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The Effects of Viscous Damping on Rogue Wave Formation and Permanent Downshift in the Nonlinear Schrödinger Equation

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THE EFFECTS OF VISCOUS DAMPING ON ROGUE WAVE FORMATION AND PERMANENT DOWNSHIFT IN THE NONLINEAR SCHRÖDINGER EQUATION

by

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ABSTRACT

This thesis investigates the effect of viscous damping on rogue wave formation and permanent downshift using the higher-order nonlinear Schrödinger equation (HONLS). The strength of viscous damping is varied and compared to experiments with only linear damped HONLS.

Stability analysis of the linear damped HONLS equation shows that instability stabilizes over time. This analysis also provides an instability criterion in the case of HONLS with viscous damping.

Numerical experiments are conducted in the two unstable mode regime using perturbations of the Stokes wave as initial data. With only linear damping permanent downshift is not observed and rogue wave formation is decreased. The addition of viscous damping leads to permanent downshift and a slight increase in rogue wave activity. Analysis of the energy and momentum gives a possible explanation for this behavior.

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CHAPTER 1
INTRODUCTION

Rogue waves are extremely steep waves that form quickly and without warning. These waves are both dangerous and mathematically interesting, prompting much study. The steepness of these waves cannot be explained by linear models, leading us to turn to nonlinear models. The model used to describe rogue waves is the nonlinear Schrödinger (NLS) equation. This model provides the nonlinearity required to form rogue waves, and has special solutions and properties suited to their study.

One such property is modulational instability. When a solution to the NLS equation is perturbed, the perturbation tends to grow in time. This growth provides a mechanism for rogue waves. One solution to the NLS equation is the plane wave, also called the Stokes wave after Sir George Stokes.

The motivation behind this thesis is to attempt to capture the phenomenon of permanent downshift. This occurs when the dominant wave number of the waveform shifts down permanently from its initial value. Permanent downshift is captured in wave tank experiments, but is not predicted by NLS[7]. We investigate whether viscous damping, a higher order effect caused by viscosity of the fluid, can create this effect. This investigation includes an analytical investigation of the stability of the Stokes wave using the higher order terms, including viscous damping, as well as numerical experiments to capture downshift.

CHAPTER 2
THE NONLINEAR SCHRÖDINGER EQUATION

To understand the physical significance of the NLS equation, we will first show how it is derived from the governing equations for water waves. We will then discuss its mathematical features.

2.1 Water Wave Derivation

To derive the NLS equation, we begin by forming a mathematical description of the system. In a physical system, the relevant variables are typically the position and velocity. In this case, the position is the position of the water's surface, η, and the velocity is the velocity at each point within the water. Water is considered to be irrotational and incompressible. Irrotational means that the curl of the velocity field is zero. This means that the velocity at each point can be written as a potential ϕ, called the velocity potential. We can find ϕ instead of finding the velocity field itself. Incompressible means that the divergence of the velocity is zero. These two conditions mean that ϕ satisfies Laplace's equation. The boundary conditions for water waves at the surface $z = \eta$ are

$$\phi_t + \frac{1}{2}(\phi_x)^2 + g\eta = 0 \tag{2.1}$$

$$\eta_t + \eta_x\phi_x = \phi_z \tag{2.2}$$

Where g is acceleration due to gravity, and we assume no y dependence. In deep water, we consider the depth to be infinite and require that the velocity approaches zero as the depth $z \to -\infty$. Since the boundary conditions are in terms of one of the variables we are solving for, we must approximate to find a new boundary condition at $z = 0$. To do this we assume that the wave steepness $\varepsilon = ak$ is small. Here a is the initial wave amplitude and k is the wave number of the waveform. Following the work of Dysthe [2], we can expand η and ϕ as

$$\phi = \epsilon^2 \bar{\phi} + \epsilon A e^{kz} e^{i(kx-\omega t)} + \epsilon^2 A_2 e^{2kz} e^{2i(kx-\omega t)} + \ldots + \text{c.c.} \tag{2.3}$$

$$\eta = \epsilon^3 \bar{\eta} + \epsilon B e^{i(kx-\omega t)} + \epsilon^2 B_2 e^{2i(kx-\omega t)} + \ldots + \text{c.c.} \tag{2.4}$$

Where $\bar{\phi}$, $\bar{\eta}$, A_i, and B_i also have expansions in powers of ϵ. The A and $\bar{\phi}$ variables depend on $\epsilon x = X$, $\epsilon z = Z$, and $\epsilon t = T$, while B and $\bar{\eta}$ depend on X and T. Equations (2.3) and (2.4) are then substituted into equations (2.1) and (2.2). Using this substitution to find a differential equation for A results in

$$\begin{aligned} 2i\omega(A_T + \frac{\omega}{2k}A_X) - \epsilon\left(\left(\frac{\omega}{2k}\right)^2 A_{XX} + 4k^4|A|^2 A\right) \\ = \epsilon^2\left(\frac{i\omega}{8k^3}A_{XXX} + 2ik^3A^2A_X^* - 12ik^3|A|^2A_X + 2\omega k\bar{\phi}_X A\right). \end{aligned} \tag{2.5}$$

The following change of variables transforms this equation into the higher order nonlinear Schrödinger (HONLS) equation used in this work:

$$\tau = \frac{-\omega\epsilon}{8}T$$

$$\chi = kX - \frac{\omega}{2}T$$

$$\bar{\phi}' = \frac{2k^2}{\omega}\bar{\phi}$$

$$u = \frac{2\sqrt{2}k^2}{\omega}A$$

giving us the equation

$$iu_\tau + u_{\chi\chi} + 2|u|^2 u + \epsilon\left(\frac{i}{2}u_{\chi\chi\chi} - 8i|u|^2 u_\chi + 2u\left[H(|u|^2)\right]_\chi\right) = 0 \tag{2.6}$$

where we have used Fourier methods to solve for $\bar{\phi}$ in terms of the Hilbert transform of $|u|^2$, defined as $H(f(x)) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(\xi)}{x-\xi}d\xi$. For convenience, we use x and t rather than τ and χ. The amplitude and mean flow can be recovered by transforming $u(\chi,\tau)$ back into $A(X,Z,T)$ and using equation (2.3) and the relation

$$\eta = \frac{\epsilon\omega}{g}((iA + \frac{\epsilon}{2k}A_x)e^{i(kx-\omega t)} - \frac{k^2\epsilon}{\omega}A^2 e^{2i(kx-\omega t)}) + \text{c.c.} \tag{2.7}$$

Incorporating viscosity into the Euler equations as in [1] and following the same steps results in the viscous damped HONLS equation, which we will refer to as dHONLS:

$$iu_t + u_{xx} + 2|u|^2 u + i\Gamma u + \epsilon\left(\frac{i}{2}u_{xxx} - 8i|u|^2 u_x + 2u[H(|u|^2)]_x + \Gamma u_x\right) = 0 \tag{2.8}$$

Where Γ is the parameter that controls the viscosity.

