Dynamic Response of a Multi-Span Curved Beam From Moving Transverse Point Loads

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DYNAMIC RESPONSE OF A MULTI-SPAN CURVED BEAM FROM MOVING TRANSVERSE POINT LOADS

by

AMANDA D. ALEXANDER

A thesis submitted in partial fulfillment of the requirements for the Honors in the Major Program in Mechanical Engineering in the College of Engineering and Computer Science and in The Burnett Honors College at the University of Central Florida Orlando, Florida

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Thesis Chair: Dr. Jeffrey L. Kauffman
Abstract

This thesis describes how to evaluate a first-order approximation of the vibration induced on a beam that is vertically curved and experiences a moving load of non-constant velocity. The curved beam is applicable in the example of a roller coaster. The present research in the field does not consider a curved beam nor can similar research be applied to such a beam. The complexity of the vibration of a curved beam lies primarily in the description of the variable magnitude of the moving load applied. Furthermore, this motion is also variable. This thesis will present how this beam will displace in response to the moving load. The model presented can be easily manipulated as it considers most variables to be functions of time or space. The model will be compared to existing research on linear beams to ensure the unique response of a curved beam.
Dedications

To Maci, for always encouraging me to do crazy things.
Acknowledgements

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Chapter 1 – Introduction and Background

The present thesis discusses the vibration of curved beams within the very popular field of structural dynamics. The first section of this chapter introduces the general content about the topic. Structural dynamics and its research and industry applications is discussed in the next section entitled Background. An outline is provided to describe the organization of the rest of the thesis after the introduction and background are summarized.

Section 1.1 – Introduction
This thesis discusses and analyzes the vibration of a curved beam traversed by a moving load. This research is applicable to the track of a roller coaster. The tracks of a roller coaster guide a car along the track at high, varying speeds. This track includes features such as tall hills, steep banks, corkscrews, and 360° loops. Beyond roller coasters, the curved beam is not frequently seen in mechanical applications. Consequently, limited research has been conducted in this rather specific field.

This research study could be described as an analysis of the structural dynamics of the curved beam. Structural dynamics is a large field of study that encompasses many classical techniques seen in physics. A structural analysis answers questions to designers about the design limits of the structure. This analysis is essential throughout the process of design as well as
afterwards to ensure that the structure can withstand the desired conditions within which it is
designed to perform.

A first-order approximation of the vibration of this curved beam is made. This first-order
approximation can affect the decisions made in designing a track based on the optimum
collection of variables to be inputted into the model. Many variables on the design of a beam are
discussed throughout this thesis in terms of previous research completed in the topic as well as
in terms of the research presented here. The manipulation of these variables will allow a designer
to experiment with the resources and constraints of the project to find the optimum design that
will bear the heaviest loads for the longest time.

The research in the field of structural dynamics is vast. Research has been conducted to
study the vibration of several different types of beams under a large variety of load and support
conditions. These studies vary in their degree of complexity and scope of the research. A review
of the state of the art is provided to present the techniques already used in the field.
Furthermore, this review confirms the sparse amount of research on curved beams, as well as
how aspects of these studies may or may not be applied to curved beams.

The techniques found in the literature of the field are tailored to be used on the unique
conditions of a roller coaster track. A first-order approximation is made to model the actual
dynamic response of a curved beam. The opportunities of advancement in this topic are
immense. The possibility of future research is discussed openly along with the applicability of the present research in the conclusions of the thesis.

Section 1.2 – Background
An analysis on the structural dynamics of a system encompasses the determination of forces and stresses within the structure as well as its deformations and deflections. Designers can create a component or structure considering these important effects to improve the capacity or lifespan of the component or structure. The design considerations that may affect the capacity or lifespan of a structure are the geometry of the structure, the location of external loads, material selection, and placement and type of support constraints.

Beyond forces and stresses, deformations and deflections of a structure under imposed conditions are important aspects of a structural analysis. The deformation of a structure may be in the form of an elastic or plastic deformation. Plastic, meaning permanent or irreversible, deformation is clearly something to be avoided in the design of a structure. The geometry and material properties play a large role on how a structure may deform instantly under a critical load or over time. Deflections can primarily be avoided in the addition of constraints. Different types of constraints differently affect the number and allocation of the degrees of freedom at a point of constraint.
One of the most popular topics in structural dynamics is the deflection of a beam under a load. The topic had been widely studied long before and since Frýba’s thorough work in the field of vibrating structures from 1972 [1]. Frýba’s work comprehensively synthesizes all of the prior research since the beginning of the nineteenth century on how a beam reacts to a moving load. Such research in structural dynamics began to increase greatly after the introduction of railway bridges in the early 19th century. Transportation structures have been required to withstand more substantial loads moving at faster speeds. Moreover, these structures are being optimized to save money, space and material which consequently make them more lightweight and slender than ever before. Therefore, the structures under the moving load are experiencing more stress as a result of these conditions.

**Section 1.2.1 – Assumptions of Beams**

Frýba’s work was succeeded by a wide array of articles finding the vibration of different types of beams under different types of conditions [2-11]. A primary assumption one must make is how a beam reacts to the imposed forces. A beam may have linear deflection, rotary deflection or shear deflection. Another difference between many researchers is the assumption of the point load function, though there is little variation in the types of loads applied. Point loads are widely assumed rather than other types of loads, such as applied moments, torsion or distributed loads. Point loads are applicable to the previously mentioned railroad tracks and bridges because they
are traversed by vehicles with wheels. Each wheel provides only one contact point between the vehicle and the surface of the beam.

Some differences between a researchers’ assumptions lie in the variation of the point load in time or space. The options include the assumptions of a point load varying with time, point load varying with location, or a non-varying point load. Another important consideration in analyzing the vibration of a beam is the number and types of support constraints as well as the definition of boundary conditions. Furthermore, the definition of the constraints and boundary conditions are important considerations. The choice of the type of constraint and its location is entirely based on the needs of the application of the structure.

The prescribed motion of the load traversing the beam is also a very important aspect in the structural analysis of a beam. The decision here lies in how it is described. Those who study the vibration of a beam in the context of the railroad track or bridge usually assume constant velocity. Though constant speed is a largely simplifying condition in structural analysis, constant speed is not an entirely realistic assumption, especially in the case of roller coaster.

Section 1.2.2 – The Curved Beam
This thesis presents a method to analyze the vibration of a curved beam. Very little has been published on the topic of curved beams, likely as a result of the lack of applications in the industry. The research pursued and discussed in this thesis provides a first-order approximation
for the vibration that a time-varying point load inflicts on a curved beam, as applied to a roller coaster track.

The field of structural dynamics and, more specifically, the topic of the vibration of beams have a large collection of features to meet many applications. This research expands upon the prior research to model the vibration of a curved beam specifically. A curved beam differs significantly from the examples previously described. Initially, a roller coaster only moves at constant velocity for extremely short periods of time. A roller coaster car will experience acceleration from gravity and centripetal forces, at the least. Therefore, the velocity profile along the track is very unique and relies strongly on its vertical and radial orientation in space. In addition, the load that is inflicted on the track by the car is constantly varying throughout its travel around a curved loop.

The roller coaster track still resembles a railroad track and bridge in several ways. The load applied to the beam can be considered a point load. A wheel of a roller coaster car has one contact point with the surface of the track as seen similarly with any contact between a vehicle’s wheels and track surface. The assumption of a multi-span beam will also be imposed. A roller coaster track can run for hundreds or thousands of feet and must support dips, banks, loops, and more. As result, the track must be supported at each end and throughout the structure to withstand fast-moving, heavy vehicles.
Section 1.3 - Summary

To review, the field of structural dynamics is concerned with how a moving force affects the behavior of a structure. The forces can vary from earthquakes to ocean waves to foot or vehicle traffic. Structures that experience dynamic loading include highway bridges, railway bridges, vehicle frames, or underwater structures. An analysis in structural dynamics encompasses many techniques commonly used in physics and engineering, including energy conservation, static equilibrium, dynamic motion, modal analysis, and differential equations, among others. The field of structural dynamics contains many useful applications. As a civil engineer, one may use structural dynamics to study the forces and stresses that in trusses that support a bridge. As a mechanical engineer, one may use techniques in structural dynamics to design the frame of a car or track of a roller coaster. Designers need to analyze structures with structural dynamics to design for strength and durability. The choice of material, geometry, location of loads, or location of constraints can greatly affect the desired strength and durability of the structure. Researchers and designers know their desired conditions and desired application so that they can proceed to analyze the structure.

The following research examines the dynamic response of a curved beam traversed by variable, moving loads. This type of dynamic analysis has been almost exclusively studied from the point of view of bridges and railroad tracks as vehicles move at high speeds across these structures. However, roller coasters also experience these conditions but yield a dynamic
response exclusive to this application. Roller coasters provide a unique platform to study considering the features that thrill-seekers experience such as tall hills, 360-degree loops, sharp banks and turns, and corkscrews. These features induce varying centripetal loads on the riders by design. Therefore, these structures must withstand transverse loads with variable magnitude and direction.

