Modeling And Design Of A Photonic Crystal Chip Hosting A Quantum Network Made Of Single Spins In Quantum Dots That Interact Via Single Photons

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MODELING AND DESIGN OF A PHOTONIC CRYSTAL CHIP HOSTING A QUANTUM NETWORK MADE OF SINGLE SPINS IN QUANTUM DOTS THAT INTERACT VIA SINGLE PHOTONS

by

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A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Optics and Photonics: CREOL and FPCE at the University of Central Florida Orlando, Florida

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Major Professors: Winston V. Schoenfeld, Michael N. Leuenberger
ABSTRACT

In this dissertation, the prospect of a quantum technology based on a photonic crystal chip hosting a quantum network made of quantum dot spins interacting via single photons is investigated. The mathematical procedure to deal with the Liouville-Von Neumann equation, which describes the time-evolution of the density matrix, was derived for an arbitrary system, giving general equations. Using this theoretical groundwork, a numerical model was then developed to study the spatiotemporal dynamics of entanglement between various qubits produced in a controlled way over the entire quantum network. As a result, an efficient quantum interface was engineered allowing for storage qubits and traveling qubits to exchange information coherently while demonstrating little error and loss in the process; such interface is indispensable for the realization of a functional quantum network. Furthermore, a carefully orchestrated dynamic control over the propagation of the flying qubit showed high-efficiency capability for on-chip single-photon transfer. Using the optimized dispersion properties obtained quantum mechanically as design parameters, a possible physical structure for the photonic crystal chip was constructed using the Plane Wave Expansion and Finite-Difference Time-Domain numerical techniques, exhibiting almost identical transfer efficiencies in terms of normalized energy densities of the classical electromagnetic field. These promising results bring us one step closer to the physical realization of an integrated quantum technology combining both semiconductor quantum dots and sub-wavelength photonic structures.
“Raise YOUR eyes high up and see. Who has created these things? It is the One who is bringing forth the army of them even by number, all of whom he calls even by name. Due to the abundance of dynamic energy, he also being vigorous in power, not one [of them] is missing.
(Isaiah 40:26)

To Jehovah God, for I have taken exquisite delight in studying His physical laws, In particular His laws of quantum mechanics.
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LIST OF ACRONYMS

AC Alternative Current
BZ Brillouin Zone
CCW Coupled Cavity Waveguide
CMOS Complementary Metal-Oxide-Semiconductor
DD Dynamical Dicke Model
DJC Dynamical Jaynes-Cummings
EPR Einstein-Podolski-Rosen
ESD Entanglement Sudden Death
ESR Electron Spin Resonance
EM Electro-Magnetic
FBZ First Brillouin Zone
FWHM Full Width at Half Maximum
GaAs Gallium Arsenide
GHZ Greenberger-Horne-Zeilinger
HH Heavy-Hole
High-Q High-Quality
InAs Indium Arsenide
InGaAs Indium Gallium Arsenide
JC Jaynes-Cummings
LCP Left Circular Polarization
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CHAPTER 1: INTRODUCTION

Today, CMOS technology is the dominant technology for microprocessors, memories and application specific integrated circuits. In fact, it is primarily responsible for the explosive growth of the information technology industry within the past 25 years by way of the personal computer and the internet. In addition, CMOS technology played, and continues to play, an important role in the development of the telecom, medical, aerospace, automobile, and consumer electronics industries. Undeniably, it is our greatest technological achievement to date.

The semiconductor industry, the engine behind the advance of CMOS technology, is driven by the supply and demand. However, the demand here take the form of a self-fulfilling forecast known as Moore’s law[1], which predicts that the number of transistors on the most complex chip would double each year. Although Moore’s Law was initially made as an observation, it now serves as a goal for the entire industry. This is related to the fact that an increasing number of transistors on a chip result in more processing power and thus superior performances. In order to satisfy Moore’s law, the transistor size must continually decrease, giving rise to more speed and reduced cost, as well as reduced power consumption since there are fewer electrons travelling shorter distance.
Size reduction however has become increasingly difficult to achieve as CMOS technology is facing several new challenges in the submicron regime [2-6]. Worst, soon important physical limits of such technology will be reached and the end of CMOS technology as we know it has to occur while entering into a quantum regime[2], that is when single particles are used to carry units of information as opposed to multiple particles.

**Need for a Quantum Technology**
Why do such physical limits exist? The main reason is that small dimensions bring forth the wave nature of the electron, which is detrimental to the switching mechanism of CMOS transistors. Consider a typical CMOS transistor structure shown in Figure 1-3, which is essentially a semiconductor box surrounded by higher potential energy dielectrics.
How is it that small dimensions bring forth the wave nature of the electrons? Reminiscent of the hydrogen atom, the energy eigenstates of the electron in such system are the result of inference effects related to the wave nature of the electron. When the size of semiconductor box is large enough, the bulk density of state is characterized by closely packed energy levels. Under these conditions, the difference in energy between eigenstates is much smaller the thermal energy $kT$ (0.026 electron-volt at room temperature) where $k$ is the Boltzmann constant and $T$ the temperature, and this means that electrons are thermally spread over what seems a “continuous” distribution of energy levels resulting in an acceptable treatment of the electron as a wave packet or particle. However, as dimensions decrease and reach few tens of nanometers, the semiconductor box is essentially a QD. As a consequence, the differences in energy between eigenstates is on the order or larger than $kT$, and energy levels are now effectively “discrete”, requiring the treatment of the electron as a wave and no longer as a particle.

How is the wave nature of the electron detrimental for CMOS technology? First, as a wave, electrons are able to tunnel through thin barriers. For normal operations, electrons in the source region must be isolated from the drain region; this sets a physical limit for the separation distance between these two n-doped regions of about 10 nm, which is referred to as the gate length. Furthermore, the creation of a channel created underneath the gate (p-doped region), which a path for the electron to flow from the source to the drain regions, is only possible if charges can be accumulated on both
sides the gate without tunneling through. This requires a minimum size of about 1 nm for the gate thickness if high-K/metal materials are considered. Second, an unwanted property of QDs is that energy is required to add or remove a single electron from the dot. It is called the charging energy, and it is analogous to the ionization energy of an atom. When this charging energy exceeds thermal energy KT at room temperature, the charging of a quantum dot runs into a Coulomb blockade. As a result, it would be impossible to create a channel or current through such transistor. Third, another feature associated with QDs is the reduction of the dielectric constant[3], which causes the binding energy of dopants to be smaller than kT, resulting in an intrinsic behavior regardless of doping density [4]. This means that it would impossible to create the necessary highly doped source and drain regions in transistors of few tens of nanometers in size. Clearly, the wave nature of the electron severely perturbs the ability of CMOS transistors to operate.

In order to continue satisfying Moore’s law and sustain technological growth, a quantum approach that take advantage of the wave nature of particles need to be considered[2]. And such novel, quantum based technology would be beneficial for speed by making use of quantum algorithms; which were found more efficient than their classical counterparts[5-8]. More importantly, from a computer architecture point of view, a register of n conventional bits can be in just one of its 2^n possible different states at a given time. In contrast, a quantum register of n qubits or quantum bits can be in a superposition of all possible register states at the same time with 2^n amplitudes. And such ability to superimpose quantum states offers the possibility to compute
simultaneously with all the amplitudes of a quantum register, resulting in the prospect of massive parallel computing and unmatched processing power. In fact, this is how Moore’s law is expected to be sustained in the quantum regime, simply by adding qubits to quantum registers[9].

**Existing Schemes**

Essential to the idea of a quantum technology is the concept of qubit. In theory, any quantized physical quantity that can be described mathematically by two states is a candidate for the physical realization of the qubit. Unlike the classical bit; the qubit can form linear combinations of its two quantum states called superpositions. A general state vector for a qubit may be written as

\[ |\psi\rangle = \alpha |0\rangle + \beta |1\rangle \] (1.1)

where \( |0\rangle \) and \( |1\rangle \) form an orthonormal basis in a two-dimensional complex vector space. In addition, the state vector of the qubit is normalized to length one, resulting in the following normalization condition

\[ |\alpha|^2 + |\beta|^2 = 1 \] (1.2)

In this section various approaches to a physical realization of the qubit are introduced.
A first approach is known as the *optical* approach and makes use of single photons, which can serve as good qubits[9]. Even though, it is possible to represent such qubit as a superposition of zero or one photon in a cavity; the qubit states normally correspond to the states of two cavities which total energy is equal to the energy of the photon. It is called the dual-rail representation. The advantages with using single photons are their lack of interaction with each other or even with most matter as well as their ability to travel long distance. They can be guided in optical fibers, delayed using phase shifters, and combined using beam splitters. In this scheme, nonlinear optical media are used to mediate the interactions between these optical qubits. Through an appropriate design of a network of fibers, shifters, beam splitter, and nonlinear optical media, it was shown that quantum computation could be performed. However, this approach has several drawbacks: (1) nonlinear materials inevitably absorb or scatter single photons, (2) a bulky apparatus and (3) a lack of robust on-demand single photon sources.

Next, the *cavity-QED* approach is in many ways similar to the optical approach[9]. It is a technique by which single atoms interact strongly with single photons. Experimentally this is made possible by placing single atoms within a small high-Q cavities resulting in a high dipole coupling between the atom and the field. Nevertheless, this approach possesses many disadvantages: (1) the need for high atom-field coupling, (2) coupling the photon(s) in and out of cavities, and (3) a lack of robust on-demand single photon sources.
Yet another scheme considers the use of ions traps, which involves controlling the spin states of an atom[2, 9]. The principle behind performing quantum computing by using traps consists in isolating and cooling the atoms until their kinetic energy is much lower than the spin energy contribution. After doing this, incident monochromatic light can be tuned to selectively induce transitions via the magnetic dipole interaction which change certain spin states depending on other spin states. However, this approach also suffers from a number of weaknesses such as (1) a weak phonon mediated spin-spin coupling, (2) decoherence, (3) difficulties in preparing states, and (4) scalability.

Again, a different method to implement physical qubits, nuclear magnetic resonance (NMR), is based on direct manipulation and detection of nuclear spin states using radiofrequency electromagnetic waves[9]. The idea is to use molecules since a strong natural spin-spin coupling is possible due to the direct dipolar coupling and indirect through-bond electron mediated interactions between neighboring nuclei as an alternative to the weak spin-spin coupling in single atoms in ion traps. In this scheme, arbitrary transforms on a spin system could in theory be realized to good fidelity using RF pulses. However, there are many limitations with this technique. (1) Because of the small nuclear magnetic moment; a large number of molecules must be present in order to produce a measurable induction signal. In turn, this causes the output of an NMR measurement to be an average over the molecules’ signals, which does not necessarily give any relevant information. (2) Another drawback is that NMR is typically applied to physical systems at room temperature, where the spin energy is much less than the thermal energy kT. At a result, it is really challenging to prepare the initial state in a
special or pure state such as the ground state, which means the initial state of the spins is nearly completely random. (3) Yet another inadequacy comes from the fact that a liquid environment is not very practical for quantum computing.

Still, another technique uses neutral atoms as qubits[2, 9]. The qubit, which is formed by either two electronic or two motional states of the atom, is confined using optical lattices. Furthermore, in order to create the entanglement between the qubits, electric or magnetic dipole interaction, or the controlled collisions between atoms can be used. It was shown that it is possible to construct large arrays of neutral atoms in an optical lattice, so in principle scalability looks viable. However, neutral atoms have many difficulties to overcome as well. For one, it is not clear how to address individual neutral atom qubits within an optical lattice since they are typically spaced so closely that spatial resolution with external lasers is difficult. Other anticipated negative aspects are external decoherence effects such as the laser intensity and the pulse timing stability.

A further approach, the solid-state implementations of qubits [2, 9] are very popular for two reasons: First, these implementations are extensions of existing semiconductor technology and possess the potential to be scaled. Second, the coherence time of the qubit is potentially long enough to achieve the error threshold required for fault tolerant quantum computation. One implementation of the solid-state qubit, the semiconductor charge qubit, uses the orbital of a single confined electron to represent the states of the qubit. Single qubit operations can be performed using external electric fields or microwave pulses. Another implementation is the spin of a confined electron known as the semiconductor quantum dot spin qubit; it provides a
natural representation of a qubit. Spins of electrons confined in semiconductor structures are particularly attractive from the point of quantum computation, since the spin degree of freedom is well isolated from the environment affecting dramatically spin coherence time. Possible methods for performing single qubit gates include applying an external, non-uniform, magnetic field, together with external electron spin resonance (ESR) pulses; and modulation of the effective electron g-factor. The readout of the final state of the spin could potentially be a difficult issue though. In 1998, it was proposed that the *nuclear spin of phosphorus donor impurities* in a silicon dioxide structure could be used as qubits. The nuclear spin relaxation time of such donors has been observed to be several hours or even days. An additional solid-state implementation consists of the *excitonic qubits*. The states of the qubit are usually represented by the absence or presence of a single exciton in a given quantum dot. Single qubit operations are performed by applying temporally shaped laser pulses to prepare the quantum dots in arbitrary superposition of zero and one exciton. Inter-qubit operations are mediated by the dipole–dipole interaction between excitons. However, it is anticipated that addressing individual quantum dots in all excitonic schemes with a laser field can be a problem.

An alternative approach is based on *superconductors* where condensed matter is effectively frozen out, leaving some form of condensate with an energy gap for the formation of quasi-particle excitations[2]. There exist two types of superconducting qubits. The first is a sub-micron size capacitor called the charge qubit, which can be thought of as a tiny grain of superconductor coupled to a larger piece of
superconductor. The second is a micron size inductor called the flux qubit, which can be thought of as a tiny ring of superconductor. However, these qubits require extremely low operating temperatures.

**Current Problem**

Traditionally, the challenge has been the physical realization of the qubit. Despite several proposed schemes and experiments, the basic requirements of quantum computation known as the DiVincenzo criteria [9] could only be partially met. These requirements are (1) a robust physical representation of quantum information, (2) a large decoherence time, (3) the ability to perform a universal family of unitary transformations, (4) the ability to prepare a fiducial initial state, and (5) the ability to perform an output measurement. Fortunately, in the light of the recent progress, the qubit now exists as a physical system. For instance, the lack of robust on-demand single photon sources was a severe hurdle for the realization of both the optical and the cavity-QED qubits. Lately, high-efficiency semiconductor single-photon sources with two-photon emission rates fifty times less than those from conventional attenuated laser sources and with repetition rates in the hundreds of megahertz have been demonstrated [10-13]. Furthermore, progress in the fabrication of nanocavities with Q factors in the millions [14-16] provide the microscopic environment for strong optical nonlinearity [17, 18] desirable for the optical qubit as well as the strong atom-field coupling resulting in Rabi oscillations needed for the cavity-QED qubit[19, 20]. Even the
predicament of efficiently coupling photons in and out of nanocavities has been resolved with the advent of cavity-waveguide couplers with efficiency as high as 90% [15, 21] as well as the dynamic switching of nanocavities Q factor in picoseconds providing much control over the trapping and releasing of photons [22]. Other successes include the ion trap and neutral atom qubits, the various solid-state qubits and the superconducting qubits. In the case of the ion trap qubit, main drawbacks have been circumvented, and even the entanglement of two trapped Ytterbium ions qubits approximately one meter apart was demonstrated[23]. As for the neutral atom qubit, they used to suffer from individual atom addressing in optical lattices as well as from decoherence; however, entanglement and gate operations between isolated qubit pairs of Rydberg atoms were shown recently[24]. Additionally, the difficulties that existed in the read out of the spin state for the already existing semiconductor quantum dot spin qubit was resolved, and it can presently be done with nanosecond time resolution[25, 26]. Finally, the superconducting qubit is clearly a reality, and both entanglement and gate operations have been verified[27, 28].

Currently, the main problem is therefore no longer the physical realization of the qubit, but rather the engineering of a practical quantum computing architecture or quantum network. This leads us to develop a new set of criteria regarding the physical realization of quantum networks, which consist of ensembles of qubits considered within a defined physical space and that can interact with one another in a controlled manner, to the end of obtaining a functional quantum technology. Furthermore, this new set of
criteria exists in addition to the DiVincenzo criteria, which are relevant for single isolated qubits.

**Practical:** Regarding this first criterion, Stanford University Professor Dr Vuckovic et al make the following statement in a recent publication: “Efficient implementation of quantum computation devices requires on chip integration of photonic circuits consisting of PC cavities and waveguides”[21]. On-chip implementation of quantum networks would allow a quantum technology to be incorporated into ordinary electronic devices, including devices that are portable, making it a practical technology. In addition, unnatural or extreme operating condition would make a technology unrealistic.

**Scalable:** Scalability usually refers to how well a hardware or software system can adapt to increased demands. For example, a scalable quantum network system would be one that can start with just a few nodes but can easily expand to thousands of nodes. Scalability is a very important feature of a quantum network because it ensures that Moore’s Law can be sustained.

**Efficient Transfer of Information:** Professor Rempe, who is another expert in quantum information processing devices at the Max-Planck Institute, noted that “a major challenge for a quantum computing architecture is the faithful transfer of information from one node to another”[29]. In a similar way that a classical computer is built from an electrical circuit containing wires and logic gates, a quantum network is also made of quantum interconnects and quantum gates. So the quantum interconnects must
preserve the integrity of the quantum information while they provide a path for the information to flow.

**Compatibility:** To operate a quantum computer, one must be able to set and control the qubits, manipulate logic gates, correct errors, remove “noise” in processing, and record the computed values. All these operations will have to be controlled using some classical computers powered by CMOS technology; therefore, it is desirable for a quantum computing architecture to be compatible with CMOS technology. Another area of compatibility to consider for a quantum network is optical communication since single photons have been shown to travel long distances in optical fibers in the past as part of quantum key distribution and quantum teleportation experiments[30-33].

**Cost:** An overlooked yet equally important criterion when considering a realistic quantum technology is the cost associated with the physical structure that is to host the quantum networks, and this includes the cost of the material and the cost of fabrication. This criterion must be satisfied for a large scale production of a quantum technology to be feasible.

Not one single physical implementation of the qubit described in the previous section even though DiVincenzo criteria compliant is able to satisfy this new set of criteria intended for quantum networks. For instance, considering the first criterion, it is difficult to envision an efficient on-chip implementation of the ion trap and the neutral atom qubits. In addition, extreme operating temperatures are required for the superconducting qubit making it somehow unpractical. Even though the next criterion of scalability seems achievable in various capacities for all qubits; they all in some aspects
fall short regarding the issue of the transfer of quantum information. On one hand, spin, excitonic, ion traps, neutral atom qubits are all excellent storage qubits due their robust representation of quantum information; as a result, they are natural candidate to form the nodes of a quantum network. However, transporting these qubits from one location to another in a quantum network without losing the quantum information encoded onto them is a daunting task that often requires complicated schemes or the cooperation of several qubits arranged in a chain[34-38]. On the other hand, the optical and cavity-QED qubits uses single photons, which can easily travel uncorrupted inside waveguides since they do not interact much with the environment or with other photons. But, they are not strong contender for locality and long-term storage and end up compromising the integrity of quantum interconnects by leaking back into them causing the stored information to be lost. As for the compatibility with CMOS technology, only the semiconductor based qubits can straightforwardly satisfy that criterion as well as optical communication by using nanostructures to achieve 850 nm, 1310 nm and 1550 nm wavelengths. These are the QD spin and excitonic qubits, the optical and cavity-QED qubits, and the superconducting qubit. At last, the same is true for the cost criterion, exotic non-semiconductor based materials that make up the ion trap or neutral atom qubits can be costly. Furthermore, there exists already a semiconductor industry that possesses the necessary equipment and the knowhow to produce on a large scale a semiconductor based quantum technology.
An Attractive Solution

As a consequence of these five new criteria, the general consensus is that various implementations of the qubit should be combined in order to obtain an efficient quantum computing architecture. This calls for qubits that are good for storage such as atoms to be used at quantum networks nodes while qubits that have desirable properties for travel such as photons to be used as quantum interconnects. Moreover, the storage qubits can map their quantum state onto the traveling qubits and vice versa by means of coherent interfaces[39-41]. To the end of realizing an efficient quantum computing architecture, this promising composite qubit approach to a quantum technology has been proposed for ion trap qubits [42] and also for neutral atoms[29]. We on the other hand have proposed a similar approach in connection with semiconductor based artificial atoms or quantum dots[43].

What does our scheme consist of? It consists of engineering a photonic crystal chip hosting a quantum network made of quantum dots spin embedded in defect cavities (storage qubits), which constitute the nodes of the network, and that interact by means of single photons (traveling qubits), which are guided through defect waveguides, to the end of mapping quantum states or even creating entanglement between these two particles. Making use of quantum dots spin to implement qubits is advantageous since they are much more robust against the influence of temperature than atoms in ion traps. And, this physical realization of qubits brings the unique benefit of long decoherence time as spin lifetimes up to 20 ms have been reported[44]. Recent progresses in semiconductor nanostructures along with the versatility of photonic crystal
in confining light provide the necessary environment for the realization of a coherent interface between QD spin and single photon qubits. Interestingly, this coherent interface, which is responsible the state of the QD spin to be mapped onto the polarization of a single photon or the entanglement between them, is itself a qubit system, the cavity-QED qubit (exchange or interaction qubit).

Our approach to a quantum network offers additional benefits with respect to the other composite qubit schemes; therefore, it naturally establishes itself as a more attractive solution. For instance, realistic on-chip implementation using photonic crystal has been shown to be plausible[45]; which is not the case both the ion trap and neutral atom qubits. In addition, even though low temperatures are desirable for minimizing decoherence for the QD spin qubit, it is nowhere near the extreme temperatures needed for the functioning of the superconducting qubit. This technology is easily scalable as additional nodes for the quantum network are generated by just creating additional cavities with embedded QDs in the photonic crystal chip. Also because it is a semiconductor based quantum technology, it is anticipated to be CMOS compatible and cost effective. The use of single photons as traveling qubits as well as the wavelengths considered makes not only the on-chip transfer of quantum information but also the long distance quantum communication by means of optical fibers efficient.
Outline

This dissertation is organized in the following chapters. Chapter 2 introduces the theoretical foundation for the qubit systems that constitute the quantum network. In Chapter 3, a numerical model that can be used to study the quantum dynamics inside the proposed quantum network node is developed. In Chapter 4, an efficient quantum interface is engineered with the aim of minimizing the phase error and photon loss during the capture and release from and onto the waveguide. Chapter 5 considers the use of a photonic crystal coupled-cavity waveguide for the high-efficiency transfer of single photons on-chip. Chapter 6 presents the dynamics of entanglement between two spatially separated quantum dot spins by means of a traveling photon. Chapter 7 discusses in details a possible physical design and anticipated performances for the proposed quantum network. Finally, Chapter 8 consists of concluding remarks and discusses future work.
CHAPTER 2: THEORY OF QUANTUM NETWORKS

A quantum network is composed of two fundamental elements: Nodes and Interconnects. Nodes are the basic building blocks of a quantum network and consist of QDs (storage qubit) embedded in nanocavities and can interact with single photons (traveling qubits) that enter the cavities creating a coherent interface (interaction qubit). Consequently, nodes can be decomposed into three subsystems: an atom subsystem describing the behavior of the quantum dot and its excess electron spin, an EM field subsystem describing a single mode quantized field trapped in a nanocavity or traveling qubit, and an interaction subsystem describing the dynamics of the coherent interface. On the other hand, interconnects provide the means for the single photons to travel from one node to another; therefore, they consist of a single subsystem, an EM field subsystem describing a single mode quantized field propagating in a waveguide.

A good place to start the mathematical treatment of our quantum network is by considering first the Jaynes-Cummings (JC) model as an idealistic approach to represent the quantum network nodes. Next, a more realistic model for the quantum network nodes is developed based on the J-C model by including an actual QD energy level structure and losses. Then, the model for the nodes is extended to a comprehensive quantum network model that includes interconnects and that is derived from dynamical Dicke and J-C models.
The Jaynes-Cummings (JC) Model

The JC Model shown in Figure 2-1 consists of a fully quantum mechanical treatment of a 2-level atom interacting with a single mode field in a cavity without losses[46, 47]. Here, $|g\rangle$ and $|e\rangle$ respectively represent the ground and excited energy levels, $\nu$ the frequency of the photon, and the atomic transition frequency.

Figure 2-1: Jaynes-Cummings Model

Hamiltonian

The Hamiltonian for the JC model has the following construction

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_{\text{Atom, dyn}} + \hat{\mathcal{H}}_{\text{Field, dyn}} + \hat{\mathcal{H}}_{\text{JC}}$$  \hspace{1cm} (2.1)
Atom Hamiltonian

When interested in the dynamics of a quantum system, the 2-level atom Hamiltonian is usually expressed in terms of the transition operators $\hat{\sigma}_{gg} = \left| g \right\rangle \langle g \left|$, $\hat{\sigma}_{ge} = \left| g \right\rangle \langle e \left|$, $\hat{\sigma}_{eg} = \left| e \right\rangle \langle g \left|$, and $\hat{\sigma}_{ee} = \left| e \right\rangle \langle e \left| [47]$. The 2-level atom Hamiltonian in terms of the transition operators, $\hat{H}_{\text{Atom,dyn}}$, is obtained from the traditional Hamiltonian $\hat{H}_{\text{Atom}}$ using the completeness relation

$$\left| g \right\rangle \langle g \left| + \left| e \right\rangle \langle e \left| = 1 \quad (2.2)$$

Thus, the 2-level atom Hamiltonian can be written as

$$\hat{H}_{\text{Atom,dyn}} = \hat{H}_{\text{Atom}}$$

$$= \left( \left| g \right\rangle \langle g \left| + \left| e \right\rangle \langle e \left| \right| g \right\rangle \langle g \left| + \left| e \right\rangle \langle e \left| \right| \hat{H}_{\text{Atom}} \right| g \right\rangle \langle e \left| + \left| e \right\rangle \langle e \left| \right| \hat{H}_{\text{Atom}} \right| e \rangle \langle e \left| \right| = \left| g \right\rangle \langle g \left| \hat{H}_{\text{Atom}} \right| g \rangle + \left| e \right\rangle \langle e \left| \hat{H}_{\text{Atom}} \right| e \rangle \right.$$  

$$= \left| g \right\rangle \langle E_g \right| + \left| e \right\rangle \langle E_e \left|$$

$$= E_g \left| g \right\rangle \langle g \left| + E_e \left| e \right\rangle \langle e \left|$$

$$= E_g \hat{\sigma}_{gg} + E_e \hat{\sigma}_{ee} = \frac{1}{2} \hbar \omega (\hat{\sigma}_{ee} - \hat{\sigma}_{gg}) + \frac{1}{2} (E_g + E_e) = \frac{1}{2} \hbar \omega \hat{\sigma}_z$$

where $\hbar \omega = E_e - E_g$ and $\hat{\sigma}_z = \hat{\sigma}_{ee} - \hat{\sigma}_{gg}$. Also, the constant energy term $\frac{1}{2} (E_g + E_e)$ is dropped since it does not contribute to the dynamics.
Field Hamiltonian

Next, the single mode free field Hamiltonian is articulated in terms of the annihilation and creation operators, $\hat{a}$ and $\hat{a}^\dagger$ [47], such as

$$\hat{H}_{\text{Field}} = \hbar \nu \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right)$$

(2.4)

In the JC model, the zero-point energy term is often dropped because it does not contribute to the dynamics of the system. The field Hamiltonian becomes

$$\hat{H}_{\text{Field, dyn}} = \hbar \nu \hat{a}^\dagger \hat{a}$$

(2.5)

Atom-Field Interaction or JC Hamiltonian

Last, the atom-field interaction Hamiltonian in the JC Model is referred to as the J-C Hamiltonian. In the electric dipole approximation, the wave-length of the photon is assumed to be much larger than the typical size of the atom, and the J-C Hamiltonian is
\[ \hat{H}_{J-C} = -\mathbf{e} \cdot \mathbf{E} \]
\[ = -e \left( \sum_{i,j} \langle i | \hat{r}^{(i)} \rangle \langle \hat{r}^{(i)} | j \rangle \cdot \mathbf{E} \right) \left( \hat{a} + \hat{a}^\dagger \right) \]
\[ = -e \left( \sum_{i,j} \left[ e \langle i | \hat{r}^{(i)} \rangle \langle \hat{r}^{(i)} | j \rangle \right] \cdot \mathbf{E} \right) \left( \hat{a} + \hat{a}^\dagger \right) \]
\[ = -\left( \sum_{i,j} \mu_{ij} \sigma_{ij} \right) \cdot \mathbf{E} \left( \hat{a} + \hat{a}^\dagger \right) \]
\[ = \hbar \sum_{i,j} \mu_{ij} \cdot \mathbf{E} \sigma_{ij} \left( \hat{a} + \hat{a}^\dagger \right) \]
\[ = \hbar \sum_{i,j} g_{ij} \sigma_{ij} \hat{a} \hat{a}^\dagger \]
\[ = \hbar g \left( \hat{\sigma}_+ + \hat{\sigma}_- \right) \hat{a} \hat{a}^\dagger \tag{2.6} \]

where \( g_{ij} = g_{ji} = g \) is the coupling strength, which is assumed to be real for simplicity, and \( \mathbf{E} \) is the electric field vector with \( \mathbf{e} \) being the unit polarization vector and \( E \) the amplitude of the field. The transition operators are defined as \( \hat{\sigma}_+ = \hat{\sigma}_{eg} = \langle e | \langle g \rangle \) and \( \hat{\sigma}_- = \hat{\sigma}_{ge} = \langle g | \langle e \rangle \). In the rotating wave approximation, the Hamiltonian becomes further be simplified to

\[ \hat{H}_{J-C} = \hbar g \left( \hat{\sigma}_+ \hat{a} + \hat{a}^\dagger \hat{\sigma}_- \right) \tag{2.7} \]

The total Hamiltonian for the Jaynes-Cummings model is thus written as
Equations of Motion

In order to derive the Equations of motion, it is convenient to work in the interaction picture. First, the Hamiltonian needs to be rewritten in the following form

\[ \hat{H} = \frac{1}{2} \hbar \omega \hat{\sigma}_z + \hbar \nu \hat{a}^{\dagger} \hat{a} + \hbar g \left( \hat{\sigma}_+ \hat{a} + \hat{a}^{\dagger} \hat{\sigma}_- \right) \]  

(2.8)

Next, the Schrödinger’s Equation in the interaction picture can be derived.
\[ i\hbar \frac{\partial}{\partial t} \ket{\psi_s} = \hat{H} \ket{\psi_s} \]

\[ i\hbar \frac{\partial}{\partial t} \ket{\psi_s} = \hat{H}(e^{\frac{i}{\hbar} \hat{H}_0 t} e^{\frac{\hbar}{i} \hat{H}_0 t}) \ket{\psi_s} \]

\[ i\hbar \frac{\partial}{\partial t} \ket{\psi_s} = e^{\frac{\hbar}{i} \hat{H}_0 t} e^{-\frac{i}{\hbar} \hat{H}_0 t} \left( e^{\frac{i}{\hbar} \hat{H}_0 t} e^{\frac{\hbar}{i} \hat{H}_0 t} \right) \ket{\psi_s} \]

\[ i\hbar \frac{\partial}{\partial t} (e^{\frac{i}{\hbar} \hat{H}_0 t} \ket{\psi_s}) = \frac{i}{\hbar} \hat{H}_0 (e^{\frac{i}{\hbar} \hat{H}_0 t} \ket{\psi_s}) = e^{\frac{i}{\hbar} \hat{H}_0 t} (\hat{H}_0 + \hat{H}_1) e^{-\frac{i}{\hbar} \hat{H}_0 t} (e^{\frac{i}{\hbar} \hat{H}_0 t} \ket{\psi_s}) \]

\[ i\hbar \frac{\partial}{\partial t} \ket{\psi_1} = e^{\frac{i}{\hbar} \hat{H}_0 t} (\hat{H}_0 + \hat{H}_1) e^{-\frac{i}{\hbar} \hat{H}_0 t} \ket{\psi_1} - \hat{H}_0 \ket{\psi_1} \]

\[ i\hbar \frac{\partial}{\partial t} \ket{\psi_1} = \left( e^{\frac{i}{\hbar} \hat{H}_0 t} \hat{H}_1 e^{-\frac{i}{\hbar} \hat{H}_0 t} \right) \ket{\psi_1} \]

\[ i\hbar \frac{\partial}{\partial t} \ket{\psi_1} = \hat{V} \ket{\psi_1} \quad (2.10) \]

Here the interaction Hamiltonian \( V \) is

\[ \hat{V} = e^{\frac{i}{\hbar} \hat{H}_0 t} \hat{H}_1 e^{-\frac{i}{\hbar} \hat{H}_0 t} \]

\[ = e^{\frac{i}{\hbar} (\hbar v a \hat{a} + \frac{1}{2} \hbar \sigma_z) t} \left[ \hbar g \left( \hat{\sigma}_+ \hat{a} + \hat{a}^\dagger \hat{\sigma}_- \right) \right] e^{-\frac{i}{\hbar} (\hbar v a \hat{a} + \frac{1}{2} \hbar \sigma_z) t} \]

\[ = \hbar g (e^{i \frac{v}{\hbar} a \hat{a}^\dagger} e^{-i \frac{v}{\hbar} a \hat{a}} e^{i \hbar \sigma_z t/2} \hat{\sigma}_+ e^{-i \hbar \sigma_z t/2}) \]

\[ + \hbar g (e^{i \frac{v}{\hbar} a \hat{a}^\dagger} e^{-i \frac{v}{\hbar} a \hat{a}} e^{i \hbar \sigma_z t/2} \hat{\sigma}_- e^{-i \hbar \sigma_z t/2}) \quad (2.11) \]

Using the following identity, we can obtain a simplified form for the interaction Hamiltonian.
\[ e^{\alpha A} B e^{-\alpha A} = B + \alpha [A, B] + \frac{\alpha^2}{2!} [A, [A, B]] + \ldots \] (2.12)

For example, if we consider the first term in the last line of Equation (2.11) that is associated with the annihilation operator, it can be expanded in the following manner.

\[ e^{iv\hat{a}^\dagger \hat{a}} e^{-iv\hat{a}^\dagger \hat{a}} = \hat{a} + iv\hat{a}^\dagger + \frac{(iv\hat{a}^\dagger)^2}{2!} + \ldots \] (2.13)

Since \[ \hat{a}^\dagger \hat{a} = \hat{a}^\dagger \hat{a} - \hat{a}^\dagger \hat{a} = -\hat{a} \] and \[ \hat{a}^\dagger \hat{a}, \hat{a}^\dagger \hat{a} \hat{a} = (\hat{a}^\dagger \hat{a} - \hat{a}^\dagger \hat{a}^\dagger) \hat{a} \], it can be shown that

\[ e^{iv\hat{a}^\dagger \hat{a}} e^{-iv\hat{a}^\dagger \hat{a}} = \hat{a} \sum_{m=0}^{\infty} \frac{(iv\hat{a}^\dagger)^m}{m!} = \hat{a} e^{-ivt} \] (2.14)

In a similar way, we can derive the simplified expression for the other terms in Equation (2.11), and they are

\[ e^{i\alpha \hat{\sigma}_z t/2} \hat{\sigma}_+ e^{-i\alpha \hat{\sigma}_z t/2} = \hat{\sigma}_+ e^{i\alpha t} \]

\[ e^{iv\hat{a}^\dagger \hat{a}^\dagger \hat{a}} e^{-iv\hat{a}^\dagger \hat{a}^\dagger \hat{a}} = \hat{a} e^{ivt} \] (2.15)

\[ e^{i\alpha \hat{\sigma}_z t/2} \hat{\sigma}_- e^{-i\alpha \hat{\sigma}_z t/2} = \hat{\sigma}_- e^{-i\alpha t} \]

The interaction Hamiltonian becomes
\[
\hat{V} = \hbar g \left( \hat{\sigma}_+ e^{i(\omega - \nu)t} + \hat{\sigma}_- e^{-i(\omega - \nu)t} \right)
\]

(2.16)

Deriving for the Equations of motion, one must substitute the general expression for the time-dependent state vector of the system back into Equation (2.10). An orthonormal basis for the overall system is nothing else than the tensor product of each individual subsystem orthonormal basis such that

\[
\left| \psi_{\text{atom}} \right> \otimes \left| \psi_{\text{field}} \right> = \left\{ \left| g \right>, \left| e \right> \right\} \otimes \left\{ \left| n \right>, \left| n+1 \right> \right\} = \left\{ \left| g, n \right>, \left| g, n + 1 \right>, \left| e, n \right>, \left| e, n + 1 \right> \right\}
\]

(2.17)

where \( \left| g, n \right> \) corresponds to the atom being in the ground state with \( n \) photons in the cavity mode, \( \left| g, n + 1 \right> \) to the atom being in the ground state with \( n + 1 \) photons in the cavity mode, \( \left| e, n \right> \) to the atom being in the excited state with \( n \) photons in the cavity mode, and \( \left| e, n + 1 \right> \) to the atom being in the excited state with \( n + 1 \) photons in the cavity mode. Eventually, it will be the case that there is only one photon in the mode when the atom is in the ground state (\( n + 1 = 1 \)), but for now an arbitrary number of photons is assumed to be in the mode. For all practical purpose, the states \( \left| g, n \right> \) and \( \left| e, n + 1 \right> \) are never populated; in other words, transition matrix elements from and to these states are always zero. Therefore, the orthonormal basis for such system is effectively expressed as \( \left\{ \left| g, n + 1 \right>, \left| e, n \right> \right\} \). The state vector of the overall system is shown below.
\[ |\psi_1(t)\rangle = C_{g,n+1}(t)|g,n+1\rangle + C_{e,n}(t)|e,n\rangle \quad (2.18) \]

In the interaction picture, \( C_{g,n+1}(t) \) and \( C_{e,n}(t) \) already correspond to the slowly varying probability amplitudes.