2.2 Conserved Quantities

The NLS is equation is integrable, meaning it has an infinite number of conserved quantities. One of these quantities is the Hamiltonian $\mathcal{H}$, defined as

$$\mathcal{H} = \int_0^L -|u_x|^2 + |u|^4 dx \tag{2.9}$$

Using this Hamiltonian, the NLS equation satisfies Hamilton's equations, meaning it is Hamiltonian. The original Dysthe equation for HONLS was not Hamiltonian. A Hamiltonian form of the HONLS equation was derived by Gramstad and Trulsen [3], which is what we use here. The HONLS system has only three conserved quantities: the Hamiltonian, energy, and momentum, defined respectively as:

$$\mathcal{H} = \int_0^L -|u_x|^2 + |u|^4 + \frac{i\epsilon}{4}(u_x u_{xx}^* - u_x^* u_{xx}) - 2i\epsilon|u|^2(u^* u_x - u u_x^*) + \epsilon|u|^2 \left[H(|u|^2)\right]_x dx \quad (2.10)$$

$$E = \int_0^L |u|^2 dx \quad (2.11)$$

$$P = i\int_0^L u^* u_x - u u_x^* dx \quad (2.12)$$

These quantities can be derived using the HONLS equation. For the Hamiltonian, we begin by multiplying the HONLS equation by u_t^*:

$$i|u_t|^2 + u_{xx}u_t^* + 2|u|^2 u u_t^* + \epsilon(\frac{i}{2}u_{3x}u_t^* - 8i|u|^2 u_x u_t^* + 2u u_t^*[H(|u|^2)]_x) = 0 \quad (2.13)$$

This equation is then added to its complex conjugate, resulting in the equation

$$u_t^* u_{xx} + u_t u_{xx}^* + 2|u|^2(u u_t^* + u^* u_t) + \epsilon(\frac{i}{2}(u_t^* u_{3x} - u_t u_{3x}^*)$$
$$- 8i|u|^2(u_t^* u_x - u_t u_x^*) + 2[H(|u|^2)]_x(u u_t^* + u^* u_t)) = 0$$

This can then be manipulated to give derivatives:

$$\frac{\partial}{\partial x}[u_t^* u_x + u_t u_x^* + \epsilon(\frac{i}{2}(u_t^* u_{xx} - u_t u_{xx}^*))] + \frac{\partial}{\partial t}[|u|^4 + \epsilon|u|^2[H(|u|^2)]_x]$$
$$- u_{xt}^* u_x - u_{xt} u_x^* + \epsilon(\frac{-i}{2}(u_{xt}^* u_{xx} - u_{xt} u_{xx}^*) - 6i|u|^2(u_t^* u_x - u_t u_x^*)$$
$$- 2i|u|^2 u_t^* u_x + 2i|u|^2 u_t u_x^* + (|u|^2)_t[H(|u|^2)]_x - |u|^2[H(|u|^2)]_{xt}) = 0$$

$$\frac{\partial}{\partial x}[u_t^* u_x + u_t u_x^* + \epsilon(\frac{i}{2}(u_t^* u_{xx} - u_t u_{xx}^* - (\frac{1}{2}u_{xt}^* u_x - u_{xt} u_x^*)) - 4i|u|^2(uu_t^* - u^* u_t) + |u|^2[H(|u|^2)]_t)]$$
$$+\frac{\partial}{\partial t}[-|u_x|^2 + |u|^4 + \epsilon|u|^2[H(|u|^2)]_x] + \epsilon(\frac{i}{4}(u_{xxt}^* u_x - u_{xxt} u_x^* - u_{xt}^* u_{xx} + u_{xt} u_{xx}^*) - 6i|u|^2(u_t^* u_x - u_t u_x^*)$$
$$+2i(u^* u_x + uu_x^*)uu_t^* + 2i|u|^2 uu_{xt}^* - 2i(u^* u_x + uu_x^*)u^* u_t - 2i|u|^2 u^* u_{xt} + (|u|^2)_t[H(|u|^2)]_x + (|u|^2)_x[H(|u|^2)]_t) = 0$$

$$\frac{\partial}{\partial x}[u_t^* u_x + u_t u_x^* + \epsilon(\frac{i}{2}(u_t^* u_{xx} - u_t u_{xx}^* - (\frac{1}{2}u_{xt}^* u_x - u_{xt} u_x^*)) - 4i|u|^2(uu_t^* - u^* u_t) + |u|^2[H(|u|^2)]_t)$$
$$+\frac{\partial}{\partial t}[-|u_x|^2 + |u|^4 + \frac{\epsilon}{4}(u_x u_{xx}^* - u_x^* u_{xx}) + \epsilon|u|^2[H(|u|^2)]_x] + -4i|u|^2(u_t^* u_x - u_t u_x^*)$$
$$-2u^2 u_x^* u_t^* - 2|u|^2 uu_{xt}^* + 2(u^*)^2 u_x u_t + 2|u|^2 u^* u_{xt} + (|u|^2)_t[H(|u|^2)]_x + (|u|^2)_x[H(|u|^2)]_t) = 0$$

$$\frac{\partial}{\partial x}[u_t^* u_x + u_t u_x^* + \epsilon(\frac{i}{2}(u_t^* u_{xx} - u_t u_{xx}^* - (\frac{1}{2}u_{xt}^* u_x - u_{xt} u_x^*)) - 4i|u|^2(uu_t^* - u^* u_t) + |u|^2[H(|u|^2)]_t)$$
$$+\frac{\partial}{\partial t}[-|u_x|^2 + |u|^4 + \frac{\epsilon}{4}(u_x u_{xx}^* - u_x^* u_{xx}) + \epsilon|u|^2[H(|u|^2)]_x] - 2i|u|^2(u_t^* u_x + uu_{xt}^* - u_t u_x^* - u^* u_{xt})$$
$$-2uu_t^* u^* u_x - 2uu_t^* uu_x^* + 2u^* u_t u^* u_x + 2u^* u_t uu_x^* + (|u|^2)_t[H(|u|^2)]_x + (|u|^2)_x[H(|u|^2)]_t) = 0$$

$$\frac{\partial}{\partial x}[u_t^* u_x + u_t u_x^* + \epsilon(\frac{i}{2}(u_t^* u_{xx} - u_t u_{xx}^* - (\frac{1}{2}u_{xt}^* u_x - u_{xt} u_x^*)) - 4i|u|^2(uu_t^* - u^* u_t) + |u|^2[H(|u|^2)]_t)$$
$$+\frac{\partial}{\partial t}[-|u_x|^2 + |u|^4 + \frac{\epsilon}{4}(u_x u_{xx}^* - u_x^* u_{xx}) + \epsilon|u|^2[H(|u|^2)]_x] - 2\epsilon i|u|^2\frac{\partial}{\partial t}(u^* u_x - u^* u_x)$$
$$-2\epsilon i(uu_t^* + u^* u_t)(u^* u_x - uu_x^*) + (|u|^2)_t[H(|u|^2)]_x + (|u|^2)_x[H(|u|^2)]_t) = 0$$

$$\frac{\partial}{\partial x}[u_t^* u_x + u_t u_x^* + \epsilon(\frac{i}{2}(u_t^* u_{xx} - u_t u_{xx}^* - (\frac{1}{2}u_{xt}^* u_x - u_{xt} u_x^*)) - 4i|u|^2(uu_t^* - u^* u_t) + |u|^2[H(|u|^2)]_t)$$
$$+\frac{\partial}{\partial t}[-|u_x|^2 + |u|^4 + \frac{\epsilon}{4}(u_x u_{xx}^* - u_x^* u_{xx}) - 2\epsilon|u|^2(u^* u_x - uu_x^*) + \epsilon|u|^2[H(|u|^2)]_x]$$
$$+(|u|^2)_t[H(|u|^2)]_x + (|u|^2)_x[H(|u|^2)]_t) = 0$$

The Hilbert transform has the property that $\int f(x)H(g(x))dx + \int g(x)H(f(x))dx = 0$. This, combined with the fact that derivatives in either variable commute with the transform, means that the non-derivative terms go to zero when integrated, as well as the $x-$derivatives which vanish due to the periodic boundary conditions. This leaves us with the expression

$$\frac{\partial}{\partial t}\int_0^L -|u_x|^2 + |u|^4 + \epsilon(\frac{1}{4}(u_x u_{xx}^* - u_x^* u_{xx}) - 2|u|^2(u^* u_x - u u_x^*) + |u|^2[H(|u|^2)]_x) = 0 \quad (2.14)$$

Indicating that the Hamiltonian is a conserved quantity of the HONLS equation.