**Section 1.4 – Outline**

Following the above discussion of the background behind the chosen research topic, the literature review and research will be presented. The state of the art review discusses the previous works that serve as a platform for the research to follow. This review can be found in Chapter 2. Next, the problem to be analyzed will be discussed in Chapter 3. A more in-depth analysis on how the problem was approached and defined is provided in Chapter 4. The model created is described in Chapter 5 along with results of a sample roller coaster track. To conclude, Chapter 6 offers a summary of the thesis, a conclusion on what was found and a discussion on the prospect of future research.
Chapter 2 – State of the Art Review

As described in the previous chapter, the vibration of structures is widely studied throughout the field of structural dynamics. A review of the relevant published works on the topic will be provided, though it is far from a comprehensive review of the field. The results and outcomes of the referenced sources led to decisions made on the assumptions to be adopted for the present research. These decisions will be discussed throughout this chapter in reference to the sources provided.

Section 2.1 – Beam Theory
Within the field of structural dynamics, the assumption of beam theory is the first concern to a researcher. To review, beam theories are differentiated between their assumptions of how a beam may deflect in response to the inflicted load. A beam may deflect linearly, for instance. In this case, linear deflection occurs up or down in a plane normal to the central axis of the beam. Beams may also deflect in shear, rather than the pure bending assumed here. Furthermore, a dynamics model of the beam under load can incorporate terms related to rotary inertia.

Beam theories are selected based on the application of the beam at hand and its expected deflection. The deflection of a Euler-Bernoulli beam occurs within one perpendicular plane relative to the neutral axis. A shear beam is one that assumes deflection in shear; i.e., the plane initially perpendicular to the neutral axis is no longer perpendicular when loaded. A Timoshenko
beam includes shear deformation and also considers the beam’s rotary inertia. A Rayleigh beam also considers rotary inertia but not shear deflection. A simple organizational model on beam theories is given in Table 1. Each beam theory is classified by its consideration of shear deflection and rotary inertia in its deflection from a load.

<table>
<thead>
<tr>
<th>Shear deflection</th>
<th>No shear deflection</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Rotary inertia</strong></td>
<td>Timoshenko</td>
</tr>
<tr>
<td><strong>No rotary inertia</strong></td>
<td>Shear</td>
</tr>
</tbody>
</table>

H. P. Lee finds the inertial effects of an accelerating mass on a Timoshenko beam [2]. The Timoshenko beam theory assumes all deflection, including linear, shear and inertial. Lee differentiates between the Euler-Bernoulli beam theory and Timoshenko beam theory in terms of the slenderness ratio. The difference of this ratio allows for the separation between the car and track to be studied on a Timoshenko beam, but the Euler-Bernoulli beam does not reflect such an occurrence. The analysis derived is extensive because the assumption of all types of
deflection is the most complicated of the assumptions on beam deflection. The author analyzed a one-span beam simply supported at its ends. The analysis of a Timoshenko beam with many spans would be even more intricate.

Oni and Omolofe approach a specific type of beam with the assumptions of rotary displacement [3]. The Rayleigh beam theory is implemented on a beam that is said to be prestressed on an elastic foundation. A prestressed structure has internal, permanent stresses that improve the function of the structure. Many bridges, buildings and underground structures are put through prestressing techniques such as pre-compression, pre-tensioning and post-tensioning. An elastic foundation is described to react with proportional reaction forces upon applied external loads. An elastic foundation, also known as a Winkler foundation, is modeled by springs under the beam usually of high stiffness. These types of conditions are beyond the scope of this research project. However, these are amongst the many conditions that are imposed on beams in practice.

Oni and Omolofe assert with citation support that if the load applied to the beam is much smaller in scale than the track, then it is safe to assume no inertial effect of the mass [3]. In their research, the inertial effects of the mass that cause rotary displacement is relevant in the actual deflection the beam may experience. The condition of the elastic foundation justifies why they chose to analyze the rotary deflection of the beam. In addition, the need to analyze a shear
deflection of the beam is hinged on the conditions imposed. A short single span beam simply supported at its ends is a typical case in which to analyze the shear deflection of the beam [3].

Furthermore, Michaltsos et al. specifically compare the response of a beam considering the mass of the load to the response of the beam when the mass of the load is not considered [4]. The article releases data of the static and dynamic displacement of the beam when the mass of the moving load is considered and when it is ignored. These results were following the same pattern but show the most inconsistency occurred with increasing velocity. The authors proved relatively consistent results between the assumption of inertial effects of the mass and the neglect of those effects.

An analysis of the dynamic response of a beam considering rotary inertia and/or shear deflection is extensive. Most researchers assume Euler-Bernoulli beam theory as a result of its simplicity and broad applicability [5 – 11]. Other researchers studying the dynamic response of a beam may be interested in finding a more accurately representative model of how a beam reacts to a load by using the Timoshenko beam theory [2, 5]. Furthermore, other researchers may be studying a beam that is under certain conditions that may cause a beam’s deflection to heavily be influenced by inertial effects of the mass [3, 4]. The present research will perform a first-order approximation on a long beam that will be supported by intermediate constraints throughout the beam. In addition, the weight of the track is comparable to the forces applied.
With the assumptions in mind, the Euler-Bernoulli beam theory is the most appropriate amongst the options.

Section 2.2 – Boundary Conditions and Intermediate Point Constraints

One of the most important pieces of information to know when performing an analysis on a structure is the imposed boundary conditions. They are often necessary to be able to begin an analysis. The purpose of a beam is to support a load. The beam itself must also be supported to perform its function. Different beams in practice will have a variety of boundary conditions that all impose different physical conditions on that beam. Intermediate point constraints are also important in many beams used in the field of transportation. Techniques used throughout the existing studies in railroad tracks or bridges that implement similar boundary conditions can be relevant to the present study on roller coaster tracks. Diversity is found between existing studies in their types of point constraints. Different types of constraints perform different functions and consequently have a unique set of support reactions.

Lee et al. defined a variety of boundary conditions to be used throughout the analysis of the beam’s vibration [5]. The authors study the effects of a load on a beam with the boundary conditions of hinged-hinged, hinged-clamped and clamped-clamped. A hinged end has zero vertical deformation and zero twist angle and bending moment. A clamped end is defined to have
zero vertical deformation, twist angle and rotation angle due to pure bending. Combinations of these boundary conditions were studied along with the variation of curvature in the beam.

Lee et al. are among few to study the dynamic response of a curved beam [5]. The types of curved beams studied are parabolic beams, sinusoidal beams and elliptic beams. Each beam shape was determined by a non-dimensional general equation to be altered by the span of the curve and height as well as characteristic constants that can alter the curvature of the beam. Differential equations were defined for the vibration of these beams, which consider rotary inertia, torsional inertia and shear deformation. Differential equations defining the vibration were able to be non-dimensionalized.

Gutierrez and Laura determine an approximate dynamic response of a beam with constant width and parabolically-varying thickness [6]. Boundary conditions are specifically set with a combination of simply-supported and clamped scenarios. The authors define boundary conditions well. The definition of the clamped boundary conditions assuming no translational or rotational motion at the supports is very useful. However, the authors only analyze one span. The application of simply supported or clamped boundary conditions with a parabolically-varying thickness is a short bridge. A parabolically-varying thickness of a beam makes the beam haunched. With these conditions, a single span beam is appropriate because that is all that is needed with the support provided by the enforced thickness and two boundary conditions.
Zheng, et al. tackle the classic structural dynamics problem using modified beam vibration functions, or Hamilton’s method [7]. The authors assert that prior research performed by Lee in 1994 [8] was done with error using the assumed modes method. The example at hand is a multi-span beam undergoing forces from moving loads. The assumed mode method does not account for zero deflection at the boundary conditions. Therefore, Lee modelled the end constraints as highly stiff springs. Zheng et al. were not satisfied by this assumption so they utilized Hamilton’s method that already assumes zero deflection at all supports including at the ends, then compared results. The method employed can account for multiple moving loads with variable velocity. The comparison between the assumed mode method, used by Lee, and Hamilton’s principle, used by Zheng et al., proved consistent patterns within the data. Zheng et al. assert that the implementation of the programming is easy and results are accurate. The description of the formulation is easy to follow and similar to formulation that will be explained in this thesis.

The use of a single span beam is inappropriate for a roller coaster track. Therefore, the implementation of intermediate point constraints is necessary for the present application. Zheng et al. define boundary conditions and intermediate point constraints particularly well while also explaining error in other methods [7].
Section 2.3 – Motion and Force

Another variable amongst the research performed in the topic of the vibration of beams is the motion and magnitude of the load causing the vibration. Constant velocity is a popular assumption amongst researchers for describing the motion to traverse a railroad track or bridge [4, 6, 9, 10]. This approach is more applicable to the function of a bridge. However, railroad tracks and, certainly, roller coaster tracks may experience loads with variable velocity. Some researchers acknowledge the possibility for a load traversing a beam to have variable velocity [2, 3, 7 – 10]. Conveniently, the velocity is left as a function of time throughout the analysis of the beam. Therefore, proper manipulation of the velocity according to the application of the beam is allowed.

