Fractal Spectral Measures In Two Dimensions

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Beng Oscar Alrud
University of Central Florida

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by

Bengt Oscar Alrud
Bachelor of Science, University of Göteborg, 1974
Licentiate of Philosophy, Chalmers University of Technology, 1995

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Dorin Ervin Dutkay

Abstract

We study spectral properties for invariant measures associated to affine iterated function systems. We present various conditions under which the existence of a Hadamard pair implies the existence of a spectrum for the fractal measure. This solves a conjecture proposed by Dorin Dutkay and Palle Jorgensen, in several special cases in dimension 2.

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CHAPTER 1
INTRODUCTION

We will study some aspects of iterated function systems of affine type (IFS). The functions are affine transformations defined on $\mathbb{R}^2$, taking values in $\mathbb{R}^2$. They are coupled by a matrix R. When such a system is iterated an infinitely number of times it may give rise to a fractal. What is a fractal? Some authors do not define fractal, but we do for the purpose of this thesis; we define it in terms of dimension. The concept of topological dimension of a set X, $\dim_T(X)$, coincides with our intuition where a line has dimension one and an open set in the plane dimension two. It is defined by a number of required properties, one of which is invariance: $\dim(\Psi(X)) = \dim(X)$ when Ψ is a homeomorphism. The Hausdorff dimension of a set X in $\mathbb{R}^2$, $\dim_H(X)$, is defined in a somewhat complicated way which is well described in the literature [Fal03]. The Hausdorff dimension is a metric dimension and so it is a parameter that describes the geometry of the set X.

Definition 1.1. *(Mandelbrot) We say that a set X in $\mathbb{R}^d$ is a fractal if $dim_T(X) < dim_H(X)$. Then the difference $dim_H(X)-$ $dim_T(X)$, the fractal degree of X, shows how fractal X is. Since $dim_T(X)$ only takes values on integers we have: A set X is fractal if $dim_H(X)$ is a non-integer value.*

In our study we will be concerned only with a subset of all fractals, the affine class. We first consider $\mathbb{R}^d$, for d a positive integer.

Definition 1.2. *Let R be a $d \times d$ expansive integer matrix. Expansive means that all its eigenvalues have absolute value strictly bigger than one. Then R must be invertible. Let B be a finite subset of $\mathbb{R}^d$, with $0 \in B$, and let N be the cardinality of B.*

Define the maps

$$\tau_b(x) = R^{-1}(x + b), \quad (x \in \mathbb{R}^d, b \in B) \tag{1.1}$$

The set of such functions is the affine iterated function system IFS associated to R and B.

The property of R being expansive implies that the τ_b's are contractions in some norm (for example, for the classical middle-third Cantor set in one dimension R^{-1} would correspond to the number 1/3) and the set B is $\{0, 2\}$ which corresponds to the left and right third; the fact that 1/2 is not in the set corresponds to the middle third being eliminated).

Next, we define the *attractor* of an iterated function system.

In general, for a contraction Ψ in a complete metric space, by the Banach fixed point theorem, Ψ has a unique fixed point x, i.e., x satisfies $x = \Psi(x)$.

We will start from $\mathbb{R}^d$ and then define another complete metric space, and another contraction Φ on it. Let $\mathcal{K}$ be the set of all non-empty compact sets in $\mathbb{R}^d$. For A in $\mathcal{K}$ we set $N_\epsilon(A) = \{x \in \mathbb{R}^d : \mathrm{dist}(x, A) \leq \epsilon\}$.

Define $d_H(A, B) = \min\{\epsilon \geq 0 : A \subset N_\epsilon(B) \text{ and } B \subset N_\epsilon(A)\}$. Then $d_H(A, B)$ becomes a metric on $\mathcal{K}$, the Hausdorff metric. We are actually looking at each compact set as a point.

This metric turns $\mathcal{K}$ into a complete metric space [YHK97]. Define Φ: $\mathcal{K} \longrightarrow \mathcal{K}$ by

$$\Phi(A) = \bigcup_{i=1}^{N} \tau_b(A). \tag{1.2}$$

According to [Hut81], the map Φ is a contraction on $\mathcal{K}$. This is proved also,. e.g., in [YHK97]. Hence there exists a unique compact set X_B, called the attractor of the IFS, such that $X_B = \Phi(X_B)$. In other words

$$X_B = \bigcup_{b \in B} \tau_b(X_B). \tag{1.3}$$

The compact X_B will contain all the iterates of the $\tau_b s$ and no other points. Also, it is enough to start from the origin, so

$$X_B = \left\{ \sum_{k=1}^{\infty} R^{-k} b_k : b_k \in B \text{ for all } k \geq 1 \right\}. \tag{1.4}$$

Definition 1.3. *The compact set X_B defined uniquely by* (1.3) *(or, equivalently by* (1.4)*) is called the attractor for the IFS $(\tau_b)_{b \in B}$.*

The attractor X_B is invariant in the following sense: starting from any x in X_B all images $\tau_b(x)$ will stay in X_B. By restriction the individual mappings τ_b induce endomorphisms in X_B and these restricted mappings we also denote by τ_b.

For this IFS there exists [Hut81] a unique invariant probability measure μ_B, which we define below.

Definition 1.4. *By [Hut81] there exists a unique probability measure on $\mathbb{R}^d$ with the property:*

$$\mu_B(E) = \frac{1}{N} \sum_{b \in B} \mu_B(\tau_b^{-1}(E)) \textit{ for all Borel subsets } E \textit{ of } \mathbb{R}^d. \tag{1.5}$$

Equivalently,

$$\int f\, d\mu_B = \frac{1}{N} \sum_{b \in B} \int f \circ \tau_b\, d\mu_B \text{ for all bounded Borel functions } f \text{ on } \mathbb{R}^d. \qquad (1.6)$$

Moreover μ_B *is supported on* X_B *and is called the invariant measure of the IFS* $(\tau_b)_{b \in B}$.

We say that μ_B *has no overlap if*

$$\mu_B(\tau_b(X_B) \cap \tau_{b'}(X_B)) = 0, \text{ for all } b \neq b' \in B. \qquad (1.7)$$

The measures μ_B, one for each IFS, are our objects of study. We now restrict our attention to the case of dimension $d = 2$.

Geometrically, an IFS in this thesis is equivalent to a pair (X, μ) where X is a compact subset of $\mathbb{R}^2$, μ is a probability measure whose support is X and determined uniquely by the initial IFS mappings.

We want to understand X and μ_B better. For some IFS's it may be possible to build inside the Hilbert space $L^2(X, \mu_B)$, where X by above is a compact subset of $\mathbb{R}^2$ and the attractor of the IFS, an orthogonal basis for this Hilbert space composed of exponential functions, i.e., a Fourier basis.

When is it possible to find such a basis for a particular IFS? As an illustration it is known [JP98] that it is not possible for the middle-third Cantor set in one dimension. Of course, the classical example when this is possible, is the unit interval with Lebesgue measure.

Existence of such a basis, we call it a Fourier basis, would make it possible to study the geometry of X and its symmetries from the associated spectral data for the IFS by using standard techniques from the theory of Fourier series.

Several papers have displayed various classes of affine IFSs for which an orthogonal Fourier basis exists in the corresponding $L^2(X_B, \mu_B)$, see e.g. [JP98, DJ06, DJ07, DJ08, DJ09b, DHS09, DJ09a, DHJ09, Str00, ŁW02, ŁW06]. But in each case one or more extra conditions must be met with in order to admit such a basis. This thesis takes a closer look at those conditions in two dimensions.

Definition 1.5. *For $\lambda \in \mathbb{R}^d$, denote by $e_\lambda(x)$ the exponential function $e_\lambda(x) := e^{2\pi i \lambda \cdot x}$.*

A Borel probability measure μ on $\mathbb{R}^d$ is called spectral, if there exists a set Λ in $\mathbb{R}^d$ such that the family of exponential functions

$$E(\Lambda) := \{e_\lambda : \lambda \in \Lambda\}$$

is an orthonormal basis for $L^2(\mu)$. In this case, the set Λ is called a spectrum of the measure μ.

Definition 1.6. *We will say that (B, L) is a Hadamard pair if B,$L \subset \mathbb{Z}^2$, $0 \in L$, $\#L = \#B = N$ and the matrix*

$$\frac{1}{\sqrt{N}}(e^{2\pi i R^{-1} b \cdot l})_{b \in B, l \in L}$$

is unitary.

If this is true we call (R, B, L) a spectral system.

Let S be the matrix R^T and define the family of functions

$$\tau_l(x) = S^{-1}(x + l) \quad (x \in \mathbb{R}^2).$$

Why is the set $(\tau_l)_{l\in L}$ introduced? This principle, to study a related dual system, is not uncommon in mathematics. To clarify this point, we underline that we are interested in the measure μ_B associated to the IFS $(\tau_b)_{b\in B}$. The main question is whether this is a spectral measure. The dual system $(\tau_l)_{l\in L}$ is only considered in order to help us in constructing the basis of exponentials.

It was proven in [DJ06] that for dimension one, the existence of a Hadamard pair is sufficient for the measure μ_B to be spectral. This was a significant improvement of earlier results, where also an analytical condition was necessary. In [DJ07], it was proved that this condition and a certain "reducibility condition" (which we will discuss below), guarantee that μ_B is a spectral measure. Dutkay and Jorgensen proposed the following conjecture.

Conjecture 1.7. *If (R, B, L) is a spectral system then the measure μ_B is spectral.*

We will study this conjecture in dimension 2, and we prove it is valid under various conditions.

We will now discuss the iteration of points under the dual system L, i.e. we consider the "dual" affine iterated function system defined by

$$\tau_l(x) := S^{-1}(x + l), \text{ where } S = R^T, \text{ and } l \in L.$$

As shown in [DJ06, DJ07] the dynamics of the dual IFS is essential in determining if μ_B is spectral.

When points are iterated by the affine maps in the IFS, some points will be periodic, resulting in a cycle.

Definition 1.8. *We say that a finite set* $C := \{x_0, x_1, \ldots, x_{p-1}\}$ *is a cycle if there exists* $l_0, l_1, \ldots, l_{p-1} \in L$ *such that* $\tau_{l_k}(x_k) = x_{k+1}$ *for* $k \in \{0, \ldots, p-1\}$*, where* $x_p := x_0$*. We say that* x_0 *is a periodic point and denote it by* $x_0 =: \wp(l_{p-1}, \ldots, l_0)$ *to indicate the participating maps. Certain cycles have a special character.*

Let m_B *be the function*

$$m_B(x) := \frac{1}{N} \sum_{b \in B} e^{2\pi i b \cdot x} \quad (x \in \mathbb{R}^2).$$

We call the cycle extreme, if $|m_B(x_i)| = 1$ *for all* $i \in \{0, \ldots, p-1\}$*.*

In dimension one a thorough study of the extreme cycles resolved the question of the measure μ_B being spectral [DJ06]. In that case it was shown that existence of a Hadamard pair is sufficient for the measure μ_B to be spectral. In addition, it was possible to compute a spectrum explicitly by analyzing the extreme cycles.

So in dimension one the conjecture is true. If we add a special condition on the matrix R and the sets B and L, the reducibility condition, it was proven in [DJ07] that the conjecture is true also in higher dimensions. But as a general fact, in higher dimensions the possibilities are much more varied. The function m_B, which would be called a filter function in signal processing, can now have infinitely many zeroes.

Also, the extreme cycles might in this case be replaced by infinite orbits. We will call them infinite invariant sets, precise definitions will follow below. The study of these invariant sets was initiated by the French researchers Cerveau, Conze and Raugi [CCR96], for a different but related purpose, and we will use their results in this thesis.

Repeating the Conjecture: if (R, B, L) is a spectral system then the measure μ_B is spectral, we intend to give several good conditions for this conjecture to be true in dimension two.

Two cases must be distinguished. Either there exists infinite minimal invariant sets or all minimal invariant sets are finite, for a particular system.

In the latter case we call the Hadamard pair (B, L) *simple*, in the former case *non-simple*. Also, we remind the readers that μ_B being spectral means that a complete orthonormal Fourier series exists for the associated space $L^2(\mu, B)$.

Among our results we mention the following:

- Whenever (B, L) is a simple Hadamard pair the measure μ_B is spectral.
- If the eigenvalues of R are not rational then (B, L) is simple and the measure μ_B is spectral.
- If the determinant of R is a prime number then μ_B is spectral.

We will also give some conditions under which a non-simple pair gives rise to a spectral measure μ_B.

To achieve the results above known facts about invariant sets are recalled in the next section and new facts about them are added.

In this case the key to achieving results is once again to focus on the function

$$m_B(x) = \frac{1}{N} \sum_{b \in B} e^{2\pi i b \cdot x} \quad (x \in \mathbb{R}^2).$$

This function arises when considering the Fourier transforms of the invariance equation for the measure μ_B

$$\int f \, d\mu_B = \frac{1}{N} \sum_{b \in B} \int f \circ \tau_b \, d\mu_B.$$

To see this, with our IFS the relation becomes (we temporarily disregard the subscript B on the measure)

$$\int f(t) \, d\mu(t) = \frac{1}{N} \sum_{b \in B} \int f(R^{-1}(t+b)) d\mu(t),$$

valid for all bounded Borel functions f.

The Fourier transform of a measure is defined by

$$\hat{\mu}(x) = \int e^{2\pi i x \cdot t} d\mu(t), \quad (x \in \mathbb{R}^2).$$

Then

$$\hat{\mu}(x) = \frac{1}{N} \sum_{b \in B} \int e^{2\pi i x \cdot R^{-1}(t+b)} d\mu(t)$$

$$= \frac{1}{N} \sum_{b \in B} \int e^{2\pi i (R^T)^{-1} x \cdot (t+b)} d\mu(t)$$

$$= \frac{1}{N} \sum_{b \in B} \int e^{2\pi i (R^T)^{-1} x \cdot t} d\mu(t) e^{2\pi i (R^T)^{-1} x \cdot b}.$$

Hence we have the useful relation

$$\hat{\mu}(x) = m_B((R^T)^{-1}x)\hat{\mu}((R^T)^{-1}x) \quad (x \in \mathbb{R}^2).$$

This result is one reason for our interest in the dual IFS:

$$\tau_l(x) = (R^T)^{-1}(x + l) \quad (x \in \mathbb{R}^2, l \in L).$$

The m_B-function and the dual IFS are also linked by the following formula:

Proposition 1.9. *Suppose (B, L) is a Hadamard pair. Then*

$$\sum_{l \in L} |m_B(\tau_l x)|^2 = 1 \quad (x \in \mathbb{R}^2),$$

which is valid irrespective of the value of x, a fact that we want to emphasize.