The energy is derived by multiplying the HONLS equation by u^*:

$$iu^* u_t + u^* u_{xx} + 2|u|^4 + \epsilon(\frac{i}{2}u^* u_{3x} - 8i|u|^2 u^* u_x + 2|u|^2[H(|u|^2)]_x) = 0$$

This is then subtracted by its complex conjugate:

$$i(u^* u_t + u u_t^*) + u^* u_{xx} - u u_{xx}^* + \epsilon(\frac{i}{2}(u^* u_{3x} + u u_{3x}^*) - 8i|u|^2(u^* u_x + u u_x^*)) = 0$$

Which is as before converted into derivatives:

$$i\frac{\partial}{\partial t}[|u|^2] + \frac{\partial}{\partial x}[u^* u_x - u u_x^* + \epsilon(\frac{i}{2}(u^* u_{xx} + u u_{xx}^*) - 4i|u|^4))]$$
$$- u_x^* u_x + u_x u_x^* + \epsilon(\frac{-i}{2}(u_x^* u_{xx} + u_x u_{xx}^*)) = 0$$

$$i\frac{\partial}{\partial t}[|u|^2] + \frac{\partial}{\partial x}[u^* u_x - u u_x^* + \epsilon(\frac{i}{2}(u^* u_{xx} + u u_{xx}^* - |u_x|^2))] = 0$$

Integrating from 0 to L and dividing out i gives the expression

$$\frac{\partial}{\partial t}\int_0^L |u|^2 dx = 0$$

Indicating that the energy is conserved.

The momentum is derived by multiplying the HONLS equation by u_x^*:

$$iu_t u_x^* + u_x^* u_{xx} + 2|u|^2 u u_x^* + \epsilon(\frac{i}{2} u_x^* u_{xxx} - 8i|u|^2|u_x|^2 + 2uu_x^*[H(|u|^2)]_x) = 0 \tag{2.15}$$

and then adding the complex conjugate of the expression:

$$i(u_t u_x^* - u_t^* u_x) + u_x^* u_{xx} + u_x u_{xx}^* + 2|u|^2(uu_x^* + u^* u_x)$$
$$+ \epsilon(\frac{i}{2}(u_x^* u_{xxx} - u_x u_{xxx}^*) + 2[H(|u|^2)]_x(uu_x^* + u^* u_x)) = 0$$

This equation is then converted into derivatives:

$$\frac{\partial}{\partial t} i[uu_x^* - u^* u_x] + \frac{\partial}{\partial x}[|u_x|^2 + |u|^4 + \epsilon(\frac{i}{2}(u_x^* u_{xx} - u_x u_{xx}^*)]$$
$$- iu_{xt} u^* + iu_{xt}^* u + \epsilon(\frac{i}{2}(u_{xx}^* u_{xx} - u_{xx} u_{xx}^*) + 2|u|_x^2 H(|u|_x^2)) = 0$$

We can substitute the HONLS equation to find the expression $i(uu_{xt}^* - u^* u_{xt})$. Since the quantity is already at order epsilon, we can ignore the higher order terms, meaning we must only substitute NLS. This gives

$$\begin{aligned}
i(uu_{xt}^* - u^* u_{xt}) &= u(u_{xxx}^* + 2|u|^2 u_x^* + 2u^*(uu_x^* + u^* u_x)) - u^*(-u_{xxx} - 2|u|^2 u_x - 2u(uu_x^* + u^* u_x)) \\
&= uu_{xxx}^* + u^* u_{xxx} + 2|u|^2(uu_x^* + u^* u_x) + 2|u|^2(uu_x^* u^* u_x) + 2|u|^2(uu_x^* + u^* u_x) \\
&= \frac{\partial}{\partial x}[uu_{xx}^* + u^* u_{xx} + 3|u|^4] - u_x u_{xx}^* - u_x^* u_{xx} \\
&= \frac{\partial}{\partial x}[uu_{xx}^* + u^* u_{xx} + 3|u|^4 - |u_x|^2]
\end{aligned}$$

When the expression is integrated, this term will vanish. It is a property of the Hilbert transform that f and $H(f)$ are orthogonal, which means that the last term will also vanish. Integrating and changing sign to agree with our convention gives:

$$\frac{\partial}{\partial t} i \int_0^L u^* u_x - uu_x^* dx = 0 \tag{2.16}$$

indicating that the momentum is conserved. Setting $\epsilon = 0$ gives the usual conserved quantities for the NLS system. This system has an infinite number of conserved quantities. This can be seen from the Lax pair associated with the NLS system, which will be discussed later.

CHAPTER 3
BEHAVIOR OF SOLUTIONS

3.1 Modulational Instability

One particular solution to the NLS equation is the plane wave:

$$u_a(t) = ae^{2ia^2t} \tag{3.1}$$

where a is the amplitude of the wave, which we may take to be real. This solution exhibits a property called modulational instability. The plane wave is perturbed by taking a nearby solution of the form

$$u(x,t) = u_a(t)(1 + \delta(A_1\phi + A_2\phi^*)) \tag{3.2}$$

where $\phi = e^{i(kx-\Omega t)}$ and δ is small. This perturbation represents a single Fourier mode of a generic perturbation. When substituted into the NLS equation and linearized, a relation between A_1 and A_2 is found. Taking the complex conjugate of this relation gives the following linear system of equations:

$$\begin{pmatrix} \Omega\phi - k^2\phi + 2a^2(\phi + \phi^*)) & -\Omega^*\phi^* - k^2\phi^* + 2a^2(\phi + \phi^*) \\ \Omega^*\phi^* - k^2\phi^* + 2a^2(\phi + \phi^*)) & -\Omega\phi - k^2\phi + 2a^2(\phi + \phi^*) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0 \tag{3.3}$$

For a nontrivial solution for A_1 and A_2 to exist, the determinant of the above matrix must be zero. Setting the determinant to zero and rearranging gives an expression for Ω:

$$\Omega = \pm k\sqrt{k^2 - 4a^2} \tag{3.4}$$

This shows that when $4a^2 > k^2$, Ω is imaginary, so there is a growth rate $\sigma = \mathrm{Im}\,\Omega = \pm k\sqrt{4a^2 - k^2}$, such that $\phi = e^{ikx\pm\sigma t}$ and $\phi^* = e^{-ikx\mp\sigma t}$. This formulation shows that when Ω is imaginary, the perturbation grows exponentially in time, indicating instability. When Ω is instead real, the perturbation is neutrally stable, as it oscillates in time rather than growing or shrinking.

This relationship between Ω and k indicates that the particular wavenumber determines whether a perturbation is unstable. Substituting the wavenumber $k_n = \frac{2\pi n}{L}$ into the inequality $4a^2 > k_n^2$ results in the criterion

$$n < \frac{aL}{\pi} \tag{3.5}$$

for instability. By varying a and L, the maximum number N for which this holds can be changed. This number defines the number of unstable modes (UMs) possible for a particular amplitude and period, which is referred to as the N unstable mode regime (UMR).

