Image Degradation Due To Surface Scattering In The Presence Of Aberrations

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IMAGE DEGRADATION DUE TO SURFACE SCATTERING
IN THE PRESENCE OF ABERRATIONS

by

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B.S. Seoul National University, 2005
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A dissertation submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy
in the College of Optics and Photonics
at the University of Central Florida
Orlando, Florida

Spring Term
2012

Major Professor: James E. Harvey
This dissertation focuses on the scattering phenomena by well-polished optical mirror surfaces. Specifically, predicting image degradation by surface scatter from rough mirror surfaces for a two-mirror telescope operating at extremely short wavelengths (9nm~30nm) is performed. To evaluate image quality, surface scatter is predicted from the surface metrology data and the point spread function in the presence of both surface scatter and aberrations is calculated.

For predicting the scattering intensity distribution, both numerical and analytic methods are considered. Among the numerous analytic methods, the small perturbation method (classical Rayleigh-Rice surface scatter theory), the Kirchhoff approximation method (classical Beckman-Kirchhoff surface scatter theory), and the generalized Harvey-Shack surface scatter theory are adopted. As a numerical method, the integral equation method (method of moments) known as a rigorous solution is discussed. Since the numerical method is computationally too intensive to obtain the scattering prediction directly for the two mirror telescope, it is used for validating the three analytic approximate methods in special cases. In our numerical comparison work, among the three approximate methods, the generalized Harvey-Shack model shows excellent agreement to the rigorous solution and it is used to predict surface scattering from the mirror surfaces.

Regarding image degradation due to surface scatter in the presence of aberrations, it is shown that the composite point spread function is obtained in explicit form in terms of convolutions of the geometrical point spread function and scaled bidirectional scattering distribution functions of the individual surfaces of the imaging system. The approximations and assumptions in this
formulation are discussed. The result is compared to the irradiance distribution obtained using commercial non-sequential ray tracing software for the case of a two-mirror telescope operating at the extreme ultra-violet wavelengths and the two results are virtually identical. Finally, the image degradation due to the surface scatter from the mirror surfaces and the aberration of the telescope is evaluated in terms of the fractional ensquared energy (for different wavelengths and field angles) which is commonly used as an image quality requirement on many NASA astronomy programs.
ACKNOWLEDGMENTS

I like the quotes that life is not a race but a journey to be savored each step of the way. It has been a long journey till where I stand now and I have tried to enjoy every single step that I took. However, some times it did turn into a race or a war. I regret that I did not enjoy every single step nor share more ideas nor talk much with people that I love.

I would like to thank my advisor, or my officiator, Dr. James Harvey. I was very lucky to work with you. First of all, I love your working hours. Also, I like the way you approach to the physical problems as well as your attitude towards the life. I hope I can open my door to others as you always have done.

I would also like to thank Andrey Krywonos. The discussions with you and your keen comments have always stimulated me.

I thank Dr. Jaisoon Kim for introducing me to the world of optics, leading me to savor diverse areas of optics. The experience I gained through working with you is my big asset.

I give enormous thanks to my parents. They have always done their best to be the best parents they could be. Father, you always have encouraged me whenever I wanted to give up. Thank you for having trust in me. Mother, you have always prayed, prayed and prayed for me. Your prayer was and is the foundation of my strength.
I love you, my wife, Nahyun Kim. Our life has just begun and we will make wonderful life together. Thank you so much for growing old with me.
# TABLE OF CONTENTS

LIST OF FIGURES .................................................................................................................. x
LIST OF ACRONYMS/ABBREVIATIONS ........................................................................... xvi
CHAPTER 1: INTRODUCTION ................................................................................................. 1
  1.1 Motivation for This Research ......................................................................................... 1
  1.2 Technical Approach of This Research ......................................................................... 3
  1.3 Organization of This Dissertation .............................................................................. 8
CHAPTER 2: TECHNICAL BACKGROUND ............................................................................. 11
  2.1 Mathematical Notation ............................................................................................... 11
    2.1.1 Plane Wave ........................................................................................................ 12
    2.1.2 Spherical Wave .................................................................................................. 16
    2.1.3 Cylindrical Wave ............................................................................................... 19
    2.1.4 Surface Statistics ............................................................................................... 20
  2.2 Integral Equations for Surface Scattering ................................................................... 21
  2.3 Integral Equations for the Equivalent Problem to Surface Scattering ......................... 28
  2.4 Scattered Field .......................................................................................................... 33
  2.5 Ensemble Average of Scattering Intensity .................................................................. 39
CHAPTER 3: NUMERICAL APPROXIMATION ..................................................................... 41
  3.1 Scattering from Two-dimensional Surfaces ................................................................. 42
    3.1.1 Incident Field ....................................................................................................... 43
    3.1.2 Surface Field ....................................................................................................... 45
    3.1.3 Scattered Field ................................................................................................... 48
  3.2 Scattering from One-dimensional Surfaces ................................................................. 49
    3.2.1 Incident Field ....................................................................................................... 50
    3.2.2 Surface Field ....................................................................................................... 51
    3.2.3 Scattered Field ................................................................................................... 56
Figure 2-1: A virtual surface and the volume $V_0$ enclosed by the surface. ........................................... 22

Figure 2-2: The volume $V_1$ is enclosed by the ideally conducting surfaces $S_{1\infty}$ and $S$. The volume $V_2$ is enclosed by the surfaces $S_{2\infty}$ and $S$. ................................................................. 26

Figure 2-3: Free current along the virtual surface in vacuum space. The direction of the current is perpendicular to the surface normal. .......................................................... 29

Figure 5-1: Scattered intensity of the IEM (asterisk), SPM (dashed line), KA (dotted line), and GHS (solid line) for (a) $\sigma_{tot}=0.01\lambda$, $l_c=0.2\lambda$, $\theta_i=30^\circ$, (b) $\sigma_{tot}=0.2\lambda$, $l_c=2\lambda$, $\theta_i=0^\circ$...... 93

Figure 5-2: Scattering intensity for (a) $\sigma_{tot}=0.1\lambda$, $l_c=0.25\lambda$, $\theta_i=0^\circ$, (b) $\sigma_{tot}=0.3\lambda$, $l_c=0.8\lambda$, $\theta_i=0^\circ$, (c) $\sigma_{tot}=0.4\lambda$, $l_c=2.5\lambda$, $\theta_i=50^\circ$ and (d) $\sigma_{tot}=\lambda$, $l_c=2\lambda$, $\theta_i=0^\circ$. The SPM is not plotted in (c) and (d). ................................................................................. 94

Figure 5-3: Error contour maps for normal incidence; (a) the SPM, (b) the KA and (c) the GHS. .................................................................................................................. 96

Figure 5-4: Contour lines of (a) rms slope and (b) rms curvature superposed on top of the error map for the KA method with normally incident light. .................................................. 97

Figure 5-5: Contour lines of rms slope superposed on top of the error map for the GHS for (a) $\theta_i=0^\circ$ and (b) $\theta_i=40^\circ$. ........................................................................................................ 98

Figure 5-6: Error contour maps for the SPM with (a) $\theta_i=20^\circ$ (d) $\theta_i=40^\circ$ (g) $\theta_i=60^\circ$, for the KA with (b) $\theta_i=20^\circ$ (e) $\theta_i=40^\circ$ (h) $\theta_i=60^\circ$, and for the GHS with (c) $\theta_i=20^\circ$ (f) $\theta_i=40^\circ$ (i) $\theta_i=60^\circ$. ............................................................................................................. 99

Figure 5-7: Error contour map for TIS predicted by Eq.(15) for $\theta_i=0^\circ$ with (a) $\sigma_{tot}$, and (b) $\sigma_{rel}$. .................................................................................................................. 101

Figure 5-8: Error contour maps of the TIS value predicted by numerically integrating the scattering function predicted by (a) the SPM, (b) the KA and (c) the GHS for normal incidence. ........................................................................ 102
Figure 5-9: Contour map of total fractional reflected energy as a function of the rms roughness and the surface correlation length for the IEM: (a) $\theta_i=0^\circ$ (b) $\theta_i=40^\circ$. ................. 104

Figure 5-10: Five incoherent intensity distributions of which sampling spacing are $\Delta x=\lambda/4, \lambda/8, \lambda/16, \lambda/32$ and $\lambda/64$ respectively. The surface length is $20\lambda$ and the $H=0.5$. The bandlimited rms roughness for the surface with $\Delta x=\lambda/64$ is $0.5\lambda$. The number of realizations is 1000. .................................................. 109

Figure 5-11: (a) Five incoherent intensity distributions of which surface length are $L=10\lambda, 20\lambda, 40\lambda, 80\lambda$ and $160\lambda$ respectively and (b) their averaged periodogram. The sampling spacing is $\Delta x=\lambda/8$ and $H=0.5$ for all the cases. The bandlimited rms roughness for the surface with $L=160\lambda$ is $0.35\lambda$. The number of realizations is 1000. .............. 111

Figure 5-12: Angular behavior between -20 to 20 degrees of scattering angles of five incoherent intensity distributions for the surfaces whose length are $L=10\lambda, 20\lambda, 40\lambda, 80\lambda$ and $160\lambda$ respectively. The bandlimited rms surface roughness for the case of $L=160\lambda$ is (a) $0.35\lambda$ (b) $0.07\lambda$. The sampling spacing is $\Delta x=\lambda/8$ and the $H=0.5$ for all the cases. The number of realizations is 1000. .................................................. 112

Figure 5-13: (a) PSD function for surface A and B. (b) The scattering intensity for surface A (asterisk) and B (solid line). The number of realization is 1000. ......................... 116

Figure 5-14: The scattering intensity for surface A (dotted line) and B (solid line). The total rms roughness of surface A is $\sigma_{tot}=0.05\lambda$ and the bandlimited rms roughness of surface B is $h_{rms}=0.05\lambda$. The slope parameter is $c=2$ for surface A and Hurst exponent is $H=0.5$ for surface B. The number of realizations is 1000. .................................................. 117

Figure 5-15: (a) The scattering intensity distributions with the various surface lengths from $L=10\lambda$ to $200\lambda$. (b) The error curves. All the calculation is done for the case of $\sigma_{tot}=0.1\lambda, c=1.2, \Delta x=\lambda/2$ ............................................................. 120

Figure 5-16: The error curves for the four cases of surface parameter sets. For all the cases, 1000 realizations are used. .......................................................................................... 122

Figure 5-17: (a) Scattering intensity distribution with the various number of sampling points per wavelength from $\Delta x=\lambda/2$ to $\Delta x=\lambda/36$. (b) Error curve (black solid) and total reflected energy (red solid). All the calculation is done for the case of $\sigma_{tot}=0.1\lambda, c=1.2, L=40\lambda$, and 1000 realizations are used. .................................................. 123

Figure 5-18: (a) Error curve and (b) total reflected energy. All the calculation is done for the case of $L=40\lambda \sim 60\lambda$ and 1000 realizations are used. ........................................... 124

Figure 5-19: Selected minimum number of the sampling points per wavelength for the 9x9 cases of the surface parameter sets ($c, \sigma_{tot}$). ............................................................. 125
Figure 5-20: Total reflected energy using the IEM for 36x36 cases of the surface parameter sets \((c, \sigma_{tot})\) for (a) \(\theta_i=0^\circ\) and (b) \(\theta_i=60^\circ\). The surface length is chosen by \(L=40\lambda \sim 60\lambda\), and \(\lambda/\Delta x\) value is picked up from the Figure 5-19........................................... 126