Proof. We have

$$m_B(\tau_l x) = \frac{1}{N} \sum_{b \in B} e^{2\pi i b \cdot (R^T)^{-1}(x+l)} = \frac{1}{N} \sum_{b \in B} e^{2\pi i R^{-1} b \cdot (x+l)}.$$

Hence

$$|m_B(\tau_l x)|^2 = \frac{1}{N} \sum_{b \in B} e^{2\pi i R^{-1} b \cdot x} e^{2\pi i R^{-1} b \cdot l} \frac{1}{N} \sum_{b' \in B} e^{-2\pi i R^{-1} b' \cdot x} e^{-2\pi i R^{-1} b' \cdot l}.$$

When summed over L this becomes

$$\frac{1}{N^2}\sum_{b,b'\in B} e^{2\pi i R^{-1}(b-b')\cdot x}\sum_{l\in L} e^{2\pi i R^{-1}(b-b')\cdot l}.$$

For each fixed pair $b \neq b'$ the sum over L is zero because the Hadamard matrix is unitary.

Hence the result follows.

$\square$

This relation can be interpreted in probabilistic terms: $|m_B(\tau_l x)|^2$ is the probability of transition from x to $\tau_l x$.

Definition 1.10. *For $x \in \mathbb{R}^2$ we call a trajectory of x a set of points $\{\tau_{\omega_n} \dots \tau_{\omega_1} x | n \geq 1\}$, where $\{\omega_n\}_n$ is a sequence of elements in L such that $m_B(\tau_{\omega_n} \dots \tau_{\omega_1} x) \neq 0$ for all $n \geq 1$.*

The union of all trajectories of x is denoted by $\mathcal{O}(x)$ and its closure $\overline{\mathcal{O}(x)}$ is called the orbit of x.

If $m_B(\tau_l x) \neq 0$ for some $l \in L$ we say that the transition from x to $\tau_l x$ is possible.

A closed subset $F \subset \mathbb{R}^2$, is called invariant if it contains the orbits of all its points. This means that, if $x \in F$ and $l \in L$ are such that $m_B(\tau_l x) \neq 0$, then it follows that $\tau_l x \in F$.

An invariant subset is called minimal if it does not contain any proper invariant subset. Since the orbit of any point is an example of an invariant set, it must be that a closed subset F is minimal if and only if $F = \overline{\mathcal{O}(x)}$ for all $x \in F$.

CHAPTER 2
A FUNDAMENTAL RESULT

Thus far we have defined and explained some preliminary ideas and facts. We also need a fundamental result from earlier research [DJ07]. To present that it is necessary to utilize more advanced concepts and they will be introduced below.

We found before that

$$\sum_{l \in L} |m_B(\tau_l x)|^2 = 1,$$

irrespective of the starting point x. Defining $Q(x) := Q_B(x) := |m_B(x)|^2$ we write this simpler as

$$\sum_{l \in L} Q(\tau_l x) = 1, \tag{2.1}$$

where $Q(\tau_l x)$ is interpreted as the probability of transition from x to $\tau_l x$.

Introduce the space Ω of all infinite sequences, $\Omega = \{\ (l_1 l_2 \dots) \mid l_k \in L \quad \text{for all} \quad k \in \mathbb{N}\}$. If the first n l_ks are fixed, all others varying freely, we have what we call an n-cylinder. The set of all n-cylinders generate a σ-algebra $\mathcal{F}_n$.

Fix $x \in \mathbb{R}^d$. (For this presentation we temporarily revert to dimension d.) The functions τ_l of a particular IFS, acting on x and its iterates, give rise to a set of paths originating at x. Each path is described by a set of indices, i.e. by a member of Ω.

The space Ω is now looked upon as a space of paths, originating at x.

Associated to Ω is a path-space measure P_x. It is defined on the σ-algebra as follows. For a function f on Ω which depends only on the first n coordinates

$$\int f\, dP_x = \sum_{\omega_1,\ldots\omega_n \in L} Q(\tau_{\omega_1}x)Q(\tau_{\omega_2}\tau_{\omega_1}x)\ldots Q(\tau_{\omega_n}\ldots\tau_{\omega_1}x)f(\omega_1,\ldots\omega_n).$$

There is a question whether this definition of P_x is well defined. For that we will define and use the Radon notation for the measure P_x:

For functions measurable on Ω

$$P_x[f] := \int_\Omega f(\omega)dP_x.$$

Now we show that P_x is well defined.

If it is understood that f depends only on the first n coordinates, we temporarily denote it by f_n, it has to be checked that $P_x[f_n]$ stays the same when f is viewed as depending on only the first $n+1$ coordinates; $f(\omega) = f(\omega_1,\ldots,\omega_n) = f(\omega_1,\ldots,\omega_n,\omega_{n+1})$. Then

$$P_x[f_{n+1}] = \sum_{\omega_1,\ldots,\omega_{n+1}} Q(\tau_{\omega_1}x)\ldots Q(\tau_{\omega_{n+1}}\ldots\tau_{\omega_1}x)f(\omega_1,\ldots,\omega_{n+1})$$

$$= \sum_{\omega_1,\ldots,\omega_n} Q(\tau_{\omega_1}x)\ldots Q(\tau_{\omega_n}\ldots\tau_{\omega_1}x)\cdot\sum_{\omega_{n+1}} Q(\tau_{\omega n+1}\tau_{\omega_n}\cdots\tau_{\omega_1}x)f(\omega_1,\ldots,\omega_n)$$

$$= \sum_{\omega_1,\ldots,\omega_n} Q(\tau_{\omega_1}x)\ldots Q(\tau_{\omega_n}\ldots\tau_{\omega_1}x)f(\omega_1,\ldots,\omega_n)$$

$$= P_x[f_n],$$

as we have claimed.

With this integral approach to the measure we now need the measure P_x given on the sets generating the σ-algebra.

When the first n components are $l_1, l_2, \ldots l_n \in L$, let $C_n(i_1,\ldots,i_n)$ be a fixed n-cylinder and for $\omega = \omega_1,\ldots,\omega_n$, let $f(\omega) = \delta_{i_1}\omega_1\ldots\delta_{i_n}\omega_n$. Then we have

$$\int f(\omega)dP_x = \int \delta_{i_1}\omega_1\ldots\delta_{i_n}\omega_n dP_x = \int \chi_{C_n}(i_1,\ldots,i_n)(\omega)dP_x = P_x(C_n).$$

Hence

$$P_x(C_n(i_1,\ldots,i_n)) = Q(\tau_{i_1}x)Q(\tau_{i_2}\tau_{i_1}x)\ldots Q(\tau_{i_n}\ldots\tau_{i_1}).$$

Definition 2.1. *Define the transfer operator*

$$Tf(x) = \sum_{l\in L} Q(\tau_l x)f(\tau_l x) \quad (x \in \mathbb{R}^d).$$

A measurable function h on $\mathbb{R}^d$ is said to be harmonic (with respect to R) if $Th = h$.

Our first aim is to construct an important harmonic function.

When F is a non-empty compact and invariant subset of $\mathbb{R}^d$, we consider those elements $N(F)$ in path space Ω such that the corresponding iterates by the τ-functions from some point x eventually end up in F;

$$N(F) := \{ \omega \in \Omega \mid \lim_{n\to\infty} d(\tau_{\omega_n} \dots \tau_{\omega_1} x, F) = 0\}.$$

The fact that the maps τ_l are contractions implies that, for all $x, y \in \mathbb{R}^d$,

$$\lim_n d(\tau_{\omega_n} \dots \tau_{\omega_1} x, \tau_{\omega_n} \dots \tau_{\omega_1} y) = 0.$$

Hence the definition of $N(F)$ does not depend on x.

The characteristic function of $N(F)$ is unaffected by a shift in the iteration from a point x. What we mean is this: If $\omega = \omega_1\omega_2\omega_3\dots$, defining $G(x,\omega) := \chi_{N(F)}(\omega)$ we have

$$G(x, \omega_1\omega_2 \dots) = G(\tau_{\omega_1} x, \omega_2\omega_3 \dots); \tag{2.2}$$

we say that G has the cocycle property.

Define $h_F(x) := P_x(G(x,\cdot))$. Observe that $h_F(x) = P_x(\chi_{N(F)}) = \int \chi_{N(F)}(\omega) dP_x = P_x(N(F))$.

Then $0 \le h_F(x) \le 1$. In [DJ07] it is proven that $h_F(x)$ is continuous.

Lemma 2.2. *Let N, Q, P_x and Ω be as above. Then for all measurable functions f on Ω which depend only on the first n coordinates*

$$\sum_{l\in L} Q(\tau_l x) P_{\tau_l x}[f(i,\cdot)] = P_x[f].$$

Proof. We have

$$\sum_{l\in L} Q(\tau_l x) P_{\tau_l x}[f(i,\cdot)]$$

$$= \sum_i \sum_{\omega_1,\ldots,\omega_n} Q(\tau_{\omega_1}\tau_i x)\ldots Q(\tau_{\omega_n}\ldots\tau_{\omega_1}\tau_i x) f(i,\omega_1,\ldots,\omega_n)$$

$$= P_x[f].$$

□

Now we prove that h_F is harmonic. By the cocycle property and the lemma

$$(Th_F)(x) = \sum_i Q(\tau_i x) h_F(\tau_i x)$$

$$= \sum_i Q(\tau_i x) P_{\tau_i x}[G(\tau_i x,\cdot)] = \sum_i Q(\tau_i x) P_{\tau_i x}[G(x, i\cdot)]$$

$$= P_x[G(x,\cdot)] = h_F(x).$$

So, for each invariant compact set F there is associated a harmonic function h_F.

In [DJ07] it is shown that there is only a finite number of minimal compact invariant subsets, and for any two of them F and G, $d(F,G) > \sigma$, where σ is a positive number and d is the distance between the sets. (An invariant set is *minimal* if it does not contain any proper invariant subset.)

We need to prove the following proposition from [DJ07], for its ideas.

Proposition 2.3. *Let $F_1, F_2, \dots F_p$ be a family of mutually disjoint closed invariant subsets of $\mathbb{R}^d$ such that there is no closed invariant set F with $F \cap \cup_k F_k = \phi$.*

Then

$$P_x(\bigcup_{k=1}^{p} N(F_k)) = 1 \quad (x \in \mathbb{R}^d).$$

Proof. Assume this is not true; for some $x \in \mathbb{R}^d$, $P_x(\bigcup_k N(F_k)) < 1$. Then defining $h(x) := P_x(\bigcup N(F_k))$ we have

$$h(x) = \sum_{k=l}^{p} h_{F_k}(x) < 1.$$

By above h is continuous and $Th = h$. Since

$$\lim_{n\to\infty} h_F(\tau_{\omega_n} \dots \tau_{\omega_1} x) = \begin{cases} 1, & if \quad \omega \in N(F_k) \\ \\ 0, & if \quad \omega \notin N(F_k) \end{cases}$$

there are some paths $\omega \notin \cup_k N(F_k)$ such that $\lim_{n\to\infty} h(\tau_{\omega_n} \dots \tau_{\omega_1} x) = 0$.

Hence the set Z of zeroes of h is not empty. Also $Th = h$ shows that Z is a closed invariant subset. Claim: Z is disjoint from $\cup_k F_k$.

If not, $Z \cap F_k \neq \phi$ for some $k \in \{1, \dots, p\}$. Then take $y \in Z \cap F_k$. Because a transition is always possible there exists $\omega \in \Omega$ such that $Q(\tau_{\omega_n} \dots \tau_{\omega_1} y) \neq 0$ for all $n \geq 1$. By invariance $\tau_{\omega_n} \dots \tau_{\omega_1} y \in Z \cap F_k$. Hence $\omega \in N(F_k)$, i.e. $\lim_{n\to\infty} h_{F_k}(\tau_{\omega_n} \dots \tau_{\omega_1} y) = 1$.

But also, $\tau_{\omega_n} \dots \tau_{\omega_1} y \in Z$, so $h(\tau_{\omega_n} \dots \tau_{\omega_1} y) = 0$ for all $n \geq 1$.

This contraction proves the claim. Hence

$$P_x(\cup_k N(F_k)) = 1.$$

$\square$

Before stating the basic result, a theorem and some definitions are needed. First, a technical definition is given. When the subspace V in question is $\{0\}$, it ensures uniqueness of paths emanating from a point x.

Definition 2.4. *For a subspace V of $\mathbb{R}^d$ we say that the hypothesis "(H) modulo V" is satisfied if for all integers $p \geq 1$ the equality $\tau_{\epsilon_1} \dots \tau_{\epsilon_p} 0 - \tau_{\eta_1} \dots \tau_{\eta_p} 0 \in V$, with $\epsilon_i, \eta_i \in L$ implies $\epsilon_i - \eta_i \in V$, $i \in \{1, \dots, p\}$.*

Remark 2.5. The hypothesis "(H) modulo V" can be rephrased as follows (assuming that V is invariant for S): take two elements $\lambda := \epsilon_p + S\epsilon_{p-1} + \cdots + S^{p-1}\epsilon_1$ and $\gamma := \eta_p + S\eta_{p-1} + \cdots + S^{p-1}\eta_1$, with all digits ϵ_i, γ_i in L. If $\lambda \equiv \gamma \bmod V$, i.e., $\lambda - \gamma \in V$ then all the digits are congruent $\bmod V$, i.e., $\epsilon_i - \eta_i \in V$ for $i \in \{1, \dots, p\}$.