Roller coaster tracks undoubtedly experience changes in velocity throughout the entire ride. The roller coaster cars travel up hills, down hills, around loops, across banked turns, etc. Acceleration and deceleration is not only necessary to allow for the car to complete the feature of the track with enough velocity but also to create the g-forces that roller coaster riders are attracted to. In addition to the enforced velocity changes of the car, the car also experiences external forces such as gravity and centripetal forces which both contribute to the forces felt by the car and the riders.
Some researchers derive analyses on the vibration of beams considering constant magnitude [2, 4, 7, 9]. Again, this assumption may be applicable to the common structural dynamics examples of the train or car on a railroad track or bridge, respectively. Other articles present analyses that allow for the variation of the magnitude of the force traversing the beam [3, 6, 8–11]. Similarly to the function for velocity, a few researchers allow this variable to be left as a function of time throughout the analysis. However, the function for the force in terms of time is not intuitive and never explained, for example in the article by Gutierrez and Laura [6]. Later in this thesis, an explanation for the derivation of the magnitude of the force in terms of time and space will be explained. This function will be determined by the geometry of the curve and how the car will experience forces such as gravity and centripetal forces throughout the curve.

Section 2.4 – Other Assumptions

Other assumptions discussed throughout the literature include the use of a point load condition and the assumption of the cross-section of the beam. A load may be applied to a structure in one of several ways. A structure may experience a point load, uniform load, varying distributed load, or coupled moment. A vehicle to traverse a beam can be modeled as a point load because the wheels are in tangent contact with the surface of the beam. A common way to mathematically represent a point load is the Dirac delta function in Equation (1), where \( P_0 \) is the magnitude of the force and \( \delta \) is the Dirac delta [3, 4, 6, 8–11].
Several authors found a need to allow for variable cross-sectional area of a beam [6, 8, 9]. These techniques were used, for example, to admit the haunched shape of a beam as seen in the studies performed by Gutierrez and Laura and Dugush and Eisenberger [6, 9]. As discussed earlier, this haunched shape or variable cross-sectional area is seen in practice for relatively short bridges to provide more stability without the use of more intermediate constraints. The cross-sectional area functions are worth noting for other applications and can be assumed to be constant when necessary.

Section 2.5 – Unique Perspectives

Though the following topics may not be applicable to the research to be explained throughout this thesis, research has been performed in recent years that examine unique conditions and techniques to this classic topic. Also, these works might contribute to future work in the example of a roller coaster. Zarfam et al. contributed much to the field by modeling the vibration of a beam in response to horizontal support excitation [10]. In other words, the article showcases how a track or bridge reacts to seismic forces. The critical velocity of the vehicle was found in simple terms of material properties and beam composition. Several 3D response spectra were given to pictorially display the displacement as a function of mass staying time and beam frequency.
Another contribution from Zarfam et al. is their definition of “mass staying time” [10]. The notion of mass staying time is unique to the field because most authors assume one point load. The critical mass staying time was found to be a simple function of the ratio between the mass of the car to the track and the natural frequency of the beam. The authors here investigated how many masses in succession and the comparison of the mass of the car to the mass of the track is very applicable to roller coasters as well as their presumed application of bridges. Roller coaster cars have many rows of wheels because the cars can extend very long to hold more passengers. In addition, multiple cars run at designated intervals all throughout one day. This idea of “mass staying time” may provide the opportunity to model how the vibration of the track after a car passes compounds on the vibration after the car before it passed. In addition, a necessary time interval could be defined between cars to avoid this compounding vibration. This idea could lead to many applications to study further.

A unique procedure was derived within an article by Wang et al. The article presents a method of determining how a car and its properties affect the vibration of the beam it traverses [11]. The authors model the car as an elastic beam suspended on two springs. This form allows the vibration of the car to be studied as well. The primary purpose of the article is to study the vibration of the car. In the introduction, the authors point out the demand for the design of lightweight cars and the consequences of a decreased structural stiffness. The present research
could be continued along the path that this article took by then determining the vibration of the roller coaster car. This determination would allow for the possible optimization of the car to withstand vibration while being as lightweight, durable and inexpensive as possible.

Section 2.6 – Summary
In conclusion, the literature review on the topic of the dynamic response of a beam with moving variable transverse point loads provided instruction on what to do, instruction on what not to do and inspiration for current and future research. A challenge arose finding current literature with similar constraints as the roller coaster problem considered here, although the vibration of a beam induced by a moving load is a classic problem in structural dynamics. However, there are several key differences in the assumptions between the classic problems of bridges and railroad tracks to the unique problem of a roller coaster track.

The primary pitfall in the literature was the absence of variable velocity traversing the beams. Though constant velocity may be applicable in the classical approach to the problem, these assumptions do not apply to the specific problem addressed here. Another concern was the lack of much investigation into the curved beam. Only Lee et al. addressed the vibration of a beam with curvature [6].

Therefore, assumptions will be made to explore the dynamic response of a curved beam traversed by variable moving loads. Based on the existing literature and a desire for a model with
first-order approximations, the Euler-Bernoulli beam theory will be used. Boundary conditions will be made assuming a multi-span beam with pins at the ends and intermediate supports. A point load of variable magnitude will be defined using the Dirac delta function. The point load will be moving at a prescribed variable velocity at every point in time and space. Methods will be used to apply all of these assumptions to a curved beam.
Chapter 3 – Problem Definition

Section 3.1 – Technical Area
The technical area under study throughout this thesis is structural dynamics. Engineers are interested in structural dynamics to determine a structure’s integrity for withstanding loads over time. A structural analysis is performed on a variety of structures throughout its design before production. An engineer may perform such an analysis on a bridge, automobile frame, or even prostheses within a body.

An understanding in many different fields leads to the analysis of a structure. Throughout design and testing, one must be familiar in material science, static conditions, dynamic conditions, and the basics of physics. A structural analysis will yield information valuable to an engineer such as the stresses within the structure upon certain loads. These loads in turn may cause deformations of the structure. Finally, a fatigue analysis of these factors may be necessary to understand how these stresses and deformations will wear on the structure over time with constant or variable forces applied.

Section 3.2 – General Problem
The general problem studied in this thesis is the dynamic response of a beam caused by moving loads. The vibration occurring as a result of a load applied to a beam is a heavily researched topic within the larger field of structural dynamics. Ever since, the travel industry has
only expanded. The amount of automotive vehicles on the roads and the weight of the cargo to be transported by trains have increased. In addition, such transportation routes are covering the entire world. Therefore, the transportation routes need to ensure safety over land and water. The structural analysis is important in ensuring this safety.

Section 3.3 – Specific Problem
The specific problem to be analyzed is the dynamic response of a curved beam by moving loads. This curved beam is seen in the application of roller coasters. As a result of the limited application of the curved beam, the topic is largely unstudied. In addition, the curved beam in the case of a roller coaster considers several complicated conditions.

A roller coaster car will experience several different forces throughout its travel around the loop of a roller coaster. Both gravitation and centripetal forces are felt by a rider in different magnitudes as their location, speed and orientation change along the track changes. Furthermore, a track must be well supported as it curves into this loop. Therefore, intermediate point constraints must be considered in the analysis.

Section 3.5 – Expected Contributions
This model will be a first-order approximation for how a loop in a roller coaster will respond to a fast-moving roller coaster car traversing it. The major contribution of understanding this model is the ease of implementation. There is very little material in the field on curved beams
but these available approaches are extensive. Furthermore, the model will also contribute
inspiration to extend research in the interesting application of a curved beam.

Section 3.6 – Novelty and Significance

The state of the art review and further inspection into the field proves a slim amount
research on the curved beam. The little research implemented on the complicated case of a
curved beam is still rather complicated. The analysis is often hard to follow and comprehensive.
A first-order approximation of the dynamic response of a curved beam from a moving load is not
available within the field.

As mentioned before, the curved beam is a complicated case in the field of structural
dynamics. A vertically curved beam on a roller coaster will route a car to travel upside-down at
high speeds. The primary difficulty in modeling the vibration of the beam in response to a load is
modeling the load itself. The load experienced by a linear beam is simple because it is consistent
and of constant magnitude. However, a roller coaster car travelling along a curved beam will
experience force from its own power in addition to gravity and centripetal forces. Therefore, the
motion of the car must be determined to model the varying magnitude.