Figure 5-21: Scattering intensity distributions by the IEM, SPM, KA and GHS for the case of (a) \(\sigma_{tot}=0.025\lambda, c=1.4, \theta_i=40^\circ\) (b) \(\sigma_{tot}=0.6\lambda, c=2.8, \theta_i=0^\circ\)........................................... 129

Figure 5-22: Scattering intensity distributions by the IEM, SPM, KA and GHS for the case of (a) \(\sigma_{tot}=0.1\lambda, c=1.4, \theta_i=40^\circ\), (b) the same to (a) but in linear scale, (c) \(\sigma_{tot}=0.3\lambda, c=2, \theta_i=0^\circ\), and (d) \(\sigma_{tot}=0.3\lambda, c=2.8, \theta_i=60^\circ\)............................................ 130

Figure 5-23: Scattering intensity distributions by the IEM, SPM, KA and GHS for the case of (a) \(\sigma_{tot}=0.15\lambda, c=2, \theta_i=0^\circ, 30^\circ, 60^\circ\) and (b) \(\sigma_{tot}=0.3\lambda, c=2.4, \theta_i=0^\circ, 30^\circ, 60^\circ\).............. 132

Figure 5-24: Map of error values for SPM (a) in logarithmic scale and (b) in linear scale for normal incident angle............................................................................................................ 133

Figure 5-25: Map of error values for the KA (a) in logarithmic scale (b) in linear scale for normal incident angle ........................................................................................................... 134

Figure 5-26: Map of error values for the GHS (a) in logarithmic scale (b) in linear scale for normal incident angle ........................................................................................................... 135

Figure 5-27: Map of error values in logarithmic scale for the SPM in the case of (a) \(\theta_i=20^\circ\) (d) \(\theta_i=40^\circ\) (g) \(\theta_i=60^\circ\), for the KA in the case of (b) \(\theta_i=20^\circ\) (e) \(\theta_i=40^\circ\) (h) \(\theta_i=60^\circ\) and for the GHS in the case of (c) \(\theta_i=20^\circ\) (f) \(\theta_i=40^\circ\) (i) \(\theta_i=60^\circ\)........................................... 136

Figure 5-28: Map of error values in logarithmic scale at normal incidence for (a) the SPM, (b) the KA and (c) the GHS............................................................................................................ 137

Figure 5-29: Map of error values in logarithmic scale for the SPM in the case of (a) \(\theta_i=20^\circ\), (d) \(\theta_i=40^\circ\), (g) \(\theta_i=60^\circ\), the KA in the case of (b) \(\theta_i=20^\circ\), (e) \(\theta_i=40^\circ\), (h) \(\theta_i=60^\circ\), and the GHS in the case of (c) \(\theta_i=20^\circ\), (f) \(\theta_i=40^\circ\), (i) \(\theta_i=60^\circ\)........................................... 138

Figure 5-30: (a) PSD functions for surface A, B and C with the same parameter \(c=2\). (b) The scattering intensity distributions predicted by the IEM for the three surfaces in log-log scale for normal incidence. ........................................................................................................... 140

Figure 5-31: Predicted intensity distributions by the SPM, KA, GHS and IEM in the case of \(\theta_i=0^\circ\) and \(\theta_i=60^\circ\) (a) for the surface A and (b) for the surface B. ........................................... 141

Figure 5-32: Predicted intensity distributions by the SPM, KA, GHS and IEM in the case of (a) \(\theta_i=0^\circ\) and \(\theta_i=60^\circ\) in linear-log scale and (b) \(\theta_i=0^\circ\) in log-log scale. ........................................... 142
Figure 6-1: Full angular scattering distribution predicted by the IEM for (a) ss, (c) sp, (e) ps, (g) pp and by the SPM for (b) ss, (d) sp, (f) ps (h) pp for the case of $\sigma_{tot}=0.001\lambda$, $l_c=0.5\lambda$, $\theta_i=0^\circ$ ................................................................. 146

Figure 6-2: The scattering distribution predicted by the four methods (a) for the unpolarized light from the two-dimensional surface having $\sigma_{tot}=0.001\lambda$, $l_c=0.5\lambda$ and (b) for the one-dimensional surface (TE for the SPM and IEM) having the same surface parameters at normal incidence ................................................................. 147

Figure 6-3: Full angular distributions predicted by the (a) IEM, (b) GHS, (c) KA, (d) SPM for the unpolarized scattering intensity from the surface having $\sigma_{tot}=0.2\lambda$, $l_c=2\lambda$ at normal incidence ................................................................. 148

Figure 6-4: The scattering distribution predicted by the four methods (a) for the unpolarized light from the two-dimensional surface with $\sigma_{tot}=0.2\lambda$, $l_c=2\lambda$ and (b) from the one-dimensional surface (TE for the SPM and IEM) having the same surface parameters at normal incidence ................................................................. 149

Figure 6-5: Full angular distributions predicted by the (a) IEM, (b) GHS and (c) KA for unpolarized scattering intensity from the surface having $\sigma_{tot}=0.6\lambda$, $l_c=2\lambda$ at normal incidence ................................................................. 150

Figure 6-6: The scattering distribution predicted by the IEM, GHS and KA (a) for the unpolarized light from the two-dimensional surface (b) from the one-dimensional surface (TE for the IEM) having the same surface parameters ................................................................. 150

Figure 6-7: Full angular distributions predicted by the (a) IEM, (b) GHS and (c) KA for the unpolarized scattering intensity from the surface having $\sigma_{tot}=0.8\lambda$, $l_c=2\lambda$ at normal incidence ................................................................. 151

Figure 6-8: The scattering distribution predicted by the IEM, GHS and KA (a) for the unpolarized light from the two-dimensional surface (b) from the one-dimensional surface (TE for the IEM) having the same surface parameters ................................................................. 151

Figure 7-1: Schematic layout of an optical imaging system consisting of a series of coaxial optical surfaces ................................................................. 152

Figure 7-2: (a) A small area in pupil plane and (b) its corresponding area in image plane ................................................................. 159

Figure 7-3: Schematic layout of a single surface optical imaging system ................................................................. 167

Figure 7-4: Schematic layout of the entrance pupil and exit pupil for a single optical surface. Vignetting is ignored ................................................................. 169
Figure 7-5: Schematic layout of a two-surface optical imaging system. If $x_1$ is equal to $x_{01}$, the system becomes conventional optical imaging system. ...................................................... 171

Figure 7-6: Ray aberration due to scattering in a single optical surface. ...................................................... 185

Figure 8-1 Schematic layout of a Cassegrain type two-mirror telescope ...................................................... 196

Figure 8-2: Geometrical spot diagram of the telescope for five different incident angles...... 197

Figure 8-3: Calculated geometrical PSF for the Cassegrain type two-mirror telescope for $4\times4$ detector size (a) for normal incidence and (b) for 0.5 degrees of incident angle. The irradiance is plotted in logarithmic scale................................. 197

Figure 8-4: Measured surface PSD of a mirror surface of the Cassegrain type two mirror telescope. The PSD is measured from four different metrology instruments for different spatial frequency bands. An abc-function has been fitted to the experimental data to characterize the surface...................................................... 199

Figure 8-5: Predicted BRDF plotted in logarithmic scale at the wavelength of $\lambda=9.4nm$ for (a) normal incidence, (b) $\theta_i=20^\circ$, (c) $\theta_i=40^\circ$ and (d) $\theta_i=60^\circ$. The number on the color bar represents the log of the BRDF values................................. 200

Figure 8-6: Calculated scattering PSF for the Cassegrain type two-mirror telescope for (a) $4\times4$ detector size and (b) $16\times16$ detector size at the wavelength of $\lambda=9.4nm$. The irradiance is plotted in logarithmic scale...................................................... 202

Figure 8-7: Rotationally symmetrical scattering PSD function at the wavelength of $\lambda=9.4nm$. The specular-specular component is omitted in this figure................................. 203

Figure 8-8: Irradiance distribution predicted by three different methods: analytic approximation method (red asterisk), non-sequential ray tracing provided by Zemax (blue solid line) and ASAP (green dotted line). The irradiance values are normalized to the incident power collected to the entrance pupil...................................................... 205

Figure 8-9: Point spread function by scattering and aberration for 0.5° incident field angle by (a) the convolution method, (b) Zemax, (c) contour map of (a), and (d) contour map of (b). ...................................................... 206

Figure 8-10: Contour map of point spread function by scattering and aberration for 0.5° incident field angle by (a) the convolution method, (b) Zemax, (c) contour map of (a), and (d) contour map of (b). ...................................................... 207

Figure 8-11: The ensquared energy for 0° and 0.5° incident field angle of the convolution method (solid line), Zemax (asterisk), and aberration free case (dotted line) centered on the Gaussian image of the point source. ...................................................... 209
Figure 8-12: Schematic shape of the filter. ................................................................. 210

Figure 8-13: (a) Direction cosines of incident light and calculated geometrical PSFs for the SUVI telescope for 2×2 detector pixel size at (b) normal incidence, (c) θ_i=0.5°, φ_i=90° (d) θ_i=0.5°, φ_i=45°, (e) θ_i=0.5°, φ_i=0° and (f) θ_i=0.5°, φ_i=-45°. The irradiance is plotted in logarithmic scale. .................................................................................. 211

Figure 8-14: The measured surface metrology data characterizing state-of-the-art EUV telescope mirror for (a) the primary and (b) the secondary mirror surface ........................................ 212

Figure 8-15: Predicted BRDFs plotted in logarithmic scale at (a) λ=93.9Å (b) 303.8Å for the primary mirror surface and at (c) λ=93.9Å and (d) 303.8Å for the secondary mirror surface at normal incidence. The number on the color bar represents the log of the BRDF values. .................................................................................. 213

Figure 8-16: Predicted BRDFs of (a) the primary mirror surface and (b) the secondary mirror surface for the six SUVI wavelengths at normal incidence. ................................. 214

Figure 8-17: Calculated scattering PSF for the SUVI telescope (a) at λ=93.9Å and (b) at 303.8Å for 4×4 detector pixel size image plane. The irradiance is plotted in logarithmic scale. .................................................................................. 215

Figure 8-18: Point spread function by surface scatter and aberrations for (a) normal incidence, (b) θ_i=0.5° with φ_i=90° at λ=93.9Å and for (c) normal incidence and (d) θ_i=0.5° with φ_i=90° at λ=303.8Å. The values in color bar denote log of the irradiance........... 216

Figure 8-19: Predicted fractional ensquared energy at normal incidence for the six SUVI wavelengths. The dotted lines denote required value ......................................................... 217

Figure 8-20: Predicted fractional ensquared energy for the six SUVI wavelengths at (a) θ_i=0.5° φ_i=90° (b) θ_i=0.5° φ_i=45° (c) θ_i=0.5° φ_i=0° and (d) θ_i=0.5° φ_i=-45°. .......................... 218
### LIST OF ACRONYMS/ABBREVIATIONS