To see this, note that

$$\tau_{\epsilon_1} \dots \tau_{\epsilon_p} 0 = S^{-p}(\epsilon_p + S\epsilon_{p-1} + \cdots + S^{p-1}\epsilon_1) = S^{-p}\lambda.$$

Similarly for γ.

Then, using the invariance of V under S, we have that $\tau_{\epsilon_1}\dots\tau_{\epsilon_p}0 - \tau_{\eta_1}\dots\tau_{\eta_p}0 \in V$ iff $\lambda - \gamma \in V$. From this we see that the two formulations of the hypothesis "(H) modulo V" are equivalent.

The hypothesis "(H) modulo V" expresses the compatibility between the mod V equivalence and the dual IFS $(\tau_l)_{l\in L}$.

Theorem 2.6. *[CCR96]. Let M be a minimal compact invariant set contained in the set of zeroes of an entire function h on $\mathbb{R}^d$.*

(i) There exists V, a proper subspace of $\mathbb{R}^d$ (possibly reduced to $\{0\}$), such that M is contained in a finite union $\mathcal{R}$ of translates of V.

(ii) This union contains the translates of V by the elements of a cycle $\{\ x_0, \tau_{l_1}x_0, \dots, \tau_{l_{m-1}}\dots\tau_{l_1}x_0\}$ contained in M, and for all x in this cycle, the function h is zero on $x + V$.

(iii) Suppose the hypothesis "(H) modulo V" is satisfied. Then

$$\mathcal{R} = \{\ x_0 + V, \tau_{l_1}x_0 + V, \dots, \tau_{l_m-1}\dots\tau_{l_1}x_0 + V\},$$

and every possible transition from a point in $M \cap (\tau_{l_q}\dots\tau_{l_1}x_0 + V)$ leads to a point in $M \cap (\tau_{l q+1}\dots\tau_{l_1}x_0 + V)$ for all $1 \le q \le m-1$, where $\tau_{l_m}\dots\tau_{l_1}x_0 = x_0$.

(iv) Since the function Q is entire, the union $\mathcal{R}$ is itself invariant.

Definition 2.7. *By saying that a Hadamard triple (R, B, L) can be reduced to $\mathbb{R}^r$ we mean that the following conditions are satisfied:*

(i) The subspace $\mathbb{R}^r \times \{0\}$ *is invariant for* $S = R^T$ *so* S *can be brought to the form*

$$S = \begin{bmatrix} S_1 & C \\ 0 & S_2 \end{bmatrix}, \qquad S^{-1} = \begin{bmatrix} S_1^{-1} & D \\ 0 & S_2^{-1} \end{bmatrix},$$

where S_1*,* C *and* S_2 *are integer matrices, the* S*-matrices are quadratic and* S_1 *is of order* r*, less than* d*.*

(ii) For all first components b_1 *of elements of* B*, the number of* $b_2 \in \mathbb{R}^{d-r}$ *such that* $(b_1, b_2) \in B$ *is* N_2*, independent of* b_1*, and for all second components* l_2 *of elements in* L*, the number of* $l_1 \in \mathbb{R}^r$ *such that* $(l_1, l_2) \in L$ *is* N_1*, independent of* l_2 *and* $N_1 N_2 = N$*.*

(iii) The invariant measure for the iterated function system

$$\tau_{r_i}(x) = (S_1^T)^{-1}(x + r_i), \quad (x \in \mathbb{R}^r),$$

where $\{r_1, \ldots, r_{N_1}\}$ *are the first components of the elements of* B*, is a spectral measure and has no overlap.*

Remark 2.8. We used here Proposition 3.2 in [DJ07] to simplify the definition.

Definition 2.9. *Two Hadamard triples* (R_1, B_1, L_1) *and* (R_2, B_2, L_2) *are conjugate if there exists an invertible integer matrix* M *whose inverse is also integer such that*

$R_2 = M R_1 M^{-1}, B_2 = M B_1$ *and* $L_2 = (M^T)^{-1} L_1$*.*

If this is the case it means that the transition between the two IFSs (τ_b) is made by M, the transition between the two IFSs (τ_l) is made by $(M^T)^{-1}$ and that the qualitative features of the two systems are the same.

We say that the Hadamard triple (R, B, L) satisfies the reducibility condition if

(i) for all minimal compact invariant subsets F, the subspace V in Theorem 2.6 can be chosen such that there exists a Hadamard triple (R', B', L') conjugate to (R, B, L) which can be reduced to $\mathbb{R}^r$, and such that the conjugating matrix M maps V onto $\mathbb{R}^r \times \{0\}$.

Here $R' = MRM^{-1}$.

(ii) for any two distinct minimal compact invariant sets F_1, F_2 the corresponding unions $\mathcal{R}_1, \mathcal{R}_2$ of the translates of the associated subspaces, given in Theorem 2.6, are disjoint.

Theorem 2.10. *Let R be an expansive $d \times d$ integer matrix, B a subset of $\mathbb{Z}^d$ with $0 \in B$.*

Assume that there exists a subset L of $\mathbb{Z}^d$ with $0 \in L$ such that (R, B, L) is a Hadamard triple which satisfies the reducibility condition. Then the invariant measure μ_B is a spectral measure.

Now a proof of this theorem is outlined. The full proof is presented in [DJ07].

Outline of the proof. Guiding line: The relation $\sum_F h_F = 1$ has to be utilized. Writing this in terms of $| \hat{\mu}_B |^2$ this relation will ultimately translate into the Parseval equality for a family of exponential functions.

Consider a minimal compact invariant set F. By Theorem 2.6 there is a subspace V, invariant for S, such that F is contained in the union of some translates of V. Since the reducibility condition is satisfied there exists a conjugated Hadamard triple (R', B', L') which can be reduced to $\mathbb{R}^r$, and such that the corresponding matrix M maps V onto $\mathbb{R}^r \times \{0\}$.

Hence we can assume that $V = \mathbb{R}^r \times \{0\}$.

Combining Theorem 2.6 with a lengthy computation it is shown that, for some cycle $\mathcal{C} := \{\ x_0, \tau_{l_1} x_0, \ldots, \tau_{l_{m-1}} \ldots \tau_{l_1} x_0\}$, with $\tau_{l_m} \ldots \tau_{l_1} x_0 = x_0$, F is contained in the union

$\mathcal{R} = \{\ x_0 + V, \tau_{l_1} x_0 + V, \ldots, \tau_{l_{m-1}} \ldots \tau_{l_1} x_0 + V\}$, and $\mathcal{R}$ is an invariant subset.

The matrix R has the form

$$R = \begin{bmatrix} A_1 & 0 \\ C & A_2 \end{bmatrix}; \quad \text{hence} \quad R^{-1} = \begin{bmatrix} A_1^{-1} & 0 \\ -A_2^{-1} C A_1^{-1} & A_2^{-1} \end{bmatrix}.$$

By induction

$$R_{-k} = \begin{bmatrix} A_1^{-k} & 0 \\ D_k & A_2^{-k} \end{bmatrix},$$

where

$$D_k := -\sum_{l=0}^{k-1} A_2^{-(l+1)} C A_1^{-(k-l)}.$$

Combining this with the fact that

$$X_B = \left\{ \sum_{k=1}^{\infty} R^{-k} b_k \mid b_k \in B \right\},$$

we will decompose X_B into two components X_1 and X_2.

Any element (x, y) in X_B can be written as:

$$x = \sum_{k=1}^{\infty} A_1^{-k} r_{i_k}, \quad y = \sum_{k=1}^{\infty} D_k r_{i_k} + \sum_{k=1}^{\infty} A_2^{-k} \eta_{i_k}.$$

If we now define

$$X_1 := \left\{ \sum_{k=1}^{\infty} A_1^{-k} r_{i_k} \mid i_k \in \{ 1, \ldots, N_1 \} \right\}$$

and let μ_1 be the invariant measure for the iterated function system

$\tau_{r_i}(x) = A_1^{-1}(x + r_i)$, $i \in \{1, \ldots, N_1\}$, where N_1 is a factor of N, then the set X_1 becomes the attractor of this iterated function system.

In this way X_B is decomposed into the detailed expressions of X_1 and X_2 and it is also accomplished to decompose the measure μ_B as a product of the measure μ_1 on X_1 and some measure μ_2.

The cycle $\mathcal{C}$ above, associated to the minimal invariant set M,

$\mathcal{C} = \{ x_0, \tau_{l_1} x_0, \ldots, \tau_{l_{m-1}} \ldots \tau_{l_1} x_0 \}$ with $\tau_{l_m} \ldots \tau_{l_1} x_0 = x_0$, is decomposed as well.

If y_0 is the second component of x_0 and $h_1, \ldots, h_m$ are the second components of $l_1, \ldots, l_m$, we arrive at $\mathcal{C}_2 = \{ y_0, \tau_{h_1} y_0, \ldots, \tau_{h_{m-1}} \ldots \tau_{h_1} y_0 \}$. This cycle is proven to be extreme.

All these facts and partial results for the components are put to work in several computations, and the Fourier transforms of the decomposed measures are computed.

Let $F_1, \ldots, F_p$ be the list of all minimal compact invariant sets. For each k there is a reduced subspace V_k and some cycle $\mathcal{C}_k$ such that $F_k \subset \mathcal{R}_k := \mathcal{C}_k + V_k$, with mutually disjoint $\mathcal{R}_k$. One of the results give, for each k, a set $\Lambda(F_K) \subset \mathbb{Z}^d$ such that

$$h_{\mathcal{R}_k}(x) = \sum_{\lambda \in \Lambda(F_k)} |\hat{\mu}_B(x+\lambda)|^2 \quad (x \in \mathbb{R}^d).$$

By the Proposition 2.3

$$\sum_{k=1}^{p} h_{R_k}(x) = 1.$$

Hence

$$\sum_{k=1} \sum_{\lambda \in \Lambda(F_k)} |\hat{\mu}_B(x+\lambda)|^2 = 1.$$

Can λ appear twice here? Fix $\lambda_0 \in \cup_k \Lambda(F_k)$ and let $x = -\lambda_0$. One term in the sum is 1, since $\hat{\mu}_B(0) = 1$, and the others 0. Thus λ cannot appear twice. We also see that $\hat{\mu}_B(-\lambda_0 + \lambda) = 0$ for $\lambda \neq \lambda_0$, which implies that $e^{2\pi i \lambda_0 \cdot x}$ and $e^{2\pi i \lambda \cdot x}$ are orthogonal in $L^2(\mu_B)$.

Recall the notation $e_x(t) = e^{2\pi i x \cdot t}$. The double sum above now turns into

$$\|e_{-x}\|_2^2 = \sum_{\lambda \in \bigcup_{k=1}^{p} \Lambda(F_k)} |< e_{-x}|e_\lambda >|^2 \quad (x \in \mathbb{R}^d).$$

Hence the closed span of the family of functions $\{\ e_\lambda \mid \lambda \in \Lambda \}$, with $\Lambda = \cup_{k=1}^{p} \Lambda(F_k)$, contains all the functions e_x.

By the Stone-Weierstrass Theorem it contains $L^2(\mu_B)$. Thus, $\{e_\lambda \mid \lambda \in \Lambda\}$ forms an orthonormal basis for $L^2(\mu_B)$.

□

Remark 2.11. Suppose now that all the minimal invariant sets are finite. Then they will have to be extreme cycles. In this case, the subspaces V in Theorem 2.6 can be taken to be the trivial one $V = 0$; hence the reducibility condition is automatically satisfied. Combining this with the results from [DJ06] and [DJ07] we obtain that the measure μ_B is spectral and a spectrum can be obtained from the extreme cycles. We make this precise in the next theorem.

Theorem 2.12. *Suppose (B, L) is a Hadamard pair and all minimal compact invariant sets are finite (hence extreme cycles). Then the measure μ_B is spectral with spectrum Λ, where Λ is the smallest subset of $\mathbb{R}^d$ that contains $-C$ for all extreme cycles C, and which has the invariance property*

$$R^T\Lambda + L \subset \Lambda.$$

Example 2.13. We illustrate some of the notions introduced above with an example in dimension one. This is the first example of a fractal measure which admits an orthonormal Fourier basis, i.e., it is a spectral measure. The example was introduced by Jorgensen and Pedersen in [JP98]. Consider the function $\sigma(x) = 4x \bmod \mathbb{Z}$. Its inverse has two branches τ_0 and τ_2.

Let $I = [0,1]$ and define on I $\tau_0(x) = \frac{x}{4}$ and $\tau_2(x) = \frac{x+2}{4}$. These mappings form an affine IFS with $R = 4$. When infinitely iterated they give rise to a minimal invariant compact set X_4, now called the quarter Cantor set.

In this example the Hausdorff dimension d_H is easily computed as

$$\frac{\log(\text{number of replicas})}{\log(\text{magnification factor})} = \frac{\ln 2}{\ln 4} = \frac{1}{2}$$

Computing the spectrum: We can write $\tau_b(x) = R^{-1}(x+b)$ with $R = 4$ and $b \in B = \{0, 2\}$. We look for a Hadamard pair (B, L). L has to be of the form $\{0, l\}$, l an integer.

There is a unique invariant probability measure μ_B such that $\mu_B = \frac{1}{2}(\mu_B \circ \tau_0^{-1} + \mu_B \circ \tau_2^{-1})$ whose support is X_4. Important is also the function

$$m_B(x) := \frac{1}{N} \sum_{b \in B} e^{2\pi i b \cdot x} = \frac{1}{2}(1 + e^{2\pi i \cdot 2x}).$$

In general, the elements of the Hadamard matrix H are

$$\frac{1}{\sqrt{N}} (e^{2\pi i R^{-1} b \cdot l})_{b \in B, l \in L}.$$

In this case

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & e^{2\pi i \frac{1}{4} 2l} \end{bmatrix}$$

which is unitary iff l is an odd integer.

