Reactive Control of Autonomous Dynamical Systems

by

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This thesis mainly consists of five independent papers concerning the reactive control design of autonomous mobile robots in the context of target tracking and cooperative formation keeping with obstacle avoidance in the static/dynamic environment.

Technical contents of this thesis are divided into three parts. The first part consists of the first two papers, which consider the target-tracking and obstacle avoidance in the static environment. Especially, in the static environment, a fundamental issue of reactive control design is the local minima problem (LMP) inherent in the potential field methods (PFMs). Through introducing a state-dependent planned goal, the first paper proposes a switching control strategy to tackle this problem. The control law for the planned goal is presented. When trapped into local minima, the robot can escape from local minima by following the planned goal. The proposed control law also takes into account the presence of possible saturation constraints. In addition, a time-varying continuous control law is proposed in the second paper to tackle this problem. Challenges of finding continuous control solutions of LMP are discussed and explicit design strategies are then proposed.

The second part of this thesis deals with target-tracking and obstacle avoidance in the dynamic environment. In the third paper, a reactive control design is presented for omni-directional mobile robots with limited sensor range to track targets while avoiding static and moving obstacles in a dynamically evolving environment. Towards this end, a multi-
objective control problem is formulated and control is synthesized by generating a potential field force for each objective and combining them through analysis and design. Different from standard potential field methods, the composite potential field described in this paper is time-varying and planned to account for moving obstacles and vehicle motion. In order to accommodate a larger class of mobile robots, the fourth paper proposes a reactive control design for unicycle-type mobile robots. With the relative motion among the mobile robot, targets, and obstacles being formulated in polar coordinates, kinematic control laws achieving target-tracking and obstacle avoidance are synthesized using Lyapunov based technique, and more importantly, the proposed control laws also take into account possible kinematic control saturation constraints.

The third part of this thesis investigates the cooperative formation control with collision avoidance. In the fifth paper, firstly, the target tracking and collision avoidance problem for a single agent is studied. Instead of directly extending the single agent controls to the multi-agents case, the single agent controls are incorporated with the cooperative control design presented in [1]. The proposed decentralized control is reactive, considers the formation feedback and changes in the communication networks. The proposed control is based on a potential field method, its inherent oscillation problem is also studied to improve group transient performance.
To my parents, my brother, and my big family
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CHAPTER 1
INTRODUCTION

The research problems considered in this thesis arise from two active research areas of mobile robotics, which are navigation and coordination in the presence of static/moving obstacles. Navigation and obstacle avoidance which basically addresses the problem of getting from A to B in a collision-free and efficient way. Meanwhile, multi-robot coordination is a somewhat broader area where the common theme is that of making heterogeneous systems act as one group and exhibit certain cooperative behaviors, e.g. maintaining a prescribed formation.

The purpose of the introduction is threefold: (1) to state the theoretical and practical importance of this research topic, (2) to give a review of existing results in above two areas and point out the existing major problems, (3) to summarize the contributions of this work.

1.1 Background and Motivation

For most real-world applications, it is desired that mobile robots can explore and move within dynamic environments. In addition, the environment is usually uncertain as complete information and future trajectories of obstacles cannot be assumed a priori. In this context, the fundamental problem arising for mobile robots is how to track moving targets where robots have limited sensor range and are simultaneously avoiding static and moving obstacles in real-time, which is illustrated in Figure 1.1. The challenges to deal with this problem are:

- Configuration space is difficult to compute;
Complexity, uncertainty, dynamic environments;

Kinematic and dynamic constraints;

Time constraints.

Besides, recent years have witnessed a boom of research motivated by the application of multi-robot systems that operate with either full autonomy or semi-autonomy, i.e., executing high level commands from a remote human operator. Many missions such as formation flight control, marine mine-sweeping, and cooperative robot reconnaissance will be implemented among distributed autonomous systems, which require formation movement capability. Some representative scenarios are illustrated in Figures 1.2, 1.3, 1.4, and 1.5.

To achieve this goal, the central and difficult issues are:

- cooperative formation movement control of multi-robots in the context of a time-varying communication topology;

- Kinematic and dynamic constraints;
Figure 1.2: Illustration of cooperative robot reconnaissance

Figure 1.3: Illustration of marine mine-sweeping

Figure 1.4: Illustration of formation flight control

- collision avoidance naturally arising in the dynamically evolving environment.
The above two areas have drawn great attention due to its practical importance and theoretical challenges. In this work, reactive control design to tackle these problem is carried out. Reactive control for a mobile robot can be defined as a mapping from the perceptual space to the control space. This mapping can be hard-coded by the user’s predefined algorithms (fuzzy logic, neutral network, and potential fields etc.), and can also be learned on line. They are capable of reacting very quickly to state changes, and this is where the name is derived from.

In what follows, a comprehensive review of the existing approaches to deal with problems in above two areas is given. And, more importantly, the existing open problems and the basic ideas proposed to solve these problems are analyzed. The promising directions to find the solution are discussed. Actually, the work described in the following chapters belongs to one of these directions.
1.2 Previous Work

Mobile robots operating in the static/dynamic environment may be simply modeled as scalar agents without any dynamics, or may be described as a double-integrator point-mass model, or may be as complicated as nonlinear dynamical systems with nonholonomic constraints. Even the control design of nonholonomic systems is complicated. A comprehensive introduction to nonholonomic systems modeling, analysis, and control can be found in [1]. Therefore, when the nonholonomic constraints are involved, the navigation and coordination problems will become more complicated. Because of this reason, most of existing approaches to solve these two problems, without additional statements, choose either scalar model or point-mass model to represent the mobile robots.

1.2.1 Target-tracking with Obstacle Avoidance in a Static Environment

In the context of stationary environments, standard approaches normally can be classified into three categories: graph method, potential field method and other physical analogies methods.

Graph methods [30, 31, are based on a geometrical cell-decomposition of the entire configuration space and generate an optimal path with respect to certain objective criteria, such as finding the shortest collision-free path. Generally, there are two types of methods for decomposing the configuration space.

- Exact cell decomposition

   Exact cell decomposition methods decompose the configuration space into cells of var-
ious shapes. Cell boundaries are generally influenced by the topological changes in the configuration space, for example, the surfaces of the obstacles. In the case of a 2-dimensional configuration space containing polygonal forbidden regions, a convex polygonal decomposition can be conducted, in which all cells are convex polygons, which is illustrated in Figure 1.6. [24] proposed a general method to decompose the configuration space into cells. In general, exact cell decomposition methods are inefficient and complicated to implement.

- Approximate cell decomposition

Compared with Exact cell decomposition methods, approximate cell decomposition methods [25] are more popular. The entire configuration space was decomposed into cells having similar shapes, but their sizes are depending on the resolution, which is illustrated in Figure 1.7. The configuration space is divided into a grid-like structure. Then the collision states of each cell are determined by an appropriate collision detection technique. Some methods use hierarchical cell resolutions, in this way, cells are labeled as in-collision, clear, or mixed. The mixed cells are further decomposed into finer resolution cells until reaching a limit on the finest resolution. These new cells are
again evaluated for their collision states. In summary, approximate cell decomposition methods can be easily applied to configuration space of arbitrary dimension. However, the number of cells increases exponentially with the dimension of the configuration space. Therefore, time of planning also increases exponentially with the dimension of the configuration space.

The main criticism of graph methods is that they require large computational resources.

In the potential field method, pioneered by Khatib [2], the target applies an attractive force to the robot while the obstacles exert a repulsive force onto the robot. The motion of the robot is determined by the resultant force. An example formula is the following and the corresponding variables are explained below:

\[
U_{\text{art}}(x) = U_a(x) + U_r(x)
\]

\[
U_a(x) = \frac{1}{2}k_p(x - x_d)^2
\]

\[
U_r(x) = \begin{cases} 
\frac{1}{2}\eta\left(\frac{1}{\rho} - \frac{1}{\rho_o}\right)^2 & \text{if } \rho \leq \rho_o \\
0 & \text{otherwise}
\end{cases}
\]
where \( x_d \) is the target position, \( x \) is the position of robot, \( k_p \), \( \eta \), and \( \rho_o \) are positive constants and \( \rho \) is shortest distance to the obstacle.

The followings are example graphs demonstrating the attractive potential field computed using \( k_p = 1 \) and the repulsive potential field computed using the values \( \eta = 1.0 \) and \( \rho_o = 2.0 \). And it is worth to note that the top of the cones actually goes to infinity but they are cut for practical demonstration purposes.

![Graphs demonstrating attractive and repulsive potential fields](image)

Figure 1.8: Illustration of attractive potential field function
Figure 1.9: Illustration of repulsive potential field function
Due to its simplicity, elegance and high efficiency, the potential field method is particularly useful. [7] pointed out some inherent limitations of potential field method, such as trapsituations due to local minima.

To avoid the disadvantages of the standard potential field method, other physical analogies methods have been proposed borrowing ideas from electromagnetics [11] or fluid mechanics [10] to build functions free of local minima, but, in general, they are computationally intensive and therefore inappropriate for dynamic environments.

1.2.2 Target-tracking with Obstacle Avoidance in a Dynamic Environment

1.2.2.1 Without Considering Nonholonomic Constraints

In the context of dynamic environments, a common technique is to add a time dimension to the state space and reduce the problem to a static one [32, 33, 34]. The major issue is that it always assumes that the trajectories of the moving obstacles are known a priori, which is often not practical in real applications.

Another approach was proposed [35, 36], which constructs repulsive potential functions by taking into account the velocity information and extending the potential field method for moving obstacle avoidance. In [36], the velocity of the obstacle is taken into account when building the repulsive potential field. But the velocity of the robot is not considered. Because the collision between the robot and obstacle depends on both the relative position and velocity between them, this method is inadequate. This issue is addressed in [35, where the repulsive potential function takes advantage of the velocity information of both the robot and the obstacle. However, it assumed that the relative velocity between the robot
and the obstacle is time-invariant in terms of position of the robot. This assumption is not practical as the relative velocity and position are actually time-varying. Thus derivatives of the relative velocity in terms of position cannot be considered zero uniformly. In [35, 36], both methods deal with the obstacle avoidance problem applied to stationary targets.

The Ge and Cui method [37] constructs repulsive and attractive potentials which take into consideration the position and velocity of the robot with respect to moving targets and obstacles. Though convergence to the target is proven, no rigorous proof of obstacle avoidance is provided.

Other than potential field methods, there are other results [41, 42, 43, 44, 45]. In [41], a method combining a Deformable Virtual Zone (DVZ)-based reactive obstacle avoidance control with path following is proposed. In [42], harmonic functions along with the panel method for obstacle avoidance in dynamic environment is utilized. In [43], the dynamic window approach to obstacle avoidance in an unknown environment is presented. With a few changes to the standard scheme, convergence to the goal position is proved. [44] presented a method to compute the probability of collision in time for linear velocities of the robot and a reactive algorithm to perform obstacle avoidance in dynamic uncertain environment. [45] gave a preliminary study of the novel collision cone approach as a viable collision detection and avoidance tool in a 2-D dynamic environment. Many of the methods are heuristic and the lack of analytical design guidelines can be problematic in real world applications. Moreover, most of these methods increased complexity and computational costs.
1.2.2.2 Considering Nonholonomic Constraints

On the one hand, obstacle avoidance has been also extensively studied at the navigation system level (path planning/trajectory planning). Compared with standard motion planning approaches such as graph methods and potential field methods, which are proposed to deal with geometrical constraints, that is, holonomic systems in the presence of static obstacles, motion planning of nonholonomic wheeled mobile robots in dynamic environments is more challenging and important for mobile robotics. Based on Reeds and Shepp’s results on shortest paths of bounded curvature without considering obstacles, nonholonomic path planners [52] are proposed. With obstacles being modeled as polygons, obstacle avoidance and curvature constraint are taken into consideration by offsetting each polygon. Then a feasible path is obtained by using a sequence of such optimal path segments as those proposed in [53]. Using ideas from fluid mechanics, in [54], a collision free path is computed which satisfies the minimum curvature constraint. This method supposes the environment is static and also known a priori. [55] proposed an analytical nonholonomic trajectory generation algorithm. A family of parameterized polynomial trajectories are firstly derived to ensure all the resulting trajectory candidates feasible. Secondly, the free parameter(s) representing the family are confined into appropriate intervals such that collision avoidance criteria are met.

On the other hand, obstacle avoidance is addressed directly in the kinematics/dynamics controller, which is normally called avoidance control. In [56, 57], the dynamic window approach is introduced. A searching space is defined, consisting only of the admissible velocities and accelerations of the robot within a small time interval. Then the commands controlling
the velocities and accelerations of the robot are computed by maximizing a performance
index function associated with target tracking and obstacle avoidance. The concepts of col-
losion cones and velocity obstacles are introduced in [45, 58] respectively, which are widely
used to design avoidance control [59, 60]. The underlying idea is that obstacle avoidance
is achieved if the robot velocity is selected such that its velocity relative to the obstacles’
motion does not enter the corresponding collision cones/velocity obstacles. Avoidance con-
trol is also proposed combining potential field methods and sliding mode control [61, 62]. A
gradient-tracking based sliding mode controller for the mobile robot is proposed to achieve
target tracking and obstacle avoidance.

1.2.3 Formation Control with Obstacle Avoidance

1.2.3.1 Formation Movement Control of Multi-robots

Most existing methods dealing with formation control normally can be classified into three
categories: behavior based, virtual structure, or leader-follower.

In the behavior based approach [67, 73], a series of primitive goal oriented behaviors (e.g.,
move-to-goal, avoid-static-obstacle, avoid-robot and maintain-formation) are proposed for
each robot. A weighting factor indicates the relative importance of the individual behaviors.
The high-level combined behavior is generated by multiplying the outputs of each primitive
behavior by its weight, then summing and normalizing the results. The advantage of behavior
based approaches is that each primitive behavior has its physical meaning and the formation
feedback can be incorporated into the group dynamics by coupling the outputs of each
individual behavior. The disadvantage is that it is difficult to formalize and analyze the
group dynamics mathematically, consequently it is difficult to study the convergence of the formation to a desired geometric configuration.

The virtual structure approach [68, 69, 70, 71] is inspired by the rigid body motion of a physical object with all points in the object maintaining a fixed geometric relationship via a system of physical constraints. The robot formation is considered as a single virtual rigid structure. Thus desired trajectories are not assigned to each single robot but to the entire formation as a whole by a trajectory generator. The formation is maintained by minimizing the error between the virtual structure and the current robot position. The advantage of virtual structure approaches is that it is quite easy and straightforward to prescribe the coordinated behavior of the whole team. The disadvantage is that the virtual structure’s position is controlled by the positions of the robots, which makes the formation itself, be the centralized control.

In the leader-follower approach [74, 75, 76], some robots are designed as leaders moving along predefined trajectories. The remaining robots are followers required to maintain a desired posture (distance and orientation) relative to their own leader. Generally, the leader-follower controls take the following forms: (1) a single leader vehicle and multiple follower vehicles or (2) a “chain” of vehicles each following the preceding vehicle (such as in automated control of highway systems). The advantage of leader-follower is the controls reduce to a tracking problem which can be designed and analyzed using standard control theoretic techniques. The disadvantage is that the formation does not tolerate leader faults, since the leader’s predefined trajectory is independent of the motion of each associated follower.
1.2.3.2 Coupling of Formation Control and Collision Avoidance

Two methods are proposed in the literature to tackle the problem. One is the aforementioned behavior based method, in which avoidance of obstacles as well as other robots is designed as primitive behaviors. As previously mentioned, it is difficult to formalize and analyze the group dynamics mathematically. Consequently, it is difficult to prove convergence to the desired formation and improve the robot’s transient performance.

The other method is leader-follower formation control based on potential field and Lyapunov direct methods (e.g., [38, 39, 40]). Potential fields and Lyapunov direct methods are utilized to solve the formation control problem with collision avoidance. Especially, potential fields yield interaction forces between neighboring robots to enforce a desired minimum space for any pair of robots. A virtual leader is a moving reference point that exerts forces on its neighboring robots by means of additional similar potential field. The purpose of the virtual leaders is to introduce the mission: to direct, herd and/or manipulate the vehicle group behavior [38]. A properly designed potential field function yields global asymptotic convergence of a group of mobile robots to a desired formation, and guarantees no collisions among the robots [39]. These two methods do not consider the obstacle avoidance issue. The leader-follower strategy essentially transforms the formation control problem into a tracking problem. Based on this, the decentralized controls are designed to achieve collision avoidance and target tracking for a single robot is proposed. It is then extended to address the problem of coordinated tracking of a group of robots [40]. This method does not consider the moving obstacle and only guarantees the tracking with a bounded error.
1.3 Statement of Contributions

Contributions of this work are briefly stated in the following paragraphs, which are sorted by chapter and provide an outline of the thesis.

Chapter 2 In this chapter, we address the fundamental problem of potential field based reactive control design, which is the local minima problem (LMP) inherent in potential field methods. Firstly, the LMP is formulated and its existence is analyzed. Through introducing a state-dependent planned goal, a switching control strategy is presented to tackle this problem. The planned goal is an augmented state. And the control law for the planned goal is provided. When trapped into local minima, the robot can escape from local minima by following the planned goal. In addition, the proposed control law also takes into account the possible saturation constraints (velocity bound and control inputs saturation). Systematic rigorous Lyapunov proof is given to show both goal convergence and obstacle avoidance of the proposed control law provided that the goal is reachable. Simulation results of escaping from classical local minima scenarios show the validity and effectiveness of the proposed controls.

Chapter 3 In this chapter, we still focus on the local minima problem (LMP). We show that there does not exist a static state feedback control to solve LMP. Then a time-varying continuous control law is presented to tackle this problem. Challenges of continuous control design to solve LMP are discussed and explicit design strategies are then proposed. More importantly, systematic rigorous Lyapunov proof is given to show both global goal convergence provided that the goal is globally reachable and obstacle
avoidance of the proposed controls. Simulation results are provided to illustrate the validity and effectiveness.

Chapter 4  In this chapter, we address the multi-objective reactive control design (tracking targets while avoiding static and moving obstacles) for point-mass vehicles with limited sensor range in a dynamic environment. The proposed control law is synthesized by generating a potential field force for each objective and combining them through analysis and design. Different from standard potential field methods, the composite potential field described in this paper is planned and time-varying to account for movement of moving obstacles and vehicle. Using rigorous Lyapunov analysis, basic conditions and key properties are derived. Simulation examples are provided to illustrate both the design process and validity of proposed control.

Chapter 5  In this chapter, we pay attention to target tracking and obstacles avoidance of a large class of mobile robots which have nonholonomic constraints. Especially, we propose the reactive control design for unicycle-type mobile robots with limited sensor range to track targets while avoiding static and moving obstacles in a dynamic environment. With the relative motion among the mobile robot, targets, and obstacles being represented in polar coordinates, kinematic control laws achieving target-tracking and obstacle avoidance are synthesized using Lyapunov technique. Furthermore, the proposed control laws also consider possible kinematic control saturation constraints. Simulation examples are included to show the effectiveness of the proposed control.

Chapter 6  In this chapter, we study a team of wheeled mobile robots to cooperatively explore in a dynamic environment to track their virtual leader(s), while avoiding static
and dynamic obstacles. A potential field based reactive control design is proposed to deal with this problem. To the best of our knowledge, the proposed design is the first systematic approach to accommodate and achieve the multiple objectives of cooperative motion, tracking virtual command vehicle(s), obstacle avoidance, and oscillation suppression. The results are demonstrated by several simulation examples including cooperative formation movement of a team of vehicles moving through urban settings in the presence of static and moving obstacles, as well as narrow passages.

Chapter 7 In this chapter, we summarize the results accomplished in this work. Based on previous research experience, we then point out possible future research topics. And the ideas to deal with these challenging future research topics are suggested.
Due to simplicity, elegance and high efficiency \([2, 3, 4, 5, 6]\), potential field methods are widely used for autonomous mobile robot navigation. In this method, the goal exerts an attractive force to the robot while the obstacles apply a repulsive force onto the robot. The composite force determines the motion of the robot. However, as pointed out in \([7]\), one well-known drawback often associated with PFM\s is the problem of local minima. Years of extensive study and investigation have gone into this major problem. In general, existing methods dealing with LMP use one of these two strategies: eliminating local minima and escaping from local minima.

On the one hand, in the eliminating local minima approach, special functions are proposed which pose a few or even no local minima. For example, the navigation function method does not render local minima \([8]\). In addition, using ideas from fluid mechanics \([9, 10]\) or electro magnets \([11]\), physical analogies methods have been put forward to build functions free of local minima. However, they are generally computationally intensive and therefore may not be practical for real-time navigation in partially known or unknown environments. The potential field based navigation problem is interpreted in the framework of game theory \([12]\). The total potential field exhibits no local minima in most cases.

On the other hand, in the escaping the local minima approach, the idea is to attract the mobile robot away from local minima when the robot is trapped into local minima. Random
walk-like techniques have been used to help potential field based motion planning techniques to escape from local minimum configurations [13, 14]. Virtual obstacles are placed nearby local minima on purpose to repel a mobile robot from local minima [15, 16]. Wall-following method steers a mobile robot to follow the current obstacle contour when it is trapped into a local minimum [17, 18]. Simulated annealing is combined with PFM, employing a random search to escape from local minima [19, 20]. Dynamic internal agent states are used to manipulate the potential field [21]. Local equilibria are transformed from stable equilibria to unstable equilibria, causing escape form local minima. Many of these proposed methods are heuristic and the lack of analytical design guidelines can be problematic in real applications. In addition, most of these methods increase complexity and computational requirements.

In this chapter, using Lyapunov-based technique, a switching control is proposed to solve the local minima problem which originates in PFM based navigation through unknown environments with obstacles of complex shape. The basic idea is to introduce a planned goal which is designed to attract the robot and thus lead to escape from the local minima in any trapping situation. Through a simple switching strategy, goal convergence and obstacle avoidance are accomplished at the same time. Basic conditions and key properties are derived in the sense of Lyapunov. Moreover, saturation of the control signal and the velocity bound are considered in the proposed control. To the best of our knowledge, the proposed design is the first systematic approach to thoroughly analyze and solve the local minima problem.

The remainder of this chapter is organized as follows: In Section 2.1, the local minima problem of PFM is formulated. In Section 2.2, the existence of local minima is analyzed. In Section 2.3, systematic Lyapunov design for this problem is proposed. In Section 2.4,
examples and their simulations to demonstrate the effectiveness of the proposed control scheme is presented. In Section 2.6, the chapter is concluded and some future research directions suggested.

2.1 Local minima: An inherent Problem of Potential Field Methods

The theoretical framework presented in this chapter relies on some basic concepts in potential field theory that are discussed subsequently.

2.1.1 Potential Field Methods

The potential field methods aim to achieve goal convergence and obstacle avoidance for a mobile robot in a static environment. Consider an autonomous mobile robot whose dynamics are given by

\[ \dot{q}_r = v_r, \quad \dot{v}_r = u_1, \]  

(2.1.1)

where \( q \overset{\Delta}{=} [x, y, z]^T \in \mathbb{R}^3 \) denotes the center position, \( v \overset{\Delta}{=} [v_x, v_y, v_z]^T \in \mathbb{R}^3 \) represents the velocity, and \( u_1 \in \mathbb{R}^3 \) is the control input. Subscripts \( r, g \) and \( o \) indicate the robot, goal and obstacle, respectively.

Let a compact set \( \Omega_{oi} \subset \mathbb{R}^3 \) represent the 3-dimensional shape of the \( i \)th obstacle, we thereby introduce the pairwise repelling set

\[ \overline{\Omega}_{oi} = \{ q_r \in \mathbb{R}^3 \mid d_i(q_r, \Omega_{oi}) < D_i \} , \]  

(2.1.2)

where \( d_i(q_r, \Omega_{oi}) \) is the minimum distance between \( q_r \) and the \( i \)th obstacle. And \( D_i > 0 \) confines the region size of \( \overline{\Omega}_{oi} \setminus \Omega_{oi} \).
Therefore, the overall avoidance region $\Omega_o$ is given by

$$\Omega_o = \bigcup_{i \in N} \Omega_{oi},$$

and the overall repelling region $\Omega_o$ is given by

$$\Omega_o = \bigcup_{i \in N} \Omega_{oi},$$

where $N$ is the total number of obstacles.

The basic idea of potential field methods is illustrated in Fig. 2.1, where the so-called attractor and repeller are placed based on the geometry location of the goal and obstacles, and the obstacles’ shape which are specified by environmental conditions. Correspondingly, the attractor and repellers are associated with attractive potential field function and repulsive potential field function defined as follows.

**Definition 1.** Let $\mathbb{R}_+ \triangleq [0, +\infty)$, a $C^2$ function $P_a : \mathbb{R}^3 \mapsto \mathbb{R}_+$, is called an attractive potential field function on $\mathbb{R}^3$ if the following conditions hold:

1. $P_a(q_g) = 0$, $\nabla P_a(q_g) = 0$;
2. $P_a(q) < \bar{P}_a$, $\|\nabla P_a(q)\| < \bar{F}_a$;
3. $\langle \nabla P_a(q), (q - q_g) \rangle > 0$, $\forall q \in B(q_g, r) \setminus q_g, r < +\infty$;

where $\bar{P}_a > 0$ is the upper bound of $P_a$ and $\bar{F}_a > 0$ is the upper bound of $\|\nabla P_a(q)\|$. $B(q_g, r)$ is the ball centered at $q_g$ with radius $r$. 

Figure 2.1: Illustration of artificial potential field method
A typical example for $P_a(\cdot)$ is that, for some $k_1 > 0$,

$$P_a(q_r) = k_1 \|q_r - q_g\|^2.$$  \hfill (2.1.3)

**Definition 2.** A $C^2$ function $P_r : \mathbb{R}^3 \mapsto \mathbb{R}_+$, is called a repulsive potential field function on $\mathbb{R}^3$ if the following conditions hold:

1. $P_r(q) \geq \bar{P}_r$ if $q \in \Omega_o$ and $P_r(q) = 0$ if $q \not\in \Omega_o$;
2. $P_r(q) \in (0, \bar{P}_r)$ and $\nabla P_r(q) \in (0, \bar{F}_r)$ if $q \in \overline{\Omega_o} \setminus \Omega_o$;
3. $\langle \nabla P_r(q), (q - q_\Omega) \rangle \leq 0$;

where $\bar{P}_r > 0$ is the upper bound of $P_r$ and $\bar{F}_r > 0$ is the upper bound of $\|\nabla P_r(q)\|$. $q_\Omega \in \Omega_o$ is the point at minimum distance from $q_r$ to $\Omega_o$.

A natural choice for $P_r(q)$ takes the following form

$$P_r(q) = \sum_{i=1}^{N} P_{ri}(q),$$  \hfill (2.1.4)

where, for some $k_2 > 0$, $P_{ri}(q)$ is given by

$$P_{ri}(q) = \begin{cases} +\infty & \text{if } d_i \leq 0, \\ 0 & \text{if } d_i \geq D_i, \\ k_2 \left( \ln \left( \frac{D_i}{d_i} \right) - \frac{D_i - d_i}{D_i} \right) & \text{otherwise}. \end{cases}$$  \hfill (2.1.5)

Using PFMs, to achieve goal-tracking and collision avoidance, the reactive control $u_1$ is generally formulated as

$$u_1 = -\nabla P_a(q_r) - \nabla P_r(q_r) - \xi(\cdot) v_r,$$  \hfill (2.1.6)

where $\xi(\cdot) > 0$ is a uniformly bounded function designed to ensure stability and damp oscillations.
Let us first give the following two definitions.