<table>
<thead>
<tr>
<th>Acronym</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACV</td>
<td>Autocovariance</td>
</tr>
<tr>
<td>ASF</td>
<td>Angle Spread Function</td>
</tr>
<tr>
<td>BRDF</td>
<td>Bidirectional Reflectance Distribution Function</td>
</tr>
<tr>
<td>BSDF</td>
<td>Bidirectional Scattering Distribution Function</td>
</tr>
<tr>
<td>GHS</td>
<td>Generalized Harvey-Shack</td>
</tr>
<tr>
<td>KA</td>
<td>Kirchhoff Approximation</td>
</tr>
<tr>
<td>OPD</td>
<td>Optical Path Difference</td>
</tr>
<tr>
<td>PSD</td>
<td>Power Spectral Density</td>
</tr>
<tr>
<td>SPM</td>
<td>Small Perturbation Method</td>
</tr>
<tr>
<td>PSF</td>
<td>Point Spread Function</td>
</tr>
<tr>
<td>RMS</td>
<td>Root Mean Square</td>
</tr>
<tr>
<td>TIS</td>
<td>Total Integrated Scatter</td>
</tr>
</tbody>
</table>
CHAPTER 1: INTRODUCTION

Everything you see is scattered light. Your eyes have been evolved to form an image of an object, to know where the object is and to distinguish how rough the surface of the object is by gathering information from the light scattered from its surface. The eyes know, but our knowledge has been limited. For a long time, scattering is considered as just noise or something cannot be analyzed, which is the exact meaning of the ‘scattering’ in Chinese. However, as technology is developing, we have come to realize the importance of surface scatter, and are trying to analyze, understand and use it for a variety of applications. This dissertation starts from a situation where surface scattering is an important issue. The following sections show why the analysis of surface scatter is required and how the analysis has been performed.

1.1 Motivation for This Research

Optical instruments play an important role in science, engineering, military applications, and manufacturing. Lenses, mirrors, gratings and other optical elements are designed and polished to control the direction of the light. But there is always some light that does not comply with our intended direction and we call it scattered light. For many applications, it does not make significant trouble because the amount of the scattered light is negligible compared to that which
does comply with our intension. However, for other advanced applications, we increasingly find that surface scatter is a dominant image degradation mechanism.

As technology is developing, people have paid attention to the light having extremely short wavelengths because extremely short wavelength light reveals what we have not seen before, or it helps in building what we have not made before. However, when short wavelength light is used, the scattering takes place in a way we have not experienced before. Due to the limitation of grinding and polishing technology, optical surfaces do not have enough smoothness compared to extremely short wavelength. This roughness introduces a large amount of undesired scattered light, and the scattered light degrades the performance of optical instruments severely. Thus, scattering analysis becomes a matter of importance when short wavelength light is used.

This dissertation starts from a situation where short wavelength light is of interest. A Cassegrain type telescope will be launched to observe the Sun, and the main wavelengths of interest are extreme ultra violet (EUV), which is quite short compared to our common experience. The two mirrors are fabricated with state-of-art technology, so they may be smooth enough for Laurent Cassegrain (Catholic priest, 1629~1693) who is the inventor of the Cassegrain telescope, but they still produce a large amount of scattered light at EUV wavelengths which severely degrades the quality of the image of the Sun. This dissertation is focused on the question how the image formed by the Cassegrain type telescope will be degraded by the surface scattering introduced by imperfect mirror surfaces.

Since Lord Rayleigh’s time (physicist, 1842~1919), light scattering has been analyzed using modern mathematics [1-3]; thus, the analysis of surface scatter has a long history. Throughout its
history, numerous researchers have tried to explain the surface scattering phenomena [4-13] but, although a century has passed after Lord Rayleigh, there is still no unique explanation. More precisely speaking, people have successfully expressed it using the symbols of modern mathematics [14] but the problem is they cannot calculate the combination of symbols. Thus, as in many other areas of physics, they calculate the problem by reforming the problem to a solvable one. This is another name for approximation. On the other side, some people try to use modern powerful computers to calculate the symbols. This method looks as though it gives quite accurate results, but as to the approximate methods, it solves only some of the problems.

None of these methods can provide a direct and firm solution to the scattering behavior from the mirror surfaces of the Cassegrain telescope type. However, it has been continuously reported that some methods give quite accurate predictions in specific situations. It is thus reasonable to consider their predictions as possible answers to our problem. Therefore, it is required to investigate systematically which possible method provides the best answer for our telescope. Then, by choosing the best method, we can accurately predict the image quality of the telescope, which is the goal of this dissertation.

1.2 Technical Approach of This Research

Solving the scattering problem is considered a boundary value problem for the electromagnetic wave with specific material properties. The exact analytic expression of the solution was
obtained more than half-century ago [14], but it is given by an integral equation which is hard to apply for a real scattering problem.

Relatively recently, with the increasing capacity and speed of computers, a numerical approach called the method of moments has been developed to calculate the integral equation numerically [15]. In this dissertation, this method is referred to as integral equation method (IEM). This method gives quite reliable results for some situations, but the problem is its accuracy is largely dependent upon the ability of the computers [16]. To achieve reasonable accuracy, a large amount of memory and much calculation time are required. Thus, this method has been used mainly for calculating the scattered field distribution from one-dimensional randomly rough surfaces because the accuracy is achieved using relatively less memory and computation time [17- 31]. Although the scattering calculation for two-dimensional random rough surfaces which follow Gaussian statistics has been reported [32- 41], it remains still a challenging task due to the limitation of the computer memory and computation time. Therefore, using current computing technology, the scattering calculation for a two-dimensional random fractal-like rough surface, which is needed for our telescope analysis, is practically impossible because the requirements for the numerical treatment are even more severe.

An alternative approach (which actually has a much longer history than the numerical one) is solving the integral equation, or equivalent equation, with reasonable approximations [42]. Many approximate methods have been suggested, however, the most widely used methods to predict surface scatter phenomena are probably the small perturbation method (SPM) [6, 43-47] and the Kirchhoff approximation (KA) method [48-53]. The SPM is known to be accurate for large
incident and scattered angles, but it has an explicit smooth surface limitation [42]. On the other hand, the KA method is known to be valid for rougher surfaces but exhibits a paraxial limitation imposed by the tangential plane approximation [53]. These two approximate methods are thus complementary, but not all-inclusive; i.e., neither of them, nor the combination of them, adequately describes scattered light behavior for moderately rough surfaces at large incident and/or scattered angles which is needed to analyze scattering phenomena by our two-mirror telescope.

More than a decade ago, an empirical modification of KA method was developed that appeared to satisfactorily combine the advantages of both the SPM and KA method without the disadvantages of either [54, 55]. However, this modified Beckman-Kirchhoff model did not gain much attention from theoretical researchers because it was obtained empirically rather than rigorously derived. In spite of that, the modified KA model has been evaluated, implemented, and referenced by researchers in the computer vision and computer animation fields who are less interested in a rigorous solution of boundary condition than merely having a surface scatter model that results in the rendering of realistic surfaces, textures, objects, and scenes under a wide variety of illumination conditions [56-61].

Recently, Krywonos, et al. described a linear system formulation of surface scatter theory [62-64] based upon a non-paraxial scalar diffraction analysis [65-69]. This method is referred to as the generalized Harvey Shack (GHS) surface scatter theory because it comes from the generalization of the original Harvey-Shack scattering model in which the scattering behavior is characterized by a surface transfer function [70]. It has been shown that the GHS theory provides identical
results to the SPM method for smooth surfaces, and essentially the same result as the modified KA model for moderately rough surfaces at arbitrary incident and scattered angles [62].

In this dissertation, the IEM, SPM, KA, and GHS are under consideration for analyzing scattering from two-dimensional mirror surfaces with a fractal-like structure. Since the rigorous solution by the IEM is not available in this case, we are forced to use predictions by one of the approximate methods. However, based on the idea that the IEM gives quite accurate results for one-dimensional ideally conducting surfaces, it is used to investigate the region of validity of the three approximate methods. Then, from the valid domain, the method expected to produce the closest answer for our situation is chosen and used for calculating the bidirectional scattering distribution function (BSDF) from the measured metrology data of the mirror surfaces.

Once the BSDF is calculated, the image quality of the telescope is evaluated by its point spread function (PSF). The PSF should include the scattering from the mirror surfaces, the aberration caused by the figure of the mirror surface and diffraction effects caused by the limited size and shape of its aperture. However, in this dissertation, the diffraction is ignored. This is justified by the fact that, for EUV wavelengths, the effects of aberrations exceed the Rayleigh’s diffraction limit by a substantial factor, thus diffraction effects become relatively insignificant [71].

However, another problem lies with the calculation of the PSF when there is surface scattering from the elements of the telescope. Every scattering prediction method assumes that the mean surface is flat, whereas the actual mean surface of the telescope is curved. Thus, an assumption is adopted that, when a ray bundle strikes a small portion of the mirror surface, the mean surface of the small portion can be considered to be locally flat, and the scattered light leaving from the
small portion is distributed as predicted by the BSDF. It is also assumed that the scattered light can be divided into many ray bundles, and each ray bundle behaves as the specular ray after leaving the small portion of the surface. Although these assumptions have not been proven in a rigorous manner, they have been used in many commercial ray-tracing software products [72-75]. And some cases are reported that the predicted results under this assumption give good agreement with experimental observations [76-78].

The technique which is based on this assumption is non-sequential ray tracing and the PSF can be obtained by using the commercial software which support this technique. However, it is a non-intuitive and time-consuming process, thus, in this dissertation, an analytical approach is developed under some assumptions and approximations and the results are compared to those obtained by commercial software.

Throughout this dissertation, the mirror is considered to be a perfect conductor (infinite conductivity). Therefore it is assumed that there is no absorption or transmission. This assumption transforms many equations to a simpler form and makes it easy to reveal the essential idea of each scattering prediction method. However, the finite conductivity possibly makes a difference in scattering behavior and the analysis of it is left for other researchers.
1.3 Organization of This Dissertation

This dissertation consists of largely two parts. The first part is for accurately predicting surface scatter and the second part is for evaluating image degradation in the presence of both surface scatter and aberrations. From Chapter 2 through Chapter 6, the scattering prediction is of interest and the image degradation due to the scattering is analyzed in Chapter 7 and 8.

In Chapter 2, after the mathematical notation is discussed, the rigorous analytic integral equations based on the Maxwell’s equations are derived for ideally conducting two-dimensional surfaces. Two physical interpretations of scattering phenomena are used and, using the two different approaches, the boundary condition represented by the integral equations are obtained separately. However, it will be shown that these two approaches give exactly the same integral equations. Lastly, the integral expression for the scattered light is represented by a superposition of plane waves.

In Chapter 3, a numerical method calculating the integral equation is introduced. After defining the incident wave in Cartesian coordinates, it is shown that the integral equation can be expressed as two coupled linear equations and, therefore, it becomes a linear system of equations for the two-dimensional ideally conducting random rough surfaces. Next, the integral equation for an one-dimensional ideally conducting random rough surface is derived by decomposing the vector quantities.

In Chapter 4, the three approximate methods which are the SPM, KA and GHS are introduced. First, the scattering intensity function is obtained by using the perturbation technique, which
results in the same analytic expression to that by the SPM. Next, the scattering intensity function is obtained by applying the Kirchhoff boundary condition. Finally, the GHS scattering theory is reviewed to calculate scattering intensity for an ideally conducting random rough surface.