According to [DJ06], for each such set L there is a spectrum $\Lambda(l)$ and a basis ONB for $L^2(X_4, \mu_B)$. Here $ONB := \{e^{2\pi i \lambda x} : \lambda \in \Lambda(l)\}$.

To find $\Lambda(1)$ we look at the extreme cycles; the cycles where $|m_B(x)| = 1$ for each x in the cycle. For $L = \{0, 1\}$ there are very few possibilities; the only extreme cycle is $\{0\}$, the iterates of τ_0 when starting from 0. By the theory the spectrum then consists of the iterated images of the correspondence $x \to 4x + l, l \in \{0, 1\}$. Hence $\Lambda(1)$ is found to be

$$\{\sum_{k=0}^{n} 4^k l_k : l_k \in \{0,1\}\} = \{0, 1, 4, 5, 16, 17, 20, 21, 24, 25, \dots\}$$

for $n = 0, 1, 2, \dots$

In this case we shall confirm the orthogonality of the exponentials by a direct computation. The general relation

$$\int f d\mu = \frac{1}{N} \sum_{b \in B} \int f(\tau_B(x)) d\mu$$

for all bounded Borel functions translates into

$$\int f d\mu = \frac{1}{2}(\int f(x/4) d\mu + \int f(x/4 + 1/2) d\mu).$$

Then

$$\int e^{2\pi i t x} d\mu(x) = \frac{1}{2}(\int e^{(1/2)\pi i t x} d\mu(x) + \int e^{(1/2)\pi i t x} e^{\pi i t} d\mu(x)).$$

Let $\hat{\mu}(t) = \int e^{2\pi i t x} d\mu(x)$ and $H(t) = \frac{1}{2}(1 + e^{\pi i t})$. Then we have the neat relation

$$\hat{\mu}(t) = H(t)\hat{\mu}(\frac{t}{4}).$$

With the assumptions, set

$$P := \{l_0 + 4l_1 + 4^2 l_2 + \cdots : l_i \in \{0, 1\}, \text{finite sums}\}.$$

Then the functions $\{e_\lambda : \lambda \in P\}$ are mutually orthogonal in $L^2(X_4, \mu)$ where

$$e_\lambda(x) := e^{2\pi i \lambda x}.$$

Indeed let $\lambda = \sum 4^k l_k$, $\lambda' = \sum 4^k l_k'$ be points in P, and assume $\lambda \neq \lambda'$. Then

$$\int \overline{e_\lambda} e_{\lambda'} d\mu = \int e^{2\pi i(\lambda' - \lambda)x} d\mu(x)$$

$$= \hat{\mu}(\lambda' - \lambda)$$

$$= \hat{\mu}(l_0' - l_0 + 4(l_1' - l_1) + \dots)$$

$$= H(l_0' - l_0)\hat{\mu}(l_1' - l_1 + 4(l_2' - l_2) + \dots).$$

If $l_0 \neq l_0'$ then $H(l_0' - l_0) = 0$ since the matrix H is unitary. If not, there is a first n such that $l_n \neq l_n'$, and then

$$\hat{\mu}(\lambda' - \lambda) = \hat{\mu}(4^n(l_n' - l_n) + 4^{n+1}(l_{n+1}' - l_{n+1}) + \dots)$$

$$= H(l_n' - l_n)\hat{\mu}(l_{n+1}' - l_{n+1} + \dots) = 0$$

since $H(l_n' - l_n) = 0$.

Spectra have also been computed for l other than 1, see [DJ06, DHS09]. If

$$l \in \{5, 7, 9, 11, 13, 17, 19, 23, 29\}$$

then one can prove that

$\Lambda(l) = l\Lambda(1) = \{l\lambda : \lambda \in \Lambda(1)\}$. However, $\Lambda(3), \Lambda(15), \Lambda(27)$, and $\Lambda(63)$ are not so easily described. For example,

$$\Lambda(3) = \{l_0 + 4l_1 + \cdots + 4^n l_n : l_k \in \{0, 3\}\} \cup \{l_0 + 4l_1 + \cdots + 4^n l_n - 1 : l_k \in \{0, -3\}\},$$

for $n = 0, 1, 2, \ldots$.

CHAPTER 3
SIMPLE HADAMARD PAIRS

Definition 3.1. *We say that the Hadamard pair* (B, L) *is simple if there are no infinite minimal compact invariant sets.*

Theorem 3.2. *If* (B, L) *is simple then the measure* μ_B *is spectral.*

Proof. Follows from [DJ07] and the spectrum is described in Theorem 2.12.

□

Theorem 3.3. *Assume the eigenvalues of the matrix* R *are not rational. Then the Hadamard pair* (B, L) *is simple and the measure* μ_B *is spectral.*

Proof. We distinguish two cases: Suppose first that the attractor $X(L)$ is contained in a finite union of some translates of a subspace V of dimension 1.

Then, in this case since m_B is an entire function, m_B restricted to any compact subset of these translates of V, in particular to $X(L)$, will have only finitely many zeros. Then one can use the results in [DJ06] to conclude that μ_B is spectral.

In the other case, $X(L)$ is not contained in a finite union of translates of a subspace. Consider M, a minimal compact invariant set. We will prove that M has to be an extreme cycle. Suppose not. By Theorem 2.6, M is contained in a finite union $\mathcal{R}$ of translates of some proper subspace V, and $\mathcal{R}$ is invariant. If $V = \{0\}$, then M coincides with the finite cycle in Theorem 2.6 and every possible transition from a point $y = \tau_{l_q} \dots \tau_{l_1} x_0$ in the cycle leads to a point $\tau_{l_{q+1}} y$ in the cycle. Then $|m_B(\tau_{l q+1} y)| = 1$ and so M would be extreme. Hence we have obtained that V has to be one-dimensional if M is not extreme.

We claim that there exists $a \in \mathbb{R}^2$ such that $m_B(a + v) = 0$ for all $v \in V$.

First there must exists some $l \in L$ and some $x \in \mathcal{R}$ such that $\tau_l x \notin \mathcal{R}$. Otherwise, $X(L)$ is contained in $\mathcal{R}$ and this would contradict our assumption. Let $x = y + v$ with $v \in V$. We have $\tau_l x = S^{-1}(y + l) + S^{-1} v$, and since V is invariant for S, it follows that $S^{-1}(y + l)$ is not in V. But then for any $x' = y + v' \in \mathcal{R}$ with $v' \in V$, we obtain $\tau_l x'$ is not in V.

Since $\mathcal{R}$ is invariant this means that $m_B(\tau_l(y + v')) = 0$ for all $v' \in V$, and therefore

$$m_B(S^{-1}(l + y) + S^{-1} v') = 0, \text{ for all } v' \in V.$$

But $S^{-1} V = V$ so we obtain our claim.

On the other hand, m_B is $\mathbb{Z}^2$-periodic. So $m_B(a + v + k) = 0$ for all $v \in V, k \in \mathbb{Z}^2$. If V is not a rational subspace (i.e., it is not spanned by a vector with rational components), then $V + \mathbb{Z}^2$ is dense in $\mathbb{R}^2$, and that would imply that m_B is constant 0, a contradiction. Hence V must be a rational subspace. Let $(p, q)^T$ be a rational vector that spans V. Since V is invariant, $(p, q)^T$ is an eigenvector for S. But, as S has integer entries, this means that

S has a rational eigenvalue. Then, since the sum of the eigenvalues is the trace of S, so an integer, both of them have to be rational.

$\square$

Lemma 3.4. *Let R be an 2×2 integer matrix with rational eigenvalues. Then the eigenvalues are integers. Let λ be one of the eiganvalues.*

There exists an integer matrix M with $\det M = 1$ such that MRM^{-1} has the form

$$MRM^{-1} = \begin{bmatrix} \lambda & n \\ 0 & q \end{bmatrix}$$

Proof. Let

$$R = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The eigenvalues λ verify the characteristic equation

$$\lambda^2 - T\lambda + D = 0,$$

where $T = \text{Trace}(R) = a + d$ and $D = \det R = ad - bc$. So

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

If the eigenvalues are rational then $T^2 - 4D$ is a perfect square. Note also that $T^2 - 4D$ is even iff T is even. Therefore, if λ is rational then λ is an integer.

Let λ be one of the two eigenvalues. Then solving the equation $Rx = \lambda x$ we obtain that R has an eigevector with rational components. Multiplying by the common denominator, we see that R has an eigenvector $(x, y)^T$ with integer components, and dividing by the larges common divisor, we can assume x and y are mutually prime.Then there exists $z, t \in \mathbb{Z}$ such that $xt + yz = 1$. Let

$$M^{-1} := \begin{bmatrix} x & -z \\ y & t \end{bmatrix}$$

Then $\det M = 1$ and M is an integer matrix. Also

$$RM^{-1} = \begin{bmatrix} \lambda x & * \\ \lambda y & * \end{bmatrix} \text{ so } MRM^{-1} = \begin{bmatrix} \lambda & * \\ 0 & * \end{bmatrix}$$

□

Corollary 3.5. *Suppose the matrix R has a prime determinant. Then the measure μ_B is spectral.*

Proof. If R has prime determinant then R cannot have rational eigenvalues, because in this case, by Lemma 3.4, it follows that the eigenvalues are integers, and since their product is $\det R$, one of them has to be ± 1 since $\det R$ is prime. But R expansive and therefore the eigenvalues are irrational so μ_B is spectral by Theorem 3.3.

$\square$

The following theorem was proved in [CHR97] in connection with the study of self-affine tiles. It provides another case when the measure μ_B is spectral.

Theorem 3.6. *[CHR97] If B is a complete set of representatives for $\mathbb{Z}^2/R\mathbb{Z}^2$ then μ_B is a spectral measure, and the spectrum is a lattice.*

Using our techniques we are able to be more specific and describe the spectrum of μ_B in the case when the system (B, L) is also simple.

Theorem 3.7. *Assume B is a complete set of representatives modulo $R\mathbb{Z}^2$ (hence also L is a complete set of representatives modulo $R^T\mathbb{Z}^2$). Assume in addition that (R, B, L) is simple. Let $\mathcal{C}$ be the set of all extreme cycle points and let Λ be the smallest subset of $\mathbb{R}^2$ that contains $-\mathcal{C}$ and with the property $R^T\Lambda + L \subset \Lambda$. Let Γ be the additive subgroup of $\mathbb{R}^2$ generated by $\mathcal{C}$ and $\mathbb{Z}^2$. Assume that Γ is a discrete lattice. Then $\Lambda = \Gamma$ and Γ is a spectrum for μ_B.*

Proof. Take $c \in \mathcal{C}$. Since c is a cycle point for $(\tau_l)_{l\in L}$ there exist some $c' \in \mathcal{C}$ and $l_0 \in L$ such that $\tau_{l_0}c' = c$. Then $R^Tc = c' + l_0$.

Since any point in Γ is of the form

$$\gamma = a + \sum_{i=1}^{p} m_i c_i,$$

for some $a \in \mathbb{Z}^2$, $c_i \in \mathcal{C}$ and $m_i \in \mathbb{Z}$ it follows that $R^T\gamma + l$ will also be in Γ. So $R^T\Gamma + L \subset \Gamma$. This implies that $\Lambda \subset \Gamma$.

To prove the reverse inclusion, we claim that for any $\gamma \in \Gamma$, there exists $l \in L$ such that $\tau_l\gamma \in \Gamma$.

To see this, take

$$\gamma = a + \sum_{i=1}^{p} m_i c_i$$

as above.

Since L is a complete set of representatives modulo $R^T\mathbb{Z}^2$, there exist $a' \in \mathbb{Z}^2$ and $l_a \in L$ such that $a = R^Ta' + l_a$. Also, for each $i \in \{1, \ldots, p\}$, since c_i is a cycle point, there exist $c_i' \in \mathcal{C}$ and $l_i \in L$ such that $\tau_{l_i}c_i = c_i'$, which implies that $c_i = R^Tc_i' - l_i$. Then

$$\gamma = R^T(a' + \sum_{i=1}^{p} m_i c_i') + l_a - \sum_{i=1}^{p} m_i l_i.$$

Using again that L is a complete set of representatives, we have that there exist $l \in L$ and $k \in \mathbb{Z}^2$ such that

$$l_a - \sum_{i=1}^{p} m_i l_i = R^Tk - l.$$

Let $\gamma' = a + k + \sum_{i=1}^{p} m_i c_i \in \Gamma$. We have $\gamma = R^T\gamma' - l$, so $\tau_l\gamma = \gamma'$.

This proves our claim.

Now take $\gamma_0 \in \Gamma$. Then $-\gamma_0 \in \Gamma$, thus there exist $l_1 \in L$ such that $\tau_{l_1}(-\gamma_0) =: -\gamma_1 \in \Gamma$. This implies that $\gamma_0 = R^T \gamma_1 + l_1$. By induction, we can find $l_1, \ldots, l_n$ such that

$-\gamma_n := \tau_{l_n} \ldots \tau_{l_1}(-\gamma_0) \in \Gamma$ and this means also that $\gamma_{n-1} = R^T \gamma_n + l_n$.

But $\tau_{l_n} \ldots \tau_{l_1}(-\gamma_0)$ converges to the attractor X_L. Therefore if we take a ball $B(0, r)$ that contains X_L, we have $\gamma_n = -\tau_{l_n} \ldots \tau_{l_1} \gamma_0 \in B(0, r) \cap \Gamma$ for n large.

Since Γ is discrete, the set $B(0, r) \cap \Gamma$ is discrete. So $-\gamma_n = \tau_{l_n} \ldots \tau_{l_1} \gamma_0$ will land in a cycle for the IFS $(\tau_l)_{l \in L}$.

We claim that this is an extreme cycle.

To see that, note first for any extreme cycle point c one has

$$N = \left| \sum_{b \in B} e^{2\pi i b \cdot c} \right| \leq \sum_{b \in B} |e^{2\pi i b \cdot c}| = N.$$

Hence we must have equality in the triangle inequality, and since $0 \in B$, we get that $e^{2\pi i b \cdot c} = 1$, which means that $b \cdot c \in \mathbb{Z}$.