Analyzing the stability of damped higher-order NLS uses the same approach with slightly different assumptions. Because linear damping is now present, we must use the damped plane wave

$$u_\Gamma = ae^{-\Gamma t}e^{ia^2\frac{1-e^{-2\Gamma t}}{\Gamma}} \tag{3.6}$$

We also insert the slightly more general substitution

$$u(x,t) = u_\Gamma(1 + \delta(\rho + i\sigma)) \tag{3.7}$$

where ρ and σ are real functions of x and t. Substituting this into HONLS and breaking into real and imaginary parts results in coupled equations for ρ and σ:

$$\sigma_t = 4e^{-2\Gamma t}\rho + \rho_{xx} + \epsilon(4ke^{-2\Gamma t}\rho + 8e^{-2\Gamma t}\sigma_x + \frac{k^2}{2}\sigma_x + 2\Gamma\rho_x) \tag{3.8}$$

$$= \epsilon(8e^{-2\Gamma t} + \frac{k^2}{2})\sigma_x + [4e^{-2\Gamma t}(1+\epsilon k) - k^2]\rho + 2\Gamma\epsilon\rho_x \tag{3.9}$$

$$\rho_t = \epsilon(8e^{-2\Gamma t} + \frac{k^2}{2})\rho_x + k^2\sigma - 2\Gamma\epsilon\sigma_x \tag{3.10}$$

We will substitute the following for ρ and σ:

$$\sigma = c_1(t)\cos kx + c_2(t)\sin kx \tag{3.11}$$

$$\rho = c_3(t)\cos kx + c_4(t)\sin kx \tag{3.12}$$

Before we do this, let's make the following substitutions:

$$A = 2a^2e^{2\Gamma t}(1+\epsilon k) \tag{3.13}$$

$$B = \epsilon(8a^2e^{-2\Gamma t} + \frac{k^2}{2}) \tag{3.14}$$

$$\gamma = 2\epsilon\Gamma \tag{3.15}$$

This simplifies the differential equation to

$$\sigma_t = B\sigma_x + (2A - k^2)\rho + \gamma\rho_x \tag{3.16}$$

$$\rho_t = B\rho_x + k^2\sigma - \gamma\sigma_x \tag{3.17}$$

Into which we will now substitute our expressions:

$$\dot{c}_1 \cos kx + \dot{c}_2 \sin kx = B(-c_1 k \sin kx + c_2 k \cos kx) + \gamma(-c_3 k \sin kx + c_4 k \cos kx) \tag{3.18}$$

$$+(2A - k^2)(c_3 \cos kx + c_4 \sin kx) \tag{3.19}$$

$$\dot{c}_3 \cos kx + \dot{c}_4 \sin kx = B(-kc_3 \sin kx + kc_4 \cos kx) - \gamma(-c_1 k \sin kx + c_2 k \cos kx) \tag{3.20}$$

$$+k^2(c_1 \cos kx + c_2 \sin kx) \tag{3.21}$$

Now we group by sines and cosines to create four equations:

$$\dot{c}_1 = Bkc_2\gamma kc_4 + (2A - k^2)c_3 \tag{3.22}$$

$$\dot{c}_2 = -Bkc_1 - \gamma kc_3 + (2A - k^2)c_4 \tag{3.23}$$

$$\dot{c}_3 = Bkc_4 - \gamma kc_2 + k^2 c_1 \tag{3.24}$$

$$\dot{c}_4 = -Bkc_3 + \gamma kc_1 + k^2 c_2 \tag{3.25}$$

We will write this as a matrix:

$$\dot{\vec{c}} = \begin{pmatrix} 0 & Bk & 2A - k^2 & \gamma k \\ -Bk & 0 & -\gamma k & 2A - k^2 \\ k^2 & -\gamma k & 0 & Bk \\ \gamma k & k^2 & -Bk & 0 \end{pmatrix} \vec{c} \tag{3.26}$$

Defining this matrix as C, we need to find the eigenvalues of C, which are determined by finding the determinant of $M = \lambda I - C$:

$$\begin{pmatrix} \lambda & -Bk & k^2 - 2A & -\gamma k \\ Bk & \lambda & \gamma k & k^2 - 2A \\ -k^2 & \gamma k & \lambda & -Bk \\ -\gamma k & -k^2 & Bk & \lambda \end{pmatrix} \tag{3.27}$$

Let's now take a moment to talk about matrices. For a $2n \times 2n$ matrix that can be written as 4 $n \times n$ blocks like this:

$$\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \tag{3.28}$$

the determinant is equal to the determinant of $M_1 M_4 - M_2 M_3$ if M_i commute with each other [4]. We will use this to our advantage. Define the following:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.29}$$

Which are of course the identity and a 90-degree rotation in $\mathbb{R}^2$. If we have two arbitrary matrices of the form $aI + bR$ and $cI + dR$, the product is:

$$(aI + bR)(cI + dR) = acI^2 + adIR + bcRI + bdR^2 = (ac - bd)I + (ad + bd)R \tag{3.30}$$

We can tell matrices of this form will commute because in $\mathbb{R}^2$ rotation matrices commute, so linear combinations of them will also. We also have the determinant formula

$$|aI + bR| = \begin{vmatrix} a & -b \\ b & a \end{vmatrix} = a^2 + b^2 \tag{3.31}$$

Our matrix M can be written as blocks in the form

$$\begin{pmatrix} X & Y - Z \\ -Y & X \end{pmatrix} \tag{3.32}$$

where

$$X = \lambda I + BkR \tag{3.33a}$$

$$Y = k^2 I + \gamma k R = k(kI + \gamma R) \tag{3.33b}$$

$$Z = 2AI \tag{3.33c}$$

Thus M is composed of 2×2 blocks that we know commute. Then we must substitute into the formula for the determinant with $M_1 = M_4 = X$, $M_2 = Y - Z$ and $M_3 = -Y$. First calculate the product $M_1 M_4 = X^2$:

$$X^2 = (\lambda I + BkR)^2 = (\lambda^2 - (Bk)^2)I + 2Bk\lambda R$$

then $M_2 M_3 = (Y - Z)(-Y) = ZY - Y^2$:

$$\begin{aligned} ZY - Y^2 &= 2AkI(kI + \gamma R) - k^2(kI + \gamma R)(kI + \gamma R) \\ &= 2Ak^2 I + 2Ak\gamma R - k^2[(k^2 - \gamma^2)I + 2k\gamma R] \\ &= k^2(2A - k^2 + \gamma^2)I + 2k\gamma(A - k^2)R \end{aligned}$$

Subtracting these, the matrix we must have the determinant of is

$$(\lambda^2 - (Bk)^2 - k^2(2A - k^2 + \gamma^2))I + (2Bk\lambda - 2k\gamma(A - k^2))R \tag{3.34}$$

which we know is

$$(\lambda^2 - (Bk)^2 - k^2(2A - k^2 + \gamma^2))^2 + (2Bk\lambda - 2k\gamma(A - k^2))^2 \tag{3.35}$$

Setting this to 0 creates an expression for λ:

$$\lambda^2 - (Bk)^2 - k^2(2A - k^2 + \gamma^2) = \pm i(2Bk\lambda - 2k\gamma(A - k^2)) \tag{3.36}$$

Which creates two quadratic equations to solve:

$$\lambda^2 - 2iBk\lambda - (Bk)^2 = (\lambda - iBk)^2 = k^2(2A - k^2 + \gamma^2) - 2ik\gamma(A - k^2) \tag{3.37a}$$

$$\lambda^2 + 2iBk\lambda - (Bk)^2 = (\lambda + iBk)^2 = k^2(2A - k^2 + \gamma^2) + 2ik\gamma(A - k^2) \tag{3.37b}$$

Setting $\gamma = 0$ eliminates the contribution of the viscous damping, allowing us to first consider linear damped HONLS. In this case, the four growth rates are

$$\lambda_1 = iBk + k\sqrt{4a^2e^{-2\Gamma t}(1 + \epsilon k) - k^2} \tag{3.38a}$$

$$\lambda_2 = iBk - k\sqrt{4a^2e^{-2\Gamma t}(1 + \epsilon k) - k^2} \tag{3.38b}$$

$$\lambda_3 = -iBk + k\sqrt{4a^2e^{-2\Gamma t}(1 + \epsilon k) - k^2} \tag{3.38c}$$

$$\lambda_4 = -iBk - k\sqrt{4a^2e^{-2\Gamma t}(1 + \epsilon k) - k^2} \tag{3.38d}$$

where we have substituted back in $A = 2a^2e^{-2\Gamma t}(1 + \epsilon k)$. In each rate, we have the instability criterion $k^2 < 4a^2e^{-2\Gamma t}(1 + \epsilon k)$. This indicates an interval of values centered around $k = 2\epsilon e^{-2\Gamma t}$ for which the mode is unstable. The width of this band narrows to 0 as $t \to \infty$, indicating that the instability disappears over time. Setting $\epsilon = 0$ recovers the usual result for linear damped NLS, $k^2 < 4a^2e^{-2\Gamma t}$, as expected.