In Chapter 5, the numerical comparison between the four predicted scattering intensity distributions by the IEM, SPM, KA and GHS is performed for one-dimensional ideally conducting random surfaces. First, surfaces having Gaussian statistics are considered and the regions of validity of the three approximate methods are obtained in terms of surface parameters which characterize the surfaces. Next, the valid domains of the three approximate methods are computed for surfaces having fractal-like structures in two-dimensional surface parameter space.

In Chapter 6, the numerical comparison of the predicted scattering intensity distributions by the IEM, SPM, KA and GHS for the two-dimensional ideally conducting surfaces is performed. The comparison is carried out for two-dimensional surfaces with Gaussian statistics and the results are compared to the domain of validity obtained from one-dimensional surfaces with Gaussian statistics.

In Chapter 7, ignoring diffraction, the PSF in the presence of both surface scattering and aberrations is obtained. First, the geometrical PSF is represented by an integral form and the new integral expression of PSF is verified to be equivalent to the expression of the geometrical PSF in other literatures under the situation where there is only aberration. Then, the new formalism is extended to the situation where there are not only aberrations but also surface scatter. Finally, with appropriate approximations, it will be shown that the geometrical PSF can be expressed by the convolution of the geometrical PSF and the scattering PSF.
In Chapter 8, using scattering prediction and the integral expression of PSF developed in the previous chapter, the PSF of the EUV two-mirror telescope is calculated. First, the BSDFs for the mirror surfaces are computed using GHS surface scattering theory and then, the scattering PSF is calculated. By convoluting the scattering PSF with the geometrical PSF, the total PSF for the telescope is obtained. Finally, the computed total PSF is compared to the results using commercial software.

Finally, the conclusion of this dissertation is presented in Chapter 9.
CHAPTER 2: TECHNICAL BACKGROUND

Optical scattering is the phenomenon of an electromagnetic wave interacting with material. In general, depending upon the nature of the material, this can includes surface scatter, bulk scatter, particulate scatter and even resonance effects such as Raman scattering. In this dissertation we will only be concerned with the light scattering behavior due to residual optical fabrication errors from clean reflective (mirror) surfaces.

There are two viewpoints to interpret this interaction. One interpretation is that the material hinders wave propagation and modifies the direction of propagation. The other interpretation is that the propagating wave activates material and the activated material radiates another propagating wave. After introducing the mathematical notation used in this dissertation, the two different interpretations of surface scatter phenomena are explained and it is shown that they produce the same surface integral equations. At the end of this chapter, the integral expression for the scattered light is represented by a superposition of plane waves.

2.1 Mathematical Notation

Before moving forward to discuss the surface scatter phenomena, the mathematical notations for describing plane waves, spherical waves and cylindrical waves are specified. These notations are
used from this chapter through Chapter 6, but it is changed in Chapter 7 and Chapter 8 and the change will be specified in Section 7.1.

2.1.1 Plane Wave

The incident wave is assumed to be monochromatic and its time dependence is assumed to be $\exp[-i\omega t]$. The unit of length is set to be the wavelength of the incident wave and it is referred to as wavelength normalized coordinates. A position in space is denoted by the vector $\mathbf{r}$ and, especially if the vector is described in Cartesian coordinates, it is denoted as the vector $\mathbf{x} = x\mathbf{\hat{x}} + y\mathbf{\hat{y}} + z\mathbf{\hat{z}}$.

If there is no source in vacuum, the electric and magnetic fields satisfies the homogeneous Maxwell’s equations given by

$$
\nabla \cdot \varepsilon_0 \mathbf{E} = 0 \\
\nabla \cdot \mu_0 \mathbf{H} = 0 \\
\n\nabla \times \mathbf{E} = i\omega \mu_0 \mathbf{H} \\
-\n\nabla \times \mathbf{H} = i\omega \varepsilon_0 \mathbf{E}
$$

(2.1)

where $\varepsilon_0$ is the constant permittivity and $\mu_0$ is the constant permeability in vacuum. Since the permittivity and the permeability are constant, these homogeneous Maxwell’s equations lead to homogeneous second order linear differential equations called as homogeneous Helmholtz equations given by
\[ \nabla^2 \mathbf{E} + k_0^2 \mathbf{E} = 0 \]
\[ \nabla^2 \mathbf{H} + k_0^2 \mathbf{H} = 0, \]

where \( k_0 = \omega / c \), the constant \( c \) is the speed of light and, in wavelength normalized coordinates, it reduces to \( k_0 = 2\pi \). The general solution to these linear differential equations is obtained by the superposition of basis functions with some coefficients. In Cartesian coordinates, the basis function is called a plane wave and is given by

\[ E(\mathbf{x}) = \exp[i \mathbf{k}_\pm \cdot \mathbf{x}]. \]

where \( \mathbf{k}_\pm = k_x \hat{x} + k_y \hat{y} + k_z(\mathbf{k}_\pm) \hat{z} \) is the wave (propagation) vector of which three components are satisfying

\[ k_z(\mathbf{k}_\pm) = \pm \left( k_0^2 - k_x^2 - k_y^2 \right)^{1/2}. \]

The plane wave is referred to as an upward wave if its wave vector has positive sign on its \( z \)-component. On the other hand, the plane wave is referred to as a downward wave if its wave vector has negative sign on its \( z \)-component. In order to distinguish the two types of plane waves, the subscript (+) for upward wave or (−) for downward wave is added to the wave vector as \( \mathbf{k}_+ \) or \( \mathbf{k}_- \). If there is no subscript, the wave vector is considered to be an upward plane wave. If the \( z \)-component has an imaginary value, the plane wave is called an evanescent wave, otherwise, it is called as propagating wave.
The direction cosine vector is denoted as \( \mathbf{s} = (\alpha, \beta, \gamma) \) and it is defined by the angles between the wave vector and the three coordinate axes. Thus the wave vector is represented by use of the direction cosine vector as

\[
\mathbf{k}_\pm = 2\pi \mathbf{s}_\pm .
\] (2.5)

If the plane wave is a propagating wave, the direction cosines can be represented in terms of the inclination angle \( 0 < \theta < \pi/2 \) and the azimuthal angle \( 0 < \varphi < 2\pi \) given by

\[
\begin{align*}
\alpha &= \sin \theta \cos \varphi \\
\beta &= \sin \theta \sin \varphi \\
\gamma &= \pm \cos \theta
\end{align*}
\] (2.6)

where the subscript \((+)\) is for upward plane waves and the subscript \((-)\) is for downward plane waves. If the subscript is omitted, it is considered to be an upward plane wave.

Using direction cosines, the electric field of an upward or downward plane wave is represented by

\[
\mathbf{E}(x) = \hat{e} \exp \left[ i 2\pi \mathbf{s}_\pm \cdot x \right],
\] (2.7)

where \( \hat{e} \) is the unit direction vector of the electric field. From Faraday's law \( \nabla \times \mathbf{E} = i\omega \mu_0 \mathbf{H} \), the corresponding magnetic field is given by

\[
\mathbf{H}(x) = \hat{h} \frac{1}{c\mu_0} \exp \left[ i 2\pi \mathbf{s}_\pm \cdot x \right],
\] (2.8)
where $\hat{h}$ is the unit vector of the magnetic field and it is given by $\hat{h} = \mathbf{s}_x \times \hat{e}$. Throughout this dissertation, incident wave is assumed to be a downward plane wave unless specified. However, for this particular plane wave, its direction cosines, $s_i$, are defined as

$$
\alpha_i \equiv -\sin \theta_i \cos \varphi_i,
\beta_i \equiv -\sin \theta_i \sin \varphi_i,
\gamma_i \equiv +\cos \theta_i,
$$

(2.9)

where $\theta_i$ is the inclination angle and $\varphi_i$ is the azimuthal angle of the incident wave. Equation (2.9) is inconsistent to the definition of the direction cosine defined in Eq. (2.6), but this one exception simplifies many expressions in later.

$\mathbf{E}(\mathbf{k}_i)$ is the spectrum of the electric field $\mathbf{E}(\mathbf{x}_i)$, and it is obtained by taking the Fourier transform as

$$
\mathbf{E}(\mathbf{k}_i) = \frac{1}{(2\pi)^3} \int d^2 k_i \mathbf{E}(\mathbf{x}_i) \exp[-i\mathbf{k}_i \cdot \mathbf{x}_i],
$$

(2.10)

and $\mathbf{E}(\mathbf{x}_i)$ can be obtained by taking the inverse Fourier transform as

$$
\mathbf{E}(\mathbf{x}_i) = \int d^2 k_i \mathbf{E}(\mathbf{k}_i) \exp[i\mathbf{k}_i \cdot \mathbf{x}_i],
$$

(2.11)

where $\mathbf{x}_i = x\hat{x} + y\hat{y}$ and $\mathbf{k}_i = k_x\hat{x} + k_y\hat{y}$ are two dimensional vectors. Using direction cosines, the spectrum and the electric field is expressed by

$$
\mathbf{E}(\mathbf{s}_i) = \mathcal{F}[\mathbf{E}(\mathbf{x}_i)] = \int d^2 \mathbf{s}_i \mathbf{E}(\mathbf{x}_i) \exp[-i2\pi\mathbf{s}_i \cdot \mathbf{x}_i],
$$

(2.12)
and

\[ E(x_i) = \mathcal{F}[E(s_i)] = \int d^2 s_i E(s_i) \exp[i2\pi s_i \cdot x_i], \]  

(2.13)

where \( s_i = \alpha \hat{x} + \beta \hat{y} \), \( \mathcal{F} \) denotes the Fourier transform operator and \( \mathcal{F}^\dagger \) denotes the inverse Fourier transform operator. Sometimes, instead of direction cosines, spatial frequencies \( k_i = 2\pi f_i \) are used to describe the spectrum where \( f_i = f_\alpha \hat{x} + f_\beta \hat{y} \). Since the unit of the length is the wavelength of interest, the amplitudes of spatial frequencies are the same as those of direction cosines. Otherwise, the relation between them is given by \( s_i = \lambda f_i \) where \( \lambda \) is the wavelength of interest.