Then for any $x \in \mathbb{Z}^2$,

$$m_B(x + c) = \frac{1}{N} \sum_{b \in B} e^{2\pi i b \cdot (x+c)} = \frac{1}{N} \sum_{b \in B} e^{2\pi i b \cdot x} = m_B(x).$$

So c is a period for m_B.

Then for any $\gamma \in \Gamma$, with γ of the form

$$\gamma = a + \sum_{i=1}^{p} m_i c_i$$

as above, with $a \in \mathbb{Z}^2$ and $c_i \in \mathcal{C}$, we have

$$|m_B(\gamma)| = |m_B(a + \sum_{i=1}^{p} m_i c_i)| = |m_B(a)| = 1.$$

This shows that the cycle where $-\gamma_n$ lands is an extreme cycle. So $\gamma_n \in -\mathcal{C} \subset \Lambda$ for some large n. Then, iterating back we have $\gamma_{n-1} = R^T\gamma_n + l_n \in \Lambda$. By induction, we get $\gamma_0 \in \Lambda$. Therefore $\Gamma \subset \Lambda$. Also, Theorem 2.12 shows that Λ is a spectrum for μ_B and this proves the last statement.

□

Corollary 3.8. *Let (R, B, L) be a Hadamard system. Let $\mathcal{C}$ be the set of all extreme cycle point and let Γ be the additive subgroup generated by $\mathcal{C}$ and $\mathbb{Z}^2$. Then $R^T\Gamma + L \subset \Gamma$ and every $\gamma \in \Gamma$ is a period for m_B, we have*

$$m_B(x + \gamma) = m_B(x), \quad (x \in \mathbb{R}^2),$$

and $b \cdot \gamma \in \mathbb{Z}$ for all $b \in B$.

Proof. Everything is contained in the proof of Theorem 3.7.

□

CHAPTER 4
NON-SIMPLE HADAMARD PAIRS

Before going into detail, first we have to prove the uniqueness of an important subspace, namely the subspace associated to infinite minimal compact invariant sets as in Theorem 2.6, see Lemma 4.6. We recall some facts about invariant sets and we prove some additional properties.

When (B, L) form a Hadamard pair, recall the notation

$$\tau_l(x) = S^{-1}(x + l) \text{ with } l \in L \text{ and } S = R^T,$$

and for a cycle starting at $x_0 : x_0 =: \wp(l_{p-1}, \dots, l_0)$ when the maps τ_{l_k} are used: $\tau_{l_k} x_k = x_{k+1}$ for $k \in \{ 0, \dots, p-1\}$ and $x_p := x_0$. In this situation we say that x_0 is a periodic point and that the cycle is extreme if $| m_B(x_i) |= 1$ for all $i \in \{ 0, \dots, p-1\}$.

Lemma 4.1. *[CCR96] Let $v = \wp(\gamma_m, \dots, \gamma_1)$ be a periodic point. Suppose there exists $l_1, \dots, l_s \in L$ such that $\tau_{l_s} \dots \tau_{l_1} v$ is again a periodic point. Then $l_1 = \gamma_m, l_2 = \gamma_{m-1}, \dots,$ so $\tau_{l_s} \dots \tau_{l_1} v$ belongs to the cycle generated by v.*

Definition 4.2. *Suppose (R, B, L) is a spectral system. We are working with the dual IFS $(\tau_l)_{l\in L}$. We say that a transition $x \to \tau_l x$ is possible if $|m_B(\tau_l x)| \neq 0$. We say that a set M*

is invariant, if for every $x \in M$ and every possible transition $x \to \tau_l x$, the point $\tau_l x$ is also in M.

Lemma 4.3. *[CCR96] If M is a compact invariant set, then one of the following conditions holds:*

(i) M contains an extreme cycle.

(ii) M contains a non-isolated cycle.

Definition 4.4. *We say that a union $\mathcal{R}$ of translates of a one-dimensional subspace V, $\mathcal{R} = \{x_0 + V, \ldots, x_m + V\}$ is associated to minimal invariant sets if $\mathcal{R}$ is invariant and contains an infinite compact minimal invariant set M. We also say that the subspace V is associated to minimal invariant sets.*

Lemma 4.5. *Let M be an infinite compact minimal invariant set. Then M is a perfect set hence uncountable.*

Proof. By Lemma 4.3, there exists a cycle C in M. At least one of the points $x_0 \in C$ has a possible transition to a point outside the cycle, $y_0 = \tau_{l_0} x_0$. Otherwise, the cycle C is extreme, and since M is minimal $M = C$, but this would contradict the fact that M is infinite. The point y_0 is in M since M is invariant, and since M is minimal, $M = \overline{\mathcal{O}(y_0)}$.

Now take a point $x \in M$. There exist points of the form $\tau_{l_n} \ldots \tau_{l_1} y_0$ as close to x as we want. Since $y_0 = \tau_{l_0} x_0 \notin C$ and x_0 is cyclic, by Lemma 4.1, it follows that these points can be chosen distinct. This proves that x is not isolated in M hence M is perfect. Since it is also compact in $\mathbb{R}^2$, it follows that M is also uncountable.

$\square$

Lemma 4.6. *Suppose there are two union of translates* $\mathcal{R} = \{x_0 + V, \dots, x_p + V\}$, $\mathcal{R}' = \{y_0 + V', \dots, y_{p'} + V'\}$ *which are both associated to minimal invariant sets. Then* $V = V'$.

Proof. Let M and M' be the infinite minimal compact invariant sets associated to $\mathcal{R}$ and $\mathcal{R}'$ respectively.

Suppose the one-dimensional subspaces V and V' are distinct. Then $\mathcal{R} \cap \mathcal{R}'$ is a finite invariant set (any two non-parallel lines intersect in a single point). Hence it has to contain an extreme cycle $\wp(\gamma_1, \dots, \gamma_m)$, and any possible transition from a point in $\mathcal{R} \cap \mathcal{R}'$ will eventually end in an extreme cycle.

By drawing a picture the truth of the fact that

$$\bigcup_{a \in \mathcal{R} \cap \mathcal{R}'} (a + V) = \mathcal{R}$$

becomes obvious.

Now take a a point in $\mathcal{R} \cap \mathcal{R}'$ and let $l_1, l_2, \dots \in L$ give possible transitions from a to $\tau_{l_1} a, \tau_{l_2} \tau_{l_1} a, \dots$

For $r \in \mathbb{N}$ consider the functions

$$f_{r,a}(v) = m_B(\tau_{l_r} \dots \tau_{l_1}(a+v)), \quad (v \in V).$$

We have that f_r is analytic and $f_r(0) \neq 0$. Therefore each function f_r has only finitely many zeroes on any compact subset of V.

Take $\omega_0 \in M$ with the property that $m_B(\tau_{l_r} \dots \tau_{l_1}(a+\omega_0)) = f_{r,a}(\omega_0) \neq 0$, for all r. This is possible because the zeroes of the functions f_r are at most countable and the set M is infinite and perfect, hence uncountable.

Then the transitions $a + \omega_0 \mapsto \tau_{l_1}(a+\omega_0) \mapsto \tau_{l_2}\tau_{l_1}(a+\omega_0) \mapsto \dots$ are all possible.

Since $\mathcal{R}$ is invariant and $a + \omega_0 \in \mathcal{R}$ we have that $\tau_{l_r} \dots \tau_{l_1}(a+\omega_0) \in \mathcal{R}$.

On the other hand $\operatorname{dist}(\tau_{l_r} \dots \tau_{l_1}(a+w_0), \tau_{l_r} \dots \tau_{l_1}(a))$ converges to 0, so $\tau_{l_r} \dots \tau_{l_1}(a+w_0)$ converges to the extreme cycle. This implies that the extreme cycle is contained in M, but this contradicts the fact that M is minimal and infinite.

$\square$

This section is about non-simple Hadamard pairs. In this case, any infinite minimal compact invariant set is contained in a union of translates of some one-dimensional subspace (Theorem 2.6). Moreover this subspace is unique (Lemma 4.6) and we call it the *subspace associated to minimal invariant sets* or *SAMIS*. We prove that, if the Hadamard pair is non-simple, then the system (R, B, L) is conjugate to a spectral system (R', B', L') where the matrix is lower triangular, and its SAMIS is $\mathbb{R} \times \{0\}$ (Proposition 4.9). In addition, the set L' can be chosen to have some extra properties (Proposition 4.13).

Thus,we have the following result:

- To solve the Conjecture 1.7 in dimension two, it is enough to study spectral systems (R, B, L) that satisfy (4.1)–(4.4).

Theorems 4.14, 4.15, 4.17 give various conditions that imply that μ_B is spectral.

Definition 4.7. *We say that two affine IFSs (R, B) and (R', B') are conjugate (through M) if there exists an integer matrix M with* $\det M = \pm 1$ *such that*

$$R' = MRM^{-1} \text{ and } B' = MB.$$

If (R, B, L) and (R', B', L') is a spectral system, then we say that they are conjugate through M if in addition $L' = (M^T)^{-1}L$.

The next proposition follows from a simple computation.

Proposition 4.8. *Let (R, B) and (R', B') be two conjugate affine IFSs through the matrix M. Then μ_B is a spectral measure with spectrum Λ iff $\mu_{B'}$ is spectral with spectrum $(M^T)^{-1}\Lambda$.*

Proposition 4.9. *Suppose (B, L) is not simple. Then the spectral system (R, B, L) is conjugate to a spectral system (R', B', L') such that R' is lower triangular and its SAMIS is $\mathbb{R} \times \{0\}$.*

Proof. From Theorem 3.3, we know the eigenvalues have to be rational. From Lemma 3.4, the eigenvalues are actually integers, the SAMIS V is actually a rational eigenspace, and we can conjugate this affine IFS to another one in such a way that the matrix R becomes lower triangular, and this eigenspace becomes $\mathbb{R} \times \{0\}$.

$\square$

Remark 4.10. By Theorem 3.3, if the eigenvalues of R are irrational then (B, L) is simple, and the measure μ_B is spectral. If the eigenvalues of R are rationals then, by Lemma 3.4 the eigenvalues are integers and we have two cases. If the pair (B, L) is simple, then the measure μ_B is spectral, by Theorem 3.2. If (B, L) is not simple, then by Proposition 4.9, the spectral system is conjugated to one that has a lower triangular matrix, and whose subspace associated to invariant sets is $\mathbb{R} \times \{0\}$. Therefore, in order to settle the conjecture for the case of dimension $d = 2$ it is enough to focus on the case when R is of the form

$$R = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$$

and the subspaces associated to invariant sets is $\mathbb{R} \times \{0\}$.

Lemma 4.11. *Suppose (B, L) is a Hadamard pair, and let $L' \subset \mathbb{Z}$, $0 \in L'$, $\#L = \#L' = N$. Assume that for every $l \in L$ there exist a unique $l'(l)$ in L' such that l is congruent to $l'(l)$ modulo S. Then (B, L') is a Hadamard pair.*

Proof. Since the sets L and L' have the same cardinality N, it follows that the map $l \mapsto l'(l)$ is a bijection. Take $l_1 \neq l_2$ in L. Then

$l_1 = l'(l_1) + Sk_1$, $l_2 = l'(l_2) + Sk_2$ for some $k_1, k_2 \in \mathbb{Z}^2$. Then we have:

$$\sum_{b\in B} e^{2\pi i b\cdot S^{-1}(l'(l_1)-l'(l_2))} = \sum_{b\in B} e^{2\pi i b\cdot (S^{-1}(l_1-l_2)-(k_1-k_2))} = \sum_{b\in B} e^{2\pi i b\cdot S^{-1}(l_1-l_2)} = 0.$$

□

Lemma 4.12. *If (B, L) is a Hadamard pair then no two distinct elements of B are congruent modulo R and no two distinct elements of L are congruent modulo R^T.*

Proof. Suppose that $b, b' \in B$ satisfy $b - b' = Rm$ for some $m \in \mathbb{Z}^d$, then

$$e^{2\pi i R^{-1} b\cdot l} = e^{2\pi i R^{-1} b'\cdot l}$$

for all $l \in L$ since $L \subset \mathbb{Z}^d$.

This means that the rows in the Hadamard matrix labeled b and b' cannot be orthogonal.

□

Proposition 4.13. *Assume*

$$R = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$$

and suppose (B, L) is not simple, and its SAMIS is $V := \mathbb{R} \times \{0\}$. Then there exists L' such that

(i) (B, L') is a Hadamard pair;

(ii) $L' \subset \{0, \dots, |a| - 1\} \times \{0, \dots, |d| - 1\}$;

(iii) The hypothesis "(H) modulo V" is satisfied (relative to L').

Proof. Assume, without loss of generality and for the rest of the section, that a and d are nonnegative. We use Lemma 4.11 and replace each $l \in L$ by some element in $\{0, \dots, a-1\} \times \{0, \dots, d-1\}$ which is congruent to it modulo S. Take $l \in L$, $l \neq 0$. Let $l = (l_1, l_2)^T$. Let $q = l_2 \bmod d$. Then there exists $y \in \mathbb{Z}$ such that $q - l_2 = dy$. Then take $p = cy + l_1 \bmod a$. Then a simple computation shows that $l'(l) := (p, q)^T$ is congruent to l modulo S. Define $L' := \{l'(l) : l \in L\}$. With Lemma 4.11, (i) follows and (ii) is clear too.

For (iii), suppose $\tau_{\epsilon_1} \dots \tau_{\epsilon_p} 0 - \tau_{\eta_1} \dots \tau_{\eta_p} 0 \in V$ with $\epsilon_i, \eta_i \in L'$. This means that

$$S^{-1}(\epsilon_1 - \eta_1) + \dots + S^{-p}(\epsilon_p - \eta_p) \in V.$$

Since V is invariant for S this implies

$$\epsilon_p - \eta_p + S(\epsilon_{p-1} - \eta_{p-1}) + \dots + S^{p-1}(\epsilon_1 - \eta_1) \in V.$$

But this means that the second component of $\epsilon_p - \eta_p$ is a multiple of d. From (ii) it follows that the second components of ϵ_p and η_p are equal so $\epsilon_p - \eta_p \in V$. Then, $S^{-1}(\epsilon_p - \eta_p)$ is in V so we can reduce the problem to $p-1$ and use induction to conclude that $\epsilon_i - \eta_i \in V$ for all i.