Substituting Equation (2.18) back into Equation (2.10), we obtain

\[ \frac{\hbar}{i} \frac{\partial}{\partial t} |\psi_1\rangle = \hat{V} |\psi_1\rangle \]

\[ \frac{\hbar}{i} \frac{\partial}{\partial t} \left( C_{g,n+1}(t)|g,n+1\rangle + C_{e,n}(t)|e,n\rangle \right) = \hat{V} \left( C_{g,n+1}(t)|g,n+1\rangle + C_{e,n}(t)|e,n\rangle \right) \quad (2.19) \]

\[ \frac{\hbar}{i} \left( C_{g,n+1}(t)|g,n+1\rangle + C_{e,n}(t)|e,n\rangle \right) = \hat{V} \left( C_{g,n+1}(t)|g,n+1\rangle + C_{e,n}(t)|e,n\rangle \right) \]

The first Equation is obtained by projecting the following Equation onto \( \langle g,n+1 \rangle \) such that
\[ \langle g,n+1| \cdot i\hbar \left( C_{g,n+1}(t)|g,n+1\rangle + C_{e,n}(t)|e,n\rangle \right) \]
\[ = \langle g,n+1| \cdot \hat{V} \left( C_{g,n+1}(t)|g,n+1\rangle + C_{e,n}(t)|e,n\rangle \right) \]
\[ = i\hbar \left( C_{g,n+1}(t) \langle g,n+1||g,n+1\rangle + C_{e,n}(t) \langle g,n+1||e,n\rangle \right) \]
\[ = C_{g,n+1}(t) \langle g,n+1||\hat{V}|g,n+1\rangle + C_{e,n}(t) \langle g,n+1||\hat{V}|e,n\rangle \]
\[ = C_{g,n+1}(t) \langle g,n+1||\hat{V}|g,n+1\rangle + C_{e,n}(t) \langle g,n+1||\hat{V}|e,n\rangle \]

Making use of the orthogonality condition \[ \langle i||j \rangle = \int \psi_i^* \psi_j \, dr = \delta_{ij} \] and parity rules of the electrical dipole interaction Equation (2.20) simplifies to

\[ C_{g,n+1}(t) = -\frac{i}{\hbar} C_{e,n}(t) \langle g,n+1||\hat{V}|e,n\rangle \] (2.21)

Next, we utilize the following rules for the atomic transition operators and the creation and annihilation operators in Equations (2.22) and (2.23) respectively to obtain the final form for the first Equation of motion in Equation (2.24).

\[ \hat{\sigma}_+ = \hat{\sigma}_{eg} = |e\rangle\langle g| \quad \hat{\sigma}_- = \hat{\sigma}_{ge} = |g\rangle\langle e| \]
\[ \hat{\sigma}_z = \hat{\sigma}_{ee} - \hat{\sigma}_{gg} \]
\[ = |e\rangle\langle e| - |g\rangle\langle g| \] (2.22)
\[ \hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle \quad \hat{a} |n\rangle = \sqrt{n} |n-1\rangle \] (2.23)

\[ \hat{a} |0\rangle = 0 \quad \hat{a}^\dagger \hat{a} |n\rangle = n |n\rangle \]

\[
\begin{align*}
C_{g,n+1}(t) & = -\frac{i}{\hbar} C_{e,n}(t) \langle g, n+1 | \hat{V} | e, n \rangle \\
& = -\frac{i}{\hbar} C_{e,n}(t) \langle g, n+1 | \hat{\sigma}_+ \hat{a} e^{i(\omega-v)t} + \hat{a}^\dagger \hat{\sigma}_- e^{-i(\omega-v)t} | e, n \rangle \\
& = -igC_{e,n}(t) \left[ e^{i(\omega-v)t} \langle g, n+1 | (\hat{\sigma}_+ \hat{a}) | e, n \rangle + e^{-i(\omega-v)t} \langle g, n+1 | (\hat{a}^\dagger \hat{\sigma}_- - 1) | e, n \rangle \right] \\
& = -igC_{e,n}(t) \left[ e^{i(\omega-v)t} \langle g | \hat{\sigma}_+ | e \rangle \langle n+1 | \hat{a} | n \rangle + e^{-i(\omega-v)t} \langle g | \hat{\sigma}_- | e \rangle \langle n+1 | \hat{a}^\dagger | n \rangle \right] \\
& = -igC_{e,n}(t) \left[ \sqrt{n} e^{i(\omega-v)t} \langle g | e \rangle \langle n+1 | n-1 \rangle + \sqrt{n+1} e^{-i(\omega-v)t} \langle g | e \rangle \langle n+1 | n+1 \rangle \right] \\
& = -ig \sqrt{n+1} e^{-i(\omega-v)t} C_{e,n}(t) \quad \text{(2.24)}
\]

The same line of reasoning can be used to derive the second Equation of motion given below

\[
C_{e,n}(t) = -ig \sqrt{n+1} e^{i(\omega-v)t} C_{g,n+1}(t) \quad \text{(2.25)}
\]
Quantum Dynamics

In order to study the quantum dynamics of such a system, one must solve the Equations of motion, which consist of the following system of coupled first-order ordinary differential Equations.

\[
\begin{align*}
\dot{C}_{g,n+1}(t) &= -ig \sqrt{n + 1} e^{-i\Delta t} C_{e,n}(t) \\
\dot{C}_{e,n}(t) &= -ig \sqrt{n + 1} e^{i\Delta t} C_{g,n+1}(t)
\end{align*}
\]

(2.26)

where \( \Delta = \omega - \nu \) corresponds to the detuning energy. A general solution for the slowly varying probability amplitudes is given below

\[
C_{g,n+1}(t) = \left\{ C_{g,n+1}(0) \left[ \cos \left( \frac{\Omega_n}{2} t \right) + i \frac{\Delta}{\Omega_n} \sin \left( \frac{\Omega_n}{2} t \right) \right] \\
- i \frac{2g}{\Omega_n} \sqrt{n + 1} C_{e,n}(0) \sin \left( \frac{\Omega_n}{2} t \right) \right\} e^{-i\Delta t/2}
\]

(2.27)

\[
C_{e,n}(t) = \left\{ C_{e,n}(0) \left[ \cos \left( \frac{\Omega_n}{2} t \right) - i \frac{\Delta}{\Omega_n} \sin \left( \frac{\Omega_n}{2} t \right) \right] \\
- i \frac{2g}{\Omega_n} \sqrt{n + 1} C_{g,n+1}(0) \sin \left( \frac{\Omega_n}{2} t \right) \right\} e^{i\Delta t/2}
\]

where
\[
\Omega_n = \sqrt{\Delta^2 + 4g^2(n+1)}
\] (2.28)

It is interesting to note that interaction frequency \( \Omega_n \) is dependent on number of photons in the mode, which is purely a quantum phenomenon.

Rabi Oscillation

It is assumed that there is only one photon in the mode (\( n = 0 \)). If initially the two-level atom is in the ground state (\( C_{g,0}(0) = 1 \) and \( C_{e,0}(0) = 0 \)), then the Equations (2.27) simplify to

\[
C_{g,1}(t) = \left[ \cos \left( \frac{\Omega_0}{2} t \right) + i \frac{\Delta}{\Omega_0} \sin \left( \frac{\Omega_0}{2} t \right) \right] e^{-i\Delta t/2}
\]

\[
C_{e,0}(t) = \left[ -i \frac{2g}{\Omega_0} \sin \left( \frac{\Omega_0}{2} t \right) \right] e^{i\Delta t/2}
\] (2.29)

where

\[
\Omega_0 = \sqrt{\Delta^2 + 4g^2}
\] (2.30)

Starting from Equation (2.29), the probabilities of finding an electron in the excited or ground states can be derived using the following identity
\[ z^2 = z\overline{z} = (a - ib)(a + ib) = a^2 - (ib)^2 = a^2 + b^2 \]  \hfill (2.31)

Thus, the probabilities are

\[
\begin{align*}
|C_{g,1}(t)|^2 &= \left[ \cos\left(\frac{\Omega_0}{2}t\right) \right]^2 + \left[ \frac{\Delta}{\Omega_0} \sin\left(\frac{\Omega_0}{2}t\right) \right]^2 \\
|C_{e,0}(t)|^2 &= \left[ \frac{2g}{\Omega_0} \sin\left(\frac{\Omega_0}{2}t\right) \right]^2
\end{align*}
\]  \hfill (2.32)

It can readily be seen that \( |C_{g,1}(t)|^2 + |C_{e,0}(t)|^2 = 1 \). Figure 2-2 shows the time evolution of these probabilities for \( \Delta = 0 \) and Figure 2-3 shows the time evolution of these probabilities for \( |\Delta| = g \).
$\Delta t = \frac{\pi}{2g}$, the electron is occupying the upper energy level, and then at $t = \frac{\pi}{g}$ it is back to the lower energy level. The electron oscillates back and forth between the excited and ground states at a frequency equal to $\frac{g}{\pi}$, which is known as Rabi oscillation.
As expected, the probabilities oscillate at the faster frequency $\Omega_0$ where

$$\Omega_0 = \sqrt{\Delta^2 + 4g^2}$$

as opposed to just $g$. Another consequence of having a detuned field is that the probabilities do not any longer oscillate with a full swing.

Spectrum

Rabi oscillation can occur both in the single-photon (or few photons) regime as well as in the case of a very large number of photons: At the single photon (or few
photons) level, a high-Q cavity mode and low temperature are required to observe Rabi oscillation [20, 48, 49]. High-Q cavities allow for the atom-photon interaction to be faster than the irreversible process due to the loss of photons out of the cavity mode while low temperature allows for the atom-photon interaction to be faster than dipole dephasing mechanisms that are responsible for the irreversible process of spontaneous emission. Once these conditions are met, the atom-photon interaction is said to be in the strong coupling regime. On the other hand, in the case of the very large number of photons, a high-intensity coherent field is sufficient to observe Rabi oscillation [50, 51]. The later case is related to the classical AC Stark Effect, and the resulting spectrum is what is known as the Mallow triplet[52, 53]. Interestingly, Rabi oscillation in the single-photon regime yields a different spectrum.

The “dressed atom” picture provides much insight in understanding the spectrum of the JC model undergoing Rabi oscillation. The particular case of the JC model when there is only a single photon in the cavity mode is considered. The “dressed” atom depicted in Figure 2-4 referred to the states of the coupled atom-field system as opposed to those of the uncoupled atom and field system. A splitting of the excited state of the coupled atom-field system occurs because of state mixing ensuing from the electric-dipole interaction; this is known as the vacuum Rabi splitting (VRS) in cavity quantum electrodynamics. A general expression for the splitting energy is \( \Delta E_n = 2\hbar g \sqrt{n} \) [46]. The factor of 2 comes from the fact the field is a standing wave in the cavity; whereas, the factor of \( \sqrt{n} \) is the result of the field being quantized.
The signature spectrum for such system is a *doublet*. In practice, to resolve the occurrence of the VRS, the line width of the coupled atom-photon system $\Gamma$ have to be narrower than the splitting $2g$. The power spectrum for single quantum dot-nanocavity systems undergoing Rabi oscillation with only a single photon in the cavity mode is given by the sum of the background radiation-mode spectrum $S_{\text{cav}}(w)$ and the cavity-mode spectrum $S_{\text{cav}}(w)$ [54, 55].
\[ S_{\text{rad}}(w) = \Gamma_b \left| \frac{(w + \omega)(w^2 + \nu^2 + iw\Gamma_{\text{cav}})}{(w^2 - \omega^2 + i\omega\gamma)(w^2 - \nu^2 + iw\Gamma_{\text{cav}}) - (\Delta^2 + 4g^2)w\nu} \right|^2 \]

\[ S_{\text{cav}}(w) = \Gamma_{\text{cav}} \left| \frac{\sqrt{\Delta^2 + 4g^2} \nu(w + \omega)}{(w^2 - \omega^2 + i\omega\gamma)(w^2 - \nu^2 + iw\Gamma_{\text{cav}}) - (\Delta^2 + 4g^2)w\nu} \right|^2 \]

(2.33)

where \( \omega \) is the frequency difference of the 2-level system, \( \nu \) the frequency of the photon, \( \Delta \) the detuning frequency, \( g \) the interaction frequency, \( \gamma \) atomic decay rate, \( \Gamma_{\text{cav}} \) the cavity decay rate, and \( \Gamma_b \) the background emission rate.

Figure 2-5 below plots the power spectrum as a function of the detuning energy. At zero detuning (\(|\Delta| = 0\)), the energy is spread evenly between the polariton branches. At large detuning (\(|\Delta| \gg 0\)); because it is the energy of the 2-level system that is scanned across the cavity mode energy, the polariton branch that is photon-like contains most of the energy.
As a consequence of recent progresses in semiconductor nanostructures along with the versatility of photonic crystal in confining and manipulating light, a single “artificial” atom or quantum dot together with a single quantum of light can be isolated within a nanocavity[46], providing the suitable environment for the realization of
quantum network nodes. Modeling quantum network nodes involves the ability to describe (1) a quantum dot, (2) a single-photon field and (3) the interaction between these two in a nanocavity. It turns out that quantum network nodes can be described by a model very similar to the JC Model as shown below in Figure 2-6. The main difference between this basic cavity QED model for the quantum network nodes and the JC model is the construction of the two-level system and the inclusion of losses, that is to say the dipole dephasing rate designated $\gamma_{12}$ and the cavity decay rate denoted $\Gamma_{cav}$.

![Figure 2-6: Quantum Network Node System](image)

The two-level system depicted in Figure 2-6 has the following new characteristics; it is 2-fold degenerate in the excited state and 4-fold degenerate in the ground state. All the ground state levels are occupied while only one electron occupies the 2-fold degenerate excited state resulting in principle in an “infinite” decay time, which is the intent here. The ability for the 2-fold degenerate excited state to retain this
electron is critical to store and process quantum information in our scheme as we will see in the rest of this chapter.

What about the losses, where do they come from? The dipole dephasing rate includes both the *longitudinal* relaxation and *transverse* relaxation. Included in the longitudinal relaxation process are the *radiative* and *non-radiative* decay rates of the excited state. The former is affected by the emission of a cavity resonant photon in the continuum (outside the cavity) or a cavity non-resonant photon (two-level system decaying to other energy levels), and the later usually relates to the two-level system being scattered into other states without the emission of a photon. The transverse relaxation process is caused by various dephasing processes resulting in a sudden phase change of the wave function without altering the population of the excited state. Often though, it is the result of the two-level system interacting with impurities or phonons. Last, the cavity decay rate solely owes its existence to the fact that real world cavities have a finite Q factor, resulting in the photon in the cavity mode eventually leaking out.

But, how exactly such basic cavity QED model for the quantum network nodes in Figure 2-6 could be used to implement the various qubits mentioned in chapter 1, namely the *storage* qubit, the *traveling* qubit and the *exchange of interaction* qubit? First, the storage qubit consists of the spin of the electron in the excited state $|e\rangle$, whose state can be expanded onto the new and degenerate eigenbasis set $\{|\uparrow\rangle, |\downarrow\rangle\}$ as depicted in Figure 2-7.
What about the traveling qubit? The polarization states of the single-photon in the cavity mode, whether in the linear or circular polarization eigenbasis, can be used to describe the states of the traveling qubit as portrayed in Figure 2-8. It turns out that the implementation of the quantum network nodes using semiconductor QD will require the representation of the traveling qubit to be in the circular polarization eigenbasis. Just like in the case of the electron spin representing the storage qubit, the number state or Fock state of the photon denoted $|1\rangle$ (one photon in the cavity mode) needs to be expanded onto a different eigenbasis set, $\{|\sigma^+,\sigma^-\rangle\}$ or $\{|x\rangle,|y\rangle\}$, which are also both degenerate if the appropriate cavity symmetry is considered.
Last, the exchange qubit takes on the form of the “dressed” states in the “dressed” atom picture, specifically $|e,0\rangle$ and $|g,1\rangle$. Due to the VRS, the states of the exchange qubit unlike the other qubits are not degenerate. However, each one of the eigenstates can be expanded in the some degenerate eigenbasis set including the one used to describe the traveling qubit. As a result, this qubit has a 2-fold degenerate excited or excitonic state and a 2-fold ground or photonic state; those are illustrated in Figure 2-9.
Figure 2-9: Cavity Exciton-Polariton as the Exchange Qubit

Notice that one excitonic state is the result of the creation of one electron in the excited state while living an empty space or “hole” in the ground state. And since that there are two degenerate ground states, the heavy and light hole bands, the result is degenerate excitonic states, $|X_1\rangle$ and $|X_2\rangle$, created from electrons from either the heavy or light hole band depicted respectively by a longer or shorter arrow for their spin (clear electrons in Figure 2-9).

How does such two-level system relate to the complex energy level structure found in semiconductor though? The next section will answer this question and discuss the resulting appropriate Hamiltonians for semiconductor QD. Also, Hamiltonians for the photon and the QD-photon interaction are introduced.
Hamiltonian

Similar to the JC Model Hamiltonian, the Hamiltonian for the quantum node system is separated into three terms that are the QD Hamiltonian, the single-photon field Hamiltonian and the QD-photon interaction Hamiltonian.

\[ \hat{H} = \hat{H}_{QD} + \hat{H}_{Photon} + \hat{H}_{QD-Photon} \] (2.34)

QD Hamiltonian

The difficulty when modeling semiconductor or other solid-state quantum system arises from the treatment of the periodic potential seen by electrons moving about the crystal as a consequence of ions being arranged in a lattice. Detailed semiconductor band structure calculations are not only beyond the scope of this dissertation but can be found in many textbook [56-58]; therefore, only few important results will be briefly summarized in this section. The typical band structure for a direct band gap semiconductor (which we consider in this dissertation) using the Kane model for the conduction band and Kohn-Luttinger model for the valence bands is shown in Figure 2-10.
Figure 2-10: Band Structure of Direct Band gap Semiconductor

Notice that the eigenbasis in which the Kane and Kohn-Luttinger Hamiltonians are diagonalized consist of the eigenstates of the total angular momentum operator $|j, m_j\rangle$. How come? In element semiconductor of the group IV or compound semiconductor of the groups III-V or II-IV, electrons in the outer shell populate the $sp^3$ orbitals. It turns out that for the cubic symmetry, at the center of its highly spherically symmetric Brillouin zone, the conduction band is very well approximated by the s-orbital
state and the valence band states by the p-orbital states. Thus, at first the idea is that
the eigenfunctions of the angular momentum operator $\hat{L}$, which are defined as $|l, m_l\rangle$, could be used as basis states. However, because an electron in a p-orbital state has an
orbital momentum of $\hbar$, that is the quantum number corresponding to the angular
momentum is $l = 1$ as opposed to $l = 0$ for s-orbital states, there is a strong interaction
of the spin with the orbital motion of the electron in the valence band states. As a
consequence, only the total angular momentum (sum of the orbital angular momentum
and the spin angular momentum) is conserved, it is thus better to transform to an
eigenstate basis of the total angular momentum operator $\hat{J} = \hat{L} + \hat{s}$ denoted $|j, m_j\rangle$ where $|l - s| \leq j \leq l + s$ and $m_j = -j, -j+1, ..., j-1, j$. In the end, the conduction band
corresponding to $l = 0$ is described by $\left|\frac{1}{2}, \pm \frac{1}{2}\right\rangle$ while the valence bands corresponding
to $l = 1$ are described by $\left|\frac{3}{2}, \pm \frac{3}{2}\right\rangle$ for the heavy-hole band, $\left|\frac{3}{2}, \pm \frac{1}{2}\right\rangle$ for the light-hole
band, and $\left|\frac{1}{2}, \pm \frac{1}{2}\right\rangle$ for the split-off band. As a result of spin-orbit coupling, the split-off
band is shifted to lower energies while the heavy-hole and light-hole bands remain
degenerate at $\Gamma$ point.

Next, the creation of a semiconductor nanostructure such as a QD result in
discrete energy levels as the electron wave function is now bound within the
nanostructure. Here, it is also assumed that the quantum dot is spherical in shape
resulting in the confining potential with symmetry approximately identical to that of the
first Brillouin zone of the semiconductor crystal. Accordingly, only a shift in the energy levels occurs while the degeneracy of the heavy-hole and light-hole bands is conserved. The importance of maintaining this degeneracy will be further explained in the next chapter. In the two-level approximation (split-off band is assumed way out of resonance with the photon), the QD system can be described by the following two-level system depicted in Figure 2-11 with a 2-fold degeneracy in the conduction band or “excited” state and a 4-fold degeneracy in the valence bands or “ground” state.

![Two-level System Approximation of a Semiconductor QD](image)

Notice that the eigenstates of the QD, \( \left| \frac{1}{2}, \pm \frac{1}{2} \right\rangle_e \) and \( \left| \frac{3}{2}, \pm \frac{3}{2} \right\rangle_{hh} \), can be used to implement the degenerate eigenstates of the storage qubit, \( \uparrow \) and \( \downarrow \) respectively, since they are states of pure spin. Similarly, the heavy-hole band states are states of pure spin also, \( \left| \frac{3}{2}, \pm \frac{3}{2} \right\rangle_{hh} \) and \( \left| \frac{1}{2}, \pm \frac{1}{2} \right\rangle_{lh} \).
\[ \begin{aligned}
\left| \frac{3}{2},-\frac{3}{2} \right\rangle_{j,m_j} &= \left| l,m \right\rangle \downarrow; \text{ however, the light-hole band states are mixed-spin states,} \\
\text{where } \left| \frac{3}{2}, \frac{1}{2} \right\rangle_{j,m_j} &\text{ corresponds to the state } \sqrt{\frac{1}{3}} \left| 1,1 \right\rangle_{l,m_l} \downarrow + \sqrt{\frac{2}{3}} \left| 1,0 \right\rangle_{l,m_l} \uparrow \text{ and } \left| \frac{3}{2}, -\frac{1}{2} \right\rangle_{j,m_j} \\
\text{to the state } \sqrt{\frac{1}{3}} \left| 1,-1 \right\rangle_{l,m_l} \uparrow + \sqrt{\frac{2}{3}} \left| 1,0 \right\rangle_{l,m_l} \downarrow. \text{ Now remains the experimental challenge of populating these energy levels such that the doubly degenerate valence bands are fully occupied while only one electron occupies one of the spin-degenerate conduction bands.}
\end{aligned} \]

Now that spherical semiconductor QD were effectively shown to be two-level systems and that the wave function of the electrons occupying their energy levels could be expressed as linear combinations of the eigenfunctions of the orbital angular momentum (spherical harmonics) and spin, how can a QD Hamiltonian be articulated in view of studying the cavity dynamics of the quantum network node system?

It is important to note that the QD as a pseudo two-level atomic system for the implementation of the quantum network node comprises five electrons occupying its energy levels as shown in Figure 2-12. The assumption is that the excess electron spin is again initialized to \( \uparrow \). Usually, a slightly different approach than using the atomic transition operators such as in the JC model would have been desirable, namely using the creation and annihilation operators for the QD energy levels within the framework of the second quantization (particle themselves are represented by quantized fields rather than just a quantization of their motion characterized by their wave function). In the
second quantization formalism, the basis that is used describes the number of particles occupying each state in a complete set of single-particle states. The creation and annihilation operators for fermions act on the abstract occupation-number Hilbert space by increasing or decreasing respectively the occupation number of the state, which can only be 0 or 1 only due to the Pauli’s exclusion principle. This approach has the benefit of greatly simplifying the discussion of many interacting particles and incorporating statistics.

![Figure 2-12: QD System for the Quantum Network Node](image)

However, we are going to ignore in this dissertation the effect of Coulomb interaction between the five electrons occupying the QD energy levels and the reservoir that is the other electrons and nuclei forming the QD. Thus, we are going to consider also transition operators to describe the QD energy levels. As a starting point for the QD Hamiltonian, let us consider an expression similar to the two-level system in the JC model.
\[ \hat{H}_{QD, dyn} = (\hat{i})\hat{H}_{QD}(\hat{i}) \]
\[ = (|g\rangle\langle g| + |e\rangle\langle e|) \hat{H}_{QD} (|g\rangle\langle g| + |e\rangle\langle e|) \]
\[ = E_g |g\rangle\langle g| + E_e |e\rangle\langle e| \]
\[ = E_g \hat{\sigma}_{gg} + E_e \hat{\sigma}_{ee} \]  

(2.35)

Then, the Hamiltonian above in terms of the atomic transition operators can be rewritten such that the 2-fold degenerate excited state and the 4-fold degenerate ground state of the QD are taken into account; the Hamiltonian becomes

\[ \hat{H}_{QD, dyn} = \hbar \omega_{hh} \hat{\sigma}_{3/2v,3/2v} + \hbar \omega_{eh} \hat{\sigma}_{3/2v,-3/2v} \]
\[ + \hbar \omega_{eh} \hat{\sigma}_{1/2v,1/2v} + \hbar \omega_{eh} \hat{\sigma}_{-1/2v,-1/2v} \]
\[ + \hbar \omega_{ec} \hat{\sigma}_{1/2c,1/2c} + \hbar \omega_{ec} \hat{\sigma}_{-1/2c,-1/2c} \]  

(2.36)

where \( \hbar \omega_{hh} = \hbar \omega_{eh} = E_g \) and \( \hbar \omega_{ec} = E_e \) and the operator indices can be related to the QD eigenstates in the following way

\[ \begin{align*}
|\frac{3}{2}, \frac{3}{2}\rangle &= 3/2v, & |\frac{3}{2}, -\frac{3}{2}\rangle &= -3/2v \\
|\frac{3}{2}, \frac{1}{2}\rangle &= 1/2v, & |\frac{3}{2}, -\frac{1}{2}\rangle &= -1/2v \end{align*} \]  

(2.37)

\[ \begin{align*}
|\frac{1}{2}, \frac{1}{2}\rangle &= 1/2c, & |\frac{1}{2}, -\frac{1}{2}\rangle &= -1/2c \\
\end{align*} \]
Photon Hamiltonian

The single-photon field can stretch to every part of the quantum network, yet it can be highly localized in cavity or a set of cavities. We will eventually discuss these two different cases and their advantages and disadvantages. At this point, since we are interested in a single cavity system or node, we will consider the highly localized case. The fact that the QD system as a pseudo two-level system can be expressed in the total angular momentum (orbital angular momentum and spin) eigenbasis, there are clearly defined optical transition rules for the electrical dipole interaction between ground and excited states, namely that the orbital angular momentum quantum number $l$ changes by $\pm 1$, and the spin is conserved as well as the parity of the envelop function. Furthermore, it ensues that in the Faraday geometry (the quantization axis of the excess electron spin is parallel to the direction of light propagation that is the z direction), which is the configuration considered for this quantum network, the empty state in the conduction band is populated by circular polarized light (either $\sigma^+$ or $\sigma^-$ depending on the excess electron spin state). We already established in Figure 2-8 that the photon number state could be expanded into the degenerate eigenbasis set of the single-photon polarization to implement the traveling qubit under the appropriate cavity design. We can thus rewrite the Hamiltonian for the single-photon field in Equation (2.5) in terms of the creation and annihilation operators for a circularly polarized field such as
\[
\hat{H}_{\text{Field,dyn}} = \hbar \nu \left( \hat{a}_{\sigma +} \hat{a}_{\sigma +} + \hat{a}_{\sigma -} \hat{a}_{\sigma -} \right)
\] (2.38)

QD-Photon Hamiltonian

Last, in similar ways to the J-C Hamiltonian in Equation (2.6), the QD-photon interaction Hamiltonian can be derived from the completeness relation

\[
\begin{align*}
\left( \left\langle \frac{3}{2} \right\rangle \left\langle \frac{3}{2} \right| + \left\langle -\frac{3}{2} \right\rangle \left\langle -\frac{3}{2} \right| + \\
\left\langle \frac{1}{2} \right\rangle \left\langle \frac{1}{2} \right| + \left\langle -\frac{1}{2} \right\rangle \left\langle -\frac{1}{2} \right| + \\
\left\langle \frac{1}{2} \right\rangle \left\langle \frac{1}{2} \right| + \left\langle -\frac{1}{2} \right\rangle \left\langle -\frac{1}{2} \right| \right) = \sum_v \left| v \right\rangle \left\langle v \right| + \sum_c \left| c \right\rangle \left\langle c \right| = 1
\end{align*}
\] (2.39)

It is thus written in the electric dipole approximation as
\[ \hat{H}_{\text{QD-Photon}} = -e \hat{r} \cdot \hat{E} \]

\[ = -e (\hat{\mathbf{r}} \hat{\mathbf{E}}) \left( \sum_v |v\rangle \langle v| + \sum_c |c\rangle \langle c| \right) \left( \hat{\Sigma}_{v,c} + \hat{\Sigma}_{c,v} \right) \]

Expanding this Hamiltonian out, and then simplifying using the rotating wave approximation, we obtain

\[ \hat{H}_{\text{QD-Photon}} = \hbar g_{3/2} \hat{\Sigma}_{3/2,v,-1/2c} + \hbar g_{-1/2} \hat{\Sigma}_{1/2,c,1/2v} + \hbar g_{1/2} \hat{\Sigma}_{1/2,v,-1/2c} \]

(2.41)

where the various coupling strengths from states with ±3/2 total angular momentum projection number to state with ±1/2 total angular momentum projection number are real and the same and can thus be denoted \( g_{3/2} \), and the various coupling strengths from states with ±1/2 total angular momentum projection number to state with ±1/2
total angular momentum projection number are also real and the same and can therefore be denoted \( g_{1/2} \).

