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Probability-Distribution Function For The Gradient Of A Passive Scalar Diffusing In Isotropic Turbulence - Mapping-Closure Model

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Probability distribution function for the gradient of a passive scalar diffusing in isotropic turbulence: Mapping-closure model

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A mapping-closure model is used to derive the probability distribution function (PDF) for the gradient of a passive scalar diffusing in a random velocity field. This PDF is non-Gaussian and shows exponential tails observed in laboratory experiments.

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I. INTRODUCTION

A passive scalar diffusing in a turbulent fluid is randomized through advection by the fluid and acquires statistical properties which are directly related to those of the random velocity field. The random convection leads to a statistical increase in the gradients of the scalar field to be halted ultimately by the smoothing action of diffusion. Obukhov [1] and Corrsin [2] invoked arguments similar to those used in Kolmogorov's [3] theory for the turbulent velocity field to surmise that the small-scale structure of the scalar field has a measure of universality and possesses statistical properties which are locally homogeneous and isotropic and depend only weakly on the large-scale features of the scalar field, provided the Reynolds number and the Peclet number of the turbulence are sufficiently high. A self-similar cascade of scalar variance to smaller scales was then postulated whereby the spectrum function $\Gamma(k)$ was given by

$$\Gamma(k) \sim \chi \varepsilon^{-1/3} k^{-5/3} , \tag{1}$$

where χ is the average rate per unit volume of the fluid at which the scalar variance is transferred in the spectral space and ε is the average rate of viscous dissipation of kinetic energy per unit mass of the fluid.

The Obukhov-Corrsin theory assumes that the scalar-variance cascade is local in the spectral space and involves a continuous loss of information. The normalized statistics of band-limited scalar fields in the inertial-convective range would then be identical and depend only on the mean dissipation rates χ and ε. However, direct numerical simulations [4] have shown that the most intense regions of the scalar gradient occur as large flat sheets. This implies that the mean dissipation rates χ and ε would exhibit spatial fluctuations, thereby indicating the buildup of intermittency in the inertial-convective range of the scalar-variance cascade [5–12]. The statistics of successively band-limited scalar fields may then be expected to become ever more non-Gaussian. Laboratory experiments [6,10,11] have shown that the statistics of the gradient of the scalar diffusing in a random velocity field are far from Gaussian and exhibit long and exponential tails.

In this paper, we propose to use the mapping-closure principle of Kraichnan [13] to derive the probability distribution function (PDF) for the gradient of a passive scalar diffusing in a random velocity field via the nonlinear mapping of a Gaussian reference field $\mathbf{X}_G(\mathbf{x})$ at each t. The model equation for the mapping function is obtained by making plausible physical assumptions about the mechanism underlying the scalar diffusion process in a turbulent fluid. (Kraichnan [13] showed that this procedure provides valid closures for single-point probability distributions of velocity gradients, thereby giving insight into the buildup of intermittency of velocity in fully developed turbulence.)

II. MAPPING-CLOSURE MODEL

The mapping-closure scheme in the following is based on the distortion of a Gaussian reference field into a dynamically evolving non-Gaussian field for the scalar gradient. The evolution is obtained from the equation describing the diffusion of the scalar T in a random velocity field $\mathbf{v}$ and is based on the competition between the diffusive relaxation and the random convection processes:

$$\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \kappa \nabla^2 T \tag{2}$$

where κ is the resistivity.

In order to motivate the rationale for modeling the diffusion of passive scalar in the Navier-Stokes turbulence, let us first consider the diffusion of a passive scalar in Burgers turbulence [13].

A. One-dimensional scalar turbulence

This problem is governed by

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} , \tag{3}$$

$$\left[\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \right] T = \kappa \frac{\partial^2 T}{\partial X^2} , \tag{4}$$

where ν is the viscosity.