The model will be useful in its ease to manipulate. The governing equation assumes
almost every variable to be a function. Therefore, the model can be used to determine the
vibration of a curved beam in addition to other types of beams with variable motion of the loads.
The cross-sectional area and material properties of the beam can be determined based on the user’s application. Furthermore, the approach to determining the magnitude of the load will be discussed in depth. Yet, a constant magnitude load can also be assumed.
Chapter 4 – Approach

The present research will be implemented through an analytical study driven by the analytical studies in the existing literature. The literature is unanimous in the governing equation of the deflection of a Euler-Bernoulli beam:

\[ pA\ddot{w} + EIw''' = P(x, t) \]  \hspace{1cm} (2)

where \( p \) is the density of the material of the track, \( A \) is the cross-sectional area, \( E \) is the Young’s modulus of the material, \( I \) is the moment of inertia, \( w \) is the transverse deflection, and \( P(x, t) \) is the distributed transverse load as a function of location and time. The track of a roller coaster is tubular, so the cross-sectional area and moment of inertia will depend on inner and outer diameters. As mentioned earlier, a transverse point load is applied and defined in terms of location and time using the Dirac delta function:

\[ P(x, t) = P_0(t)\delta(x - x_c(t)) \]  \hspace{1cm} (1)

where \( P_0(t) \) is the point load as a function of time and \( x_c \) is the point load’s location. A representative track will be made to define the track position and orientation in space. Velocity will also be defined along the track considering the potential and kinetic energy at each point in time and space. Finally, the point load will vary based on the centripetal acceleration and
gravitational force at each point along the track. Therefore, the location, velocity and force of the car on the track will all be defined along the representative track.

Assuming synchronous motion of the beam, \( w(x, t) \) can be separated into the product of two functions that are each in terms of only one of the independent variables. Looking ahead to a modal analysis of the beam, the deflection can be written as:

\[
    w(x, t) = \sum_{n=1}^{\infty} W_n(x) T_n(t) \text{ for } n = 1, 2 \ldots
\]

where \( W_n(x) \) is the \( n \)th mode shape in terms of space and \( T_n(t) \) is the modal amplitude in time. As a solution to the eigenvalue problem, these mode shapes are consequently orthogonal and can be scaled to be orthonormal with respect to mass. Therefore, the deflection function can be evaluated again within the original partial differential equation, Equation (2), to find:

\[
    T_m'' + \omega_m^2 T_m = \int_0^L P(x, t) W_m(x) \, dx \text{ for } m = 1, 2 \ldots
\]

where \( \omega_m \) is the natural frequency and right hand side is now defined as the modal force. The Dirac delta function, Equation (1), describes the point force so now the modal force is:

\[
    \int_0^L P(x, t) W_m(x) \, dx = P_o(t) \int_0^L \delta(x-x_c(t)) W_m(x) \, dx \text{ for } m = 1, 2 \ldots
\]
and owing to the properties of the Dirac delta function, Equation (4) becomes:

$$\ddot{T}_m + \omega_m^2 T_m = P_o(t)W_m(x_c(t)) \text{ for } m = 1,2 \ldots$$  \hspace{1cm} (6)

Therefore, Equation (6) can be used to describe the frequency and modal amplitude of the beam at every point in time and space. Numerical solutions will be obtained after submitting a representative geometry and the material properties of the beam gathered from research into the practices used in the field.

**Section 4.1 – Representative Track**

The representative track of the roller coaster is defined in a piecewise manner. The pieces of the track, given in Figure 1, include a hill, dip, incline, inversion, decline, and straight. The hill is approached by the roller coaster car with zero velocity at the start of the hill. The hill declines at a slope of $\theta$. In order to ensure continuity, the dip follows after the hill for an arc angle of $\theta$, as well. The incline then sends the roller coaster car along the track from $-90^\circ$ to $0^\circ$ at a radius of $r_1$. The car is then inverted for an arc angle of $180^\circ$ at a radius of $r_2$. The decline is a reflection of the incline; sending the car down the track from $180^\circ$ to $270^\circ$ at a radius of $r_1$. The straight piece of the track continues from the end of decline. This track follows the clothoid configuration which is defined by different radii used in a single loop.
With these parameters, the location of the car along the track will be defined for all points in time. In other words, the arc length of the entire track will be found. This approach is advantageous because it will appear as an unwrapped linear beam, the car’s location can now be defined as a function and the velocity along the unwrapped track is always tangential velocity.

![Figure 1: Roller Coaster Track Profile](image)

Only gravitational and centripetal acceleration is considered, so according to the law of energy conservation, the hill must be higher than the top of the inversion piece of the track. (Forces applied by the roller coaster itself will be discussed in the further research section.) Another observation of the track profile is the radius of the curves. The inversion curve must be about half the radius of the incline/decline because the roller coaster car needs to whip around
this curve. The car will not complete the curve if it is too long and/or approached at too slow of a speed.

Figure 2: Tubular Roller Coaster Track

The track’s cross-sectional area will be assumed to be tubular, as shown in Figure 2. The material assumed to be used on the track is made of A618 structural steel. The properties of this beam are described in Table 2.
Table 2: Beam Properties

<table>
<thead>
<tr>
<th>Geometric Properties</th>
<th>Material Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cross-sectional Area, $A = \pi (r_o^2 - r_i^2)$</td>
<td>$\frac{0.196}{in^2}$</td>
</tr>
<tr>
<td>Moment of Inertia, $I = \frac{1}{4}\pi (r_o^4 - r_i^4)$</td>
<td>$39.845 \text{ in}^4$</td>
</tr>
</tbody>
</table>

Section 4.2 – Velocity Profile

The velocity profile will be determined for the curve for every point in time and space. Assuming no friction, drag or other losses, the tradeoff between kinetic and potential energy can be used by the conservation of energy, Equation (7). Therefore, the roller coaster car’s velocity at every point in time can be determined easily with simply the knowledge of the initial velocity and height above some common reference datum. That is, the sum of kinetic energy and potential energy is assumed constant:

$$\frac{1}{2}mv_i + mgh_i = \frac{1}{2}mv_f + mgh_f \quad (7)$$
Section 4.3 – Point Load

The velocity profile along the curved track allows for the calculation of the centripetal force at every point in time. Therefore, the load of roller coaster car on the track is described by Equation (8) in terms of its mass $m$, function of tangential velocity $v_t$ in time, radius $r$ and orientation $\phi$.

$$P_0(t) = F_g + F_c(t) = mg \sin \phi + \frac{v_t(t)^2}{r}$$  \hspace{1cm} (8)

Section 4.4 – Alternate Track Profile

Before the track mentioned above was chosen, a cycloid was contemplated. The cycloid is a commonly used configuration for the loop of a roller coaster. A cycloid shape is given by the shape of a line traced by following the movement of a point on a circle that rolls along a flat surface. In other words, a pen, held on one vertex of circle, will trace a cycloid curve as the circle rolls along a flat surface. This is depicted in Figure 3 [12].

A curve called the prolate cycloid more closely resembles how a roller coaster would route a track. A prolate cycloid is defined as the shape of a line traced by a point at a radius larger than the radius of a circle that rolls on a flat surface. To compare to a cycloid curve, a prolate cycloid
curve is created by a pen at a larger radius than the circle. This track path is depicted in Figure 4 [12].

![Prolate Cycloid Curve](image)

**Figure 4: Prolate Cycloid Curve**

A prolate cycloid is defined by the functions in Equation (9). These functions determine distance, \( x \), and height, \( y \), in terms of time where \( a \) is the ratio between the radius to the pen (greater than the radius of the circle) and the radius of the rolling circle. A roller coaster may follow the track of an inverted prolate cycloid like the one depicted in Figure 5.

\[
\begin{align*}
x(t) &= t + a \sin(t) \\
y(t) &= a \cos(t)
\end{align*}
\]

(9)

![Inverted Prolate Cycloid Curve](image)

**Figure 5: Inverted Prolate Cycloid Curve**
The prolate cycloid provided favorable and unfavorable properties for the ease of calculation of some information needed. First, the governing equation of the cycloid track is parameterized function not a piecewise function, like the clothoid track. This parameterized form creates more difficulty in the evaluation of a tangential velocity. Since all sections of the clothoid curve could be written in terms of the arc length, velocity at any point is, by definition, the tangential velocity. This tangential velocity is related to the radius of the curve to determine centripetal force on the track. This type of curve also presents another difficulty: the radius of curvature of the cycloid track is always changing. Not only is the radius changing, but the center vertex of the curvature is also translating in the x-direction.

One of the only advantages of this type of track profile is the easy determination of the height, $h$, which is, in this case, always defined by the function $y(t)$. This variable is needed in the evaluation for the conservation of energy. For the clothoid curve, the initial and final height of each section is determined manually by geometry instead. Also, the transition between the sections of the clothoid roller coaster loop, described in Section 4.1 – Representative Track are more continuous and gradual in the cycloid loop instead. The evaluation of the cycloid curve proved to be beyond the scope of the project but it will be discussed further in the future work section.
Chapter 5 – Simulation and Evaluation

The approach previously presented will be implemented and results will be provided in this chapter. Also, the two conditions in which the hypothesis must be validated are the uniqueness and accuracy of the model.

Section 5.1 – Numerical Simulation

The analysis performed yields a function representing the track, defined boundary conditions, a function of the force, a modal analysis, and a graph representing deflection of the track at any specified location and time. Due to the curvature of the roller coaster track, the track profile is not a function. Therefore, the location will always be represented in terms of arc length.

Section 5.1.1 – Track Profile

The first step in understanding the deflection of a roller coaster is to know the profile of the track. A function needs to be prescribed for where the roller coaster car is at any point in time. As mentioned before, this will be given in terms of arc length. Therefore, the position of the car in terms of time, \( x_c(t) \), needs to be determined.