The total Hamiltonian for the quantum network node is thus written as

\[
\dot{\hat{H}} = \hat{H}_{QD,dyn} + \hat{H}_{Field,dyn} + \hat{H}_{QD-Photon} \\
= \left( \hbar \omega_{hh} \hat{\sigma}^{3/2}_{3v,3/2v} + \hbar \omega_{hh} \hat{\sigma}^{-3/2}_{3v,-3/2v} \right) \\
+ \left( \hbar \omega_{lh} \hat{\sigma}^{1/2}_{1/2v,1/2v} + \hbar \omega_{lh} \hat{\sigma}^{-1/2}_{1/2v,-1/2v} \right) \\
+ \left( \hbar \omega_{lh} \hat{\sigma}^{1/2}_{1/2c,1/2c} + \hbar \omega_{hc} \hat{\sigma}^{-1/2}_{1/2c,-1/2c} \right) \\
+ \left( \hbar g_{3/2} \hat{a}^{\dagger}_{\sigma} \hat{\sigma}^{3/2}_{3v,1/2c} + \hbar g_{3/2} \hat{\sigma}^{3/2}_{3v,3/2v} \right) \\
+ \left( \hbar g_{3/2} \hat{a}^{\dagger}_{\sigma} \hat{\sigma}^{1/2}_{1/2c,3/2v} + \hbar g_{3/2} \hat{\sigma}^{1/2}_{1/2c,3/2v} \right) \\
+ \left( \hbar g_{1/2} \hat{a}^{\dagger}_{\sigma} \hat{\sigma}^{1/2}_{1/2c,1/2c} + \hbar g_{1/2} \hat{\sigma}^{1/2}_{1/2c,1/2c} \right) \\
+ \left( \hbar g_{1/2} \hat{a}^{\dagger}_{\sigma} \hat{\sigma}^{1/2}_{1/2c,1/2c} + \hbar g_{1/2} \hat{\sigma}^{1/2}_{1/2c,1/2c} \right)
\]

(2.42)

**Equations of Motion**

For the quantum network nodes, we will consider two approaches to deriving the Equations of motion for the probability amplitudes associated with the state vector, namely the Schrödinger’s Equation formalism and the density matrix formalism based on the Heisenberg picture of quantum mechanics. What are the advantages and disadvantages of these two approaches?

Recalling that inside the quantum network node, the cavity-QED qubit (exchange or interaction qubit) provides the coherent interface needed to either (1) map the state of the excess electron spin onto the polarization of a single photon or (2) entangle these
two particles. If one is interested in the mapping operation only, the Schrödinger’s Equation formalism accurately describe the process only using a system of \( m \) second order ODE where \( m \) is the size of the state vector. On the other hand, only the density matrix formalism is useful when describing such an ensemble of systems (in our case a spin qubit and photon qubit) with regards to entanglement and quantum decoherence. However, using the density matrix formalism, a system of \( m^2 \) second order ODE needs to be solved.

Schrödinger’s Equation Formalism

In order to derive the Equations of motion, it is convenient to work in the interaction picture just as in the case of the JC model. First, the Hamiltonian needs to be rewritten in the following form

\[
\hat{H}_0 = \hbar \omega_{hh} \hat{\sigma}_{3/2v,3/2v} + \hbar \omega_{hh} \hat{\sigma}_{-3/2v,-3/2v} + \hbar \omega_{hh} \hat{\sigma}_{1/2v,1/2v} + \hbar \omega_{hh} \hat{\sigma}_{-1/2v,-1/2v} + \hbar \omega_c \hat{\sigma}_{1/2c,1/2c} + \hbar \omega_c \hat{\sigma}_{-1/2c,-1/2c} + \hbar \nu \left( \hat{a}_{\sigma}^\dagger \hat{a}_{\sigma}^\dagger + \hat{a}_{\sigma} \hat{a}_{\sigma} \right)
\]

(2.43)

\[
\hat{H}_1 = \hbar g_{3/2} \left( \hat{a}_{\sigma}^\dagger \hat{\sigma}_{-3/2v,1/2c} + \hat{a}_{\sigma} \hat{\sigma}_{3/2v,1/2c} + \hat{\sigma}_{-1/2c,-3/2v} \hat{a}_{\sigma}^\dagger + \hat{\sigma}_{1/2c,3/2v} \hat{a}_{\sigma} \right) + \hbar g_{1/2} \left( \hat{a}_{\sigma}^\dagger \hat{\sigma}_{-1/2v,1/2c} + \hat{a}_{\sigma} \hat{\sigma}_{1/2v,-1/2c} + \hat{\sigma}_{-1/2c,-1/2v} \hat{a}_{\sigma}^\dagger + \hat{\sigma}_{1/2c,1/2v} \hat{a}_{\sigma} \right)
\]

Using a similar approach as in section 0, the interaction Hamiltonian in the interaction picture becomes
\[
\hat{V} = \hbar g_{3/2} \left( \hat{a}_{\sigma_+}^{\dagger} \sigma_{-3/2V, -1/2c}^{\dagger} + \hat{a}_{\sigma_-}^{\dagger} \sigma_{3/2V, 1/2c}^{\dagger} \right) e^{-i(\omega - \nu)t} \\
+ \hbar g_{5/2} \left( \sigma_{-1/2c, -3/2V}^{\dagger} \hat{a}_{\sigma^+}^{\dagger} + \sigma_{1/2c, 3/2V}^{\dagger} \hat{a}_{\sigma_-}^{\dagger} \right) e^{i(\omega - \nu)t} \\
+ \hbar g_{3/2} \left( \hat{a}_{\sigma_+}^{\dagger} \sigma_{-1/2c, -3/2V}^\dagger + \hat{a}_{\sigma_-}^{\dagger} \sigma_{1/2c, 3/2V}^\dagger \right) e^{-i(\omega - \nu)t} \\
+ \hbar g_{5/2} \left( \sigma_{1/2c, -1/2V}^{\dagger} \hat{a}_{\sigma^+}^{\dagger} + \sigma_{-1/2c, 1/2V}^{\dagger} \hat{a}_{\sigma_-}^{\dagger} \right) e^{i(\omega - \nu)t} 
\]

(2.44)

Next, general expression for the time-dependent state vector of the system is

\[
|\psi(t)\rangle = C_{\downarrow, \sigma_z^-} \langle \downarrow, \sigma_z^- | + C_{\downarrow, \sigma_z^+} \langle \downarrow, \sigma_z^+ | + C_{\downarrow, \sigma_{lh}^\dagger} \langle \downarrow, \sigma_{lh}^\dagger | + C_{\downarrow, \sigma_{hh}^\dagger} \langle \downarrow, \sigma_{hh}^\dagger | + C_{\uparrow, \sigma_z^-} \langle \uparrow, \sigma_z^- | + C_{\uparrow, \sigma_z^+} \langle \uparrow, \sigma_z^+ | + C_{\uparrow, \sigma_{lh}^\dagger} \langle \uparrow, \sigma_{lh}^\dagger | + C_{\uparrow, \sigma_{hh}^\dagger} \langle \uparrow, \sigma_{hh}^\dagger | (2.45)
\]

where \( |\downarrow, \sigma_z^- \rangle \) corresponds to the excess electron spin being down with a left circularly polarized (LCP or \( \sigma_z^- \)) photon in the cavity, \( |\downarrow, \sigma_z^+ \rangle \) to the excess electron spin being down with a right circularly polarized (RCP or \( \sigma_z^+ \)) photon in the cavity, \( |\downarrow, \sigma_{lh}^\dagger \rangle \) to the excess electron spin being down with a light-hole exciton, \( |\downarrow, \sigma_{hh}^\dagger \rangle \) to the excess electron spin being down with a heavy-hole exciton, \( |\uparrow, \sigma_z^- \rangle \) to the excess electron spin being up with a LCP photon in the cavity, \( |\uparrow, \sigma_z^+ \rangle \) to the excess electron spin being up with a RCP photon in the cavity, \( |\uparrow, \sigma_{lh}^\dagger \rangle \) to the excess electron spin being up with a light-hole exciton, and \( |\uparrow, \sigma_{hh}^\dagger \rangle \) to the excess electron spin being up with a heavy-hole exciton. In the interaction picture, the various time-dependent coefficients already
correspond to the *slowly varying probability amplitudes*. Let us assume for the sake of simplicity that the spin of the excess electron is initialized to the up state $|\uparrow\rangle$, this results in the following general state vector

$$|\psi(t)\rangle = C_{\uparrow,\sigma_z^-(i)}|\uparrow,\sigma_z^-angle + C_{\uparrow,\sigma_z^+(i)}|\uparrow,\sigma_z^+angle + C_{\uparrow,X_{lh}}(t)|\uparrow,X_{lh}\rangle + C_{\uparrow,X_{hh}}(t)|\uparrow,X_{hh}\rangle \quad (2.46)$$

Using the same line of reasoning as in the case of the JC model, one can derive the subsequent Equations of motion

$$\dot{C}_{\uparrow,\sigma_z^-} = -ig_{1/2}e^{-i(\omega-v)t}C_{\uparrow,X_{lh}} \quad (2.47)$$

$$\dot{C}_{\uparrow,\sigma_z^+} = -ig_{3/2}e^{-i(\omega-v)t}C_{\uparrow,X_{hh}}$$

$$\dot{C}_{\uparrow,X_{lh}} = -ig_{1/2}e^{i(\omega-v)t}C_{\uparrow,\sigma_z^-}$$

$$\dot{C}_{\uparrow,X_{hh}} = -ig_{3/2}e^{i(\omega-v)t}C_{\uparrow,\sigma_z^+}$$

Without going into mathematical details of the dipole interaction, it can be shown just from selection rule (magnetic quantum number changes by $\Delta l = \pm 1$ while the spin is conserved) why the Equation of motion for the probability amplitude of the LCP component of the field is only dependent on the probability amplitude of the light hole exciton and not that of the heavy hole exciton whereas the Equation of motion for the probability amplitude of RCP component of the field is only dependent on the probability amplitude of the heavy hole exciton and not that of the light hole exciton. Also, this can
be observed in the case of the Equations of motion for the probability amplitudes of the light and heavy holes. This is illustrated in Figure 2-13 below.

![Figure 2-13: Dipole Selection Rules in the Quantum Network Nodes](image)

Clearly, a transition from the heavy hole state \( \left| \frac{3}{2}, -\frac{3}{2} \right\rangle_{hh} \) to the conduction band state \( \left| \frac{1}{2}, -\frac{1}{2} \right\rangle_{j,mj} \) is only possible by means of a RCP or photon. A transition from the light hole band state \( \left| \frac{1}{2}, -\frac{1}{2} \right\rangle_{j,mj} \) to the conduction band state \( \left| \frac{3}{2}, -\frac{3}{2} \right\rangle_{hh} \) is only possible by means of a LCP photon. However, notice that only the first component of the light hole state \( \sqrt[3]{1} \left| \frac{1}{2}, -\frac{1}{2} \right\rangle_{j,mj} \) really has a nonzero transition matrix element though. Furthermore, there cannot be a transition from the
heavy hole state \( |\frac{3}{2}, \frac{3}{2}\rangle_{j,mj} \) to the conduction band state \( |1,1\rangle_{l,mf} |\uparrow\rangle \) the spin would not be conserved. Similarly, a transition from the light hole state \( |\frac{1}{2}, -\frac{1}{2}\rangle_{j,mj} \) to the conduction band state \( |0,0\rangle_{l,mf} |\downarrow\rangle \) is not allowed since the spin in the first term would not be conserved and the magnetic quantum number in second term would not change by ±1.

So the coupling strength associated with the single photon field and the heavy hole charged exciton can be expressed as [46, 47]

\[
g_{3/2} = \frac{\hat{\mu}_v \cdot \vec{E}}{\hbar} = \frac{\mu_v E_{\text{vac}}}{\hbar} \tag{2.48}
\]

where the magnitude of the vacuum field can be written

\[
E_{\text{vac}} = \left( \frac{\hbar \nu}{2 \varepsilon_0 V_0} \right)^{1/2} \tag{2.49}
\]

thus \( g_{3/2} \) can be calculated if the dipole moment associated with the QD heavy hole electrons \( \mu_v \), the frequency of the photon \( \nu \) and the cavity mode volume \( V_0 \) are known. As for the coupling strength associated with the single photon field and the light hole
charged exciton can be expressed as \( s_{1/2} = \sqrt{\frac{1}{3}} s_{3/2} \); this is directly related to the dipole selection rules and the oscillator strength linked to light hole excitons as discussed in the previous paragraph (i.e. only \( \frac{1}{\sqrt{3}} |l, \downarrow \rangle \) has a nonzero matrix element). This factor of \( \sqrt{\frac{1}{3}} \) in the transition strength between the heavy and light hole excitons will prove to be crucial to the scheme that is used in this research to map out quantum information from a qubit onto another or entangle qubits.

Density Matrix Formalism

This powerful and elegant formulation of quantum mechanics that is the density matrix formalism [47, 59] is useful when describing quantum entanglement. Quantum entanglement is one of the most fundamental properties of quantum mechanics and a key ingredient in quantum teleportation. Thus, understanding quantum entanglement has the benefit on one hand of gaining valuable insight into a fundamental process in quantum mechanics and on the other hand allowing us to gain control over the transfer of quantum information from one node to another within the quantum network by means of quantum teleportation.

What exactly is the density matrix though? Traditionally, quantum mechanics is formulated using the language of state vectors. Thus, any quantum system is described
by a state vector $|\Psi\rangle$ from which all possible information about the system can be extracted. A general form for a state vector can be written as

$$|\Psi\rangle = \sum_n c_n |\psi_n\rangle \quad (2.50)$$

where $c_n$ and $|\psi_n\rangle$ are the probability amplitudes and the eigenvectors of some complete set of operators. For such a system, the expectation value of an operator $\hat{O}$ represented by a matrix $O_{nn'}$ can be expressed as

$$\langle \hat{O} \rangle = \langle \Psi | \hat{O} |\Psi\rangle = \sum_{n'} \sum_n \langle \psi_{n'} | \hat{O} |\psi_n\rangle c_n^* c_{n'} = \sum_{n'} \sum_n O_{n'n} c_n^* c_{n'} \quad (2.51)$$

Now, consider the density matrix $\rho_{nn'}$ or (density operator $\hat{\rho}$) as it appears in the calculation of $\langle \hat{O} \rangle$

$$c_n^* c_{n'} = \sum_n \sum_{n'} \langle \psi_n | \Psi \rangle \langle \psi_{n'} | \Psi \rangle = \sum_n \sum_{n'} \langle \psi_n | \hat{\rho} |\psi_{n'}\rangle = \rho_{nn'} \quad (2.52)$$

The density matrix here is what is referred to as the "pure" state density matrix, and it is simply defined as

$$\hat{\rho} = |\Psi\rangle \langle \Psi| \quad (2.53)$$

Equation (2.51) can then be rewritten
\[ \langle \hat{O} \rangle = \langle \Psi | \hat{O} | \Psi \rangle \]
\[ = \sum_{n} \sum_{n'} O_{n'n} c_{n'}^{*} c_{n} \]
\[ = \sum_{n} \sum_{n'} O_{n'n} \rho_{nn'} \]
\[ = \sum_{n} \rho_{nn} \]
\[ = \text{Tr} \left[ \hat{O} \hat{\rho} \right] \]  

(2.54)

A common application of the "pure" state density matrix is found in the treatment of a two-level quantum system interacting with a EM field leading to what is referred to in quantum optics as the optical Bloch Equations. The optical Bloch Equations are the Equations of motion for the diagonal (probabilities of being in upper or lower states) and off-diagonal (atomic polarization) elements of the density matrix.

However, as it is often the case in an experimental setting, the state vector of a quantum system is not always known. We may only know only that the system is described by an ensemble of quantum states \( \{|\Psi_{i}\rangle\} \) with probabilities \( p_{i} \), then the density matrix is said to be a "mixed" state density matrix and is defined as

\[ \hat{\rho} = \sum_{i} p_{i} |\Psi_{i}\rangle \langle \Psi_{i}| = \sum_{i} p_{i} \hat{\rho}_{i} \]  

(2.55)

The expectation value of an operator \( \hat{O} \) is thus
\[
\langle \hat{O} \rangle = \sum_i p_i \langle \psi_i | \hat{O} | \psi_i \rangle \\
= \sum_{n'} \sum_n \sum_i p_i \langle \psi_i | n' \rangle \langle n' | \hat{O} | n \rangle \langle n | \psi_i \rangle \\
= \sum_{n'} \sum_n \langle n' | \hat{O} | n \rangle \sum_i p_i \langle n | \psi_i \rangle \langle \psi_i | n' \rangle \\
= \sum_{n'} \sum_n O_{nn'} \rho_{nn'} \\
= \text{Tr} \left[ \hat{O} \hat{\rho} \right] 
\]

The “mixed” state density matrix is useful also in the analysis of quantum noise which has the effect of disturbing the state of a quantum system and therefore introducing an uncertainty in the knowledge of that state. In this dissertation, we will rely heavily on the density matrix formalism (both the “pure” and also “mixed” state density matrices when considering errors in the initialization of the qubit states) as we attempt to describe the quantum dynamics within the quantum network.

Yet, the most powerful use of the density matrix is when it is used to describe a composite quantum system such as our quantum network nodes made of multiple qubit systems. First, observable quantities for subsystems of a composite system can be easily evaluated by means of the reduced density matrix. For instance, consider a composite system in the product state \( \rho^{A \otimes B} = \psi \otimes \sigma \) where \( \psi \) is the density matrix for subsystem A and \( \sigma \) is the density matrix for subsystem B. The reduced density matrix for subsystem A, \( \rho^A \), can be obtained from a partial trace over subsystem B, \( \text{Tr}_B \), such that
\[ \rho_A = Tr_B(\rho^{4\otimes B}) = \nu Tr(\sigma) = \nu \]  

Second and most importantly, using the density matrix to describe a composite quantum system proves itself indispensable to the analysis of quantum entanglement useful in quantum teleportation. In this subsequent case, the density matrix is able to describe correlations between observable physical properties of possibly remote subsystems. For instance, the entanglement of two qubits forming a composite system represented by the density matrix \( \rho^{1\otimes2} \) can be computed directly as either the Von Neumann entropy[60] or the normalized linear entropy[61] of the reduced density matrix of either of the two qubits. Equations (2.58) and (2.59) below show respectively the Von Neumann and normalized linear entropies.

\[
S_{VN} = -Tr_1\left[Tr_2\rho^{1\otimes2}\log_2\left(Tr_2\rho^{1\otimes2}\right)\right] \tag{2.58}
\]

\[
S_{NL} = 2\left[1 - Tr_1\left(Tr_2\rho^{1\otimes2}\right)^2\right] \tag{2.59}
\]

Whether it is calculated from the Von Neumann entropy or the normalized linear entropy, the entanglement between two qubits ranges from 0 to 1, where 1 means that they are maximally entangled.

Moreover, we are also interested in the time evolution of our quantum network node as a composite qubit system and which is described by a density matrix. Starting
from the Schrödinger’s Equation $|\psi\rangle = -\frac{i}{\hbar} \hat{H} |\psi\rangle$, the Equation of motion of the density matrix can be derive in the following manner

$$\frac{d}{dt} \left( \hat{\rho}^{\otimes 2} \right) = \frac{d}{dt} \left( \sum_i p_i |\Psi_i^{\otimes 2}\rangle \langle \Psi_i^{\otimes 2}| \right)$$

$$= \sum_i p_i \left( \frac{\partial}{\partial t} \left( |\Psi_i^{\otimes 2}\rangle \langle \Psi_i^{\otimes 2}| \right) + |\Psi_i^{\otimes 2}\rangle \langle \Psi_i^{\otimes 2}| \frac{\partial}{\partial t} \left( |\Psi_i^{\otimes 2}\rangle \langle \Psi_i^{\otimes 2}| \right) \right)$$

$$= \sum_i p_i \left( \left( -\frac{i}{\hbar} \hat{\mathbf{H}} |\Psi_i^{\otimes 2}\rangle \langle \Psi_i^{\otimes 2}| + |\Psi_i^{\otimes 2}\rangle \langle \Psi_i^{\otimes 2}| \frac{i}{\hbar} \hat{\mathbf{H}} \right) \right)$$

$$= -\frac{i}{\hbar} \sum_i p_i \left( \hat{\mathbf{H}} \left( |\Psi_i^{\otimes 2}\rangle \langle \Psi_i^{\otimes 2}| + \langle \Psi_i^{\otimes 2}| \langle \Psi_i^{\otimes 2}| \hat{\mathbf{H}} \right) \right)$$

$$= -\frac{i}{\hbar} \left[ \hat{\mathbf{H}}, \hat{\rho}^{\otimes 2} \right]$$

This is often referred to as the Liouville or Von Neumann Equation of motion for the density matrix where $\left[ \hat{\mathbf{H}}, \hat{\rho}^{\otimes 2} \right]$ is the commutator of the Hamiltonian and the density matrix, defined as $[A, B] = AB - BA$. What’s more, longitudinal relaxation (population decay) and transverse relaxation (pure dephasing) can be included in the Equation of motion for the density matrix by the addition of a phenomenological relaxation matrix $\hat{\mathbf{\Gamma}}$. Equation (2.60) becomes

$$\frac{d}{dt} \left( \hat{\rho}^{\otimes 2} \right) = -\frac{i}{\hbar} \left[ \hat{\mathbf{H}}, \hat{\rho}^{\otimes 2} \right] - \frac{1}{2} \left\{ \hat{\mathbf{\Gamma}}, \hat{\rho}^{\otimes 2} \right\}$$

where $\left\{ \hat{\mathbf{\Gamma}}, \hat{\rho}^{\otimes 2} \right\}$ is the anticommutator of the relaxation matrix and the density matrix,
defined as \( \{A, B\} = AB + BA \). This will allow us to study the effect of relaxation on the quantum dynamics inside the quantum network.

**Quantum Dynamics**

The assumption here is that the detuning energy is \( \Delta \geq 0 \) so that the photon energy is smaller than the transition energy. As a result, the cavity polariton created during strong interaction of the single photon with the two-level system is mostly in its photonic state rather than its excitonic state. Why is this assumption made? Whether the context of the interaction in the quantum network node is the *mapping of quantum state* (Schrödinger’s Formalism) or *quantum entanglement* (Density matrix Formalism), the single photon must eventually exit the cavity and propagate down a waveguide to a specified destination, another quantum network node or a detector. If the cavity polariton has a large probability to be in its excitonic state at the end of the interaction, then the single photon may be lost. However, there is upper limit on the value of the detuning energy. As the detuning increases, the interaction energies decrease, and the cavity polariton becomes more sensitive to various decoherence processes in addition to the need for extremely high-Q cavities to prevent the photon from escaping the cavity before the specified interaction time.
Rabi Oscillation

From the Schrödinger’s Equation formalism, the Equations of motion for the probability amplitudes $C_{\uparrow, \sigma_z}^{-}(t)$, $C_{\uparrow, \sigma_z}^{+}(t)$, $C_{\uparrow, \text{Xlh}}(t)$, and $C_{\uparrow, \text{Xhh}}(t)$ can be solved exactly subject to certain initial conditions[60]; general solutions are shown below.

\begin{align*}
C_{\uparrow, \sigma_z}^{-}(t) &= \left[ \cos\left(\frac{\Omega_{3/2}t}{2}\right) + \frac{i\Delta}{\Omega_{3/2}} \sin\left(\frac{\Omega_{3/2}t}{2}\right) \right] C_{\uparrow, \sigma_z}^{+}(0) \left( e^{-i\Delta t} \right) \\
C_{\uparrow, \sigma_z}^{+}(t) &= \left[ \cos\left(\frac{\Omega_{1/2}t}{2}\right) + \frac{i\Delta}{\Omega_{1/2}} \sin\left(\frac{\Omega_{1/2}t}{2}\right) \right] C_{\uparrow, \sigma_z}^{-}(0) \left( e^{-i\Delta t} \right) \\
C_{\uparrow, \text{Xlh}}(t) &= \left[ \cos\left(\frac{\Omega_{1/2}t}{2}\right) - \frac{i\Delta}{\Omega_{1/2}} \sin\left(\frac{\Omega_{1/2}t}{2}\right) \right] C_{\uparrow, \text{Xlh}}(0) \left( e^{-i\Delta t} \right) \\
C_{\uparrow, \text{Xhh}}(t) &= \left[ \cos\left(\frac{\Omega_{1/2}t}{2}\right) \sin\left(\frac{\Omega_{1/2}t}{2}\right) \right] C_{\uparrow, \sigma_z}(0) \left( e^{-i\Delta t} \right)
\end{align*}

where \( \Omega_{3/2} = \sqrt{\Delta^2 + 4g_{3/2}^2} \) with \( g_{3/2} \) being the coupling strength involving a heavy hole.
electron, \( \Omega_{1/2} = \sqrt{\Delta^2 + 4g_{1/2}^2} \) with \( g_{1/2} = \frac{g_{3/2}}{\sqrt{3}} \) being the coupling strength involving a light hole electron, and \( \Delta = \omega - \nu \) being the detuning frequency.

The state of the traveling qubit is modified as the single photon enters a periodic cycle of absorption-emission inside the cavity; whereas, the spin of the excess as the storage qubit is assumed for now to remain mostly unaffected. In fact, the state of the qubit only determines whether the electron that undergoes Rabi oscillation the cavity exciton-polariton system is from the heavy-hole valence band or the light-hole valence band. Here, we will mainly concern ourselves with the probability amplitudes for finding a photon in the cavity which is either right circularly polarized \( C_{\uparrow,\sigma_z^+}^{(t)} \) or left circularly polarized \( C_{\uparrow,\sigma_z^-}^{(t)} \). One reason why we are ignoring the probability amplitudes for obtaining excitons, \( C_{\uparrow,X_{hh}}^{(t)} \) and \( C_{\uparrow,X_{lh}}^{(t)} \), is because they perfectly mirror the probability amplitudes \( C_{\uparrow,\sigma_z^+}^{(t)} \) and \( C_{\uparrow,\sigma_z^-}^{(t)} \). In addition, it is our aim that the system is operated such that a photon rather than an exciton is obtained at the conclusion of the interaction.

Assuming the initial conditions are the following, \( C_{\uparrow,X_{hh}}^{(0)} = 0 \), \( C_{\uparrow,\sigma_z^+}^{(0)} = \frac{1}{\sqrt{2}} \), \( C_{\uparrow,X_{lh}}^{(0)} = 0 \), and \( C_{\uparrow,\sigma_z^-}^{(0)} = \frac{1}{\sqrt{2}} \), then the probability amplitudes of interest as defined in Equation (2.62)-(2.65) simplify to
Using Equation (2.31), the probabilities of finding a photon in the cavity with a right circular polarization or left circular polarization can be derived.

\[
C_{\uparrow, \sigma_z^+} (t) = e^{-\frac{i \Delta t}{2}} \left[ \cos \left( \frac{\Omega_{3/2} t}{2} \right) + \frac{i \Delta}{\Omega_{3/2}} \sin \left( \frac{\Omega_{3/2} t}{2} \right) \right] C_{\uparrow, \sigma_z^+} (0)
\]

(2.66)

\[
C_{\uparrow, \sigma_z^-} (t) = e^{-\frac{i \Delta t}{2}} \left[ \cos \left( \frac{\Omega_{1/2} t}{2} \right) + \frac{i \Delta}{\Omega_{1/2}} \sin \left( \frac{\Omega_{1/2} t}{2} \right) \right] C_{\uparrow, \sigma_z^-} (0).
\]

It can be readily seen as expected that the energy exchange cycle between the right circular polarized component of the photon and the heavy hole exciton is many times higher than that of between the left circular polarized component of the photon and the light hole exciton. How many times higher exactly? It depends on the values of \(\Omega_{3/2}\), \(\Omega_{1/2}\), and \(\Delta\). However, there is an important restriction that we wish to impose on the chosen values of \(\Omega_{3/2}\), \(\Omega_{1/2}\), and \(\Delta\). For a given interaction time \(t_{\text{interaction}}\), the resulting probabilities \(\left| C_{\uparrow, \sigma_z^+} (t) \right|^2\) and \(\left| C_{\uparrow, \sigma_z^-} (t) \right|^2\) must be required to match their initial values such that
\[ \left| C_{\uparrow,\sigma_z}^{+}(t_{\text{interaction}}) \right|^2 = \left| C_{\uparrow,\sigma_z}^{+}(0) \right|^2 = \frac{1}{2} \]

\[ \left| C_{\uparrow,\sigma_z}^{-}(t_{\text{interaction}}) \right|^2 = \left| C_{\uparrow,\sigma_z}^{-}(0) \right|^2 = \frac{1}{2} \]

(2.68)

This ensures that the polariton or exciton-photon system created during the strong interaction of the photon with the QD is in its photonic state rather than its excitonic state at the conclusion of an operation performed inside the quantum cavity node. This is further discussed in Chapter 4.: Figure 2-14 shows the time evolution of these probabilities when \( \Delta = 0 \), which does not seem to satisfy Equation (2.68) for any time \( t \) in the time interval considered. It does mean that the desired conditions could not be met for bigger time interval.
It is noteworthy to mention that the condition imposed by Equation (2.68) applies mostly when the detuning energy is on the same order of magnitude of the interaction energies, that is $|\Delta| \leq g_{3/2}$. This constraint can be relaxed when one considers the case where $|\Delta| \gg g_{3/2}$. Under this condition, the probabilities are asymptotically approach 1 for all $t$. This is illustrated in Figure 2-15, the detuning energy here is set approximately to $\Delta = 10g_{3/2}$.
Figure 2-15: Rabi Oscillation in Quantum Network Node with $\Delta = 10g_{3/2}$

As a conclusion, there are really two valid approaches one can explore when considering the interaction in the quantum network node so as not to lose the photon. The first approach consists of keeping the detuning energy approximately on the same order of magnitude as the interaction energies ($|\Delta| \leq g_{3/2}$) while requiring Equation (2.68) to be satisfied. The second approach favors very large detuning energies in comparison to interaction energies, which relaxes the constraints of Equation (2.68).
Spectrum

In theory, the JC model predicts a doublet as a result of the VRS that occurs in a case of a single photon in a cavity mode interacting strongly with an atom. In practice, however, both doublets [20, 62, 63] and triplets [19, 49, 55] are been reported when for instance an InAs or GaAs QD is embedded in a nanocavity. The observance of a triplet has been attributed to either the renormalization of the exciton energy via Coulomb interaction [19], or nonlinearities on the single-photon scale [49], or even to the presence of in-plane decay, which can result in unexpected interference effects [55]. In any case, they are different from the mallow triplet in that the third peak is no center frequency.

The quantum network node system considered here yet yields a different theoretical result altogether. Among the peculiarities of the system is the configuration of the quantum dot, which is negatively charged and thus exhibiting Pauli blocking. Additionally, the assumption of a circular shape for the QD is distinct from the above cases; it prevents the splitting of the degenerate heavy-hole and light-hole bands, which then allows us to maintain the 2-level system approximation for both bands. Last, the Faraday geometry imposes that the quantization axis of the excess electron spin be parallel to the direction of light propagation. Under these specific conditions, the theory predicts a quadruplet at low detuning energies ($\Delta \lesssim g$) and a doublet at high detuning energies ($\Delta \gg g$).

In order to observe or resolve the predicted spectrum for our quantum network node system, the single photon must be strongly coupled to the charged QD inside the
quantum network nodes. Therefore, using a commonly accepted condition for the strong coupling regime\cite{19}, we establish the following condition for the strong coupling regime inside the quantum network node

\[ 2\Omega_{3/2} \gg \frac{\left(\Gamma_{\text{cav}} + \gamma\right)}{2} \]

\[ 2\Omega_{1/2} \geq \frac{\left(\Gamma_{\text{cav}} + \gamma\right)}{2} \]

(2.69)

where \( \Gamma_{\text{cav}} \) is the cavity decay rate and \( \gamma \) is the dipole dephasing rate. Equations for the cavity decay rate and the dipole dephasing rate are

\[ \Gamma_{\text{cav}} = \frac{\omega_{\text{ph}}}{Q} \]

(2.70)

\[ \gamma = \gamma_{\text{dot}} + \gamma_{\text{enh}} = \gamma_{\text{dot}} + F_p \cdot \gamma_o = \gamma_{\text{dot}} + \left(\frac{3Q\lambda^3}{4\pi^2 n^3 V}\right) \cdot \gamma_o \]

where \( \gamma_{\text{dot}} \) is the sum of the QD non-radiative dephasing rate and the radiative decay rate outside the cavity (\( \gamma_o \)), and \( \gamma_{\text{enh}} \) is the enhanced rate due to the Purcell Effect. As a result, expressions for the power spectrum in Equation (2.33) become in the case of the quantum network nodes.
where \( \omega \) is the frequency difference of the 2-level system, \( \nu \) the frequency of the photon, \( \Delta \) the detuning frequency, \( g \) the interaction frequency, \( \gamma \) atomic decay rate, \( \Gamma_{\text{cav}} \) the cavity decay rate, and \( \Gamma_{\text{b}} \) the background emission rate.

The signature spectrum for our system as a function of detuning energy is shown for the first time in Figure 2-16. Each polariton branch is further split in two resulting in four distinct peaks at low detuning energies. However, in order to resolve this additional splitting, cavity Qs in few tens of thousand will be needed (e.g. This calculation assumed a Q of 50,000).
An interesting outcome of the dynamic in the quantum network node is the time evolution of the single photon polarization, which is used to represent the state of the traveling qubit. It is assumed that the single photon polarization is initially linear along the x direction or $|\uparrow\rangle$ while the photon propagates in the z direction. Since we are
considering the Faraday geometry, the direction of quantization for the excess spin in the conduction band (storage qubit) is also along z. Because linear polarized light is nothing but a balanced superposition of right and left-hand circular polarized light such that \( |\psi_p\rangle = |\uparrow\rangle = \frac{|\sigma_z^+\rangle + |\sigma_z^\downarrow\rangle}{\sqrt{2}} \), the right and left circular components of the polarization accumulate different phases, \( s^h \) and \( s^l \) respectively, for any specified interaction time as dictated by the optical transition rules for the electrical dipole interaction in effect inside the node. Under these conditions, the linear polarization of the photon will rotate as it interacts strongly with the charged QD. This is illustrated in Figure 2-17 and Figure 2-18 below.