The velocity gradient $\partial u / \partial x$ and the scalar gradient

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$\partial T/\partial x$, then, obey

$$\left[\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right]\frac{\partial u}{\partial x}=-\left[\frac{\partial u}{\partial x}\right]^2+\nu\frac{\partial^3 u}{\partial x^3}\ , \tag{5}$$

$$\left[\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right]\frac{\partial T}{\partial x}=-\frac{\partial u}{\partial x}\frac{\partial T}{\partial x}+\kappa\frac{\partial^3 T}{\partial x^3} \tag{6}$$

The first term on the right-hand side of Eq. (5) describes the self-straining of $\partial u/\partial x$. This leads to an enhancement (relaxation) of negative (positive) $\partial u/\partial x$ and therefore the formation of sawtooth waves with shock fronts whose steepness is limited by the dissipative term on the right-hand side of Eq. (5) [13].

The first term on the right-hand side of Eq. (6) describes the convective stretching of $\partial T/\partial x$. It leads to a relaxation (enhancement) of a positive (negative) $\partial T/\partial x$ when it is in the same direction as $\partial u/\partial x$ and an enhancement (relaxation) of a positive (negative) $\partial T/\partial x$ when it is in the direction opposite to that of $\partial u/\partial x$. This enhancement is limited by the dissipative term on the right-hand side of Eq. (6).

In the simplest mapping closure for one-dimensional scalar turbulence, $\partial u/\partial x$ and $\partial T/\partial x$ may be taken to evolve according to

$$\frac{\partial u}{\partial x}=J(t)\frac{du^0}{d\xi}\ , \tag{7}$$

$$\frac{\partial T}{\partial x}=I(t)\frac{dT^0}{d\xi}\ , \tag{8}$$

where $J(t)$ and $I(t)$ are the effective intensification ratios of velocity gradient and temperature gradient, respectively, of local flow structures. In Eqs. (7) and (8), the dependences of the mapping functions J and I on x and ξ are not explicitly shown. $u^0(\xi)$ and $T^0(\xi)$ are multivariate Gaussian, statistically homogeneous reference fields.

Using (7) and (8), we obtain from Eqs. (5) and (6)

$$\frac{\partial J}{\partial t}=-\frac{du^0}{d\xi}J^2+\nu k_d^2 J^3\ , \tag{9}$$

$$\frac{\partial I}{\partial t}=-\frac{du^0}{d\xi}JI+\kappa\hat{k}_d^2 I^3\ , \tag{10}$$

where k_d and $\hat{k}_d$ are the characteristic dissipation wave numbers for the fluctuations $du^0/d\xi$ and $dT^0/d\xi$, respectively,

$$k_d^2\equiv\frac{\left\langle\left[\frac{d^2u_0}{d\xi^2}\right]^2\right\rangle}{\left\langle\left[\frac{du^0}{d\xi}\right]^2\right\rangle}\ ,$$

$$\hat{k}_d^2\equiv\frac{\left\langle\left[\frac{d^2T^0}{d\xi}\right]^2\right\rangle}{\left\langle\left[\frac{dT^0}{d\xi}\right]^2\right\rangle}\ .$$

The first terms on the right-hand side of Eqs. (9) and (10) come from the self-stretching term $-(\partial u/\partial x)^2$ and the convective-stretching term $-(\partial u/\partial x)(\partial T/\partial x)$ in Eqs. (5) and (6). The second terms on the right-hand side of Eqs. (9) and (10) describe the viscous decay of $\partial u/\partial x$ with the decay constant proportional to the square of the stretching ratio J^2 and the diffusive decay of $\partial T/\partial x$ with the decay constant proportional to the square of the stretching ratio I^2, respectively.