Returning to the case of viscous damping is more tricky. Solving the quadratics in Equation (3.37) now requires us to take the square root of a general complex number of the form $z = x + iy$.

The eigenvalues now satisfy

$$\lambda = \pm iBk + \sqrt{x \mp iy} \tag{3.39a}$$

$$\lambda = \pm iBk - \sqrt{x \mp iy} \tag{3.39b}$$

where

$$x = k^2(2A + \gamma^2 - k^2), \quad y = 2\gamma k(A - k^2) \tag{3.40}$$

In general, the square root of $x \pm iy$ will include a real part since $y \neq 0$. This indicates an instability present in all modes. We can examine the behavior of this instability as $t \to \infty$. In this limit, we have

$$x \to k^2(k^2 - \gamma^2), \quad y \to -k^2(2\gamma k) \tag{3.41}$$

thus

$$\sqrt{x \pm iy} \to k\sqrt{\gamma^2 - k^2 \pm 2i\gamma k} = k\sqrt{(\gamma \pm ik)^2} = k(\gamma \pm ik) \tag{3.42}$$

This shows that as $t \to \infty$, the real part of λ goes to $\pm\gamma k$, thus there are always two of the four λ that contribute to growth of the perturbation. This growth is multiplied by $\gamma = 2\epsilon\Gamma$, indicating that it is quite small.

3.2 Spatially Periodic Breathers

The nonlinear Schrödinger equation has a few classes of special solutions. The type that is relevant to this work is the spatially periodic breather (SPB) solution. These solutions are constructed using a Bäcklund transformation on a Lax pair associated with the NLS equation. In this section we will define the Lax Pair and its associated spectrum, and then use the spectrum of the plane wave to construct the new SPB solutions.

3.2.1 Spectrum of the Lax Pair

To find the SPB solutions, we begin with the Lax pair

$$\mathcal{L}^{(x)} = \begin{pmatrix} -i\lambda & u \\ u^* & i\lambda \end{pmatrix} \tag{3.43}$$

$$\mathcal{L}^{(t)} = \begin{pmatrix} i|u|^2 - 2i\lambda^2 & -u_x + 2i\lambda u \\ u_x^* + 2i\lambda u^* & -i|u|^2 + 2i\lambda^2 \end{pmatrix} \tag{3.44}$$

with the corresponding Lax equations

$$\mathcal{L}^{(x)}\phi = \phi_x \tag{3.45}$$

$$\mathcal{L}^{(t)}\phi = \phi_t \tag{3.46}$$

where the compatibility condition is $\phi_{xt} = \phi_{tx}$[6]. This system is compatible when u solves the NLS equation. The spectrum of (3.43) is the values of λ for which ϕ is bounded for all x. To find this spectrum, we must solve (3.45). This expression is a differential equation with periodic boundary conditions, so we can use Floquet theory to solve it. This means we must find the monodromy matrix $M(\lambda)$, defined by

$$M(\lambda) = \Phi(L) \tag{3.47}$$

where $\Phi(x)$ is the fundamental solution matrix of the system, chosen to satisfy $\Phi(0) = I$. Floquet's theorem states that

$$\Phi(x + L) = \Phi(x)M(\lambda) \tag{3.48}$$

and that for any solution ϕ to (3.43), there is a Floquet multiplier $m(\lambda) = e^{ip(\lambda)x}$ such that $\phi(x + L) = m(\lambda)\phi(x)$. Writing this solution in terms of the fundamental matrix Φ gives the equation

$$\Phi(x + L)\begin{pmatrix} A \\ B \end{pmatrix} = m(\lambda)\Phi(x)\begin{pmatrix} A \\ B \end{pmatrix} \tag{3.49}$$

and substituting (3.48) results in

$$\Phi(x)M(\lambda)\begin{pmatrix} A \\ B \end{pmatrix} = m(\lambda)\Phi(x)\begin{pmatrix} A \\ B \end{pmatrix} \tag{3.50}$$

which simplifies to the eigenvalue equation for $M(\lambda)$. This indicates that the boundedness of the eigenfunctions ϕ is determined by the eigenvalues $m(\lambda)$. Since $M(\lambda)$ is a 2 by 2 matrix, its eigenvalues can be expressed in terms of its trace and its determinant, satisfying the equation

$$m^2 - (\operatorname{Tr} M)m + \det M = 0 \tag{3.51}$$

We define the discriminant $\Delta = \operatorname{Tr} M$, and we will show that $\det M = 1$.

Recall that $\Phi(x)$ is the fundamental solution matrix of the system described by (3.43). We can call the two solutions that it contains ϕ and $\tilde{\phi}$. The determinant of $\Phi(x)$ will also be a function of x:

$$\det \Phi(x) = \phi_1\tilde{\phi}_2 - \phi_2\tilde{\phi}_1 \tag{3.52}$$

We can then take the derivative of this function with respect to x, using (3.43) to simplify:

$$\begin{aligned}
\frac{\partial}{\partial x}\det\Phi(x) &= \phi_{1x}\tilde{\phi}_2 + \phi_1\tilde{\phi}_{2x} - \phi_{2x}\tilde{\phi}_1 - \phi_2\tilde{\phi}_{1x} \\
&= (u\phi_2 - i\lambda\phi_1)\tilde{\phi}_2 + (-u^*\tilde{\phi}_1 + i\lambda\tilde{\phi}_2)\phi_1 - (-u^*\phi_1 + i\lambda\phi_2)\tilde{\phi}_1 - (u\tilde{\phi}_2 - i\lambda\tilde{\phi}_1)\phi_2 \\
&= (u-u)\tilde{\phi}_2\phi_2 + (u^* - u^*)\tilde{\phi}_1\phi_1 + (i\lambda - i\lambda)\phi_1\tilde{\phi}_2 + (i\lambda - i\lambda)\phi_2\tilde{\phi}_1 \\
&= 0
\end{aligned}$$

This means that $\det\Phi(x)$ is constant, so $\det M(\lambda) = \det\Phi(L) = \det\Phi(0) = 1$. Now that we have these terms, we can write the eigenvalues using the quadratic formula:

$$m(\lambda) = \frac{\Delta \pm \sqrt{\Delta^2 - 4}}{2} \tag{3.53}$$

This information is enough to find the spectrum of the plane wave, which is used to create the SPBs.