Figure 2-17: Clockwise Rotation of the Linear Polarization
Figure 2-18: Counter-clockwise Rotation of the Linear Polarization

The interaction Hamiltonian is an electrical dipole interaction with Pauli blocking action which results in the conditional rotation of the single photon linear polarization. In other words, if the spin of the excess electron is initialized to $|\uparrow\rangle$, then the single photon polarization rotates towards the positive y direction. However, if the spin of the excess electron is initialized to $|\downarrow\rangle$, then the single photon polarization rotates towards the negative y direction. This mechanism can effectively be used to encode or map the state of the excess electron spin onto the polarization of a single photon.

It is advantageous to rewrite the expressions for the single photon polarization shown in Figure 2-17 and Figure 2-18, $|\psi_p\rangle = |\psi^\alpha\rangle$ where $|\psi^\alpha\rangle = \alpha |\uparrow\rangle + \beta |\leftrightarrow\rangle$ and $|\psi_p\rangle = |\psi^\beta\rangle$ where $|\psi^\beta\rangle = \alpha |\uparrow\rangle - \beta |\leftrightarrow\rangle$, in terms of the angles $\theta$ and $\phi$ such that the relationship between the accumulated phases, $s_0^{hh}$ and $s_0^{lh}$, and the probability amplitudes, $\alpha$ and $\beta$, is explicit. $\theta$ is just a general phase accumulated by both circular
components while \( \varphi \) is the relative phase corresponding to the angle by which the polarization has rotated. As a result, the single-photon polarization vector \( \vec{\xi} \) becomes

\[
\vec{\xi} = e^{-i\varphi} \left( \sigma_z^+ e^{-i\varphi} + \sigma_z e^{i\varphi} \right) \sqrt{2}
\]

(2.72)

where \( \theta = \frac{(s_0^{hh} + s_0^{ll})}{2} \) and \( \varphi = \frac{(s_0^{hh} - s_0^{ll})}{2} \). Similarly, an expression in terms of the angles \( \theta \) and \( \varphi \) can be derived for the linear polarization eigenbasis.

\[
\vec{\xi} = e^{-i\varphi} \left( \cos \varphi |\uparrow\rangle + \sin \varphi |\leftarrow\rangle \right)
\]

(2.73)

where \( \cos \varphi = \alpha \) and \( \sin \varphi = \beta \). It is important to note that the expressions in Equations (2.72) and (2.73) are valid whether the excess electron spin is \( |\uparrow\rangle \) or \( |\downarrow\rangle \); only the sign of \( \varphi \) changes.

In order to obtain an expression for the rate of rotation of the single-photon linear polarization, we need to derive explicitly for the time dependence of phases \( \theta(t) \) and \( \varphi(t) \), which amount to solving for the time-dependent expressions of \( s_0^{hh}(t) \) and \( s_0^{ll}(t) \). This done by taking Equation (2.66), which consists of solutions to the equations of motion for the probability amplitudes of the quantum field, and then solving for the complex phase. The following expressions are obtained [64]
\[ s_{0}^{hh}(t) = \tan^{-1}\left[ \frac{\Delta}{\Omega_{3/2}} \tan\left(\frac{\Omega_{3/2} t}{2}\right)\right] \]

\[ s_{0}^{lh}(t) = \tan^{-1}\left[ \frac{\Delta}{\Omega_{1/2}} \tan\left(\frac{\Omega_{1/2} t}{2}\right)\right] \]

Equation (2.74) is plotted in Figure 2-19 for small detuning energy (\( \Delta \leq \frac{3}{2} \)) and in Figure 2-20 for a large detuning energy (\( \Delta \gg \frac{3}{2} \)).
For a small detuning energy, the rate of rotation of the single-photon polarization vector is nonlinear with respect to time, even chaotic although periodic. There exist regions of acceleration, deceleration, and yet at times rotation occurs in the opposite direction. On the other hand, for a very large detuning energy, the rate of rotation of the single-photon polarization vector is quasi linear with respect to time. However, the rate of rotation of the polarization is very small, thus many more interaction cycles are needed to achieve the same phase $\varphi$.

This rotation of a single photon linear polarization as a result of its interaction with a 2-level system is called the Conditional or Single-photon Faraday Effect. This
fully quantum effect has some similarities with the classical Faraday Effect. Mainly, the rotation of linearly polarized light is a result of an existing broken symmetry between left and right component of circularly polarized light. However, the Single-photon Faraday Effect does not require an external magnetic field to break the symmetry. Here, the symmetry is broken by means of the magnetic moment or spin of the excess electron such that the Pauli blocking principle prevents the left and right components of circularly polarized light from coupling to the same valence bands.

Entanglement

What is quantum entanglement? Quantum Entanglement is a well established quantum property that has no counterpart in classical physics; it occurs when the total wave function of the mixed system cannot be written in any basis, as a direct product of independent substates (i.e. tensor product). As a result, the system of the two entangled qubits individually represented by states $|0\rangle$ and $|1\rangle$ has four new computational basis states designated $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$.

Why is quantum entanglement important? One important characteristic of quantum mechanics is that the information contained in a quantum two-level system or qubit cannot be fully copied (i.e. no-cloning theorem[9]). This limitation constitutes an essential difference between classical and quantum information. For quantum networks, an operation that can partially replace copying and provide the ability to transfer quantum information from one system to another is much needed. And this is because
the interchange of resources, namely qubits, is indispensable to perform quantum computation and to process quantum information. There exist several methods to interchange resources; however, we now single out one method referred to as quantum teleportation. This concept of quantum teleportation is fundamental in this dissertation because it constitutes the theoretical foundation for quantum networks made of single spins in quantum dots that interact via single photons.

How can entanglement be created between the single conduction band electron spin of the quantum dot and the single photon within the quantum network node? Simply put, this can be done by means of the Single-photon Faraday Effect. The excess electron spin was assumed at that time to be initialized to $\uparrow$ in the preceding section. But, what happens when the spin must be initialized to $\psi_e = \frac{1}{\sqrt{2}}(\uparrow + \downarrow)$; that is in a perfect superposition of its eigenstates. This is depicted in Figure 2-21. Under these conditions, it is unclear in which direction does the polarization of the single photon rotate. This uncertainty in the state of the individual qubit is a foremost characteristic of quantum entanglement. Transforming to a Bell state eigenbasis, we thus have the following maximally entangled Bell state.

$$
|\psi_{ep}\rangle = e^{i\phi_{hh}} |\psi_{X_{hh}}\rangle + e^{i\phi_{lh}} |\psi_{X_{lh}}\rangle
$$

$$
= e^{i\phi_{hh}} \left( \frac{1}{2} \left( |\sigma^+\rangle |\uparrow\rangle + |\sigma^-\rangle |\downarrow\rangle \right) + e^{i\phi_{lh}} \frac{1}{2} \left( |\sigma^-\rangle |\uparrow\rangle + |\sigma^+\rangle |\downarrow\rangle \right) \right)
$$

$$
= e^{-i\theta} \left( |\leftrightarrow\rangle |\uparrow\rangle \cdot e^{-i\phi} + |\leftrightarrow\rangle |\downarrow\rangle \cdot e^{i\phi} \right) \cdot \frac{1}{\sqrt{2}}
$$

(2.75)
What phase $\varphi$ accumulated during the QD-Photon interaction resulting in the Single Photon Faraday Effect corresponds to the maximally entangled Bell state shown above? Without solving for the dynamics of entanglement, which would require solving for the density matrix describing the composite qubit system or quantum network node (i.e. this being the goal of this research); one can consider instead the standard parameterization approach[43]. The objective here is not to solve for the entanglement dynamics but to solve for the condition of maximum entanglement. One important assumption here is that the detuning energy is much larger than the interaction energy ($\Delta \gg g$). Consider the following general state vector in the quantum network node

$$|\psi_s\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$$

$$|\psi_p\rangle = |\uparrow\rangle \quad \sigma_z^+ |\uparrow\rangle = \frac{1}{\sqrt{2}}(|\varphi^\uparrow\rangle + |\varphi^\downarrow\rangle)$$

Figure 2-21: Spin State Initialization for Entanglement
where the subscripts \( e_p \) on \( \psi_{e_p} \) stand for electron and photon, \( \varphi \) for a phase related to the state of the linear polarization of the single photon with basis states \( |\downarrow\rangle \) and \( |\leftrightarrow\rangle \), \( a \) and \( b \) for coefficients associated with the state of the excess electron spin, \( |\uparrow\rangle \) and \( |\downarrow\rangle \).

The \( a \) and \( b \) coefficients are defined as

\[
a = \cos(\vartheta/2)e^{-i\chi/2}
\]

\[
b = \sin(\vartheta/2)e^{i\chi/2}
\]

Thus, an expression for the density matrix can be derived, and it is shown below.

\[
[\rho] = \langle \psi_{e_p} | \psi_{e_p} \rangle = \\
\begin{bmatrix}
| \uparrow, \downarrow \rangle & | \uparrow, \leftrightarrow \rangle & | \downarrow, \downarrow \rangle & | \downarrow, \leftrightarrow \rangle \\
\end{bmatrix}
\begin{bmatrix}
a \sin^2 \varphi & a \cos \varphi \sin \varphi & -\sqrt{ba} \sin^2 \varphi & \sqrt{ba} \cos \varphi \sin \varphi \\
a \sin \vartheta \cos \varphi & a \cos^2 \varphi & -\sqrt{ba} \sin \varphi \cos \varphi & \sqrt{ba} \cos^2 \varphi \\
-\sqrt{ab} \sin^2 \varphi & -\sqrt{ab} \cos \varphi \sin \varphi & b \sin^2 \varphi & -b \cos \varphi \sin \varphi \\
\sqrt{ab} \sin \varphi \cos \varphi & \sqrt{ab} \cos^2 \varphi & -b \sin \varphi \cos \varphi & b \cos^2 \varphi \\
\end{bmatrix}
\begin{bmatrix}
| \uparrow, \downarrow \rangle & | \uparrow, \leftrightarrow \rangle & | \downarrow, \downarrow \rangle & | \downarrow, \leftrightarrow \rangle \\
\end{bmatrix}
\]

(2.78)
Using this parameterization approach, the normalized linear entropy takes the following form

\[
E_L = 2 \left[ 1 - \text{Tr}_e \left( \text{Tr}_p [\rho] \right)^2 \right]
\]

\[
= 2 \left[ 1 - \text{Tr}_e \left( \begin{bmatrix}
    a & \sqrt{ba} \cos \varphi \\
    \sqrt{ba} \cos \varphi & b
  \end{bmatrix} \right)^2 \right]
\]

\[
= 2 \left[ 1 - \text{Tr}_e \left( \begin{bmatrix}
    a^2 + ab \cos^2 \varphi & \sqrt{ba}^2 \cos \varphi + b^2 a \cos \varphi \\
    a^2 \sqrt{b} \cos \varphi + \sqrt{ab}^2 \cos \varphi & b^2 + ab \cos^2 \varphi
  \end{bmatrix} \right) \right]
\]

\[
= 2(1 - a^2 - b^2 - 2ab \cos^2 \varphi)
\]

(2.79)

Figure 2-22 shows a plot of the linear entropy as a function of $\vartheta$ and $\varphi$. Clearly, the maximum entanglement occurs when $\vartheta = \frac{\pi}{2}$ and $\varphi = \pm \frac{\pi}{2}$. The entanglement between the excess electron spin and the single photon is greatest at what would correspond to a 45 degrees rotation of the linear polarization of the single photon.
Towards a Comprehensive Quantum Network Model

A comprehensive model for quantum networks must include the description of the interface that allows the basic building blocks or nodes to be joined together. Consider an arbitrary rendering of the most basic configuration for a quantum network shown in Figure 2-23, which consists of collection of two nodes joined together by a single interconnect or waveguide.
The key element to obtaining a functional quantum network is the quantum interface that allows for deterministic interaction between stationary and flying qubits in the quantum network node[40, 41, 65]. The quantum interface is responsible for the generation of the flying qubit from the node to the continuum along with the complementary operation, which is the reception of the flying qubit from the continuum into the node. Proper dynamic control of that interface is essential for implementing desired functions of the quantum network such as transfer, swap and entanglement creation for qubits at distant nodes. For the sake of simplicity, we will not consider quantum network comprising more than two nodes. Furthermore, a 2-nodes quantum network is all we need to learn how to design an efficient quantum interface.
**2-Nodes Quantum Network**

A schematic of the 2-Nodes quantum network is shown in Figure 2-24. It consists of two quantum nodes similar to our original model depicted in Figure 2-7 except for coupling parameters, $\Gamma_1$ and $\Gamma_2$, which are added to describe the coherent coupling of the single photon from network nodes to the continuum and vice versa.

![Figure 2-24: 2-Nodes Quantum Network Model](image)

The Hamiltonian for the 2-Nodes Quantum Network Model is given by
\[
\hat{H}_{\text{QNet}} / \hbar = \sum_k \omega_k \hat{a}^+_k \hat{a}_k + \omega_{c1} \left( \hat{a}^+_\sigma \hat{a}_{\sigma+} + \hat{a}^+_\sigma^* \hat{a}_{\sigma-} \right) + \omega_{c2} \left( \hat{a}^+_\sigma \hat{a}_{\sigma+} + \hat{a}^+_\sigma^* \hat{a}_{\sigma-} \right) \\
+ \sum_{k \sigma} \Gamma_1 \left( \hat{a}^+_k + \hat{a}_k \right) \left( \hat{a}^+_\sigma + \hat{a}_\sigma \right) + \sum_{k \sigma} \Gamma_2 \left( \hat{a}^+_k + \hat{a}_k \right) \left( \hat{a}^+_\sigma + \hat{a}_\sigma \right) \\
+ \omega_{c1,h} \hat{\sigma}^-_{3/2v,3/2v} + \omega_{c1,h} \hat{\sigma}^-_{-3/2v,-3/2v} \\
+ \omega_{c1,h} \hat{\sigma}^+_{1/2v,1/2v} + \omega_{c1,h} \hat{\sigma}^-_{-1/2v,-1/2v} \\
+ \omega_{c1,e} \hat{\sigma}^-_{1/2v,1/2v} + \omega_{c1,e} \hat{\sigma}^+_{-1/2v,-1/2v} \\
+ \omega_{c2,h} \hat{\sigma}^-_{3/2v,3/2v} + \omega_{c2,h} \hat{\sigma}^-_{-3/2v,-3/2v} \\
+ \omega_{c2,h} \hat{\sigma}^+_{1/2v,1/2v} + \omega_{c2,h} \hat{\sigma}^-_{-1/2v,-1/2v} \\
+ \omega_{c2,e} \hat{\sigma}^+_{1/2v,1/2v} + \omega_{c2,e} \hat{\sigma}^-_{-1/2v,-1/2v} \\
+ g_{c1,3/2} \left( \hat{a}^+_\sigma \hat{\sigma}^+_{3/2v,1/2c} + \hat{a}^-_{\sigma} \hat{\sigma}^-_{1/2c,3/2v} + \hat{a}^+_\sigma \hat{\sigma}^-_{-3/2v,-1/2c} + \hat{a}^-_{\sigma} \hat{\sigma}^+_{1/2c,-3/2v} \right) \\
+ g_{c1,1/2} \left( \hat{a}^+_\sigma \hat{\sigma}^+_{1/2v,1/2c} + \hat{a}^-_{\sigma} \hat{\sigma}^-_{-1/2c,1/2v} + \hat{a}^+_\sigma \hat{\sigma}^-_{-1/2v,1/2c} + \hat{a}^-_{\sigma} \hat{\sigma}^+_{1/2c,-1/2v} \right) \\
+ g_{c2,3/2} \left( \hat{a}^+_\sigma \hat{\sigma}^+_{3/2v,1/2c} + \hat{a}^-_{\sigma} \hat{\sigma}^-_{1/2c,3/2v} + \hat{a}^+_\sigma \hat{\sigma}^-_{-3/2v,-1/2c} + \hat{a}^-_{\sigma} \hat{\sigma}^+_{1/2c,-3/2v} \right) \\
+ g_{c2,1/2} \left( \hat{a}^+_\sigma \hat{\sigma}^+_{1/2v,1/2c} + \hat{a}^-_{\sigma} \hat{\sigma}^-_{-1/2c,1/2v} + \hat{a}^+_\sigma \hat{\sigma}^-_{-1/2v,1/2c} + \hat{a}^-_{\sigma} \hat{\sigma}^+_{1/2c,-1/2v} \right) \quad (2.80)
\]

**Increasing the Number of Nodes**

A quantum network node by itself constitutes a system of 64 first-order linear ordinary differential equations (ODEs). When a 2-Node quantum network is considered, the result is a system of at least 1024 first-order linear ODEs, omitting the waveguide states. Furthermore, when taking into account the dynamic coupling between the node and the waveguide allowing the photon to hop back and forth between these two, a system of at least 1024 first-order linear ODEs with time-dependent coefficients is obtained. So, for an arbitrary number of nodes \( N \), the result is a density matrix with \( 2^N \)
states and a system of $2^N \times 2^N$ first-order linear ODEs. For each added node, the number of state doubles per node in the density matrix. Scalability from a modeling standpoint is not very good. In fact, the computational requirement becomes so stringent that it is virtually impossible to model quantum networks with more than few nodes using one of the dynamical models unless using a supercomputer. This dilemma is further discussed in Chapter 3.
CHAPTER 3: NUMERICAL MODEL

Modeling the quantum dynamics inside quantum networks is an essential step towards their physical realization. Such model is expected to lead to a thorough understanding of not only the coherent interaction between a QD and a single photon but also the entanglement dynamics between such particles produced inside quantum networks in a controlled way, in particular the fidelity or amount of such entanglement. For that reason, it is necessary to find a quantum model that is both accurate and also easy to interpret.

Numerical Method

Earlier, the need for a numerical approach to solving the full density matrix for the most basic component of quantum networks, namely a node, resulting in a system of 64 differential equations, was already established. Further, for each added node in the quantum network, the number of state doubles in the density matrix, and for N states the density matrix, the result is a $N^2$ system of differential equations. Since Matlab provides a set of functions aimed at solving numerically large system of differential equations as well as a set of graphical functions that can be helpful for visualization, it is used as the platform for modeling the dynamics inside our quantum network.
The first step to being able to model the quantum dynamics inside our quantum network is to derive a master equation. A master equation consists of a collection of differential equations describing the evolution in time of the probability of occupying each distinct set of states describing a system (i.e. a quantum network). The Liouville-Von Neumann Equation (2.61) is the most general form of a master equation for a system whose states are described in term of a density matrix \( \hat{\rho} \) and whose interactions are described according to the Hamiltonian matrix \( \hat{H} \). However, because relaxation processes are more complicated than just the anticommutator of a single relaxation matrix \( \hat{\Gamma} \) with the density matrix \( \hat{\rho} \), we redefine the general form of the master equation to a form more suitable for the modeling of our quantum network in Equation (3.1) using two relaxation matrices \( \hat{W} \) and \( \hat{\gamma} \).

\[
\frac{d\rho_{mm'}}{dt} = -\frac{i}{\hbar} \sum_k \left( H_{mk} \rho_{km'} - \rho_{mk} H_{km'} \right) + \delta_{mm'} \sum_{k \neq m} \rho_{kk} W_{mk} - \gamma_{mm'} \rho_{mm'} \quad (3.1)
\]

where \( W_{mk} \) are transitions rates affecting the diagonal elements of the density matrix and \( \gamma_{mm'} \) are decoherence rates affecting both diagonal and off-diagonal elements of the density matrix. The matrix \( \hat{\rho} \) is a \( N \times N \) matrix with \( m = 1, 2, ..., N \) and \( m' = 1, 2, ..., N \).
Matlab Differential Equation Solver

Matlab is able to solve initial value problems for ordinary differential equations using at least eight different solvers (ode23, ode45, ode113, ode15s, ode23s, ode23t, ode23tb, ode15i). All these various solvers require the systems of equations be expressed as a matrix equation in the following form.

\[ \hat{M}y' = \hat{\mathcal{L}}y \]  \hspace{1cm} (3.2)

where \( \hat{M} \) is a mass matrix, \( y' \) is vector whose elements are derivatives of dependent variables \( y_1, y_2, \ldots, y_n \) with respect to a single variable \( t \) or time, and \( \hat{\mathcal{L}} \) is a matrix describing the system of ODEs. Because our problem does not involve a mass matrix, we can rewrite Equation (3.2) using the identity matrix instead.

\[
\begin{bmatrix}
y'
\end{bmatrix}
= \begin{bmatrix}
\mathcal{L}
\end{bmatrix}
\begin{bmatrix}
y
\end{bmatrix}
\]  \hspace{1cm} (3.3)

However, our system of equations is currently expressed in terms of the subsequent matrix equation
\[
\begin{bmatrix}
\hat{\rho}' \\
\hat{\mathbf{H}}
\end{bmatrix}
= \begin{bmatrix}
\hat{\mathbf{H}} \\
\hat{\rho}
\end{bmatrix}
- \begin{bmatrix}
\hat{\rho} \\
\hat{\mathbf{H}}
\end{bmatrix}
+ \delta \sum
\begin{bmatrix}
\hat{\rho} \\
\hat{\mathbf{W}}
\end{bmatrix}
- \begin{bmatrix}
\hat{\gamma} \\
\hat{\rho}
\end{bmatrix}
\]  

(3.4)

where \( \delta \sum \begin{bmatrix} \hat{\rho} \end{bmatrix} \hat{\mathbf{W}} = \delta_{mm'} \sum_{k \neq m} \rho_{kk} W_{mk} \) and \( \begin{bmatrix} \hat{\gamma} \end{bmatrix} \begin{bmatrix} \hat{\rho} \end{bmatrix} = (\hat{\gamma} \cdot \hat{\rho})_{mm'} = \gamma_{mm'} \cdot \rho_{mm'} \). So, in order to solve for it numerically, Equation (3.4) must be rewritten in the form of Equation (3.3). This means rearranging the density matrix \( \hat{\rho} \) into a column vector \( y \), and the commutator along with the relaxation matrices \( [\hat{\mathbf{H}}, \hat{\mathbf{W}}] + \hat{\mathbf{W}} - \hat{\gamma} \) into a matrix \( \hat{\mathcal{L}} \) such that

\[
\begin{bmatrix}
\hat{\rho}
\end{bmatrix}
\xrightarrow{\mathcal{L}_{mm'}[1]}
\begin{bmatrix}
y_{[mm']}[1]
\end{bmatrix}
\]  

(3.5)

\[
\begin{bmatrix}
[\hat{\mathbf{H}}, \hat{\mathbf{W}}]_{mm'} + \hat{\mathbf{W}} - \hat{\gamma}
\end{bmatrix}
\xrightarrow{\mathcal{L}_{mm'}[nn']}
\begin{bmatrix}
\hat{\mathcal{L}}_{mm'}[nn']
\end{bmatrix}
\]  

(3.6)

These operations amount to a transformation to Liouville space where \( \hat{\mathcal{L}}_{mm'}[nn'] \), is a
superoperator (tetradic matrices) acting on that space with dimension $N^2$. In practice, Equation (3.6) is broken down into the following three matrix transformations.

\[
\begin{bmatrix}
\hat{L}^m_{m'} \\
\hat{H}_{m'}
\end{bmatrix} \rightarrow T_L \rightarrow 
\begin{bmatrix}
\hat{H}_{mm'}^{mm'}[nn']
\end{bmatrix}
\tag{3.7}
\]

\[
\begin{bmatrix}
\hat{W}_{mm'} - \hat{t}_{mm'}
\end{bmatrix}_{m=m'} \rightarrow T_L \rightarrow 
\begin{bmatrix}
\hat{L}^1_{mm'}[nn']
\end{bmatrix}
\tag{3.8}
\]

\[
\begin{bmatrix}
\hat{t}_{mm'}
\end{bmatrix}_{m=m'} \rightarrow T_L \rightarrow 
\begin{bmatrix}
\hat{L}^2_{mm'}[nn']
\end{bmatrix}
\tag{3.9}
\]

such that

\[
\hat{L}^2_{mm'}[nn'] = \hat{H}_{mm'}^{mm'}[nn'] + \hat{L}^1_{mm'}[nn'] + \hat{L}^2_{mm'}[nn']
\tag{3.10}
\]

As a result, the following matrix equation is obtained, which can then be fed to any of the Matlab differential equation solvers.
\[
\begin{bmatrix}
y' \\
y
\end{bmatrix} = 
\begin{bmatrix}
\hat{\mathcal{L}}
\end{bmatrix}
\begin{bmatrix}
y
\end{bmatrix}
\]  

(3.11)

For this project, however, ode45 is the solver of choice. Ode45 is based on an explicit Runge-Kutta (4,5) formula, known as the Dormand-Prince pair[66]. It is a single step solver in computing the vector \( y \); it needs only the solution at the immediately preceding time point.

**Matrix Transformation Algorithms**

This section discusses how to carry out these operations for arbitrary quantum network structures that are described in terms their density, Hamiltonian, and relaxation matrices.

\[\hat{\rho}_{mm'} \xrightarrow{T_1} y_{[mm']}[t] \] Transformation

The first operation transforming \( \hat{\rho}_{mm'} \) to \( y_{[mm']}[t] \) is quite straightforward for an arbitrary size square matrix. The essence of the algorithm is to create a column vector \( y \) whose number of elements are \( m \times m' \) and then fill it up using one row of the density
matrix at the time. For example, consider the simple case of a 2x2 density matrix \( \hat{\rho}_{[2],[2]} \), we get the following column vector \( y_{[4][1]} \) in Liouville space.

\[
\begin{pmatrix}
\rho_{11} & \rho_{12} \\
\rho_{21} & \rho_{22}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\rho_{11} \\
\rho_{12} \\
\rho_{21} \\
\rho_{22}
\end{pmatrix}
\]

\( (3.12) \)

\[
\left[ \hat{H}_{\cdot \cdot} \right]_{mm'} \xrightarrow{T_L} \left[ \hat{\rho} \right]_{[nn'][mm']}
\]

Transformation

The second operation transforming an arbitrary size square matrix \( \left[ \hat{H}_{\cdot \cdot} \right]_{mm'} \) to \( \left[ \hat{\rho} \right]_{[nn'][mm']} \) is more complex. Before generating algorithms for problems of arbitrary size, it is useful to work out a simple example again. Let assume that the density and Hamiltonian matrices are 2x2 matrices as shown below.

\[
\hat{\rho} = \begin{bmatrix}
\rho_{11} & \rho_{12} \\ 
\rho_{21} & \rho_{22}
\end{bmatrix} \quad \hat{H} = \begin{bmatrix}
H_{11} & H_{12} \\ 
H_{21} & H_{22}
\end{bmatrix}
\]

\( (3.13) \)

The commutator of the Hamiltonian and the density matrix in matrix form is expressed as
\[ \left[ \hat{\mathbf{H}}, \hat{\rho} \right] = \begin{bmatrix} [H_{12}\rho_{21} - H_{21}\rho_{12}] & [H_{11}\rho_{12} - H_{12}\rho_{22}] \\ [H_{21}\rho_{11} - H_{22}\rho_{21}] & [-H_{12}\rho_{11} - H_{22}\rho_{12}] \end{bmatrix} \]  \tag{3.14}

Next, the flowing operation \( \left[ \hat{\mathbf{H}}, \cdot \right]_{2\times2} \rightarrow \left[ \hat{\mathcal{H}} \right]_{4\times4} \) is performed on the Hamiltonian \( \hat{\mathbf{H}} \) such that

\[
\frac{d}{dt} \hat{\rho}_{mm'} = -\frac{i}{\hbar} \left[ \hat{\mathbf{H}}, \hat{\rho} \right]_{mm'} \quad \leftrightarrow \quad \frac{d}{dt} \psi_{[mm']} = \hat{\mathcal{H}}_{[mm']} \psi_{[mm']}.
\]

\[
\hat{\mathbf{H}} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \quad \rightarrow \quad \hat{\mathcal{H}} = \begin{bmatrix} 0 & -H_{21} & H_{12} & 0 \\ -H_{12} & H_{11} - H_{22} & 0 & H_{12} \\ H_{21} & 0 & H_{22} - H_{11} & -H_{21} \\ 0 & H_{21} & -H_{12} & 0 \end{bmatrix} \]  \tag{3.15}

The algorithm for an arbitrary square matrix \( \hat{\mathcal{H}} \) of size \( N^2 \) by \( N^2 \) is considered next. The strategy adopted to fill this matrix up consists of dividing the matrix \( \hat{\mathcal{H}} \) into smaller quadrants or matrices \( \hat{\mathbf{L}}_{ij} \) of size \( N \times N \).
The off-diagonal matrices $\hat{L}_{ij}$ ($i \neq j$) identified with the indices $i$ and $j$ are filled with element from the original Hamiltonian $[\hat{H}]_{mm}$, using the same indices such that matrix $\hat{L}_{ij}$ are themselves diagonal matrices filled with elements $H_{ij}$. Therefore

$$\hat{L}_{i\times j} = \begin{bmatrix} H_{ij} & 0 & 0 & 0 \\ 0 & H_{ij} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & H_{ij} \end{bmatrix}$$

(3.17)

As for the diagonal matrices ($\hat{L}_{ij}$ ($i = j$)), they are constructed such that
This can be easily verified using the simple example we consider earlier. For instance, Equation (3.15) can be rewritten in terms of smaller diagonal and off diagonal matrices.

\[ L_{i=j} = \begin{bmatrix}
H_{ij} - H_{i,1} & -H_{1,2} & -H_{i,3} & \cdots & -H_{i,n} \\
-H_{2,1} & H_{ij} - H_{2,2} & -H_{2,3} & \cdots & -H_{2,n} \\
-H_{3,1} & -H_{3,2} & H_{ij} - H_{3,3} & \ddots & -H_{3,n} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-H_{m,1} & -H_{m,2} & -H_{m,3} & \cdots & H_{ij} - H_{m,n}
\end{bmatrix} \quad (3.18) \]

where the diagonal matrices are

\[ L_{11} = \begin{bmatrix}
0 & -H_{21} \\
-H_{12} & H_{11} - H_{22}
\end{bmatrix} \quad L_{22} = \begin{bmatrix}
H_{22} - H_{11} & -H_{21} \\
-H_{12} & 0
\end{bmatrix} \quad (3.20) \]

and the off-diagonal matrices are

\[ L_{12} = \begin{bmatrix}
H_{12} & 0 \\
0 & H_{12}
\end{bmatrix} \quad L_{21} = \begin{bmatrix}
H_{21} & 0 \\
0 & H_{21}
\end{bmatrix} \quad (3.21) \]
Finally, the relaxation matrices $\hat{W}$ and $\hat{\gamma}$ also need to be transformed. In the original master equation, $\hat{W}$ and $\hat{\gamma}$ are $N \times N$ matrices and are written as

$$
\hat{W}_{mm'} = \begin{bmatrix}
W_{11} & W_{12} & W_{13} & \cdots & W_{1m'} \\
W_{21} & W_{22} & W_{23} & \cdots & W_{2m'} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
W_{m1} & W_{m2} & \cdots & W_{m(m'-1)} & W_{mm'}
\end{bmatrix}
$$

(3.22)

and

$$
\hat{\gamma}_{mm'} = \begin{bmatrix}
\gamma_{11} & \gamma_{12} & \gamma_{13} & \cdots & \gamma_{1m'} \\
\gamma_{21} & \gamma_{22} & \gamma_{23} & \cdots & \gamma_{2m'} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\gamma_{m1} & \gamma_{m2} & \cdots & \gamma_{m(m'-1)} & \gamma_{mm'}
\end{bmatrix}
$$

(3.23)

$\hat{W}_{mm'} - \hat{\gamma}_{mm'} \xrightarrow{\mathcal{T}_1} \hat{\Gamma}_1 [mm'][nn']$ Transformation

$\hat{\Gamma}_1$ is built both from relaxation matrices $\hat{W}$ and $\hat{\gamma}$ in connection with the diagonal elements of the density matrix. When $m = m'$, the term relating to the transition rates in Equation (3.1) in matrix form is written
\[ \delta \sum \left[ \hat{\rho} \right] \left[ W \right] = \delta_{mm'} \sum_{k \neq m} \rho_{kk} W_{mk} \]

\[
\begin{bmatrix}
\sum_k \rho_{kk} W_{1k} & 0 & \cdots & 0 \\
0 & \sum_k \rho_{kk} W_{2k} & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \sum_k \rho_{kk} W_{mk}
\end{bmatrix}
\]

(3.24)

whereas the term relating to the decoherence rates in Equation (3.1) in matrix form is written

\[
\begin{bmatrix}
\dot{\rho} \\
\dot{\rho}
\end{bmatrix} = \gamma_{mm} \rho_{mm}
\]

\[
= \rho_{mm} \sum_k W_{km}
\]

\[
\begin{bmatrix}
\rho_{11} \sum_k W_{1k} & 0 & \cdots & 0 \\
0 & \rho_{22} \sum_k W_{2k} & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \rho_{kk} \sum_k W_{km}
\end{bmatrix}
\]

(3.25)

Since
Combining Equations (3.24) and (3.25), we get the following new term where \( \hat{W} \) and \( \hat{\gamma} \) are forming the matrix \( \hat{W}' \)

\[
\gamma_{mm} = \frac{1}{2} \sum_k (W_{km} + W_{km'}) = \frac{1}{2} \sum_k (W_{km} + W_{km}) = \sum_{k \neq m} W_{km} \quad (3.26)
\]

\[
[\hat{\rho}] [\hat{W}'] = \delta \sum [\hat{\rho}] [\hat{W}] - [\hat{\gamma}] [\hat{\rho}]
\]

\[
= \delta_{mm'} \sum_{k \neq m} \rho_{kk} W_{mk} - \rho_{mm} \sum_k W_{km}
\]

\[
= \begin{bmatrix}
\sum_k \rho_{kk} W_{1k} - \rho_{11} \sum_k W_{k1} & 0 & \cdots & 0 \\
0 & \sum_k \rho_{kk} W_{2k} - \rho_{22} \sum_k W_{k2} & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \sum_k \rho_{kk} W_{mk} - \rho_{kk} \sum_k W_{km}
\end{bmatrix}
\]

(3.27)