**Definition 3.** For a point $q \in \mathbb{R}^3$, $q_g$ is said to be $\tau$ reachable if and only if there exists a path $l(q, q_g, \tau)$ connecting $q$ and $q_g$ and also satisfying the following condition

$$\Psi \cap \Omega_\omega = \emptyset,$$

where $\Psi = \{q' \in \mathbb{R}^3 ||q' - q|| < \tau, \forall q_l \in l(q, q_g, \tau)\}$. And $\tau > 0$ indicates the safety degree of the collision free path $l(q, q_g, \tau)$.

The concept of $\tau$ reachable is exemplified in Fig. 2.2. It is straightforward that, if $q_g$ is $\tau_1$ reachable, then $q_g$ is $\tau_2$ reachable provided $\tau_1 > \tau_2$.

**Definition 4.** A point $q^* \in \mathbb{R}^3 \setminus q_g$ is said to be a stationary point if and only if it satisfies the following equation

$$-\nabla P_a(q^*) = \nabla P_r(q^*). \quad (2.1.7)$$

If $q^*$ is a stationary point and there also exists some $r > 0$ such that $P_a(q^*) + P_r(q^*) < P_a(q) + P_r(q) \forall q \in B(q^*, r)$, then $q^*$ is said to be a local minimum. Otherwise $q^*$ is a saddle point.

**Problem:** Composite potential functions generally exhibit stationary points. As will be discussed in Section 2.2, in most cases, some of stationary points are local minima. Thus system (2.1.1) under control (2.1.6) may have more than one stable equilibrium point other than $q_g$. Therefore, under certain initial conditions, the vehicle will be trapped into a lo-
A local minimum example. (b) Vector field of artificial potential field.

cal minimum (LMP) instead of converging to its goal, which is the so-called local minima problem illustrated in Fig. 2.3.

In order to have a well-defined problem, we assume the following throughout the chapter:

**Assumption 1.** The mobile robot is represented by a 3-D sphere with the center at $q_r(t)$ and of radius $R$. Meanwhile, the range of its sensors is also described by a sphere centered at $q_r(t)$ and of radius $R_s$.

**Assumption 2.** The mobile robot has the following two saturation constraints:

1. Velocity bound, i.e. $\|v_r(t)\| \leq \bar{v}_r$;
2. Control input saturation, i.e. $\|u_1(t)\| \leq \bar{a}_r$.

The control objectives and main contributions of this chapter can be summarized as follows:

1. Goal convergence, i.e. using PFMs, $\lim_{t \to +\infty} q_r(t) \to q_g$ provided that goal $q_g$ is $\tau$ reachable;
2. Obstacle avoidance, i.e. $q_r(t) \notin \Omega_o$, $\forall t \geq t_0$ provided that $q_r(t_0) \notin \Omega_o$. 

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2.2 Stability Analysis of Equilibrium Points of Composite Potential Field Functions

Although composite potential functions generally present stationary points, they can be either saddle points or local minima on the potential surface. In this section, Theorem 2.2.1 is proposed, providing a geometrical method to identify whether an existing stationary point is a local minimum or not. Let us first begin with some definitions:

**Definition 5.** Surfaces $S_a(\kappa_a)$ and $S_r(\kappa_r)$ are said to be the equipotential surfaces of potential functions defined by

$$S_a(\kappa_a) \triangleq \{ q \in \mathbb{R}^3 | P_a(q) = \kappa_a \} \ (\kappa_a > 0),$$

and

$$S_r(\kappa_r) \triangleq \{ q \in \mathbb{R}^3 | P_r(q) = \kappa_r \} \ (\kappa_r > 0).$$

**Definition 6.** Curves $C_a(\kappa_a)$ and $C_r(\kappa_r)$ are said to be the level curves of potential functions if

$$P_a(q) = \kappa_a \ \forall q \in C_a(\kappa_a),$$

and

$$P_r(q) = \kappa_r \ \forall q \in C_r(\kappa_r).$$

In order to check if $q^*$ is a local minimum, compared with computing the composite potential field function’s Hessian matrix at $q^*$, the following theorem provides us with a simple criterion.

**Theorem 2.2.1.** At a stationary point $q^*$, suppose the straight line connecting $q_g$ to $q^*$ is normal to the equipotential surfaces $S_a(P_a(q^*))$ and $S_r(P_r(q^*))$. In addition, $S_a(P_a(q^*))$ and $S_r(P_r(q^*))$ are convex at the stationary point $q^*$. Let $u$ be surfaces’ unit-normal vector and $T$ to be their tangent vector lying in the tangent plane of the surfaces orthogonal to $u$. The induced level curves $C_a(P_a(q^*))$ and $C_r(P_r(q^*))$ are obtained by intersecting the surfaces with the plane containing $T$ and $u$. Let $k_{aq}$ and $k_{rq}$ be the curvature of $C_a(P_a(q^*))$ and $C_r(P_r(q^*))$, respectively. Then $q^*$ is not a local minimum if and only if, for some $T$, its associated $k_{aq}$ and $k_{rq}$ satisfy $k_{aq} < k_{rq}$.

**Proof.** Let us introduce the coordinate system (see Fig. 2.4) in which the origin is $q_g$ and $q_gq^*$ represents the positive $y$ axis.
Figure 2.4: Equipotential surfaces tangency at the stationary point

Since \( S_r (P_r (q^*)) \), \( S_a (P_a (q^* - q_g)) \) and \( S_c (P_c (q^* - q_o)) \) are convex at \( q^* \) and \( q_g q^* \) is normal to \( S_a (P_a (q^* - q_g)) \) at \( q^* \), it is clear that \( q^* \) has the following properties,

\[
\begin{align*}
\frac{\partial P_a}{\partial x} &= \frac{\partial P_r}{\partial x} = 0, \\
\frac{\partial P_a}{\partial y} &= -\frac{\partial P_r}{\partial y} > 0, \\
\frac{\partial^2 P_a}{\partial x \partial y} &= \frac{\partial^2 P_r}{\partial x \partial y} = 0, \\
-\frac{\partial^2 P_a}{\partial y^2} - \frac{\partial^2 P_r}{\partial y^2} &< 0.
\end{align*}
\]

(2.2.1)

For the implicit function \( P_a (q) = \kappa_a \), we have

\[
\frac{dy}{dx} \big|_{q^*} = -\left. \frac{\partial P_r}{\partial x} \frac{\partial P_a}{\partial y} \right|_{q^*} = 0.
\]

(2.2.2)

And

\[
\frac{d^2 y}{dx^2} \big|_{q^*} = -\frac{\partial^2 P_a}{\partial x^2} + \left( \frac{\partial^2 P_a}{\partial y^2} \frac{\partial y}{\partial x} \right) \frac{dy}{dx} - \left( \frac{\partial^2 P_a}{\partial x \partial y} \right)^2 \big|_{q^*}
\]

\[
= -\frac{\partial^2 P_a}{\partial x^2} \big|_{q^*}.
\]

(2.2.3)

In addition, \( C_a (P_a (q^*)) \) is convex, which implies \( \frac{\partial^2 y}{\partial x^2} < 0 \). Hence combining (2.2.1) and (2.2.3) yields

\[
\frac{\partial^2 P_a}{\partial x^2} \big|_{q^*} > 0.
\]

(2.2.4)

Moreover, it follows from (2.2.2), (2.2.3), and (2.2.4) that

\[
k_{aq} = \left. \frac{\partial^2 P_a}{\partial x^2} \right|_{q^*}.
\]

(2.2.5)

Similarly, we have

\[
\frac{\partial^2 P_r}{\partial x^2} \big|_{q^*} < 0,
\]

(2.2.6)
and
\[ k_{rq} = \frac{\partial^2 P_r}{\partial x^2} \bigg|_{q^*} \]  \hspace{1cm} (2.2.7)

Now we study the Hessian matrix \( H(P_a(\cdot) + P_r(\cdot)) \) to identify whether \( q^* \) is a local minimum or not, which is given by
\[
[H]_{2\times2} = \begin{bmatrix}
-\frac{\partial^2 P_a}{\partial x^2} & -\frac{\partial^2 P_r}{\partial x^2} \\
-\frac{\partial^2 P_a}{\partial y \partial x} & -\frac{\partial^2 P_r}{\partial y \partial x} - \frac{\partial^2 P_r}{\partial y^2}
\end{bmatrix}.
\]

It follows from (2.2.1), at the stationary point \( q^* \), we can obtain the eigenvalues as follows,
\[ \lambda_1 = -\frac{\partial^2 P_a}{\partial x^2} - \frac{\partial^2 P_r}{\partial x^2} \quad \text{and} \quad \lambda_2 = -\frac{\partial^2 P_a}{\partial y^2} - \frac{\partial^2 P_r}{\partial y^2} < 0. \]  \hspace{1cm} (2.2.8)

Substituting (2.2.5) and (2.2.7) into \( \lambda_1 \), we can rewritten \( \lambda_1 \) as
\[ \lambda_1 = \frac{\partial P_a}{\partial y} (k_{rq} - k_{aq}) \]  \hspace{1cm} (2.2.9)

Following form (2.2.8) and (2.2.9), we can conclude that \( q^* \) is not a local minimum if and only if \( k_{aq} < k_{rq} \).

To validate Theorem 2.2.1, we present two examples in which the potential functions are given by (2.1.3) and (2.1.4) with \( k_1 = 1, k_2 = 100 \) and \( D_i = 10 \).

**Example 1.** Stationary point \( q^* \) is a saddle point

The goal is placed at the origin and a circular obstacle is centered at \((0, 18, 0)\) with radius being 3. In this setting, \( q^* \) is solved to be \((0, 22.7986, 0)\). Given any tangent vector \( T \), the induced level curves \( C_a(\kappa_a) \) and \( C_r(\kappa_r) \) can be shown in Fig. 2.5.

Furthermore, computing the 2nd order partial derivatives of \( P_a \) and \( P_r \), the Hessian matrix at point \( q^* \) is then given by
\[
[H]_{2\times2} = \begin{bmatrix}
7.5025 & 0 \\
0 & -32.9123
\end{bmatrix}.
\]

Two eigenvalues are 7.5025 and -32.9123, denoting the point \( q^* \) is a saddle point. On the other hand, \( k_{aq} = 0.0439 \) and \( k_{rq} = 0.2084 \). Invoked by Theorem 2.2.1, we can have the same conclusion.
Example 2. **Stationary point** $q^*$ **is a local minimum**

Compared with example 1, the spherical obstacle is replaced by a cubic obstacle with the length being 8 (along $x$ axis), the width being 6 (along $y$ axis) and the height being 4 (along $z$ axis). And $q^*$ is solved to be $(0, 22.7986, 0)$. Given any tangent vector $T$, the induced level curves $C_a(\kappa_a)$ and $C_r(\kappa_r)$ are shown in Fig. 2.6. Similarly, the Hessian matrix at point $q^*$
is

$$[H]_{2\times2} = \begin{bmatrix} -2 & 0 \\ 0 & -32.9123 \end{bmatrix}$$

The two eigenvalues are $-2$ and $-32.9123$. On the other hand, $k_{aq} = 0.0439$ and $k_{rq} = 0$.

Invoked by Theorem 2.2.1, the point $q^*$ is identified as a local minimum.

Referring to above examples, it can be easily seen that obstacle shape plays an important role in the existence of local minima, upon which we make the following remark.

**Remark 2.2.1.** Suppose any $\Omega_{oi} (i = 1, \cdots, N)$ is a sphere on $\mathbb{R}^3$ and $\Omega_{oi} \cap \Omega_{oj} = \emptyset (i \neq j)$, which is a common assumption with the multitude of potential function based collision avoidance approaches already published. We can chose $D_i$ to ensure $\overline{\Omega_{oi}} \cap \overline{\Omega_{oj}} = \emptyset$. Thus, any stationary point $q^*$ can be exemplified by Fig. 2.5. From the geometric viewpoint, $q_r$ is closer to $q^*$ than $q_g$, which implies, for any tangent vector $T$, its associated $k_{aq}$ and $k_{rq}$ satisfying $k_{aq} < k_{rq}$. Invoked by Theorem 2.2.1, $q^*$ is not a local minimum. However, for arbitrary shaped obstacle, it can not be guaranteed that all stationary points are local minima.

### 2.3 A Planned Potential Field based Reactive Control Solving the Local Minima Problem

In the context of unknown environment, local minima can not be identified a priori. Considering this fact, we propose a two-stage switching control scheme with state and control magnitude constraints being addressed as well. Afterwards, goal convergence and obstacle avoidance of this switching control system are synthesized using the Lyapunov technique. The detailed design and proofs are discussed as follows.

#### 2.3.1 Switching Control Law

A planned goal $q_p(t)$ is introduced. When the robot is trapped into local minima $q^*$, the planned goal is designed to instantaneously switch from $q_g$ to $q^*$ and then move along the
equipotential surface $S_r(P_r(q^*))$ until $P_a(q_p(t)) < P_a(q^*)$. At the same, the robot is required to track $q_p(t)$ and ensure the tracking error is uniformly bounded for the purpose of obstacle avoidance.

Before proceeding to the control law, we first introduce the following definition:

**Definition 7.** $q_r(t)$ is said to be trapped into local minima if and only if the following conditions hold:

1. $q_r(t) \in \overline{\Omega_o}\setminus\Omega_o$;
2. $\|v_r(t)\| \leq \sigma$ and $\|u_1\| \leq \sigma$;

where $\sigma > 0$ is a very small number.

We begin with the following augmented system dynamical model given by

$$
\begin{align*}
\dot{q}_r &= v_r \\
\dot{v}_r &= u_1 \\
\dot{q}_p &= u_2 
\end{align*}
$$

(2.3.1)

Then the two-stage switching control is given as follows:

1. Stage A: normal mode to track goal and avoid obstacles

$$
\begin{align*}
u_1 &= -\nabla P_a(q_r) - \nabla P_r(q_r) - \xi(\cdot)v_r,
\end{align*}
$$

and

$$
q_p(t_{s^{B\rightarrow A}}) = q_g \quad \text{and} \quad u_2 = 0,
$$

where $t_{s^{B\rightarrow A}}$ is the switching time from stage B to A.

2. Stage B: escape from local minima

$$
\begin{align*}
u_1 &= -k_a(q_r - q_p) - \xi v_r,
\end{align*}
$$

(2.3.2)
and

\[ q_p(t_s^{A \rightarrow B}) = q_r(t_s^{A \rightarrow B}), \]

\[ u_2 = \delta^2 \cdot \frac{\nabla P_r(\dot{q}_p)}{\|\nabla P_r(\dot{q}_p)\|} \min (\xi - \gamma, k_a \gamma) \left( \frac{1}{(\gamma(1 + \xi) + k_a)} (\|v_r\| + \|q_r - q_p\| + 1) \right), \tag{2.3.3} \]

where \( t_s^{A \rightarrow B} \) is the switching time from stage A to stage B. Gains \( \gamma > 0, k_a > 0 \) and the nonlinear damping function \( \xi(\cdot) > \gamma \).

Specifically, gain \( \delta \) is chosen to satisfy the following conditions:

\[ \delta < \frac{\sqrt{\lambda_{\min}(Q)}}{\sqrt{\lambda_{\max}(Q)}} \min (\varepsilon, \bar{v}_r), \tag{2.3.4} \]

and

\[ \delta = 0 \text{ when } P_a(q_p(t_s^{A \rightarrow B})) - P_a(q_p(t)) \geq \sigma, \tag{2.3.5} \]

where \( Q \) is a positive definite matrix which is give by

\[ Q = \begin{bmatrix} 1 & \gamma \\ \gamma & k_a + \gamma \xi \end{bmatrix}. \]

And \( \varepsilon \) is chosen to satisfy the following condition

\[ \Phi \cap \Omega_o = \emptyset, \tag{2.3.6} \]

where \( \Phi \triangleq \{ q \in \mathbb{R}^3 \mid d(q, S_r(P_r(q_p(t_s^{A \rightarrow B})))) \leq \varepsilon \}. \)

For instance, \( \varepsilon \) can be analytically solved when we use the repulsive potential field function (2.1.4) and chose \( D_i = D_j \ (i, j = 1, 2 \cdots N) \). Otherwise, \( \varepsilon \) can be conservatively chosen to be a small positive number.

3. Design of \( \nabla P_r(q_p) \)

Let \( T \) be tangent vector lying in the tangent plane of \( S_r(P_r(q_p(t_s^{A \rightarrow B}))) \) at the point
Then using the right hand rule, the direction of $\nabla P^\perp (q_p)$ is determined by $\nabla P^\perp (q_p) \times T$.

And also

$$\|\nabla P^\perp (q_p)\| = \|\nabla P^\perp (q_p)\|.$$ 

In particular, for the 2-D case, $T$ can be either $[0, 0, 1]^T$ or $[0, 0, -1]^T$.

4. Switching strategies

(a) When $t = t_0$, choose control design for stage A. Then we continue to (2);

(b) Choose control design for stage A. If $q_r(t)$ is trapped into local minima, then we switch to (3);

(c) Choose control design for stage B. And we switch back to (2) at time $t_{s \rightarrow B}$ only when $V(t_{s \rightarrow B}) < V(t_{s \rightarrow A})$.

To avoid the saturation of state and control action aforementioned in assumption 2, some stronger conditions are needed. These conditions are presented in the following lemma.

**Lemma 2.3.1.** Consider system (2.1.1) under controls (2.1.6) and (2.3.2) and suppose that:

1. $(k_a + \xi)\bar{v}_r < a_r$;
2. $\bar{F}_a + \bar{F}_r \leq \min (\xi\bar{v}_r, a_r - \xi\bar{v}_r)$.

Then $\|v_r(t)\| < \bar{v}_r$ and $\|u_1(t)\| < a_r$, $\forall t \geq t_0$.

**Proof.** let us consider the Lyapunov candidate,

$$V_T (t) = \frac{1}{2} [v_r, q_r - q_p] Q [v_r, q_r - q_p]^T.$$ (2.3.7)
Calculating the derivatives on both sides, we have
\[
\dot{V}_T = \dot{v}_r^T v_r + \gamma (v_r - v_p)^T v_r + \gamma (q_r - q_p)^T \dot{v}_r \\
+ (k_a + \gamma \xi) (q_r - q_p)^T (v_r - v_p).
\]

In stage B ($t \geq t_s^{A-B}$), it follows from (2.3.2) that
\[
\dot{V}_T = [\langle k_a (q_r - q_p) - \xi v_r \rangle] v_r + \gamma (v_r - v_p)^T v_r \\
+ \gamma (q_r - q_p)^T [-k_a (q_r - q_p) - \xi v_r] \\
+ (k_a + \gamma \xi) (q_r - q_p)^T (v_r - v_p) \\
= - (\xi - \gamma) \| v_r \|^2 - k_a \gamma \| q_r - q_p \|^2 \\
- \gamma v_r^T v_p - (k_a + \gamma \xi) (q_r - q_p)^T v_p \\
\leq - \min (\xi - \gamma, k_a \gamma) (\| v_r \|^2 + \| q_r - q_p \|^2) \\
+ (\gamma \| v_r \| + (k_a + \gamma \xi) \| q_r - q_p \|) \| v_r \|. \tag{2.3.8}
\]

Substituting (2.3.3) into (2.3.8) we can obtain
\[
\dot{V}_T \leq \min (\xi - \gamma, k_a \gamma) (\| v_r \|^2 - \| q_r - q_p \|^2 + \delta^2). \tag{2.3.9}
\]

Recalling $q_p (t_s^{A-B}) = q_r (t_s^{A-B})$ and $v_r (t_s^{A-B}) \leq \sigma$, then we have
\[
(\| v_r (t) \|^2 + \| q_r (t) - q_p (t) \|^2)_{t=t_s^{A-B}} \leq \sigma^2. \tag{2.3.10}
\]

Hence, it follows from (2.3.7), (2.3.9), (2.3.10) and (2.3.4)
\[
\| v_r (t) \|^2 + \| q_r (t) - q_p (t) \|^2 < \min (\varepsilon, \bar{v}_r), \tag{2.3.11}
\]

which indicates $\| v_r (t) \| < \bar{v}_r$ and $\| q_r (t) - q_p (t) \| < \bar{v}_r$ in stage B. Furthermore, since

\((k_a + \xi) \bar{v}_r < \bar{a}_r\), recalling (2.3.2), we have $\| u_1 (t) \| < \bar{a}_r$ in stage B.

On the other hand, in stage A, let the Lyapunov candidate be
\[
E (t) = \frac{1}{2} \| v_r (t) \|^2. \tag{2.3.12}
\]

Taking time derivative yields
\[
\dot{E} = -\xi \| v_r \|^2 - (\nabla P_a + \nabla P_r)^T v_r. \tag{2.3.13}
\]

Substituting $\bar{F}_a + \bar{F}_r \leq \min (\xi \bar{v}_r, \bar{a}_r - \xi \bar{v}_r)$ into (2.3.13), we can obtain $\| v_r (t) \| \leq \bar{v}_r$ in stage A. Furthermore, recalling (2.1.6), we also have $\| u_1 (t) \| \leq \bar{a}_r$ in stage A.

The proof is completed by summarizing these two stages.
2.3.2 Global Convergence and Obstacle Avoidance

We state the definitions and lemmas as follows.

**Definition 8.** A point \( q_{kr} \in C_r(\kappa_r) \) is said to be a subgoal of \( C_r(P_r(q^*)) \) if and only if
\[
P_a(q_{kr}) \leq P_a(q), \quad \forall q \in C_r(\kappa_r).
\]

**Definition 9.** A point \( q_{kr} \in S_r(\kappa_r) \) is said to be a subgoal potential point of \( S_r(P_r(q^*)) \) if and only if
\[
P_a(q_{kr}) \leq P_a(q), \quad \forall q \in S_r(\kappa_r).
\]

When it comes to obstacle avoidance in the 3-D environment, we make the following assumption:

**Assumption 3.** \( \Omega_i \ (i = 1, \ldots, N) \) are convex on \( \mathbb{R}^3 \).

The following lemma studies the relationship between local minima and subgoal.

**Lemma 2.3.2.** Let \( q_i^* \ (i = 1, 2, \ldots) \) be all the possible local minima which \( q_r \) may run into. If \( q_g \) is \( \tau \) reachable, as long as \( D_i \) in (2.1.2) is sufficient small, then we can draw the following conclusions:

1. In a 2-D environment, none of \( q_i^* \) is a subgoal of \( C_r(P_r(q_i^*)) \);
2. In a 3-D environment, for any possible tangent vector \( T \), let \( C_r(P_r(q_i^*)) \) be obtained by intersecting \( S_r(P_r(q_i^*)) \) with the plane containing \( T \) and \( \nabla P_r(q_i^*) \). Suppose assumption 3 holds, none of \( q_i^* \) is a subgoal of \( C_r(P_r(q_i^*)) \).

Furthermore, \( D_i \) can be conservatively chosen to be \( 0.5\tau \).

**Proof.** Let \( D_i = 0.5\tau \). Therefore, we obtain
\[
\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset \quad (i \neq j). \tag{2.3.14}
\]

Let us introduce the coordinate system (see Fig. 2.7) in which the origin is \( q_g \) and \( q_g q_i^* \) represents the positive \( y \) axis.

We first prove (1), using proof by contradiction, suppose there exists a local minimum \( q_i^* \) being the subgoal of its associated level curve \( C_r(P_r(q_i^*)) \). It follows from (2.3.14) that \( C_r(P_r(q_i^*)) \) is either enclosed by the \( i \)th obstacle or vice versa, which is depicted by Fig. 2.7. Considering \( q_i^* \) is the subgoal of \( C_r(P_r(q_i^*)) \), we conclude that \( C_r(P_r(q_i^*)) \) is enclosed by the \( i \)th obstacle, implying \( q_g \) is not \( \tau \) reachable. Thus, \( q_i^* \) can not be the subgoal of \( C_r(P_r(q_i^*)) \).
In order to prove (2), for any possible tangent vector $T$, let $C_r(P_r(q^*_i))$ and $\Gamma$ obtained by intersecting $S_r(P_r(q^*_i))$ and $\Omega_o$ respectively with the plane containing $T$ and $\nabla P_r(q^*_i)$. Under assumption 3, it follows from (2.3.14) that $C_r(P_r(q^*_i))$ encloses $\Gamma$. As illustrated in Fig. 2.7, $q^*_i$ can not be the subgoal of $C_r(P_r(q^*_i))$.

\[ \text{Definition 10. Let } q^*_i \text{ be the } i\text{th local minimum, then centripetal potential threshold } P_c \text{ is defined by} \]

\[ P_c = \min \left( P_a(q^*_i) + P_r(q^*_i) \right) \quad (i = 1 \cdots m) , \quad (2.3.15) \]

where $m$ denotes the total number of local minima.

Notwithstanding the existence of local minima, a mobile robot may still converge to the goal position $q_g$ under certain initial conditions. The following theorem is proposed, estimating the region of attraction of goal position $q_g$.

\[ \text{Lemma 2.3.3. Let } V = P_a(q_r) + P_r(q_r) + 0.5\|v_r\|^2, \text{ we define } M = \{(q_r,v_r) | V(q_r,v_r) < P_c \}. \]

If $q_g \not\in \Omega_o$ and the initial states $(q_r(t_0),v_r(t_0)) \in M$, system (2.1.1) under control (2.1.6) converges asymptotically to $q_g$, implying $M$ is a region of attraction of goal position $q_g$.

\[ \text{Proof. Considering the Lyapunov function } V, \text{ it follows from (2.1.1) and (2.1.6) that} \]

\[ \dot{V} = \dot{v}_r^T v_r + \nabla P_a^T(q_r) v_r + \nabla P_r^T(q_r) v_r \]

\[ = (-\nabla P_a(q_r) - \nabla P_r(q_r) - \xi(\cdot) v_r)^T v_r \]

\[ + \nabla P_a^T(q_r) v_r + \nabla P_r^T(q_r) v_r \]

\[ = -\xi(\cdot)\|v_r\|^2. \quad (2.3.16) \]

Recalling from (2.3.15), it is straightforward that

\[ \dot{V} < 0, \quad \forall \ (q_r,v_r) \in M \setminus \{q_g,0\} \quad \text{and} \quad \dot{V}(q_g,0) = 0. \]
Hence, we can conclude asymptotic convergence of \([q_r(t) - q_g] \to 0\) invoked by LaSalle’s invariant set theorem [23] under the condition \(q_g \not\in \Omega_o\) and \((q_r(t_0), v_r(t_0)) \in M\). Thus, \(M\) is an attraction region of \(q_g\).