A dynamic analysis of the track shown in Figure 1 was treated as a piecewise function. The law of conservation of energy, governed by Equation (7), was used to determine the arc length function in terms of time for each piece.
Each curved piece of the track was found to be described by the function in Equation (10) where \( x_c \) is the position of the car, \( s_{n-1} \) is the initial arc length of the section (or final arc length of the section before), which is a function of time, and \( A, B, C, \) and \( D \) are variables for each section, given in Table 3. An example of the evaluation of the equation of motion is provided in Figure 6.

\[
\frac{1}{2} m (\dot{x}_c^2 - v_3^2) = -mgr_2 \sin (\varphi) \\
\dot{x}_c^2 = v_3^2 - 2gr_2 \sin \left( \frac{x_c}{r_2} - \frac{s_3}{r_2} \right) \\
\dot{x}_c = \sqrt{A + B \sin (C x_c + D)}
\]

\[
\varphi = \frac{x_c - s_3}{r_2} \\
A = \frac{v_3^2}{r_2} \\
B = -2gr_2 \\
C = \frac{1}{r_2} \\
D = -\frac{s_3}{r_2}
\]

Figure 6: Inversion Section Evaluation

\[
\dot{x}_c^2 = A + B \sin (C x_c + D) \quad (10)
\]
Table 3: Variables for Each Curved Track Section

<table>
<thead>
<tr>
<th></th>
<th>Section 2 – Dip</th>
<th>Section 3 – Incline</th>
<th>Section 4 – Inversion</th>
<th>Section 5 – Decline</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$v_1^2 - 2gr_1 \cos(\theta)$</td>
<td>$v_2^2 - 2gr_1$</td>
<td>$v_3^2$</td>
<td>$v_4^2$</td>
</tr>
<tr>
<td>B</td>
<td>$2gr_1$</td>
<td>$2gr_1$</td>
<td>$-2gr_2$</td>
<td>$2gr_1$</td>
</tr>
<tr>
<td>C</td>
<td>$\frac{1}{r_1}$</td>
<td>$-\frac{1}{r_1}$</td>
<td>$\frac{1}{r_2}$</td>
<td>$\frac{1}{r_1}$</td>
</tr>
<tr>
<td>D</td>
<td>$\frac{\pi}{2} - \theta - \frac{s_1}{r_1}$</td>
<td>$\frac{\pi}{2} + \frac{s_2}{r_1}$</td>
<td>$-\frac{s_3}{r_2}$</td>
<td>$-\frac{s_4}{r_1}$</td>
</tr>
</tbody>
</table>

Each $v_n$ and $s_n$ value refers to the final velocity and final arc length of the $n^{th}$ section, respectively. Variables $r_1$ and $r_2$ are the radii of the dip/incline/decline and inversion. For now, Equation (10) is a differential equation. Instead, it is desired to have a single function of arc length versus time, $x_c(t)$. This was accomplished via numerical integration using the following form of Equation (10):

$$\frac{dx_c}{dt} = \sqrt{A + B \sin(C x_c + D)} \quad (11)$$

Section 1 refers to the hill and Section 6 refers to the straight. These sections, being relatively simpler, were computed separately and are given by Equation (12) and Equation (13), respectively.

$$x_c(t) = \sqrt{2gh} t, \text{HILL} \quad (12)$$
\[ x_c(t) = v_5 t, \text{STRAIGHT} \]  \hspace{1cm} (13)

Each section was computed assuming that the time was zero and then shifting the time appropriately to stitch together all of the sections. After numerical integration, each section was superimposed onto one graph, Figure 7. The final arc length of each section, found in the dynamic analysis, was used to find the corresponding time on the graph. Therefore, the arc length of the roller coaster track traveled at time \( t \) is now corresponding to a certain time period and a function \( x_c(t) \), graphed by Figure 7 and is prescribed for all points in time. As discussed previously, this analysis unwraps the track and treats it as if it were a flat surface.

![Figure 7: Car’s Position versus Time](image)
Section 5.1.2 – Boundary Conditions

The boundary conditions were prescribed according to trends seen in practice. These boundary conditions are depicted in Figure 8. Dense trusses typically support the hill so five equally-spaced points of zero deflection pins were applied to the track. Also, pins were applied to the end of the dip, beginning of the straight, halfway across the straight, and at the end of the track. These supports are low to the ground so that are assumed to act as pins. Tall supports typically attach to the track halfway along the incline and decline and at each side of the inversion, as seen in Figure 9. These tall supports were given relative stiffness values according to Equation (14) where $k$ is the stiffness, $F$ is the applied force of the roller coaster car on the track at that point and $\delta$ is the displacement of the track at that point. The deflection for the points at the beginning and end of the inversion were given a value of 1 foot and the halfway points on the incline and decline were given a value of 0.5 feet.

\[
k = \frac{F}{\delta}
\]  

(14)
Figure 8: Boundary Conditions on the Track

Figure 9: Supports on Roller Coaster [13]
Section 5.1.3 – Force Function

As mentioned before, the force of the roller coaster car onto the track is variable throughout the ride. Assuming no friction or drag, the trade-off between potential and kinetic energy according to the conservation of energy equation, Equation (7), was considered to determine the velocity at any point in time along the track. Due to the curvature of the track, centripetal force is a large factor in the contribution of force to the track of the roller coaster. The centripetal force on the track is the square of the velocity at that point divided by the radius of curvature, Equation (15). The forces that the car inflicts on the curved part of the track is determined by Figure 10, which shows the direction and relative magnitude of gravitational, centripetal and resultant force. The resultant force is the final force of interest, with the normal component as the value used for $P_0(t)$, as in Equation (1).

$$F_{\text{centripetal}} = \frac{v^2}{r}$$  \hspace{1cm} (15)
Each section of the track has a unique force applied to the track according to the piecewise function in Equation (16). The car’s position, \( x_c \), is determined by solving the differential equation from Equation (11) for all applicable sections and graphing them in sequence with all of the sections, as shown in Figure 7. Then, the velocity, \( \dot{x}_c \), is graphed for the
centripetal force evaluation in Figure 11. Both $x_c$ and $\dot{x}_c$ are graphed versus time in order to determine at what time each section is traded off into the next. Since the initial and final arc length and speed are already known, an inspection of the graph can yield the time steps that will define the piecewise function.

\[
F(t) = \begin{cases} 
mg \cdot \cos(\theta) & \text{if } 0 \leq t < 3.61 \text{ s} \quad \text{HILL} \\
mg \cdot \cos \left( \theta - \frac{s_2 - s_1}{r_1} \right) + m \frac{s_2^2}{r_1} & \text{if } 3.61 \leq t < 4.16 \text{ s} \quad \text{DIP} \\
mg \cdot \cos \left( \frac{s_3 - s_2}{r_1} \right) + m \frac{s_3^2}{r_1} & \text{if } 4.16 \leq t < 5.34 \text{ s} \quad \text{INCLINE} \\
-mg \cdot \sin \left( \frac{s_4 - s_3}{r_2} \right) + m \frac{s_4^2}{r_2} & \text{if } 5.34 \leq t < 6.79 \text{ s} \quad \text{INVERSION} \\
mg \cdot \sin \left( \frac{s_5 - s_4}{r_1} \right) + m \frac{s_5^2}{r_1} & \text{if } 6.79 \leq t < 7.98 \text{ s} \quad \text{DECLINE} \\
\frac{mg}{mg} & \text{if } 7.98 \leq t < 9.09 \text{ s} \quad \text{STRAIGHT}
\end{cases}
\]

\[\text{Figure 11: Speed versus Time}\]
The force can now be determined at any point in time or space. The force is graphed in terms of arc length in Figure 12. The sample track for the model does not have ideally gradual transitions between sections which is evident by the steep increase or decrease in force between sections, for example near 150, 280, 360, and 400 feet along the track. This steep change is caused by the sudden change in radius of curvature. A more accurate model will need to have a more gradual introduction of a radius as the roller coaster car approaches the inversion and continued throughout the loop.

Figure 12: Force versus Arc Length

Section 5.1.4 – Modal Analysis

With the track and forcing function fully defined, these terms were used to develop a finite element model for numerical simulation of the vibration response. This sets up and solves
the eigenvalue problem which produces the natural frequencies of vibration and their corresponding mode shapes.

The first ten mode shapes are depicted in Equation (13). These mode shapes show how the beam will react to excitation. The node locations at boundary conditions are evident in most cases. Node locations defined with zero deflection show mode shapes converging to zero. However, where there are flexural supports, deflection does not go back to zero. This is more easily seen in Figure 14. In this figure, Nodes 1 and 4 refer to the constraints modelled by shorter springs in the middle of the incline and decline, respectively. Nodes 2 and 3 refer to the constraints at the beginning and end of the inversion, respectively. These nodes were longer and more flexible which is apparent by the more movement around the node location when compared to the stiffer, shorter constraints.
Figure 13: First Ten Mode Shapes

Figure 14: Mode Shapes around Flexural Supports
Section 5.1.5 – Location, Time and Deflection

A three-dimensional surface plot was created to display deflection in terms of both time and space. This plot in Figure 15 can be manipulated to inspect deflection at any point in time within the ride and/or any point along the track. More interestingly, a contour plot in provided in Figure 16 which shows the car’s position versus time (similarly in Figure 7) on top of the colored contour of the original 3D plot.