### 2.1.2 Spherical Wave

If there are sources in the vacuum, the electric and magnetic fields satisfy the inhomogeneous Maxwell’s equations given by

\[
\begin{align*}
\nabla \cdot \varepsilon_0 E &= \rho \\
\nabla \cdot \mu_0 H &= 0 \\
\n\nabla \times E &= i\omega \mu_0 H \\
\n-\nabla \times H &= i\omega \varepsilon_0 E - J
\end{align*}
\]

(2.14)

where \( \rho \) is a free charge and a \( J \) is a free current. These inhomogeneous Maxwell’s equations lead to inhomogeneous second order linear differential equations referred to as inhomogeneous Helmholtz equations given by
\[ \nabla^2 \mathbf{E} + k_0^2 \mathbf{E} = -i \omega \mu_0 \mathbf{J} + \nabla \rho \epsilon_0, \]
\[ \nabla^2 \mathbf{H} + k_0^2 \mathbf{H} = -\nabla \times \mathbf{J}. \quad (2.15) \]

The solution to these differential equations is obtained by taking the convolution integral of the source term and the Green’s function which is the solution to the equation, in Cartesian coordinates, given by

\[ \nabla^2 g(\mathbf{x} | \mathbf{x}') + k_0^2 g(\mathbf{x} | \mathbf{x}') = -\delta(\mathbf{x} | \mathbf{x}'), \quad (2.16) \]

Where \( g(\mathbf{x} | \mathbf{x}') \) is the Green’s function and it is given by [79]

\[ g(\mathbf{x} | \mathbf{x}') = \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} \exp\left[i k_0 |\mathbf{x} - \mathbf{x}'|\right] \]
\[ = \frac{1}{(2\pi)^2} \int d^2k_z \frac{i}{2k_z(k_\perp)} \exp\left[i (k_\perp \cdot (\mathbf{x}_\perp - \mathbf{x}'_\perp))\right] \exp\left[i k_z(z - z')\right], \quad (2.17) \]

where \( \mathbf{x}' \) is the source position and \( \mathbf{x} \) is an observing position. The first equation is known as the (diverging only) spherical wave and the second equation is the Fourier space representation of the Green’s function. Using direction cosines, the Green’s function is written by

\[ g(\mathbf{x} | \mathbf{x}') = \int d^2s \frac{i}{4\pi\gamma} \exp\left[i 2\mu s_\perp \cdot (\mathbf{x} - \mathbf{x}')\right], \quad (2.18) \]

for the region \( z - z' > 0 \) and

\[ g(\mathbf{x} | \mathbf{x}') = \int d^2s \frac{i}{4\pi\gamma} \exp\left[i 2\mu s_\perp \cdot (\mathbf{x} - \mathbf{x}')\right], \quad (2.19) \]
for the region \( z - z' < 0 \). When the observing point is located above the source, the Green’s function is represented by the superposition of only upward plane waves. On the other hand, the Green’s function is the superposition of only downward plane waves if the position of observing point is below the source. Taking the Gradient on the both sides of the Eqs. (2.18) and (2.19), it is obtained

\[
\nabla g(x | x') = - \int d^2 s l \frac{s_z}{2\gamma} \exp[i2\pi s_z \cdot (x - x')],
\]

(2.20)

where \( s_z \) is for the region \( z - z' > 0 \) and \( s_z \) is for the region \( z - z' < 0 \). Especially, the \( z \)-component of the \( \nabla g(x | x') \) becomes

\[
\hat{z} \cdot \nabla g(x | x') = \frac{\partial}{\partial z} g(x | x')
\]

\[
= -\frac{1}{2} \int d^2 s l \exp[i2\pi s_z \cdot (x - x')] = -\frac{1}{2} \mathcal{F}'\{\exp[i2\pi \gamma z (z - z')]\},
\]

(2.21)

which is the inverse Fourier transform of the exponent term. The term \( z \)-component of \( \nabla g(x | x') \) can be directly calculated from the first equation in Eq.(2.17) as

\[
-2 \frac{\partial}{\partial z} g(x | x') = \left( \frac{1}{2\pi |x - x'|} - i \right) \frac{|z - z'|}{|x - x'|} g(x | x').
\]

(2.22)

Combining equations (2.21) and (2.22), and if the observing position is far from the source position \( |x - x'| >> 1 \), it is approximately

\[
\mathcal{F}'\{\exp[i2\pi \gamma z (z - z')]\} \approx -i\gamma z g(x | x'),
\]

(2.23)
where the subscript (+) is for \( z - z' > 0 \) and the subscript (-) is for \( z - z' < 0 \). The term \( \gamma_\perp \) is called as obliquity factor and the term \((-i)\) is equivalent to a \( 3\pi/2 \) phase delay.

### 2.1.3 Cylindrical Wave

If the distribution of the source is symmetric about the y-axis, the electric and magnetic field does not depend on the y variables. In this situation, the two-dimensional Green’s function is the solution to the differential equation given by

\[
\nabla^2 g(x_\perp | x'_\perp) + k_0^2 g(x_\perp | x'_\perp) = -\delta(x_\perp | x'_\perp),
\]

and it is given by

\[
g(x_\perp | x'_\perp) = \frac{i}{4} H_0^{(i)}(k_0 |x_\perp - x'_\perp|),
\]

where \( x_\perp = x\hat{x} + z\hat{z} \) is a two dimensional vector lying on the x-z plane and \( H_0^{(i)} \) is the zeroth order Hankel function of the first kind. This Green’s function is referred to as a cylindrical wave and it is useful when analyzing scattering one-dimensional surfaces.


2.1.4 Surface Statistics

We assume that the space is divided by a boundary surface S. It is also assumed that the space above the boundary surface is the vacuum and that the boundary surface is ideally conducting. The incident wave is a downward plane wave and it is scattered by the boundary surface. Since the surface is perfectly conducting, there is no field in the conducting region.

In Cartesian coordinates, the surface profile function is denoted by $\zeta(x)$. It is assumed that the surface profile function is continuous, single valued and differentiable to the spatial variable $x$ and $y$ at least once. Also, it is assumed that the surface heights are normally distributed which means the density function of the height is a Gaussian probability density function. The mean surface is set to be zero $\langle \zeta(x) \rangle = 0$ where $\langle \cdots \rangle$ denotes average over the $x$-$y$ plane, and the rms roughness is defined by standard deviation of the Gaussian probability density function. It is assumed that the surface is statistically stationary which is expressed by

$$\langle \zeta(x_1)\zeta(x_1 + x'_1) \rangle = C(x'_1), \quad (2.26)$$

where $x_1$ and $x'_1$ are two vectors lying on the $x$-$y$ plane and $C(x'_1)$ is the autocorrelation function of the surface profile function. If the surface is statistically stationary, the autocorrelation function depends only on the distance of the two variables, $x'_1$. In addition, the surface profile is assumed to be ergodic which is expressed by

$$\langle \zeta(x_1)\zeta(x_1 + x'_1) \rangle = \lim_{A \to \infty} \frac{1}{A} \int_{-\infty}^{\infty} d^2 x_1 \zeta(x_1)\zeta(x_1 + x'_1), \quad (2.27)$$
where $A$ is the area of the surface. If the surface is ergodic, the spatial average over the infinite plane converges to the ensemble average.

The power spectral density function is defined by the Absolute-Square of the Fourier transform of the surface profile function expressed by

$$W(s_l) = |\mathcal{F}[\zeta(x_l)]|^2. \quad (2.28)$$

Comparing equations (2.26), (2.27) and (2.28), the relation between autocorrelation function and surface PSD function is given by

$$W(s_l) = \mathcal{F}[C(x_l)]. \quad (2.29)$$

Therefore, the statistical properties of the random rough surfaces are characterized by either its auto-correlation function or its surface PSD function.

### 2.2 Integral Equations for Surface Scattering

In this section, the integral equations for surface scattering, especially from an ideally conducting two-dimensional rough surface, are derived using the vector Green’s theorem.

Consider a virtual volume $V_0$ in vacuum and the volume is enclosed by a composite surface $S_0$ given by $S_0 = S_{1\infty} + S_{2\infty}$ as shown in Figure 2-1.
It is assumed that there is no source in the volume. The electric field \( \mathbf{E} = \mathbf{E}(\mathbf{r}) \) and the magnetic field \( \mathbf{H} = \mathbf{H}(\mathbf{r}) \) inside this volume are satisfying the homogeneous Maxwell’s equations and, from the homogeneous (scalar) Helmholtz equations, the fields are satisfying the homogeneous vector Helmholtz equations given by

\[
\begin{align*}
\nabla \times \nabla \times \mathbf{E} - k_0^2 \mathbf{E} &= 0 \\
\nabla \times \nabla \times \mathbf{H} - k_0^2 \mathbf{H} &= 0.
\end{align*}
\]

(2.30)

where the vector identity \( \nabla (\nabla \cdot \mathbf{E}) = \nabla \times \nabla \times \mathbf{E} + \nabla^2 \mathbf{E} \) is used. Applying vector Green’s theorem, the electric field inside the volume \( V \) satisfies the equation

\[
\int_{V_0} d\mathbf{v}' \left( \hat{\mathbf{a}} g \right) \cdot \nabla' \times \nabla' \times \mathbf{E}' - \mathbf{E}' \cdot \nabla' \times \nabla' \times \left( \hat{\mathbf{a}} g \right) = \int_{S_0} d\mathbf{a}' \hat{n}' \cdot \left[ \left( \hat{\mathbf{a}} g \right) \times \nabla' \times \mathbf{E}' - \mathbf{E}' \times \nabla' \times \left( \hat{\mathbf{a}} g \right) \right]_{r \in S_0},
\]

(2.31)
where \( g = g(r \mid r') \) is the Green’s function, \( \hat{n}' = \hat{n}(r') \rvert_{r=r_0} \) is the unit normal vector directed inward from the surface \( S_0 \) and \( \hat{a} = \hat{a}(r) \) is an unit vector in arbitrary direction. Also, simplified notations \( \mathbf{E}' = \mathbf{E}(r') \), \( \mathbf{H}' = \mathbf{H}(r') \) are used. Before simplify the equation, it is obtained that

\[
\nabla \times (\hat{a} g) = (\nabla g) \times \hat{a} \\
\n\nabla \times \nabla \times (\hat{a} g) = \hat{a} k_0^2 g + \nabla (\hat{a} \cdot \nabla g) + \hat{a} \delta(r - r').
\]

Substituting the identities, the left side of Eq.(2.31) becomes

\[
(LS) = - \int_{V_0} dv' \left[ \mathbf{E}' \cdot \hat{a} \delta(r - r') + \mathbf{E}' \cdot \nabla' \left( \hat{a} \nabla' g \right) \right] \\
= - \int_{V_0} dv' \left[ \mathbf{E}' \cdot \hat{a} \delta(r - r') + \nabla' \left[ (\hat{a} \cdot \nabla' g) \mathbf{E}' \right] - (\hat{a} \cdot \nabla' g) \nabla' \cdot \mathbf{E}' \right], \quad (2.33)
\]

\[
= - \int_{V_0} dv' \left[ \mathbf{E}' \cdot \hat{a} \delta(r - r') \right] + \int_{S_0} d\alpha' \hat{n}' \cdot \left[ (\hat{a} \cdot \nabla' g) \mathbf{E}' \right]
\]

where the vector identity \( A(\nabla \psi) = \nabla (\psi A) - \psi (\nabla A) \), free space Gauss law \( \nabla \cdot \mathbf{E} = 0 \) and divergence theorem are used. Since the term \((\hat{a} \cdot \nabla' g)\) is a scalar quantity, Eq.(2.33) can be further simplified to

\[
(LS) = - \hat{a} \cdot \mathbf{E} + \hat{a} \cdot \int_{S_0} d\alpha' \left[ (\nabla' g) (\hat{n}' \cdot \mathbf{E}') \right], \quad (2.34)
\]

for the case where \( r \) is inside of the volume \( V_0 \) and

\[
(LS) = \hat{a} \cdot \int_{S_0} d\alpha' \left[ (\nabla' g) (\hat{n}' \cdot \mathbf{E}') \right], \quad (2.35)
\]

for the case where \( r \) is outside of the volume \( V_0 \).
On the other hand, substituting the identities given in Eq. (2.32), the right side of Eq. (2.31) can be simplified to

\[
(RS) = \int_{S_0} da' \hat{n}' \cdot \left[ (g \hat{a}) \times (i \omega \mu_0 \mathbf{H}') - \mathbf{E}' \times \left[ (\nabla' g) \times \hat{a} \right] \right]_{r \in S_0},
\]

\[
= \int_{S_0} da' \left[ -i \omega \mu_0 g \hat{n}' \cdot (\mathbf{H}' \times \hat{a}) + \hat{n}' \cdot \left[ \mathbf{E}' \times \left[ \hat{a} \times (\nabla' g) \right] \right] \right]_{r \in S_0},
\]

\[
= \int_{S_0} da' \left[ -i \omega \mu_0 g \hat{a} \cdot (\hat{n}' \times \mathbf{H}') + (\hat{n}' \times \mathbf{E}') \cdot \left[ \hat{a} \times (\nabla' g) \right] \right]_{r \in S_0},
\]

where the vector identity \( \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \) and Faraday's law are used. Applying the identity once more, the right side of the equation is further simplified to

\[
(RS) = \hat{a} \cdot \int_{S_0} da' \left[ -i \omega \mu_0 g (\hat{n}' \times \mathbf{H}') + (\nabla' g) \times (\hat{n}' \times \mathbf{E}') \right]_{r \in S_0}.
\]