For example, consider a one-dimensional case as shown in Fig. 2.8. It is clear that

\[
M = \{ (q_r, v_r) \mid P(q_r) + 0.5 \|v_r\|^2 < P(q_1^*) \}.
\]

Suppose \(v_r(i(t_0) = 0(i = 1, 2, 3, 4)\), as stated in the above proof, \(q_{r1}\) converges to \(q_g\) while \(q_{r3}\) and \(q_{r4}\) get stuck in \(q_1^*\) and \(q_2^*\), respectively.

**Remark 2.3.1.** Although \((q_{r2}(t_0), v_{r2}(t_0)) \not\in M\), \(q_{r2}\) still converges to \(q_g\), which indicates \((q_r(t_0), v_r(t_0)) \in M\) and \(q_g \not\in \Omega_o\) is only a sufficient condition for convergence to \(q_g\).

Now we prove here the following theorem. We first consider the case: \(\bar{P}_a, \bar{P}_r \to +\infty\) and no saturation constraints. The saturation constraints are addressed in the subsequent remark. Choosing \(V\) as a Lyapunov function, the proposed two-stage switching control is proved to ensure that \(V\) is decreasing over time, which implies goal convergence and obstacle avoidance.

**Theorem 2.3.4.** Suppose assumptions 1 and 3 hold. If \(q_r(t_0) \not\in \Omega_o\) and the initial conditions \(q_r(t_0)\) and \(v_r(t_0)\) are bounded. Consider system (2.1.1) under controls (2.1.6) and (2.3.2), as long as \(q_o\) is \(\tau\) reachable and \(D_i\) in (2.1.2) is sufficiently small, then we can draw the following conclusions:
1. \( \lim_{t \to +\infty} \| q_r(t) - q_g \| = 0; \)

2. \( q_r(t) \not\in \Omega_o, \forall t \geq t_0. \)

**Proof.** In stage B, since \( q_g \) is \( \tau \) reachable and \( D_i \) is sufficiently small, invoked by Lemma 2.3.2, there must exist time \( t^* > t_s^{A \to B} \) such that

\[
P_a(q_p(t_s^{A \to B})) - P_a(q_p(t^*)) \geq \sigma.
\]

Note that \( q_p \) under control \( u_2, \)

\[
P_r(q_p(t)) = P_r(q_p(t_s^{A \to B})) \quad \forall t \geq t_s^{A \to B}.
\]

Furthermore, following from (2.3.5) that

\[
u_2(t) = 0, \forall t > t^*.
\]

Following from (2.3.8) and (2.3.18), \((q_r(t), v_r(t))\) will exponentially converge to \((q_p(t^*), 0)\).

We rewrite the Lyapunov function stated in Lemma 2.3.3

\[
V = P_a(q_r) + P_r(q_r) + \frac{\| v_r \|^2}{2}.
\]

Therefore, there must exist \( t_s^{B \to A} > t^* \) such that

\[
V(t_s^{B \to A}) < V(t_s^{A \to B}).
\]

In the meantime, recalling from (2.3.11), we can claim

\[
q_r(t) \not\in \Omega_o, \forall t \leq t_s^{A \to B}. \tag{2.3.20}
\]

On the other hand, in stage A, since \( q_g \) is \( \tau \) reachable and \( D_i \) is sufficiently small, we can ensure \( q_g \not\in \Omega_o. \) As already proved in Lemma 2.3.3, \( V \) is nonincreasing and \( q_r(t) \) will either converge to some local minima \( q^*_i \) or to goal \( q_g. \) That is, stage A either switches to stage B or converges to goal \( q_g. \) If stage A switches to stage B, it has been proved in (2.3.19), stage B will switch back to stage A with \( V \) decreasing. Therefore, in either case, as \( t \to +\infty, \)

\((q_r, v_r)\) will enter into the region of attraction of \( q_g \) and thus converge to \( q_g. \) Meanwhile, \( q_r(t_0) \not\in \Omega_o \) and the initial conditions \( q_r(t_0) \) and \( v_r(t_0) \) are bounded, thus \( V(t_0) \) is bounded. \( V \) is nonincreasing in stage A, thus a robot can avoid obstacles in stage A. As proved in (2.3.20), a robot can also avoid obstacles in stage B. Therefore, \( q_r(t) \not\in \Omega_o, \forall t \geq t_0. \)

\( \square \)

**Remark 2.3.2.** Taking into account assumption 2, we can achieve the same control objectives as stated in Theorem 2.3.4 as long as the following additional conditions hold:

1. \((k_a + \xi)\tilde{v}_r < \tilde{a}_r;\)
2. $\bar{F}_a + \bar{F}_r \leq \min (\xi \bar{v}_r, \bar{a}_r - \xi \bar{v}_r)$;

3. $\bar{P}_r > \bar{P}_a + 0.5 \bar{v}_r^2$.

Suppose the first two conditions hold, as already proved in Lemma 2.3.1, $\|v_r(t)\| \leq \bar{v}_r$ and $\|u_1(t)\| \leq \bar{a}_r$, $\forall t \geq t_0$. We then can redo the proof of Theorem 2.3.4. And instead of choosing $\bar{P}_a$, $\bar{P}_a \rightarrow +\infty$, it is straightforward that the third condition is sufficient to prove obstacle avoidance.

### 2.4 Simulation Results

This section describes the simulation results as follows, showing the effectiveness of the proposed control in Sections 4. For these simulations, the potential functions are given by (2.1.3) and (2.1.4). The nonlinear damping function $\xi(\cdot)$ is simply chosen to be a constant function. The parameters used for these simulations are: $R = 0.5 \text{ m}$, $R_s = 2 \text{ m}$, $\gamma = 1$, $k_a = 6$, $\sigma = 0.01$, $\xi(\cdot) = 3.16$, $k_1 = 1$, $k_2 = 100$ and $D_i = 1.5$. And the bound on the linear velocity is $2 \text{ m/s}$.

#### 2.4.1 Goal tracking and obstacle avoidance in a 2-D environment

There are some rectangular, circular and even concave obstacles scattered in a $45 \text{ m} \times 45 \text{ m}$ workspace. We set the initial conditions of three mobile robots be: $q_{r1}(t_0) = (1, 1)^T$, $q_{r2}(t_0) = (25, 5)^T$, $q_{r3}(t_0) = (7, 36)^T$ and $v_{r1}(t_0) = v_{r2}(t_0) = v_{r3}(t_0) = (0, 0)^T$. The goal $q_g$ is located at $(27.5, 28)^T$. In this simulation, we do not consider the collision avoidance among three robots. The simulation result is shown in Fig. 2.9 and 2.10.
As illustrated in Fig. 2.9, under the standard PFMs (2.1.6), three robots are trapped into their local minima. While, by applying the proposed control, three mobile robots successfully escape from their local minima, which is shown in Fig. 2.10.
Given a $45 \text{ m} \times 45 \text{ m} \times 45 \text{ m}$ 3-D space scattered with six cubic and four spherical obstacles, the initial conditions of the mobile robot are:

$$q_r(t_0) = (10, 10, 0)^T \quad \text{and} \quad v_r(t_0) = (0, 0, 0)^T.$$ 

The goal $q_g$ is located at $(20, 20, 42)^T$. The simulation result is shown in Fig. 2.11.

2.5 Experiment

In this section, the proposed control algorithm is experimentally implemented and validated on a robotic platform. The experiment results demonstrates the correctness and the ease with which the control algorithm can be implemented. In addition, compared with the open loop path planning methods, the successful experimental results indicate a tolerance to position and orientation disturbances mainly due to dead-reckoning from wheel motion derived from encoder readings.
2.5.1 Experimental platform and implementation

Experiment was conducted in the School of Electrical Engineering and Computer Science Control and Robotics Laboratory at the University of Central Florida. As shown in Fig. 2.12, the overall experimental system consists of a host computer and an AmigoBot robot, which can communicate with each other via RS-232 serial connection.

AmigoBot from MobileRobots Inc. is a differential-drive mobile robot. Let \((x, y, \theta)\) denote the Cartesian position and orientation of the robot, then the kinematic equations of the robot are

\[
\dot{x} = v \cos \theta, \quad \dot{y} = v \sin \theta, \quad \dot{\theta} = \omega,
\]

(2.5.1)

where \(v\) is linear velocity of the robot and \(\omega\) is angular velocity of the robot. Considering that controls (2.1.6) and (2.3.2) require double-integrator dynamics, one challenge to implementing the proposed algorithms on our platform is to calculate the commands \(v\) and \(\theta\). To
focus on the main issue, let the robot’s commands \( v \) and \( \theta \) to be of the form

\[
v = \min(\nabla, \sqrt{u_{1x}^2 + u_{1y}^2}), \quad \theta = \tan\left(\frac{u_{1y}}{u_{1x}}\right),
\]

(2.5.2)

where \( \nabla \) is the speed limit.

### 2.5.2 Experimental Result

In this experiment, goal position, obstacle configurations and the robot’s initial position are illustrated in metric unit in Fig. 2.12. And the initial orientation \( \theta(t_0) = 0 \). Compared with the parameters we used in the simulation section, only the following changes were made: \( D_i = 0.4 \)m and the bound on the linear velocity \( \nabla = 0.1 \) m/s. The experimental results are shown in Fig. 2.13 with nine photos, (a)-(i), which are taken during the experiment. Figure 2.13(a) shows the initial experimental settings. Under the control (2.1.6), the robot moves to its goal, as shown in Fig. 2.13(b). Fig. 2.13(c) shows the robot switches from stage A to stage B to escape the local minima. Then under the control (2.3.2), Fig. 2.13(d), Fig. 2.13(e) and Fig. 2.13(f) show the robot can successfully track the planned goal to avoid the obstacles. As expected, the robot switches back to stage A to converge its goal which is shown in Fig. 2.13(g). Fig. 2.13(h) shows the robot detects another local minima and switches to stage B again. As shown in Fig. 2.13(i), the robot converges to its goal position. Since the robots rely on integration of encoder data to determine its position and orientation, as observed from Fig. 2.13(i), the robot does not precisely arrive its goal position, which implies the proposed control algorithm can be a better choice than open loop path planning algorithms which require large computational resources and sensitive to the measurement noises.
2.6 Conclusions

In this chapter, we proposed a systematic approach to solve the local minima problem inherent in PFMs. In the proposed control, saturation of state and control input is addressed as well. Examples through simulations and experiment confirm the effectiveness of Lyapunov design of two-stage switching control for the point-mass robot proposed in Section 2.3. Future research will consider more complex dynamic models to accommodate a larger class of mobile robots.
Figure 2.13: Experimental result
CHAPTER 3
DESIGN OF A TIME-VARYING CONTINUOUS CONTROL TO SOLVE THE POTENTIAL FIELD LOCAL MINIMA PROBLEM

The objective of this chapter is to systematically analyze the theoretical difficulty of control design to solve LMP and present a time-varying continuous control law to solve this problem. In particular, challenges of finding continuous control solutions of LMP are discussed and explicit design strategies are then proposed. Moreover, systematic rigorous Lyapunov proof is given to show both global goal convergence provided that the goal is globally reachable and obstacle avoidance of the proposed controls. Simulation results are given to illustrate the validity and effectiveness of the proposed control.

The contribution of this chapter is twofold: (1) to systematically analyze the LMP and point out there does not exist a static state feedback control to solve LMP, (2) to propose a Lyapunov-based time-varying continuous control to solve the local minima problem. The idea is to identify the attraction region of PFM and then present a time-varying continuous control to ensure the mobile robot will get into the attraction region whenever the goal is globally reachable. Using Lyapunov technique, basic conditions and key properties are derived.

The remainder of this chapter is organized as follows: In Section 3.1, the local minima problem of PFM is formulated and analyzed. In Section 3.2, the obstructions to continuous control design for solving LMP is presented, the attractive region is identified and our time-varying continuous Lyapunov design for this problem is proposed. In Section 3.3, simu-
lation examples to demonstrate the effectiveness of the proposed control scheme is presented.

Section 3.4 concludes the chapter and suggests some future research directions.

3.1 Problem Formulation

The potential field methods are proposed to achieve goal convergence and obstacle avoidance for a mobile robot in a static environment. Consider an autonomous mobile robot whose dynamics are given by

\[ \dot{q}_r = v_r, \quad \dot{v}_r = u, \]  

(3.1.1)

where \( q = [x, y]^T \in \mathbb{R}^2 \) denotes the center position, \( v = [v_x, v_y]^T \in \mathbb{R}^2 \) represents the velocity, and \( u \in \mathbb{R}^2 \) is the control input. Subscripts \( r, g \) and \( o \) indicate the robot, goal and obstacle, respectively.

To achieve goal-tracking and collision avoidance, the PFM based reactive control \( u \) is generally formulated as

\[ u = -\nabla P_a (q_r) - \nabla P_r (q_r) - \xi (\cdot) v_r, \]  

(3.1.2)

where \( \xi (\cdot) > 0 \) is a uniformly bounded function designed to ensure stability and damp oscillations. And functions \( P_a (\cdot) \) and \( P_r (\cdot) \) represent attractive potential field function and repulsive potential field function, respectively.

**Definition 11.** A point \( q^* \in \mathbb{R}^3 \setminus q_g \) is defined to be a stationary point if and only if it satisfies the following equation

\[ -\nabla P_a (q^*) = \nabla P_r (q^*). \]  

(3.1.3)

If \( q^* \) is a stationary point and there also exists some \( r_1 > 0 \) such that \( P_a (q^*) + P_r (q^*) < P_a (q) + P_r (q) \ \forall q \in B (q^*, r_1) \), then \( q^* \) is said to be a local minimum. Otherwise \( q^* \) is a saddle
point. The difference between those two types of stationary points can be vividly illustrated in Figures 3.1 and 3.2.

We now present an important result [29] which suggests the necessary condition of global convergence for a general autonomous nonlinear system.

**Proposition 1.** Consider nonlinear deterministic control systems of the form

$$\dot{x} = f(x) + g(x) u_d,$$  \hspace{1cm} (3.1.4)

where $x \in X \subset \mathbb{R}^n$ is the state, $u_d \in U \subset \mathbb{R}^m$ is the input, $f : X \leftarrow \mathbb{R}^n$ with $f(0) = 0$, and $g : X \rightarrow \mathbb{R}^n \times \mathbb{R}^m$ with $g(0) = 0$. If the state space $X$ is not contractible, no $C^1$ feedback law exists such that the origin of (3.1.4) becomes globally asymptotically stable.
For system (3.1.1) under controls (3.1.2), using proof by contradiction, then we can claim that composite potential functions inevitably yield stationary point(s) invoked by Proposition 1. In most cases, some of stationary points are local minima. Therefore, under certain initial conditions, the vehicle will be trapped into a local minimum instead of converging to its goal, which is the so-called local minima problem (LMP).

In order to have a well-defined problem, we firstly introduce the following symbols:

Let a compact set \( \Omega_{oi} \subset \mathbb{R}^2 \) represent the 2-dimensional shape of the \( i \)th obstacle, we thereby introduce the pairwise repelling set

\[
\overline{\Omega}_{oi} = \{ q_r \in \mathbb{R}^2 | d_i(q_r, \Omega_{oi}) < D_i \},
\]

where \( d_i(q_r, \Omega_{oi}) \) is the minimum distance between \( q_r \) and the \( i \)th obstacle. And \( D_i > 0 \) indicates the region size of \( \overline{\Omega}_{oi} \backslash \Omega_{oi} \).

Then the overall avoidance region \( \Omega_o \) is given by

\[
\Omega_o = \bigcup_{i \in N} \Omega_{oi},
\]

and the overall repelling region \( \overline{\Omega}_o \) is given by

\[
\overline{\Omega}_o = \bigcup_{i \in N} \overline{\Omega}_{oi},
\]

where \( N \) is the total number of obstacles.

And then we assume the following throughout the chapter:

**Assumption 4.** The mobile robot is represented by a 2-D circle with the center at \( q_r(t) \) and of radius \( R \). The range of its sensors is also a circle centered at \( q_r(t) \) and of radius \( R_s \).

**Assumption 5.** Without loss of generality, let the \( i \)th obstacle \( \Omega_{oi} \) be either a circle or a convex/concave polygon.
Definition 12. The goal position \( q_g \) is said to be globally reachable if and only if there exists a collision free path \( \ell \) such that \( d(q, \Omega'_o) > R, \forall q \in \ell \), where \( \Omega'_o \) is modified \( \Omega_o \) and its definition is given shortly in Section 3.2.

The control objectives of this chapter is to design continuous PFM s based state feedback control \( u(\cdot) \) to achieve:

1. global goal convergence, i.e. \( q_r(t) \rightarrow q_g \) as \( t \rightarrow +\infty \) provided that goal \( q_g \) is globally reachable;

2. obstacle avoidance, i.e. \( d(q_r(t), \Omega'_o) > R, \forall t \geq t_0 \) provided that \( d(q_r(t_0), \Omega'_o) > R \) and \( q_r(t_0), v_r(t_0) \), and \( q_g \) are bounded.

3.2 Time-varying Continuous Control Design

The idea of navigation function or density function methods is to construct special potential field functions of which all the stationary points are saddle points. Invoked by Proposition 1, there are also two alternative options to tackle this problem. One option is to use static discontinuous feedback control. Actually, the escaping the local minima approach reviewed in Chapter 2 belongs to this category. Another option is to propose time-varying continuous feedback control which our research result belongs to.

3.2.1 Obstructions to Achieve Continuous Global Convergence

In this section, we discuss the challenges of finding continuous control solutions of LMP.
3.2.1.1 Sensor Constraints

Taking into account assumption 4, the robot has a limited sensor range $R_s$ which denotes

$$d_i(q_r, \Omega_{\alpha_i}) = \min(d_i(q_r, \Omega_{\alpha_i}), R_s).$$

To avoid the discontinuity of derivatives caused by the sensor constraints, we can employ the following smooth function

$$g(s) = \frac{\int_a^b f(x) \, dx}{\int_a^b f(x) \, dx},$$

where $f(x)$ is also a smooth function which is defined by, for positive numbers $a$, $b$ and $0 \leq a < b$,

$$f(x) = \begin{cases} \exp\left(-\frac{1}{x-a} + \frac{1}{x-b}\right) & \text{if } x \in (a,b), \\ 0 & \text{if } x \notin (a,b). \end{cases}$$

Then to avoid the discontinuity caused by the sensor constraint, by choosing $a = 0$ and $b = R_s$, we use $\text{dis}_i$ as an alternate measurement of the distance between the robot $q_r(t)$ and $\Omega_{\alpha_i}$, which is given by

$$\text{dis}_i = R_s (1 - g(d_i(q_r, \Omega_{\alpha_i}))).$$

3.2.1.2 Topological Obstructions

As pointed out in section 2.1, the presence of obstacles make $\mathbb{R}^2/\Omega_o$ not contractible and consequently prevents the global convergence to the goal under static continuous state feedback control.

In addition, the shape of obstacles will cause discontinuity of derivatives. Before proceeding to illustrate this issue, let us first introduce the following definitions,
Definition 13. Curves $C_a (\kappa_a)$ and $C_r (\kappa_r)$ are said to be the level curves of potential functions if

$$P_a (q) = \kappa_a \forall q \in C_a (\kappa_a),$$

and

$$P_r (q) = \kappa_r \forall q \in C_r (\kappa_r).$$

Consider a concave polygon obstacle which is illustrated in Figure 3.3, at point $q$, $C_r (P_r (q))$ is not differentiable. In order to avoid this type of discontinuity, we enlarge the $i$th obstacle $\Omega_{oi}$ by connecting the neighboring concave edges with an arc whose radius of curvature is a given positive constant $r_f$.

Definition 14. $\Omega'_o$ is said to be modified $\Omega_o$ if $\Omega_o$ is modified to ensure that any pair of neighboring concave edges of $\Omega_o$ are connected with an arc whose radius of curvature being a given positive constant $r_f$.

Assumption 6. For any two obstacles $\Omega'_{oi}$ and $\Omega'_{oj}$, the Hausdorff distance $h (\Omega'_{oi}, \Omega'_{oj}) \geq (r + 2R)$, where $r > 0$.

3.2.2 Control Law Development

The idea of designing the time-varying continuous control is illustrated in Figure 3.4. A virtual goal $q_d (t)$ is proposed to guide the robot to reach its goal $q_g$ provided that $q_g$ is globally reachable. The robot is then required to track its virtual goal $q_d (t)$ instead of the real goal $q_g$. To ensure the distance between the robot ant its virtual goal is less than a positive constant $D_s (D_s > R)$, a virtual circular obstacle $q_o (t)$ centers at $q_d (t)$ and of radius $D_s$ is introduced to encircle the robot, which moves with the virtual goal ($v_o (t) = v_d (t)$).
3.2.2.1 Augmented Dynamical Model

Compared with system (3.1.1), the following augmented system is proposed whose dynamical model is given by,

\[
\begin{align*}
\dot{q}_r &= v_r \\
\dot{v}_r &= u \\
\dot{q}_d &= v_d
\end{align*}
\tag{3.2.4}
\]

where \( q_d(t) \) is the virtual goal and \( q_d(t_0) = q_r(t_0) \). \( u \) and \( v_d \) are the controls to be designed. Therefore, in order to achieve the multi-objective of goal tracking and obstacle avoidance, the control design objectives is converted into

1. for \( q_r(t) \), \( \| q_r(t) - q_d(t) \| < (D_s - R) \), \( \forall t \geq t_0 \) and \( \lim_{t \to +\infty} \| q_r(t) - q_d(t) \| = 0 \);

2. for \( q_d(t) \), \( d(q_d(t), \Omega'_{o}) > D_s \), \( \forall t \geq t_0 \) as long as \( d(q_d(t_0), \Omega'_{o}) > D_s \) and \( q_d(t) \to q_g \) as \( t \to +\infty \) provided that \( q_g \) is globally reachable.

In the subsequent sections, the control law design and its proof are explored in detail.
3.2.2.2 Control Design

Firstly, let us introduce the following definitions.

**Definition 15.** $P_r(q_d, t)$ is said to be adjoint time-varying repulsive potential field function of $P_r(q_d)$ if and only if, the following conditions hold:

1. \[ \left\langle \frac{\partial P_r(q_d)}{\partial q_d}, \frac{\partial P_r(q_d, t)}{\partial q_d} \right\rangle = \left\| \frac{\partial P_r(q_d)}{\partial q_d} \right\| \left\| \frac{\partial P_r(q_d, t)}{\partial q_d} \right\|, \]

2. $\Omega_o \subset \Omega_o(t)$ and $\Omega_o(t) \not\subset \Omega_o$.

For example, let $P_r(q_d)$ given by

\[
P_r(q_d) = \begin{cases} 
+\infty & \text{if } d \leq 0, \\
0 & \text{if } d \geq D, \\
k_r \left( \ln \left( \frac{D}{d} \right) - \frac{D-d}{D} \right) & \text{otherwise},
\end{cases}
\]  

(3.2.5)

where $d = d_i(q_r, \Omega_{oi})$. And $D > 0$, defining the confined set $\Omega_{oi}$. The repulsive force is “active” only if $d < D$. Then we consider the following function $P_r(q_d, t)$ which is given by

\[
P_r(q_d, t) = \begin{cases} 
+\infty & \text{if } d \leq 0, \\
0 & \text{if } d \geq D(t), \\
k_r \left( \ln \left( \frac{D(t)}{d} \right) - \frac{D(t)-d}{D(t)} \right) & \text{otherwise},
\end{cases}
\]  

(3.2.6)

The only difference between (3.2.5) and (3.2.6) is that constant $D$ turns into a smooth time function $D(t)$. An typical choice of $D(t)$ is given by

\[
D(t) = D + \Delta D(1 - e^{-(t-t_0)}),
\]  

(3.2.7)

where $\Delta D$ is a very small positive number. It is straightforward to verify $P_r(q_d, t)$ is adjoint time-varying repulsive potential field function of $P_r(q_d)$. 

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**Definition 16.** $P_r^*(q_d)$ is said to be adjoint repulsive potential field function of $P_r(q_d,t)$ if and only if, the following conditions hold:

1. $\left\langle \frac{\partial P_r^*(q_d)}{\partial q_d}, \frac{\partial P_r(q_d,t)}{\partial q_d} \right\rangle = \left\| \frac{\partial P_r^*(q_d)}{\partial q_d} \right\| \left\| \frac{\partial P_r(q_d,t)}{\partial q_d} \right\|$;

2. $\overline{\Omega}_o(t) \subset \overline{\Omega}_o^*$ and $\overline{\Omega}_o^* \not\subset \overline{\Omega}_o(t)$.

The above defined repulsive potential functions are exemplified in Figure 3.5. $\Omega_o''$ is an enlarged version of $\Omega_o'$ such that $\forall q \in Bd(\Omega_o''), d(q, \Omega_o') \geq D_s$. $\Omega_o''$ and $\overline{\Omega}_o''$ are associated with $P_r(q_d)$. Referring to system (3.2.4), $q_d(t)$ is required to avoid $\Omega_o''$. And $\overline{\Omega}_o''(t_o) = \overline{\Omega}_o''(t_0)$ is time-varying avoidance region, which is designed to be expanding over time and relates to $P_r(q_d,t)$. $\overline{\Omega}_o^*$ represents the largest avoidance region, which comes with $P_r^*(q_d)$.

![Figure 3.5: Illustration of the above defined repulsive potential field functions.](image)

**Definition 17.** Smooth function $P_a^*(t)$ is said to be adjoint attractive potential field function if and only if, the following conditions hold:

1. $P_a^*(t_0) = P_a(q_d(t_0))$;

2. assume at the time $t_{in}$, $q_d(t)$ enters into the set $\overline{\Omega}_o''(t)$, given a pair of very small positive numbers $\varepsilon$ and $t_\delta$, $\|P_a^*(t) - P_a(q_d(t_{in}))\| \leq \varepsilon$, $\forall t \geq t_{in} + t_\delta$ unless $q_d(t)$ moves out of the set $\overline{\Omega}_o''(t)$ at $t_{out}$;
3. assume at the time $t_{\text{out}}$, $q_d(t)$ leave the set $\Omega_o''(t)$, given a pair of very small positive numbers $\varepsilon$ and $t_\delta$, \[ \|P_a^*(t) - P_a(q_d(t_{\text{out}}))\| \leq \varepsilon, \forall t \geq t_{\text{out}} + t_\delta \] unless $q_d(t)$ enters the set $\Omega_o''(t)$ at $t_{\text{in}}$.

Note that $P_a^*(t)$ is designed to store the attractive potential value $P_a(q_d(t))$ when either $q_d(t)$ gets into $\Omega_o''(t)$ at $t_{\text{in}}$ or $q_d(t)$ leaves $\Omega_o''(t)$ at $t_{\text{out}}$.

Let us consider the following smooth function of time

$$ \phi(t) = A \left(1 - e^{-k_e f_1(t-t_0)}\right), \quad (3.2.8) $$

where $A \neq 0$, $k_e > 0$ is a positive gain. And function $f_1(\cdot)$ is a nonnegative monotonous smooth function chosen to satisfy the following conditions:

$$ f_1(s) = \begin{cases} 
0 & \text{if } s \leq 0, \\
(0, c) & \text{if } s > 0. (c > 1) 
\end{cases} \quad (3.2.9) $$

For example, functions $f_1(\cdot)$ can be simply chosen to be

$$ f_1(s) = \begin{cases} 
0 & \text{if } s \leq 0, \\
\exp\left(k_1 - \frac{1}{k_2 s^s}\right) & \text{if } s > 0, \quad (3.2.10) 
\end{cases} $$

where $k_1 = \ln(c)$ and $k_2 > 0$.