3.2.2 Spectrum of the Plane Wave

To create the SPBs, the Bäcklund transformation is applied to the plane wave, so we will find its spectrum, following [6]. Since the spectrum is conserved in time, we use $u_a(x,t)$ at any time, in particular $t = 0$, which leaves $u_a(x,0) = a$. Inserting this into (3.43) gives the system of equations

$$\phi_{1x} = a\phi_2 - i\lambda\phi_1 \tag{3.54}$$

$$\phi_{2x} = -a\phi_1 + i\lambda\phi_2 \tag{3.55}$$

which can be solved by taking a derivative and substituting:

$$\begin{aligned}\phi_{1xx} &= a\phi_{2x} - i\lambda\phi_{1x} \\ &= -a^2\phi_1 + ia\lambda\phi_2 - ia\lambda\phi_2 - \lambda^2\phi_1 \\ &= -(\lambda^2 + a^2)\phi_1\end{aligned}$$

This suggests a linear combination of sines and cosines, with a frequency $k = \sqrt{\lambda^2 + a^2}$:

$$\phi_1 = A\cos kx + B\sin kx \tag{3.56}$$

Solving for ϕ_2 gives

$$a\phi_2 = (i\lambda A + kB)\cos kx + (i\lambda B - kA)\sin kx \tag{3.57}$$

From these relations we can find our fundamental solution matrix. Setting $\phi_1(0) = 1$, $\phi_2(0) = 0$ fixes $A = 1$ and $B = \frac{-i\lambda}{k}$. Instead setting $\phi_1(0) = 0$, $\phi_2(0) = 1$ gives $A = 0$ and $B = \frac{a}{k}$. This results in the fundamental solution matrix

$$\Phi(x) = \begin{pmatrix} \cos kx - \frac{i\lambda}{k}\sin kx & \frac{a}{k}\sin kx \\ -\frac{a}{k}\sin kx & \cos kx + \frac{i\lambda}{a}\sin kx \end{pmatrix} \tag{3.58}$$

With this, we can find $\Delta = \operatorname{Tr} M = \operatorname{Tr}\Phi(L) = \cos kL - \frac{i\lambda}{k}\sin kL + \cos kL + \frac{i\lambda}{k}\sin kL = 2\cos kL = 2\cos\sqrt{a^2+\lambda^2}L$. To have a bounded solution, the Floquet multiplier $m(\lambda)$ must have a modulus of one. Inserting our expression for Δ results in

$$m(\lambda) = \frac{2\cos kL \pm \sqrt{4\cos^2 kL - 4}}{2} = \cos kL \pm \sqrt{\cos^2 kL - 1} \tag{3.59}$$

For real values of k, this expression simplifies to $m(\lambda) = e^{\pm ikL}$, which always has a modulus of one. Since $k = \sqrt{\lambda^2 + a^2}$, k is real for all real values of lambda. This is the continuous spectrum.

To find the discrete spectrum λ_j, we set $\Delta = 2\cos kL = \pm 2$. The obvious solutions that appear are $kL = j\pi$. To rule out any complex solutions we set $kL = \rho + i\eta$ and use the identity $\cos\rho + i\eta = \cosh\eta\cos\rho - i\sinh\eta\sin\rho$. The imaginary part vanishes when $\rho = n\pi$ or $\eta = 0$. Choosing either option leads to the other when considering $\cos\rho = \pm 1$. Thus the only possible values of λ for which $\Delta = \pm 2$ are those that satisfy $kL = L\sqrt{\lambda^2 + a^2} = j\pi$, or $\lambda^2 = (\frac{j\pi}{L})^2 - a^2$.

3.2.3 Constructing the SPB Solutions

Once we have the spectrum, we can begin to construct the SPBs with the Bäcklund transform. The Bäcklund transform in general relates two PDEs. In this case it is an auto-Bäcklund transform as it relates the NLS equation to itself using two solutions. In particular, given a solution to the NLS equation q and an eigenfunction ϕ with eigenvalue ν satisfying the Lax equations with the potential q, another solution Q can be constructed of the form

$$Q = q - 2(\nu - \nu^*)\frac{\phi_1\phi_2^*}{|\phi|^2} \tag{3.60}$$

We will use the plane wave ae^{2ia^2t} to find ϕ and ν and use these to construct the SPB, following the constructing used in [5]. The spectrum derived earlier contains only real or imaginary λ, thus ν will be either real or imaginary. If we choose a real point of the spectrum, we gain the unhelpful solution $Q = q$, so we must choose an imaginary value.

We begin by solving the time equation. With the plane wave substituted, the system is

$$\begin{pmatrix} ia^2 - 2i\lambda^2 & 2i\lambda a e^{2ia^2 t} \\ 2i\lambda a e^{-2ia^2 t} & -ia^2 + 2i\lambda^2 \end{pmatrix} \phi = \phi_t \tag{3.61}$$

To solve this, we make the substitution

$$\phi = \begin{pmatrix} e^{ia^2 t} & 0 \\ 0 & e^{-ia^2 t} \end{pmatrix} \hat{\phi} \tag{3.62}$$

Where we now have

$$\begin{aligned} \phi_t &= \begin{pmatrix} ia^2 e^{ia^2 t} & 0 \\ 0 & -ia^2 e^{-ia^2 t} \end{pmatrix} \hat{\phi} + \begin{pmatrix} e^{ia^2 t} & 0 \\ 0 & e^{-ia^2 t} \end{pmatrix} \hat{\phi}_t \\ &= \begin{pmatrix} e^{ia^2 t} & 0 \\ 0 & e^{-ia^2 t} \end{pmatrix} \left(\begin{pmatrix} ia^2 & 0 \\ 0 & -ia^2 \end{pmatrix} \hat{\phi} + \hat{\phi}_t \right) \end{aligned} \tag{3.63}$$

Also note that we can write

$$\begin{pmatrix} ia^2 - 2i\lambda^2 & 2i\lambda a e^{2ia^2 t} \\ 2i\lambda a e^{-2ia^2 t} & -ia^2 + 2i\lambda^2 \end{pmatrix} = \begin{pmatrix} e^{ia^2 t} & 0 \\ 0 & e^{-ia^2 t} \end{pmatrix} \begin{pmatrix} ia^2 - 2i\lambda^2 & 2i\lambda a \\ 2i\lambda a & -ia^2 + 2i\lambda^2 \end{pmatrix} \begin{pmatrix} e^{-ia^2 t} & 0 \\ 0 & e^{ia^2 t} \end{pmatrix} \tag{3.64}$$

Thus for $\hat{\phi}$ we can form the time-independent system

$$\hat{\phi}_t = \begin{pmatrix} -2i\lambda^2 & 2i\lambda a \\ 2i\lambda a & 2i\lambda^2 \end{pmatrix} \hat{\phi} \tag{3.65}$$

This system has eigenvalues μ satisfying

$$\mu^2 + 4\lambda^4 + 4\lambda^2 a^2 = 0 \tag{3.66}$$

If we let $k = \sqrt{\lambda^2 + a^2}$ as previously, the solution will be a linear combination of $\hat{\phi}^+$ and $\hat{\phi}^-$, defined as

$$\hat{\phi}^{\pm} = \vec{c}(x)e^{\pm 2ik\lambda t} \tag{3.67}$$

Using these functions in time, we move on to the spatial equation. The system is

$$\phi_x = \begin{pmatrix} -i\lambda & iae^{2ia^2t} \\ iae^{-2ia^2t} & i\lambda \end{pmatrix}\phi \tag{3.68}$$

We can again write

$$\begin{pmatrix} -i\lambda & iae^{2ia^2t} \\ iae^{-2ia^2t} & i\lambda \end{pmatrix} = \begin{pmatrix} e^{ia^2t} & 0 \\ 0 & e^{-ia^2t} \end{pmatrix}\begin{pmatrix} -i\lambda & ia \\ ia & i\lambda \end{pmatrix}\begin{pmatrix} e^{-ia^2t} & 0 \\ 0 & e^{ia^2t} \end{pmatrix} \tag{3.69}$$