□

Proposition 4.13 allows us to make the following assumptions which we assume to hold throughout this section:

$$R = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \tag{4.1}$$

Either (B, L) is simple or it is a non-simple Hadamard pair and its SAMIS is $V = \mathbb{R} \times \{0\}$;
(4.2)

$$L \subset \{0, \dots, a-1\} \times \{0, \dots, d-1\}; \tag{4.3}$$

The hypothesis "(H) modulo V" is satisfied. (4.4)

Theorem 4.14. *Assume (4.1)–(4.4) hold. Define $B_1 := \mathrm{proj}_1(B) = \{b_1 : (b_1, b_2) \in B$ for some $b_2\}$; for $b_1 \in B_1$ let $B_2(b_1) := \{b_2 : (b_1, b_2) \in B\}$ and define the Laurent polynomials*

$$p_{b_1}(z) = \sum_{b_2 \in B_2(b_1)} z^{b_2}, \quad (b_1 \in B_1).$$

Suppose the polynomials p_{b_1}, $b_1 \in B_1$ have no common zero of the form $e^{2\pi i \frac{k}{d(d^j - 1)}}$ with $k \in \mathbb{Z}, j \in \mathbb{N}$. Then the measure μ_B is spectral.

Proof. We prove (B, L) is simple hence μ_B is spectral, by Theorem 2.12. If not, since $\mathbb{R} \times \{0\}$ is its SAMIS then, there exists a cycle point $x_0 = (x_1, y_1)^T$ as in Theorem 2.6 (iii). Then it is easy to see that y_1 is a cycle point for the IFS $\tau_{l_2} : x \mapsto d^{-1}(x + l_2)$ where $l_2 \in \text{proj}_2(L)$.

This means that for some $\eta_1, \ldots, \eta_j \in \text{proj}_2(L)$ we have

$$y_1 = \tau_{\eta_j} \ldots \tau_{\eta_1} y_1 = d^{-1}\eta_j + \cdots + d^{-j}\eta_1 + d^{-j}y_1.$$

Then

$$y_1 = \frac{\eta_1 + \cdots + d^{j-1}\eta_j}{d^j - 1}.$$

Consider now the union $\mathcal{R}$ of translates of $V = \mathbb{R} \times \{0\}$ as in Theorem 2.6(iii). For one of the translates, which we can relabel $x_0 + V = \{(x, y_1)^T : x \in \mathbb{R}\}$, there exists some $l = (l_1, l_2)^T$ such that $\tau_l(x_0 + V) = \{(x, d^{-1}(y_1 + l_2))^T : x \in \mathbb{R}\}$ is not contained in $\mathcal{R}$, hence it is disjoint from it. Otherwise, the whole attractor $X(L)$ will be contained in $\mathcal{R}$, and in this case m_B has only finitely many zeroes on $X(L)$ so we can use the results in [DJ06].

Since $\mathcal{R}$ is invariant, this means that $m_B((x, d^{-1}(y_1 + l_2))^T) = 0$. Then

$$\sum_{b_1 \in B_1} \sum_{b_2 \in B_2(b_1)} e^{2\pi i (b_1 x + b_2 d^{-1}(y_1 + l_2))} = 0, \quad (x \in \mathbb{R})$$

This implies that for all $b_1 \in B_1$

$$\sum_{b_2 \in B_2(b_1)} e^{2\pi i b_2 d^{-1}(y_1 + l_2)} = 0$$

and this contradicts the nonexistence of a common zero for the polynomials p_{b_1} of the given form. The contradiction show that (B, L) has to be simple so the measure is spectral.

□

Theorem 4.15. *Suppose* $\det R$ *is a product of 2 (not necessarily distinct) prime numbers. Then* μ_B *is a spectral measure.*

Proof. We can assume that (4.1)–(4.4) hold. Also, with the notation in Theorem 4.14 we can assume there exists $b_1 \in B_1$ such that $\#B_2(b_1) \geq 2$; otherwise $p_b(z)$ has only one term so it cannot have zeroes on the unit circle, and the result follows from Theorem 4.14.

Define $L_2 := \mathrm{proj}_2(L)$ and for $l_2 \in L_2$ let $L_1(l_2) := \{l_1 : (l_1, l_2)^T \in L\}$.

Lemma 4.16. *We can assume there exist* $l_2 \in L_2$ *such that* $\#L_1(l_2) \geq 2$*; otherwise the measure* μ_B *is spectral.*

Proof. Suppose $\#L_1(l_2) = 1$ for all $l_2 \in L_2$. Take $\mathcal{R}$ as in Theorem 2.6(iii). We know that each possible transition from a point $(x, y_1)^T$ in $\mathcal{R}$ will lead to a point $(x', y_2)^T$ in $\mathcal{R}$ and y_2 is independent of x. Suppose this transition is done using a map τ_{l_0} with $l_0 = (l_1, l_2)^T$. The assumption then implies that, using instead $\tau_{l'}$ with $l' \neq l$, the second coordinate of this point will not be y_2.

But this contradicts Theorem 2.6(iii). So for all $l' \neq l_0$, $\tau_{l'}(x, y_1)^T$ is outside $\mathcal{R}$. Therefore $m_B(\tau_{l'}(x, y_1)^T) = 0$.

But, since

$$\sum_{l \in L} |m_B(\tau_l(x, y_1)^T)|^2 = 1,$$

this implies that $|m_B(\tau_{l_0}(x, y_1)^T)| = 1$, and using the triangle inequality and the fact that $0 \in B$, this implies that $b \cdot (x, y_1) \in \mathbb{Z}$. Since x is arbitrary, this implies in turn that $B_1 = \{0\}$, which means that $X(B)$ is actually one-dimensional, contained in $\{0\} \times \mathbb{R}$, and we can apply the results in [DJ06].

□

Resuming the proof of the theorem, since $\det R$ is a product of two primes, we can assume a and d are prime.

First, take $b_1 \in B_1$, such that there exist $b_2 \neq b_2'$ in $B_2(b_1)$. Using Lemma 4.12, b_2 and b_2' are not congruent modulo d. Apply the Hadamard property to the rows corresponding to $(b_1, b_2), (b_1, b_2') \in B$:

$$\sum_{(l_1, l_2)^T \in L} e^{2\pi i \frac{b_2 - b_2'}{d} \cdot l_2} = 0.$$

Then

$$p_{L_2}(e^{2\pi i\frac{b_2-b_2'}{d}}) = \sum_{l_2\in L_2} \#L_1(l_2)e^{2\pi i\frac{b_2-b_2'}{d}\cdot l_2} = 0,$$

where $p_{L_2}(z) = \sum_{l_2\in L_2} \#L_1(l_2)z^{l_2}$. But since p_{L_2} has integer coefficients it follows that p_{L_2} is divisible by the minimal polynomial for $e^{2\pi i\frac{b_2-b_2'}{d}}$ which is the cyclotomic polynomial $\Phi_d(z) = 1 + z + \cdots + z^{d-1}$, since d is prime. But $L_2 \subset \{0, \ldots, d-1\}$ according to our assumptions. Therefore p_{L_2} is a constant multiple of Φ_d. This means that $L_2 = \{0, \ldots, d-1\}$ and $\#L_1(l_2)$ is independent of $l_2 \in L_2$. We also have $d \cdot \#L_1(l_2) = N$.

Now, using Lemma 4.16, take $l_2 \in L_2$ and $l_1 \neq l_1'$ in $L_1(l_2)$. Apply the Hadamard property to the columns corresponding to (l_1, l_2) and (l_1', l_2) in L:

$$\sum_{(b_1,b_2)^T\in B} e^{2\pi i b_1\cdot\frac{l_1-l_1'}{a}} = 0.$$

Then

$$p_{B_1}(e^{2\pi i\frac{l_1-l_1'}{a}}) = \sum_{b_1\in B_1} \#B_2(b_1)e^{2\pi i(b_1 \bmod a)\cdot\frac{l_1-l_1'}{a}} = \sum_{b_1\in B_1} \#B_2(b_1)e^{2\pi i b_1\cdot\frac{l_1-l_1'}{a}} = 0,$$

where $p_{B_1}(z) = \sum_{b_1\in B_1} \#B_2(b_1)z^{b_1 \bmod a}$. We might have two different b_1, b_1' in B_1 such that $b_1 \equiv b_1' \bmod a$.

We write further

$$p_{B_1}(z) = \sum_{k=0}^{a-1}\left(\sum_{b_1\in B_1, b_1 \bmod a=k} \#B_2(b_1)\right) z^k.$$

Since p_{B_1} has integer coefficients, it follows that p_{B_1} is divisible by the minimal polynomial for $e^{2\pi i\frac{l_1-l_1'}{a}}$ which is the cyclotomic polynomial $\Phi_a(z) = 1 + z + \cdots + z^{a-1}$, since a is prime. Therefore p_{B_2} is a constant multiple of Φ_a. This means that $B_1 \bmod a = \{0, \ldots, a-1\}$ and $\sum_{b_1 \bmod a = k} \#B_2(b_1)$ is independent of $k \in \{0, \ldots, a-1\}$. Hence

$$a \cdot \left(\sum_{b_1 \bmod a = k} \#B_2(b_1) \right) = \sum_{i=0}^{a-1} \left(\sum_{b_1 \bmod a = i} \#B_2(b_1) \right) = \sum_{b_1 \in B_1} \#B_2(b_1) = \#B = N.$$

We have $d \cdot \#L_1(l_2) = N = a \cdot (\sum_{b_1 \bmod a = k} \#B_2(b_1))$. If $a \neq d$, then a divides $\#L_1(l_2)$ and since $N \leq ad$ it follows that $N = ad$. But this implies that B is a complete set of representatives for $\mathbb{Z}^2/R\mathbb{Z}^2$. Using Theorem 3.6 it follows that μ_B is spectral.

If $a = d$ then take $(l_1, l_2) \neq 0$ in L. Using the Hadamard property we have

$$0 = \sum_{b \in B} e^{2\pi i R^{-1} b \cdot l} = \sum_{(b_1, b_2) \in B} e^{2\pi i \frac{ab_1 l_1 - cb_1 b_2 + ab_2 l_2}{a^2}}.$$

Thus, we have a sum of $\#B = N$ roots of order a^2 of unity. Since a is prime, using [LL00] we get that N is divisible by a. Therefore $N = a$ or $N = a^2$. If $N = a^2 = \det R$, then B is a complete set of representatives for $\mathbb{Z}^2/R\mathbb{Z}^2$, and with Theorem 3.6, we get that μ_B is spectral.

If $N = a$ then we obtain that $\#L_1(l_2) = 1$ for all $l_2 \in L_2$. But this contradicts the assumption of Lemma 4.16, so μ_B is spectral.

$\square$

Theorem 4.17. *Assume* (4.1)–(4.4) *hold. Define* $B_1 = \mathrm{proj}_1(B)$*,* $B_2(b_1) := \{b_2 : (b_1, b_2)^T \in B\}$ *for* $b_1 \in B_1$*,* $L_2 := \mathrm{proj}_2(L)$*,* $L_1(l_2) := \{l_1 : (l_1, l_2)^T \in L\}$ *for* $l_2 \in L_2$*. If* $\#B_2(b_1) = N_2$

independent of $b_1 \in B_1$ *and* $\#L_1(l_2) = N_1$ *independent of* $l_2 \in L_2$ *and* $N_1 N_2 = N$ *then* μ_B *is a spectral measure.*

Proof. Theorem 4.18 below guarantees that the IFS $\tau_{b_1}(x) = a^{-1}(x + b_1)$, $b_1 \in B_1$ has no overlap. Then the result follows from [DJ07, Proposition 3.2 and Theorem 3.8].

□

Theorem 4.18. *Let* R *be an integer,* $|R| > 1$ *and let* $\mathcal{D}$ *be a set of integers such that no two distinct elements of* $\mathcal{D}$ *are congruent modulo* R*. Consider the IFS* $\tau_d(x) = R^{-1}(x + d)$*,* $d \in \mathcal{D}$ *and let* $X(\mathcal{D})$ *be its attractor and* $D := \log_{|R|}(\#\mathcal{D})$*. Then the Hausdorff measure of* $X(\mathcal{D})$ *satisfies* $0 < \mathcal{H}^D(X(\mathcal{D})) < \infty$*, the invariant measure* $\mu_{\mathcal{D}}$ *of the IFS* $(\tau_d)_{d\in\mathcal{D}}$ *is the renormalized Hausdorff measure* $\mathcal{H}^D$ *restricted to* $X(\mathcal{D})$ *and the measure* $\mu_{\mathcal{D}}$ *has no overlap.*

Proof. Let $N := \#\mathcal{D}$. Since the elements of $\mathcal{D}$ are incongruent modulo R we can enlarge it to a set $\tilde{\mathcal{D}} \supset \mathcal{D}$ which is a complete set of representatives for $\mathbb{Z}/R\mathbb{Z}$. We denote by $X(\tilde{\mathcal{D}})$ the attractor of the IFS associated to $\tilde{\mathcal{D}}$.

By [Ban91, Theorem 1], the attractor $X(\tilde{\mathcal{D}})$ has non-empty interior $\mathrm{int}(X(\tilde{\mathcal{D}})) \neq \emptyset$.

Then

$$\cup_{d\in\mathcal{D}}\tau_d(\mathrm{int}(X(\tilde{\mathcal{D}}))) \subset \cup_{d\in\tilde{\mathcal{D}}}\tau_d(\mathrm{int}(X(\tilde{\mathcal{D}}))) \subset \mathrm{int}\left(\cup_{d\in\tilde{\mathcal{D}}}\tau_d(X(\tilde{\mathcal{D}})\right)) = \mathrm{int}(X(\tilde{\mathcal{D}})).$$

This means that the Open Set Condition is satisfied for the IFS $(\tau_d)_{d\in\mathcal{D}}$.