To ensure $\|\phi(t) - A\| \leq \varepsilon, \forall t \geq t_0 + t_\delta$ iff the following condition holds, for $\varepsilon < A$

$$ k_e \geq -\ln\left|\frac{\varepsilon}{A}\right| \frac{1}{f_1(t_\delta)}. \quad (3.2.11) $$

Hence, with properly chosen $k_e$, $P_a^*(t)$ can be constructed using a series of time functions $(3.2.8)$. 

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Definition 18. Let point \( q_{gc} \in Bd(\Omega^r_o) \) and \( \|q_{gc} - q_o\| = dist(q_g, \Omega^r_o) \), where \( dist(q_g, \Omega^r_o) = \inf_{q \in \Omega^r_o} \|q_g - q\| \) then \( q_{gc} \) is named the repulsive contact point of \( q_g \). Let point \( q_{gb} \) be the cross-point of line \( q_{gc}q_g \) and level curve \( C_r(0) \) of \( P^*_r(q_d) \), then \( q_{gb} \) is called the repulsive boundary point of \( q_g \). Similarly, we can define the repulsive contact point \( q_{dc}(t) \) of \( q_d(t) \) and the repulsive boundary point \( q_{db}(t) \) of \( q_d(t) \). Then we define \( \cos \theta = \frac{\langle q_g - q_{gc}, q_g - q_{db} \rangle}{\|q_g - q_{gc}\| \|q_g - q_{db}\|} f_1 \left( \frac{\langle q_{db} - q_{dc}, q_g - q_{db} \rangle}{\|q_{db} - q_{dc}\| \|q_g - q_{db}\|} \right) \).

The potential field based Lyapunov control is designed to be

\[
    u = -\nabla P_a(q_r - q_d) - \nabla P_r(q_r - q_o) - k (v_r - v_d) + \dot{v}_d, \tag{3.2.12}
\]

where \( k > 0 \) is a positive constant. And \( P_r(q_r - q_o) \) represents the repulsive potential function yielded only by the virtual obstacle \( q_o(t) \).

\[
    v_d = -f_1 \left( R_s \left( 1 - g \left( d \left( q_d(t), \Omega^r_o(t) \right) \right) \right) \right) \cdot \nabla P_a(q_d) \\
    - \left( f_1 \left( h_1(q_d,t) \right) + f_1 \left( h_2(q_d,t) \right) \right) \cdot \nabla P_r(q_d,t) \\
    - f_1 \left( -h_1(q_d,t) \right) f_1 \left( -h_2(q_d,t) \right) \cdot \nabla P_r^*(q_d), \tag{3.2.13}
\]

where \( h_1(\cdot) = P_a(q_d) - P^*_a(t) + \lambda_1 - \gamma_1(t) \) and \( h_2(\cdot) = \lambda_2 - \cos \theta - \gamma_2(t) \). And \( \lambda_1 > 0, \lambda_2 > 0 \) are positive numbers. \( \gamma_1(t) \) and \( \gamma_2(t) \) are smooth functions (of time), which are designed to satisfy the following conditions:

1. assume at the time \( t_{attr} \), \( P_a(q_d(t)) - P^*_a(t) + \lambda_1 \leq \frac{\lambda_1}{4}, \) \( (\lambda_2 - \cos \theta) \leq \frac{\lambda_1}{4} \) and \( q_d(t) \in \Omega^r_o \), given a pair of very small positive numbers \( \varepsilon \) and \( \delta \), then \( \| \gamma_1(t) - \frac{\lambda_1}{4} \| \leq \varepsilon, \forall t \geq t_{attr} + t_{\delta} \) unless \( q_d(t) \) leave the set \( \Omega^r_o \) at the time \( t_{free} \). Also \( \| \gamma_2(t) - \frac{\lambda_1}{4} \| \leq \varepsilon, \forall t \geq t_{attr} + t_{\delta} \) unless \( q_d(t) \) leave the set \( \Omega^r_o \) at the time \( t_{free} \).

2. assume at the time \( t_{free} \), \( q_d(t) \) leave the set \( \Omega^r_o \), given a pair of very small positive numbers \( \varepsilon \) and \( \delta \), then \( \| \gamma_1(t) \| \leq \varepsilon, \forall t \geq t_{free} + t_{\delta} \) unless \( P_a(q_d) - P^*_a(t) + \lambda_1 \leq \frac{\lambda_1}{4}, \) \( (\lambda_2 - \cos \theta) \leq \frac{\lambda_1}{4} \) and \( q_d(t) \in \Omega^r_o \) at the time \( t_{attr} \). Also \( \| \gamma_2(t) \| \leq \varepsilon, \forall t \geq t_{free} + t_{\delta} \) unless \( P_a(q_d) - P^*_a(t) + \lambda_1 \leq \frac{\lambda_1}{4} \) and \( (\lambda_2 - \cos \theta) \leq \frac{\lambda_1}{4} \) at the time \( t_{attr} \).
Similar to $P_a^v(t)$, following from (3.2.8), the constructions of $\gamma_1(t)$ and $\gamma_2(t)$ are straightforward.

In general, there are two available options to choose $\nabla P_r(q_d,t)^\perp$, which are given by

$$\nabla P_r(q_d,t)^\perp = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \nabla P_r(q_d,t).$$  \hspace{1cm} (3.2.14)

Or

$$\nabla P_r(q_d,t)^\perp = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \nabla P_r(q_d,t).$$  \hspace{1cm} (3.2.15)

### 3.2.3 Proof of Obstacle Avoidance

The obstacle avoidance problem is to ensure that the robot will not enter the given compact set $\Omega'_o$ provided that certain initial conditions hold. The following theorem provides the basic result.

**Theorem 3.2.1.** If assumptions 4 and 5 hold, then system (3.2.4) under controls (3.2.12) and (3.2.13) is collision-free provided that $d(q_r(t_0), \Omega'_o) > R$ and $q_r(t_0), v_r(t_0)$, and $q_g$ are bounded.

**Proof.** The obstacle avoidance of the proposed control is proved through the following two steps. On one hand, consider the Lyapunov function

$$V_1 = P_a (q_r - q_d) + P_r (q_r - q_o) + \frac{1}{2} \|v_r - v_d\|^2.$$  \hspace{1cm} (3.2.16)

Differentiating both sides of $V_1$, we can obtain

$$\dot{V}_1 = \nabla P_a^T(\cdot)(v_r - v_d) + \nabla P_r^T(\cdot)(v_r - v_o)$$

$$= \nabla P_a^T(\cdot)(v_r - v_d) + \nabla P_r^T(\cdot)(v_r - v_o)$$

$$- (\nabla P_a^T(\cdot) + \nabla P_r^T(\cdot)) (v_r - v_d)$$

$$- k \|v_r - v_d\|^2$$

$$= -k \|v_r - v_d\|^2,$$  \hspace{1cm} (3.2.17)
which implies $V_1$ is nonincreasing. Since $d(q_r(t_0), \Omega'_o) > R$, $q_r(t_0)$ and $v_r(t_0)$ are bounded, $V_1(t_0)$ is bounded. Hence, we can claim $V_1(t)$ is bounded $\forall t > t_0$. Therefore, the virtual obstacle avoidance is proved. Considering $\|q_r(t_0) - q_d(t_0)\| = 0$, thus the robot can be uniformly confined within a neighboring ball centered at $q_d(t)$ which is written into the following inequality

$$\|q_r(t) - q_d(t)\| < (D_s - R), \quad \forall t \geq t_0. \quad (3.2.18)$$

On the other hand, Let us consider the following Lyapunov function

$$V_2 = P_r(q_d, t) \quad (3.2.19)$$

Differentiating $V_2$ on both sides

$$\dot{V}_2 = \frac{\partial P_r(q_d, t)}{\partial q_d}^T v_d + \frac{\partial P_r(q_d, t)}{\partial t} \quad (3.2.20)$$

Therefore, it follows from (3.2.13)

$$\dot{V}_2 \leq \frac{\partial P_r(q_d, t)}{\partial t} \quad (3.2.21)$$

Note that the integration of $\frac{\partial P_r(q_d, t)}{\partial t}$ over time can be bounded by a positive constant $\overline{P}$, which is

$$\|P_r(q, t) - P_r(q, t_0)\| \leq \overline{P}, \quad \forall (q, t) \in \mathbb{R}^2 \times [0, +\infty) \quad (3.2.22)$$

Hence, as long as $P_r(q_d, t_0) < +\infty$, we can draw the conclusion that

$$P_r(q_d, t) < +\infty, \quad \forall t \geq t_0, \quad (3.2.23)$$

which indicates that

$$d(q_d(t), \Omega'_o) > D_s. \quad (3.2.24)$$

The proof of obstacle avoidance is completed by combining (3.2.18) and (3.2.24).

\[\square\]

### 3.2.4 Proof of Target Global Convergence

The tracking problem is to ensure that the robot will converge to the target position $q_g$ provided that $q_g$ is globally reachable. In what follows, the target global convergence of the proposed control is proved through the following two steps. And the following theorem provides the basic result.
Lemma 3.2.2. If assumptions 4, 5, and 6 hold, considering system (3.2.4) under control (3.2.13), $q_d(t)$ enters into the set $\Omega''_o(t)$ at the time $t_{in}$ then there exists $t_{attr}$ and small positive numbers $\Delta_1, \Delta_2$ such that $P_a(q_d) \leq P^*_a(t) - \Delta_1$ and $\cos \theta \geq 1 - \Delta_2$ at the time $t_{attr}$ provided that $q_g$ is globally reachable. Moreover, by selecting $\lambda_1 = \frac{2}{3} \Delta_1$ and $\lambda_2 = 1 + \frac{1}{3} \Delta_1 - \Delta_2$, $q_d(t)$ will then asymptotically converge to $q_g$.

Proof. At the time $t_{in}$, $q_d(t)$ enters into the set $\Omega''_o(t)$. If assumptions 4, 5, and 6 hold, we can choose $\Omega^o_{oi} \cap \Omega^o_{oj} = \emptyset$. Therefore, considering system (3.2.4) under control (3.2.13), $q_d(t)$ will then move along the contour line of $P_a(P_r(q_d), t)$. As illustrated in Figure 3.6, if the goal is globally reachable, there must exist $t_{attr}$ and small positive numbers $\Delta_1, \Delta_2$ such that $P_a(q_d) \leq P^*_a(t) - \Delta_1$ and $\cos \theta \geq 1 - \Delta_2$ at $t_{attr}$.

Given a small positive number $\lambda_1$, considering $v_d(t)$ is bounded, therefore we can chose $t_{\delta}$ small enough to ensure

$$\|P_a(q_d(t_{attr} + t_{\delta})) - P_a(q_d(t_{attr}))\| < \frac{\lambda_1}{4}. \quad (3.2.25)$$

And

$$\|\cos \theta (t_{attr} + t_{\delta}) - \cos \theta (t_{attr})\| < \frac{\lambda_1}{4}. \quad (3.2.26)$$

Furthermore, we choose $\lambda_1 = \frac{4}{3} \Delta_1$ and $\lambda_2 = 1 + \frac{1}{3} \Delta_1 - \Delta_2$. And substitute (3.2.25) and (3.2.26) into (3.2.13), we can claim

$$h_1(q_d, t) < 0 \quad \text{and} \quad h_2(q_d, t) < 0, \ (t = t_{attr} + t_{\delta}). \quad (3.2.27)$$

On the other hand, note the compact set $S$ enclosed by $q_d(q_g, q_{gb}, q_{db}, q_{dc}, B_d(\Omega''_o))$ and also satisfying the condition $\cos \theta \geq (1 - \Delta_2 - \frac{\lambda_1}{4})$. It follows from (3.2.26), we have

$$q_d(t_{attr} + t_{\delta}) \in S.$$  

Moreover, following from the definition of $\cos \theta$, it is straightforward that if $\cos \theta \geq (1 - \Delta_2 - \frac{\lambda_1}{4})$

$$f_1(\langle q_{db} - q_{dc}, q_g - q_{db} \rangle) > 0. \quad (3.2.28)$$

Figure 3.6: Illustration of attainable and non attainable goal.
Hence, for any point \( q \in S \), following from (3.2.28), we have

\[
\langle \nabla P^*_r(q), \nabla P_a(q - q_g) \rangle \geq 0,
\]  
(3.2.29)

where the equality holds iff \( P^*_r(q, t) = 0 \).

Note that \( q_d(t_{\text{attr}} + t_{\delta}) \in S \). It follows from (3.2.27) and (3.2.29), \( q_d(t) \) under the control (3.2.13) will then asymptotically converge to \( q_g \) when \( t \geq t_{\text{attr}} + t_{\delta} \). And the proof is straightforward by simply choosing Lyapunov candidate \( V_3 = P_a(q_d - q_g) \).

**Theorem 3.2.3.** If assumptions 4, 5 and 6 hold, considering system (3.2.4) under controls (3.2.12) and (3.2.13), \( \lim_{t \to +\infty} q_r(t) \to q_g \) provided that goal \( q_g \) is globally reachable.

**Proof.** We will firstly prove the robot will asymptotically converge to its virtual goal \( q_r(t) \). Then we will prove \( q_r(t) \) will converge to the goal position \( q_g \) provided that \( q_g \) is globally reachable.

On one hand, it is straightforward to check that the following conditions hold

1. \( V_1 \) is lower bounded;
2. \( \dot{V}_1 \) is negative semi-definite;
3. \( \ddot{V}_1 \) is finite.

Invoked by Barbalat’s lemma, we can conclude

\[
\lim_{t \to \infty} \|v_r - v_d\| \to 0
\]  
(3.2.30)

Recall that

\[
\ddot{V}_1 = -2k (\dot{v}_r - \dot{v}_d)^T (v_r - v_d).
\]  
(3.2.31)

Substitute (3.2.30) into (3.2.31), we can obtain

\[
\lim_{t \to \infty} \|\dot{V}_1\| \to 0
\]  
(3.2.32)

Compute \( \dddot{V}_1 \), we have

\[
\dddot{V}_1 = -2k (\ddot{v}_r - \ddot{v}_d)^T (v_r - v_d) - 2k \|\dot{v}_r - \dot{v}_d\|^2
\]  
(3.2.33)

Furthermore \( \dddot{V}_1 \) can be computed as

\[
\dddot{V}_1 = -2k (\dddot{v}_r - \dddot{v}_d)^T (v_r - v_d) - 6k (\ddot{v}_r - \ddot{v}_d)^T (\dot{v}_r - \dot{v}_d).
\]  
(3.2.34)

Note that \( \dot{v}_r, \ddot{v}_r, \dddot{v}_r, \dot{v}_d, \dddot{v}_d \) and \( \dddot{v}_d \) are bounded, then \( \dddot{V}_1 \) is bounded. Hence \( \dddot{V}_1 \) are uniformly continuous. Again invoked by Barbalat’s lemma, we can obtain

\[
\lim_{t \to \infty} \|\dddot{V}_1\| \to 0
\]  
(3.2.35)
Therefore, following from (3.2.30), (3.2.33), and (3.2.35), it can be concluded that

$$\lim_{t \to \infty} \| \dot{v}_r - \dot{v}_d \| \to 0 \quad (3.2.36)$$

Substitute (3.2.30) and (3.2.36) into system (3.2.4), as \( t \to \infty \), we can claim the behavior of system (3.2.4) as follows

1. $$- \frac{\partial P_a}{\partial (q_r - q_d)} - \frac{\partial P_r}{\partial q_r} = 0;$$
2. $$v_r = v_d.$$

Following from (3.2.12), behavior 1 guarantees that the robot will converge to its virtual goal as \( t \to \infty \), which is

$$\lim_{t \to \infty} \| q_r(t) - q_d(t) \| = 0.$$

Behavior 2 says the robot will move with its virtual goal as \( t \to \infty \).

Therefore, up to now, we prove the robot will asymptotically converge to its virtual goal \( q_r(t) \).

On the other hand, we will prove the virtual goal \( q_r(t) \) will converge to the goal position \( q_g \) provided that \( q_g \) is globally reachable. We will consider the following three stages:

1. stage 1: \( q_d(t) \notin \overline{\Omega}_o(t) \)
   In this stage, control (3.2.13) reduced to be
   
   $$v_d = -f_1 \left( R_s \left( 1 - g \left( d \left( q_d(t), \overline{\Omega}_o(t) \right) \right) \right) \right) \cdot \nabla P_a(q_d)$$
   $$-f_1 \left( -h_1(q_d, t) \right) f_1 \left( -h_2(q_d, t) \right) \cdot \nabla P_r^*(q_d), \quad (3.2.37)$$

   When \( q_d(t) \notin \overline{\Omega}_o \), then control (3.2.13) can be further simplified to be
   
   $$v_d = -f_1 \left( R_s \left( 1 - g \left( d \left( q_d(t), \overline{\Omega}_o(t) \right) \right) \right) \right) \cdot \nabla P_a(q_d). \quad (3.2.38)$$

   When \( q_d(t) \in \overline{\Omega}_o \), we investigate the following two cases. If \( t = t_0 \), since \( P_o^*(t_0) = P_a(q_d(t_0)) \), (3.2.37) turns into (3.2.38). Otherwise \( q_d(t) \) enters into \( \overline{\Omega}_o \) from \( \overline{\Omega}_o(t) \). Invoked by lemma 1, we have
   
   $$\langle \nabla P_r^*(q_d), \nabla P_o(q_d) \rangle \geq 0. \quad (3.2.39)$$

   Combine above analysis, choosing \( V_3 = P_o(q_d - q_g) \) as Lyapunov candidate, we can claim \( q_r(t) \) either asymptotically converge to \( q_g \) or transits from stage 1 to stage 2 during the process of converging to \( q_g \).

2. stage 2: \( q_d(t) \in \overline{\Omega}_o(t) \) and \( h_1(\cdot) \geq 0 \) or \( h_2(\cdot) \geq 0 \)
   At time \( t_{in} \), \( q_d(t) \) enters into \( \overline{\Omega}_o(t) \). Invoked by Lemma 1, as long as the goal \( q_g \) is globally reachable, there exists \( t_{attr} \) and \( t_{d} \) such that
   
   $$h_1(q_d, t) < 0 \quad \text{and} \quad h_2(q_d, t) < 0, \quad (t = t_{attr} + t_{d}).$$

   Hence \( q_d(t) \) will transit from stage 2 to stage 3 as long as \( q_g \) is globally reachable.
(3) stage 3: \( q_d(t) \in \overline{\Omega}'_o(t) \) and \( h_1(\cdot) < 0 \) and \( h_2(\cdot) < 0 \)

Invoked by Lemma 1, in this stage \( q_d(t) \) will then asymptotically converge to \( q_g \). Therefore, there exists \( t_{out} \) when \( q_d(t) \) leaves \( \overline{\Omega}'_o(t) \) and \( t_{free} \) when \( q_d(t) \) leaves \( \overline{\Omega}^*_o \). Since \( \overline{\Omega}_{out} \cap \overline{\Omega}^*_{oj} = \emptyset \), \( q_d(t) \) will transit from stage 3 to stage 1. It is clear that \( P^*_a(t_{out}) < P^*_a(t_{in}) \).

Combining the analysis of the above three stages, the adjoint at tractive potential field function \( P^*_a(t) \) will keep decreasing as long as \( q_g \) is globally reachable. Thus we can claim \( q_d(t) \rightarrow q_g \) as \( t \rightarrow +\infty \). Since we had already proved \( \lim_{t \rightarrow \infty} ||q_r(t) - q_d(t)|| = 0 \), the proof of global goal convergence is done. \( \square \)

### 3.3 Simulation Validation

This section describes the simulation results to show the effectiveness of the proposed control.

For these simulations, the attractive potential functions are given by (3.3.1). For some \( k_a > 0 \),

\[
P_a(q_r) = k_a ||q_r - q_g||^2.
\]

(3.3.1)

And the repulsive potential functions are given by (3.2.5) and (3.2.6). The parameters used for these simulations are: \( R = 0.1 \text{ m}, R_s = 3 \text{ m}, r_f = 2.5 \text{ m}, r = 3 \text{ m}, k_a = 2, k_r = 1, k_1 = 1e - 14, k_2 = 1e + 14, k = 3.16, D_i = 1 \text{ m}, D_s = 1 \text{ m}, t_\delta = 0.1 \text{ s}, \varepsilon = 1e - 10, \Delta_1 = 1, \text{ and } \Delta_2 = 0.5. \) And \( \lambda_1 = \frac{4}{3} \Delta_1 = \frac{3}{4} \) and \( \lambda_2 = 1 + \frac{1}{3} \Delta_1 - \Delta_2 = \frac{5}{6}. \) There are some rectangular concave, circular obstacles scattered in a \( 40 \text{ m} \times 40 \text{ m} \) workspace. We set the initial conditions of the mobile robot be : \( q_r(1) = (10, 5)^T \) and \( v_r(1) = (0, 0)^T \). The goal \( q_g \) is located at \( (15, 39)^T \). The simulation result is shown in Figure 3.7. By applying the proposed control, the mobile robot successfully escapes from their local minima, which is shown in Figure 3.7. Meanwhile, the control \( v_d(t) \) is shown in Figure 3.8.

In the above simulation, we select

\[
\nabla P_r(q_d, t)^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \nabla P_r(q_d, t).
\]
If we chose another option which is
\[ \nabla P_r(q_d, t) \perp = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \nabla P_r(q_d, t). \]

Then the simulation result is shown in Figure 3.9. Compare Figure 3.7 and Figure 3.9, we can chose either of the above two options. But, for \( \nabla P_{r1}(q_d, t)^\perp \), the consistence of our choice matters.
3.4 Conclusions

In this chapter, the theoretical basis of LMP and theoretical difficulties of control design to solve LMP are addressed. Challenges of finding continuous control solutions of LMP are discussed and explicit design strategies are then proposed. Simulation examples confirm the effectiveness of Lyapunov design of time-varying continuous control for the point-mass robot proposed in Section 3.2. Another advantage of the proposed control is that it can greatly suppresses the oscillation which is another major issue of potential field methods. Future research will consider more complex dynamic models to accommodate a larger class of mobile robots.
This chapter addresses the reactive control design for point-mass vehicles with limited sensor range to track targets while avoiding static and moving obstacles in a dynamic environment. The proposed control law is synthesized by generating a potential field force for each objective and combining them through analysis and design. Different from standard potential field methods, the composite potential field put forward in this chapter is time-varying and planned to account for movement of moving obstacles and vehicle. Using rigorous Lyapunov analysis, basic conditions and key properties are derived. Simulation examples are included to illustrate both the design process and performance of proposed control.

4.1 Introduction

In real-world applications, mobile robots are required to explore and move within dynamic environments. Moreover, the environment is usually uncertain and trajectories of obstacles cannot be assumed \textit{a priori}. In this context, the problem that arises for mobile robots is how to track moving targets where robots have limited sensor range and simultaneously avoid static and moving obstacles in real-time.

In the context of stationary environments, existing path planning methods are classified into three categories: graph method, potential field method and other physical analogies methods. Graph methods \cite{30, 31} are based on a geometrical cell-decomposition of the
entire configuration workspace and yield an optimal path with respect to certain objective
criteria, for example finding the shortest collision-free path. The main disadvantage of graph
methods is that they require much computational resources. In the potential field method,
the target applies an attractive force to the robot while the obstacles exert a repulsive force
onto the robot. The composite force determines the movement of the robot. Because of its
simplicity, elegance and high efficiency, the potential field method is particularly popular.
Some inherent issues of potential field method have been pointed out in [7], such as trap
situations due to local minima. To avoid the disadvantages of the standard potential field
method, other physical analogies methods have been proposed using ideas from fluid me-
chanics [10] or electromagnetics [11] to build functions free of local minima, but they are
generally computationally intensive and therefore inappropriate for dynamic environments.

In the context of dynamic environments, a common technique is to add a time dimension
to the state space and reduce the problem to a static one [32, 33, 34]. The major issue is that
it always assumes that the trajectories of the moving obstacles are known \textit{a priori}, which is
often not practical in real applications.

Another approach was proposed [35, 36], which constructs repulsive potential functions
by taking into account the velocity information and extending the potential field method
for moving obstacle avoidance. In [36], the velocity of the obstacle is taken into account
when building the repulsive potential field. But the velocity of the robot is not considered.
Because the collision between the robot and obstacle depends on both the relative position
and velocity between them, this method is inadequate. This issue is addressed in [35, where
the repulsive potential function takes advantage of the velocity information of both the
robot and the obstacle. However, it assumed that the relative velocity between the robot and the obstacle is time-invariant in terms of position of the robot. This assumption is not practical as the relative velocity and position are actually time-varying. Thus derivatives of the relative velocity in terms of position cannot be considered zero uniformly. In, both methods deal with the obstacle avoidance problem applied to stationary targets.

The Ge and Cui method [37] constructs repulsive and attractive potentials which take into consideration the position and velocity of the robot with respect to moving targets and obstacles. Though convergence to the target is proven, no rigorous proof of obstacle avoidance is provided.

Other than potential field methods, there are other results [41, 42, 43, 44, 45]. In [41], a method combining a Deformable Virtual Zone (DVZ)-based reactive obstacle avoidance control with path following is proposed. In [42], harmonic functions along with the panel method for obstacle avoidance in dynamic environment is utilized. In [43], the dynamic window approach to obstacle avoidance in an unknown environment is presented. With a few changes to the standard scheme, convergence to the goal position is proved. [44] presented a method to compute the probability of collision in time for linear velocities of the robot and a reactive algorithm to perform obstacle avoidance in dynamic uncertain environment. [45] gave a preliminary study of the novel collision cone approach as a viable collision detection and avoidance tool in a 2-D dynamic environment. Many of the methods are heuristic and the lack of analytical design guidelines can be problematic in real world applications. Moreover, most of these methods increased complexity and computational costs.
In this chapter, we propose a reactive control to achieve target-tracking and moving obstacles avoidance. The proposed control combines planned potential field method and nonlinear damping. A desired trajectory is designed to resolve the potential conflict between target-tracking and collision avoidance. The planned potential functions are proposed based upon relative positions among the robot, the desired trajectory, and obstacles. At the same time, the nonlinear damping is designed to ensure stability and damp oscillation. Generalized potential functions are proposed which have no stable local minima. We present a theorem to analyze the stability property of the equilibrium point of the potential functions. More importantly, rigorous Lyapunov proof of target tracking and obstacle avoidance is given.

4.2 Problem Formulation

Consider a single point-mass agent whose dynamical model is given by

\[
\dot{q}_r = v_r, \quad \dot{v}_r = u, \tag{4.2.1}
\]

where \(q \triangleq [x, y]^T\) denotes the center position, \(v \triangleq [v_x, v_y]^T\) represents the velocity, and \(u\) is the control input. Thus we can define the states \(S(t) = (q(t), v(t))\). Subscripts \(r, g\) and \(o\) indicate the vehicle, target and obstacle respectively.