Before coming up with an algorithm that can transform \( \hat{W}' \) to \( \hat{\Gamma}_1 \) for a problem of arbitrary size, it is useful to work out a simple example. It is assumed for the moment that \( \hat{\rho}, \hat{W}, \) and \( \hat{\gamma} \) are 2x2 matrices such that \( m = 1, 2 \) and \( m' = 1, 2 \) and therefore \( N = 2 \).
\[
\hat{\rho} = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}, \quad \hat{W} = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}, \quad \hat{\gamma} = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} = \begin{bmatrix} W_{21} & 0 \\ 0 & W_{12} \end{bmatrix}
\]

(3.28)

From the master equation, the terms related to transition and decoherence rates are therefore expressed in this example as

\[
\delta \sum \left[ \hat{\rho} \right] \left[ \hat{W} \right] - \left[ \gamma \right] \left[ \hat{\rho} \right] = \delta \sum_{k \neq m} \rho_{kk} W_{mk} - \rho_{mm} \sum_k W_{km}
\]

\[
= \begin{bmatrix} \rho_{22} W_{12} - \rho_{11} W_{21} & 0 \\ 0 & \rho_{11} W_{21} - \rho_{22} W_{12} \end{bmatrix}
\]

(3.29)

In Liouville space, the term \( \delta \sum \left[ \hat{\rho} \right] \left[ \hat{W} \right] - \left[ \gamma \right] \left[ \hat{\rho} \right] \) is rewritten \( \gamma^T_{1[N]} \hat{\Gamma}_{11} [I] [I] [I] \) such that

\[
\begin{bmatrix} -W_{21} & 0 & 0 & W_{21} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ W_{12} & 0 & 0 & -W_{12} \end{bmatrix}
\]

(3.30)

Thus, in order to obtain \( \hat{\Gamma}_1 \) for an arbitrary size problem, the strategy is to create a square matrix of size \( N^2 \times N^2 \) such that the term \( \delta \sum \left[ \hat{\rho} \right] \left[ \hat{W} \right] - \left[ \gamma \right] \left[ \hat{\rho} \right] \) is rewritten \( \gamma^T_{1[N]} \hat{\Gamma}_{11} [mm][nn'] \), and then fill \( \hat{\Gamma}_1 \). The notation for matrix indices in state space is as follows \( i = 1, 2, \cdots, N \) and \( j = 1, 2, \cdots, N \). Their counterpart in Liouville space are
\( I = 1, 2, \cdots, N^2 \) and \( J = 1, 2, \cdots, N^2 \). The key is to identify the columns (or \( J \)'s) in \( \hat{\Gamma} \) that needed to be filled, and then fill out the appropriate rows (or \( I \)'s) with the appropriate rates. The columns or \( J \)'s to be filled are the ones whose indices are matching the \( J \)'s in \( y^T(1, J) \) that correspond to diagonal terms in the original density matrix \( \hat{\rho}_{mm'} \), namely \( \rho_{m=1,m'=1} \to y_{1,J=1} \) and \( \rho_{m=2,m'=2} \to y_{1,J=4} \) as shown in the example above. A general expression for the columns to be filled is

\[
(3.31)
\]

It turns out that the set of indices for rows and columns to be filled are identical (\( I = J \)); therefore determining the indices for the columns to be filled also give the ones for the rows. Therefore, the algorithm to generate elements of \( \hat{\Gamma} \) for a problem of an arbitrary size is as the following

\[
\hat{\Gamma}(I,J) = \begin{cases} \hat{W}(i,j) & \text{(For } I \neq J) \\ -\gamma(i,j) & \text{(For } I = J) \end{cases}
\]

\[
(3.32)
\]

where \( I = (i-1) \cdot N + i \) and \( J = (j-1) \cdot N + j \) with \( i = 1, 2, \ldots, N \) and \( j = 1, 2, \ldots, N \). In Liouville space, when considering only the diagonal terms of the density matrix, \( \hat{\Gamma} \) can be written as
\[ \hat{W} = \begin{bmatrix} -\sum_{k \neq 1} W_{k1} & W_{21} & W_{31} & \cdots & W_{k1} \\ W_{12} & -\sum_{k \neq 2} W_{k2} & W_{32} & \cdots & W_{k2} \\ W_{13} & W_{23} & -\sum_{k \neq 3} W_{k3} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & W_{k(m-1)} \\ W_{1m} & W_{2m} & \cdots & W_{(k-1)m} & -\sum_{k \neq m} W_{km} \end{bmatrix} \] (3.33)

\[ \hat{\gamma}_{mm'} \xrightarrow{\mathcal{L}} \hat{\Gamma}_2 [mm'][nn'] \] Transformation

Unlike \( \hat{\Gamma}_1 \), \( \hat{\Gamma}_2 \) is associated with the case when \( m \neq m' \) in the master equation (See Equation (3.4)). This means that \( \delta \sum [\hat{\rho} \hat{W}] = 0 \); therefore, only \(-[\hat{\gamma}] [\hat{\rho}]\) contributes to the master equation. Furthermore, \( \hat{\Gamma}_2 \) is generated from only the decoherence rates \( \gamma_{mm'} \), which are off-diagonal elements of \( \hat{\gamma} \) such that

\[ \hat{\gamma}_{m\neq m'} = \frac{1}{2} \sum_k (W_{mk} + W_{km'}) \] (3.34)

These rates are included in the master equation by means of a Hadamard or Schur product between the relaxation matrix \( \hat{\gamma} \) and the density matrix \( \hat{\rho} \) such that
\[
[\hat{\gamma}][\rho] = \gamma_{mm'} \rho_{mm'}
\]

\[
= \begin{bmatrix}
0 & \gamma_{12}\rho_{12} & \gamma_{13}\rho_{13} & \cdots & \gamma_{1m'}\rho_{1m'} \\
\gamma_{21}\rho_{21} & 0 & \gamma_{23}\rho_{23} & \cdots & \gamma_{2m'}\rho_{2m'} \\
\gamma_{31}\rho_{31} & \gamma_{32}\rho_{32} & 0 & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \gamma_{(m-1)m'}\rho_{(m-1)m'} \\
\gamma_{m1}\rho_{m1} & \gamma_{m2}\rho_{m2} & \cdots & \gamma_{m(m'-1)}\rho_{m(m'-1)} & 0
\end{bmatrix}
\]

(3.35)

Before coming with an algorithm that can transform \( \hat{\gamma} \) to \( \hat{\Gamma}_1 \) for a problem of arbitrary size, it is useful to work out a simple example. It is assumed for the moment that \( \hat{\rho} \) and \( \hat{\gamma} \) are 2x2 matrices such that \( m = 1, 2 \) and \( m' = 1, 2 \) and therefore \( N = 2 \).

\[
\hat{\rho} = \begin{bmatrix}
\rho_{11} & \rho_{12} \\
\rho_{21} & \rho_{22}
\end{bmatrix}
\]

(3.36)

\[
\hat{\gamma}_{ij} = \begin{bmatrix}
0 & \gamma_{12} \\
\gamma_{21} & 0
\end{bmatrix} = \begin{bmatrix}
0 & W_{12} \\
W_{21} & 0
\end{bmatrix}
\]

From the master equation, the terms related to decoherence rates are therefore expressed as
\[
\begin{bmatrix}
\hat{y} \\
\hat{\rho}
\end{bmatrix}
= \gamma_{mm'} \rho_{mm'} =
\begin{bmatrix}
0 & \rho_{12} W_{12} \\
\rho_{21} W_{21} & 0
\end{bmatrix}
\]  

(3.37)

In Liouville space, the term \( \begin{bmatrix} \hat{y} \\ \hat{\rho} \end{bmatrix} \) is rewritten \( \hat{\Gamma}_2(4,4) \gamma_{[1][1]} \) such that

\[
\begin{bmatrix}
\hat{\Gamma}_2(I,J) \\
y(I,1)
\end{bmatrix}
\begin{bmatrix}
\rho_{11} \\
\rho_{12} \\
\rho_{21} \\
\rho_{22}
\end{bmatrix}
\]

(3.38)

Consequently, in order to obtain \( \hat{\Gamma}_2 \) for an arbitrary size problem, the strategy is to create first a vector \( \gamma_{[mm'][1]} \) from \( \hat{y}_{mm'} \) whose number of elements is \( N^2 \) and then fill it up by concatenating the rows of the density matrix at the time. Next, \( \hat{\Gamma}_2 \), a square matrix of size \( N^2 \times N^2 \), is generated by filling its diagonal elements with elements from the vector \( \gamma_{[mm'][1]} \). Evidently, the matrix \( \hat{\Gamma}_2 \) always consists of a diagonal matrix. The master equation in Equation (3.4) now takes on the following form

\[
\begin{bmatrix}
y'
\end{bmatrix}
= \hat{\mathcal{H}} \begin{bmatrix} y \\ y' \end{bmatrix} + \begin{bmatrix}
y
\end{bmatrix} \begin{bmatrix}
\gamma_{11}' \\
\gamma_{12}' \\
\gamma_{21}' \\
\gamma_{22}'
\end{bmatrix} \begin{bmatrix}
\hat{\Gamma}_1 \\
\hat{\Gamma}_2 \\
y
\end{bmatrix} - \begin{bmatrix}
y
\end{bmatrix} \begin{bmatrix}
\hat{\Gamma}_2 \\
y
\end{bmatrix}
\]

(3.39)
**Graphical Representation of Qubits**

The qubits are subsequently visualized by means a vector on either the Bloch sphere (spin) or the Poincare sphere (photon). On the other hand, entanglement is visualized by means of the von Neumann Entropy or the Linear Entropy. In addition, the probability of creating an exciton during the interaction is also monitored.

**The Bloch Sphere**

As discussed in Chapter 1, a general state vector for a qubit or two-level quantum system may be written as

\[
|\psi\rangle = \alpha |0\rangle + \beta |1\rangle
\]  

(3.40)

where \(|0\rangle\) and \(|1\rangle\) form an orthonormal basis in a two-dimensional complex vector space. The Bloch sphere named after famous physicist Felix Bloch consists of an elegant way to graphically represent the “pure” state space of a qubit as a point on a unit sphere[9]. Thus, any qubit with general state vector \(|\psi\rangle\) can be written in terms of \(\theta\) and \(\phi\) corresponding to polar angles of the spherical coordinates in Euclidean space \(\mathbb{R}^3\) such that
\[ |\psi\rangle = \cos\frac{\theta}{2} |0\rangle + e^{i\phi} \sin\frac{\theta}{2} |1\rangle \]  \hspace{1cm} (3.41)

with \(0 \leq \theta \leq \pi\) and \(0 \leq \phi \leq 2\pi\).

![Bloch Sphere](image)

Figure 3-1: Bloch Sphere

The expression of the general state of the qubit in Equation (3.40) in terms of Cartesian coordinates \((x, y, z)\) can be obtained from the density matrix for the qubit system shown below.
\[ \rho = |\psi\rangle\langle\psi| = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} = \begin{bmatrix} \alpha^2 & \beta^* \alpha \\ \alpha^* \beta & \beta^2 \end{bmatrix} \]  

(3.42)

Next, using Pauli's matrices, the coordinates are derived

\[ x = Tr[\rho \sigma_x] \]
\[ = Tr\left[ \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right] \]
\[ = Tr\left[ \begin{bmatrix} \rho_{12} & \rho_{11} \\ \rho_{22} & \rho_{21} \end{bmatrix} \right] \]
\[ = \rho_{12} + \rho_{21} \]
\[ = \alpha^* \beta + \beta^* \alpha \]  

(3.43)

\[ y = Tr[\rho \sigma_y] \]
\[ = Tr\left[ \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \right] \]
\[ = Tr\left[ \begin{bmatrix} i\rho_{12} & -i\rho_{11} \\ i\rho_{22} & -i\rho_{21} \end{bmatrix} \right] \]
\[ = i(\rho_{12} - \rho_{21}) \]
\[ = \frac{1}{i}(\alpha^* \beta - \beta^* \alpha) \]  

(3.44)
Additionally, a “pure” state of a qubit lies on the surface of the sphere, whereas a “mixed” state such as in the case the qubit being entangled to another would result in a point inside the Bloch sphere.

**The Poincare Sphere**

The Poincare sphere was introduced by Henry Poincare in 1892 as a mean of representing polarized light graphically. Figure 3-2 shows the Poincare sphere. The upper and lower poles represent circularly polarized light, whereas points on the equator represent linearly polarized light. Points that are neither at the poles nor on the equator represent elliptical polarized light.
Any point on the surface of the Poincare sphere represent a unique state of polarization, and just as in the case of the Bloch sphere in Euclidian space $\mathbb{R}^3$, it can be represented in terms of polar angles in spherical coordinates or in Cartesian coordinates by $S_1$, $S_2$, and $S_3$ known as the Stokes parameters. $S_1$ corresponds to the probability of obtaining a photon polarized along the $\uparrow$ direction minus the probability of obtaining a photon polarized along the $\leftrightarrow$ direction. $S_2$ corresponds to the probability of obtaining a photon polarized along the $\uparrow$ or $+45^\circ$ direction minus the probability of obtaining a photon polarized along the $\downarrow$ or $-45^\circ$ direction. $S_3$ corresponds to the
probability of obtaining a photon polarized along the $\sigma^+$ or RCP direction minus the probability of obtaining a photon polarized along the $\sigma^-$ or LCP direction.

Even though the single photon inside our quantum network is linearly polarized; its state of polarization is expressed in terms of the circular polarization eigenbasis, which is more suitable in describing the single-photon Faraday Effect. Therefore, the correspondence between the orthonormal basis $|\sigma^+\rangle$ and $|\sigma^-\rangle$ and the Stokes parameters $(S_1, S_2, S_3)$ needs to be shown more explicitly. This is shown in Equations (3.47), (3.48), and (3.49). Assuming the following general state vector for a single photon is $|\psi\rangle = \alpha|\sigma^+\rangle + \beta|\sigma^-\rangle$, the following density matrix is obtained

$$\rho = |\psi\rangle \langle \psi| = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} = \begin{bmatrix} \alpha^2 & \beta^* \alpha \\ \alpha^* \beta & \beta^2 \end{bmatrix}$$  \hspace{1cm} (3.46)$$

Because we are considering the circular polarization eigenbasis, the quantization axis is along the $z$ axis in Euclidian space $\mathbb{R}^3$, which allows us to use the same configuration for the Pauli’s matrices as in the case of the Bloch sphere in solving for the Stokes parameters below.
\[ S_1 = Tr[\rho \sigma_x] \]
\[ = Tr\begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]
\[ = Tr\begin{bmatrix} \rho_{12} & \rho_{11} \\ \rho_{22} & \rho_{21} \end{bmatrix} \]
\[ = \rho_{12} + \rho_{21} \]
\[ = \alpha^* \beta + \beta^* \alpha \]  
\hspace{1cm} (3.47)

\[ S_2 = Tr[\rho \sigma_y] \]
\[ = Tr\begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \]
\[ = Tr\begin{bmatrix} i\rho_{12} & -i\rho_{11} \\ i\rho_{22} & -i\rho_{21} \end{bmatrix} \]
\[ = i(\rho_{12} - \rho_{21}) \]
\[ = \frac{1}{i}(\alpha^* \beta - \beta^* \alpha) \]  
\hspace{1cm} (3.48)

\[ S_3 = Tr[\rho \sigma_z] \]
\[ = Tr\begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \]
\[ = Tr\begin{bmatrix} \rho_{11} & -\rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} \]
\[ = \rho_{11} - \rho_{22} \]
\[ = \alpha^2 - \beta^2 \]  
\hspace{1cm} (3.49)
CHAPTER 4: ENGINEERING THE QUANTUM INTERFACE

Quantum network nodes represent the basic building block of the quantum network, and they are envisioned here to be photonic crystal defect nanocavities encapsulating single quantum dots. In fact, quantum network nodes are the elements in the quantum networks that are responsible the manipulation of quantum information encoded both onto the storage and flying qubits. The successful implementation of quantum networks therefore depends on such manipulation to be efficiently achieved. This calls for a carefully engineered quantum interface, namely one resulting in minimum loss and error. They are two important aspects to a quantum interface as depicted in Figure 4-1: Quantum Interface: (1) the information exchange between the qubits inside the cavity and (2) the loading/unloading of the flying qubit from/into the waveguide. In order to engineer an efficient quantum interface, we will first consider a cavity system and study various aspect of the spin-photon exchange. Then, we will consider the challenge of loading/unloading of the flying qubit by looking at a system consisting of a cavity coupled to a waveguide.
Controlling the Spin-Photon Exchange Dynamics

Various operations can be achieved inside quantum network nodes; we will consider two of them: (1) quantum state mapping and (2) electron-photon entanglement. The effects of decoherence and fabrication errors are also considered in this section. Independent of the operation performed, the “pure” state density matrix describing the combined spin-photon system inside a cavity must be defined.
Spin-Photon Density Matrix

The spin-photon density matrix is shown in Equation (4.1). It contains time-dependent variables $\rho_{ij}$, which describe the probability of occupying each distinct set of states in the cavity system, which are $|\downarrow, \sigma_z^\rightarrow\rangle$ corresponding to the excess electron spin being down with a LCP photon in the cavity, $|\downarrow, \sigma_z^+\rangle$ to the excess electron spin being down with a RCP photon in the cavity, $|\downarrow, X_{lh}\rangle$ to the excess electron spin being down with a LH exciton (trion), $|\downarrow, X_{hh}\rangle$ to the excess electron spin being down with a HH exciton (trion), $|\uparrow, \sigma_z^\rightarrow\rangle$ to the excess electron spin being up with a LCP photon in the cavity, $|\uparrow, \sigma_z^+\rangle$ to the excess electron spin being up with a RCP photon in the cavity, $|\uparrow, X_{lh}\rangle$ to the excess electron spin being up with a LH exciton (trion), and $|\uparrow, X_{hh}\rangle$ to the excess electron spin being up with a HH exciton (trion).
\[ \dot{\rho} = \ket{\psi(t)}\bra{\psi(t)} = \]

\[ \begin{bmatrix}
\rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} & \rho_{15} & \rho_{16} & \rho_{17} & \rho_{18} \\
\rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} & \rho_{25} & \rho_{26} & \rho_{27} & \rho_{28} \\
\rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} & \rho_{35} & \rho_{36} & \rho_{37} & \rho_{38} \\
\rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} & \rho_{45} & \rho_{46} & \rho_{47} & \rho_{48} \\
\rho_{51} & \rho_{52} & \rho_{53} & \rho_{54} & \rho_{55} & \rho_{56} & \rho_{57} & \rho_{58} \\
\rho_{61} & \rho_{62} & \rho_{63} & \rho_{64} & \rho_{65} & \rho_{66} & \rho_{67} & \rho_{68} \\
\rho_{71} & \rho_{72} & \rho_{73} & \rho_{74} & \rho_{75} & \rho_{76} & \rho_{77} & \rho_{78} \\
\rho_{81} & \rho_{82} & \rho_{83} & \rho_{84} & \rho_{85} & \rho_{86} & \rho_{87} & \rho_{88}
\end{bmatrix} \begin{bmatrix}
\downarrow, \sigma_z^+ \\
\downarrow, \sigma_z^+ \\
\downarrow, X_{lh} \\
\downarrow, X_{lh} \\
\uparrow, \sigma_z \\
\uparrow, \sigma_z^+ \\
\uparrow, X_{lh} \\
\uparrow, X_{hh}
\end{bmatrix} \]

(4.1)

**Cavity System Hamiltonian**

In order to solve for the quantum dynamics inside our quantum network node though, we need first the Hamiltonian for the cavity system in matrix form as well as the density matrix \( \dot{\rho} \), and the relaxation matrices \( \dot{W} \) and \( \dot{\gamma} \). For now though, we consider only the cavity system without the effect of decoherence. The full Hamiltonian matrix \( \dot{H} \) in the rotating frame is
\[ \hat{H} = \]

\[
\begin{bmatrix}
\Delta & 0 & 0 & V_{hh} & 0 & 0 & 0 & 0 \\
0 & \Delta & V_{lh} & 0 & 0 & 0 & 0 & 0 \\
0 & V_{lh} & E_c & 0 & 0 & 0 & 0 & 0 \\
V_{hh} & 0 & 0 & E_c & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \Delta & 0 & V_{lh} & 0 \\
0 & 0 & 0 & 0 & \Delta & 0 & V_{hh} & 0 \\
0 & 0 & 0 & 0 & V_{lh} & 0 & E_c & 0 \\
0 & 0 & 0 & 0 & V_{hh} & 0 & E_c & 0 
\end{bmatrix}
\begin{bmatrix}
\downarrow, \sigma_z \\
\downarrow, \sigma_z^+ \\
\downarrow, \chi_{lh} \\
\downarrow, \chi_{hh} \\
\uparrow, \sigma_z \\
\uparrow, \sigma_z^+ \\
\uparrow, \chi_{lh} \\
\uparrow, \chi_{hh} 
\end{bmatrix}
\]

where \( E_c \) is the energy of the excess electron, \( \Delta \) the detuning energy, \( V_{hh} = \hbar g_{3/2} \) the interaction energy associated with the HH excitons, and \( V_{lh} = \hbar g_{1/2} \) the interaction energy associated with the LH excitons.

**Quantum State Mapping**

As previously mentioned, Pauli blocking action inside the cavity system results in the conditional rotation of the single photon linear polarization, which can effectively be
used to encode or map the “pure” state of the excess electron spin onto the polarization of a single photon. The expression “pure” state here refers to either $|\uparrow\rangle$ or $|\downarrow\rangle$ rather than a superposition giving rise to entanglement. If the spin of the excess electron is initialized to $|\uparrow\rangle$ (+$z$ direction in the Faraday configuration), then the single photon polarization rotates clockwise when facing the +$z$ direction. However, if the spin of the excess electron is initialized to $|\downarrow\rangle$ (-$z$ direction in the Faraday configuration), then the single photon polarization rotates counterclockwise when facing the +$z$ direction. There are two regimes of interest when studying the dynamics inside the cavity, namely the small and large energy detuning regimes with respect to the QD and the field.

**CASE 1**: Small $\Delta$, the spin is initialized to $|\uparrow\rangle$, the photon is initialized to $|\beta\rangle$

Assuming a GaAs/InGaAs QD with emission wavelength $\lambda_{QD}$ of $1.182\,\mu m$, which corresponds to the frequency $\omega_{QD}=1.594 \cdot 10^{15}\,rad/\,s$ such that the energy of the excess electron in the Hamiltonian is $E_{c}=\hbar \omega_{QD}$. The detuning energy is selected to be $\Delta=78\,\mu eV$, which corresponds to the optimized value $\Delta=0.86 g_{3/2}$ so as to have a photon back into the cavity once the polarization has rotated by 90 degrees or $2\phi=\pi$ as shown in Figure 4-26 and Figure 2-19. Next, assuming the dipole moment associated with the InGaAs QD for interactions involving heavy-hole electrons is $\mu_{vc}=29\,D$, the frequency of the photon $\omega=\omega_{QD}-\Delta=1.594 \cdot 10^{15}\,rad/\,s$, and the cavity mode volume $V_{0}=0.039\,\mu m^{3}$, the interaction frequency involving the heavy-hole valence
band $g_{3/2}$ was found to be $132 \cdot 10^9 \text{rad/s}$ leading to the interaction energy in the Hamiltonian $V_{hh} = \hbar g_{3/2}$ and the interaction frequency involving the light-hole valence band $g_{1/2}$ to be $\frac{g_{3/2}}{\sqrt{3}} = 76 \cdot 10^9 \text{rad/s}$ corresponding to the interaction energy $V_{lh} = \hbar g_{1/2}$.

Last the density matrix was initialized to

$$\hat{\rho}(t = 0) =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

The results are shown in the next three figures (Figure 4-2, Figure 4-3, and Figure 4-4). In the small $\triangle$ regime, it takes approximately 130 picoseconds to rotate the single photon linear polarization from $\rightharpoonup$ to $\leftrightarrow$. The Poincare sphere shows intermediate polarization states with various elicitities that are situated inside the unit
sphere. However, this is not the result of entanglement between the “pure” state spin and the single photon polarization since the linear entropy of the electron-photon system in fact remains zero (Please refer to Section CHAPTER 0:), but rather the consequence of a strong absorption of the photon as depicted by the large swings in the probability amplitudes.

Figure 4-2: Bloch Sphere for Small $\Delta$, Spin $|\uparrow\rangle$, Photon $|\downarrow\rangle$
Figure 4-3: Poincare Sphere for Small $\Delta$, Spin $|\uparrow\rangle$, Photon $|\downarrow\rangle$
Figure 4-4: Probability Amplitudes for Small $\Delta$, Spin $|\uparrow\rangle$, Photon $|\downarrow\rangle$

**CASE 2**: Large $\Delta$, the spin is initialized to $|\uparrow\rangle$, the photon is initialized to $|\downarrow\rangle$

All the assumptions are the same as in the case above except that now the detuning energy is selected to be $\Delta = 1.5 \text{meV}$ resulting only in virtual absorption so as to have a photon inside the cavity at all time until the polarization has rotated by 90 degrees or $2\varphi = \pi$ as shown in Figure 2-15 and Figure 2-20. The resulting single photon frequency, interaction frequency involving the heavy-hole valence band, and interaction frequency involving the light-hole valence band are respectively $\omega = 1.592 \cdot 10^{15} \text{rad/s}$,
\[ g_{3/2} = 132 \cdot 10^9 \text{ rad} / s, \text{ and } g_{1/2} = 76 \cdot 10^9 \text{ rad} / s. \] Last, \( \hat{\rho} \) is initialized again to Equation (4.3).

The results are shown in the next three figures (Figure 4-5, Figure 4-6, and Figure 4-7). In the large \( \Delta \) regime, it takes approximately 1.3 nanoseconds (i.e. about 10 times the amount of time it took in the small \( \Delta \) regime) to rotate the single photon linear polarization from \( |\downarrow\rangle \) to \( |\leftrightarrow\rangle \). The Poincare sphere shows that intermediate polarization states are always linearly polarized and situated on the surface of the unit sphere. Clearly, there is not any entanglement present between the “pure” state spin and the single photon polarization, which is confirmed by the fact that the linear entropy of the electron-photon system remains zero at all times.
Figure 4-5: Bloch Sphere for Large $\Delta$, Spin $\uparrow\rangle$, Photon $\uparrow\uparrow\rangle$
Figure 4-6: Poincare Sphere for Large $\Delta$, Spin $|\uparrow\rangle$, Photon $|\Downarrow\rangle$
Unlike quantum state mapping, it is desirable to have the excess electron spin initialized to a superposition of its eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$. Then, the amount of entanglement between the excess electron spin and the single photon polarization can be worked out from the density matrix.
\[ \hat{\rho} = |\psi(t)\rangle \langle \psi(t)| = \]

\[
\begin{bmatrix}
\rho_{\downarrow\downarrow} & \rho_{\downarrow\uparrow} & \rho_{\uparrow\downarrow} & \rho_{\uparrow\uparrow} \\
\rho_{\downarrow\uparrow} & \rho_{\uparrow\downarrow} & \rho_{\downarrow\uparrow} & \rho_{\uparrow\downarrow} \\
\rho_{\uparrow\downarrow} & \rho_{\downarrow\uparrow} & \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\
\rho_{\uparrow\uparrow} & \rho_{\downarrow\downarrow} & \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\
\end{bmatrix}
\]

\[ |\downarrow, \sigma_z \rangle \langle \downarrow, \sigma_z | \quad |\downarrow, X_{lh} \rangle \langle \downarrow, X_{hh} | \quad |\uparrow, \sigma_z \rangle \langle \uparrow, \sigma_z | \quad |\uparrow, X_{lh} \rangle \langle \uparrow, X_{hh} | \]

Tracing the density matrix over the cavity polariton results in the spin reduced density matrix such that
\[ \text{TrPolariton} \hat{\rho} = \hat{\rho}_{\text{red.spin}} \]
\[ = \text{TrPhoton} \hat{\rho} + \text{TrExciton} \hat{\rho} \]
\[ = \langle \sigma_z^+ | \hat{\rho} | \sigma_z^+ \rangle + \langle \sigma_z^- | \hat{\rho} | \sigma_z^- \rangle + \langle X_{hh} | \hat{\rho} | X_{hh} \rangle + \langle X_{lh} | \hat{\rho} | X_{lh} \rangle \]
\[ = \begin{bmatrix} \text{Tr} \hat{\rho}_{\alpha\alpha}^{\uparrow\uparrow} & \text{Tr} \hat{\rho}_{\alpha\alpha}^{\uparrow\downarrow} \\ \text{Tr} \hat{\rho}_{\alpha\alpha}^{\downarrow\uparrow} & \text{Tr} \hat{\rho}_{\alpha\alpha}^{\downarrow\downarrow} \end{bmatrix} + \begin{bmatrix} \text{Tr} \hat{\rho}_{XX}^{\uparrow\uparrow} & \text{Tr} \hat{\rho}_{XX}^{\uparrow\downarrow} \\ \text{Tr} \hat{\rho}_{XX}^{\downarrow\uparrow} & \text{Tr} \hat{\rho}_{XX}^{\downarrow\downarrow} \end{bmatrix} \]
\[ = \begin{bmatrix} \text{Tr} \hat{\rho}_{\alpha\alpha}^{\uparrow\uparrow} + \text{Tr} \hat{\rho}_{XX}^{\uparrow\uparrow} & \text{Tr} \hat{\rho}_{\alpha\alpha}^{\uparrow\downarrow} + \text{Tr} \hat{\rho}_{XX}^{\uparrow\downarrow} \\ \text{Tr} \hat{\rho}_{\alpha\alpha}^{\downarrow\uparrow} + \text{Tr} \hat{\rho}_{XX}^{\downarrow\uparrow} & \text{Tr} \hat{\rho}_{\alpha\alpha}^{\downarrow\downarrow} + \text{Tr} \hat{\rho}_{XX}^{\downarrow\downarrow} \end{bmatrix} \]
\[ = 2 \times 2 \]

where

\[ \hat{\rho}_{\alpha\alpha}^{\uparrow\uparrow} = \begin{bmatrix} \rho_{\sigma_z^+ \sigma_z^+}^{\uparrow\uparrow} & \rho_{\sigma_z^+ \sigma_z^-}^{\uparrow\uparrow} \\ \rho_{\sigma_z^- \sigma_z^+}^{\uparrow\uparrow} & \rho_{\sigma_z^- \sigma_z^-}^{\uparrow\uparrow} \end{bmatrix} \quad \hat{\rho}_{\alpha\alpha}^{\uparrow\downarrow} = \begin{bmatrix} \rho_{\sigma_z^+ \sigma_z^+}^{\uparrow\downarrow} & \rho_{\sigma_z^+ \sigma_z^-}^{\uparrow\downarrow} \\ \rho_{\sigma_z^- \sigma_z^+}^{\uparrow\downarrow} & \rho_{\sigma_z^- \sigma_z^-}^{\uparrow\downarrow} \end{bmatrix} \]
\[ \hat{\rho}_{\alpha\alpha}^{\downarrow\uparrow} = \begin{bmatrix} \rho_{\sigma_z^+ \sigma_z^+}^{\downarrow\uparrow} & \rho_{\sigma_z^+ \sigma_z^-}^{\downarrow\uparrow} \\ \rho_{\sigma_z^- \sigma_z^+}^{\downarrow\uparrow} & \rho_{\sigma_z^- \sigma_z^-}^{\downarrow\uparrow} \end{bmatrix} \quad \hat{\rho}_{\alpha\alpha}^{\downarrow\downarrow} = \begin{bmatrix} \rho_{\sigma_z^+ \sigma_z^+}^{\downarrow\downarrow} & \rho_{\sigma_z^+ \sigma_z^-}^{\downarrow\downarrow} \\ \rho_{\sigma_z^- \sigma_z^+}^{\downarrow\downarrow} & \rho_{\sigma_z^- \sigma_z^-}^{\downarrow\downarrow} \end{bmatrix} \]

and

\[ (4.5) \]
\[ (4.6) \]
Next, the 2x2 matrix $\text{Tr}_{\text{Polariton}} \rho$ needs to be squared. As a result, we obtain

$$\hat{\rho}_{XX}^{\uparrow \downarrow} = \begin{bmatrix} \rho_{X_{hh}X_{hh}}^{\uparrow \uparrow} & \rho_{X_{hh}X_{lh}}^{\uparrow \uparrow} \\
\rho_{X_{lh}X_{hh}}^{\uparrow \uparrow} & \rho_{X_{lh}X_{lh}}^{\uparrow \uparrow} \end{bmatrix} \quad \hat{\rho}_{XX}^{\uparrow \uparrow} = \begin{bmatrix} \rho_{X_{hh}X_{hh}}^{\uparrow \uparrow} \\
\rho_{X_{lh}X_{lh}}^{\uparrow \uparrow} \end{bmatrix}$$

(4.7)

$$\hat{\rho}_{XX}^{\downarrow \downarrow} = \begin{bmatrix} \rho_{X_{hh}X_{hh}}^{\uparrow \downarrow} & \rho_{X_{hh}X_{lh}}^{\uparrow \downarrow} \\
\rho_{X_{lh}X_{hh}}^{\uparrow \downarrow} & \rho_{X_{lh}X_{lh}}^{\uparrow \downarrow} \end{bmatrix} \quad \hat{\rho}_{XX}^{\downarrow \uparrow} = \begin{bmatrix} \rho_{X_{hh}X_{hh}}^{\downarrow \uparrow} \\
\rho_{X_{lh}X_{lh}}^{\downarrow \uparrow} \end{bmatrix}$$


\[
\left[\text{Tr}_{\text{Polariton}} \hat{\rho}\right]^2 = \begin{bmatrix}
\text{Tr} \hat{\rho}_{\sigma \sigma}^{\uparrow \uparrow} + \text{Tr} \hat{\rho}_{XX}^{\uparrow \uparrow} & \text{Tr} \hat{\rho}_{\sigma \sigma}^{\uparrow \downarrow} + \text{Tr} \hat{\rho}_{XX}^{\downarrow \uparrow} \\
\text{Tr} \hat{\rho}_{\sigma \sigma}^{\uparrow \downarrow} + \text{Tr} \hat{\rho}_{XX}^{\downarrow \downarrow} & \text{Tr} \hat{\rho}_{\sigma \sigma}^{\downarrow \downarrow} + \text{Tr} \hat{\rho}_{XX}^{\downarrow \downarrow}
\end{bmatrix}^2
\]

\[
= \begin{bmatrix}
\text{Tr} \hat{\rho}_{\sigma \sigma}^{\uparrow \uparrow} + \text{Tr} \hat{\rho}_{XX}^{\uparrow \uparrow} & \text{Tr} \hat{\rho}_{\sigma \sigma}^{\uparrow \downarrow} + \text{Tr} \hat{\rho}_{XX}^{\downarrow \uparrow} \\
\text{Tr} \hat{\rho}_{\sigma \sigma}^{\uparrow \downarrow} + \text{Tr} \hat{\rho}_{XX}^{\downarrow \downarrow} & \text{Tr} \hat{\rho}_{\sigma \sigma}^{\downarrow \downarrow} + \text{Tr} \hat{\rho}_{XX}^{\downarrow \downarrow}
\end{bmatrix}
\]

(4.8)

Finally, we take the trace of this square matrix over the spin such that
The normalized linear entropy is then

\[ S_{NL} = 2[1 - \eta] \] (4.10)

When \( S_{NL} = 0 \), the storage qubit and the traveling qubit are not entangled. On the other hand, when \( S_{NL} = 1 \), the two qubits are maximally entangled

**CASE 1:** Small \( \Delta \), Spin is initialized to \( \alpha |\downarrow\rangle + \beta |\uparrow\rangle \), Photon is initialized to \( |\uparrow\rangle \).