Using [Hut81, Theorem 5.3.1 (ii)], we can conclude that $0 < \mathcal{H}^D(X(\mathcal{D})) < \infty$.

For any Borel subset E of $\mathbb{R}$ and $d \in D$, we have

$\mathcal{H}^D(\tau_d^{-1}(E)) = \mathcal{H}^D(RE-d) = \mathcal{H}^D(RE) = R^D\mathcal{H}^D(E) = N\mathcal{H}^D(E)$. Similarly $\mathcal{H}^D(\tau_d(E)) = \frac{1}{N}\mathcal{H}^D(E)$.

We have

$$\mathcal{H}^D(X(\mathcal{D})) = \mathcal{H}^D\left(\cup_{d\in\mathcal{D}}\tau_d(X(\mathcal{D}))\right) \leq \sum_{d\in\mathcal{D}}\mathcal{H}^D(\tau_d(X(\mathcal{D}))) = \frac{1}{N}\cdot N\mathcal{H}^D(X(\mathcal{D})) = \mathcal{H}^D(X(\mathcal{D})).$$

Since we must have equality, this implies that $\mathcal{H}^D(\tau_d(X(\mathcal{D}))\cap\tau_{d'}(X(\mathcal{D}))) = 0$ for distinct $d, d' \in \mathcal{D}$, which means that there is no overlap (other than on sets of measure zero).

Then we also have, for any Borel set E:

$$\mathcal{H}^D(E\cap X(\mathcal{D})) = \sum_{d\in\mathcal{D}}\mathcal{H}^D(E\cap\tau_d(X(\mathcal{D}))) = \sum_{d\in\mathcal{D}}\frac{1}{N}\mathcal{H}^D(\tau_d^{-1}(E)\cap X(\mathcal{D})).$$

This proves that $\mathcal{H}^D$ restricted to $X(\mathcal{D})$ is invariant for the IFS, but since $\mu_{\mathcal{D}}$ is the unique measure with this property, all statements in the theorem have been proven.

$\square$

CHAPTER 5
EXAMPLES

We begin by studying some examples where the matrix R has determinant 2. Such matrices were completely classified in [LW95]. We include the result here.

Now, introduce $C_1 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$, $C_2 = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$, $C_3 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $C_4 = \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}$.

We say that two matrices A and B are conjugate if there exists a matrix $P \in M_2(\mathbb{Z})$ with $|\det P| = 1$ such that $PAP^{-1} = B$. We then write $A \sim B$. For the general case $|\det A| = 2$ we have the following lemma from [LW95].

Lemma 5.1. *[LW95]*

Let $A \in M_2(\mathbb{Z})$ be expansive. If $\det A = -2$, then A is conjugate to C_1. If $\det A = 2$, then A is conjugate to

one of the matrices $C_2, \pm C_3, \pm C_4$.

To gain a better understanding of these matrices we shall need the full proof.

Proof.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Following [LW95] we define the *weight* $p(A)$ of A to be $p(A) := -a_{11}a_{22}$.

The assumptions $|\lambda_1| > 1$, $|\lambda_2| > 1$, $|\lambda_1\lambda_2| = 2$ imply that $|\lambda_1| < 2$, $|\lambda_2| < 2$

and then $|a_{11} + a_{22}| = |\lambda_1 + \lambda_2| < 4$. Since the common sum is an integer $|a_{11} + a_{22}| \leq 3$. But 3 is

not possible, so we actually have $|a_{11} + a_{22}| \leq 2$. Squaring this we obtain $a_{11}a_{22} \leq 1$, which can be written

as $p(A) \geq -1$.

We will use induction on the weight $p(A)$ to prov that $A \sim B$ for some matrix

$$B = \begin{bmatrix} 0 & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Base case $p(A) = -1$. In this case $|a_{11}| = |a_{22}| = 1$ and $a_{12}a_{21} = -p(A) - \det A = -1$ or 3, hence $|a_{12}| = 1$ or

$|a_{21}| = 1$. We may assume, without loss of generality, that $|a_{21}| = 1$. Attempting $P = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}$ we have

$$PAP^{-1} = \begin{bmatrix} a_{11} + \lambda a_{21} & * \\ * & * \end{bmatrix}.$$

Now choose $\lambda = -\text{sign}(a_{11}a_{21})$ and we are done.

The case $p(A) = 0$. Here $a_{11} = 0$ or $a_{22} = 0$. If $a_{11} = 0$ we are done. Suppose $a_{22} = 0$. Then

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & a_{21} \\ a_{12} & a_{11} \end{bmatrix}.$$

Assume now that the hypothesis is true when the weight $p(A) < k, k \in \mathbb{N}$.

Suppose $p(A) = k$. Claim: the hypothesis then is true in this case as well.

It must be that $|a_{21}| \leq |a_{11}|$ or $|a_{12}| \leq |a_{22}|$, because, if not true, we would have

$$|\det A| = |a_{21}a_{12} - a_{11}a_{22}| \geq (|a_{11}| + 1)(|a_{22}| + 1) - |a_{11}||a_{22}| \geq 3,$$

which is not possible.

We now assume, without loss of generality, that $|a_{21}| \leq |a_{11}|$. Let $\lambda = -\text{sign}(a_{11}a_{21})$ and consider

$$A_1 = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} a_{11} + \lambda a_{21} & * \\ * & a_{22} - \lambda a_{21} \end{bmatrix}.$$

Here

$$p(A_1) = -(a_{11} + \lambda a_{21})(a_{22} - \lambda a_{21}) = p(A) + a_{21}^2 + \lambda a_{21}a_{11} - \lambda a_{21}a_{22}.$$

Since $a_{11}a_{22} = -k < 0$

$$-\lambda a_{21}a_{22} = \text{sign}(a_{11}a_{21})a_{21}a_{22} < 0$$

in all cases. Hence

$$p(A_1) < p(A) + a_{21}^2 - \text{sign}(a_{11}a_{21})a_{21}a_{11} \leq p(A).$$

Observe now that a completely general matrix was used in the initial discussions; their conclusions therefore hold for A_1. Since we have shown that $p(A_1) < k$ the hypothesis is true for A_1. Hence $A_1 \sim B$ for some $B = \begin{bmatrix} 0 & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$. Since $A \sim A_1$ we then have that $A \sim B$. This proves our claim and ends the induction.

Assume now that $\det A = -2$. Then $b_{12}b_{21} = 2$. From

$$|\lambda_1| > 1, |\lambda_2| > 1, \lambda_1\lambda_2 = -2$$

follows $-2 < \lambda_1 < -1, 1 < \lambda_2 < 2$ (if λ_1 is smaller).

Then we infer that $b_{22} = \lambda_1 + \lambda_2 = 0$. Whatever is the combination of b_{12} and b_{21} it is always true that $B \sim C_1$. Take e.g. $b_{12} = -1, b_{21} = -2, P = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then

$$PBP^{-1} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}.$$

At last, assume that $\det A = 2$. Then $b_{12}b_{21} = -2$. Now we can only deduce that $|b_{22}| \le 2$.

Let $D = \begin{bmatrix} 0 & 2 \\ -1 & 2 \end{bmatrix}$. By taking P to be one of the matrices

$$I, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

we will have $PBP^{-1} = C_2, \pm D$, or $\pm C_4$.

Finally, with $Q = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, we have that $C_3 = QDQ^{-1}$.

□

Example 5.2. Let $R = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$. Then

$$R^{-1} = \begin{bmatrix} 0 & 1 \\ 0.5 & 0 \end{bmatrix}.$$

We want $B = \{(0,0)^T, (b_1, b_2)^T\}$ to be a complete set of representatives modulo $R\mathbb{Z}^2$. This means that

$(0,0)^T$ and $(b_1, b_2)^T$ should not be congruent modulo R; there must not be a solution in $\mathbb{Z}$ to $(b_1, b_2)^T = R(x, y)^T$. In other words $R^{-1}(b_1, b_2)^T \notin \mathbb{Z}^2$, so we must have that $(1/2)b_1 \notin \mathbb{Z}$ or $b_2 \notin \mathbb{Z}$.

We can therefore choose $b_1 = 1$, $b_2 = 0$. Hence let

$$B = \{(0,0)^T, (1,0)^T\}.$$

The attractor of the affine IFS (R, B) is shown in Figure 5.1.

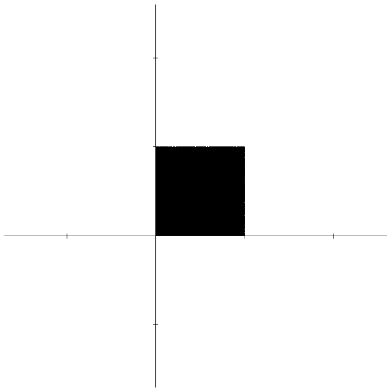

Figure 5.1: X_B

If the dual IFS corresponds to $L = \{(0,0)^T, (l_1, l_2)^T\}$, then the Hadamard matrix

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & e^{2\pi i R^{-1}b\cdot l} \end{bmatrix}$$

equals

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & e^{i\pi l_2} \end{bmatrix}$$

which is unitary iff l_2 is odd. Take

$$L = \{(0,0)^T, (0,1)^T\}.$$

The attractor of the affine IFS $(R^T, L) := X_L$ is shown in Figure 5.3.

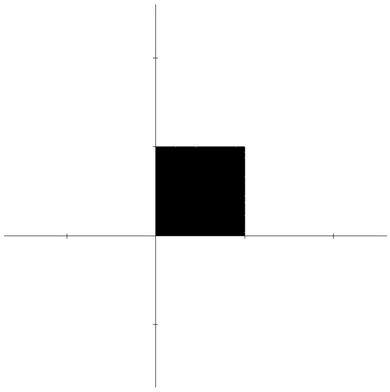

Figure 5.2: X_L

To find the spectrum Λ we refer to the theorem on determinants whose absolute value is a prime number, saying that such a system must be simple. Since this is the case with the matrix R, the system (R, B, L) is simple, by that theorem. Therefore looking at the extreme cycles will give us the spectrum. We have that

$$|m_B(x, y)| = |1/2(1 + e^{2\pi ix})| = 1$$

iff $x \in \mathbb{Z}$, while y is arbitrary.

The extreme cycle points must belong to the attractor X_L, which in this example is just the closed filled unit square. Among the four points there with $x \in \mathbb{Z}$, (0,0) and (1,1) are the only cycle points (fixpoints) and they are also extreme. All of this is

very easily checked. By Theorem 3.7 the spectrum Λ is the lattice generated by the extreme cycles and $\mathbb{Z}^2$. Therefore $\Lambda = \mathbb{Z}^2$.

Example 5.3. Let $R = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$.

Then

$$R^{-1} = \begin{bmatrix} 0 & -1 \\ 0.5 & 0 \end{bmatrix}.$$

We want $B = \{0, (b_1, b_2)^T\}$ to be a complete set of representatives modulo $R\mathbb{Z}^2$. This is the same as asking for the equation $R(x, y)^T = (b_1, b_2)^T$ to have no solution in $\mathbb{Z}$. Equivalently $R^{-1}(b_1, b_2)^T \notin \mathbb{Z}^2$.

Hence $0.5b_1 \notin \mathbb{Z}$ or $-b_2 \notin \mathbb{Z}$. We can then take $b_1 = 1$, $b_2 = 0$.

Hence let $B = \{0, (1, 0)^T\}$.

The attractor of the affine IFS (R, B) is shown in Figure 5.3.

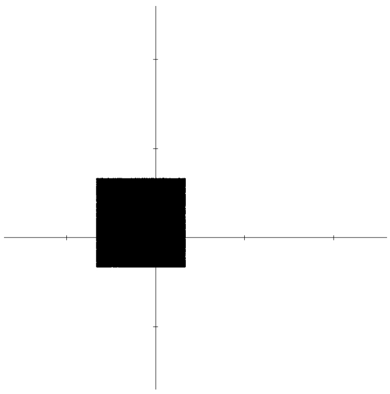

Figure 5.3: X_B

Let $L = \{0, (l_1, l_2)\}$ be the dual IFS. The Hadamard matrix will once more become

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & e^{i\pi l_2} \end{bmatrix},$$

which is unitary iff l_2 is an odd number.

Take

$$L = \{0, (0,1)^T\}.$$

The attractor of the affine IFS $(R^T, L) := X_L$ is shown in Figure 5.4.

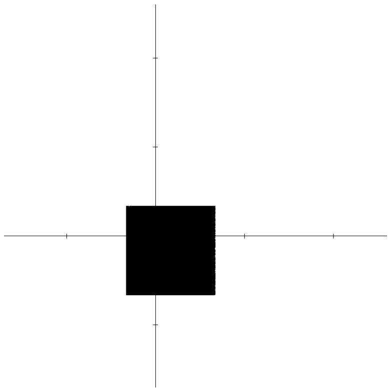

Figure 5.4: X_L

Since the determinant of R is 2, a prime number, the system (R, B, L) is simple. Therefore it is enough to consider the extreme cycles in order to find the spectrum. Since B is the same set as in the previous example we obtain that a point $(x, y)^T$ is extreme iff $x \notin \mathbb{Z}$, while y is arbitrary.

Since any cycle must be contained in the attractor X_L, and since this is a filled square contained in $(-0.4, 0.7) \times (-0.7, 0.4)$,

(see Figure 5.4) we must have that the extreme points are of the form $(0, y)$ for some y.

Now, from an extreme point we must be able to reach an extreme point, which may possibly be the same point, by some τ_l.

That is to say that $\tau_l(0,y)^T = (0,y_1)^T$ for some $l = (0,l_2)$ and y_1. Hence $y = -l_2 = 0$ or -1 and $y_1 = 0$.

In conclusion, the origin is the only extreme point.

Since the spectrum Λ is generated by the extreme points and $\mathbb{Z}^2$, we have found that $\Lambda = \mathbb{Z}^2$.