Given the initial configurations \(S_r(t_0) = (q_r(t_0), v_r(t_0))\), as shown in Figure 4.1, the objective of this chapter can be summarized as follows:

- tracking the specified target \(S_g(t) = (q_g(t), v_g(t))\);
- avoiding the \(n\) obstacles \(S_{oi} = (q_{oi}(t), v_{oi}(t))\) \((i = 1, 2, \cdots n)\).

We make the following assumptions without loss of generality:
Figure 4.1: Illustration of target-tracking with collision avoidance in a 2D dynamic environment

- The agent is represented by a 2-D circle centered at $q_r(t)$ and of radius $R$. The range of its sensors is also described by a circle centered at $q_r(t)$ and of radius $R_s$;

- The $i$th static/moving obstacle is represented by a circle centered at $q_{oi}(t)$ and of radius $R_{oi}$.

### 4.3 Target-tracking and Collision Avoidance for a Single Agent

In this section, using Lyapunov-type analysis, we derive a nonlinear reactive control to guarantee collision avoidance and tracking of a target for a single robot. To achieve these two design objectives at the same time, two potential field functions are used to generate the reactive forces. Specifically, let us consider the following composite potential function:

$$P(q_r - q_o, q_r - q_g) = P_a(q_r - q_g) + P_r(q_r - q_o), \quad (4.3.1)$$

where $P_a(\cdot)$ is the attractive potential function and $P_r(\cdot)$ is the repulsive potential function, which satisfy the properties that
Figure 4.2: Typical potential field functions (a: attractive potential field function; b: contour lines of attractive potential field function; c: repulsive potential field function; d: contour lines of repulsive potential field function)

\[
\begin{align*}
\left\{ \begin{array}{l}
P_a(0) = 0, \quad & \nabla P_a(q_r - q_g)|_{(q_r - q_g)=0} = 0, \\
0 < P_a(q_r - q_g) < \infty & \text{if } \|q_r - q_g\| \text{ is nonzero and finite}, \\
\|\nabla P_a(q_r - q_g)\| < +\infty & \text{if } \|q_r - q_g\| \text{ is finite}, \\
\end{array} \right.
\end{align*}
\]

and

\[
\begin{align*}
\left\{ \begin{array}{l}
P_r(q_r - q_o) = +\infty & \text{if } (q_r - q_o) \in \Omega_o, \\
P_r(q_r - q_o) = 0 & \text{if } (q_r - q_o) \notin \overline{\Omega_o}, \\
P_r(q_r - q_o) \in (0, \infty) & \text{if } (q_r - q_o) \in \overline{\Omega_o} \text{ but } (q_r - q_o) \notin \Omega_o, \\
\lim_{(q_r - q_o) \to \Omega_o} \|\nabla P_r(q_r - q_o)\| = +\infty & \text{if } (q_r - q_o) \notin \Omega_o,
\end{array} \right.
\end{align*}
\]

where \( \Omega_o \subset \mathbb{R}^2 \) is a compact set representing the 2-dimensional shape of the obstacle, \( \overline{\Omega_o} \) is the compact set which is an enlarged version of \( \Omega_o \) and in which repulsive force becomes active. \( \Omega_o \) and \( \overline{\Omega_o} \) will move with the center \( q_o \). The above defined attractive potential function and repulsive potential function are exemplified by Figure 4.2.
Should sets $\Omega_{oj}$ and $\Omega_{ok}$ overlap for some $j \neq k$; the two obstacles can be combined into one obstacle. Thus we can assume the following throughout the chapter without loss of generality:

**Assumption 7.** $\Omega_{oj} \cap \Omega_{ok}$ be empty for $j \neq k$.

Let the vehicle control be a reactive control of the form

$$u = -\nabla P_a (q_r - q'_g) - \nabla P_r (q_r - q_{oi}) - \xi(q_r - q'_g)(v_r - v'_g) + \dot{v}_g,$$  \hspace{1cm} (4.3.4)

where the terms $\nabla P_a(\cdot)$ and $\nabla P_r(\cdot)$ are the standard reactive control components, $\xi(\cdot) > 0$ is a uniformly bounded function designed to ensure stability and damp oscillations. As shown in (4.3.4), a desired trajectory is introduced, denoted by $q'_g(t)$ to resolve the potential conflict between target-tracking and collision avoidance, which is given as follows:

- $(q_r - q_{oi}) \notin \Omega_{oi}$

$$\lim_{t \to \infty} q'_g = q_g, \quad \lim_{t \to \infty} v'_g = v_g, \quad \text{and} \quad \lim_{t \to \infty} \dot{v}'_g = \dot{v}_g.$$  

To satisfy the above conditions, an obvious choice for $q'_g$, $v'_g$ and $\dot{v}'_g$ is that,

$$q'_g = q_g, \quad v'_g = v_g, \quad \text{and} \quad \dot{v}'_g = \dot{v}_g.$$

- $(q_r - q_{oi}) \in \Omega_{oi}$

$$q'_g = \begin{cases} 
q_g & \text{if } (q_g - q_{oi}) \notin \Omega_{oi}, \\
q^* + \varepsilon \cdot (q_g - q_{oi}) & \text{otherwise,}
\end{cases}$$

$$v'_g = v_{oi}, \quad \text{and} \quad \dot{v}'_g = \dot{v}_{oi},$$
$q_g'$ is reset when the robot reaches the $\text{bd}(\Omega_{oi})$. At time $t'$, the robot reaches the $\text{bd}(\Omega_{oi})$. Then $q_g'(t') = q_g(t')$, if $(q_g(t') - q_{oi}(t')) \not\in \Omega_{oi}$. Otherwise, we first draw a line connecting $q_{oi}(t')$ to $q_g(t')$. $q^*$ denotes the crosspoint of the extension line $q_{oi}(t')q_g'(t')$ and $\text{bd}(\Omega_{oi})$. $\varepsilon$ is a very small positive constant. Then we pick $q_g'(t') = q^* + \varepsilon \cdot (q_g - q_{oi})$ to ensure $(q_g'(t') - q_{oi}(t')) \not\in \Omega_{oi}$. Furthermore, $v'_g = v_{oi}$, $i'_g = i_{oi}$ as long as $(q_r - q_{oi}) \in \Omega_{oi}$. This strategy is depicted in Figure 4.3. In the same logic, we can specify the initial configuration $q_g'(0)$, $v_g'(0)$, and $i_g'(0)$.

![Figure 4.3: Strategy to choose desired trajectory when $(q_r - q_{oi}) \in \Omega_{oi}$](image)

(a:$(q_g(t') - q_{oi}(t')) \not\in \Omega_{oi}$; b:$(q_g(t') - q_{oi}(t')) \in \Omega_{oi}$)

### 4.3.1 Generalized Differentiable Potential Functions

In this section, we propose a generalized potential functions whose gradients exist everywhere.

The attractive potential function $P_a (q_r - q_g)$ is given by

$$P_a (q_r - q_g) = \frac{k_a}{2} \|q_r - q_g\|^2.$$  

(4.3.5)

Accordingly, the attractive force can be given as follows,

$$-\nabla P_a (q_r - q_g) = k_a (q_g - q_r).$$  

(4.3.6)
On the other hand, the repulsive potential function $P_r(q_r - q_o)$ is given by

$$P_r(q_r - q_o) = \begin{cases} +\infty & \text{if } d \leq 0, \\ 0 & \text{if } d \geq D, \\ k_r \left( \ln \left( \frac{d}{D} \right) - \frac{D-d}{D} \right) & \text{otherwise}, \end{cases} \quad (4.3.7)$$

where $d = (\|q_r - q_o\| - R - R_{oi})$, which is the minimum distance between the agent and the $i$th obstacle. And $D > 0$, defining the confined set $\Omega_{oi}$. The repulsive force is “active” only if $d < D$, which is written into

$$-\nabla P_r(q_r - q_o) = \begin{cases} +\infty & \text{if } d \leq 0, \\ 0 & \text{if } d \geq D, \\ k_r \left( \frac{1}{d} - \frac{1}{D} \right) \frac{q_r - q_o}{\|q_r - q_o\|} & \text{otherwise}. \end{cases} \quad (4.3.8)$$

**Remark 3.1:** It is straightforward from (4.3.8) that, given $k_r$, the smaller the value of $D$ is chosen, $-\nabla P_r(q_r - q_o)$ becomes steeper. Hence, an effective way to prevent large acceleration inputs is to increase $D$. Meanwhile, a smaller $D$ means less chance of entering into $\Omega_{oi}$, which is beneficial for target-tracking.

### 4.3.2 Stability Analysis of Equilibrium Point

In this section, Theorem 1 is proposed, providing a geometrical method to analyze the stability property of equilibrium points yielded by the composite potential function.

**Definition 19.** A point in the composite potential function (4.3.1), point $q^* \in \mathbb{R}^2$ is defined to be a stationary point if and only if it satisfies the following equation

$$-\nabla P_a(q^* - q_g) = \nabla P_r(q^* - q_o).$$

**Definition 20.** Curves $C_a(K_a)$ and $C_r(K_r)$ are said to be the level curves of potential functions defined by

$$C_a(K_a) \triangleq \{ q \in \mathbb{R}^2 | P_a(q - q_g) = K_a \} \quad (K_a > 0),$$

and
and

\[ C_r (K_r) \triangleq \{ q \in \mathbb{R}^2 | P_r (q - q_o) = K_r \} \quad (K_r > 0) \]

**Theorem 4.3.1.** Upon the attractor-repeller form potential function (4.3.1), at the stationary point \( q^* \), let \( K_{aq} \) to be the curvature of the level curve \( C_a (P_a(q^* - q_o)) \) and \( K_{rq} \) to be the curvature of the level curve \( C_r (P_r(q^* - q_o)) \). The level curves \( C_a (P_a(q^* - q_o)) \) and \( C_r (P_r(q^* - q_o)) \) are convex at the stationary point \( q^* \). Suppose the straight line connecting \( q_o \) to \( q^* \) is normal to the level curves \( C_a (K_a) \) and \( C_r (K_r) \). Then \( q^* \) is saddle point if and only if \( K_{aq} < K_{rq} \).

**Proof.** Since the straight line connecting \( q_o \) to \( q^* \) is normal to the level curves \( C_a (K_a) \) and \( C_r (K_r) \), let us introduce the coordinate system (see Fig. 4.4) in which the origin is \( q_o \) and \( q_o q^* \) represents the positive \( y \) axis.

![Figure 4.4: Level curves tangency at the equilibrium point](image)

Therefore, the stationary point has the following properties (we do not consider the trivial case \( -\nabla P_a(\cdot) = \nabla P_r(\cdot) = 0 \), in which the stationary point is the target),

\[
\begin{align*}
\frac{\partial P_a}{\partial x} &= \frac{\partial P_r}{\partial x} = 0, \\
\frac{\partial P_a}{\partial y} &= -\frac{\partial P_r}{\partial y} > 0, \\
\frac{\partial^2 P_a}{\partial x \partial y} &= \frac{\partial^2 P_r}{\partial x \partial y} = 0, \\
-\frac{\partial^2 P_a}{\partial y^2} &= \frac{\partial^2 P_r}{\partial y^2} < 0.
\end{align*}
\] (4.3.9)

For the implicit function \( P_a (q - q_o) = K_a \), we have

\[
\frac{dy}{dx} \bigg|_{q^*} = -\frac{\partial P_a}{\partial x} \bigg|_{q^*} = 0.
\] (4.3.10)

And

\[
\frac{d^2 y}{dx^2} \bigg|_{q^*} = -\frac{\partial^2 P_a}{\partial y^2} \bigg|_{q^*} - \left( \frac{\partial^2 P_a}{\partial y \partial x} \frac{dy}{dx} \right)^2 \bigg|_{q^*} - \frac{\partial^2 P_a}{\partial y^2} \bigg|_{q^*}.
\] (4.3.11)
In addition, $C_a(P_a(q - q_g))$ is convex, which implies $\frac{\partial^2 y}{\partial x^2} < 0$. Hence combining (4.3.9) and (4.3.11) yields

$$\frac{\partial^2 P_a}{\partial x^2} |_{q^*} > 0.$$  \hspace{1cm} (4.3.12)

Moreover, it follows from (4.3.10), (4.3.11), and (4.3.12) that

$$K_{aq} = \frac{\partial^2 P_a}{\partial x^2} |_{q^*}.$$  \hspace{1cm} (4.3.13)

Similarly, we have

$$\frac{\partial^2 P_r}{\partial x^2} |_{q^*} < 0,$$  \hspace{1cm} (4.3.14)

and

$$K_{rq} = \frac{\partial^2 P_r}{\partial x^2} |_{q^*}.$$  \hspace{1cm} (4.3.15)

Now considering the following system model,

$$\dot{x} = -\frac{\partial P_a}{\partial x} - \frac{\partial P_r}{\partial x},$$

$$\dot{y} = -\frac{\partial P_a}{\partial y} - \frac{\partial P_r}{\partial y}.$$  

Correspondingly, the Jacobian matrix $[J]_{2\times2}$ is given by,

$$[J]_{2\times2} = \begin{bmatrix} -\frac{\partial^2 P_a}{\partial x^2} - \frac{\partial^2 P_r}{\partial x^2} & -\frac{\partial^2 P_a}{\partial x \partial y} - \frac{\partial^2 P_r}{\partial x \partial y} \\ -\frac{\partial^2 P_a}{\partial y^2} - \frac{\partial^2 P_r}{\partial y^2} & \frac{\partial^2 P_a}{\partial y^2} - \frac{\partial^2 P_r}{\partial y^2} \end{bmatrix}.$$  

It follows from (4.3.9), at the stationary point $q^*$, we can obtain the eigenvalues as follows,

$$\lambda_1 = -\frac{\partial^2 P_a}{\partial x^2} - \frac{\partial^2 P_r}{\partial x^2} \text{ and } \lambda_2 = -\frac{\partial^2 P_a}{\partial y^2} - \frac{\partial^2 P_r}{\partial y^2} < 0.$$  \hspace{1cm} (4.3.16)

Substituting (4.3.13) and (4.3.15) into $\lambda_1$, we can rewritten $\lambda_1$ as

$$\lambda_1 = \frac{\partial P_a}{\partial y} (K_{rq} - K_{aq}).$$  \hspace{1cm} (4.3.17)

Following form (2.2.1), (4.3.16), and (4.3.17), we can conclude that $q^*$ is saddle point if and only if $K_{aq} < K_{rq}$. 

Under the potential functions (4.3.5) and (4.3.7), level curves $C_a(K_a)$ and $C_r(K_r)$ are concentric circles with centers $q_g$ and $q_o$ respectively. From the geometric viewpoint, the obstacle is closer to the local minimum than the target, which implies $K_{aq} < K_{rq}$. Invoked by Theorem 1, it is the saddle point. Thus we can assume the following throughout the chapter:

**Assumption 8.** Composite potential field function (4.3.1) has only one stable local minimum, which is the target.
4.3.3 Tracking of a Target and Obstacle Avoidance

The tracking problem is to ensure that the agent will converge to the target position $q_g$ provided the target is reachable. And the obstacle avoidance problem is to ensure that the agent will not enter the given compact set $\Omega_{oa}$ provided that certain initial conditions hold.

The following theorem provides the basic result.

**Theorem 4.3.2.** Suppose that potential field function (4.3.1) satisfies properties (4.3.2) and (4.3.3). If assumptions 7 and 8 hold, as long as $(q_r(t_0) - q_{oa}(t_0)) \notin \Omega_{oa}$ and the initial conditions $S_r(t_0) = (q_r(t_0), v_r(t_0))$ are finite, then system (4.2.1) under control (4.3.4) is collision-free provided that $v_g(t)$ and $v_{oa}(t)$ are uniformly bounded. Furthermore, after a finite time instant $t^*$, if $[q_g(t) - q_{oa}(t)] \notin \overline{\Omega}_{oa}$ for all $t \geq t^*$, $q_r(t)$ converges asymptotically to $q_g(t)$. If $[q_g(t) - v_{oa}(t)]$ stays in or intermittently returns to $\overline{\Omega}_{oa}$, there is no convergence of $[q_r(t) - q_g(t)] \to 0$.

**Proof.** considering the following Lyapunov function candidate

$$V_1(t) = \frac{1}{2} \|v_r - v_g\|^2 + P(q_r - q_g, q_r - q_{oa}).$$

Let us consider the case $[q_r(t) - q_{oa}(t)] \in \overline{\Omega}_{oa}$. Under assumption 1, it follows from (4.2.1) and (4.3.4) that

$$\dot{V}_1 = (v_r - v'_g)^T (\dot{v}_r - \dot{v}'_g) + (v_r - v'_g)^T \nabla P_a (q_r - q_g') + (v_r - v_{oa})^T \nabla P_r (q_r - q_{oa})$$

$$= (v_r - v'_g)^T (-\nabla P_a (q_r - q_g') - \nabla P_r (q_r - q_{oa}))$$

$$- \xi (q_r - q_g') (v_r - v'_g)$$

$$+ (v_r - v_{oa})^T \nabla P_r (q_r - q_{oa})$$

$$= -\xi (q_r - q_g') \|v_r - v'_g\|^2$$

$$+ (v'_g - v_{oa})^T \nabla P_r (q_r - q_{oa})$$

$$(4.3.18)$$

which is negative semi-definite. Therefore, provided that $(q_r(t_0) - q_{oa}(t_0)) \notin \Omega_{oa}$ and the initial conditions $S_r(t_0) = (q_r(t_0), v_r(t_0))$ are finite, $P(q_r(t) - q'_g(t), q_r(t) - q_{oa}(t))$ will remain bounded as long as $v_g(t)$ and $v_{oa}(t)$ are uniformly bounded. As proved subsequently, $v_g(t)$ is required to be uniformly bounded, which ensures $v_r(t)$ remains bounded provided that the initial conditions are finite when $[q_r(t) - q_{oa}(t)] \notin \overline{\Omega}_{oa}$. Thus collision avoidance is guaranteed.

It follows from (4.3.18), $[q_r(t) - q'_g(t)] \to 0$ under the assumption 8 invoked by LaSalle’s invariant set theorem [22]. Hence, using proof by contradiction, we can conclude that no
convergence of \([q_r(t) - q_g(t)] \to 0\) can be achieved if \([q_g(t) - q_{oi}(t)]\) stays in or intermittently returns to \(\Omega_o\).

Furthermore, from the geometric viewpoint, the transient process to track the target and avoid collision can be illustrated in Figure 4.5. As shown in Figure 4.5, once the robot is in the set \(\Omega_o\), it will asymptotically converge to its desired trajectory \(q'_g\). Thus, unless \([q_g(t) - q_{oi}(t)]\) stays in or intermittently be in \(\Omega_{oi}\) which implies \([q_g(t) - q_{oi}(t)] \notin \Omega_{oi}\) for all \(t \geq T^* (T^* > t^*)\).

To show asymptotic convergence under the condition \([q_g(t) - q_{oi}(t)] \notin \Omega_{oi}\) for all \(t \geq T^* (T^* > t^*)\), we note that after \(T^*\), the tracking error dynamics of system (4.2.1) under control (4.3.4) reduces to

\[
\dot{e}_1 = e_2, \quad \dot{e}_2 = -\nabla P_a(e_1) - \xi(e_1)e_2,
\]

where \(e_1 = q_r - q'_g\) and \(e_2 = v_r - v'_g\). Adopting the simple Lyapunov function

\[
V_2(t) = P_a(e_1) + \frac{1}{2}\|e_2\|^2.
\]

We have

\[
\dot{V}_2 = e_2^T \nabla P_a(e_1) + e_2^T [-\nabla P_a(e_1) - \xi(e_1)e_2] = -\xi(e_1)\|e_2\|^2,
\]

which implies asymptotic stability of \(e_1\) and \(e_2\) under the assumption 8 invoked by LaSalle’s invariant set theorem. Considering \(q'_g \to q_g\), \(v'_g \to v_g\), and \(v_g \to v_g\) as \(t \to \infty\), asymptotic convergence can be concluded.

4.4 Simulations

This section describes the simulation results of a differential drive vehicle to validate the proposed controls.
4.4.1 Model and vehicle control for differential drive vehicle

Consider the following kinematic and dynamic model of a differential drive vehicle (as shown in Figure 4.6),

\[
\begin{align*}
\dot{x} &= V \cos \theta \\
\dot{y} &= V \sin \theta \\
\dot{\theta} &= \omega \\
\dot{V} &= \frac{F}{M}
\end{align*}
\]

(4.4.1)

where \( \theta \) is the orientation, \( V \) is the linear velocity, \( \omega \) is the angular velocity, \( F \) is the applied force and \( M \) is the mass.

![Figure 4.6: Relevant variables for the unicycle (top view)](image)

Figure 4.6: Relevant variables for the unicycle (top view)

Consider the following dynamic compensator [46]:

\[
\begin{align*}
\omega &= \frac{u_2 \cos \theta - u_1 \sin \theta}{V} \\
F &= M (u_1 \cos \theta + u_2 \sin \theta)
\end{align*}
\]

(4.4.2)

Substituting (4.4.2) into (4.4.1) yields the transformed system (4.2.1). Note the following facts: (1) the sign of linear velocity \( V \) will determine forward or backward motion of the vehicle; (2) transformation (4.4.2) is singular at \( V = 0 \), i.e., when the mobile robot is not moving. We take the following two measures to cope with the singularity problem.
• Set the initial linear velocity to be nonzero;
• Let \( V(k+1) = \begin{cases} 
\delta & \text{if } V(k) + \dot{V}(k) T < \delta \\
V(k) + \dot{V}(k) T_s & \text{otherwise}
\end{cases} \),

where \( T_s \) is the sampling period, \( k = 0, 1, 2 \cdots \), and \( \delta \) is a very small positive constant.

4.4.2 Simulation results

In these simulation settings, the potential functions are given by (4.3.5) and (4.3.7). The nonlinear damping function \( \xi(\cdot) \) is simply chosen to be a constant function. The parameters used for these simulations are: \( R = 1 \) m, \( R_s = 2 \) m, \( R_{ai} = 1 \) m, \( k_a = 100 \), \( k_r = 20 \), \( D = 1 \) m, \( \xi(\cdot) = 80 \), \( \varepsilon = 0.1 \), \( \delta = 0.1 \) m/s, \( r = 0.6 \) m and \( L = 1.821 \) m. In addition, the initial location of the vehicle is \((1,1)\), \( v(0) = 1 \) m/s, \( \dot{v}(0) = 0 \) m/s\(^2\), and \( \omega(0) = 0 \) rad/s. And the bounds on the angular velocity of both wheels is \( \frac{50}{3} \) rad/s.

4.4.2.1 Target-tracking and collision avoidance with static obstacles

There are three static obstacles \((2,7,1)\), \((10,10,1)\), and \((22,19,1)\). The simulation result is shown in Figure 4.7.

4.4.2.2 Target-tracking and collision avoidance with moving obstacles

Compared with example 1, three moving obstacles of radius being 1 are also considered. The simulation result is shown in Figure 4.8.

\(^1\)Data format: (center position, radius). For example, (2,7) denotes the center position. The radius is 1 m.
Figure 4.7: Collision avoidance with static obstacles

Figure 4.8: Collision avoidance with moving obstacles
4.5 Conclusions

In this chapter, a systematic approach is proposed to achieve virtual command vehicle tracking and collision avoidance. Simulation examples confirm the effectiveness of Lyapunov design of multi-objective control for the point-mass agent proposed in Section 3. Future research will consider more complex dynamical models to accommodate a larger class of mobile robots. In addition, investigation of the oscillation issues inherent in the potential field method and improving the overall performance will be addressed.
CHAPTER 5
REACTIVE TARGET TRACKING WITH OBSTACLE AVOIDANCE CONTROL OF UNICYCLE-TYPE MOBILE ROBOTS IN A DYNAMIC ENVIRONMENT

This chapter studies the reactive control design to track targets while avoiding static and moving obstacles for unicycle-type mobile robots with limited sensor range in a dynamic environment. The relative motion among the mobile robot, targets, and obstacles is formulated in polar coordinates. And then kinematic control laws achieving both target-tracking and obstacle avoidance are designed using Lyapunov based technique, and more importantly, the proposed control laws also consider possible kinematic control saturation constraints. Simulation examples are provided to illustrate the effectiveness of the proposed control.

5.1 Introduction

For most real-world applications, it is a basic requirement that mobile robots can safely explore and move within dynamic environments. Because mobile robots are often subject to nonholonomic constraints, to achieve this goal, the central and difficult studies can be classified into two regimes.

5.1.1 Target Tracking Control of Nonholonomic Systems

The target tracking control objective is to make the system asymptotically follow a desired trajectory. The desired trajectory must be feasible, that is, it has been planned to satisfy nonholonomic constraints. It is worth to note that, when a desired trajectory stops at some
point in the configuration space, the tracking control problem reduces to the stabilization problem, also called the regulation problem.

Control of nonholonomic systems has drawn great attention due to its practical importance and theoretical challenges. A comprehensive introduction to nonholonomic systems modeling, analysis, and control can be found in [1]. Traditionally, modeling and control synthesis of nonholonomic mobile robots are discussed in cartesian coordinates. Especially, the chained form has been used as a canonical form in analysis and control design for nonholonomic systems. By Brockett’s theorem [47], nonholonomic systems can not be asymptotically stabilized around a fixed point under any smooth (or even continuous) time-independent state feedback control law in cartesian coordinates. Therefore, tracking control problem and regulation problem are usually treated separately using different approaches.

Most existing methods dealing with regulation problem use one of these two strategies: time-implicit but discontinuous feedback control laws [48] and time-varying continuous controls [49]. Tracking control designed using the backstepping method is shown to ensure global asymptotic stability [50]. A linear time varying output feedback tracking control is proposed to ensure global exponential stability under the conditions that reference input is continuously differentiable, non-vanishing, and Lipschitz with respect to time. In [51], a unifying design framework is proposed by investigating uniform complete controllability of time varying systems. The proposed controls are globally asymptotically stabilizing, in simple closed forms, time varying and smooth, and near-optimal.
5.1.2 Obstacle Avoidance of Nonholonomic Systems

On the one hand, obstacle avoidance has been also extensively studied at the navigation system level (path planning/trajectory planning). Compared with standard motion planning approaches such as graph methods and potential field methods, which are proposed to deal with geometrical constraints, that is, holonomic systems in the presence of static obstacles, motion planning of nonholonomic wheeled mobile robots in dynamic environments is more challenging and important for mobile robotics. Based on Reeds and Shepp’s results on shortest paths of bounded curvature without considering obstacles, nonholonomic path planners [52] are proposed. With obstacles being modeled as polygons, obstacle avoidance and curvature constraint are taken into consideration by offsetting each polygon. Then a feasible path is obtained by using a sequence of such optimal path segments as those proposed in [53]. Using ideas from fluid mechanics, in [54], a collision free path is computed which satisfies the minimum curvature constraint. This method supposes the environment is static and also known a priori. [55] proposed an analytical nonholonomic trajectory generation algorithm. A family of parameterized polynomial trajectories are firstly derived to ensure all the resulting trajectory candidates feasible. Secondly, the free parameter(s) representing the family are confined into appropriate intervals such that collision avoidance criteria are met.