Using the equations (2.34) and (2.37), since \( \hat{a} \) is an arbitrary direction vector, the Eq. (2.31) reduces to

\[
\mathbf{E} = \int_{S_0} da' \left[ i \omega \mu_0 g (\hat{n} \times \mathbf{H}') - (\nabla' g) \times (\hat{n} \times \mathbf{E}') + (\nabla' g) (\hat{n}' \cdot \mathbf{E}') \right]_{r \in S_0},
\]

for the case where \( \mathbf{r} \) is inside of the volume \( V_0 \) and

\[
0 = \int_{S_0} da' \left[ i \omega \varepsilon_0 g (\hat{n} \times \mathbf{E}') - (\nabla' g) \times (\hat{n} \times \mathbf{H}') + (\nabla' g) (\hat{n}' \cdot \mathbf{H}') \right]_{r \in S_0},
\]

for the case where \( \mathbf{r} \) is outside of the volume \( V_0 \). Similarly, it is obtained

\[
\mathbf{H} = \int_{S_0} da' \left[ -i \omega \varepsilon_0 g (\hat{n} \times \mathbf{E}') - (\nabla' g) \times (\hat{n} \times \mathbf{H}') + (\nabla' g) (\hat{n}' \cdot \mathbf{H}') \right]_{r \in S_0},
\]

for the case where \( \mathbf{r} \) is inside of the volume \( V_0 \) and

\[
0 = \int_{S_0} da' \left[ i \omega \varepsilon_0 g (\hat{n} \times \mathbf{H}') - (\nabla' g) \times (\hat{n} \times \mathbf{E}') + (\nabla' g) (\hat{n}' \cdot \mathbf{E}') \right]_{r \in S_0},
\]
\[ 0 = \int_{S_0} da' \left[ -i\omega\varepsilon_0 g(\hat{n} \times E') - (\nabla' g) \times (\hat{n} \times H') + (\nabla' g) (\hat{n} \cdot H') \right]_{r = S_0}, \quad (2.41) \]

for the case where \( \mathbf{r} \) is outside of the volume \( V_0 \) for the magnetic field. The equations (2.38) through (2.41) are called as Stratton-Chu equation for free space [14] and especially equations (2.38) and (2.40) are known as Huygens’ Principle and equations (2.39) and (2.41) are known as extinction theorem [79] for free space. Those equations state that the electric and magnetic field inside the volume is described in terms of their distributions on the boundary surface.

If there is a plane wave traveling through the volume and the sources generating the plane wave are located far away outside of the volume \( V_0 \), the total electric and magnetic field inside the volume is, from the linearity of the Helmholtz equation, given by the superposition of the incident field and the field satisfying equations (2.38) and (2.40) as \( \mathbf{E}_t = \mathbf{E}_i + \mathbf{E} \) and \( \mathbf{H}_t = \mathbf{H}_i + \mathbf{H} \) where \( \mathbf{E}_i = \mathbf{E}_i(\mathbf{r}) \) and \( \mathbf{H}_i = \mathbf{H}_i(\mathbf{r}) \) are the electric and magnetic fields of the incident plane wave respectively.

Now, consider the situation illustrated in Figure 2-2. The volume \( V_0 \) is split into vacuum region \( V_1 \) and conducting region \( V_2 \) by the boundary surface denoted as \( S \). The volume \( V_1 \) is enclosed by the surface \( S_1 \) given by \( S_1 = S + S_{1\infty} \) and the volume \( V_2 \) is enclosed by the surfaces \( S \) and \( S_{2\infty} \).
Figure 2-2: The volume $V_1$ is enclosed by the ideally conducting surfaces $S_{1\infty}$ and $S$. The volume $V_2$ is enclosed by the surfaces $S_{2\infty}$ and $S$.

Applying the Stratton-Chu equations, the electric and magnetic fields inside the volume $V_1$ satisfies

$$
E_1 = E_i + \int_S da \left[ i \omega \mu_0 g (\mathbf{n} \times \mathbf{H}'_i) - (\nabla' g) \times (\mathbf{n}' \times E'_i) + (\nabla' g) (\mathbf{n}' \cdot E'_i) \right]_{r \in S},
$$

$$
H_1 = H_i - \int_S da \left[ i \omega \epsilon_0 g (\mathbf{n} \times E'_i) + (\nabla' g) \times (\mathbf{n}' \times H'_i) - (\nabla' g) (\mathbf{n}' \cdot H'_i) \right]_{r \in S},
$$

where the simplified notation $E_j = E_j(r)_{r \in V_j}$ and $H_j = H_j(r)_{r \in V_j}$ are used. Similarly, the electric and magnetic fields outside the volume $V_1$ but inside the volume $V_0$ satisfies the integral equations given by

$$
E_2 = E_i + \int_S da \left[ i \omega \mu_0 g (\mathbf{n} \times \mathbf{H}'_i) - (\nabla' g) \times (\mathbf{n}' \times E'_i) + (\nabla' g) (\mathbf{n}' \cdot E'_i) \right]_{r \in S},
$$

$$
H_2 = H_i - \int_S da \left[ i \omega \epsilon_0 g (\mathbf{n} \times E'_i) + (\nabla' g) \times (\mathbf{n}' \times H'_i) - (\nabla' g) (\mathbf{n}' \cdot H'_i) \right]_{r \in S},
$$

(2.42)
Due to the fact that the tangential component of the electric field and the normal component of the magnetic field are continuous at the boundary between the vacuum and the ideal conductor, the boundary condition is imposed by

\[
\hat{n} \times (E_1 - E_2) = 0 \\
\hat{n} \cdot (H_1 - H_2) = 0.
\] (2.44)

The fields inside the volume \( V_2 \) are, together with the fact that there is fields are vanished inside of the ideal conductor without any skin depth, given by

\[
E_2 = 0 \\
H_2 = 0.
\] (2.45)

Therefore, the fields inside the vacuum region satisfy

\[
E_s = +\int_S da'[i \omega \mu_0 g (\hat{n}' \times H'_1) + (\nabla' g) (\hat{n}' \cdot E'_1)]_{r \leq S}, \quad H_s = -\int_S da' [i \omega \mu_0 g (\hat{n}' \times H'_1)]_{r \leq S},
\] (2.46)

where \( E_s = E_i - E_r \) is called the scattered electric field and \( H_s = H_i - H_r \) is called the scattered magnetic field. Similarly, the fields outside the volume \( V_i \) but inside the volume \( V_0 \) satisfy

\[
E_i = -\int_S da'[i \omega \mu_0 g (\hat{n}' \times H'_1) + (\nabla' g) (\hat{n}' \cdot E'_1)]_{r > S}, \quad H_i = +\int_S da' [i \omega \mu_0 g (\hat{n}' \times H'_1)]_{r > S}.
\] (2.47)

Equation (2.47) states that the incident field distribution is reformed at the boundary and Eq.(2.46) states that the propagation of the reformed field is the scattered field.
In conclusion, using the vector Green’s theorem, we derived the Stratton-Chu equation and it is applied to the scattering problem from an ideally conducting surface. The scattered fields are related to the fields at the boundary surface and the relation is given by the surface integral equations.

2.3 Integral Equations for the Equivalent Problem to Surface Scattering

In this section, the integral equations for surface scattering from an ideally conducting two-dimensional surface are derived using the vector potentials. Let us consider the situation illustrated in Figure 2-3. We are interested in the vacuum space inside the volume $V_0$. Let assume that an incident downward wave propagates through the volume, but there is no conducting boundary surface. Instead, there is a free charge $\rho$ and a current $J$ along the virtual surface $S$. For convenience, the space above the virtual surface is referred to as region $V_1$ and the space below the virtual surface is referred to as region $V_2$. It is also assumed that the current $J$ satisfies $\mathbf{n} \cdot \mathbf{J} = 0$ and the charge satisfies the continuity equation given by $0 = \nabla \cdot \mathbf{J} - i\omega \rho$. Finally, it is assumed that, due to the distribution of the sources, there is no field in region $V_2$. 
Before moving forward, it must be understood that, since it is assumed that the charge and the current exist only on the surface $S$, the surface charge must have the form of $\rho = \rho(r)\delta(r-r')|_{r \in S}$ and the surface current also must have the form of $J = J(r)\delta(r-r')|_{r \in S}$.

These conditions are justified by the following. Assume that the charge and current are distributed in a thin volume of which its center is the surface $S$. As shrinking the depth of the thin volume, the densities of the sources must be increased. Finally, if we confine the sources in the ideal surface without any thickness, its density must be infinite.