All the assumptions are the same as in the case 1 in Section CHAPTER 0: except for the density matrix, which is now initialized to
\[
\hat{\rho}(t = 0) = \\
\begin{align*}
\begin{bmatrix}
\langle \downarrow \sigma_z \rangle & \langle \downarrow \sigma_z \rangle & \langle \downarrow X_{lh} \rangle & \langle \downarrow X_{hh} \rangle & \langle \uparrow \sigma_z \rangle & \langle \uparrow \sigma_z \rangle & \langle \uparrow X_{lh} \rangle & \langle \uparrow X_{hh} \rangle \\
\alpha \alpha & \alpha \alpha & 0 & 0 & \alpha \beta & \alpha \beta & 0 & 0 \\
\alpha \alpha & \alpha \alpha & 0 & 0 & \alpha \beta & \alpha \beta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\beta \alpha & \beta \alpha & 0 & 0 & \beta \beta & \beta \beta & 0 & 0 \\
\beta \alpha & \beta \alpha & 0 & 0 & \beta \beta & \beta \beta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\end{align*}
\]

The results are shown in the next four figures (Figure 4-8, Figure 4-9, Figure 4-10, and Figure 4-11). In the small \( \Delta \) regime, there are 3 peaks or maxima of magnitude 1 in the calculated electron-photon system entropy, which means that there are 3 different conditions for maximum entanglement. This is not what was predicted in section CHAPTER 0: using the standard parameterization approach. These peaks occur at 71, 93, and 116 picoseconds. However, there is a trade-off since the probability of creating an exciton is nearly 0.4 or 40% for the first 2 peaks and 45% for the last peak. This is not desirable from an engineering point of view since the aim would be to
have a high probability of releasing the photon back into the quantum network waveguides once maximally entangled. Entanglement is also confirmed by the fact that the Poincare sphere shows intermediate polarization states inside the unit sphere.

Figure 4-8: Bloch Sphere for Small $\Delta$, Spin $\alpha\downarrow + \beta\uparrow$, Photon $\downarrow$
Figure 4-9: Poincare Sphere for Small $\Delta$, Spin $\alpha \downarrow + \beta \uparrow$, Photon $|\downarrow\rangle$
Figure 4-10: Entanglement for Small $\Delta$, Spin $\alpha \downarrow + \beta \uparrow$, Photon $\$$. 
Figure 4-11: Prob. Amplitudes for Small $\Delta$, Spin $\alpha \downarrow + \beta \uparrow$, Photon $\downarrow$

**CASE 2:** Large $\Delta$, the spin is initialized to $\uparrow$, the photon is initialized to $\downarrow$

All the assumptions are the same as in the cases above, and $\hat{\rho}$ is initialized again to Equation (4.11). The results are shown in the next four figures (Figure 4-12, Figure 4-13, Figure 4-14, and Figure 4-15). In the large $\Delta$ regime, there are only peak or maxima of magnitude 1 in the calculated electron-photon system entropy, as predicted using the standard parameterization approach. These peaks occur at 62.5 picoseconds. Furthermore, the probability of creating an exciton is effectively always zero. This is great from an engineering point of view since there will always be a high
probability of releasing a photon back into the quantum network waveguides once maximally entangled.

Figure 4-12: Bloch Sphere for Large $\Delta$, Spin $\alpha \downarrow + \beta \uparrow$, Photon $\dagger$
Figure 4-13: Poincare Sphere for Large $\Delta$, Spin $\alpha \downarrow + \beta \uparrow$, Photon $\downarrow$
Figure 4-14: Entanglement for Large $\Delta$, Spin $\alpha \downarrow + \beta \uparrow$, Photon $|\uparrow\rangle$
Recently, the hyperfine coupling with the nuclear spins of the semiconductor host material (~$10^5$ nuclei in a single quantum dot), which acts as a random magnetic field (up to a few Tesla in InAs quantum dots), has been identified as the ultimate limit, at low temperature, to the electron spin relaxation or decoherence in quantum dots. Consequently, assuming low temperature conditions, only the various decoherence
processes associated with hyperfine interactions will be considered; where as, those associated with phonon interactions are ignored.

First, the Fermi contact term is considered. It relates to the direct interaction of the nuclear dipole with the spin dipoles and is only non-zero for states with a finite electron spin density at the position of the nucleus (those with unpaired electrons in the conduction band). Therefore, the Fermi contact hyperfine interaction does not affect the trion states, just the following states $|\downarrow, \sigma_z^\pm\rangle$, $|\downarrow, \sigma_z^+\rangle$, $|\uparrow, \sigma_z^\pm\rangle$, and $|\uparrow, \sigma_z^+\rangle$. A typical longitudinal relaxation time for semiconductors is on the order of $T_i = 1\, ns$ [67].

Second, due to the $p$ symmetry of the Bloch wavefunctions in the valence band, the coupling of the nuclei with holes can be generally neglected because the Fermi contact interaction vanishes. The coupling constants of the hole-nuclear interaction are only one order of magnitude smaller than the coupling constants of the electron-nuclear interaction. One recent reference work determined the longitudinal relaxation time for hole-nuclear interaction to be $T_i = 14\, ns$ [68]. Furthermore, all trion states are affected, namely $|\downarrow, X_{ih}\rangle$, $|\downarrow, X_{hh}\rangle$, $|\uparrow, X_{ih}\rangle$, and $|\uparrow, X_{hh}\rangle$. It is also interesting to note that there are not any dark states for trions.

Last, radiative recombination rates of trions are also included in our model of decoherence. In the case of InGaAs quantum dots with an emission wavelength of $\lambda = 950\, nm$, a radiative recombination lifetime of $\tau_{X^-} = 0.84\, ns$ for negatively charged excitons was reported [69] as well as $\tau_{X^-} = 1.24\, ns$ in the case of InAs quantum dots with an emission wavelength of $\lambda = 850\, nm$ [70]. On the other hand, GaAs quantum dots
spin relaxation by phonon-assisted Dresselhaus spin-orbit scattering was estimated $T_1 = 34 \mu s$ at $T = 4.5K$. This illustrates the fact that these types of decoherence processes can be ignored at low temperature.

**CASE 1**: Small $\Delta$, Spin initialized to $\alpha \frac{\downarrow}{\downarrow} + \beta \frac{\uparrow}{\uparrow}$, Photon initialized to $\frac{\downarrow}{\downarrow}$

The “small” detuning energy is again selected to be $\Delta = 78 \mu eV$. The results are shown in the next four figures. In the small $\Delta$ regime, for all case of the excess electron spin being in a superposition of its eigenstates, up to 35% entanglement is obtained within only the first 80 picoseconds. Afterward, all entanglement is lost. Moreover, the dynamics of entanglement is as expected in that for the lower amounts of superposition in the excess spin state, maximum entanglement is smaller in magnitude.
Figure 4-16: Bloch Sphere for Small $\Delta$ and Decoherence

- $\alpha = 0.5, \beta = 0.5$
- $\alpha = 0.4, \beta = 0.6$
- $\alpha = 0.3, \beta = 0.7$
- $\alpha = 0.2, \beta = 0.8$
- $\alpha = 0.1, \beta = 0.9$
- $\alpha = 0, \beta = 1$
Figure 4-17: Poincare Sphere for Small $\Delta$ and Decoherence

- $\alpha = 0.5, \beta = 0.5$
- $\alpha = 0.4, \beta = 0.6$
- $\alpha = 0.3, \beta = 0.7$
- $\alpha = 0.2, \beta = 0.8$
- $\alpha = 0.1, \beta = 0.9$
- $\alpha = 0, \beta = 1$
Figure 4-18: Entanglement for Small $\Delta$ and Decoherence
CASE 2: Large $\Delta$, Spin initialized to $\alpha|\downarrow\rangle + \beta|\uparrow\rangle$, Photon initialized to $|\uparrow\rangle$

Also, the "large" detuning energy is selected to be $\Delta = 1.5 \text{meV}$. No entanglement is realized. And, the decoherence rate is just faster than the rate of entanglement between the QD spin and photon. Here again the dynamics of entanglement is unexpected as lower amounts of superposition in the excess electron spin states results in a larger maximum for the entanglement.
Figure 4-20: Bloch Sphere for Large $\Delta$ and Decoherence
Figure 4-21: Poincare Sphere for Large $\Delta$ and Decoherence
Figure 4-22: Entanglement for Large $\Delta$ and Decoherence
Some entanglement between an electron spin qubit within a quantum dot and a single-photon qubit interacting inside a high-Q nanocavity can be obtained even in the presence of severe decoherence processes. Whether or not such amount of entanglement for such short periods of time can be useful in a real physical system remains to be seen. For a certainty, the performance of such scheme will have to be improved. How?

First, performances could be increased by considering a different material system. Indium atoms have $I_9/2$ spin and arsenic atom have $I_3/2$ spin; which make InAs quantum dots bad candidate as far as decoherence is concern. Other semiconductors
with lower or without nucleus spin could be used. The dephasing times of the II-VI compounds are 3-10 times larger than dephasing times for III-V compounds[71], however, for wurzite-type semiconductors to have similar optical selection rules as in the case of zinc-blende-type semiconductors, propagation along the c axis is needed[72]. Second, improvements in performances could also be obtained performing manipulations on the nuclear system[73], or using hole spin in the valence band instead of the electron spin in the conduction band as storage qubit since it is less influenced by the nucleus spin. Third, Press et al recently introduced an all-optical spin echo technique that could in theory increase the decoherence time of a single quantum dot electron spin from nanoseconds to several microseconds[74]. In order for this concept to work in our scheme, the magnetic field would have to be applied in the faraday geometry instead of the Voigt geometry causing the spin to precess about the quantization axis of the spin which also happens to coincide with the direction of propagation of the flying qubit. In turn, this requires the rotating pulses, which are used to coherently rotate the spin so as to prevent decoherence, to be propagating in a direction perpendicular to this quantization axis.

**Controlling the Cavity-Waveguide Coupling Dynamics**

The primary challenge here is to be able to unload the photon from a node, which consists of in essence a single cavity mode with an embedded QD, into a waveguide and doing so with minimum or no loss and error in the phase. Various approaches to
solving this problem have been proposed in the literature such as the time symmetric scheme and the adiabatic scheme, which are both resonant Raman processes[40, 41, 65]. We consider an entirely different approach based on dynamically switching the quality factor of the cavity node[22].

**Minimizing the Phase Error**

Because dynamically switching the quality factor of the cavity does not instantaneously stop the photon from interacting with the QD so as to be unloaded onto a waveguide, a time window over which the interaction can realistically be stop must be considered. The consequence is an uncertainty in the phase or phase error. In order to reduce the phase error during that period of time, there is a need to keep the rate of rotation associated with the linear polarization of the single photon at a minimum. How can this be done?

As mentioned earlier in Chapter CHAPTER 2:, the rate of rotation of the single-photon polarization vector is nonlinear with respect to time for a small detuning energy exhibiting accelerating and decelerating regions. The minimization of the phase error is achieved when the desired accumulated phase coincides with a region of minimum change in the relative phase; that is a region with a horizontal slope as shown in Figure 4-24. A standard relative phase $2\varphi(t_{\text{interaction}})$ to be achieved before the single-photon is coupled back into a waveguide and routed to its next destination is $\pi$ that is

$\varphi(t_{\text{interaction}}) = \pi/2$ corresponding to a $90^\circ$ rotation of its linear polarization, which is also
the condition for maximum entanglement. As for $t_{\text{interaction}}$, it is the interaction time needed to achieved a $90^\circ$ rotation of the linear polarization. And because the rate of rotation of the single-photon polarization essentially corresponds to the slope of the relative phase $\varphi(t)$, a horizontal slope means a slow rate of rotation of the linear polarization and therefore a minimum phase error.

![Figure 4-24: Minimizing the Phase Error for $2\varphi(t_{\text{interaction}}) = \pi$](image)

Assuming a GaAs/InGaAs QD with emission wavelength $\lambda_{QD}$ of $1.182 \mu m$, an interaction frequency involving the heavy-hole valence band of $g_{3/2} = 132 \times 10^9 \text{rad/s}$, and an interaction frequency involving the light-hole valence band equal to
\[ g_{1/2} = 76 \cdot 10^9 \text{rad/s}, \quad 2\varphi = \pi \text{ coincides with a horizontal slope (no significant change in the relative phase) when the small detuning energy is } \Delta = 0.86g_{3/2} \text{ (Solved graphically).} \]

The interaction time is \[ t_{\text{interaction}} = 130 \text{ps}, \text{ and the time window corresponding to a horizontal slope is approximately } \tau = 20 \text{ps} \text{ (turquoise region).} \] The time window parameter \( \tau \) can also be understood as the maximum FWHM of the optical pulse needed to switch the cavity node Q factor to release the single photon from it while maintaining an insignificant change in the phase error. Noda demonstrated such capability by switching a cavity Q factor from 12,000 to 3,000 in few picoseconds\[22\]. If a \( \tau = 4 \text{ps} \) optical pulse is considered along with a change in Q factor magnitude from 10,000 to 1,000 for our cavity node allowing for an escape probability of 92\% after only few picoseconds, the result would be a phase error of \( \pm 0.022 \text{ rad/s} \text{ or } \pm 0.315^\circ \) for a 90\(^\circ\) rotation of the linear polarization as opposed to \( \pm 30^\circ \) or more for the worst case scenario\[64\].

On the other hand, for a very large detuning energy, the rate of rotation of the single-photon polarization is slow and steady with respect to time; and thus cannot be further minimized. The figure below shows the rate of rotation of the linear polarization when the detuning energy is \( \Delta = 10g_{3/2} \) under the same conditions mentioned in the previous paragraph.
The rate of rotation of the polarization is very small, thus many more interaction cycles are needed to achieve a $\pi/2$ phase for $\phi$. The phase error in this regime ($|\Delta| \gg g_{3/2}$) is typically constant and always remains well below the worst case scenario when the detuning energy is small ($|\Delta| \leq g_{3/2}$). However, the trade-off is that the number of interaction cycles needed to achieve the same phase $2\phi = \pi$ increases dramatically.
Minimizing the Loss

Even under the assumption of a perfectly isolated system, if the polariton or exciton-photon system created during the strong interaction of the photon with the QD is in its excitonic state rather than its photonic state at the conclusion of their interaction inside the quantum cavity node, then the photon is lost and won’t couple to the waveguide. To ensure that the photon is in the cavity at time $t = t_{\text{interaction}}$, the following equations must be satisfied.

\[
\left| C_{\uparrow, \sigma_z^+} (t_{\text{interaction}}) \right|^2 = \left| C_{\uparrow, \sigma_z^+} (0) \right|^2 = \frac{1}{2}
\]

\[
\left| C_{\uparrow, \sigma_z^+} (t_{\text{interaction}}) \right|^2 = \left| C_{\uparrow, \sigma_z^-} (0) \right|^2 = \frac{1}{2}
\]

(4.12)

In order to achieve the following condition $\left| C_{\uparrow, \sigma_z^+} (t_{\text{interaction}}) \right|^2 = \frac{1}{2}$, it can be easily deduced from Equation (2.67) that $\Omega_{3/2} t_{\text{interaction}} = j 2\pi$ where $j$ is a positive integer number ($j = 0, 1, 2...$). Similarly, in order to achieve the following condition $\left| C_{\uparrow, \sigma_z^-} (t_{\text{interaction}}) \right|^2 = \frac{1}{2}$, it must be that $\Omega_{1/2} t_{\text{interaction}} = k 2\pi$ where $k$ is a positive integer number such that $\frac{j}{\sqrt{3}} < k \leq j$. The boundaries for $k$ are a result of the fact that for $\lim_{\Delta \to 0} \frac{\Omega_{1/2}}{\Delta} = \frac{\Omega_{3/2}}{\sqrt{3}}$, resulting in $k = j$ and also the fact that for $\lim_{\Delta \to 0} \frac{\Omega_{1/2}}{\Delta} = \frac{\Omega_{3/2}}{\sqrt{3}}$, resulting in...
being bounded by $\frac{j}{\sqrt{3}}$. Since $t_{\text{interaction}}$ is the same for both probabilities as required, then $\frac{j2\pi}{\Omega_{3/2}} = \frac{k2\pi}{\Omega_{1/2}}$. Solving for the detuning energy, we finally obtain

$$\Delta = \pm 2g_{3/2}\sqrt{\frac{k^2 - \left(j^2 / 3\right)}{j^2 - k^2}}$$

(4.13)

The parameters $k$ and $j$ were selected to be 2 and 3 respectively. The resulting value of the detuning energy is approximately $\Delta = 0.86g_{3/2}$. Equation (2.68) is then satisfied periodically as depicted in Figure 4-26. This value for the detuning energy ($\Delta = 0.86g_{3/2}$) happens to coincide with the value we solved for graphically in the previous section for minimum phase error. This is the case because the zeros for the phases accumulated by the RCP and LCP components of the single photon field are integer numbers of the zeros for the probability amplitudes. The detuning energy ($\Delta = 0.86g_{3/2}$) thus corresponds to the first zero or flat slope for the relative phase, the second zero for the LCP polarization component ($k = 2$) and the third zero for the RCP polarization component ($j = 3$).
Furthermore, it is noteworthy to mention that the condition imposed by Equation (4.12) applies mostly when the detuning energy $|\Delta| \leq g_{3/2}$. This constraint can be relaxed when one considers the case where $|\Delta| \gg g_{3/2}$ since the probability amplitudes for the single photon field always 1 for all $t$. This is illustrated in Figure 2-15, the detuning energy here is set approximately to $\Delta = 10 g_{3/2}$.
CHAPTER 5: ENGINEERING SINGLE-PHOTONS TRANSPORT

Recently, the on-chip generation and transfer of microwave single photons have been demonstrated in connection with superconducting qubits via transmission line cavities[75-78]. In addition, the generation and transfer of single photons on photonic crystal chips using a $25 \mu m$ long defect waveguide structure has been studied by Vuckovic et al showing 12% transfer efficiency with quantum dots inside the nodes and 49% transfer efficiency without[79].

In this research, we consider use of a photonic crystal coupled-cavity waveguide (CCW) for the transfer of single photons on-chip. Such approach to an on-chip quantum network present several advantages. For example, since the photons we use have a wavelength of around $1 \mu m = 1 \cdot 10^{-4} cm$ against $1 cm$ or more as in the case of microwave photons, the surface area of a quantum network using a photonic crystal coupled-cavity waveguide would be about $(10^{-4})^2 = 10^8$ times smaller and thus much more suitable for on-chip integration. In addition, the transfer of microwave photons was shown to take few hundred nanoseconds; whereas, the transfer of optical photons inside the photonic crystal chip is anticipated to take no more than few tens of picoseconds.

Furthermore, CCWs offer a truly unique and sophisticated control over the transport of single photons[80]. Because modes of CCWs resemble those of the high-Q cavity modes and possess the same field symmetries, these devices can be used to make bends with no reflection. In addition, they can dramatically slow down optical waves, and because of their versatile dispersion properties (both positive and negative
dispersion are achievable), they allow for a great deal of control over a single-photon pulse propagation. Moreover, since each mode is strongly localized, the guided mode is composed of a linear combination of these individual bound modes. This renders the propagating mode easy to model quantum-mechanically[81], therefore allowing for their guiding behavior to be optimized with respect to the fidelity of quantum operations inside the quantum network. And, once their guiding behavior has been characterized with a few parameters, namely the coupling coefficient $\kappa$ and resonant frequency $\omega_c$, their physical structure can be constructed to emulate their intended behavior.

**System: Two Cavity Connected Via a Waveguide**

The device under investigation is depicted in Figure 5-1. It consists of two cavities, cavity 1 and cavity $N$, linked to one another by means of a CCW, consisting of cavity 2 through cavity $N-1$. In this specific example, $N$ is 12. The coefficient $\Gamma$ describes the weak coupling between the cavities forming the waveguide.

![Figure 5-1: Two high-Q Cavities Connected Via a CCW](image)

A general Hamiltonian for this system is derived from the tight-binding Hamiltonian and it is shown in Equation (5.1). $\hat{a}^\dagger_j$ and $\hat{a}_j$ are the creation and the
annihilation for the field in the $j$th cavity, $\omega_c$ the resonant cavities frequency, $V_{C_1-W}(t)$ the time-dependent interaction between the first cavity and the waveguide, and $V_{W-C_N}(t)$ the time-dependent interaction between the waveguide and the last cavity.

$$\hat{H} = \sum_j \hbar \omega_c \hat{a}_j^\dagger \hat{a}_j + \sum_2^{N-2} \Gamma \left(\hat{a}_j^\dagger \hat{a}_{j+1} + \hat{a}_{j+1}^\dagger \hat{a}_j\right) + V_{C_1-W}(t) + V_{W-C_N}(t) \quad (5.1)$$

Our key concern is the engineering of the interaction between cavities and the waveguide structure $V_{C_1-W}(t)$ and $V_{W-C_N}(t)$: First, we consider the waveguide by itself, and design its characteristics. Then, both the cavities and the waveguide are considered; and the system is engineered such that the photon is unloaded into the waveguide and transferred to its destination with maximum efficiency.

**Waveguide Hamiltonian**

First, it is useful to consider a CCW by itself. These structures can easily be included within fully quantum-mechanical models using the phenomenological description of the tight-binding Hamiltonian, which is simply written down in the basis of creation and annihilation operators that move photons from one quasi-mode to another. This allows for a deeper understanding of the underlying physics and the identification and characterization of features that are truly critical to the behavior of the quantum
network using only a few parameters. The Hamiltonian that can describe such a waveguide is exactly the tight-binding Hamiltonian in Equation (5.2).

\[ \hat{H}_{TB} = \sum_j \hbar \omega_c \hat{a}_j \hat{a}_j^\dagger + \sum_j \Gamma \left( \hat{a}_j^\dagger \hat{a}_{j+1} + \hat{a}_{j+1}^\dagger \hat{a}_j \right) \]  

(5.2)

In the Wannier representation, essentially a real-space picture of localized orbitals, this Hamiltonian matrix is written as

\[ \begin{pmatrix}
E_c & \Gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Gamma \\
\Gamma & E_c & \Gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \Gamma & E_c & \Gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \Gamma & E_c & \Gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Gamma & E_c & \Gamma & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \Gamma & E_c & \Gamma & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \Gamma & E_c & \Gamma & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \Gamma & E_c & \Gamma & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \Gamma & E_c & \Gamma & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Gamma & E_c & \Gamma & 0 \\
\Gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Gamma & E_c & \Gamma
\end{pmatrix} \begin{pmatrix}
\langle 1_n \rvert \\
\langle 2_n \rvert \\
\langle 3_n \rvert \\
\langle 4_n \rvert \\
\langle 5_n \rvert \\
\langle 6_n \rvert \\
\langle 7_n \rvert \\
\langle 8_n \rvert \\
\langle 9_n \rvert \\
\langle 10_n \rvert \\
\langle 11_n \rvert \\
\langle 12_n \rvert
\end{pmatrix} \]

(5.3)

where state \( \langle 1_n \rvert \) corresponds to the Wannier function localized in the first cavity forming the CCW, and state \( \langle 2_n \rvert \) corresponds to the Wannier function localized in the second cavity forming the waveguide, and so on. The diagonal matrix element for each
site is $E_c$, this is the energy of the resonant mode for each cavity. Periodic boundary conditions are expressed at the waveguide ends as $\hat{a}_1^\dagger = \hat{a}_{12}^\dagger$ and $\hat{a}_1 = \hat{a}_{12}$.

**Dispersion, Group Velocity, and GVD**

It is assumed that the wavelength of the single photon $\lambda_{ph}$ of 1.182 $\mu$m, which is a realistic wavelength for a GaAs/InGaAs based QD emitter. This wavelength corresponds to the cavity resonant frequency $\omega_c = 1.594 \cdot 10^{15}$ rad/s, where $E_c = \hbar \omega_c$. Next, assuming a quality factor $Q = 1000$ for each cavity of the CCW, the coupling rate between neighboring cavities is thus calculated to be $\Gamma = 1.594 \cdot 10^{12}$ rad/s, where $\Gamma$ is defined as $\omega_c / Q$. The definition of $\Gamma$ assumes that coupling to the nearest neighbor cavity is the only loss channel, i.e., out-of-plan losses are ignored.

The dispersion relation of this CCW is solved from the diagonalization of the Hamiltonian matrix. Although working in a real space representation, the problem is fully equivalent to the reciprocal space-based Bloch representation. In fact, for $N$ sites, $N$ $k$ points exist in reciprocal space and are defined as $k = 2\pi K / Na$ where $K$ is a quantum number such that $K = -N/2 + 1, \ldots, N/2$ and $a$ the lattice constant of the periodic waveguide. As a result, the same energy eigenvalues are obtained. They are plotted in red for the normalized positive $k$ points of the first Brillouin zone in Figure 5-2.
The dispersion obtained numerically matches very well the theoretical dispersion for CCWs in Equation (5.4) in the linear dispersion approximation. Under this approximation, we assume weak coupling, which means that photons may only leak into the nearest neighbor cavity[82-86]. The tight-binding model yields the optical carrier frequencies

\[
\omega_k = \omega_c \left[ 1 + \kappa \cos(ka) \right]
\]

(5.4)

where \( \omega_c \) the resonant frequency of a single cavity, \( \kappa \) the coupling coefficient between
cavities, $k$ the Bloch wave number, and $a$ the lattice constant. For this waveguide, the coupling coefficient was found to be $\kappa = -0.002$.

An expression for the group velocity $v_g$ can be derived from Equation (5.4) as follows:

$$v_g = \nabla_k \omega_k = -\omega_c \kappa a \sin(ka)$$

(5.5)

The group velocity $v_g$ normalized over $c$ (the speed of light in vacuum) is plotted in Figure 5-3 throughout the normalized coupled cavity waveguide band. It is obtained by taking the derivative of obtained energy eigenvalues with respect to $k$. At the edges of the waveguide band, that which corresponds to when $k$ is 0 or 1, the group velocity tends toward zero.
This result has two important consequences. First, the group velocity dispersion is ill defined when \( k \) is 0 or 1 according to the standard definition in Equation (5.6). Second, the group velocity dispersion may be either positive or negative depending on which band edge \( k \) is closest to.

\[
GVD = -\left(\frac{2\pi c}{\lambda^2}\right) \frac{d}{d\omega} \left( \frac{1}{v_g(k)} \right)
\]  

(5.6)

The group velocity dispersion (GVD) for the 12 cavities forming the waveguide is plotted in Figure 5-4 within the CCW band. For practical applications, Mookherjea
derived more appropriate definitions of GVD that satisfy a small fractional change of the GVD coefficient over the range of frequencies of interest[87].

![Figure 5-4: Group Velocity Dispersion](image)

**Single-Photon Propagation**

A single-photon pulse is shown propagating for the waveguide composed of 12 cavities in Figure 5-5. The time-dependence for the coupled-cavity waveguide is obtained numerically using the Liouville or Von Neumann Equation (similar to the approach discussed in[88]) in conjunction with the tight-binding Hamiltonian to solve for the time evolution of the density matrix whose states correspond to the Wannier
functions localized in the cavities forming the waveguide. It is assumed that there are neither any out-of-plane losses or material absorption causing the CCW mode to decay nor any scattering resulting in a sudden change of the phase.

Figure 5-5: Traveling Single Photon Pulse

The single photon pulse can be seen traveling from cavity 1 to Cavity 12 in about 9 picoseconds before being reflected. An oscillatory structure at the trailing edge of the pulse can be noticed. We believe this feature has to do with the single-photon pulse defined initially at a precise point in space and time. The consequence is a wideband single-photon pulse, and therefore higher-order terms can no longer be neglected resulting in envelope distortion. In a recent publication[88], we showed how the
coherent exchange between a photon qubit and an electron spin qubit in a QD is affected by decoherence resulting from hyperfine interactions. Although these decoherence processes prevented the interacting qubits from reaching maximum entanglement, there were at least partially entangled. It is important to note that these interactions were taking place over a longer period of time, about 1 to 2 orders of magnitude longer than the time needed for a photon to propagate down 12 cavities. In addition, a novel scheme based on ultrafast optical spin echo shows that decoherence times on the order of microseconds are achievable[74]. Therefore, it is anticipated that the propagation time of the photon won’t be much of an issue over small to medium range distances.

Figure 5-5 also depicts the single-photon pulse propagation in the linear region of the CCW; therefore, the single-photon frequency is tuned to the middle of the waveguide band (This corresponds to a normalized frequency equal to 1 in Figure 5-2). As a result, ignoring the oscillatory structure at the trailing edge of the pulse, a pulse propagates mostly unchanged in shape. On the other hand, when the single-photon pulse propagation is determined by the flat region of the CCW dispersion corresponding to one of the edges of the transmission band, there exists a considerable slowing of the group velocity. Figure 5-6 depicts the travel time from cavity 1 to cavity 12 as a function of coupling coefficient $\kappa$. 
Mechanisms for Dynamically Switching Cavity Q Factor

The careful engineering of the dynamic coupling between nearest neighbor cavities provide the necessary control for loading and loading the single photon from and onto the waveguide, and therefore it is key element in achieving high-efficiency transfer of single-photon on-chip. Dynamically coupling the end cavities to the waveguide can be realized by either switching the magnitude of the coupling coefficient or by switching the nearest neighbor cavity resonant frequency.
Switching the Coupling Constant Coefficient

The first way to dynamically couple end cavities to the waveguide is done by setting their resonant frequency (being single mode) to be the same for all cavities including the end cavities while varying the coupling coefficients between the end cavities and the waveguide in time in order to load and unload the photon from and onto the waveguide. This is depicted in Figure 5-7.

![Figure 5-7: Unloading/Loading Using Time-Varying Coupling Coefficients](image)

The corresponding Hamiltonian is described in (5.7).

\[
\hat{H} = \sum_j \hbar \omega_j \hat{a}_j^{\dagger} \hat{a}_j + \sum_{j=1}^{N-2} \Gamma \left( \hat{a}_j^{\dagger} \hat{a}_{j+1} + \hat{a}_{j+1}^{\dagger} \hat{a}_j \right) \\
+ \Gamma_1(t) \left( \hat{a}_1^{\dagger} \hat{a}_2 + \hat{a}_2^{\dagger} \hat{a}_1 \right) + \Gamma_2(t) \left( \hat{a}_{N-1}^{\dagger} \hat{a}_N + \hat{a}_N^{\dagger} \hat{a}_{N-1} \right) 
\]  

Switching the Cavity Resonant Frequency

The second way to dynamically couple end cavities to the waveguide is done by setting coupling coefficients to be the same for all cavities including the end cavities
while varying the resonant modes of cavity 2 and cavity \( N - 1 \) in time in order to load and unload the photon from and onto the waveguide. This is depicted in Figure 5-8.

![Figure 5-8: Unloading/Loading Using Time-Varying Resonant Frequencies](image)

The corresponding Hamiltonian is described in Equation (5.8).

\[
\hat{H} = \sum_{j \neq 2, N-1} \hbar \omega_c \hat{a}_j \hat{a}_j + \hbar \omega_2(t) \hat{a}_2^+ \hat{a}_2 + \hbar \omega_{N-1}(t) \hat{a}_{N-1}^+ \hat{a}_{N-1} + \sum_{j} \Gamma \left( \hat{a}_j^+ \hat{a}_{j+1} + \hat{a}_{j+1}^+ \hat{a}_j \right)
\]

(5.8)

In practice, both of these approaches can be realized by means of a spatial modulation of the refractive index within the quantum network. In order for these loading and unloading operations to be feasible in a functioning quantum network though, they must be performed at speeds much greater than usual decoherence possesses therefore requiring the assistance of ultrafast optical pulses. Methods for the dynamical tuning of refractive index are based on nonlinear effects, carrier injection by linear absorption of an optical pump (free-carrier plasma dispersion effect), carrier injection using a PIN diode, or thermal tuning via optical heating. We are interested in carrier injection by linear absorption of an optical pump; which many groups have shown to be a viable method. For example, Tanaka et al demonstrated a change in cavity Q from...
12,000 to 3,000 in 4ps[22]. Lipson’s group at Cornell University used a 100fs pump pulse to generate a 18ps index change pulse [89] and also demonstrated how a 1.5ps pump pulse corresponding to a 25ps index change pulse could change the Q factor of a cavity from 60000 to 17000[90]. Tanabe et al generated a 14ps pump pulse resulting in a change in photon lifetime from 320ps to 70ps inside a high Q cavity[91]. Carrier injection using PIN diode is an interesting alternative and allows for both the injection and extraction of carrier simultaneously. The resulting shape of the free carrier index change looks a lot like a square pulse. Gardes et al showed a 7ps rise and fall time in the index change was possible[92].