Thus substituting $\hat{\phi}$ gives the equation

$$\begin{pmatrix} -i\lambda & ia \\ ia & i\lambda \end{pmatrix}\hat{\phi} = \hat{\phi}_x \tag{3.70}$$

And further substituting (3.67) gives

$$\begin{pmatrix} -i\lambda & ia \\ ia & i\lambda \end{pmatrix}\vec{c} = \vec{c}_x \tag{3.71}$$

The eigenvalues of this system satisfy

$$\mu^2 + \lambda^2 + a^2 = \mu^2 + k^2 = 0 \tag{3.72}$$

meaning that $\vec{c}$ has solutions

$$\vec{c}^{\pm}(x) = \vec{d}^{\pm}e^{\pm ikx} \tag{3.73}$$

With $\vec{d}^{\pm}$ satisfying

$$\begin{pmatrix} -i\lambda \mp ik & ia \\ ia & i\lambda \mp ik \end{pmatrix} \vec{d} = 0 \tag{3.74}$$

fixing it to be of the form

$$\vec{d}^{\pm} = d \begin{pmatrix} a \\ \lambda \pm k \end{pmatrix} \tag{3.75}$$

When λ is imaginary, as it will be when the SPB is constructed, this expression can be made slightly more simple by introducing a parameter p defined by $\cos p = \frac{k}{a}$. We then have $k = a\cos p$ and $\lambda = \sqrt{k^2 - a^2} = a\sqrt{\cos^2 p - 1} = ai \sin p$.

Reuniting the constituents of ϕ gives

$$\phi^{\pm} = \begin{pmatrix} e^{ia^2 t} & 0 \\ 0 & e^{-ia^2 t} \end{pmatrix} \begin{pmatrix} a \\ \lambda \pm k \end{pmatrix} e^{\pm i(kx + 2k\lambda t)} = \begin{pmatrix} a e^{ia^2 t} \\ (\lambda \pm k) e^{-ia^2 t} \end{pmatrix} e^{\pm i(kx + 2k\lambda t)} \tag{3.76}$$

To construct the SPB, we choose a complex point in the spectrum, denoted ν, and its associated eigenfunction

$$\phi = c_1 \begin{pmatrix} a e^{ia^2 t} \\ (\nu + k) e^{-ia^2 t} \end{pmatrix} e^{i(kx + 2k\nu t)} + c_2 \begin{pmatrix} a e^{ia^2 t} \\ (\nu - k) e^{-ia^2 t} \end{pmatrix} e^{-i(kx + 2k\nu t)} \tag{3.77}$$

For simplicity, we define $\frac{c_1}{c_2} = e^{\rho} e^{i\beta}$ and $\cos p = \frac{a}{k}$. This allows us to instead write

$$\phi = c_2 \begin{pmatrix} a e^{ia^2 t} (e^{\rho} e^{i\beta} e^{i(kx + 2k\nu t)} + e^{-i(kx + 2k\nu t)}) \\ a e^{-ia^2 t} (e^{ip} e^{i(kx + 2k\nu t)} - e^{-ip} e^{-i(kx + 2k\nu t)}) \end{pmatrix} \tag{3.78}$$

Substituting this into 3.60 and making the substitutions $\sigma = -4ik\nu$ and $\tau = \rho - \sigma t$ gives us the commonly used representation of the SPB

$$Q(x,t) = a e^{2ia^2 t} \left(\frac{i \sin 2p \tanh \tau + \cos 2p - sinp \cos(2kx + \beta) \operatorname{sech} \tau}{1 + sinp \cos(2kx + \beta) \operatorname{sech} \tau} \right) \tag{3.79}$$

These SPB solutions are used to model rogue waves because they are steeply peaked.

CHAPTER 4
NUMERICAL EXPERIMENTS

This chapter will discuss the numerical experiments conducted to investigate how viscous damping affects rogue wave formation and downshifting.

4.1 Diagnostics

The first diagnostic used in this study is the wave strength:

$$S(t) = \frac{U_{max}(t)}{H_s(t)} \tag{4.1}$$

where $U_{max}(t)$ is the maximum amplitude of the waveform and $H_s(t)$ is the significant wave height, which is found by identifying the mesh points with the highest amplitude and averaging the highest third. This diagnostic is used to quantify whether a rogue wave has appeared: when $S(t) >= 2.2$, a rogue wave has occured at time t.

Next are the two diagnostics associated with downshifting: the spectral peak k_p, and the spectral mean k_m, defined as

$$k_p = k \text{ such that } |u_k| \text{ is the Fourier mode maximal amplitude} \tag{4.2}$$

$$k_m = -\frac{1}{2}\frac{P}{E} \tag{4.3}$$

where P and E are the momentum and energy as defined earlier. Downshifting occurs when the spectral peak changes from its original value $k_p = 0$ to a lower mode. The spectral mean is used to track whether the concentration of energy between Fourier modes is changing. When the spectral mean is decreasing, this indicates that downshifting may occur, which is then checked via the spectral peak.

All of these diagnostics are tracked at each point of time in the simulation. Permanent downshifting occurs when k_p no longer shifts back up to a higher mode.

4.2 Numerical Integrator

The numerical integrator used in these experiments uses the fast Fourier transform to calculate the spatial derivatives and Hilbert transform used in the equation. The solution is evolved in time using the fourth order Runge-Kutta method. The accuracy of this method can be confirmed by checking the conservation of the conserved quantities of the HONLS equation. This integrator conserves the energy and momentum with an accuracy of $O(10^{-12})$ on the time scales used, as demonstrated in Figure 4.1. Simpson's rule is used to calculate the energy and momentum.

Once the integrator has reached the specified end time, the program, written in Matlab, generates plots for the waveform and diagnostics, including the energy and momentum.

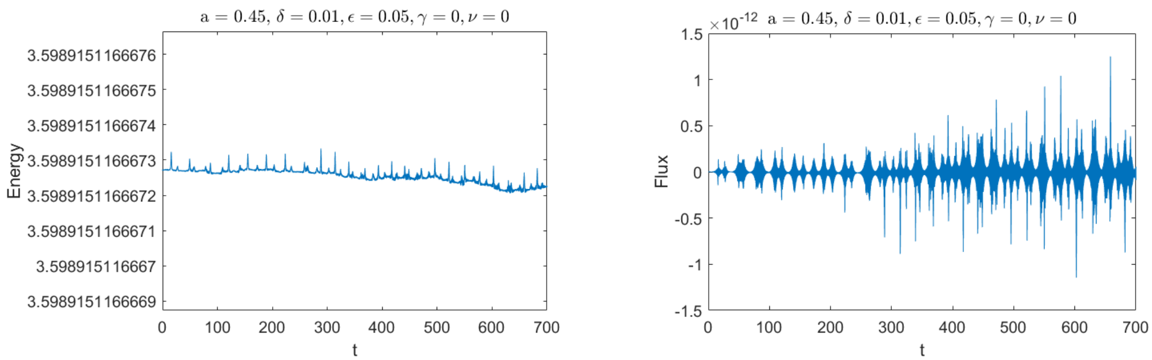

Figure 4.1: Energy (left) and momentum (right) as tracked by the integrator using HONLS with typical initial conditions.