![Figure 15: Deflection versus Time and Arc Length](image)
Figure 16: Contour Plot

The contour shows an interesting phenomenon. The unraveled track is superimposed on this plot to show the location of the car in time. Notice that the first area of yellow occurs at a time earlier than the roller coaster car actually traverses the track. The vibration of the track travelled along the track faster than the car. This can be observed upon further inspection of Figure 17.

The deflection occurs as a solid in this figure because it is a cross-sectional view of surface plot; therefore, it shows the deflection of all of the arc length locations at that time. On this plot, it can be observed that the highest deflection, occurring during the inversion, is oscillating about
an offset instead of the point of zero deflection. This behavior occurs due to the vibration of the track before contact with the roller coaster car. In other words, the beam is vibrating in result of travelling frequencies. The beam is deflected further when the car travels across, then it vibrates about that offset deflection. This phenomenon is interesting because it may affect design decisions. A designer should be alerted that this vibration is traveling so supports need to be stronger which means stronger material, larger cross-section, etc.

As discussed previously, roller coasters are traversed by many riders in cars all day every day. Therefore, it is advantageous for a designer to know how often cars can be run. Schedule will be affected by how long it takes for the beam, or a section of the beam, to return back to a

Figure 17: Deflection versus Time
steady state of zero deflection. Figure 18 shows the deflection at four locations (top left – middle of hill / top right – middle of dip / bottom left – start of inversion / bottom right – end of inversion) throughout the length of the ride. In each case, the highest point of deflection occurs at the time that it traversed by the car. The hill faces a very small amount of deflection which dissipates to nearly zero throughout the end of the ride. The dip deflects on a larger scale but can also be observed to dissipate throughout the ride of this roller coaster car. At the start of the inversion,
the deflection doubles and appears to begin to converge to a steady state. However, for this location and the end of the inversion, steady state is not achieved within the duration of the ride. Again, the deflection doubles when the car travels to the end of the inversion section of the ride. It can also be observed here that there is more vibration before the car travels across the point on the beam (where the highest peak in deflection occurs) when compared to the deflection occurring before the start of the inversion.

Section 5.2 – Evaluating Success

This model can provide valuable information in the first stages of designing a roller coaster. If several preliminary features of the roller coaster are determined, the model can be run to give a designer an idea of the amount of deflection it shall withstand. For example, a designer might be tasked with designing a roller coaster that reaches a certain height, makes the riders feel a certain amount of g-forces, or will travel at a certain velocity. With this information available, the model can be run to approximate the deflection of the track and its mode shapes. This can aid design by determining how many and what kind of boundary conditions to impose, how high to make the roller coaster, or how large the radius of the loop needs to be, on average, in certain sections.
A downfall of the model is the sudden change of radius between sections. This discontinuity is evident by the steep change in force entering and leaving the inversion, when the track changes radius. The change in radius affects the calculation of centripetal force. As observed from Figure 19, the centripetal force contributes more to the resultant force than gravity. These steep changes are apparent here, as well. The best course of action to alleviate these effects is to implement a curve with more sections with decreasing radius of curvature in steps or evaluating a curve that is defined by a gradual change in curvature.

![Figure 19: Force Contribution](image)
Section 5.3 - Summary

A prescribed track was defined to provide for functions for the velocity and magnitude of forces in terms of time and space. A modal analysis of this track was also determined. The model allows for any mode shape to be graphed as well as several mode shapes on a single graph. This analytical study of the track of a roller coaster provides a first-order approximation model on the expected dynamic response of the track that is transversely loaded with the point loads of variable magnitude. It was discussed how the hypothesis was upheld throughout the research. The use of this model to estimate the vibration of a curved beam is certainly a better estimation than approximating its vibration on a linear beam model.
Chapter 6 – Conclusion

Section 6.1 – Summary of Findings

This thesis reviews the literature of the vibration of beams, presents a model for the estimation of the vibration of a curved Euler-Bernoulli beam and discusses results of the model and its utility. First, the literature of the vast field of the vibration of beams is reviewed briefly. All of the cited references contributed to an understanding of the research being conducted in beam theory over the years, ever since the transportation industry started to boom in the early 19th century.

Several conditions and assumptions must be made when conducting research on the vibration of a beam. First, one must decide which beam theory to use. This decision is based on its geometry, constraints and the types of force conditions that the beam encounters or for the purpose of achieving a different level of accuracy. Euler-Bernoulli beam theory is the most simplified beam theory. Shear beams and Rayleigh beams admit shear deflection and consider rotary inertia, respectively. A Timoshenko beam is the most comprehensive beam theory, containing the shear and Rayleigh models.

The next aspect of the beam to consider is the application of boundary conditions and intermediate point constraints. A beam can be constrained by different mechanisms that apply various combinations of deflection constraints such as zero deflection and zero slope of
deflection. This step is very important in defining the actual conditions of the beam in practice. Then, the force on the beam and the possible motion of the force on the beam must be considered. Particularly in the transportation industry, these forces and their motion are important considerations. The definition of these forces needs to be determined based on a representation of the actual conditions that the beam will face. Other assumptions include the cross-section of the beam and type of force (point load, distributed load, etc.). These assumptions also need to mimic the situation that the research concerns.

The references listed all offer methods to evaluate the deflection of a beam under various conditions described above, as well as unique perspectives that were discussed. The vibration of beams has been studied in response to horizontal support excitation, as in the case of seismic forces [10]. Another source was discussed because it modelled the car traversing the beam to be a beam itself, suspended by two springs as models for the wheels [11]. These topics were discussed not for their relevance to this particular thesis but for potential future work to be expanded upon the results given in this thesis.

Next, the problem at hand was described in detail. Upon the literature review, many assumptions were made in order to make the research in the field applicable to the case of a roller coaster. The research was then narrowed down to a general problem of the dynamic response of a beam, in particular. This field mostly pertains to the transportation industry in
answering the question of how a beam (road, railroad track, etc.) responds to moving loads. Finally, the specific problem of the dynamic response of a curved beam by moving loads was discussed. This topic has been shown to be largely unstudied in the field, at least with the level of detail provided here.

The most challenging portion of the research conducted was defining the representative track. The first track to be researched was called the cycloid curve. A cycloid is characterized by following a point on the edge of a circle rolling on a flat surface. A prolate cycloid which is characterized by following a point outside the diameter of a circle that rolls on a flat surface. A roller coaster may send the riders on the track of an inverted prolate cycloid. This curve has changing radius throughout the track as well as a changing center vertex. This track proved to be difficult to work with and much beyond the scope of this thesis.

The track chosen is a clothoid loop. An estimated version of a clothoid loop is the superposition of portions of a circles with different radii. The incline into the loop has a large radius, then it is followed by a semicircle with about half the radius of the first portion of the loop. This section forms the inverted section of the roller coaster. This smaller radius allows for the roller coaster car to whip around the loop while maintaining contact with the track. The key for the inverted portion of the roller coaster to maintain contact with the track is for the centripetal acceleration to be greater than the gravitational acceleration. Since centripetal
acceleration is a function of the velocity of the car, the inverted section of the loop has to be small so that speed is maintained throughout this loop. The inverted section is followed by a decline which has the original, large radius of curvature. On this section, the roller coaster car gains more speed lost from the inversion in order to continue onto other features in the roller coaster to follow.

Deflection was observed to occur most at the inversion section of the roller coaster, as expected. This section of the track is the most unsupported part of the structure of a roller coaster. An interesting observation in the deflection was that vibration of the track preceded contact with the roller coaster car. This early vibration caused the vibration after contact with the car to be offset from the point of zero deflection. Little deflection occurred along the hill of the roller coaster, along the dip and the straight due to the greater amount of supports. This was also found to dissipate quickly. However, deflection in the inversion was not only of higher magnitude but also did not reach a steady state during the duration of the ride.

Section 6.2 – Concluding Statements
The biggest shortcoming of the representative track was the estimation of the superimposed curves of different curvature. Gravitational force appeared smooth and consistent between sections of the track. However, the transition from the incline to the inversion and then inversion to decline of the track showed extreme spikes in force. These discontinuities occurred
due to the sudden change of radius, which affects the centripetal force. The centripetal force also contributes more heavily to the force felt by the track than the gravitational acceleration. Therefore, it is shown in a graph of the force versus location of the car (Figure 12) that the car would experience sharp, and dangerous, changes in force on such a simplified track.

The very aspect that made the inverted prolate cycloid curve too difficult to work with was the aspect that would have avoided the shortcoming in the estimated clothoid loop. In order to avoid the very sudden changes in force on the track, the loop should have had a more gradual change in the radius of curvature of the track.