B. Navier-Stokes scalar turbulence

Let s be a transverse component of $\partial v_i/\partial x_j$ and X be a component of $\partial T/\partial x_i$ and have Gaussian distributed initial values s_0 and X_0, respectively. The initial states s_0 and X_0 are then assumed to evolve via local distortion through the action of nonstochastic effective stretching functions $J(s_0,t)$ and $I(X_0,t)$, according to

$$s=J(s_0,t)s_0\ ,\quad X=I(X_0,t)X_0\ . \tag{11}$$

The stretching functions $J(s_0,t)$ and $I(X_0,t)$ are then postulated to evolve, in turn, according to

$$\frac{\partial J}{\partial t}=|s_0|J^2-\nu k_d^2 J^3\ , \tag{12}$$

$$\frac{\partial I}{\partial t}=|s_0|JI-\kappa\hat{k}_d^2 I^3\ , \tag{13}$$

where ν is the viscosity. Here, Eq. (12) is due to Kraichnan [13]. The first terms on the right-hand sides in Eqs. (12) and (13) describe the growths due to inertial and convective stretching while the second terms describe the viscous and diffusive decays. The absolute values in Eqs. (12) and (13) reflect the symmetry between positive and negative values of the transverse velocity gradient. k_d and $\hat{k}_d$ are the characteristic dissipation wave numbers for the fluctuations s_0 and X_0, respectively.

We then have in the stationary state,

$$J=\frac{|s_0|}{\nu k_d^2}\ ,\quad I^2=\frac{|s_0|J}{\kappa\hat{k}_d^2}\ . \tag{14}$$

Using (14), we then have form (11)

$$|s|=\frac{|s_0|^2}{\nu k_d^2}\ ,\quad |X|=\frac{|s_0||X_0|}{\sqrt{\nu\kappa k_d^2\hat{k}_d^2}}\ . \tag{15}$$

If we assume that the PDF's of s_0 and (s_0,X_0) are given by

$$P(s_0)=\frac{1}{\sqrt{2\pi\sigma_1^2}}e^{-s_0^2/2\sigma_1^2}\ , \tag{16}$$

$$P(s_0,X_0)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\times\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{s_0^2}{\sigma_1^2}-\frac{2\rho s_0X_0}{\sigma_1\sigma_2}+\frac{X_0^2}{\sigma_2^2}\right]\right\}\ , \tag{17}$$

where

$$\sigma_1^2\equiv\langle s_0^2\rangle\ ,\quad \sigma_2^2\equiv\langle X_0^2\rangle\ ,\quad \rho\equiv\frac{\langle s_0X_0\rangle}{\sigma_1\sigma_2}\ .$$

The PDF's of s and X are then given by

$$P(s)=P(s_0)\frac{\partial s_0}{\partial s} , \tag{18}$$

$$P(X)=P(s_0,X_0)\frac{\partial |s_0 X_0|}{\partial X} . \tag{19}$$

Thus,

$$P(s)=\left[\frac{\nu k_d}{\pi\sigma_1^2|s|}\right]^{1/2} e^{-|s|/2\sigma_1^2/\nu k_d^2} , \tag{20}$$

$$P(X)=\frac{\sqrt{\nu\kappa k_d^2 \hat{k}_d^2}}{\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \times \exp\frac{\rho|X|}{\sigma_1\sigma_2(1-\rho^2)/\sqrt{\nu\kappa k_d^2\hat{k}_d^2}} \times K_0\left[\frac{|X|}{\sigma_1\sigma_2(1-\rho^2)/\sqrt{\nu\kappa k_d^2\hat{k}_d^2}}\right] , \tag{21}$$

where $K_0(x)$ is the modified Bessel function of the second kind. Here, (20) is due to Kraichnan [13]. The derivation of (21) follows the usual procedure to calculate the PDF's of functions of random variables [14] and is briefly sketched in the Appendix.

Noting that

$$P(X)\sim\frac{(\nu\kappa k_d^2\hat{k}_d^2)^{1/4}}{\pi\sqrt{\sigma_1\sigma_2}}\frac{1}{\sqrt{2\pi X}} \times\exp\left[\frac{|X|}{\sigma_1\sigma_2(1+\rho)/\sqrt{\nu\kappa k_d^2\hat{k}_d^2}}\right] , \text{ as } |X \Longrightarrow \infty| , \tag{22}$$

we see that $P(X)$, like $P(s)$, shows exponential tails.