On the other hand, obstacle avoidance is addressed directly in the kinematics/dynamics controller, which is normally called avoidance control. In [56, 57], the dynamic window approach is introduced. A searching space is defined, consisting only of the admissible velocities and accelerations of the robot within a small time interval. Then the commands
controlling the velocities and accelerations of the robot are computed by maximizing an performance index function associated with target tracking and obstacle avoidance. The concepts of collision cones and velocity obstacles are introduced in [45, 58] respectively, which are widely used to design avoidance control [59, 60]. The underlying idea is that obstacle avoidance is achieved if the robot velocity is selected such that its velocity relative to the obstacles’ motion does not enter the corresponding collision cones/velocity obstacles. Avoidance control is also proposed combining potential field methods and sliding mode control [61, 62]. A gradient-tracking based sliding mode controller for the mobile robot is proposed to achieve target tracking and obstacle avoidance. In addition, potential field based formation control of multiple mobile robots are studied in [40, 63].

5.1.3 Outline of coupling between the above two areas

In this chapter, we focus on reactive control solutions to position tracking and obstacle avoidance of a class of most studied nonholonomic systems: unicycle-type mobile robots. The polar representation is utilized to design the controls due to the following two reasons: (1) The polar representation can naturally provide a better measure of progress towards a target position and the distance to the obstacle; (2) By introducing the polar coordinate transformation, it is shown that control laws can be readily developed. The polar representation approach has been firstly introduced in [64], adopted for solving regulation problem [65], and employed to handle the tracking control problem [66]. To the best of our knowledge, however, we are the first to combine polar coordinate transformation and Lyapunov-like analysis to solve the aforementioned multi-objective control problem (position tracking and
obstacle avoidance). The main contributions of this chapter are: (1) We derive an easily implementable reactive control algorithm to solve the position tracking and obstacle avoidance of unicycle-type; (2) In the proposed control algorithm, we also take into consideration the possible kinematic control saturation constraints.

The remainder of this chapter is organized as follows: In Section 2, the problems of tracking target and avoiding the static/moving obstacles is formulated. In Section 3, a novel reactive control design is proposed for a single unicycle-type mobile robot to achieve target-tracking and collision avoidance based upon polar description of the relative motions. In Section 4, examples and their simulations is presented to demonstrate the effectiveness of proposed controls. In Section 5, the chapter is concluded and some future research directions are suggested.

5.2 Problem Formulation

Unicycle-type mobile robot is subject to the nonholonomic no-slip kinematics constraint of form

\[ A(q) \dot{q} = 0 \quad \text{with} \quad A(q) = \begin{bmatrix} -\sin \theta & \cos \theta & 0 \end{bmatrix}, \]

where \( q = \begin{bmatrix} x & y & \theta \end{bmatrix}^T \) are states defined in the configuration space. \((x, y)\) is the center position and \( \theta \) is the orientation.

A basis \( G(q) \) of the null space of \( A(q) \) is given by

\[
G(q) = \begin{bmatrix} \cos \theta & 0 \\ -\sin \theta & 0 \\ 0 & 1 \end{bmatrix}.
\]
Therefore, the kinematic model can be represented as follows

\[ \dot{q} = G(q)U, \quad (5.2.1) \]

where \( U = \begin{bmatrix} v & \omega \end{bmatrix}^T \) is called kinematic control. In particular, \( v \) is the linear velocity and \( \omega \) is the angular velocity. Forward or backward motion of the vehicle is determined by the sign of linear velocity \( v \). The target and obstacles are also supposed to satisfy the kinematic model \( (5.2.1) \). Subscripts \( r, g \) and \( o \) indicate the vehicle, target and obstacle respectively.

![Image of target-tracking with collision avoidance in a 2D dynamic environment](image)

Figure 5.1: Illustration of target-tracking with collision avoidance in a 2D dynamic environment

Referring to Figure 5.1, \( X_GO_GY_G \) is the global inertial reference frame. The relative motion between the robot and its target in polar coordinates is represented by,

\[ \rho_{rg} = \sqrt{(x_r - x_g)^2 + (y_r - y_g)^2}, \quad (5.2.2) \]

and

\[ \phi_{rg} = \text{atan2}(y_g - y_r, x_g - x_r). \quad (5.2.3) \]

Clearly, \( \rho_{rg} \) represents the distance between the robot and target. And \( \phi_{rg} \) is the line-of-sight angle.
In order to derive the tracking error kinematics in polar coordinates, differentiating (5.2.2) and (5.2.3) on both sides, we can obtain

\[
\dot{\rho}_{rg} = -\cos \phi_{rg} \left( \dot{x}_r - \dot{x}_g \right) - \sin \phi_{rg} \left( \dot{y}_r - \dot{y}_g \right),
\]

(5.2.4)

and

\[
\dot{\phi}_{rg} = -\cos \phi_{rg} \left( \dot{y}_r - \dot{y}_g \right) + \sin \phi_{rg} \left( \dot{x}_r - \dot{x}_g \right) / \rho_{rg}.
\]

(5.2.5)

Substituting (5.2.1) into (5.2.4) and (5.2.5), following from the trigonometric identities, the error system in the polar coordinates can be written as

\[
\begin{aligned}
\dot{\rho}_{rg} &= -v_r \cos (\theta_r - \phi_{rg}) + v_g \cos (\theta_g - \phi_{rg}) \\
\dot{\phi}_{rg} &= -v_r \sin(\theta_r - \phi_{rag}) + v_g \sin(\theta_g - \phi_{rag}) / \rho_{rg} \\
\dot{\theta}_r &= \omega_r,
\end{aligned}
\]

(5.2.6)

Meanwhile, the relative motion between the robot and the ith obstacle in the polar coordinates can be formulated as

\[
\begin{aligned}
\dot{\rho}_{oir} &= -v_{oi} \cos (\theta_{oi} - \phi_{oir}) + v_r \cos (\theta_r - \phi_{oir}) \\
\dot{\phi}_{oir} &= -v_{oi} \sin(\theta_{oi} - \phi_{oir}) + v_r \sin(\theta_r - \phi_{oir}) / \rho_{oir} \\
\dot{\theta}_r &= \omega_r,
\end{aligned}
\]

(5.2.7)

In what follows, (5.2.6) and (5.2.7) will be used to design the controller for the robot.

Without loss of generality, \(\theta, \phi_{oir},\) and \(\phi_{rg}\) fall into \([-\pi, \pi]\).

In order to have a well-defined problem, we assume the followings throughout the chapter:

**Assumption 9.** The mobile robot under consideration is represented by a 2-D circle with the center at \((x_r(t), y_r(t))\) and of radius \(R_r\). The range of its sensors is also described by a circle centered at \((x_r(t), y_r(t))\) and of radius \(R_s\). Meanwhile, the ith obstacle is represented by a 2-D circle with the center at \((x_{oi}(t), y_{oi}(t))\) and of radius \(R_{io}\).
Assumption 10. The mobile robot has the following velocity saturation constraints:

\[ |v_r| \leq \overline{v}_r \quad \text{and} \quad |\omega_r| \leq \overline{\omega}_r, \quad (5.2.8) \]

where \( \overline{v}_r > 0 \) is the maximum linear velocity. And \( \overline{\omega}_r \) is the maximum angular velocity. Furthermore, the robot is supposed to have a superior maneuvering capability given by

\[ \overline{v}_r \geq k_1 \overline{v}_g \quad \text{and} \quad \overline{v}_r \geq k_2 \overline{v}_{oi} \quad (k_1, k_2 > 2). \quad (5.2.9) \]

Then the control objectives of this chapter can be summarized as follows:

- Globally uniform bounded position tracking, i.e. provided that \( \lim_{t \to +\infty} \rho_{oig}(t) > R_{io} \), there exists \( D > 0 \) such that \( \lim_{t \to +\infty} \rho_{rg}(t) \leq D, \forall (x_r(t_0), y_r(t_0)) \in \mathbb{R}^2 \);

- Obstacle avoidance, i.e. there exists \( D_o > 0 \) such that \( \rho_{oir}(t) > (R_{io} + R_r), \forall t \geq t_0 \) provided that \( \rho_{oir}(t_0) \geq D_o \).

### 5.3 Target-tracking and Obstacle Avoidance Control Law Synthesis

In this section, using Lyapunov-type analysis, we derive a reactive switching control that achieve collision avoidance and tracking of a target for a single robot. Specifically, the kinematic control \( v_r \) is chosen to be a constant speed \( \overline{v}_r \). The reason is twofold: (1) observing form (5.2.6) and (5.2.7), kinematic control \( \omega_r \) plays a vital role as to multi-objective of target tracking and obstacle avoidance; and (2) this type of control is simple from theoretical development aspect and requires less control effort as well. The detailed design and proofs are presented as follows.

Firstly, let us begin with some definitions:

**Definition 21.** Let \( \mathbb{R}_+ \triangleq [0, +\infty) \), a \( C^1 \) function \( P_a : \mathbb{R}_+ \to \mathbb{R}_+ \), is called an attractive potential field function on \( \mathbb{R}_+ \) if the following conditions hold:

1. \( P_a(\rho_{rg}) = 0 \) and \( \nabla P_a(\rho_{rg}) = 0 \) iff \( \rho_{rg} = 0 \);
2. $P_a(\rho_{rg}) < \bar{P}_a$ and $0 < \nabla P_a(\rho_{rg}) < \bar{F}_a$ when $\rho_{rg} \neq 0$;

where $\bar{P}_a > 0$ is the upper bound of $P_a(\rho_{rg})$ and $\bar{F}_a > 0$ is the upper bound of $\|\nabla P_a(\rho_{rg})\|$.

**Definition 22.** A $C^1$ function $P_r: \mathbb{R}_+ \mapsto \mathbb{R}_+$, is called a repulsive potential field function on $\mathbb{R}_+$ if the following conditions hold:

1. $P_r(\rho_{oir}) \geq \bar{P}_r$ if $\rho_{oir} \leq (R_{io} + R_r)$;
2. $P_r(\rho_{oir}) \in [0, \bar{P}_r)$ and $\nabla P_r(q) \in (-\bar{F}_r, 0]$ if $\rho_{oir} > (R_{io} + R_r)$;

where $\bar{P}_r > 0$ is the upper bound of $P_r$ and $\bar{F}_r > 0$ is the upper bound of $\|\nabla P_r(\rho_{oir})\|$.

In what follows, the properties of above defined potential functions will be used for Lyapunov proof.

### 5.3.1 Target-tracking Control Design

![Figure 5.2: Illustration of target-tracking control strategy](image)

As shown in Figure 5.2, with consideration of the angular velocity saturation $\varpi_r$, the idea of tracking control design is to steer orientation angle $\theta_r$ to track line-of-sight $\phi_{rg}$ as soon as possible. Therefore, the robot’s tracking control is given by

$$
\begin{align*}
  v_r &= \begin{cases} 
    \bar{v}_r & \text{if } \rho_{rg} \leq D, \\
    0 & \text{otherwise},
  \end{cases} \\
  \omega_r &= \text{sat} \left( -k \Phi (\theta_r - \phi_{rg}) + \dot{\phi}_{rg, \varpi_r} \right),
\end{align*}
$$

(5.3.1)
where $D$ is the position tracking bound, $k$ is a positive constant gain. And function $\Phi(\alpha)$ determines the direction of rotation movement, which is designed to be

$$
\Phi(\alpha) = \begin{cases} 
2\pi + \alpha & \text{if } \alpha \leq -\pi, \\
\alpha - 2\pi & \text{if } \alpha > \pi, \\
\alpha & \text{otherwise.}
\end{cases}
$$

(5.3.2)

And $\text{sat}(u, \bar{u})$ is a saturation function defined as

$$
\text{sat}(u, \bar{u}) \triangleq \text{sgn}(u(t)) \cdot \min(|u(t)|, \bar{u}).
$$

(5.3.3)

**Theorem 5.3.1.** Consider system (5.2.1) under control (5.3.1) and suppose assumptions 9 and 10 hold. Let $D > \frac{(\overline{\pi}_r + \overline{\pi}_g)}{\omega_r}$, then there exists $k > 0$ such that $\lim_{t \to +\infty} \rho_{rg}(t) \leq D, \forall (x_r(t_0), y_r(t_0)) \in \mathbb{R}^2$. Moreover, the tracking error bound $D$ can be reduced by increasing the value of the angular velocity saturation $\omega_r$.

**Proof.** To prove the tracking error is uniformly bounded by $D$, we restrict our attention to the case $\rho_{rg}(t) > D$. In this case, it follows from (5.2.6) that

$$
\dot{\phi}_{rg} \leq \frac{(\overline{\pi}_r + \overline{\pi}_g)}{D}.
$$

(5.3.4)

To avoid the saturation of control input $\omega_r$, we can chose

$$
k \leq \frac{\omega_r - (\overline{\pi}_r + \overline{\pi}_g)}{\pi}.
$$

(5.3.5)

Substituting (5.3.4) and (5.3.5) into (5.3.1), we can claim that $\omega_r$ avoids to violate the saturation bound. Hence

$$
\omega_r = -k\Phi(\theta_r - \phi_{rg}) + \dot{\phi}_{rg}.
$$

(5.3.6)

Consider the function $\Theta(\alpha)$ given by

$$
\Theta(\alpha) = \begin{cases} 
2\pi + \alpha & \text{if } \alpha \leq -\pi, \\
2\pi - \alpha & \text{if } \alpha > \pi, \\
\alpha & \text{otherwise.}
\end{cases}
$$

(5.3.7)

Differentiating (5.3.7) on both sides and following from (5.3.6), it is straightforward to verify that

$$
\dot{\Theta}(\theta_r - \phi_{rg}) = -k\Theta(\theta_r - \phi_{rg})
$$

(5.3.8)
Then let us consider the Lyapunov candidate

\[ V = P_a(\rho_{rg}) + \frac{1}{2} \bar{F}_a(\bar{v}_r + \bar{v}_g) \left( \frac{3}{\pi} \right)^2 \Theta^2(\theta_r - \phi_{rg}). \]  

(5.3.9)

It follows from (5.2.6), (5.3.1), and (5.3.8) that

\[
\dot{V} = \frac{\partial P_a}{\partial \rho_{rg}} (- \bar{v}_r \cos(\theta_r - \phi_{rg}) + v_g \cos(\theta_g - \phi_{rg})) \\
- \bar{F}_a(\bar{v}_r + \bar{v}_g) \left( \frac{3}{\pi} \right)^2 \Theta^2(\theta_r - \phi_{rg}).
\]  

(5.3.10)

In the case \( \Theta(\theta_r - \phi_{rg}) \geq \frac{\pi}{3} \), following from (5.3.10) and (5.3.7), we can obtain

\[
\dot{V} \leq \frac{\partial P_a}{\partial \rho_{rg}} (\bar{v}_r + v_g) - \bar{F}_a(\bar{v}_r + \bar{v}_g) < 0.
\]  

(5.3.11)

On the other hand, in the case \( \Theta(\theta_r - \phi_{rg}) < \frac{\pi}{3} \), following from (5.3.10), under assumption \( 2 (\bar{v}_r > 2 \bar{v}_g) \), we can obtain

\[
\dot{V} \leq \frac{\partial P_a}{\partial \rho_{rg}} (- \frac{1}{2} \bar{v}_r + v_g) - \bar{F}_a(\bar{v}_r + \bar{v}_g) \left( \frac{3}{\pi} \right)^2 \Theta^2(\theta_r - \phi_{rg}) \\
< 0.
\]  

(5.3.12)

Adding (5.3.11) and (5.3.12) together, we can show that \( \dot{V} < 0 \) as long as \( \rho_{rg}(t) > D \). Noting (5.3.5), \( D \) can be chosen smaller by increasing \( \bar{\omega}_r \), which implies a smaller tracking error bound \( D \) can be achieved with a larger \( \bar{\omega}_r \).

Remark 5.3.1. Considering the regulation problem, without loss of generality, let us choose \( q_g = [0, 0, 0] \). Thus (5.2.6) reduces to

\[
\begin{align*}
\dot{\rho}_{rg} &= -v_r \cos(\theta_r - \phi_{rg}) \\
\dot{\phi}_{rg} &= -v_r \sin(\theta_r - \phi_{rg}) / \rho_{rg} \\
\dot{\theta}_r &= \omega_r
\end{align*}
\]  

(5.3.13)

Compared with (5.2.1), (5.3.13) is still a nonholonomic system defined in polar coordinates, which can also be rewritten into the standard driftless nonholonomic form as

\[
\begin{bmatrix}
\dot{\rho}_{rg} \\
\dot{\phi}_{rg} \\
\dot{\theta}_r
\end{bmatrix} =
\begin{bmatrix}
- \cos(\theta_r - \phi_{rg}) & 0 & -v_r \\
- \sin(\theta_r - \phi_{rg}) / \rho_{rg} & 0 & 0 \\
\rho_{rg} & 0 & 1
\end{bmatrix}
\begin{bmatrix}
v_r \\
\omega_r
\end{bmatrix}.
\]  

(5.3.14)
Correspondingly, we redesign the tracking control which is given by

\[ v_r = \text{sat} \left( \min \{ \rho_{rg}, 1 \} \cdot k_v, \overline{v}_r \right), \]
\[ \omega_r = \text{sat} \left( -k\Phi (\theta_r - \phi_{rg}) + \dot{\phi}_{rg}, \overline{\omega}_r \right), \]  
(5.3.15)

where \( k_v < \overline{\omega}_r \) is a positive constant.

Considering (5.3.13) under control (5.3.15), let \( k \leq \frac{\overline{\omega}_r - k_v}{\pi} \). Asymptotic position tracking can be concluded by essentially the same proof as shown above.

### 5.3.2 Obstacle Avoidance Control Design

As shown in Figure 5.3, with consideration of the angular velocity saturation \( \overline{\omega}_r \), the idea of avoidance control design is to steer orientation angle \( \theta_r \) to track line-of-sight \( \phi_{oir} \) as soon as possible. Therefore, the robot’s avoidance control is given by

\[ v_r = \overline{v}_r \]
\[ \omega_r = \text{sat} \left( -k_o \Phi (\theta_r - \phi_{oir}) + \dot{\phi}_{oir}, \overline{\omega}_r \right), \]  
(5.3.16)

where \( k_o \) is a positive constant.

![Figure 5.3: Illustration of obstacle avoidance control strategy](image)

**Theorem 5.3.2.** Consider system (5.2.1) under control (5.3.16) and suppose assumptions 9 and 10 hold. There exists a pair \((D_o, k_o)\) such that \( \rho_{oir}(t) > (R_{io} + R_r), \forall t \geq t_0 \) provided that \( \rho_{oir}(t_0) \geq D_o + (R_{io} + R_r) \).

**Proof.** Following from (5.3.8), we have

\[ \Theta (\theta_r(t) - \phi_{oir}(t)) = \Theta (\theta_r(t_0) - \phi_{oir}(t_0)) e^{-k_o(t-t_0)}. \]  
(5.3.17)
In order to prove the obstacle avoidance, let us consider the worst case that \(\Theta (\theta_r (t_0) - \phi_{\text{air}} (t_0)) = \pi\), \(v_r = \overline{v}_r\), and \(v_{\text{oi}} = \overline{v}_{\text{oi}}\). Let \(\tau = \frac{\ln 3}{k_o}\), it follows from (5.3.17) that

\[
\Theta (\theta_r (t) - \phi_{\text{air}} (t)) \leq \frac{\pi}{3} \quad (t \geq t_0 + \tau). \tag{5.3.18}
\]

Under the assumption 2 \((\overline{v}_r > 2 \overline{v}_{\text{oi}})\), substituting (5.3.18) into (5.2.7), we can obtain

\[
\dot{\rho}_{\text{air}} (t) < 0 \quad (t \geq t_0 + \tau). \tag{5.3.19}
\]

Therefore, following from (5.3.16) and (5.3.19), a conservative design to achieve obstacle avoidance boils down to solve a pair \((D_o, k_o)\) to satisfy the following inequality

\[
D_o - \frac{\ln 3}{k_o} (v_r + v_{\text{oi}}) > 0. \tag{5.3.20}
\]

To avoid the saturation of control action \(\omega_r\), following from (5.2.7) and (5.3.16), \((D_o, k_o)\) should also satisfy the inequality

\[
k_o \pi + \frac{\overline{v}_r + \overline{v}_{\text{oi}}}{D_o - \frac{\ln 3}{k_o} (\overline{v}_r + \overline{v}_{\text{oi}})} < \overline{\omega}_r. \tag{5.3.21}
\]

For an example, \((D_o, k_o)\) can be given by

\[
k_o = \frac{\overline{\omega}_r}{2\pi} \quad \text{and} \quad D_o = (2 + 2\pi \ln 3) \frac{v_r + v_{\text{oi}}}{\overline{\omega}_r}. \tag{5.3.22}
\]

Let us consider the Lyapunov candidate as

\[
E = P_r (\rho_{\text{air}}) + \frac{1}{2} \Theta^2 (\theta_r - \phi_{\text{air}}). \tag{5.3.23}
\]

It follows from (5.2.7), (5.3.8), and (5.3.16) that

\[
\dot{E} = \frac{\partial P_r}{\partial \rho_{\text{air}}} (-v_{\text{oi}} \cos (\theta_{\text{oi}} - \phi_{\text{air}}) + v_r \cos (\theta_r - \phi_{\text{air}}))
- k_o \Theta^2 (\theta_r - \phi_{\text{air}}). \tag{5.3.24}
\]

Note that the set \(\{(\rho_{\text{air}}, \phi_{\text{air}}, \theta_r) \mid \Theta (\theta_r - \phi_{\text{air}}) \leq \frac{\pi}{3}\}\) is an invariant set (from 5.3.18). Following from (5.3.24), \(E\) is nonincreasing. Thereby we can come to the conclusion \(\rho_{\text{air}} (t) > (R_{\text{io}} + R_r), \forall t \geq t_0\) provided that \(\rho_{\text{air}} (t_0) \geq D_o + (R_{\text{io}} + R_r)\). \qed
5.3.3 Switching Strategy

In the above two subsections, we suggest tracking control and avoidance control respectively. And the switching strategy is that the avoidance control (5.3.16) is activated when $\rho_{oir} \leq D_o + (R_{io} + R_r)$; otherwise, the tracking control (5.3.1) is activated. Therefore, it is obvious that the transient process will occur when $\Theta (\varphi_{rg} - \varphi_{oir}) > \frac{\pi}{2}$ and $\rho_{oir} \leq D_o + (R_{io} + R_r)$. Similar to the local minima problem inherent in potential field methods, a natural problem of the proposed switching strategy is to identify whether the transient process will last forever or not. A graphical exploration is given below to address this issue.

As shown in Figure 5.4, to avoid the obstacle, the bold arrows indicate the desired orientation angles when $\rho_{oir} \leq D_o + (R_{io} + R_r)$. Otherwise, the mobile robot tries to align its heading with the thin arrows which point to the target. Therefore, the worst case is $\Theta (\varphi_{rg} - \varphi_{oir}) = \pi$ and $\rho_{oir} \leq D_o + (R_{io} + R_r)$. Considering the following simulation scenario in which $(x_r(t_0), y_r(t_0)) = (0, 0)$, $(x_o(t), y_o(t)) = (15, 15)$, and $(x_g(t), y_g(t)) = (25, 25)$. The obstacle and target are static. The simulation parameters are the same as the parameters...
discussed in Section 5. As shown in Figure 5.5. The robot will make a detour and converge to the goal successfully.

![Figure 5.5: Illustration of transient process caused by switching](image)

**Remark 5.3.2.** Now we discuss the extended applicability of the proposed algorithms to the differential drive vehicle, whose control inputs are the angular velocities $\omega_R$ and $\omega_L$ of the right and left wheel respectively. Let $r$ be the wheel radius and $d$ be the axis length. Then a one-to-one mapping into the driving and steering velocities $v_r$ and $\omega_r$ is given by

$$v_r = r (\omega_R + \omega_L)/2, \quad \omega_r = r (\omega_R - \omega_L)/d.$$  \hfill (5.3.25)

In view of the bounded velocity of the motors, each wheel can achieve a maximum angular velocity $\Omega$. Therefore, following from (5.3.25), to extend the controls (5.3.1) and (5.3.16) to differential drive vehicles boils down to the following condition:

$$r (\omega_R + \omega_L) = 2 \cdot v_r; \quad r (\omega_R - \omega_L) \in [0, d \cdot \omega_r].$$

which can be further simplified into

$$\overline{v}_r \geq \frac{d \cdot \omega_r}{2}, \quad \Omega \geq \frac{2v_r + d \cdot \omega_r}{2r}.$$  

### 5.4 Simulations

This section describes the simulation results of a unicycle mobile robot. The parameters used for these simulations are: $R_r = 1 \text{ m}$, $R_s = 2 \text{ m}$, $R_{oi} = 1 \text{ m}$, $k = k_o = \frac{3}{\pi}$, $D = 0.1 \text{ m}$, and $D_o = 1 \text{ m}$. Moreover, the initial location of the vehicle is $(0, 0)$, $\theta_r(0) = 0 \text{ rad}$, $v_r(0) = 0.2$
m/s, and $\omega_r(0) = 0$ rad/s. And the bounds on the linear velocity and angular velocity are $\bar{v}_r = 0.2$ m/s and $\bar{\omega}_r = 6$ rad/s.

The desired trajectory $(x_g(t), y_g(t))$ is given by

$$
\begin{align*}
x_g(t) &= \frac{\sqrt{2}}{2} x'(t) + \frac{\sqrt{2}}{2} \sin (0.15x'(t)) \\
y_g(t) &= \frac{\sqrt{2}}{2} x'(t) - \frac{\sqrt{2}}{2} \sin (0.15x'(t))
\end{align*}
$$

where $x'(t) = 0.09t$. It can be easily verified that $\bar{v}_g < 0.1$ m/s.

### 5.4.0.1 Target Tracking and Collision Avoidance with Static Obstacles

There are three static obstacles (4,4,1), (15,15,1), and (25,20,1) in the workspace. The simulation result is illustrated in Figure 5.6.

![Figure 5.6: Collision avoidance with static obstacles](image)

When $T = 200$ s, the target falls behind the robot. Therefore, the conflict between target tracking and obstacle avoidance results in the circular-like movement of the unicycle robot.

---

1 Data format: (center position, radius). For example, (4,4) denotes the center position. The radius is 1 m.
5.4.0.2 Target Tracking and Collision Avoidance with Moving Obstacles

Compared with example 1, three moving obstacles of radius being 1 m are also taken into consideration. The simulation result is shown in Figure 5.7. In this simulation, \( v_{oi}(t) = 0.1 \) m/s.

