_\ell(S_t, p) \leq (t-1)p + 2$. It remains to show that $r_\ell(S_t, p) > (t-1)p + 1$. Let $G = K_{(t-1)p+1}$. Let x be a vertex of G and let $A_1, \ldots, A_{t-1}$ be a partition of $V(G \setminus x)$ with $|A_1| = \cdots = |A_{t-1}| = p$. Furthermore, let $G_i = G[A_i \cup \{x\}]$ and for $i \neq j$, let $G_{i,j}$ be the complete bipartite subgraph of G with parts A_i and A_j. Then every vertex of G belongs to exactly $t-1$ of these subgraphs which partition

$E(G)$. By Theorem 5.0.1, $\chi'_\ell(G_{i,j}) = p$. Since p is an odd prime, by 5.0.2, $\chi'_\ell(G_i) = p$. By Lemma 3.0.2, $r_\ell(S_t, p) > (t-1)p + 1$, as desired. □

Remark 5.0.11. *If $\chi'_\ell(K_{p+1}) = p$ for every odd p, then the above proof shows that $r_\ell(S_t; p) = (t-1)p + 2$ for all $t \geq 3$.*

To end this section we look at the list Ramsey numbers for subdivided stars. Since $S_n^1 = S(n-1, 1)$, we have $r_\ell(S_n^1; 2) = r(S_n^1; 2)$ for $n \geq 2$

Proposition 5.0.12. *For $n \geq 5$, $r_\ell(S_n^2; 2) = r(S_n^2; 2)$*

Proof. Let $n \geq 5$. Then by Proposition 4.0.14, $r(S_n^2; 2) = r(S_n; 2)$. We know $r_\ell(S_n; 2) = r(S_n; 2)$. This implies, $r_\ell(S_n^2; 2) \leq r_\ell(S_n; 2)$. Since $S_n \subseteq S_n^2$, $r_\ell(S_n; 2) \leq r_\ell(S_n^2; 2)$. Thus, $r_\ell(S_n; 2) = r_\ell(S_n^2; 2)$, and so $r_\ell(S_n^2; 2) = r(S_n^2; 2)$. □

Proposition 5.0.13. *Let $n \geq 10 - \varepsilon$ where $\varepsilon = 1$ if n is odd and $\varepsilon = 0$ otherwise. Then $r_\ell(S_n^3; 2) = r(S_n^3; 2)$*

Proof. Let $n \geq 10 - \varepsilon$ where ε by defined as above. Then by Proposition 4.0.15, $r(S_n^3; 2) = r(S_n; 2)$. We know $r_\ell(S_n; 2) = r(S_n; 2)$. This implies, $r_\ell(S_n^3; 2) \leq r_\ell(S_n; 2)$. Since $S_n \subseteq S_n^3$, $r_\ell(S_n; 2) \leq r_\ell(S_n^3; 2)$. Thus, $r_\ell(S_n; 2) = r_\ell(S_n^3; 2)$, and so $r_\ell(S_n^3; 2) = r(S_n^3; 2)$. □

CHAPTER 6: FUTURE WORK

I plan on continuing to work on Conjecture 3.0.4 and the following problems.

Question 6.0.1. *For even $k \geq 2$, determine $r(S(n,m);k)$ for all sufficiently large n.*

The above question is analogous to Corollary 4.0.8 but for k even. The lower bounds given in this paper rely on the properties of K_k where k is odd, so constructions for even k require new ideas.

Question 6.0.2. *Can the bound given by Theorem 4.0.5 be improved?*

When $m = f(n)$ is where $f(n) \leq n$ is an increasing function, it is conjectured that $r(S(n,m);k) \sim (n + f(n) + 2)k$ as $k \to \infty$.

Now we discuss problems about subdivided stars. The family of graphs we call subdivided stars is currently nonexistent in the literature. Other than resolving Conjecture 4.0.17, I plan to study subdivided stars in the multicolor Ramsey and list Ramsey number setting.

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