The scalar gradient PDF (21) is compared in Fig. 1 with the experimental data of Antonia *et al.* [6] and Tong and Warhaft [11], and the agreement is seen to be very good.

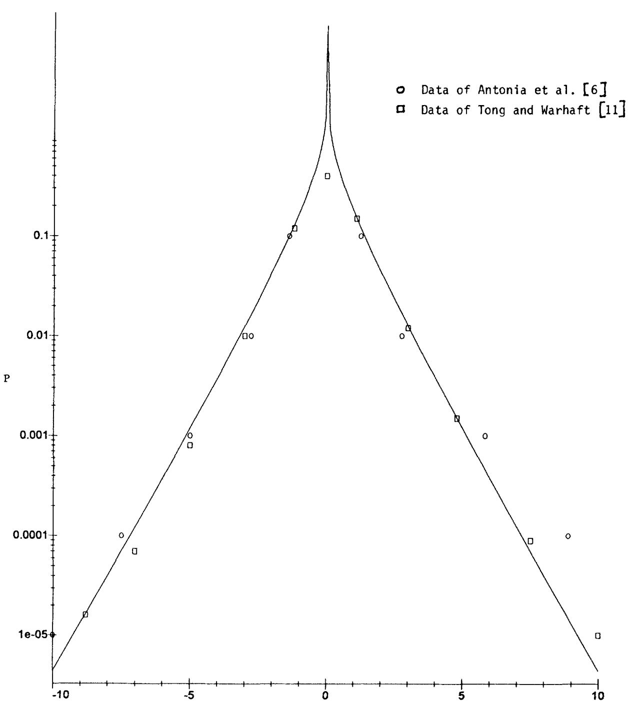

FIG. 1. Comparison of the calculated PDF (21) with the experimental data (the parameters ρ and $\sigma_1\sigma_2/\sqrt{\nu\kappa k_d^2\hat{k}_d^2}$ have been taken to be 0.6 and 0.59, respectively).

III. DISCUSSION

The sharp maxima exhibited by the scalar-gradient PDF (21) suggests the occurrence of (a) small regions where the scalar gradient X becomes very large, and (b) a strong mixing in large regions of the flow. These results are supported by the direct numerical simulations of Pumir [15], which showed that the structure of the scalar field in physical space exhibits relatively well-mixed domains separated by very narrow sheetlike regions where the scalar gradients are very large.

It may be noted that exponential tails on the PDF can be obtained via a mapping function satisfying averaged equations, as in the above development, or, alternatively, by considering the effects of stochastic strain on the mapping function [16]. It is remarkable, however, that these simple mapping models are able to generate, from Gaussian fields via self-straining and convective-stretching acting over the order of a singly eddy-turnover time of dissipation-range scales, intermittencies that closely resemble those of real turbulence at large Reynolds and Peclet numbers.

Incidentally, PDF's involving modified Bessel functions, like that in (21), have been given previously for the kinetic energy dissipation ε [17]; here, the velocity gradient was taken to be conditionally Gaussian distributed with the variance Γ distributed.

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APPENDIX

Let x and y be two random variables, and let $z=xy$. The probability distribution function $F_z(z)$ is determined by the probability of the event $\{xy\le z\}$, and is given by

$$F_z(z)=\int_{D_z}\int f_{xy}(x,y)dx\,dy , \tag{A1}$$

where $f_{xy}(x,y)$ is the joint probability density function of the random variables x and y, and the region D_z is given by

$$\{D_z:(x,y)|xy \le z\}$$

or

$$\left\{D_z:(x,y)|x \lessgtr \frac{z}{y}, y \gtrless 0\right\}.$$

Thus,

$$F_z(z)=\int_0^{\infty}\int_{-\infty}^{z/y} f_{xy}(x,y)dx\,dy + \int_{-\infty}^{0}\int_{z/y}^{\infty} f_{xy}(x,y)dx\,dy \,, \tag{A2}$$

from which the probability density function $f_z(z)$ is given by

$$f_z(z)=\int_0^{\infty}\frac{1}{y}f_{xy}\left[\frac{z}{y},y\right]dy - \int_{-\infty}^{0}\frac{1}{y}f_{xy}\left[\frac{z}{y},y\right]dy$$

or

$$f_z(z)=2\int_0^{\infty}\frac{1}{y}f_{xy}\left[\frac{z}{y},y\right]dy \,. \tag{A3}$$