![Figure 5.7: Collision avoidance with moving obstacles](image)

5.5 Conclusions

In this chapter, we proposed a reactive control solution for unicycle-type robot to achieve virtual command vehicle tracking and collision avoidance. Examples through simulation confirm the effectiveness of Lyapunov design of multi-objective control for the unicycle-type robot proposed in Section 3. Future research will consider more complex nonholonomic vehicle models to accommodate a larger class of mobile robots.
In this chapter, we propose a nonlinear control design for a team of wheeled mobile robots to cooperatively explore in a dynamic environment to track their virtual leader(s), while avoiding static and dynamic obstacles. The multi-objective control problem is firstly formulated, and then the control is synthesized by generating a potential field force for each objective and incorporating them through analysis and design. To the best of our knowledge, our proposed design is the first systematic approach to accommodate and achieve the multiple objectives of cooperative motion, tracking virtual command vehicle(s), obstacle avoidance, and oscillation suppression. Using rigorous Lyapunov analysis and theoretical proof, basic conditions and key properties are derived. The validity and effectiveness are illustrated by several simulation examples including cooperative motion of a team of vehicles moving through urban settings with static and moving obstacles, as well as narrow passages.

6.1 Introduction

Recent years have witnessed a boom of real applications such as cooperative robot reconnaissance [67], marine mine-sweeping [72], and formation flight control [69, 76] which are implemented with distributed autonomous multi-agent systems, which require formation movement capability. To achieve this goal, the central and difficult issues are:

- cooperative formation movement control of multi-robots;
• collision avoidance arising in the dynamic environment;

• coupling between the above two areas.

6.1.1 Formation Movement Control of Multi-robots

Most existing methods dealing with formation control can be classified into: behavior based, virtual structure, or leader-follower.

The idea of behavior based approach [67, 73] is that, a number of basic goal oriented behaviors (e.g., move-to-goal, avoid-static-obstacle, avoid-robot and maintain-formation) are proposed to each robot. A weighting factor is introduced to indicate the relative importance of the above proposed individual behaviors. The high-level combined behavior is actually generated by multiplying the outputs of each primitive behavior by its weight, then summing and normalizing the results. The advantage of behavior based approaches is that each primitive behavior has its physical meaning and the formation feedback can be incorporated into the group dynamics by coupling the outputs of the related individual behavior. The drawback is that it is difficult to formalize and analyze the group dynamics mathematically, consequently it is difficult to study the convergence of the formation to a desired geometric configuration.

The idea of virtual structure approach [68, 69, 70, 71] is inspired by the rigid body movement of a physical object with all points in the object maintaining a fixed geometric relationship due to a system of physical constraints. The robot formation is considered as a single virtual rigid structure. Thus, instead of assigning desired trajectories to each single robot, the entire formation as a whole by a trajectory generator is specified. The formation
is maintained by minimizing the error between the virtual structure and the current robot position. The advantage of virtual structure approaches is that it is quite easy and straightforward to prescribe the coordinated behavior of the whole team. The weak point is that the virtual structure's position is determined by the positions of the robots, which makes the formation control itself, be the centralized control.

The idea of the leader-follower approach [74, 75, 76] is that, some robots are selected as leaders moving along the predefined reference trajectories while the rest robots are said to be followers and are desired to maintain a expected posture (distance and orientation) relative to their own leaders. In general, the leader-follower controls take the following forms: (1) a single leader vehicle and multiple follower vehicles or (2) a “chain” of vehicles each following the preceding vehicle (such as in automated control of highway systems). In the leader-follower approach, the controls reduce to a tracking problem which can be designed and analyzed using standard control theoretic techniques. Because the leader’s predefined trajectory is independent of the motion of each associated follower, the disadvantage of the leader-follower approach is that the formation does not tolerate leader faults.

6.1.2 Collision Avoidance

Obstacle avoidance is a fundamental issue in mobile robotics area. Extensive studies had been conducted at the navigation system level (path planning). The objective of obstacle avoidance always comes with target tracking. Most existing methods of path planning can be classified into two strategies: graph methods and potential field methods. Graph methods are based on a geometrical cell-decomposition of the whole configuration space and generate
an optimal path with respect to certain objective criteria, for an instance finding the shortest collision-free path. The main criticism to graph methods is that these methods require much computational cost. In the potential field method, the target applies an attractive force to the robot while the obstacles exert a repulsive force onto the robot. The composite force determines the movement of the robot. Because of its simplicity, elegance and high efficiency, the potential field method is particularly popular. [7] point out some inherent issues of potential field method, including the following: (1) trap situations due to local minima; (2) no passage between closely spaced obstacles; (3) oscillations in the presence of obstacles; and (4) oscillations in narrow passages.

On the other hand, obstacle avoidance is addressed directly in the kinematics/dynamics controller, which is normally called avoidance control. Given a dynamical system, avoidance control is defined as a control design which ensures that every trajectory that origins from outside of the prescribed avoidance set will never enter into the set. The potential field method and Lyapunov technique are applied in the design of avoidance controls. Leitmann and his coworkers [78, 79] pioneered and extensively studied the problem of avoidance control for a single dynamical system. In [78], sufficient conditions were given to avoid the set for all the time. In their later work [79], two special cases of avoidance problems are considered: the set must be avoided during a prescribed time interval (finite-time avoidance), or the set must be avoided for all time after some prescribed time interval (ultimate avoidance). Sufficient conditions are presented for these two kinds of avoidance. A generalization of avoidance control for multi-agent dynamic systems is studied in [77]. Sufficient conditions are provided for a class of nonlinear dynamic systems with a special decomposed structure.
6.1.3 Coupling of Formation Control and Collision Avoidance

Two frameworks are presented in the literature to tackle the problem. One is the aforementioned behavior based method, in which collision avoidance of obstacles and other robots is designed as one of primitive behaviors. As mentioned previously, it is mathematically difficult to formalize and analyze the group dynamics. Consequently, it is difficult to prove convergence to the desired formation and improve the robot’s transient performance.

The other one is leader-follower formation control based on potential field and Lyapunov direct methods (e.g., [38, 39, 40]). In particular, potential fields yield interaction forces between neighboring robots to ensure a desired minimum distance for each pair of robots. A virtual leader acts as a moving reference point that applies forces on its neighboring robots. The aim of the virtual leaders is to manipulate the vehicle group behavior [38]. A properly designed potential field function yields global asymptotic convergence of a group of mobile robots to a desired formation, and guarantees no collisions among the robots [39]. However, these two methods do not consider the obstacle avoidance issue. The leader-follower strategy essentially transforms the formation control problem into a tracking problem. Based on this fact, the decentralized controls are proposed to achieve target tracking and collision avoidance for a single robot. It is then extended to address the problem of coordinated tracking of a group of robots [40]. The moving obstacle is not considered in this method. And this method only ensures the tracking with a bounded error.
6.1.4 Outline of this chapter

This chapter addresses cooperative formation control with collision avoidance. Firstly, the target tracking and collision avoidance problems are investigated for a single agent. Instead of directly extending the single agent controls to the multi-agents case, we combine it with the cooperative control design proposed in [80]. The proposed decentralized control is reactive and allows topological changes of the communication networks. Since the proposed control is based on a potential field method, its inherent oscillation problem is also studied to improve group transient performance.

This chapter is organized as follows: Section 2 formulates the problem of achieving a specified formation among a team of mobile robots, tracking their leader(s), avoiding the static/moving obstacles as well as each other, and suppressing the excessive oscillations. Unifying time-varying potential field, nonlinear damping, and velocity-scaled force control, Section 3 proposes a novel analytical control design for a single point-mass agent to achieve target tracking and collision avoidance. Using rigorous Lyapunov analysis and theoretical proof, basic conditions and key properties are derived. Section 4, extends the results for networked agents through incorporating an existing cooperative control design [80]. Section 5 presents examples and their simulations to illustrate the design process and its effectiveness of proposed controls. Finally, Section 6 concludes the chapter and suggests some future research directions.
6.2 Problem Formulation

Consider a collection of point-mass agents whose dynamics are given by

\[
\dot{q}_{\mu} = v_{\mu}, \quad \dot{v}_{\mu} = u_{\mu}, \quad (\mu = 1, \ldots, m)
\]

(6.2.1)

where \( q \triangleq [x, y]^T \) denotes the center position, \( v \triangleq [v_x, v_y]^T \) represents the velocity, and \( u \) is the control input. Thus we can define the states \( S(t) = (q(t), v(t)) \). Subscripts \( r, g \) and \( o \) indicate the vehicle, goal and obstacle respectively.

Figure 6.1: Illustration of cooperative formation movement with collision avoidance (Three agents are required to maintain a triangular formation and track the goal in the presence of obstacles)

Considering the initial configurations \( S_{r\mu}(t_0) = (q_{r\mu}(t_0), v_{r\mu}(t_0)) \), as shown in Figure 6.1, the objective of this chapter can be summarized as follows:

- tracking the specified virtual leader \( S_{g\mu}(t) = (q_{g\mu}(t), v_{g\mu}(t)) \);
- avoiding the \( n \) obstacles \( S_{oi} = (q_{oi}(t), v_{oi}(t)) \) \((i = 1, 2, \cdots n)\);
• avoiding the remaining \((m - 1)\) agents \(S_{rj} = (q_{rj}(t), v_{rj}(t))\),
\((j = 1, 2, \ldots, \mu - 1, \mu + 1, \ldots, m)\);

• suppressing the oscillation of the system trajectory.

Without loss of generality, we make the following assumptions:

• The \(\mu\)th agent under consideration is represented by a 2-D circle with the center at \(q_{r\mu}(t)\) and of radius \(R\). The range of its sensors is also described by a circle centered at \(q_{r\mu}(t)\) and of radius \(R_s\).

• The \(i\)th static/moving obstacle will be represented by a convex object of any shape (such as circle, ellipse, or polygon).

### 6.3 Target Tracking and Collision Avoidance for a Single Agent

First, using Lyapunov-type analysis, a decentralized feedback control is derived for a single robot, which guarantees collision avoidance and tracking of a virtual leader. Then in Section 4, this result is extended to the case of networked agents through incorporating cooperative control [80]. We thereby propose a novel cooperative formation control design with collision avoidance.

To accomplish above design objectives, two potential field functions are used to generate reactive forces. Specifically, consider the following composite potential function:

\[
P(q_r - q_o, q_r - q_g) = P_a(q_r - q_g) + P_r(q_r - q_o),
\]  
(6.3.1)

where \(P_a(\cdot)\) is the attractive potential function and \(P_r(\cdot)\) is the repulsive potential function. Intuitively and necessarily, potential functions should have the properties that
Figure 6.2: Typical attractive potential function versus repulsive potential function
(a: attractive potential field function; b: contour lines of attractive potential field function; c: repulsive potential field function; d: contour lines of repulsive potential field function)

\[
\begin{align*}
\begin{cases}
P_a(0) &= 0, \quad \nabla P_a(s) \big|_{s=0} = 0, \\
0 < P_a(s) < \infty & \text{if } s \neq 0 \text{ and } \|s\| \text{ is finite}, \\
\|\nabla P_a(s)\| &< +\infty \text{ if } \|s\| \text{ is finite},
\end{cases}
\end{align*}
\]

(6.3.2)

and

\[
\begin{align*}
\begin{cases}
P_r(s) &= +\infty \quad \text{if } s \in \Omega_o, \\
P_r(s) &= 0 \quad \text{if } s \not\in \overline{\Omega}_o, \\
P_r(s) &\in (0, \infty) \quad \text{if } s \in \overline{\Omega}_o \text{ but } s \not\in \Omega_o, \\
\lim_{s \to \Omega_o} \|\nabla P_r(s)\| &= +\infty \quad \text{if } s \not\in \Omega_o,
\end{cases}
\end{align*}
\]

(6.3.3)

where \( \Omega_o \subset \mathbb{R}^2 \) is a compact set representing the 2-dimensional shape of the obstacle, \( \overline{\Omega}_o \) is the compact set which is an enlarged version of \( \Omega_o \) and in which repulsive force becomes
active. The above defined attractive potential function and repulsive potential function are exemplified by Figure 6.2.

Furthermore, commonsense dictates that an additional detour force could be used to easily drive vehicle to make a detour and reach its goal. Therefore, a novel conception “unit detour force vector” \( T(q_r - q_o) \) is introduced, which has the properties that

\[
\nabla P_r^T(s) T(s) = 0, \quad -\nabla P_a^T(s) T(s) \geq 0, \quad \text{and} \quad \|T(s)\| = 1. \quad (6.3.4)
\]

Let the vehicle control to be a reactive control of the form

\[
\begin{align*}
  u_\mu &= -\nabla P_a (q_r - q_g) - \nabla P_r (q_r - q_o) + f_d (q_r - q_o) T(q_r - q_o) \\
  &\quad - \xi (q_r - q_g) (v_r - v_g) + \dot{v}_g - \dot{\eta} (q_r - q_o) \|v_g - v_o\|^2 \\
  &\quad - 2\eta (q_r - q_o) (v_g - v_o)^T (\dot{v}_g - \dot{v}_o),
\end{align*}
\]

where the term \( \nabla P_a (q_r - q_g) \) and \( \nabla P_r (q_r - q_o) \) are the standard reactive control components, \( \xi(\cdot) > 0 \) is a locally uniformly bounded function designed to ensure stability and damp oscillations, \( f_d (s) > 0 (f_d (s) = 0, \text{if } s \notin \bar{\Omega}_o) \) is a locally uniformly bounded function designed to render detouring force in the vicinity of the obstacle, and \( \eta(\cdot) \) is the force to resolve the potential conflict between goal tracking and collision avoidance. Vector function \( \eta(\cdot) \) has the properties that

\[
\eta(s) \Delta \begin{bmatrix} \eta_1(s) \\ \eta_2(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{if } s \notin \bar{\Omega}_o, \quad (6.3.6)
\]

and

\[
\lim_{s \notin \Omega_o, s \to \Omega_o} \frac{\eta^T(s) \nabla P_r(s)}{\|\nabla P_r(s)\|} = +\infty. \quad (6.3.7)
\]
6.3.1 Tracking of a Virtual Leader

The tracking problem is to ensure that the \(\mu\)th agent will converge to the goal position \(q_{g\mu}(t)\) of the \(\mu\)th virtual leader. We proposed the following lemma which provides the basic result.

**Lemma 1:** The state of system (6.2.1) under control (6.3.5) converges asymptotically to that of the virtual vehicle provided that, after a finite time instant \(t^*\), \([q_g(t) - q_o(t)] \notin \bar{\Omega}_o\) for all \(t \geq t^*\). If \([q_g(t) - q_o(t)]\) stays in or intermittently returns to \(\bar{\Omega}_o\), there is no convergence of \([q_r(t) - q_g(t)]\) → 0.

**Proof.** It follows from (6.2.1), (6.3.1) and (6.3.5) that the tracking error system is

\[
\begin{align*}
\dot{e}_{1\mu} &= e_{2\mu} \\
\dot{e}_{2\mu} &= -\nabla P_a(e_{1\mu}) - \nabla P_r(e_{1\mu} + q_{g\mu} - q_o) \\
&\quad + f_d(e_{1\mu} + q_{g\mu} - q_o) T(e_{1\mu} + q_{g\mu} - q_o) - \xi(e_{1\mu}) e_{2\mu} \\
&\quad - \left[ \nabla \eta_1(e_{1\mu} + q_{g\mu} - q_o) T + \nabla \eta_2(e_{1\mu} + q_{g\mu} - q_o) T \right] (e_{2\mu} + v_{g\mu} - v_{oi}) \| v_{g\mu} - v_{oi} \|^2 \\
&\quad - 2 \left[ \nabla \eta_1(e_{1\mu} + q_{g\mu} - q_o) T + \nabla \eta_2(e_{1\mu} + q_{g\mu} - q_o) T \right] (v_{g\mu} - v_{oi})^T (\dot{v}_{g\mu} - \dot{v}_{oi}).
\end{align*}
\]

where \(e_{1\mu} = q_r - q_{g\mu}\) and \(e_{2\mu} = v_r - v_{g\mu}\). It is straightforward to verify that, if \([q_{g\mu}(t) - q_o(t)] \in \bar{\Omega}_o\), \(e_{1\mu} = e_{2\mu} = 0\) is not an equilibrium point of the error system and hence no convergence can be achieved.

Furthermore, from the geometric point of view, the resultant force vector field yielded by the proposed control can be shown in Figure 6.3 when \((q_r - q_o) \notin \bar{\Omega}_o\). As we had mentioned in Section 2, the obstacle is assumed to be a convex object. As shown in Figure 6.3, once the robot is in the set \(\bar{\Omega}_o\), it will be repelled away from the obstacle and then make a detour to converge to its goal. Finally, the agent will not stay in or intermittently be in \(\bar{\Omega}_o\) which implies \([q_{g\mu}(t) - q_o(t)] \notin \bar{\Omega}_o\) for all \(t \geq t^*(t^* > t^*)\).

In this case, by properties (6.3.3) and (6.3.6), the system error reduces to

\[
\dot{e}_{1\mu} = e_{2\mu}, \quad \dot{e}_{2\mu} = -\nabla P_a(e_{1\mu}) - \xi(e_{1\mu}) e_{2\mu}.
\]

Let us consider the Lyapunov candidate

\[
L_1(t) = P_a(e_{1\mu}) + \frac{1}{2} \| e_{2\mu} \|^2.
\]

It follows that

\[
\dot{L}_1 = e_{2\mu}^T \nabla P_a(e_{1\mu}) + e_{2\mu}^T [-\nabla P_a(e_{1\mu}) - \xi(e_{1\mu}) e_{2\mu}]
\]

\[
= -\xi(e_{1\mu}) \| e_{2\mu} \|^2.
\]
which is negative semi-definite. Using LaSalle’s invariant set theorem [22], asymptotic stability of $e_{1\mu}$ and $e_{2\mu}$ can be concluded.

\section{6.3.2 Obstacle Avoidance}

The obstacle avoidance problem is to ensure that the agent will not enter the given compact set $\Omega_o$ provided that certain initial conditions hold. The following lemma provides the basic result.

\textbf{Lemma 2:} Suppose that potential field function (6.3.1) satisfies properties (6.3.2) and (6.3.3). Then, as long as the initial condition is not in set $\Omega_o$, system (6.2.1) under control (6.3.5) is collision-free provided that $v_o(t)$ and $v_g(t)$ are uniformly bounded and that $q_o(t)$ is uniformly bounded.

\textbf{Proof.} Let us choose the following Lyapunov candidate:

\[ V_1(t) = \frac{1}{2} \left\| v_{r\mu} - v_{g\mu} + \eta (q_{r\mu} - q_{o\mu}) \right\| v_{g\mu} - v_{o\mu} \right\|^2 + P(q_{r\mu} - q_{g\mu}, q_{r\mu} - q_{o\mu}). \]
It follows from (6.2.1) and (6.3.5) that

\[
\begin{align*}
\dot{V}_1 &= \left[ v_{\mu} - v_{\eta} + \eta (q_{r\mu} - q_{oa}) \|v_{\mu} - v_{\eta}\|^2 \right]^T \left[ \dot{v}_{\mu} + \dot{\eta} (q_{\eta} - q_{oa}) \|v_{\mu} - v_{\eta}\|^2 \right] \\
&\quad - \dot{v}_{\eta} + 2\eta (q_{r\mu} - q_{oa}) (v_{\mu} - v_{\eta})^T (\dot{v}_{\eta} - \dot{\eta}) \\
&\quad + (v_{\mu} - v_{\eta})^T \nabla P_a (q_{r\mu} - q_{\eta}) + (v_{\mu} - v_{\eta})^T \nabla P_r (q_{r\mu} - q_{oa}) \\
&= \left[ v_{\mu} - v_{\eta} + \eta (q_{r\mu} - q_{oa}) \|v_{\mu} - v_{\eta}\|^2 \right]^T [-\nabla P_a (q_{r\mu} - q_{\eta}) - \nabla P_r (q_{r\mu} - q_{oa}) + f_d (q_{r\mu} - q_{oa}) T (q_{r\mu} - q_{oa}) - \xi (q_{r\mu} - q_{\eta}) (v_{\mu} - v_{\eta})] \\
&\quad + (v_{\mu} - v_{\eta})^T \nabla P_a (q_{r\mu} - q_{\eta}) + (v_{\mu} - v_{\eta})^T \nabla P_r (q_{r\mu} - q_{oa}) \\
&= \omega (q_{r\mu} - q_{oa}, v_{\mu} - v_{\eta}, v_{\mu} - v_{\eta}) \\
&\quad - \xi (e_{\mu}) \|e_{\eta}\|^2 - \eta^T (q_{r\mu} - q_{oa}) \nabla P_r (q_{r\mu} - q_{oa}) \|v_{\mu} - v_{\eta}\|^2 ,
\end{align*}
\]

(6.3.8)

where

\[
\omega (q_{r\mu} - q_{oa}, q_{r\mu} - q_{oa}, v_{\mu} - v_{\eta}, v_{\mu} - v_{\eta}) \\
= (v_{\mu} - v_{\eta})^T \nabla P_r (q_{r\mu} - q_{oa}) + (v_{\mu} - v_{\eta})^T f_d (q_{r\mu} - q_{oa}) T (q_{r\mu} - q_{oa}) \\
- \eta^T (q_{r\mu} - q_{oa}) \nabla P_a (q_{r\mu} - q_{\eta}) \|v_{\mu} - v_{\eta}\|^2 \\
- \eta^T (q_{r\mu} - q_{oa}) \xi (q_{r\mu} - q_{\eta}) (v_{\mu} - v_{\eta}) \|v_{\mu} - v_{\eta}\|^2 \\
+ \eta^T (q_{r\mu} - q_{oa}) f_d (q_{r\mu} - q_{oa}) T (q_{r\mu} - q_{oa}) \|v_{\mu} - v_{\eta}\|^2 .
\]

Recall that \(\|v_{\mu} - v_{\eta}\|\) and \(\|q_{oa}(t)\|\) are uniformly bounded. It follows from (6.3.2) that \(\nabla P_a (q_{r\mu} - q_{oa})\) and \(\xi (q_{r\mu} - q_{\eta})\) are uniformly bounded for \((q_{r\mu} - q_{oa}) \in \bar{\Omega}_o\). Hence, there exist constants \(c_1, c_2 \geq 0\) such that

\[
|\omega (q_{r\mu} - q_{oa}, q_{r\mu} - q_{oa}, v_{\mu} - v_{\eta}, v_{\mu} - v_{\eta})| \\
\leq c_1 \|\nabla P_r (q_{r\mu} - q_{oa})\| + c_2 \|\eta^T (q_{r\mu} - q_{\eta})\| \|v_{\mu} - v_{\eta}\| .
\]

Therefore, we know from (6.3.7) and (6.3.8) that,

\[
\lim_{(q_{r\mu} - q_{oa}) \to \Omega_o} \dot{V}_1 < 0 .
\]

Similarly, we also have

\[
\lim_{v_{\mu} \to \infty} \dot{V}_1 < 0 .
\]

We can claim \(V_1(t)\) is finite in any finite region for any finite initial conditions \((q_r(t_0), q_q(t_0), q_o(t_0), v(t_0), v_{\eta}(t_0))\). Thus, \(V_1(t)\) will stay finite under the initial and collision-free conditions mentioned in the lemma.

\[
\square \quad \square
\]

6.3.3 Oscillation Suppression of Potential Field Methods

In this section, firstly the nature of the inherent oscillation problem of potential field methods is investigated. Then the proposed control is illustrate why it is a remedy for this problem.
6.3.3.1 Oscillation Analysis

The causes of oscillations can be classified into the following three types:

1. *Potential field functions, especially the interaction between the attractive potential field function (attractor) and repulsive potential field function (repeller).*

When the distance between an obstacle and the goal is small compared to the distance between either and the vehicle, the two similar but opposite force fields can cause oscillation.

2. *Insufficient damping, especially in the nonlinear setting.*

Damping design serves two purposes: (1) Stabilize the system; (2) Suppress the oscillation.

For instance, if $\xi(\cdot) > 0$ is set to be zero, then the system (1) will never converge to the goal unless the initial condition is trivially given by $(q_{r, \mu}(t_0) = q_{g, \mu}(t_0), v_{r, \mu}(t_0) = v_{g, \mu}(t_0))$.

3. *Sampling and the gradient descent method.*

In the potential filed methods, a robot takes the negative gradient direction to determine a vector that points toward the target. Whenever we consider a discrete system and the potential contour is not perfectly circular, solutions tend to exhibit oscillation, in particularly, in proximity to obstacles or in narrow passages.

6.3.3.2 Quasi-Monotone Convergence

We begin with the following definitions.

**Definition 23.** Let $\alpha(t)$ be a scalar function. Function $\alpha(t)$ is (strictly) monotone decreasing over an interval if $\alpha(t_2) \leq \alpha(t_1)$ ($\alpha(t_2) < \alpha(t_1)$) for any $t_2 > t_1$ within the interval. Function $\alpha(t)$ is (strictly) monotone increasing if $-\alpha(t)$ is (strictly) monotone decreasing. Function $\alpha(t)$ is (strictly) monotone if $\alpha(t)$ is either (strictly) monotone increasing or (strictly) monotone decreasing. Function $\alpha(t)$ is (strictly) monotone convergent if it is (strictly) monotone and if $\lim_{t \to \infty} \alpha(t) = 0$. 

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Definition 24. Let $\alpha(t)$ be a scalar function. $\alpha(t)$ is one-swing quasi-monotone over an interval if the interval can be divided into two subintervals over each of which $\alpha(t)$ is monotone. Function $\alpha(t)$ is called to be one-swing quasi-monotone convergent if it is one-swing quasi-monotone and if $\lim_{t \to \infty} \alpha(t) = 0$.

Therefore, any monotone function is also one-swing quasi-monotone. The converse generally does not always hold true. The nontrivial case of $\alpha(t)$ being one-swing quasi-monotone convergent is that $\alpha(t)$ is convergent while $|\alpha(t)|$ is monotone increasing for $t \in [t_0, t_1]$ and monotone decreasing for $t \in [t_1, \infty)$. For dynamic systems of order higher than one, quasi-monotone convergence is generally best achievable upon successfully suppressing all the oscillations. The following lemma provides such a result.

**Lemma 3:** Suppose that differentiable function $\beta(\cdot) \in \mathbb{R}^2$ exists to satisfy the following properties:

$$
\beta(-s) = -\beta(s), \quad \beta^T(s)\beta(s) = \frac{P_a(s)}{2}, \quad \frac{\partial \beta(s)}{\partial s} = \frac{\xi(s)}{2}I.
$$

(6.3.9)

Assume that after a finite time instant $t^*$, $[q_g(t) - q_o(t)] \not\in \Omega_o$ for all $t \geq t^*$. Then, the error between the state of system (6.2.1) under control (6.3.5) and that of the virtual vehicle is one-swing quasi-monotone convergent.