It is still asserted that the estimated clothoid representative track is a good first-order approximation of the track of a roller coaster and its resulting deflection. The model is customizable to fit the needs a roller coaster with specified dimensions and presents the framework necessary to consider more advanced tracks including the prolate cycloid. Also, the boundary conditions can be manipulated easily, either to match an existing structure or during design to meet vibration requirements. A modal analysis can be run of this representative track in order to create a basic understanding of its vibratory response.

**Section 6.3 – Future Research**
There are vast opportunities for future research on the topic of the vibration of curved beams. This thesis describes a model that ran a modal analysis of a first-order approximation of
a representative track. The representative track proved to be the most difficult portion of the research and the part where the most approximations were made. The first area for improvement of the model is the definition of a more accurate representative track. This track should have a more gradual curve with a variable radius of curvature.

Also, the first approximation made was the assumption of a Euler-Bernoulli beam. This beam theory assumes only linear deflection. More complex beam theories, such as shear, Rayleigh or Timoshenko, may slightly improve the accuracy of the numerical simulations, as might nonlinear strain-displacement relations. Another assumption made was the constant tubular cross-sectional area of the track. Steel roller coasters typically have a three-tube design with webs and struts between the tubes at intermediate points. Furthermore, roller coaster cars may also have a three-wheel design that has contact to the tube on the top, bottom and outside. Even more, some roller coasters even have seating that hangs from the track rather than the traditional car on top of the track, though this likely has little effect.

All of these aforementioned aspects introduce tremendous complexity to the problem. The cross-sectional area is truly variable along the length of the track. Also, there are greater points of contact that are not necessarily applied to the top of a cross-section. This complexity can require consideration of additional dimensions of beam motion to understand the actual vibratory deflection of the roller coaster track. These conditions are the reality of the complex
design of a modern steel roller coaster and research on these interesting structures can be conducted with a variety of assumptions to fit the needs of a researcher or customer. More information can be found in ASTM standards governing standard practices for amusement park rides [14, 15, 16]. These provisions will allow a researcher or designer to create a better estimation of roller coaster characteristics in practice.
APPENDIX A: MAIN CODE
% DEFINING CONSTANTS OF ROLLER COASTER

r1 = 60; % radius of dip, incline, decline (ft)
r2 = 25; % radius of inversion (ft)
h = 105; % height of hill (ft)
theta = pi / 4; % grade of hill
g = 32.2; % gravity (ft/sec^2)
m = 500; % estimated mass of train (lb)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% FINDING EQUATIONS OF MOTION
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Section 1 - Hill
s0 = 0; % initial arc length
v0 = 0; % initial velocity
s1 = h / sin(theta); % final arc length
v1 = sqrt(2 * g * h); % final velocity

% Section 2 - Dip
A2 = v1^2 - 2 * g * r1 * cos(theta);
B2 = 2 * g * r1;
C2 = 1 / r1;
D2 = pi/2 - theta - s1 / r1;
s2 = s1 + (r1 * theta); % final arc length
v2 = sqrt(v1^2 + (2 * g * r1 * (1 - cos(theta)))); % final velocity

% Section 3 - Incline
A3 = v2^2 - 2 * g * r1;
B3 = 2 * g * r1;
C3 = -1 / r1;
D3 = pi/2 + s2 / r1;
s3 = s2 + (r1 * (pi/2)); % final arc length
v3 = sqrt(v2^2 - 2 * g * r1); % final velocity

% Section 4 - Inversion
A4 = v3^2;
B4 = -2 * g * r2;
C4 = 1 / r2;
D4 = -s3 / r2;
s4 = s3 + (r2 * pi); % final arc length
v4 = v3; % final velocity

% Section 5 - Decline
A5 = v4^2;
B5 = 2 * g * r1;
C5 = 1 / r1;
D5 = -s4 / r1;
s5 = s4 + (r1 * (pi / 2));  % final arc length
v5 = sqrt(v4^2 + (2 * g * r1));  % final velocity

% Section 6 - Flat
s6 = s5 + 100;
v6 = v5;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% DEFINING CONSTANTS OF EQUATIONS OF MOTION
A = [nan A2 A3 A4 A5];
B = [nan B2 B3 B4 B5];
C = [nan C2 C3 C4 C5];
D = [nan D2 D3 D4 D5];
%EOM--> (ds/dt)^2 = A+B*cos((C*s)+D)
%Sol--> s = sqrt(A+B*cos((C*s)+D))
x = [s0 s1 s2 s3 s4 s5 s6];
v = [v0 v1 v2 v3 v4 v5 v6];
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% FINDING TIME INTERVALS
z = 1;
t = 0:.001:5;  s = 1/2*g*sin(theta)*t.^2;
ind = find(s>x(z+1),1);
tfinal = t(ind);
tsave = t(1:ind);  ssave = s(1:ind);
for z=2:5
    tsfinal = tfinal + linspace(0,(s6 - s5)/v6,101);
s = s5 + v6*(tsfinal - tfinal);
tsave = [tsave tsfinal];  ssave = [ssave s];
end
figure(1); subplot(311); plot(tsave,ssave)
xlabel('Time (s)'); ylabel('Distance along track (ft)')

subplot(312); plot(tsave(1:end-1),diff(ssave)./diff(tsave))
xlabel('Time (s)'); ylabel('Speed (ft/s)')

svel = diff(ssave)./diff(tsave);
tvel = tsave(1:end-1) + diff(tsave)/2;

subplot(313); plot(tvel(1:end-1),(diff(svel)./diff(tvel)/32.2))
xlabel('Time (s)'); ylabel('Acceleration (ft/s/s)')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%STEPS IN ARC LENGTH, FINAL VELOCITY, TIME
s_steps = [0 148.4924 195.6163 289.8641 368.4039 462.6517 562.6517];
vf_steps = [0 82.2314 88.8467 63.4802 88.8467 88.8467];
t_steps = [0 3.611 4.156 5.343 6.790 7.977 9.092];

%FORCE FUNCTION
for i = 1:length(tsave);
    % Section 1 - Hill
    if t_steps(1) <= tsave(i) && tsave(i) < t_steps(2)
        Fg(i) = m*g*cos(theta);
        Fc(i) = 0;
    % Section 2 - Dip
    elseif t_steps(2) <= tsave(i) && tsave(i) < t_steps(3)
        Fg(i) = m*g*cos(theta-((ssave(i)-s1)/r1));
        Fc(i) = m*((A2+B2*sin(C2*ssave(i)+D2))/r1);
    % Section 3 - Incline
    elseif t_steps(3) <= tsave(i) && tsave(i) < t_steps(4)
        Fg(i) = m*g*cos((ssave(i)-s2)/r1);
        Fc(i) = m*((A3+B3*sin(C3*ssave(i)+D3))/r1);
    % Section 4 - Inversion
    elseif t_steps(4) <= tsave(i) && tsave(i) < t_steps(5)
        Fg(i) = -m*g*sin((ssave(i)-s3)/r2);
        Fc(i) = m*((A4+B4*sin(C4*ssave(i)+D4))/r2);
    % Section 5 - Decline
    elseif t_steps(5) <= tsave(i) && tsave(i) < t_steps(6)
        Fg(i) = m*g*sin((ssave(i)-s4)/r1);
        Fc(i) = m*((A5+B5*sin(C5*ssave(i)+D5))/r1);
    % Section 6 - Flat
    else
        Fg(i) = m*g;
        Fc(i) = 0;
    end

end
F = Fg + Fc;
figure(2); plot(ssave,F/m)
xlabel('Arc Length (ft)'); ylabel('Force/Mass (kip/lb)');

figure(3); subplot(211); plot(ssave,Fg/m)
title('Gravitational Force');
ylabel('Force/Mass (kip/lb)');
subplot(212); plot(ssave,Fc/m)
title('Centripetal Force');
xlabel('Arc Length (ft)'); ylabel('Force/Mass (kip/lb)');

%BOUNDARY CONDITIONS
s_bc = [s0 (s1-s0)/5 2*(s1-s0)/5 3*(s1-s0)/5 4*(s1-s0)/5 s1 s2 ... s2+(s3-s2)/2 s3 s4 s4+(s5-s4)/2 s5 s5+(s6-s5)/2 s6];

ind = find(ssave>s_bc(8),1);
F1 = F(ind);
ind = find(ssave>s_bc(9),1);
F2 = F(ind);
ind = find(ssave>s_bc(10),1);
F3 = F(ind);
ind = find(ssave>s_bc(11),1);
F4 = F(ind);

delta = [0 0 0 0 0 0 0 0.5 1 1 0.5 0 0 0];
k = [0 0 0 0 0 0 F1/delta(8) F2/delta(9) F3/delta(10) F4/delta(11) 0 ... 0 0];