Let x and y be jointly normal with the joint probability density function given by

$$f_{xy}(x,y)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \times \exp -\frac{1}{2(1-\rho^2)}\left[\frac{x^2}{\sigma_1^2}-\frac{2\rho xy}{\sigma_1\sigma_2}+\frac{y^2}{\sigma_2^2}\right], \tag{A4}$$

where

$$\sigma_1^2 \equiv \langle x^2\rangle \,, \quad \sigma_2^2 \equiv \langle y^2\rangle \,, \quad \rho \equiv \frac{\langle xy\rangle}{\sigma_1\sigma_2} \,.$$

Using (A4), (A3) becomes

$$f_z(z)=\alpha\int_0^{\infty}\frac{1}{y}e^{-\left[ay^2+\frac{b(z)}{y^2}\right]}dy \,, \tag{A5}$$

where

$$\alpha \equiv \frac{2}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}e^{\rho/[\sigma_1\sigma_2(1-\rho^2)]} \,,$$

$$a \equiv \frac{1}{2\sigma_2^2(1-\rho^2)} \,, \quad b(z) \equiv \frac{z^2}{2\sigma_1^2(1-\rho^2)} \,.$$

On evaluating the integral, (A5) leads to

$$f_z(z)=\alpha K_0(2\sqrt{ab(z)}) \,, \tag{A6}$$

where $K_0(x)$ is the modified Bessel function of the second kind.

[1] A. M. Obukhov, Izv. Akad. Nauk. SSSR Ser. Georg. Geofiz. **13**, 58 (1949).
[2] S. Corrsin, J. Appl. Phys. **22**, 469 (1951).
[3] A. N. Kolmogorov, Dokl. Akad. Nauk. SSSR **30**, 301 (1941).
[4] G. R. Ruetsch and M. R. Maxey, Phys. Fluids A **3**, 1587 (1991).
[5] R. A. Antonia and C. W. van Atta, J. Fluid Mech. **67**, 273 (1975).
[6] R. A. Antonia, E. J. Hopfinger, Y. Gagne, and F. Anselmet, Phys. Rev. A **30**, 2704 (1984).
[7] R. R. Prasad, C. Meneveau, and K. R. Sreenivasan, Phys. Rev. Lett. **61**, 74 (1988).
[8] C. Meneveau, K. R. Sreenivasan, P. Kailasnath, and M. S. Fan, Phys. Rev. A **41**, 894 (1990).
[9] L. C. Andrews and B. K. Shivamoggi, Phys. Fluids A **2**, 105 (1994).
[10] Jayesh and Z. Warhaft, Phys. Fluids A **4**, 2292 (1992).
[11] C. Tong and Z. Warhaft, Phys. Fluids **6**, 2165 (1994).
[12] B. K. Shivamoggi, Phys. Lett. A **168**, 47 (1992).
[13] R. H. Kraichnan, Phys. Rev. Lett. **65**, 575 (1990).
[14] A. Papoulis, *Probability, Random Variables and Stochastic Processes* (McGraw-Hill, New York, 1965).
[15] A. Pumir, Phys. Fluids **6**, 2118 (1994).
[16] R. H. Kraichnan (private communication).
[17] L. C. Andrews, R. L. Phillips, B. K. Shivamoggi, J. K. Beck, and M. Joshi, Phys. Fluids A **1**, 999 (1989).