**Proof.** Considering the state transformation

$$
z_1 = e_1, \quad z_2 = e_2 + \beta(e_1),
$$

The error dynamics can be rewritten as, as long as $[q_g(t) - q_o(t)] \not\in \Omega_o$

$$
\dot{z}_1 = -\beta(z_1) + z_2,
$$

$$
\dot{z}_2 = -\frac{\partial P_a(z_1)}{\partial z_1} + \frac{\partial \beta(z_1)}{\partial z_1} [z_2 - \beta(z_1)] - \xi(z_1)[z_2 - \beta(z_1)]
$$

$$
= -\frac{\partial P_a(z_1)}{\partial z_1} - \frac{\partial \beta(z_1)}{\partial z_1} \beta(z_1) + \xi(z_1)\beta(z_1)
$$

$$
- \left[\xi(z_1)I - \frac{\partial \beta(z_1)}{\partial z_1}\right] z_2.
$$

It follows from (6.3.9) that

$$
\dot{z}_1 = -\beta(z_1) + z_2,
$$

$$
\dot{z}_2 = \frac{\xi(z_1)}{2} z_2,
$$

from which one-swing quasi-monotone convergence can be concluded using Lemma 4. \qed
Remark 1: On one hand, the set $\bar{\Omega}_o$ can be chosen small such that the impact of the obstacle is confined to a small area to suppress the oscillation. On the other hand, the $\bar{\Omega}_o$ can not be too small due to the numerical calculation.

Remark 2: The term $f_d T$ in (6.3.5) generates the detour force which can greatly decrease the chance of oscillation, while speeding up the convergence to the target.

Remark 3: The controller sampling rates must be small to suppress the oscillation.

Lemma 4: Consider the differential equations

$$\dot{\alpha}_1(t) = -f_1(\alpha_1) + \alpha_2(t), \quad \dot{\alpha}_2(t) = -f_2(\alpha_2),$$

(6.3.10)

where functions $f_i(s)$ have the properties that $f_i(0) = 0$ and $df_i(s)/ds > 0$ for all $s \neq 0$.

Then, solution $\alpha_1(t)$ is one-swing quasi-monotone convergent.

Proof. It follows from the property of $f_2(\cdot)$ that

$$\frac{d\alpha_2^2}{dt} = -2\alpha_2 f_2(\alpha_2) < 0$$

and hence $\alpha_2(t)$ is strictly monotone convergent. Using the property of $f_1(\cdot)$, we know that

$$\epsilon(t) = f_1^{-1}(\alpha_2(t))$$

is well defined, that $\epsilon(t)$ is also strictly monotone convergent (either $\dot{\epsilon}(t) \geq 0$ with $\epsilon(t_0) \leq 0$ or $\dot{\epsilon}(t) \leq 0$ with $\epsilon(t_0) \geq 0$), and that

$$\dot{\alpha}_1(t) = -f_1(\alpha_1) + f_1(\epsilon(t))$$

(6.3.11)

$$\frac{d}{dt}[\alpha_1(t) - \epsilon(t)] = -[f_1(\alpha_1) - f_1(\epsilon(t))] - \dot{\epsilon}(t).$$

(6.3.12)

Choose $q > 1$ and function $h_2(s)$ such that $h_2(s)$ is a strictly monotone increasing function with $h_2(0) = 0$ and that

$$|h_2(s)f_2(s)|^{\frac{1}{q}} \geq |s|.$$  

Let $p > 1$ be the constant such that $1/p + 1/q = 1$, and select $h_1(s)$ such that $h_1(s)$ is a strictly monotone increasing function with $h_1(0) = 0$ and that

$$|f_1(s)|^{\frac{1}{p}} \geq \frac{1}{p}|h_1(s)|^{\frac{1}{q}}.$$  

Consider Lyapunov function

$$V = \frac{1}{p} \int_0^{\alpha_1} h_1(s)ds + \frac{2}{q} \int_0^{\alpha_2} h_2(s)ds.$$
It follows from Holder’s inequality $a^p/p + b^q/q \geq ab$ that, along the solution of (6.3.10),

$$
\dot{V} = - \frac{1}{p} h_1(\alpha_1) f_1(\alpha_1) + \frac{1}{p} h_1(\alpha_1) \alpha_2 - \frac{2}{q} h_2(\alpha_1) f_2(\alpha_2) \\
\leq - |h_1(\alpha_1) f_1(\alpha_1)|^{\frac{2}{3}} |h_2(\alpha_2) f_2(\alpha_2)|^{\frac{1}{3}} + \frac{1}{p} h_1(\alpha_1) \alpha_2 \\
- \frac{1}{q} h_2(\alpha_1) f_2(\alpha_2) \\
\leq - \frac{1}{q} h_2(\alpha_1) f_2(\alpha_2) \leq 0,
$$

from which convergence of $\alpha_2(t)$ can be concluded. One-swing quasi-monotone convergence is analyzed below by studying four distinct cases.

**Case 1:** $\dot{\epsilon}(t) \leq 0$ and $\alpha_1(t_0) \geq \epsilon(t_0) \geq 0$. In this case, we know from (6.3.12) that $\alpha_1(t) \geq \epsilon(t)$ for all $t \geq t_0$ and consequently from (6.3.11) that $\alpha_1(t)$ is monotone decreasing (and convergent).

**Case 2:** $\dot{\epsilon}(t) \leq 0$ and $\alpha_1(t_0) < \epsilon(t_0)$. In this case, we know from (6.3.11) that $\alpha_1(t)$ is monotone increasing while $\epsilon(t)$ is monotone decreasing. Hence, there exists time instant $t_1 \in [t_0, \infty]$ such that $\alpha_1(t_1) = \epsilon(t_1)$, that $\alpha_1(t) < \epsilon(t)$ for $t \in [t_0, t_1)$, and that evolution of $\alpha_1(t)$ over $[t_1, \infty)$ becomes that in case 1. Hence, $\alpha_1(t)$ is one-swing quasi-monotone convergent.

**Case 3:** $\dot{\epsilon}(t) \geq 0$ and $\alpha_1(t_0) \leq \epsilon(t_0) \leq 0$. This case is analogous to case 1 except that $\alpha_1(t)$ is monotone increasing (and convergent).

**Case 4:** $\dot{\epsilon}(t) \geq 0$ and $\alpha_1(t_0) > \epsilon(t_0)$. This case is parallel to case 2 except that, while convergent, $\alpha_1(t)$ is first monotone decreasing and then monotone increasing.

The proof is completed by summarizing all the cases. 

### 6.4 Cooperative Formation Control of Networked Agents with Collision Avoidance

#### 6.4.1 Cooperative Control for Networked Systems of Canonical Form

Consider a group of networked dynamic systems given by the following canonical form

$$
\dot{X}_i = A_i X_i + B_i U_i, \quad Y_i = C_i X_i, \quad \dot{\eta}_i = g_i(\eta_i, X_i), \quad (6.4.1)
$$

where $i = 1, \cdots, q$, $l_i \geq 1$ is an integer, $X_i \in \mathbb{R}^{l_i \times m}$, $\eta_i \in \mathbb{R}^{n_i - l_i \times m}$, $I_{m \times m}$ is the $m$ dimensional identity matrix, $\otimes$ denotes the Kronecker product, $J_k$ is the $k$th order Jordan canonical form
given by

\[
J_k = \begin{bmatrix}
-1 & 1 & 0 & \cdots & 0 & 0 \\
0 & -1 & 1 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots & \\
0 & 0 & \cdots & -1 & 1 & 0 \\
0 & 0 & \cdots & -1 & 1 & \\
0 & 0 & \cdots & 0 & -1 \\
\end{bmatrix} \in \mathbb{R}^{k \times k},
\]

where \( A_i = J_i \otimes I_{m \times m} \in \mathbb{R}^{(l_i \times m) \times (l_i \times m)} \), \( B_i = \begin{bmatrix} 0 \\ I_{m \times m} \end{bmatrix} \in \mathbb{R}^{(l_i \times m) \times m} \), \( C_i = \begin{bmatrix} I_{m \times m} & 0 \end{bmatrix} \in \mathbb{R}^{m \times (l_i \times m)} \), \( Y_i \in \mathbb{R}^m \) is the output, \( U_i \in \mathbb{R}^m \) is the cooperative control law to be designed, and subsystem \( \dot{\eta}_i = g_i(\eta_i, X_i) \) is input-to-state stable.

We consider the general case where exchange of output information among the vehicles occurs only intermittently and locally. To capture this information flow, let us define the following sensing/communication matrix and its corresponding time sequence \( \{t^*_{k} : k = 0, 1, \ldots\} \)

\[
S(t) = \begin{bmatrix}
S_1(t) \\
S_2(t) \\
\vdots \\
S_q(t)
\end{bmatrix} = \begin{bmatrix}
s_{11}(t) & s_{12}(t) & \cdots & s_{1q}(t) \\
s_{21}(t) & s_{22}(t) & \cdots & s_{2q}(t) \\
\vdots & \vdots & \ddots & \vdots \\
s_{q1}(t) & s_{q2}(t) & \cdots & s_{qq}(t)
\end{bmatrix}, \quad \left\{ \begin{array}{l}
S(t) = S(t^*_k), \quad \forall t \in [t^*_k, t^*_k+1] \\
S(k) \overset{\Delta}{=} S(t^*_k)
\end{array} \right. ,
\]

(6.4.2)

where \( s_{ii}(t) \equiv 1; s_{ij}(t) = 1 \) if the \( j \)th vehicle is known to the \( i \)th vehicle at time \( t \), and \( s_{ij}(t) = 0 \) otherwise, and \( t^*_0 \overset{\Delta}{=} t_0 \). Time sequence \( \{t^*_k\} \) and the corresponding changes in the row \( S_i(t) \) of matrix \( S(t) \) are detectable instantaneously and locally at the \( i \)th vehicle, but they are not predictable, prescribed or known \textit{a priori} or modeled in any way.
Cooperative controls proposed in this chapter are in the class of linear, piecewise constant, local feedback controls with feedback gain matrices $G_i(t) \triangleq [G_{i1}(t), \ldots, G_{iq}(t)]$, where $i = 1, \ldots, q$,

\[ G_{ij}(t) = G_{ij}(t^*_k), \quad \forall t \in [t^*_k, t^*_{k+1}) ; \]

\[ G_{ij}(k) \triangleq G_{ij}(t^*_k) = \frac{s_{ij}(t^*_k)}{\sum_{\eta=1}^{q} s_{i\eta}(t^*_k)} K_c, \quad j = 1, \ldots, q; \quad (6.4.3) \]

where $s_{ij}(t)$ are piecewise-constants as defined in (6.4.2) and $K_c \in \mathbb{R}^{m \times m}$ is a constant, nonnegative, and row stochastic matrix. That is, cooperative controls are of form

\[ U_i \triangleq \sum_{j=1}^{q} G_{ij}(t)[s_{ij}(t)y_j] = G_i(t)Y \quad (6.4.4) \]

where $Y = [Y^T_1, \ldots, Y^T_q]^T$. Although $S(t)$ is not known \textit{a priori} nor can it be modelled, $S(t)$ is piecewise constant, diagonally positive and binary, and the value of row $S_i(t)$ is known at time $t$ to the $i$th vehicle. The above choice of the feedback gain matrix block $G_{ij}(t)$ in terms of $s_{ij}(t)$ ensures that matrices $G_i(t)$ are row stochastic and that control is always local and implementable with only available information.

**Theorem 1:** Consider dynamics system in (6.4.1) and under cooperative control (6.4.4). Then systems of (6.4.1) exhibit a single cooperative behavior as,

\[ X_{ss} = 1_{N_q}cX(t_0) = c_01_{N_q}, \quad \text{and} \quad Y_{ss} = c_01_m, \quad c \in \mathbb{R}^{1 \times N_q}, \quad c_0 \in \mathbb{R}, \quad (6.4.5) \]

where $N_q = m \sum_{i=1}^{q} l_i$ provided that

i) Gain matrix $K_c$ is chosen to be irreducible and row stochastic.

ii) Systems in (6.4.1) have a sequentially complete sensing/communication.

**Proof.** Please refer to [80] for a detailed proof.
The single cooperative behavior described in (6.4.5) does not necessarily mean that, if 

\[ Y_{ss}^d = c_0^d 1_m, \]

the desired behavior represented by constant \( c_0^d \) is achieved. In order to ensure 

\( c_0 = c_0^d \) in (6.4.5), we must employ an adaptive version of cooperative control (6.4.4). To 
this end, a virtual vehicle representing a hand-off operator is introduced as

\[
\dot{X}_0 = -X_0 + U_0, \quad Y_0(t) = X_0(t), \quad U_0 = K_c X_0(t),
\]

where \( X_0 \in \mathbb{R}^m \) with \( X_0(t_0) = c_0^d 1_m \). Communication from the virtual vehicle to the 
physical vehicles is also intermittent and local, thus we can introduce the following augmented 
sensor/communication matrix and its associated time sequence \( \{\bar{t}_s^k : k = 0, 1, \cdots \} \) as:

\[
\bar{S}(t) = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
s_{q0}
\end{bmatrix} \in \mathbb{R}^{(q+1) \times (q+1)}, \\
\bar{S}(t) = \bar{S}(\bar{t}_s^k), \quad \forall t \in [\bar{t}_s^k, \bar{t}_s^{k+1})
\]

\[
\bar{S}(k) \triangleq \bar{S}(\bar{t}_s^k), \quad \text{(6.4.6)}
\]

Accordingly, cooperative control is modified from (6.4.4) to the following adaptive version:

\[
U_i(t) = \sum_{j=0}^{q} \sum_{\eta=0}^{q} s_{ij}(t) s_{ij}(t) K_c [s_{ij}(t) Y_j], \quad i = 1, \cdots, q, \quad \text{(6.4.7)}
\]

where \( s_{ij}(t) \) are piecewise-constant entries of (6.4.6). Applying Theorem 1 to the resulting 
augmented closed loop system renders the following corollary.

**Corollary 1:** Under the adaptive version cooperative control (6.4.7) with irreducible 
and row stochastic matrix \( K_c \), systems of (6.4.1) exhibit the desired cooperative behavior 
\( Y_{ss}^d \), i.e.,

\[
X_{ss} = 1_{L_q+1} \otimes Y_{ss}^d, \quad \text{and} \quad Y_{i,ss} = Y_{ss}^d, \quad (L_q = \sum_{i=1}^{q} l_i),
\]

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if \( Y^d_{ss} = c^d_01_m \) for \( c^d_0 \in \mathbb{R} \) and if their augmented sensor/communication sequence \( \{ \bar{S}(k) \} \) defined by (6.4.6) is sequentially complete.

### 6.4.2 Formation Calculation

A formation is defined in a coordinate frame that moves with the desired trajectory relative to some other, fixed, coordinate frame. Let \( o_j(t) \in \mathbb{R}^3 \) \((j = 1, 2, 3)\) be the orthonormal vectors which form the moving frame \( F(t) \). Let \( O_\mu = [x_\mu, y_\mu, z_\mu] \in \mathbb{R}^3 \) be the location of the \( \mu \)th agent and \( O_d = [x_d(t), y_d(t), z_d(t)] \in \mathbb{R}^3 \) be any desired trajectory of the origin of the moving frame. A formation consists of \( m \) agents in \( F(t) \), denoted by \( \{O_1, \cdots, O_m\} \), where

\[
O_\mu = d_{\mu 1}(t) o_1(t) + d_{\mu 2}(t) o_2(t) + d_{\mu 3}(t) o_3(t), \quad \mu = 1, \cdots, m, \tag{6.4.8}
\]

with \( d_\mu(t) = [d_{\mu 1}(t), d_{\mu 2}(t), d_{\mu 3}(t)] \in \mathbb{R}^3 \) being the coordinate values of the \( \mu \)th agent in the formation. The desired position for the \( \mu \)th agent is then

\[
O^d_\mu(t) = O_d(t) + d^d_{\mu 1} o_1(t) + d^d_{\mu 2} o_2(t) + d^d_{\mu 3} o_3(t). \tag{6.4.9}
\]

where the constant vector \( d^d_{\mu} = [d^d_{\mu 1}, d^d_{\mu 2}, d^d_{\mu 3}] \in \mathbb{R}^3 \) is the desired relative position for the \( \mu \)th agent in the formation. Meanwhile, \( o_j(t) \) \((j = 1, 2, 3)\) generally can be chosen based on the attitude angles of the virtual leader (yaw \( \psi \), pitch \( \theta \), roll \( \phi \)). Certainly, the choice of \( o_j(t) \) \((j = 1, 2, 3)\) is not unique. For example, \( o_j(t) \) can also be determined by the velocity vector of the virtual leader.
6.4.3 Mapping from Formation Control Problem to Cooperative Control Problem

Through state transformations, the formation control problem for (6.2.1) can be recast as the cooperative control design problem (6.4.1). Let the transformation be

\[ X_\mu = O_\mu(t) - O^d_\mu(t) , \]  

(6.4.10)

Then we introduce the canonical model with \( X_\mu = [X_{\mu1}, X_{\mu2}, X_{\mu3}]^T \in \mathbb{R}^3 \), \( U_\mu \in \mathbb{R}^3 \), and \( Y_\mu \in \mathbb{R}^3 \).

\[ \dot{X}_\mu = \lambda^* (A_\mu X_\mu + B_\mu U_\mu) \, , \quad Y_\mu = C_\mu X_\mu . \]  

(6.4.11)

where \( A_\mu, B_\mu \) and \( C_\mu \) are given by

\[
A_\mu = \begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}, \quad B_\mu = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad C_\mu = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} .
\]

To this end, if we can design the cooperative control \( U_\mu \) such that states \( X_\mu \) for all \( \mu \) converge to the same steady state \( X_{ss} \), then it follows from Corollary 1 that

\[ O_\mu \rightarrow X_{ss} + O^d_\mu(t) , \]

from which it can be seen that the desired formation is achieved for the whole group, while the agents move along the desired trajectory shape.
6.4.4 Vehicle Level Control for Nonholonomic Agents

Consider the following kinematic and dynamic model of a unicycle,

\[
\begin{align*}
\dot{x} &= v \cos \theta \\
\dot{y} &= v \sin \theta \\
\dot{\theta} &= \omega \\
\dot{v} &= \frac{F}{M}
\end{align*}
\]

(6.4.12)

where \( \theta \) is the orientation, \( v \) is the linear velocity, \( \omega \) is the angular velocity, \( F \) is the applied force and \( M \) is the mass. The top view of the unicycle is shown in Figure 6.4.

![Figure 6.4: Relevant variables for the unicycle (top view)](image)

Consider the following dynamic compensator:

\[
\begin{align*}
\omega &= \frac{u_2 \cos \theta - u_1 \sin \theta}{v} \\
F &= M \left( u_1 \cos \theta + u_2 \sin \theta \right)
\end{align*}
\]

(6.4.13)

Note the following facts:

1. The sign of linear velocity \( v \) will determine forward or backward motion of the vehicle.
2. Transformation (6.4.13) is singular at \( v = 0 \), i.e., when the mobile robot is not moving.
Substituting (6.4.13) into (6.4.12) yields the following transformed system:

\[
\begin{align*}
\ddot{x} &= u_1 \\
\ddot{y} &= u_2.
\end{align*}
\]  

(6.4.14)

6.4.5 Formation Control with Collision Avoidance

Considering the dynamic system (6.4.11), where we aim to make \( \| X_i - X_j \| \to 0 \) while at the same time ensure that \( \| O_i - O_j \|^2 \geq \rho_s \) for some positive constant \( \rho_s \). The following condition is then imposed:

\[
\| O_i^d(t) - O_j^d(t) \|^2 \geq \rho_s.
\]

To address the collision avoidance problem, let us consider the control to be given by

\[
U_\mu^* = U_\mu + \sum_{j=1,j\neq \mu}^n \left[ -\nabla P_r (O_{r\mu} - O_{rj}) - \dot{\eta} (O_{r\mu} - O_{rj}) \| v_{g\mu} - v_{rj} \|^2 \\
+ f_d (O_{r\mu} - O_{rj})^T - \eta (O_{r\mu} - O_{rj}) (v_{g\mu} - v_{rj})^T (v_{g\mu} - v_{rj}) \right] \\
+ \sum_{l=1}^n \left[ -\nabla P_r (O_{r\mu} - O_{ol}) - \dot{\eta} (O_{r\mu} - O_{ol}) \| v_{g\mu} - v_{ol} \|^2 \\
+ f_d (O_{r\mu} - O_{ol})^T - \eta (O_{r\mu} - O_{ol}) (v_{g\mu} - v_{ol})^T (v_{g\mu} - v_{ol}) \right].
\]  

(6.4.15)

6.5 Simulation

In this section, two simulation scenarios are presented to show the effectiveness of the proposed control, respectively.

A. Target Tracking and Collision Avoidance for A Single Agent
There are only two rectangular static obstacles, (3000,11000,2000,6000)\(^1\) and (3000,3000,2000,6000).

The initial location of the virtual leader is (6000,-2000) with the following waypoints: (3000,3000), (0,8000), and (-2000,10000). The initial location of the unicycle vehicle is (7000,-1000). The simulation result is shown in Figure 6.5.

![Demonstration of Target Tracking with Collision Avoidance](image)

**Figure 6.5**: Experiment on a single unicycle vehicle

**B. Cooperative Formation Control of Networked Agents with Collision Avoidance**

Three agents are required to execute the formation movement with the desired triangular formation shown in Figure 6.6. The initial location of the virtual leader is (1850,-1000) with the following waypoints: (1800,3000), (1900,11000), and (1850,15000). The initial location of the above three agents are: (1850,-960), (1930,-1040), and (1770,-1040). There are six rectangular static obstacles, (700,3000,2000,6000), (700,11000,2000,6000), (3000,3000,2000,6000), (3000,11000,2000,6000), (3000,3000,2000,6000), (3000,11000,2000,6000).

\(^1\)Data format: (center position, width, length). For example, (3000,11000) denotes the center position. The width is 2000 and the length is 6000.
Figure 6.6: Illustration of the desired formation 

(3000,11000,2000,6000), (1850,7000,60,150), and (1752.5,13700,60,150). In addition, one circular moving obstacle of radius being 10 is also considered. The simulation result considering only the static obstacles is shown in Figure 6.7. The simulation result considering all obstacle, static and moving, is shown in Figure 6.8. Figure 6.8 is zoomed in to shown the successful avoidance of the moving obstacle, depicted in Figure 6.9.

Figure 6.7: Cooperative formation movement with avoiding static obstacles
Figure 6.8: Cooperative formation movement with avoiding static/moving obstacles

Figure 6.9: Cooperative formation movement with avoiding static/moving obstacles (enlarged view)
6.6 Conclusions

In this chapter, a systematic approach is proposed to achieve multiple objectives of cooperative motion, tracking of virtual command vehicle(s), collision avoidance and oscillation suppression. Simulation example A is provided to confirm the effectiveness of Lyapunov Design of multi-objective control for the single agent proposed in Section 3. Rigorous proof of incorporation of the proposed control for the single agent with the cooperative formation control is still needed. The effectiveness of the incorporation has been validated by the simulation example B. In addition, we plan to consider a variety of different feedback controllers such as dynamic, adaptive types of controllers to improve the overall system performance.
CHAPTER 7
CONCLUSIONS AND FUTURE WORK

7.1 Summary of Main Contributions

In this work, we study reactive control design of autonomous dynamical systems to propose real-time solutions of target tracking with obstacles avoidance for a single agent and cooperative formation control with collision avoidance for multi-agent systems in static/dynamic environments. The main contributions of this work can be summarized as:

(1) We address the local minima problem (LMP) inherent in potential field methods. We formulate the LMP and analyze stability property of equilibria yielded by composite potential field functions. We show that there does not exist a static state feedback control to solve LMP. Then we propose a switching control strategy and a time-varying continuous control law to tackle this problem, respectively. For the switching control strategy, the possible saturation constraints (velocity bound and control inputs saturation) are also taken into account. For the time-varying continuous control scheme, challenges of finding continuous control solutions of LMP are discussed and explicit design strategies are then proposed. Systematic rigorous Lyapunov proof is given to show both goal convergence and obstacle avoidance of the proposed control law as long as the goal is reachable.

(2) To deal with avoidance of moving obstacles, different from standard potential field methods, we propose a time-varying and planned potential filed function to account for moving obstacles and vehicle motion. Then the multi-objective reactive control design (tracking targets while avoiding static and moving obstacles) for point-mass vehicles are presented
based on above potential field function. Rigorous Lyapunov analysis are provided to show tracking the target and avoiding moving obstacle.

(3) Considering most mobile robots have nonholonomic constraints, the reactive control design of nonholonomic systems are studied. We begin with the simple nonholonomic vehicle model “unicycle-type mobile robots”. With the relative motion among the mobile robot, targets, and obstacles formulated in polar coordinates, kinematic control laws achieving target-tracking and obstacle avoidance are synthesized using Lyapunov based technique, and more importantly, the proposed control laws also account for possible kinematic control saturation constraints. To the best of our knowledge, we are the first to combine polar coordinate transformation and Lyapunov-like analysis to solve the multi-objective control problem (position tracking and obstacle avoidance) for unicycle-type mobile robots.

(4) The advanced topic cooperative formation control with collision avoidance is also addressed in this work. A potential field based reactive control design is proposed to deal with this problem. To the best of our knowledge, the proposed design is the first systematic approach to accommodate and achieve the multiple objectives of cooperative motion, tracking virtual command vehicle(s), obstacle avoidance, and oscillation suppression. The results are illustrated by several simulation examples including cooperative formation movement of a team of vehicles moving through urban settings with static and moving obstacles, as well as narrow passages.
7.2 Future Directions and Possible Extensions

Looking back on the past five years of research, we deal with the local minima problem to address the obstacle avoidance in a static environment for point-mass vehicle. Then we focus on the topic of moving obstacle avoidance for point-mass vehicle. To step further, we tackle with the moving obstacle avoidance for nonholonomic vehicles. Especially, we begin with the simple unicycle-type vehicle. Our future goal and also the most challenging topic is the cooperative formation control with collision avoidance among a team of robots with nonholonomic constrains in the presence of moving obstacles.

To solve this problem, we have to divide this hard problem into three phases: (1) Cooperative avoidance control for a team of mobile robots in the absence of obstacles; (2) Cooperative avoidance control for a team of mobile robots in the presence of static obstacles; (3) Cooperative avoidance control for a team of mobile robots in the presence of moving obstacles.

For each phase, to achieve the multi-objective (target tracking, collision avoidance, formation movement etc), the proposed reactive control law has to be synthesized for each objective. In most cases, these objectives are contradictory to each other. According to our experience from past years of research, kinds like the local minima problem generally existing in the static environment, in the dynamic environment, the naturally arising key issue for the control design is to deal with transient process caused by these contradictory objectives.

So far, the popular approach reported is path-planning with MILP (Mixed-Integer Linear Programming). The idea is to formulate the multi-objective control problem into a group linear inequalities and find the solution. This approach is time-consuming.
Based on our extensive studies on the reactive multi-objective control design, we had better not using the pure reactive control to handle such a complicated problem. We had found a hierarchy control to tackle with this problem. The proposed hierarchy control has three levels: (1) Cooperative coordination control—based on current states, calculated the way points command to each agent; (2) Motion planning control—taking the way points command, calculated the collision free trajectory for each agent, considering the possible constraints (nonholonomic constraints, velocity constraints etc); (3) vehicle level control—tracking the trajectory yielded by the motion planning control to ensure the agent is uniformly within certain range of the planned trajectory.
APPENDIX A
PUBLICATIONS LIST


LIST OF REFERENCES