**Transfer Efficiencies**

On-chip on-demand single-photon transfer stipulates that the photon is unloaded onto a waveguide from a cavity 1 at a time \( t_1 \) allowing it to travel down the waveguide before being loaded into a cavity 2 at a time \( t_2 \). Considering the two approaches mentioned in the previous section, a critical question to be answered is what transfer efficiencies can be obtained.

First, we consider the case of time-varying coupling coefficients. The unloading of the photon onto the waveguide is achieved by dynamically switching the magnitude of the coupling coefficient between cavity 1 and cavity 2 starting at \( t_1 \). Perhaps, this can be realized using an approach similar to Noda’s[22]. Then, the loading of the photon
from the waveguide into cavity $N$ is achieved by once again dynamically switching the
magnitude of coupling coefficient between cavity $N-1$ and cavity $N$ starting at $t_2$.

Design parameters of interest for both switching functions are their shape, the
range of coupling strength over which they are varied, the time over which they should
be varied. There are also few design constraints. The temporal width of $\Gamma_1(t)$ or FWHM
should be larger than the natural cavity decay into the CCW so as to allow the photon to
escape yet smaller than round trip time. Similarly, properties of $\Gamma_2(t)$ will greatly
depend on factors such as the CCW length, the group velocity, or the group velocity
dispersion. Figure 5-9 shows the switching functions $\Gamma_1(t)$ and $\Gamma_2(t)$ qualitatively.

![Switching Functions](image)

Figure 5-9: Switching Functions $\Gamma_1(t)$ and $\Gamma_2(t)$
It was found that the optimum shape of switching functions $\Gamma_1(t)$ and $\Gamma_2(t)$ is a Gaussian profile. In a CCW with constant coupling between its cavities, when a photon is allowed to propagate freely, though highly localized initially (effectively represented by a delta function in space), it eventually exhibits a distribution in space that happens to be well-approximated by a Gaussian. It may be that the distribution could be also approximated by other functions such as a squared hyperbolic secant for example. The authors believe it is worth further investigating the mechanism behind the broadening associated with the probability of finding the photon in space. In any case, this is certainly related to the fact the coupled cavities have a finite $Q$ which introduces an uncertainty in the time over which the photons actually hops from one cavity to the next. That being said, the context of Figure 5-9 is slightly different as it relates not just to a waveguide with constant coupling coefficient but to time-varying coupling coefficients. In other words, the coupling constant between two quasi-bound modes, that are degenerate in frequency, is varied. And the cavity $Q$ depends on the coupling $\Gamma(t)$ where $Q = \omega / \Gamma(t)$. Providing that the switching function $\Gamma(t)$ has a Gaussian shape, the $Q(t)$ will also have a Gaussian shape, thus allowing for minimum reflections at the target cavity boundaries since the incoming photon also has a Gaussian-like probability distribution in time.

As far as the range over which the coupling strength should be varied for $\Gamma_1(t)$, it is assumed that there is not any coupling, initially, between cavity 1 and cavity 2, yielding $\Gamma_1(0) = 0$. However, the maximum of the Gaussian shaped switching function is
designed to be $\Gamma_{\text{max}} = 1.594 \cdot 10^{12} \text{rad/s}$, which corresponds to the regular coupling rate between neighboring cavities of the CCW. The minimum FWHM for $\Gamma_1(t)$ that allows the entire photon to leak out of the cavity was found to be $3 \text{ ps}$. The characteristics of $\Gamma_2(t)$ are engineered so as to maximize the transfer efficiency.

Figure 5-10 shows transfer efficiencies for various FWHM for $\Gamma_2(t)$ and a large range of $t_2$. These transfer efficiencies are calculated assuming $\Gamma_1(t)$ with a FWHM equal to $3 \text{ ps}$, a peak coupling rate equal to $1.594 \cdot 10^{12} \text{rad/s}$, and $t_1$ equal to $1.5 \text{ ps}$. A maximum transfer efficiency of $93\%$ is obtained for $\Gamma_2(t)$ with a FWHM equal to $2.75 \text{ ps}$ and a starting time $t_2$ equal to $7.25 \text{ ps}$. The transfer efficiency is defined as the ratio of the probability of finding a photon in cavity $N$ following its capture (once $\Gamma_2(t)$ goes back to zero) over the probability of finding a photon in Cavity 1 before its release (when $\Gamma_1(0) = 0$), which is always unity.
Next, the case of time-varying resonant frequencies is considered. The unloading of the photon onto the waveguide is achieved by dynamically switching the resonant frequency of cavity 2 from $\omega_c + \Delta$ to $\omega_c$ starting at $t_1$. Then, the loading of the photon from the waveguide into cavity $N$ is achieved by dynamically switching the resonant frequency of cavity $N-1$ from $\omega_c + \Delta$ to $\omega_c$ starting at $t_2$ and subsequently from $\omega_c$ back to $\omega_c + \Delta$.

Design parameters of interest for both switching functions, $S_1(t)$ and $S_2(t)$, are their shape or time-dependence, the range of frequencies over which the cavity resonant frequencies are varied, the time window over which resonant frequencies
should be varied. As far as design constraints, it is desirable that the amount of detuning $\Delta$ for the resonant frequency of the “barrier” cavities to be larger than the waveguide bandwidth to avoid any significant coupling between the waveguide and the end cavities. Also, the duration of $S_1(t)$, which is the time during which the resonant frequency of cavity 2 is switched from $\omega_c + \Delta$ to $\omega_c$, should be large enough for the photon to escape, yet smaller than the round trip time. Similarly, $S_2(t)$ needs to be large enough for the photon to be captured. Figure 5-11 shows $S_1(t)$ and $S_2(t)$ qualitatively.

Figure 5-11: Switching Functions $S_1(t)$ and $S_2(t)$
It was found that the optimum shape for $S_1(t)$ and $S_2(t)$ is a square profile. In the case of the time-varying frequency scheme, the coupling is mostly dictated by $\omega$ where $\omega \gg \Gamma$, so until the frequency of the “barrier” cavity matches the frequencies of adjacent cavities, the probability of tunneling through is insignificant. That explains why a square profile is more appropriate. Earlier, in the case of the time-varying coupling coefficient scheme, the difference was that all the $\omega$’s were the same; therefore, the probability of tunneling through was only depending on $\Gamma(t)$. In fact, much higher transfer efficiencies were achieved for the time-varying frequency scheme with a square shape switching function with the prescribed duration (about 75%) compared to a Gaussian shape switching function with a wide range of duration (no more than 10%). The authors believe that the transfer efficiency could further be improved if the switching function was switched on and off adiabatically, resulting in a rounded square shape. In addition, the switching function duration is now chosen to correspond to approximately twice the photon lifetime in the “barrier” cavity since the photon has to both enter and exit the “barrier” cavity before it can reach the target cavity.

As far as the range over which the resonant frequency of the “barrier” cavities should be varied, we assume that each cavity in the CCW has a Q of 1000 resulting in a coupling coefficient of $\Gamma = 1.594 \cdot 10^{12} \text{rad/s}$ at the wavelength of interest and a bandwidth of $BW = \omega \pm \kappa = 1 \pm 0.002$ in normalized units of frequency. In our case, this corresponds to $BW = 1.59 \cdot 10^{15} \pm 3.18 \cdot 10^{12} \text{rad/s}$. Consequently, we designed the detuning parameter to be $\Delta = 3.24 \cdot 10^{12} \text{rad/s}$ such that $\Delta \geq |\kappa|$. Therefore, by switching
“barrier” cavity frequencies from $\omega_c$ to $\omega_c + \Delta$, we are able to prevent coupling between the end cavities and the CCW.

Figure 5-12 shows transfer efficiencies for various FWHM for $S_2(t)$ and a large range of $t_2$. These transfer efficiencies are calculated assuming $S_1(t)$ with a FWHM equal to 4.875 ps and $t_1$ equal to 1.5 ps. A maximum transfer efficiency of 75% is obtained for $S_2(t)$ with a FWHM equal to 3.985 ps and a starting time $t_2$ equal to 5.7 ps.

![Figure 5-12: Transfer Efficiency Using Time-Varying Resonant modes](image)

Figure 5-12: Transfer Efficiency Using Time-Varying Resonant modes
CHAPTER 6: ENGINEERING MULTIPARTITE ENTANGLEMENT

Why is multipartite entanglement important for quantum networks? Entanglement is not only unique in that it is purely a quantum effect, but it also a key ingredient for being able to transfer quantum information encoded onto one spin to another spin located elsewhere within the quantum network. This process by which this information is transferred is referred to as quantum teleportation. In this section, we focus on engineering the efficient generation of entanglement between two quantum dot spins by means of a photon, known as a Greenberger-Horne-Zeilinger (GHZ) state.

**Dynamical Models Basics**

When studying multi-qubit systems as in the case of the most basic quantum network, it is advantageous to introduce dynamical models. These models are deemed dynamical because of their taking into account both the time-dependent and the area-confined nature of the interactions between qubits. In other words, various qubits interact at different times and at different locations within the quantum network for different amounts of time. Two models in particular are of interest when considering a 3-qubit system: the *dynamical Jaynes-Cummings* model[93], which can describe a series of sequential interaction between 2 qubits within the 3-qubit system, and the *dynamical Dicke* model[94], which can describe the simultaneous interaction of 3 or more qubits.
The Dynamical Jaynes-Cummings (DJC) Model

The Jaynes-Cummings Model shown consists of a fully quantum mechanical treatment of a 2-level atom or QD interacting with a single mode field in a cavity without losses. Recall the Jaynes-Cummings Hamiltonian from Equation (2.8)

\[
\hat{H} = \frac{1}{2} \hbar \omega \hat{\sigma}_z + \hbar \nu \hat{a}^\dagger \hat{a} + \hbar g (\hat{\sigma}_+ \hat{a} + \hat{a}^\dagger \hat{\sigma}_-) \quad (6.1)
\]

When considering the dynamical Jaynes-Cummings model[95], the JC Hamiltonian is modified to

\[
\hat{H} = \frac{1}{2} \hbar \omega \hat{\sigma}_z + \hbar \nu \hat{a}^\dagger \hat{a} + f(t, t', \tau) \left[ \hbar \left( \hat{\sigma}_+ \hat{a} + \hat{a}^\dagger \hat{\sigma}_- \right) \right] \quad (6.2)
\]

where \( f(t, t', \tau) \) is the time-dependent coupling of the QD with the single photon field defined as

\[
f(t, t', \tau) = \left[ \Theta(t - \tau) - \Theta(t - \tau - t') \right] g(t) \quad (6.3)
\]

This time-window function describes the situation where a photon is injected at time \( t' \) inside the cavity and interacts for a time \( \tau \) with the QD with the time-dependent coupling strength \( g(t) \) before exiting the cavity.
The Dynamical Dicke (DD) Model

The Dicke model provides a description of the interaction between a collection of 2-level systems with a radiation field taking place in a perfect cavity while assuming that the radiation field is quasi-monochromatic and that the 2-level systems are identical with no direct interactions between or among themselves. The Hamiltonian of the Dicke model in the electric dipole and rotating-wave approximations is

\[
\hat{H}_D = \frac{1}{2} \hbar \omega \hat{S}_z + \hbar v \hat{a}^\dagger \hat{a} + \hbar g \left( \hat{S}_+ \hat{a} + \hat{a}^\dagger \hat{S}_- \right)
\]

(6.4)

where \( \hat{S}_z = \sum_j S_z(j) \) is the operator of atomic inversion, \( \hat{S}_+ = \sum_j S_+(j) \) the collective atomic raising operator, \( \hat{S}_- = \sum_j S_-(j) \) the collective atomic lowering operator. We set \( j = 2 \) since we are considering a 2-Node quantum network.

Another assumption is that the spatial dimensions of the system are smaller than the wavelength of the field so that both of the 2-level atoms experience the same field. This may not be true for our quantum network as the QDs could be separated by a distance larger than the wavelength for reasons we will shortly discuss. One way to circumvent this restraint is by assuming that the QDs even though far apart inside a long cavity are located within a wavelength distance from the cavity mirrors at both ends. This ensures that both the QDs are located at a field maximum rather than a field minimum.
The dynamical Dicke Hamiltonian shown below is obtained by introducing both a
time-dependent coupling strength and a time-window function just as in the case of the
dynamical JC Hamiltonian [95].

$$\hat{H}_{DD}(t, \tau) = \frac{1}{2} \hbar \omega \hat{S}_z + \hbar v \hat{a}^\dagger \hat{a} + f(t, t', \tau) \left\{ \hbar \left( \hat{S}_x \hat{a} + \hat{a}^\dagger \hat{S}_x \right) \right\}$$ (6.5)

**Dynamical 2-Nodes Quantum Network Models**

In this section, the dynamic of the 2-Nodes quantum network is described of in
terms of the modified DJC model and modified DD model. These two approaches prove
fundamentally different and lend themselves to either a particle picture of the field or a
wave picture of the field. Regardless, the system exhibit quantum properties such as
entanglement, although entanglement dynamics is expected to change.

**Quantum Networks as a Modified DJC Model**

When the dynamic of the 2-Nodes quantum network is described of in terms of
the modified DJC model, the photon interacts first with the QD inside the first node at
time $t_1$, and then it interacts with the other QD inside the second node at a subsequent
time $t_2$. This is illustrated in Figure 6-1. This is the particle picture of the field as the field
is highly localized within subsystems of the quantum network. In order to achieve such scheme, coupling from nodes to the continuum must be highly deterministic in nature.

The perturbed part of the Hamiltonian is thus

\[
\begin{align*}
    t_1 < t & \leq t_1 + \tau_1 \\
    t_1 + \tau_1 & \leq t \leq t_2 \\
    t_2 < t & \leq t_2 + \tau_2
\end{align*}
\]

Figure 6-1: 2-Node Quantum Network as a Modified DJC Model
Quantum Networks as a Modified DD Model

When the dynamic of the 2-Nodes quantum network is described in terms of the modified DD model, the photon interacts simultaneously with both QDs inside the first node at time \( t' \). This is depicted in Figure 6-2. This is the wave picture of the field as the field is spread over the entire coupled cavity system. All the traditional assumptions made in the Dicke model are satisfied in the specified coupled cavity.
system. First, the radiation field is quasi-monochromatic. Second, the QDs are identical and do not directly interact with one another since they physically separated by 6 cavities. Third, the system can be engineered so that the QDs experience the same field.

Figure 6-2: 2-Nodes Quantum Network as a Modified DD Model

The perturbed part of the Hamiltonian is thus
Calculating Entanglement

Comparing these two models serves to enhance our understanding of the relationships between entanglement and entropy during the generation, transfer, or sharing of entanglement. How though to measure entanglement in a 3-qubit system? We will follow a similar approach found in [95]. First, the full density matrix for the 2-Nodes quantum network must be generated. There are a total of 64 states, 16 for the
first node, 32 for the photonic channel, and 16 for the second node. Second, the linear entropy of each subsystem can be calculated such that

\[
S_{NL,\text{spin}} = 1 - Tr_{\text{spin}} \left( \hat{\rho}_{\text{spin}}^{\text{red}} \right)^2
\]

\[
S_{NL,\text{spin2}} = 1 - Tr_{\text{spin2}} \left( \hat{\rho}_{\text{spin2}}^{\text{red}} \right)^2
\]

\[
S_{NL,\text{photon}} = 1 - Tr_{\text{photon}} \left( \hat{\rho}_{\text{photon}}^{\text{red}} \right)^2
\]

where

\[
\hat{\rho}_{\text{spin}}^{\text{red}} = \left\{ Tr_{\text{polariton}} Tr_{\text{spin2}} \hat{\rho} \right\}_{\text{Node1}} + \left\{ Tr_{\text{photon}} Tr_{\text{spin2}} \hat{\rho} \right\}_{\text{CCW}} + \left\{ Tr_{\text{polariton}} Tr_{\text{spin2}} \hat{\rho} \right\}_{\text{Node2}}
\]

\[
= \left\{ Tr_{\text{photon}} Tr_{\text{spin2}} \hat{\rho} \right\}_{\text{Node1}} + \left\{ Tr_{\text{photon}} Tr_{\text{spin2}} \hat{\rho} \right\}_{\text{CCW}} + \left\{ Tr_{\text{photon}} Tr_{\text{spin2}} \hat{\rho} \right\}_{\text{Node2}}
\]

(6.8)

\[
\hat{\rho}_{\text{spin2}}^{\text{red}} = \left\{ Tr_{\text{spin1}} Tr_{\text{polariton}} \hat{\rho} \right\}_{\text{Node1}} + \left\{ Tr_{\text{spin1}} Tr_{\text{photon}} \hat{\rho} \right\}_{\text{CCW}} + \left\{ Tr_{\text{spin1}} Tr_{\text{polariton}} \hat{\rho} \right\}_{\text{Node2}}
\]

\[
= \left\{ Tr_{\text{spin1}} Tr_{\text{photon}} \hat{\rho} \right\}_{\text{Node1}} + \left\{ Tr_{\text{spin1}} Tr_{\text{photon}} \hat{\rho} \right\}_{\text{CCW}} + \left\{ Tr_{\text{spin1}} Tr_{\text{photon}} \hat{\rho} \right\}_{\text{Node2}}
\]

(6.9)
\[
\hat{\rho}_{\text{photon}}^{\text{red}} = \left\{ \text{Tr}_{\text{spin}_1} \text{Tr}_{\text{spin}_2} \hat{\rho}_{\text{Node}_1} \right\} + \left\{ \text{Tr}_{\text{spin}_1} \text{Tr}_{\text{spin}_2} \hat{\rho}_{\text{CCW}} \right\} + \left\{ \text{Tr}_{\text{spin}_1} \text{Tr}_{\text{spin}_2} \hat{\rho}_{\text{Node}_2} \right\}
\] (6.11)

Last, entanglement between the subsystems can be derived.

\[
E_{\text{spin}_1,\text{spin}_2} = S_{NL,\text{spin}_1} + S_{NL,\text{spin}_2} - S_{NL,\text{photon}}
\]

\[
E_{\text{spin}_1,\text{photon}} = S_{NL,\text{spin}_1} + S_{NL,\text{photon}} - S_{NL,\text{spin}_2}
\] (6.12)

\[
E_{\text{spin}_2,\text{photon}} = S_{NL,\text{spin}_2} + S_{NL,\text{photon}} - S_{NL,\text{spin}_1}
\]

**GHZ Channel**

In this section, we will first consider the dynamics of the spin1-photon-spin2 entanglement without decoherence. In addition, it is assumed that the system under investigation is operated in the small detuning regime since maximum entanglement is achieved 10x faster. This is necessary as the size of the time-dependent Hamiltoninan in Liouville space grows dramatically when a larger time range is considered resulting in memory problems.
**Device Structure**

The GHZ channel is depicted in Figure 6-3. It consists of two cavities, cavity 1 and cavity 6, linked to one another by means of a 4 coupled cavity forming a waveguide. Cavity 1 and Cavity 6 have embedded quantum dots. It is assumed that the quantum dots' emission wavelength are $\lambda_{QD} = 1.182 \mu m$. The QD energy corresponding to this wavelength is detuned from the cavity resonant by $\Delta = 75 \mu eV$ and later by $\Delta = 1.5 meV$, resulting in the following cavity resonant frequency $\omega_c = 1.594 \cdot 10^{15} \text{rad/s}$, where $E_c = \hbar \omega_c$. Next, assuming a quality factor $Q = 1000$ for each cavity of the CCW, the coupling rate between neighboring cavities is thus calculated to be $\Gamma = 1.594 \cdot 10^{12} \text{rad/s}$, where $\Gamma$ is defined as $\omega_c/Q$. As for the end cavities, a quality factor $Q = \infty$ assumed until it is switched to $Q = 1000$ in order to trap or release the photon from and into the waveguide.

![Figure 6-3: GHZ Channel](image)

Source Cavity

Target Cavity

1 2 3 4 5 6
Hamiltonian

Still in the Wannier representation, this Hamiltonian matrix in the rotating frame is written as

\[
\hat{H}_{\text{GHZ}} = \begin{pmatrix}
\langle 1_c | & \langle 1_c | & \langle 2_w | & \langle 3_w | & \langle 4_w | & \langle 5_w | & \langle 6_c | & \langle 6_c | \\
\end{pmatrix}
\] (6.13)

where \([C_1], [C_2], [C_3], [C], [C_{12}], [C_{13}], [C_{14}], [W], [W_1], [W_2], [W_{12}], [W_{13}], [R]\) are all 8x8 submatrices. \([C_1]\) and \([C_2]\) describe the source cavity (cavity 1). \([W_1]\) and \([W_2]\) and their transposed matrices, \([W_1]'\) and \([W_2]'\), are describing coupling coefficient between the source cavity and the CCW. The CCW is itself described by matrices. \([C_3], [C], [C_{12}],\) that are couple to one another with a constant coupling coefficient described by matrices \([W]\). \([C_{13}]\) and \([C_{14}]\) matrices describe the target cavity (cavity 6). \([W_{12}]\) and \([W_{13}]\) and their transposed matrices, \([W_{12}]'\) and \([W_{13}]'\), are describing coupling coefficient between the target cavity and the CCW. These matrices are shown in Equations (6.14) though (6.26).
where $E$ is the resonant frequency of the source cavity, $E_x$ is the emission frequency from the quantum dot exciton, $V_{lh}$ the interaction frequency between the photon the light hole exciton, and $V_{hh}$ the interaction frequency between the photon the heavy hole exciton.

$$\begin{bmatrix}
| \langle \downarrow \sigma_i \downarrow \rangle & \langle \downarrow \sigma_i \uparrow \rangle & \langle \downarrow l h_i \downarrow \rangle & \langle \downarrow \sigma_i \uparrow \rangle & \langle \downarrow l h_i \uparrow \rangle & \langle \downarrow h h_i \uparrow \rangle \\
E & 0 & 0 & V_{hh} & 0 & 0 & 0 & 0 \\
0 & E & V_{lh} & 0 & 0 & 0 & 0 & 0 \\
0 & V_{lh} & E_x & 0 & 0 & 0 & 0 & 0 \\
V_{hh} & 0 & 0 & E_x & 0 & 0 & 0 & V_{hh} \\
0 & 0 & 0 & 0 & E & 0 & V_{lh} & 0 \\
0 & 0 & 0 & 0 & 0 & E & 0 & 0 \\
0 & 0 & 0 & V_{lh} & 0 & E_x & 0 & V_{hh} \\
0 & 0 & 0 & 0 & V_{lh} & 0 & E_x & 0 \\
0 & 0 & 0 & 0 & V_{hh} & 0 & E_x & 0
\end{bmatrix} = \begin{bmatrix}
| \langle \downarrow \sigma_i \downarrow \rangle \\
| \langle \downarrow \sigma_i \uparrow \rangle \\
| \langle \downarrow l h_i \downarrow \rangle \\
| \langle \downarrow \sigma_i \uparrow \rangle \\
| \langle \downarrow l h_i \uparrow \rangle \\
| \langle \downarrow h h_i \uparrow \rangle \\
\end{bmatrix}$$  \hspace{1cm} (6.14)

where $E$ is the resonant frequency of the source cavity, $E_x$ is the emission frequency from the quantum dot exciton, $V_{lh}$ the interaction frequency between the photon the light hole exciton, and $V_{hh}$ the interaction frequency between the photon the heavy hole exciton.

$$\begin{bmatrix}
| \langle \uparrow \sigma_i \downarrow \rangle & \langle \uparrow \sigma_i \uparrow \rangle & \langle \uparrow l h_i \downarrow \rangle & \langle \uparrow \sigma_i \uparrow \rangle & \langle \uparrow l h_i \uparrow \rangle & \langle \uparrow h h_i \uparrow \rangle \\
E & 0 & 0 & V_{lh} & 0 & 0 & 0 & 0 \\
0 & E & 0 & V_{hh} & 0 & 0 & 0 & 0 \\
V_{lh} & 0 & E_x & 0 & 0 & 0 & 0 & 0 \\
0 & V_{hh} & 0 & E_x & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & E & 0 & V_{lh} & 0 \\
0 & 0 & 0 & 0 & 0 & E & 0 & V_{hh} \\
0 & 0 & 0 & V_{lh} & 0 & E_x & 0 & V_{hh} \\
0 & 0 & 0 & 0 & V_{lh} & 0 & E_x & 0 \\
0 & 0 & 0 & 0 & V_{hh} & 0 & E_x & 0
\end{bmatrix} = \begin{bmatrix}
| \langle \uparrow \sigma_i \downarrow \rangle \\
| \langle \uparrow \sigma_i \uparrow \rangle \\
| \langle \uparrow l h_i \downarrow \rangle \\
| \langle \uparrow \sigma_i \uparrow \rangle \\
| \langle \uparrow l h_i \uparrow \rangle \\
| \langle \uparrow h h_i \uparrow \rangle \\
\end{bmatrix}$$  \hspace{1cm} (6.15)
from the quantum dot exciton, \( V_{lh} \) the interaction frequency between the photon the light hole exciton, and \( V_{hh} \) the interaction frequency between the photon the heavy hole exciton.

\[
\begin{pmatrix}
\langle \downarrow \sigma_2 \downarrow \rangle & \langle \downarrow \sigma_2 ^+ \downarrow \rangle & \langle \downarrow \sigma_2 ^+ \uparrow \rangle & \langle \uparrow \sigma_2 ^\downarrow \rangle & \langle \uparrow \sigma_2 ^+ \downarrow \rangle & \langle \uparrow \sigma_2 ^+ \uparrow \rangle \\
\end{pmatrix}
\]

\[
[C_3] =
\begin{pmatrix}
V_{e1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & V_{e1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & V_{e1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & V_{e1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & V_{e1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & V_{e1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & V_{e1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & V_{e1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & V_{e1}
\end{pmatrix}
\]

(6.16)

where \( V_{e1} \) is the resonant frequency of the cavity nearest to the source cavity. \( V_{e1} \) can be time-dependent if this cavity is used as a barrier.
where $E$ is the resonant frequency of the cavities inside the CCW.
\[
\begin{pmatrix}
\langle \downarrow \sigma_5 \downarrow \rangle & \langle \downarrow \sigma_5 \downarrow \rangle & \langle \downarrow \sigma_5 \uparrow \rangle & \langle \downarrow \sigma_5 \uparrow \rangle & \langle \uparrow \sigma_5 \downarrow \rangle & \langle \uparrow \sigma_5 \downarrow \rangle & \langle \uparrow \sigma_5 \uparrow \rangle & \langle \uparrow \sigma_5 \uparrow \rangle \\
Ve2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & Ve2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & Ve2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & Ve2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Ve2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & Ve2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & Ve2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & Ve2
\end{pmatrix}
\]

(6.18)

where \( Ve2 \) is the resonant frequency of the cavity nearest to the target cavity. \( Ve2 \) can be time-dependent if this cavity is used as a barrier.

\[
\begin{pmatrix}
\langle \downarrow \sigma_6 \downarrow \rangle & \langle \downarrow \sigma_6 \downarrow \rangle & \langle \downarrow lh_6 \downarrow \rangle & \langle \downarrow lh_6 \downarrow \rangle & \langle \downarrow \sigma_6 \uparrow \rangle & \langle \downarrow \sigma_6 \uparrow \rangle & \langle \downarrow lh_6 \uparrow \rangle & \langle \downarrow lh_6 \uparrow \rangle \\
E & 0 & 0 & V_{hh} & 0 & 0 & 0 & 0 \\
0 & E & V_{lh} & 0 & 0 & 0 & 0 & 0 \\
0 & V_{lh} & E_x & 0 & 0 & 0 & 0 & 0 \\
V_{hh} & 0 & 0 & E_x & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & E & 0 & V_{lh} & 0 \\
0 & 0 & 0 & 0 & 0 & E & 0 & V_{hh} \\
0 & 0 & 0 & 0 & V_{lh} & 0 & E_x & 0 \\
0 & 0 & 0 & 0 & 0 & V_{hh} & 0 & E_x
\end{pmatrix}
\]

(6.19)

where \( E \) is the resonant frequency of the target cavity, \( E_x \) is the emission frequency from the quantum dot exciton, \( V_{lh} \) the interaction frequency between the photon the light
hole exciton, and $V_{hh}$ the interaction frequency between the photon the heavy hole exciton.

$$[C_{14}] = \begin{bmatrix}
\langle \uparrow \sigma_6 \downarrow \rangle & \langle \uparrow \sigma_6 \downarrow \rangle & \langle \uparrow \text{lh}_6 \downarrow \rangle & \langle \uparrow \text{hh}_6 \downarrow \rangle & \langle \uparrow \sigma_6 \uparrow \rangle & \langle \uparrow \text{lh}_6 \uparrow \rangle & \langle \uparrow \text{hh}_6 \uparrow \rangle \\
E & 0 & 0 & V_{hh} & 0 & 0 & 0 & 0 \\
0 & E & V_{lh} & 0 & 0 & 0 & 0 & 0 \\
0 & V_{lh} & E_c & 0 & 0 & 0 & 0 & 0 \\
V_{hh} & 0 & 0 & E_c & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & E & 0 & V_{lh} & 0 \\
0 & 0 & 0 & 0 & 0 & E & 0 & V_{hh} \\
0 & 0 & 0 & 0 & V_{lh} & 0 & E_c & 0 \\
0 & 0 & 0 & 0 & 0 & V_{hh} & 0 & E_c
\end{bmatrix} (6.20)$$

where $E$ is the resonant frequency of the target cavity, $E_x$ is the emission frequency from the quantum dot exciton, $V_{lh}$ the interaction frequency between the photon the light hole exciton, and $V_{hh}$ the interaction frequency between the photon the heavy hole exciton.
where $V_1$ is the coupling coefficient between the cavity nearest and the source cavity. $V_1$ can be time-dependent if the time-varying coupling coefficient scheme is used to unload and load the photon into the source/target cavities.

$$[W_1] = \begin{bmatrix} V_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & V_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & V_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & V_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & V_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & V_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & V_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & V_1 & 0 \end{bmatrix} \begin{bmatrix} \downarrow \sigma^- \\ \downarrow \sigma^+ \\ \downarrow \sigma^+ \\ \uparrow \sigma^- \\ \uparrow \sigma^+ \\ \uparrow \sigma^+ \\ \downarrow \sigma^- \\ \downarrow \sigma^+ \end{bmatrix} = (6.21)$$

where $V_1$ is the coupling coefficient between the cavity nearest and the source cavity. $V_1$ can be time-dependent if the time-varying coupling coefficient scheme is used to unload and load the photon into the source cavity.

$$[W_2] = \begin{bmatrix} 0 & 0 & 0 & 0 & V_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & V_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & V_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & V_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & V_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & V_1 \end{bmatrix} \begin{bmatrix} \downarrow \sigma^- \\ \downarrow \sigma^+ \\ \downarrow \sigma^+ \\ \uparrow \sigma^- \\ \uparrow \sigma^+ \\ \uparrow \sigma^+ \\ \downarrow \sigma^- \\ \downarrow \sigma^+ \end{bmatrix} = (6.22)$$

where $V_1$ is the coupling coefficient between the cavity nearest and the source cavity. $V_1$ can be time-dependent if the time-varying coupling coefficient scheme is used to unload and load the photon into the source cavity.
where \( V_2 \) is the coupling coefficient between the cavity nearest and the target cavity.

\( V_2 \) can be time-dependent if the time-varying coupling coefficient scheme is used to unload and load the photon into the target cavity.

\[
[W_{12}] = \begin{bmatrix}
V_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & V_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & V_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & V_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & V_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & V_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(6.23)

\[
[W_{13}] = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
V_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & V_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & V_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & V_2 & 0 & 0
\end{bmatrix}
\]

(6.24)

where \( V_2 \) is the coupling coefficient between the cavity nearest and the target cavity.
\( V_2 \) can be time-dependent if the time-varying coupling coefficient scheme is used to unload and load the photon into the target cavity.

\[
\begin{bmatrix}
V & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & V & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & V & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & V & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & V & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & V & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & V
\end{bmatrix}
\]

(6.25)

where \( W \) is the coupling coefficient between the cavities inside the CCW. This coefficient is always constant.

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(6.26)
GHZ Channel with Time-Varying Coupling Coefficient Scheme

In this scheme, matrices $[W_1]$, $[W_2]$, $[W_{12}]$, and $[W_{13}]$ are time-dependent while matrices $[C_3]$ and $[C_{12}]$ are time-independent. This is shown in Equation (6.27).

$$
\hat{H}_{GHZ} = 
\begin{pmatrix}
\langle 1_c | & \langle 1_c | & \langle 2_w | & \langle 3_w | & \langle 4_w | & \langle 5_w | & \langle 6_w | & \langle 6_c | & \langle 6_c | \\
| C_1 | & | R | & | W_1(t) | & | R | & | R | & | R | & | R | & | 1_c | \\
| R | & | C_2 | & | W_2(t) | & | R | & | R | & | R | & | R | & | 1_c | \\
| W_1(t)' | & | W_2(t)' | & | C_3 | & | W | & | R | & | R | & | R | & | 2_w | \\
| R | & | R | & | W | & | C | & | W | & | R | & | R | & | 3_w | \\
| R | & | R | & | W | & | C | & | W | & | R | & | R | & | 4_w | \\
| R | & | R | & | R | & | W | & | C_{12} | & | W_{12}(t) | & | W_{13}(t) | & | 5_w | \\
| R | & | R | & | R | & | R | & | R | & | W_{12}(t)' | & | C_{13} | & | R | & | 6_w | \\
| R | & | R | & | R | & | R | & | W_{13}(t)' | & | R | & | C_{14} | & | 6_w | \\
\end{pmatrix}
$$

(6.27)

GHZ Channel with Time-Varying Resonant Frequency Scheme

In this scheme, matrices $[C_3]$ and $[C_{12}]$ are time-dependent while matrices $[W_1]$, $[W_2]$, $[W_{12}]$, and $[W_{13}]$ are time-independent. This is shown in Equation (6.28).
Spin 1-Photon-Spin 2 Entanglement Dynamics

For the sake of simplicity, entanglement is only considered in the DJC model in this section. More complex engineering is indeed needed for the implementation of the DD model such as the dynamic manipulation of multiple cavity resonances simultaneously. Because the DD model requires the single photon simultaneously interacts with both QDs in order to create a 3-particle entanglement or GHZ state, this means that the photon field would have to be spread evenly over all cavities before a flattening the dispersion relation takes place over the CCW system resulting in the photon being effectively “stopped”. The challenge then would be to achieve the strong coupling regime needed for coherent interaction between the qubits given the large effective modal volume. On the other hand, in the DJC model, the single photon sequentially interacts with QD 1 located in the source cavity and then interacts at a later
time with QD 2 at the target cavity in order to create a 3-particle entanglement. Each node provides the required strong coupling environment in between the capture and release of the photon. In addition, the unloading/loading scheme considered for the calculations in this section is based on the time-varying coupling coefficients simply because it results in higher single photon transfer efficiencies.