The free charge and current generate a secondary electric field $E_s$ and a magnetic field $H_s$, and the total fields in the both regions are given by the summation of the incident and secondary fields. Since all the fields vanish in region $V_2$, the electric and magnetic field satisfy
\[ 0 = \mathbf{E}_i + \mathbf{E}_s, \]
\[ 0 = \mathbf{H}_i + \mathbf{H}_s. \]  

(2.48)

for the region \( V_2 \) and the boundary condition is given by

\[
\hat{n} \cdot (\mathbf{E}_i + \mathbf{E}_s) = \varepsilon_0^{-1} \rho \\
\hat{n} \cdot (\mathbf{H}_i + \mathbf{H}_s) = 0 \\
\hat{n} \times (\mathbf{E}_i + \mathbf{E}_s) = 0, \\
\hat{n} \times (\mathbf{H}_i + \mathbf{H}_s) = \mathbf{J}
\]  

(2.49)

for the region \( V_1 \). Since the secondary fields are generated by the free charge and current, they satisfy the inhomogeneous Maxwell’s equations, the two coupled inhomogeneous Helmholtz equations given by

\[
\nabla^2 \mathbf{E}_s + k_0^2 \mathbf{E}_s = -i\omega\mu_0 \mathbf{J} + \nabla \rho / \varepsilon_0 \\
\nabla^2 \mathbf{H}_s + k_0^2 \mathbf{H}_s = -\nabla \times \mathbf{J}
\]  

(2.50)

To solve the inhomogeneous Helmholtz equation, the vector potentials \( \mathbf{A} \) and the scalar potentials \( \Phi \) are introduced as

\[
\mathbf{E}_s = i\omega \mathbf{A} - \nabla \Phi \\
\mu_0 \mathbf{H}_s = \nabla \times \mathbf{A}
\]  

(2.51)

and the two potentials are chosen to satisfy the Lorentz condition given by \( \nabla \cdot \mathbf{A} - i\omega\varepsilon_0 \mu_0 \Phi = 0 \). Substituting Eq.(2.51) into Eq.(2.50), the two uncoupled inhomogeneous Helmholtz equations are obtained by
\[
\begin{align*}
\nabla^2 \mathbf{A} + k_0^2 \mathbf{A} &= -\mu_0 \mathbf{J} \\
\nabla^2 \Phi + k_0^2 \Phi &= -\varepsilon_0^{-1} \rho.
\end{align*}
\] (2.52)

The solutions to these equations are well known and are given in integral form by [80]

\[
\begin{align*}
\mathbf{A} &= \int_S da' g(\mu_0 \mathbf{J'})_{r'\in S} \\
\Phi &= \int_S da' g(\varepsilon_0^{-1} \rho')_{r'\in S},
\end{align*}
\] (2.53)

where the simplified notation \( \mathbf{J'} = \mathbf{J}(r') \) and \( \rho' = \rho(r') \) are used and the volume integral is reduced to a surface integral because of the delta function in the source terms. Using the vector and scalar potentials, the solution to the Eq.(2.50) is written by

\[
\begin{align*}
\mathbf{E}_s &= +\int_S da' \left[ g(i\omega\mu_0 \mathbf{J'}) + (\nabla' g)(\varepsilon_0^{-1} \rho') \right]_{r'\in S} \\
\mathbf{H}_s &= -\int_S da' \left[ (\nabla' g) \times \mathbf{J'} \right]_{r'\in S},
\end{align*}
\] (2.54)

where the identities \( \nabla(g\rho') = g\nabla\rho' + \rho\nabla g \), \( \nabla \times (g\mathbf{J'}) = g\nabla \times \mathbf{J'} + \nabla g \times \mathbf{J'} \) and \( \nabla g = -\nabla' g \) are used. Together with \( \mathbf{E}_2 = \mathbf{E}_i + \mathbf{E}_s \) and \( \mathbf{H}_2 = \mathbf{H}_i + \mathbf{H}_s \), substituting Eq.(2.54) into the condition for region \( V_2 \) given in Eq.(2.48) gives

\[
\begin{align*}
\mathbf{E}_i &= -\int_S da' \left[ g(i\omega\mu_0 \mathbf{J'}) + (\nabla' g)(\varepsilon_0^{-1} \rho') \right]_{r'\in S} \\
\mathbf{H}_i &= +\int_S da' \left[ (\nabla' g) \times \mathbf{J'} \right]_{r'\in S},
\end{align*}
\] (2.55)

for the case where \( r \in V_2 \). Equation (2.55) is the integral equation which the free charge and current satisfy. Equations (2.54) and (2.55) can be interpreted as follows. If the virtual surface is ideally conducting, the incident fields induce free charge and current, and they are forced to exist
on the surface only. These induced sources generate secondary fields which cancel the incident field inside the conductor as shown in Eq.(2.55). And the generated field in upper region is called the scattered field in Eq.(2.54).

In the view of Eq.(2.49), it can be replaced that \( \mathbf{J} = \mathbf{n} \times \mathbf{H} \) and \( \rho = \varepsilon_0 \mathbf{n} \cdot \mathbf{E} \). Then the equations (2.54) and (2.55) are changed to

\[
\mathbf{E}_s = + \int_S da' [i \omega \varepsilon_0 \mathbf{g} \mathbf{n} \times (\mathbf{H}')] + (\mathbf{V}' \cdot \mathbf{n}) \mathbf{E}' \right]_{r = \infty},
\]

\[
\mathbf{H}_s = - \int_S da' [\mathbf{V}' \times (\mathbf{n} \times \mathbf{H}')]_{r = \infty},
\]

for the region \( V_1 \) and

\[
\mathbf{E}_i = - \int_S da' [i \omega \varepsilon_0 \mathbf{g} \mathbf{n} \times (\mathbf{H}')] + (\mathbf{V}' \cdot \mathbf{n}) \mathbf{E}' \right]_{r = \infty},
\]

\[
\mathbf{H}_i = + \int_S da' [\mathbf{V}' \times (\mathbf{n} \times \mathbf{H}')]_{r = \infty},
\]

for the region \( V_2 \). Equations (2.56) and (2.57) are thus identical to equations (2.46) and (2.47) respectively [81].

In some applications, another form of Eq.(2.57) has been considered. Recalling the curl of the gradient of any scalar field is always zero, taking curl curl to Eq.(2.57) leads to

\[
0 = k_0^2 \mathbf{E}_{i,2} + \nabla \times \nabla \times \int_S da' [g (i \omega \mu_0 \mathbf{J}')]_{r = \infty},
\]

\[
0 = k_0^2 \mathbf{H}_{i,2} - \nabla \times \nabla \times \int_S da' [(\mathbf{V}' g) \times \mathbf{J}]_{r = \infty}.
\]

32
where the vector homogeneous Helmholtz equation is applied to the incident fields. Equation (2.58) has no dependency on the surface charge density function and this form is known as the extended boundary condition for a perfect conductor [81-83].

Equations (2.54) and (2.55) will be used to solve our scattering problem and will be referred to as the Stratton-Chu equation in this dissertation. Even though the charge and current are mathematically treated as a free charge and current, it was assumed that there is no boundary surface in this section. The sources are thus referred to as an induced surface charge and current, and the virtual surface will be called a boundary surface in the rest of this dissertation.

In conclusion, we derived the integral equations for surface scatter using vector potentials and it was shown that the results are identical to the integral equations calculated by using the vector Green’s theorem. The apparent approach looks different, but the essential physics in the two approaches are the same.

2.4 Scattered Field

In this section, the scattered field is represented by the superposition of upward plane waves using the integral equation derived in the previous section. If the induced surface current $\mathbf{J}$ is obtained from the boundary condition, the scattered field can be calculated by using Eq.(2.54). Let us consider the integral equation for the scattered magnetic field for the region $V_i$ given by
\[ \mathbf{H}_s(\mathbf{r})_{\mathbb{R} \forall \mathbf{V}_1} = -\int_S da'[\nabla' g \times \mathbf{J}(\mathbf{r}')]_{x'=S, x=V_1}. \] (2.59)

For the rest of this section, it is assumed that \( \mathbf{r} \in \mathbf{V}_1 \). In Cartesian coordinates, by taking the Gradient of the Fourier space representation of the Green’s function with respect to the prime coordinates, the \( \nabla' g \) term in the previous equation becomes

\[ \nabla' g = \int d^2s' \left[ \frac{s}{2\gamma} \exp[i2\pi s \cdot (x - x')] \right]_{x' \in S'}, \] (2.60)

where the direction cosine vector \( s = s_+ \) is for upward plane waves and the subscript (+) is dropped. Substituting Eq.(2.60) into Eq.(2.59), the scattered magnetic field is written by

\[ \mathbf{H}_s(\mathbf{r}) = -\int d^2s' \left( \frac{1}{\gamma s} \mathbf{x} \times \mathbf{E}(s_+) \right) \exp[i2\pi s \cdot x], \] (2.61)

where

\[ \mathbf{E}(s_+) = \frac{1}{2} \int_{S'} |n| d^2x' \mathbf{J}(x'_+) \exp[-i2\pi s \cdot x']_{x'=S'}, \] (2.62)

and the infinitesimal area of the boundary surface \( da' = |n| d^2x' \) is used. Equation (2.62) states that only the perpendicular components of \( \mathbf{E}(s_+) \) to the direction cosine vector \( s \) contribute to the scattered field. From the free space Ampere's law \( \nabla \times \mathbf{H} = -i\omega \varepsilon_0 \mathbf{E} \), the electric field is obtained by
\[ \mathbf{E}_i(\mathbf{r}) = \frac{k_0}{\omega \varepsilon_0} \int d^2 s_i \left\{ \frac{1}{\gamma} \left[ (\mathbf{s}_i \times \mathbf{E}(\mathbf{s}_i)) \times \mathbf{s}_i \right] \right\} \exp \left[ i2\pi s_i \cdot \mathbf{x} \right], \]  

(2.63)

where \( k_0 = 2\pi \) and \( \mathbf{e} = \hat{\mathbf{h}} \times \mathbf{s} \) are used. The amplitude of the time-averaged Poynting vector \( \mathbf{S}(\mathbf{s}_i) \) for each upward plane wave is given by

\[ \langle \mathbf{S}(\mathbf{s}_i) \rangle = \frac{\varepsilon_0 c}{2} \left| \mathbf{E}(\mathbf{s}_i) \right|^2 = \frac{\mu_0 c}{2} \left| \mathbf{H}(\mathbf{s}_i) \right|^2 = \frac{\eta}{2} \left| \mathbf{s} \times \mathbf{E}(\mathbf{s}_i) \right|^2, \]  

(2.64)

where \( \eta = \left( \frac{\mu_0}{\varepsilon_0} \right)^{1/2} \) is the vacuum impedance. The power per unit surface area for a plane wave is given by the Poynting vector of the plane wave projected to the mean surface normal direction which is given by \( P(\mathbf{s}_i) = \mathbf{S}(\mathbf{s}_i) \cdot \hat{z} = |\mathbf{S}(\mathbf{s}_i)|\gamma \). Thus, the total scattered power per unit surface area is given by

\[ P = \frac{\eta}{2} \int d\Omega \left| \mathbf{s} \times \mathbf{E}(\mathbf{s}_i) \right|^2, \]  

(2.65)

where \( d\Omega \) is the solid angle and the relation \( d^2 s_i = \gamma d\Omega \) is used. If the magnetic field of the incident plane wave is assumed to be \( \mathbf{H}_i = \exp \left[ -i2\pi s_i \cdot \mathbf{x} \right] \), the total incident power per unit surface area is given by \( P_i = \eta \gamma_i / 2 \). The total scattered power per unit area per unit solid angle divided by the total incident power per unit area is, therefore, given by

\[ \frac{1}{P_i} \frac{dP}{d\Omega} = \frac{\left| \mathbf{s} \times \mathbf{E}(\mathbf{s}_i) \right|^2}{\gamma_i}, \]  

(2.66)
which is the scattering intensity or the differential reflection coefficient. In direction cosine space, it is obtained that

\[
\frac{L}{E} = \frac{1}{P} \frac{dP(s)}{d^2s} = \left| \mathbf{s} \times \mathbf{E}(s) \right|^2, \\
\gamma, \gamma
\]

where \( L \) represents the Radiance and \( E \) represents the Irradiance. Regarding surface scattering from an ideal conductor, the quantity \( L/E \) is the bidirectional reflectance distribution function (BRDF) commonly referred to in radiometry.