4.3 Experiments

Initial data used for the experiments are perturbations of the plane wave, of the form

$$u(x, 0) = a(1 + \delta \cos \mu x) \tag{4.4}$$

where a is the amplitude of the initial waveform and $\mu = \frac{2\pi}{L}$. The value L refers to the spatial period, which in our set of experiments was chosen to be $4\pi\sqrt{2}$. The values of a and δ are $.45$ and $.01$, respectively. This combination of values gives

$$\frac{aL}{\pi} = 2.5456 \tag{4.5}$$

Using the criterion in equation (3.5), this puts us in the two unstable mode regime. This regime provides more rogue wave activity than the one unstable mode regime, allowing us to see the relationship between the rogue waves and the dominant frequency more clearly.

Different values of the viscous damping coefficient, Γ, were investigated to find an appropriate range. The values ranged from 0.00025 to 0.008, with a particular interest in the 0.0025 to 0.0035 range. Values of Γ above 0.008 quickly damp the plane wave, preventing any rogue wave behavior.

The time of last rogue wave and time of permanent downshift were recorded as a function of Γ for dHONLS. Results of simulations with viscous damping were compared to simulations with only linear damping for the same value of Γ.

4.4 Results

Simulations of dHONLS show that permanent downshift does occur for values of Γ in the relevant range, as seen in figure 4.2. Values below $.0025$ did not exhibit permanent downshift when integrated to $t = 700$. A test run of $\Gamma = .0001$ was integrated to $t = 10,000$. This experiment showed that the spectral peak still varied between $-1, 0$ and 1 as late as $t = 9,000$, shown in figure 4.3.

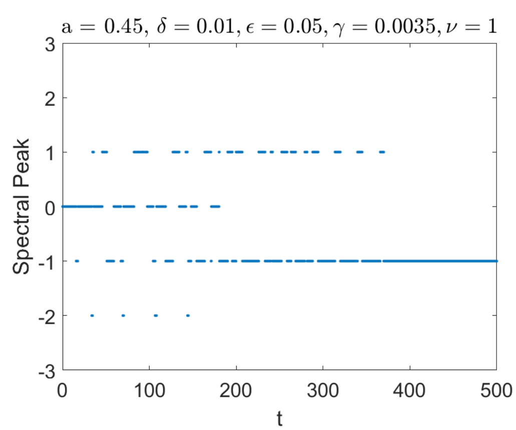

Figure 4.2: Spectral peak for viscous damped HONLS with $\Gamma = 0.0035$.

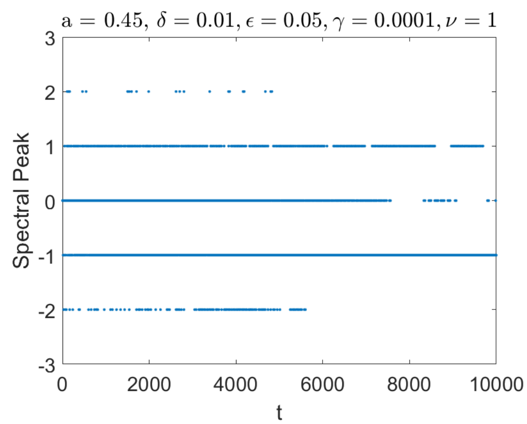

Figure 4.3: Spectral peak for viscous damped HONLS with Γ = .0001.

Time of last rogue wave and time of permanent downshift decreased for increasing values of Γ, show in figure 4.4. It also held that the last rogue wave occurred before permanent downshift. This confirms previous findings that rogue waves do not occur after permanent downshift.

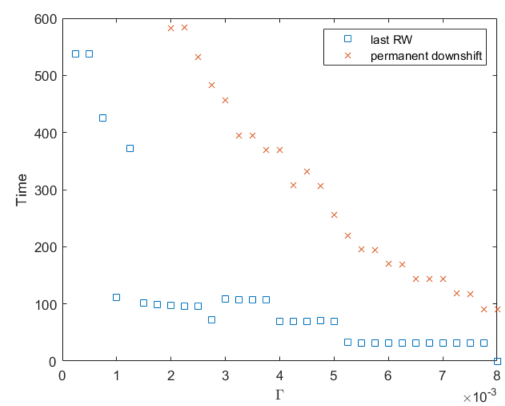

Figure 4.4: Time of last rogue wave and time of permanent downshift, if present, as a function of Γ.

For the same value of Γ, simulations with viscous damping seem to have more rogue wave activity. An example is shown in Figure 4.5. Figure 4.6 shows the number of rogue waves for different values of Γ. In all but one of the values of Γ, viscous damping had equal or higher number of rogue waves.

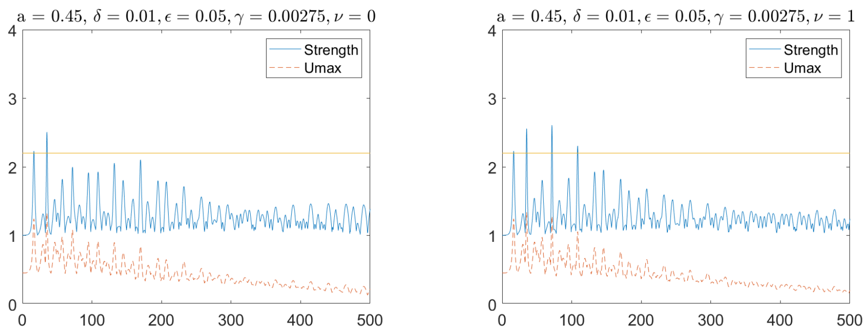

Figure 4.5: Wave strength for HONLS with linear damping only (left) and with viscous damping (right) for Γ = 0.00275

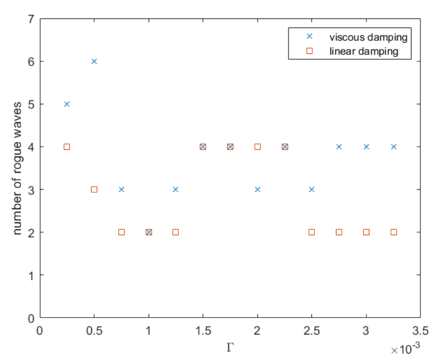

Figure 4.6: Number of rogue waves for viscous and linear damped HONLS as a function of Γ

A possible reason for this rogue wave behavior is how the viscosity effects the energy and momentum. In dHONLS, the derivatives of energy and momentum are

$$\frac{dE}{dt} = -2\Gamma(E - \epsilon P) \tag{4.6}$$

$$\frac{dP}{dt} = -2\Gamma(P - 4\epsilon Q) \tag{4.7}$$

where

$$Q = \int_0^L |u_x|^2 dx \tag{4.8}$$

The order ϵ effects provide a slight decrease to the rate at which E and P decay.

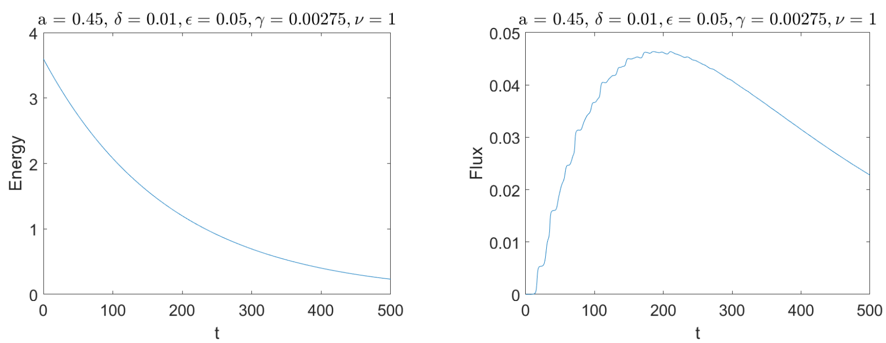

Figure 4.7: Energy (right) and momentum (left) for dHONLS with $\Gamma = 0.00275$

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