The first step is to design switching timing for maximized transfer efficiency for a CCW system made up of 6 cavities. In order words, the time-dependence associated with submatrices \( [W_1], [W_2], [W_{12}], \) and \( [W_{13}] \) in Equation (6.27) must be engineered properly so that the single photon can be released from the initial cavity and subsequently captured at the destination cavity with the highest probability. Figure 6-4 shows \( \Gamma_1(t) \) and \( \Gamma_2(t) \), which represents the time dependence of \( [W_1] \) and \( [W_2] \), has a Gaussian profile, and its magnitude is varied from 0 to \( 1.594 \cdot 10^{12} \text{ rad/s} \) corresponding to a Q factor of 1000. \( \Gamma_1(t) \) is switched on at \( t_1 = 0.1 \text{ ps} \), and its FWHM (Full Width at Half Maximum) was designed to be \( 2.0168 \text{ ps} \). On the other hand, \( \Gamma_2(t) \) represents the time dependence of \( [W_{12}] \) and \( [W_{13}] \). \( \Gamma_2(t) \) also has a Gaussian profile, and its magnitude varies from 0 to \( 1.594 \cdot 10^{12} \text{ rad/s} \), which corresponds to a Q factor of 1000. \( \Gamma_2(t) \) is switched on at \( t_2 = 1.71 \text{ ps} \), and its FWHM was designed to be \( 2.1176 \text{ ps} \).
A maximum transfer efficiency of 96% was obtained for the timing considered above. Figure 6-5 shows the probability amplitude of finding the photon in target cavity equaled to 0.96 as opposed to 0 for the probability amplitude of finding the photon in source cavity at the end of all switching, i.e. after 11\(\text{ps}\), and thus the photon is effectively trapped in the target cavity.
CASE 1: Large Detuning ($\Delta = 1.5\text{meV}$)

Figure 6-6 shows the dynamics of entanglement between the photon and the QD spin 1 taking place in the source cavity. Maximum entanglement occurs after an interaction time of 650\,ps if the spin happens to be initialized in a perfect superposition of its eigenstates; this is when the photon is chosen to be released. Once the photon is captured in the target cavity, a similar interaction time is considered between the photon and the QD spin 2 resulting in a GHZ state.
The next figure shows the single photon field components and its sum as it interacts with the QD spin1 qubit in the source cavity, propagates down the CCW, and then interacts again with the QD spin 2 qubit in the target cavity. It can readily be seen that the photon is always in its photonic form as the probability of creating an exciton is quite insignificant due to the large detuning in energies between the photon and the QDs.
Figure 6-7: Photon Field Components in the Large Detuning Regime
Figure 6-8 shows the GHZ entanglement dynamics which is reflected in the spin1-spin2 subsystem entanglement. As expected, in the DJC model, the GHZ state is not created until the photon is released from the source cavity, captured in the target cavity, and then starts interacting with the QD spin2 qubit. The Bloch and Poincare spheres also show evidence of entanglement as the magnitudes associated with each qubits shrink to zero.

![figure 6-8](image)

**Figure 6-8: GHZ Entanglement Dynamics in the Large Detuning Regime**

The individual linear entropy, which are excellent measure of how individual elements are entangled to the rest of the system, where used to calculate the intrinsic entanglement plotted above. These individual entropies are shown below.
CASE 2: Small Detuning ($\Delta = 75 \mu eV$)

In the small detuning regime, there are three conditions for maximum entanglement if the spin happens to be initialized in a perfect superposition of its eigenstates. We will consider the 2nd peak, which occurs after $93 \text{ ps}$ of interaction in the source cavity. This is shown in Figure 6-10. Once the photon is released and then recaptured in the target cavity, a similar interaction time is considered between the photon and the QD spin2 resulting in a GHZ state.
However, the drawback for operating in the small detuning regime is that the photon is absorbed more strongly, and thus there is a 30% chance of creating an exciton at that specified interaction time, i.e. $93\, \text{ps}$, given by the fact that the total probability of finding a photon in the cavity is approximately 0.7. This is depicted in Figure 6-11.
Figure 6-11: Photon Field Components in the Small Detuning Regime
One consequence is that the transfer efficiency from the source cavity to the target cavity is lowered. This can be seen from the oscillation in the photon field components in the target cavity where the maximum magnitude of the photon is considerably smaller than it was in the source cavity.

Figure 6-12 shows the GHZ entanglement dynamics which is reflected in the spin1-spin2 subsystem entanglement.

![Figure 6-12: GHZ Entanglement Dynamics in the Small Detuning Regime](image)

The various intrinsic entanglement between pairs of qubits and the rest of the system exhibit interesting features. First, $E_{\text{spin2-phot}}$ and $E_{\text{spin1-spin2}}$ become negative
during the time the photon is the source cavity and thus separated from the QD spin2 qubit. Second, $E_{\text{spin1-photon}}$ drops below 0.4 from almost a maximized entangled state during the time the photon is the target cavity and thus separated from the QD spin1 qubit. At the same time, their individual linear entropy is always positive and between 0 and 0.5 shown in Figure 6-13.

These negative values in the intrinsic entanglement are referred to as entanglement sudden death (ESD), which is a feature of nonlocal quantum coherence[96, 97]. It indicates that the density matrix $\hat{\rho}$ is both mixed and separable. A mixed-state quantum state described by the density matrix refers to the fact that we are dealing with a statistical ensemble of pure states, i.e. cannot be described by a single ket vector. On the other hand, the density matrix is said to be mixed and separable if it can be written as a product state of its subsystems, where one or more subsystems themselves constitute a nonseparable composite quantum state. It is exactly the case in the DJC model where two QDs spins couple independently to the photon within their respective environments. It is interesting to note that even in the large detuning regime, this phenomenon can be observed to some extent as $E_{\text{spin1-spin2}}$ becomes slightly negative during the time the photon is the source cavity. We confirm Han et al's conclusion that that the energy transfer between the atom and the photon has a direct bearing on entanglement sudden death[97].
In the future, other interaction time in the small detuning regime resulting in maximum entanglement should be considered. Since, the entanglement sudden death phenomenon is dependent on the initial state, more interesting dynamics should be expected. Furthermore, the GHZ entanglement dynamics should be considered with time varying resonant frequency scheme even though transfer efficiencies are smaller for the reason that this scheme is the most experimentally feasible. In addition, various decoherent processes can be added to the such model. Some examples are hyperfine interactions for each quantum dot, phonon interactions, various scattering mechanism.
for the photon due to the QDs or surface roughness, and even modal dispersion in the waveguide resulting in additional rotation of the linear polarization.
CHAPTER 7: PHYSICAL DESIGN

The idea of a quantum technology that can be implemented on a chip is very attractive because of several long term benefits such as more computing power and also speed using quantum algorithms. Stanford University Professor Dr Vuckovic et al made the following statement in a recent publication: “Efficient implementation of quantum computation devices requires on chip integration of photonic circuits consisting of photonic crystal cavities and waveguides”[21]. However, a functional quantum technology based on a photonic crystal chip has not been successfully designed nor fabricated to date. In this chapter, a promising photonic crystal chip is designed using PWE and FDTD numerical techniques bringing us ever closer to the physical realization of a functional quantum technology implemented on a photonic crystal chip.

Why a Photonic Crystal Backbone?

Photonic crystal structures are ideal because of their unmatched ability to both control the flow of light even realizing sharp turn without significant losses and strongly confine light in small volume providing a unique environment for strong coupling between a single photon and a QD. Furthermore, since photons in photonic crystal structures with small features have a small wavelength, photonic crystal devices are much more suitable for large scale on-chip integration. What’s more, nodes and connecting waveguides are easily added just by introducing defects.
Origin of the Photonic Band Gap

A good starting point in understanding the origin of photonic band gaps [98] is Maxwell’s equations.

\[
\begin{align*}
\nabla \cdot \mathbf{D} &= \rho \\
\nabla \cdot \mathbf{B} &= 0 \\
\n\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\n\nabla \times \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t} + j
\end{align*}
\]

(7.1)

Next, few assumptions are needed. First, the mixed dielectric medium composing a photonic crystal structure is assumed to be free of charges and currents; therefore, \( \rho = 0 \) and \( j = 0 \). Second, the linear regime is assumed so that all higher-order terms of the dielectric constant can be neglected. Third, the material is assumed to be macroscopic and isotropic so that there isn’t any dependence of the dielectric constant on the direction of propagation of the EM wave. Forth, material dispersion is ignored, which allows for the frequency dependence of the dielectric constant to be dropped. Fifth, the material is assumed transparent; therefore, the dielectric constant can be treated as real and positive. Equation (7.1) can therefore be rewritten as
\[ \nabla \cdot \varepsilon(r) \mathbf{E}(r,t) = 0 \]

\[ \nabla \cdot \mathbf{H}(r,t) = 0 \quad (7.2) \]

\[ \nabla \times \mathbf{E}(r,t) = -\mu_0 \frac{\partial \mathbf{H}(r,t)}{\partial t} \]

\[ \nabla \times \mathbf{H}(r,t) = \varepsilon_0 \varepsilon(r) \frac{\partial \mathbf{E}(r,t)}{\partial t} \]

It is useful mathematically to expand the electric field and magnetic fields into a set of harmonic modes such that \( \mathbf{E}(r,t) = \mathbf{E}(r)e^{j\omega t} \) and \( \mathbf{H}(r,t) = \mathbf{H}(r)e^{j\omega t} \). Equation (7.2) becomes

\[ \nabla \cdot \varepsilon(r) \mathbf{E}(r) = 0 \]

\[ \nabla \cdot \mathbf{H}(r) = 0 \quad (7.3) \]

\[ \nabla \times \mathbf{E}(r) = -j\omega \mu_0 \mathbf{H}(r) \]

\[ \nabla \times \mathbf{H}(r) = j\omega \varepsilon_0 \varepsilon(r) \mathbf{E}(r) \]

The two curl equations can easily be decoupled by taking the curl of them and rearranging them as shown below. The result of this operation is known as the wave equation.
$$\nabla \times (\nabla \times \mathbf{E}(r)) = \nabla \times (-j \omega \mu_0 \mathbf{H}(r))$$
$$= -j \omega \mu_0 \nabla \times (\mathbf{H}(r))$$
$$= -j \omega \mu_0 \left(j \omega \varepsilon_0 \mathbf{E}(r)\right)$$
$$= \frac{\omega^2}{c^2} \mathbf{E}(r)$$

$$\nabla \times \left(\frac{1}{\varepsilon(r)} \nabla \times \mathbf{H}(r)\right) = \nabla \times (j \omega \varepsilon_0 \mathbf{E}(r))$$
$$= j \omega \varepsilon_0 \left(\nabla \times \mathbf{E}(r)\right)$$
$$= j \omega \varepsilon_0 \left(-j \omega \mu_0 \mathbf{H}(r)\right)$$
$$= \frac{\omega^2}{c^2} \mathbf{H}(r)$$

(7.4)

Only the wave equation in terms of the magnetic field is retained to perform calculations for the simple reason that the operator associated with the magnetic field $\nabla \times \frac{1}{\varepsilon(r)} \nabla \times$ is Hermitian and the one associated with the electric field $\nabla \times \nabla \times$ is not. It ensues that the eigenvalues are always positive and real. Furthermore, the electric field can always be recovered using the curl equation associated with the magnetic field in Equation (7.3).

In the presence of a periodic media such as photonic crystal, the Bloch-Floquet theorem allows us to rewrite the magnetic field as a plane wave but modulated by a periodic function of periodicity proportional to the lattice vector $\mathbf{R}$ such that
\[ H_k(r) = e^{ikr}u_k(r) \]  \hspace{1cm} (7.5)

where \( u_k(r) = u_k(r + R) \) and \( R = la_1 + ma_2 + na_3 \) with \((l, m, n)\) being integer numbers and \((a_1, a_2, a_3)\) primitive lattice vectors.

Substituting Equation (7.5) into the wave equation for the magnetic field, we obtain

\[
\nabla \times \frac{1}{\varepsilon(r)} \nabla \times H(r) = \frac{\omega(k)^2}{c^2} H(r)
\]

\[
\nabla \times \frac{1}{\varepsilon(r)} \nabla \times e^{ikr}u_k(r) = \frac{\omega(k)^2}{c^2} e^{ikr}u_k(r)
\]

\[
(ik + \nabla) \times \frac{1}{\varepsilon(r)} (ik + \nabla) \times u_k(r) = \frac{\omega(k)^2}{c^2} u_k(r)
\]  \hspace{1cm} (7.6)

Equation (7.6) is the master equation of photonic crystal, it tells everything we need to know about the field in the photonic crystal. This eigenvalue problem has a new Hermitian operator defined as \((ik + \nabla) \times \frac{1}{\varepsilon(r)} (ik + \nabla) \times\), and we can now solve for \( u_k(r) \).

As a result of \( u_k(r) = u_k(r + R) \), this eigenvalue problem can be viewed as being restricted to a finite volume, which leads for an infinite set of modes or bands discretely spaced in frequency for each \( k \) value resulting in band gaps. Therefore, photonic band gaps are created as the result of a periodicity in the refractive index much in the same way semiconductor band gaps are created as the result of a periodicity in the atomic potential.
Numerical Methods for the Design of Photonic Crystals

In this section, two numerical approaches that are employed in this research to study and thereafter design photonic crystal structures with particular optical properties are considered, namely the plane wave expansion (PWE) method and the finite difference time domain (FDTD) method.

**Plane Wave Expansion Method**

The master equation in Equation (7.6) consists of an eigenvalue problem as mentioned before. It can be successfully solved numerically using any standard spectral method; however, the spectral method of choice in this research is the plane wave expansion method. The principle is to approximate the unknown periodic function $u_k(r)$ as a series expansion in a complete basis set of smooth functions whose solutions are known (i.e. Fourier series). The series is truncated in order to have a finite number of terms.

What kind of information can be obtained from the PWE method?[99, 100]. It allows for the computation of the eigenvalues and eigenvectors. The eigenvalues are used to construct photonic band diagrams, which provide the transmission and dispersive properties of a photonic crystal. The eigenvectors on the other hand describe the field profiles of Bloch modes.
How is the PWE method implemented? Recalling that in an infinite periodic crystal and using the Bloch-Floquet theorem, the unknown periodic function part of the Bloch mode representing the magnetic field can be expanded into its Fourier series such as

\[ u_k(r) = \sum_G K_H(G) \exp([i(G) \cdot r]) \]  

\[ K_H(G) = \frac{1}{V} \int u_k(r) \exp(-iG \cdot r) d^3r \]  

(7.7)

where \( r \) is the position vector, \( k \) is the wave vector of the plane wave, \( G \) is the reciprocal lattice vector, and \( K_H(G) \) are Fourier coefficients of the magnetic field. The reciprocal lattice vector is defined as \( G = l'b_1 + m'b_2 + n'b_3 \) so that \( a_i \cdot b_j = 2\pi\delta_{ij} \) where \( (l', m', n') \) are integer numbers, \( a_i \) the primitive lattice vectors, and \( b_j \) the primitive reciprocal lattice vectors. Next, the dielectric function is also expanded into a Fourier series and it is rewritten as

\[ \frac{1}{\varepsilon(r)} = \sum_G K_\varepsilon(G) \exp(iG \cdot r) \]  

\[ K_\varepsilon(G) = \frac{1}{V} \int \frac{1}{\varepsilon(r)} \exp(-iG \cdot r) d^3r \]  

(7.8)

where \( K_\varepsilon(G) \) are Fourier coefficients of the dielectric function. Substituting Equations
(7.7) and (7.8) into (7.6), and then Fourier transforming both side of Equation (7.6), we get

\[
\sum_G (k + G) \cdot (k + G') \frac{1}{K_0 (G - G')} K_H (k + G) = -\frac{\omega^2}{c^2} K_H (k + G')
\]

(7.9)

This equation represents an infinite set of linear equations because of the infinite number of Fourier coefficients \( K_H (k + G) \). This infinite set of equation is truncated by solving this problem for a finite set of plane waves \( G \) since the contribution of the terms associated with larger \( |G| \) values are negligible. Typically, for this research project, we selected between 16 and 256 plane waves for each direction of the Cartesian coordinate system. Furthermore, all possible \( k \) values inside a photonic crystal do not have to be solved for in order to generate a band diagram. One special properties of Bloch states is that one Bloch state with wave vector \( k \) is identical to another Bloch state with wave vector \( k + G \). As a consequence, only \( k \) values existing in the range \( -\frac{\pi}{a} \) to \( \frac{\pi}{a} \) must be considered when solving for band diagrams therefore reducing the size of the problem (note: \( a \) is the lattice constant of the photonic crystal). This region of nonredundant \( k \) values is known as the Brillouin zone (BZ). Rotational symmetry can create additional redundancy for wave vectors within the Brillouin zone itself; therefore, the problem can be reduced even further by considering a smaller region within the Brillouin zone referred to as the irreducible Brillouin zone (IBZ). Therefore, only a sequence of Bloch wave vectors corresponding to the perimeter of the IBZ and hence
connecting all key points of symmetry are needed in the computation of eigenvalues and eigenvectors. Typically, for this research project, we selected between 8 and 12 \( k \) points along each direction of the IBZ. The figure below shows the first BZ (blue region) and IBZ (green region) for both 2D square and triangular lattices.

Figure 7-1: Irreducible Brillouin Zone for Square and Triangular Lattices

**Finite-Difference Time Domain Method**

The principle behind this time-domain technique is that the electric and magnetic fields are propagated in both space and time by iterating the Maxwell’s equations in Equation (7.1). In a nutshell, this is done by taking the central difference approximation for both the temporal and spatial derivative of the time dependant Maxwell’s curl equations. Because it is a fully vectorial method, a rigorous solution to Maxwell’s equations can be obtained.
What kind of information can be obtained from the FDTD method? In addition to being able to compute the eigenvalues needed to generate photonic crystal band diagrams along with transmission and reflection spectra as in the case with frequency-domain techniques, the FDTD method can be used to model nonlinear or active media as well as losses. However, this powerful technique presents some disadvantages typically related to computational time and computer resources. First, structures with sharp resonances will require a long computational time in order to resolve spectral features. Similarly, it takes a considerable amount of time to reach the steady-state response of a structure subject to a time-harmonic source. Furthermore, depending on the desired spatial resolution, a significant amount of computational time and computer resources may be needed.

How is the FDTD method implemented? The traditional FDTD algorithm is implemented by generating field-update equations based on the Yee algorithm[101, 102]. In this scheme, the spatial derivative in the Maxwell’s curl equations are replaced by discretized operators. This calls for the discretization of the problem. The physical structure is therefore defined on a spatial grid or Yee grid, which means that material properties such as the refractive index are assigned to each grid point corresponding to a specific location in the Cartesian space. Then the propagation of the fields is achieved by computing the fields at each grid point at discrete time intervals using values from the neighboring grid points recorded at the previous time interval. The time interval at which calculations must be performed is chosen carefully so that numerical stability is maintained. It is interesting to note that the Yee grid naturally enforces the divergence
conditions of the electric and magnetic fields. The 1D field-update equations for both the electric and magnetic fields under the assumptions of an isotropic, lossless, source free, spatially-dependent homogeneous medium are shown below.

\[
E^{t+\frac{1}{2}}(x) = E^{t-\frac{1}{2}}(x) + \left[ \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \right] \frac{\Delta t}{\Delta x} \left( H'(x + \frac{1}{2}) - H'(x - \frac{1}{2}) \right) - \Delta t \left( \frac{\partial P}{\partial t} \right)'(x)
\]

\[
H^{t+1}(x + \frac{1}{2}) = H'(x + \frac{1}{2}) - \left[ \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \right] \frac{\Delta t}{\Delta x} \left( E^{t+\frac{1}{2}}(x + 1) - E^{t+\frac{1}{2}}(x) \right)
\]

\[\text{(7.10)}\]

**Engineering the Photonic Crystal Structure of the Quantum Network**

Although 3D structure with a complete photonic band gap would strongly confine light in all directions and therefore desirable for hosting a quantum network, a reliable fabrication process for 3D structures still remains to be achieved. An alternative approach relies on photonic crystal slabs. However, they only strongly confine light in the plane of the slab using Distributed Bragg Reflection (DBR) while vertical confinement is weakly achieved by means of total internal reflection (TIR). Furthermore, the effective indices seen by both a TE and TM wave propagating in a slab waveguide are quite different making it a challenge to obtain an overlap of the photonic band gap. For the sake of simplicity, a 2D structure and a TE polarized photon are assumed. The objective is to emulate their intended behavior characterized quantum mechanically and
therefore to achieve similar or higher transfer efficiencies in terms of normalized energy
densities of the classical electromagnetic fields.

**Device Structure**

The envisioned system will consist of 8 coupled cavities. The single photon is
initially trapped in the source cavity by means of barrier cavities whose resonant
frequencies are detuned from the source cavity resonant frequency as a result of an
optically induced local change in the refractive index using off resonant excitation[22].
The same is true for the target cavity and its barrier cavities. By engineering the timing
associated with the dynamic switching of these various barrier cavities resonant
frequencies, the photon can be transported from the source to the target cavity and then
trapped there. We choose to implement L3s cavities, where L3 denotes a defect line
cavity of three missing holes while the s denotes that the holes on the ends of the cavity
are reduced in size. This size reduction enables the designer to sculpt the cavity
resonance but is not considered variable for the current study. The CCW will be a chain
of these cavities as defects in a triangular array of air hole circles of \( r = 0.4a \) with 2 hole
spacers \( (r = 0.3a) \) between each cavity as shown in Figure 7-2. The variable \( a \) is the
lattice spacing of holes in the crystal.
Implementation of our PWE scheme with the above device shows that light will be non-dispersive and therefore strongly confined along the M crystal direction but will exhibit dispersion and therefore propagate along the K crystal direction, which is our intended direction of CCW propagation as shown in Figure 7-3. The cavity supercell is shown in the inset.
Figure 7-3: Dispersion of Targeted Mode Over the First Brillouin Zone

By orthogonalizing the k-vectors calculated in the PWE scheme so as to consider only the K direction, we may increment the PWE's eigenvalue calculations along only the direction of propagation to simulate the CCW structure's dispersion relation, shown in Figure 7-4. On the left, we see that many modes exist in the crystal's photonic band gap, some with high confinement that result in no dispersion and others that allow coupled-cavity resonance for propagation that result in the sinusoidal dispersion. The mode of interest is selected and zoomed in upon for the figure to the right.
The resulting spectral signature of this mode dispersion is shown in Figure 7-5 and will result in as many peaks over the range of resonant frequencies as there are coupled cavities in the chain. The left figure represents the spectra inside a single L3s source cavity isolated from any other cavities, while the figure on the right represents the spectra through a 8 cavity coupled L3s2 system. Interestingly, there are only 7 peaks; this is due to the fact that the last barrier cavity does not contribute to the propagating mode as its only purpose is to stop the photon.
Finally, it is important for the spatial field mode profiles of our cavity and CCW to be similar in order to maximize coupling between the two structures. Using both PWE and FDTD simulation methods, we verify the spatial field dependence shown in Figure 7-6 of our mode of choice. On the left, a PWE calculation of the allowed mode corresponding to the dispersive mode in Figure 7-4 matches well with the FDTD simulation of the propagating mode on the right, both showing the out of plane field component (Hy).
Photon Trapping Mechanism

Relying on FDTD and the analysis of spectra, the design of our cavity is made through observation of the behavior of cavity resonance with changes to surrounding regions. First, an isolated L3s cavity is simulated to establish a basis for the types of modes this sort of cavity is likely to support. This sort of isolated defect is unable to be readily released into a CCW chain, and so a heterostructured cavity is implemented by including index-switchable regions to act as barrier cavities to confine light away from the CCW portion of the device. Because the resonances of identical cavities shift in materials of different refractive index, shown in Figure 7-7, it is possible to use this resonance mismatch as a confining mechanism. Using a standard index of 3.4 (red) as our base, to comply with GaAs substrates, switching the barrier regions of our heterostructured cavity to an index of 3.3 (yellow) offers negligible overlap between “barrier” cavity resonances and the waveguide bandwidth (grey).
By implementing switched index cavities as barriers for the generation of a high-Q cavity, the mode of interest for both barrier-open (left) and barrier-closed (right) three cavity segments are shown in Figure 7-8. The fact that only peak shows when the barriers are closed is proof that the cavity is optically isolated. The spatial mode profile of this High-Q mode is also shown (bottom).
While the heterostructured cavity is greatly diminished in Q-factor when compared to a single well isolated L3 cavity, it still represents a functional and well confined optical mode that is effectively isolated from the neighboring CCW chain. With a Q-factor just under 2000, this proof of concept cavity represents an area for future research in optimization if strong coupling is desired. Now, as we look toward coupling this cavity mode to the CCW chain, it should be noted that the cavity resonance of...
Isolated Heterostructured High-Q Cavity is now centered to the CCW spectral feature. With these tools, the release and capture of an optical cavity mode is now outlined.

**Timing Design**

To perform release of the cavity mode into the CCW chain, one cavity barrier is index switched (opened) by optically induced carrier injection. Now matching the chain region, the cavity mode is allowed to couple to the CCW resonance and propagate from cavity to cavity down the chain. As the energy propagates down the chain, it becomes distributed amongst the coupled cavities and must be collected in a two step process by which the propagating mode is stopped by the outside secondary cavity barrier and allowed to accumulate before the inside secondary cavity barrier is activated, trapping the mode in the secondary heterostructured cavity. The full three step release and capture process is diagrammed here in Figure 7-9. A shows the confined cavity with both barriers activated with an index of 3.3. B shows the opening of the inside cavity by index switching to n=3.4. With the second cavity’s outside barrier closed, C, the energy accumulates and is then trapped by closing the inner barrier of the second cavity in D.
Through the switching protocol described above, transfer of energy from the optical mode in one high-Q cavity, through a chain of 5 coupled cavities, to a second high-Q cavity was achieved in a 2D FDTD scheme with 77% efficiency. Confinement at the first cavity, propagation between cavities and capture in the second cavity is shown in Figure 7-10. Energy density (top) and field evolution (bottom) are shown to illustrate the process.
Figure 7-10: On-Chip Single Photon Transfer
CHAPTER 8: CONCLUSION

In this chapter, concluding remarks regarding the prospect of a quantum technology based on a photonic crystal chip hosting a quantum network made of quantum dot spins interacting via single photons are considered. To this end, a summary of the main results is presented; then several ways to improve this work are discussed.

Summary

First, the dynamics of entanglement between a QD spin qubit and a single photon qubit inside a quantum network work node was investigated with and without decoherence resulting from hyperfine interactions. In the small detuning regime, for all case of the excess electron spin being in a superposition of its eigenstates, 35% entanglement can only be obtained within the first 200 picoseconds. In the large detuning regime, a smaller amount of entanglement is realized, namely, 25%. And, it lasts only within the first 300 picoseconds.

Second, the loading and unloading of the photon from and onto waveguide connecting nodes were considered to end of minimizing the phase error and losses. Using a Q switching pulse $\tau = 4\, ps$ to unload a single photon, the resulting phase error would be no more $\pm 0.022\, rad / s$ or $\pm 0.315^\circ$ for a $90^\circ$ rotation of the linear polarization.
Third, we have demonstrated that in theory using a quantum model that single photons can be transferred efficiently on-chip from a high-Q cavity to another using coupled cavity waveguides if the dynamic coupling between nearest neighbor cavities is carefully engineered. As far as single-photon trapping and releasing mechanisms are concerned, dynamically switching coupling coefficients between “end” cavities and the waveguide yields much higher single photon transfer efficiencies (93% versus 75%) than dynamically switching the resonant frequencies of the “barrier” cavities.

Fourth, a feasible physical implementation using a photonic crystal backbone was proposed. The single-photon trapping and releasing mechanism based on dynamically switching the resonant frequencies of the “barrier” cavities was able to be implemented in a practical way within our classical FDTD model. Until then, though, the two models shown in this work yield strikingly similar single-photon transfer efficiencies (~75%), which reinforces the versatility of CCWs and the advantage of using such structures to implement photonic quantum networks.

**Future Work**

In the future, further improvement can be made. First of all, the capability of Stokes high-performance computer cluster must be improved if quantum network with more nodes are to be considered. For example, the entire time-dependent Hamiltonian in Liouville space for a system consisting of a source and target cavity linked by means of a 10 coupled cavities waveguide is 3-dimension matrix of size $112^2 \times 112^2 \times 400$, 

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whose individual element is a complex double number. This single variable requires approximately 1 TB or 1000GB of memory. Given a minimum 16 GB per nodes (only few are 32GB) and only 16 workers with 1 worker per nodes, we have in theory 256GB available in the cluster. To solve this problem, we would need at least 64 workers, each operating on one node with at least 16 GB. For our current problem, a system consisting of a source cavity and a target cavity linked via a 4 coupled cavities waveguide results in a Hamiltonian matrix of size $64^2 \times 64^2 \times 100$ requiring approximately 216 GB of memory on the cluster.

Second, regarding the heavy-hole light-hole band splitting for electrons in semiconductors, it is well known from Kane’s $k \cdot p$ theory [103] that the band with total spin $J=3/2$ is split into a heavy-hole band with energy $E_{hh} = \hbar^2 k^2 / 2m_{hh}$ and a light-hole band with energy $E_{lh} = \hbar^2 k^2 / 2m_{lh}$. The heavy-hole and light-hole masses are given by

$$m_{hh} = m / (\gamma_1 - 2\gamma_2) \quad \text{and} \quad m_{lh} = m / (\gamma_1 + 2\gamma_2),$$

respectively, where $\gamma_1$ and $\gamma_1$ are the Luttinger parameters[103-105]. In three dimensions the two bands are degenerate at the $\Gamma$ point, where $k=0$. In two dimensions the confinement and the strain lead to a $k$-independent splitting $\Delta_{2D} / \hbar = \omega_{hh} - \omega_{lh}$ between the heavy-hole and the light-hole band. It would useful to consider various quantum dot materials and symmetries and see how these affect energy levels of the two-level system in our modified Jaynes-Cummings model as well as the quantum dynamics of the single photon Faraday Effect.

Third, Quantum entanglement, although being one of the most important properties of quantum system, it is also the one of the most fragile and elusive. As
discussed earlier, decoherence resulting from hyperfine interactions can greatly affect
the entanglement between the QD spin qubits and the single photon qubit; therefore, it
must be protected. Approaches vary from considering different material systems to
developing coherence protection schemes. First, performances could be increased by
considering a different material system. Indium atoms have $\frac{19}{2}$ spin and arsenic atom
have $\frac{13}{2}$ spin; which make InAs quantum dots bad candidate as far as decoherence is
concern. Other semiconductors with lower or without nucleus spin could be used. The
dephasing times of the II-VI compounds are 3-10 times larger than dephasing times for
III-V compounds[71], however, for wurzite-type semiconductors to have similar optical
selection rules as in the case of zinc-blende-type semiconductors, propagation along
the c axis is needed[72]. Second, improvements in performances could also be
obtained performing manipulations on the nuclear system[73], or using hole spin in the
valence band instead of the electron spin in the conduction band as storage qubit since
it is less influenced by the nucleus spin. Third, Press et al recently introduced an all-
optical spin echo technique that could in theory increase the decoherence time of a
single quantum dot electron spin from nanoseconds to several microseconds[74]. In
order for this concept to work in our scheme, the magnetic field would have to be
applied in the faraday geometry instead of the Voigt geometry causing the spin to
precess about the quantization axis of the spin which also happens to coincide with the
direction of propagation of the flying qubit. In turn, this requires the rotating pulses,
which are used to coherently rotate the spin so as to prevent decoherence, to be
propagating in a direction perpendicular to this quantization axis.
Fourth, the design of the physical structure can be greatly improved too. Cavities with larger quality factor are desirable as well as a larger dynamics range in switching the Q factor. It can be readily seem that finer linewidth reduce the range by which refractive index needs to be switched in order to trap or release the photon. The endeavor to realize this system in a realistic form of a 2D finite photonic crystal slab (PCS) presents two additional hurdles: increased numerical cost, and out of plane loss mechanisms. It is expected that the increased numerical cost may be greatly alleviated by using 2D approximation methods for TE (even) polarization. However, due to the low group velocity of CCW modes and the zero group velocity of the high-Q state, careful engineering of the Fourier components of the field distributions in this system will be necessary to avoid coupling to out of plane radiative modes. Last but not least, an important improvement is achieving a complete bandgap in 2D slab photonic crystal structure. Although many approaches are discussed in the literature, the reality is that much more is needed to open the TM bandgap wider. This is critical for the realization of a functional quantum network[106].
LIST OF REFERENCES