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% MODAL ANALYSIS SYSTEM PARAMETERS
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
E = 297000e3; % Young's Modulus of A618 steel (psi)
rho = 0.284; % density of A618 steel (lb/in3)
do = 4; % outside diameter of track (in)
di = 3.5; % inner diameter of track (in)
I = (pi/64)*(do^4-di^4); % moment of inertia of track (in4)
A = (pi*(do/2)^2)-(pi*(di/2)^2); % cross sectional area of track (in2)
s0 = 0;
s1 = h / sin(theta);
s2 = s1 + (r1 * theta);
s3 = s2 + (r1 * (pi/2));
s4 = s3 + (r2 * pi);
s5 = s4 + (r1 * (pi / 2));
s6 = s5 + 100;
\[
\begin{align*}
\text{s\_bc} &= \begin{bmatrix} s0+1 & (s1-s0)/5 & 2*(s1-s0)/5 & 3*(s1-s0)/5 & 4*(s1-s0)/5 & s1 & s2 \\
& & s2+(s3-s2)/2 & s3 & s4 & s4+(s5-s4)/2 & s5 & s5+(s6-s5)/2 & s6 \end{bmatrix}; \\
\rho A &= \rho*A; \quad EI = E*I; \quad L = s6; \\
\text{Nel} &= 563; \quad \% \text{number of elements} \\
\text{s\_bc} &= \text{round}(\text{s\_bc}); \\
\text{Ndof} &= 2*\text{Nel}+2; \\
\end{align*}
\]

\% Finite Element Method, cantilever beam example
\% Created by Jeffrey L. Kauffman  <JLKauffman@ucf.edu>

\% LOCAL ELEMENT MATRICES
\text{Le}l = L/\text{Nel};  \% element length
\text{ke}l = EI / \text{Le}l^3 * ... \% element stiffness matrix
\begin{bmatrix}
12 & 6*Le & -12 & 6*Le \\
6*Le & 4*Le^2 & -6*Le & 2*Le^2 \\
-12 & -6*Le & 12 & -6*Le \\
6*Le & 2*Le^2 & -6*Le & 4*Le^2
\end{bmatrix};
\text{me}l = \rho A * \text{Le}l / 420 * ... \% element mass matrix
\begin{bmatrix}
156 & 22*Le & 54 & -13*Le \\
22*Le & 4*Le^2 & 13*Le & -3*Le^2 \\
54 & 13*Le & 156 & -22*Le \\
-13*Le & -3*Le^2 & -22*Le & 4*Le^2
\end{bmatrix};

\% GLOBAL MATRICES
\text{K} = \text{zeros}(2*\text{Nel}+2,2*\text{Nel}+2);  \quad \text{M} = \text{K}; \\
\text{for } n=1:\text{Nel} \% loop over each element
\quad \text{i1} = 2*n - 1;  \quad \text{i2} = 2*n + 2; \% indices instead of assembly matrix
\quad \text{K(i1:i2,i1:i2)} = \text{K(i1:i2,i1:i2)} + \text{ke}l; \\
\quad \text{M(i1:i2,i1:i2)} = \text{M(i1:i2,i1:i2)} + \text{me}l;
\text{end}

\% DISCRETE SPRINGS
\text{spring\_nodes} = [\text{s\_bc}(8) \text{s\_bc}(9) \text{s\_bc}(10) \text{s\_bc}(11)]; \\
\text{ksp}ring = [135400 80560 33590 135520]; \\
\text{for } \text{ik}=1:\text{length(spring\_nodes)} \\
\quad \text{K(2*spring\_nodes(ik)-1,2*spring\_nodes(ik)-1)} = ... \\
\quad \text{K(2*spring\_nodes(ik)-1,2*spring\_nodes(ik)-1)} + \text{ksp}ring(ik); \\
\text{end}

\% CONSTRAINED MATRICES (APPLY BCS)
\text{disp\_BCs} = [\text{s\_bc}(1) \text{s\_bc}(2) \text{s\_bc}(3) \text{s\_bc}(4) \text{s\_bc}(5) \text{s\_bc}(6) \text{s\_bc}(7) ... \\
\text{s\_bc}(12) \text{s\_bc}(13) \text{s\_bc}(14)]; \quad \% \text{node number where disp} = 0
\text{slope\_BCs} = []; \quad \% \text{node number were slope} = 0
\text{qBCs} = [2*\text{disp\_BCs}-1 2*\text{slope\_BCs}];
Kc = K;  Kc(qBCs,:) = [];  Kc(:,qBCs) = [];  
Mc = M;  Mc(qBCs,:) = [];  Mc(:,qBCs) = [];  
\% zero disp & slope at five nodes  
Nc = (2*Nel + 2) - length(qBCs);  \% fewer degrees of freedom  

\% EIGENVALUE PROBLEM  
[v,d] = eig(Kc,Mc);  \% solve eigenvalue problem  
[omgr,ind] = sort(sqrt(diag(d)));  \% find omega_n and sort modes  

omgr';  \% sorted & normalized using mass  
for r=1:Nc  \% sorted & normalized using mass  
   Phic(:,r) = v(:,ind(r))/sqrt(v(:,ind(r))'*Mc*v(:,ind(r))));  \% constrained e'vecs, column-wise  
end  

Phi = [Phic; nan(length(qBCs),Nc)];  
for iq = 1:length(qBCs)  
   Phi(qBCs(iq)+1:end+1,:) = Phi(qBCs(iq):end,:);  
   Phi(qBCs(iq),:) = 0;  
end  

\% PLOT MODE SHAPES  
mode = 2;  \% mode shape to plot  
xp = linspace(0,L,101);  \% array of x to plot exact shapes  
xnode = linspace(0,L,Nel+1);  \% array of x at the nodes  

% figure(4); plot(xnode,Phi(1:2:end,mode)/sign(Phi(3,mode)),'o-r')  
% plot FE mode shapes, disp only  
for s=2:length(xp)  
   el = find(xp(s)<=[Le1:Le1:L],1);  \% find element number of global xp  
   xel = xp(s) - (el-1)*Le1;  \% get local xel from global xp  
   w(s) = [1 - 3*(xel/Le1).^2 + 2*(xel/Le1).^3;  
      Le1*(xel/Le1 - 2*(xel/Le1).^2 + (xel/Le1).^3);  
      3*(xel/Le1).^2 - 2*(xel/Le1).^3;  
      Le1*(-(xel/Le1).^2 + (xel/Le1).^3)]'*...  
         Phi(2*el-1:2*el+2,mode);  
% w = [N(x)]'{q} local  
end  
% figure(5); plot(xp,w/sign(w(2)),'-k') \% plot FE mode shapes, interpolated  

% PLOT SEVERAL MODE SHAPES  
Phi = Phic;  
for iq = 1:length(qBCs)  
   Phi(qBCs(iq)+1:end+1,:) = Phi(qBCs(iq):end,:);  
   Phi(qBCs(iq),:) = 0;  
end
\begin{verbatim}
xnode = linspace(0,L,Nel+1);       % array of x at the nodes

for mode=1:10
    nodaldispplot(mode,:) = Phi(1:2:end,mode)/sign(Phi(3,mode));
end

figure(6); plot(xnode,nodaldispplot(1:5,:))

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% PLOT DEFINITION VERSUS TIME AND SPACE

time = linspace(0,tsave(end-1),10001);
w = zeros(Nel+1,length(time));
for ia=1:30,
    [t,alpha] = ode45(@SDOF_int,time,[0 0],[],tsave,F,ssave,...
        Phi(1:2:end,ia),Lei,omgr(ia),.02);
    w = w + Phi(1:2:end,ia)*alpha(:,1)';
end

figure(7); surf(time,[1:Nel+1],w); shading flat
xlabel('Time (s)'); ylabel('Arc Length (feet)'); zlabel('Deflection')

figure(8);contourf(time,[1:Nel+1],w,[-1:.075:-.025 .025:.075:2],...
    'LineColor','none')
hold on; plot(tsave,ssave,'m','LineWidth',3)
xlabel('Time (s)'); ylabel('Arc Length (feet)');
\end{verbatim}
APPENDIX B: DETERMINING CAR’S POSITION
function ds = myfunc_sdot(t, s, AA, BB, CC, DD)
% (s')^2 = A + B*sin(C*s+D)
ds = size(s);
ds = sqrt(AA+BB*sin(CC*s+DD));
function dx = SDOF_int(t,x,tsave,F,ssave,Phi,Lel,wr,zr)
dx = zeros(size(x));
% Interpolate to find value of the force at the current t
ind = find(t<tsave,1)-1;
force = F(ind) + (t-tsave(ind)) / (tsave(ind+1)-tsave(ind)) * (F(ind+1)-F(ind));
% Interpolate to find location of the car at the current t
xcar = ssave(ind) + (t-tsave(ind)) / (tsave(ind+1)-tsave(ind)) * (ssave(ind+1)-ssave(ind));
% Interpolate to find displacement of the mode shape at the current car position
indx = fix(xcar/Lel)+1;
Wr = Phi(indx) + rem(xcar,Lel) / Lel * (Phi(indx+1)-Phi(indx));
% Modal force
Nr = force*Wr;
% EOM
dx(1) = x(2);
dx(2) = Nr - 2*zr*wr*x(2) - wr^2*x(1);
References