Example 5.4. Let

$$R = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Then

$$R^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

We want $B = \{0, (b_1, b_2)^T\}$, to be a complete set of representative modulo $\mathbb{Z}^2$. This means that the equation $R(x,y)^T = (b_1,b_2)^T$ should have no solution in $\mathbb{Z}$. Equivalently $R^{-1}(b_1,b_2)^T \notin \mathbb{Z}^2$. This means

$\frac{b_1-b_2}{2} \notin \mathbb{Z}$ or $\frac{b_1+b_2}{2} \notin \mathbb{Z}$. Therefore we can take $(b_1,b_2)^T = (1,0)^T$ so

$$B = \left\{0, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}.$$

The attractor of the affine IFS (R, B) is given in Figure 5.5.

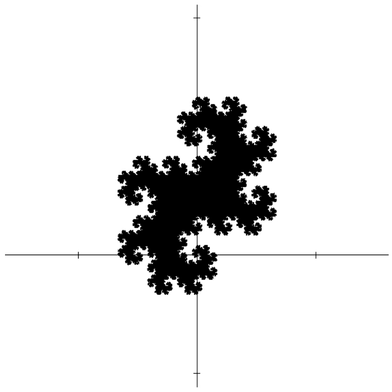

Figure 5.5: X_B

Next, we need the set $L = \{0, (l_1, l_2)^T\}$. We require the Hadamard condition so we want the matrix

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & e^{2\pi i (1/2, 1/2)^T \cdot (l_1, l_2)^T} \end{bmatrix}$$

to be unitary. This condition is satisfied if $l_1 + l_2$ is odd. Therefore $(l_1, l_2)^T = (1, 0)^T$ will verify the conditions, so we can take

$$L = \left\{ 0, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$$

The attractor of the affine IFS (R^T, L) is given in Figure 5.6.

Since the determinant of R is 2 which is a prime number, the system (R, B, L) is simple. Therefore we just have to look for the extreme cycles. The function m_B is

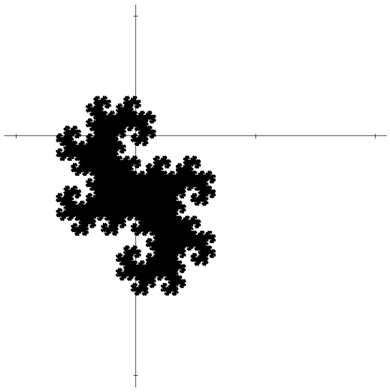

Figure 5.6: X_L

$$m_B(x,y) = \frac{1}{2}(1 + e^{2\pi i x}), \quad ((x,y) \in \mathbb{R}^2).$$

Then $|m_B(x,y) = 1$ iff $x \in \mathbb{Z}$ and y is arbitrary.

Since any cycle is contained in the dual attractor X_L, and since X_L is contained in $(-1,1) \times (-2,1)$ (see Figure 5.6), we have that any cycle point (x_0, y_0) is of the form $(0, y)$ with $y \in (-2, 1)$.

One of the transitions of the extreme cycle point (x_0, y_0) will lead to another extreme cycle point. Therefore we must have that for some $l \in L$, $\tau_l(0, y)$ is of the form $(0, y')$. This means that

$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} l_1 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y' \end{bmatrix}.$$

Then $l_1 + y = 0$ so either $l_1 = 0$, $y = 0$ or $l_1 = 1$, $y = -1$. In the first case we obtain the trivial extreme cycle $\{0\}$. In the second case we obtain the extreme cycle $\{(0, -1)\}$.

Thus all the extreme cycles in this example are

$$\{0\} \text{ and } \{(0, -1)\}.$$

Since B is a complete set of representatives modulo $R\mathbb{Z}^2$, by Theorem 3.7 the spectrum Λ will be the lattice generated by the extreme cycles and $\mathbb{Z}^2$. Therefore

$$\Lambda = \mathbb{Z}^2.$$

Example 5.5. Let

$$R = \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}.$$

Then

$$R^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix}$$

We want $B = \{0, (b_1, b_2)^T\}$, to be a complete set of representatives modulo $R\mathbb{Z}^2$. This means that the equation $R(x, y)^T = (b_1, b_2)^T$ should have no solution in $\mathbb{Z}$. Equivalently $R^{-1}(b_1, b_2)^T \notin \mathbb{Z}^2$. This means

$\frac{b_1 - 2b_2}{2} \notin \mathbb{Z}$ or $\frac{b_1}{2} \notin \mathbb{Z}$. Therefore we can take $(b_1, b_2)^T = (1, 0)^T$ so

$$B = \left\{ 0, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$$

The attractor of the affine IFS (R, B) is given in Figure 5.7.

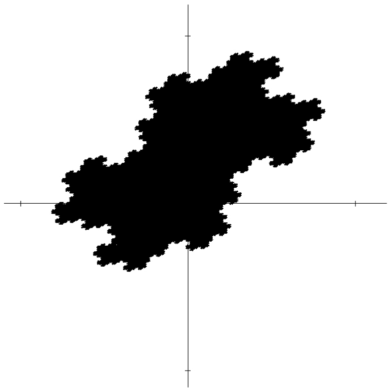

Figure 5.7: X_B

Next, we need the set $L = \{0, (l_1, l_2)^T\}$. The Hadamard condition implies that the matrix

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & e^{2\pi i (1/2,1/2)^T \cdot (l_1,l_2)^T} \end{bmatrix}$$

has to be unitary. This condition is satisfied if $l_1 + l_2$ is odd. Therefore $(l_1, l_2)^T = (1, 0)^T$ will verify the conditions, so we can take

$$L = \left\{ 0, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$$

The attractor of the affine IFS (R^T, L) is given in Figure 5.8.

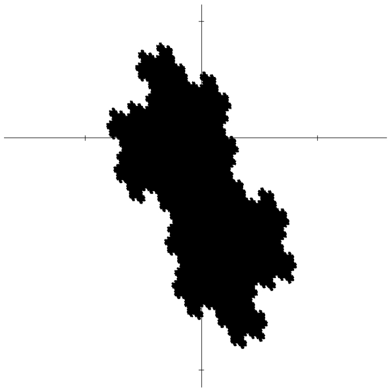

Figure 5.8: X_L

Since the determinant of R is 2 which is a prime number, the system (R, B, L) is simple. Therefore we just have to look for the extreme cycles. The function m_B is again

$$m_B(x, y) = \frac{1}{2}(1 + e^{2\pi ix}), \quad ((x, y) \in \mathbb{R}^2).$$

Then $|m_B(x, y) = 1$ iff $x \in \mathbb{Z}$ and y is arbitrary.

Since any cycle is contained in the dual attractor X_L, and since X_L is contained in $(-1, 1) \times (-1, 1)$ (see Figure 5.7), we have that any cycle point (x_0, y_0) is of the form $(0, y)$ with $y \in (-1, 1)$.

One of the transitions of the extreme cycle point (x_0, y_0) will lead to another extreme cycle point. Therefore we must have that for some $l \in L$, $\tau_l(0, y)$ is of the form $(0, y')$. This means that

$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} l_1 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y' \end{bmatrix}.$$

Then $l_1 + y = 0$ so either $l_1 = 0$, $y = 0$ or $l_1 = 1$, $y = -1$. In the first case we obtain the trivial extreme cycle $\{0\}$. In the second case we obtain the extreme cycle $\{(0, -1)\}$.

Thus all the extreme cycles in this example are

$$\{0\} \text{ and } \{(0, -1)\}.$$

Since B is a complete set of representatives modulo $R\mathbb{Z}^2$, by Theorem 3.7 the spectrum Λ will be the lattice generated by the extreme cycles and $\mathbb{Z}^2$. Therefore

$$\Lambda = \mathbb{Z}^2.$$

Remark 5.6. Looking at the picture of the attractor X_L it seems that this tiles $\mathbb{R}^2$ by $\mathbb{Z} \times 2\mathbb{Z}$ and not by $\mathbb{Z}^2$. We check that this is the case by showing that the spectrum of X_L is the dual lattice $\mathbb{Z} \times \frac{1}{2}\mathbb{Z}$.

For this, we turn the Example 5.5 around and take R^T for the matrix R, L for the set B and vice versa.

Example 5.7. Let

$$R = \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}.$$

Then

$$R^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}$$

We saw in Example 5.5 that we can take

$$B = \left\{ 0, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} = L.$$

Since the determinant of R is 2 which is a prime number, the system (R, B, L) is simple. Therefore we just have to look for the extreme cycles. The function m_B is

$$m_B(x, y) = \frac{1}{2}(1 + e^{2\pi i x}), \quad ((x, y) \in \mathbb{R}^2).$$

Then $|m_B(x, y) = 1$ iff $x \in \mathbb{Z}$ and y is arbitrary.

Since any cycle is contained in the dual attractor X_L, and since X_L is contained in $(-1, 1) \times (-2, 1)$ (see Figure 5.8), we have that any cycle point (x_0, y_0) is of the form $(0, y)$ with $y \in (-2, 1)$.

One of the transitions of the extreme cycle point (x_0, y_0) will lead to another extreme cycle point. Therefore we must have that for some $l \in L$, $\tau_l(0, y)$ is of the form $(0, y')$. This means that

$$\frac{1}{2}\begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} l_1 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y' \end{bmatrix}.$$

Then $l_1 - 2y = 0$ so either $l_1 = 0$, $y = 0$ or $l_1 = 1$, $y = 1/2$. In the first case we obtain the trivial extreme cycle $\{0\}$. In the second case we obtain the extreme cycle $\{(0, 1/2)\}$.

Thus all the extreme cycles in this example are

$$\{0\} \text{ and } \{(0, 1/2)\}.$$

Since B is a complete set of representatives modulo $R\mathbb{Z}^2$, by Theorem 3.7 the spectrum Λ will be the lattice generated by the extreme cycles and $\mathbb{Z}^2$. Therefore

$$\Lambda = \mathbb{Z} \times \frac{1}{2}\mathbb{Z}.$$

Example 5.8. We present here a non-simple system (R, B, L).

Consider the expansive matrix:

$$R = \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix}.$$

Let

$$B = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$$

Let

$$L = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}.$$

Then the matrix $\frac{1}{2}(e^{2\pi i R^{-1} b \cdot l})_{b \in B, l \in L}$ is seen to be unitary.

Hence (R, B, L) is a Hadamard triple.

By analyzing the extreme cycles, i.e. those cycles where $\mid m_B(x, y) \mid = 1, \mid m_{B_1}(x) \mid = 1$ and $\mid m_{B_2}(y) \mid = 1$, it is possible to compute the spectrum Λ of μ_B.

For example, we have

$$1 = \quad \mid m_B(x, y) \mid \quad = \quad \mid \frac{1}{4}(1 + e^{2\pi i x} + e^{2\pi i 3y} + e^{2\pi i (x+3y)}) \mid .$$

This is only possible if all exponentials are equal to 1. Hence the cycle is extreme iff $x \in \mathbb{Z}$ and $y \in \mathbb{Z}/3$.

Now, if (x_0, y_0) is a point of an m_B-cycle (extreme) and $(l_1, l_2) \in L$ then $\tau_{(l_1, l_2)}(x_0, y_0)$ is also one of the points in the cycle.

With $\tau_l(z) = S^{-1}(z + l)$ we have

$$\tau_{(l_1, l_2)}(x_0, y_0) = \tfrac{1}{16} \begin{bmatrix} 4 & -1 \\ 0 & 4 \end{bmatrix} \left(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \right).$$

Hence $\frac{1}{4}(x_0 + l_1) - \frac{1}{16}(y_0 + l_2) \in \mathbb{Z}$ and $\frac{1}{4}(y_0 + l_2) \in \mathbb{Z}/3$.

The point (x_0, y_0) must also belong to the attractor X_L of the IFS $(\tau_l)_{l \in L}$.

Since the rectangle

$$\left[-1/4,\ \ 2/3\right] \times \left[0,\ \ 2/3\right]$$

is invariant for all $\tau_l, l \in L$, this means that the attractor is a subset of that rectangle, and so (x_0, y_0) must also satisfy

$-1/4 \leq x_0 \leq 2/3, 0 \leq y \leq 2/3.$

Combining these facts we conclude that the only extreme m_B-cycle is $\{(0,0)\}$.

In two dimensions the situation is more complex. It is not enough only to consider the cycles, we need to find a proper vector space whose translates by the elements of the m_B-cycle (here the origin) is invariant with respect to $S = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$.

The set of eigenvectors of S is invariant, hence $V = \{(x, 0)\}$ fits very well. Now the measure μ_1 on the first component corresponds to the IFS $\tau_0 = \frac{x}{4}$ and $\tau_1 = \frac{x+1}{4}$. This means that $R = 4$ and $B = \{0, 1\}$ and then $L = \{0, 2\}$ gives a Hadamard pair. For the associated m_{B_1}-function $\mid \frac{1}{2}(1 + e^{2\pi i x}) \mid = 1$ it means that x must be an integer, and so the cycle is $\{0\}$.

Still, it gives a contribution $\Lambda(0)$ to the spectrum, as in earlier examples. Analyzing the second component a new IFS appears with respect to which the translated vector space $2/3+V$ is invariant. It gives a contribution $\Lambda(2/3)$ to the spectrum. But $\Lambda(0)$ and $\Lambda(2/3)$ both contributes to both components. The system (R, B, L) satisfies the reducibility condition. The result of all this is the following.

The spectrum $\Lambda_B = \Lambda(0) \cup \Lambda(2/3)$, where

$$\Lambda(0) = \left\{ \left(\sum_{k=0}^{n} 4^k a_k + \sum_{k=0}^{n} k 4^{k-1} b_k, \sum_{k=0}^{n} 4^k b_k \right) \mid a_k, b_k \in \{0, 2\} \right\}$$

and

$$\Lambda(2/3) = \left\{ \left(\sum_{k=0}^{n} 4^k a_k, -2/3 - \sum_{k=0}^{m} 4^k b_k \mid a_k, b_k \in \{0, 2\}, n, m \in \mathbb{N} \right\} .$$

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