Sometimes, analyzing the polarization of the scattered wave is required. If the scattered electric field vector lies on the plane defined by the direction cosine vector \( \mathbf{s} \) and the \( \hat{z} \) direction, it is referred to as TM polarized scattering light. If the electric field vector is perpendicular to the direction cosine and to the magnetic field vector, it is referred to as TE polarized scattering light.

Let us introduce the two unit direction vectors \( \hat{\mathbf{s}}_+ = \hat{\mathbf{s}}_+(s) \) and \( \hat{\mathbf{p}}_+ = \hat{\mathbf{p}}_+(s) \) given by

\[
\hat{\mathbf{s}}_+(s) = \frac{\mathbf{s} \times \hat{z}}{|\mathbf{s} \times \hat{z}|} = \rho^{-1}(\beta\hat{x} - \alpha\hat{y}), \\
\rho
\]

and

\[
\hat{\mathbf{p}}_+(s) = \mathbf{s} \times \hat{\mathbf{s}}_+ = \rho^{-1}[\alpha\gamma\hat{x} + \beta\gamma\hat{y} - \rho^2\hat{z}], \\
\rho
\]

where \( \rho = (\alpha^2 + \beta^2)^{1/2} \). The vector \( \hat{\mathbf{s}}_+ \) is lying on the plane defined by the direction cosine vector \( \mathbf{s} \) and the \( \hat{z} \) direction and the vector \( \hat{\mathbf{p}}_+ \) is perpendicular to the direction cosine and to
the $\mathbf{\hat{s}}_{s+}$. Together with the direction cosine, the three vectors form orthonormal right handed coordinates. If the direction of the scattered electric field is parallel to the $\mathbf{\hat{s}}_{s+}$, it is TE polarized. On the other hand, if the direction of the scattered electric field is parallel to the $\mathbf{\hat{p}}_{s+}$, it is TE polarized. Using these new coordinates, the $\mathbf{E}(\mathbf{s}_i)$ can be decomposed by

$$\mathbf{E}(\mathbf{s}_i) = \mathbf{E}_s(\mathbf{s}_i)\mathbf{\hat{s}}_{s+} + \mathbf{E}_p(\mathbf{s}_i)\mathbf{\hat{p}}_{s+} + \mathbf{\xi}(\mathbf{s}_i)s,$$  \hspace{1cm} (2.70)

where the first and second components are $\mathbf{E}_s(\mathbf{s}_i) = \mathbf{E}(\mathbf{s}_i) \cdot \mathbf{\hat{s}}_{s+}$ and $\mathbf{E}_p(\mathbf{s}_i) = \mathbf{E}(\mathbf{s}_i) \cdot \mathbf{\hat{p}}_{s+}$, and the last component is $\mathbf{\xi}(\mathbf{s}_i) = \mathbf{E}(\mathbf{s}_i) \cdot \mathbf{s}$. Using this vector decomposition, the Eq.(2.63) is rewritten by

$$\mathbf{E}_s(\mathbf{r}) = \frac{k_0}{\omega \varepsilon_0} \int d^2 s \left[ \frac{1}{\gamma} \left( \mathbf{E}_s(\mathbf{s}_i)\mathbf{\hat{s}}_{s+} + \mathbf{E}_p(\mathbf{s}_i)\mathbf{\hat{p}}_{s+} \right) \right] \exp[i2\pi\mathbf{s} \cdot \mathbf{x}],$$  \hspace{1cm} (2.71)

for the scattered electric field and

$$\mathbf{H}_{s,1}(\mathbf{x}) = \int d^2 s \left[ \frac{1}{\gamma} \left( \mathbf{E}_s(\mathbf{s}_i)\mathbf{\hat{p}}_{s+} - \mathbf{E}_p(\mathbf{s}_i)\mathbf{\hat{s}}_{s+} \right) \right] \exp[i2\pi\mathbf{s} \cdot \mathbf{x}],$$  \hspace{1cm} (2.72)

for the scattered magnetic field. The total scattered power per unit area per unit solid angle divided by the total incident power per unit area is, therefore, rewritten by

$$\frac{1}{P} \frac{dP}{d\Omega} = \frac{\left| \mathbf{E}_s(\mathbf{s}_i) \right|^2 + \left| \mathbf{E}_p(\mathbf{s}_i) \right|^2}{\gamma},$$  \hspace{1cm} (2.73)

where the first term in the right side corresponds to TE polarized scattered light and the second term corresponds to TM polarized scattered light.
The same method is applied to describe the polarization states of a downward plane wave. Let us introduce the two unit direction vectors \( \hat{\sigma}_{s-} = \hat{\sigma}_{s-}(s_\parallel) \) and \( \hat{\sigma}_{p-} = \hat{\sigma}_{p-}(s_\parallel) \) given by

\[
\hat{\sigma}_{s-}(s_\parallel) = \frac{s_\perp \times \hat{z}}{|s_\perp \times \hat{z}|} = \rho^{-1}(\beta\hat{x} - \alpha\hat{y}) = \hat{\sigma}_{s+}(s_\parallel), \tag{2.74}
\]

and

\[
\hat{\sigma}_{p-}(s_\parallel) = s_\perp \times \hat{\sigma}_{s-} = \rho^{-1}[\alpha\hat{x} - \beta\hat{y} - \rho^2\hat{z}] = -\hat{\sigma}_{p+}(s_\parallel) - 2\rho\hat{z}. \tag{2.75}
\]

Note that the \( z \)-component of a downward direction cosine has negative sign. If the direction of an downward electric field is parallel to the \( \hat{\sigma}_{s-} \), it is TE polarized and, meanwhile, if it is parallel to the \( \hat{\sigma}_{p+} \), it is TM polarized scattered light.

In conclusion, the scattering intensity is calculated from the integral expression of the scattered field by decomposing the scattered fields to upward plane waves. The expression of the scattering intensity can be applied to any surface profile as long as the profile function is single-valued, continuous, and differentiable at least once.
2.5 Ensemble Average of Scattering Intensity

In this section, the ensemble average of the scattered field is introduced and the coherent specular wave and incoherent scattering wave are distinguished. The ensemble average of the scattering intensity can be expressed by

\[
\left\langle \frac{1}{P_i} \frac{\partial P}{\partial \Omega} \right\rangle = \frac{1}{\gamma_i} \left[ \left| s \times \mathbf{E}(s_i) \right|^2 \right] = \frac{1}{\gamma_i} \left[ \left| s \times \mathbf{E}(s_i) \right|^2 \right] + \frac{1}{\gamma_i} \left[ \left\langle \left| s \times \mathbf{E}(s_i) \right|^2 \right\rangle - \left| s \times \mathbf{E}(s_i) \right|^2 \right].
\] 

(2.76)

The first term on the right side of Eq. (2.76) corresponds to coherent specular wave and the second term on the right side of the equation corresponds to incoherent scattering wave. Since, it can be interpreted that all refracted or reflected light, including specular light, is scattered light, the terminology ‘scattered field’ will be used for the refracted or reflected fields by both randomly rough surfaces or surfaces having a deterministic profile including flat surfaces. However, when regarding radiant intensity from random rough surfaces, only the second term in the Eq. (2.76) will be referred to as scattering intensity such as

\[
\left\langle \frac{1}{P_i} \frac{\partial P}{\partial \Omega} \right\rangle_s = \frac{1}{\gamma_i} \left[ \left\langle \left| s \times \mathbf{E}(s_i) \right|^2 \right\rangle - \left| s \times \mathbf{E}(s_i) \right|^2 \right].
\] 

(2.77)

And the coherent light corresponding to the first term in Eq. (2.76) will be referred to as specular light such as

\[
\left\langle \frac{1}{P_i} \frac{\partial P}{\partial \Omega} \right\rangle_p = \frac{1}{\gamma_i} \left| s \times \mathbf{E}(s_i) \right|^2.
\] 

(2.78)
Throughout this chapter, the surface scatter phenomenon is described in terms of surface integral equations. First, the Stratton-Chu equations are derived using the vector Green’s theorem and it is applied to scattering from ideally conducting rough surfaces. The same integral equation is derived using vector potentials by solving the equivalent problem. The scattering intensity is represented in terms of the power carried by propagating upward plane waves and the ensemble average of the scattering intensity is introduced for analyzing surface scatter from ideally conducting randomly rough surfaces.
CHAPTER 3: NUMERICAL APPROXIMATION

Relatively recently, together with the growing capacity and speed of computers, many numerical approaches to the scattering problem have been developed [16]. The numerical methods are categorized in either the integral or the differential method. In this dissertation, the integral methods are taken as our numerical approximation.

Many integral methods are based on the integral equations obtained in the previous chapter [16]. To solve the integral equation numerically, matrix inversion is needed and this process requires large amount of computer memory and much calculation time. The difference between most integral methods lies in the method of matrix inversion. Some integral methods use different basis functions to describe the surface current. We will use an integral equation method (IEM) based upon the integral equation discussed in Chapter 2 which represents the surface current in real space. We do not distinguish numerical methods having different matrix inversion algorithms because their result must converge to the same answer if each inversion method is treated correctly.

The limitation of the IEM lies in the manner of numerical treatment such as the finite number of sampling points and the precision of computations. Thus, as long as the numerical aspects are adequately handled, the results from this method are considered to be rigorously obtained. However, to solve the integral equations, a large number of sampling points are required and it is computationally intensive. Also, to calculate ensemble average of the scattered field from a randomly rough surface, many realizations are required and it is time consuming process.
The algorithm of the IEM for calculating scattered field from a one-dimensional random rough surface is stabilized and the speed of the calculation is reasonably fast. However, even though its results are continuously reported, calculating the scattered field from a two-dimensional random rough surface remains a challenging task [34]. In this chapter, the algorithms for calculating the scattered field using the IEM for both one and two-dimensional random rough surfaces are introduced. For convenience, the scattering from one-dimensional rough surfaces is referred to as two-dimensional scattering and the scattering from a two-dimensional rough surface is referred to as three-dimensional scattering because the two-dimensional Green’s function is used in the former case and the three dimensional Green’s function is used in the other case.

3.1 Scattering from Two-dimensional Surfaces

In this section, a numerical approach for solving the three-dimensional scattering problem is described. First, a finite size beam with a Gaussian amplitude distribution is taken our incident wave to remove edge effect. Then, the singularity of the Stratton-Chu equation is removed and it is converted into a system of linear equations using the method of moments. Finally, the fractional scattered intensity and ensemble average of it is expressed.
3.1.1 Incident Field

In Chapter 2, it was assumed that the incident wave was a monochromatic plane wave of which direction cosines are given by $s_i$. However, in the IEM, a single plane wave cannot be adopted as the incident wave because the limited sampled surface size produces undesired edge effects. To reduce the edge effect, a finite-size beam is used instead of the infinite-size plane wave. The finite-size beam can be obtained by the superposition of plane waves, where the spectrum of the beam is chosen to be a Gaussian function given by

$$G(s_i) = \frac{w^2}{4\pi} \exp\left\{ -\frac{w^2}{4} \left[ 2\pi (s_i + s_{i1}) \right]^2 \right\}, \quad (3.1)$$

where $w$ is the half-width of the Gaussian spectrum. Using this Gaussian incident beam, the incident electric field is represented by

$$E_i(x) = \frac{\omega H_0}{k_0} \int d^2 s_i \hat{e}_i G(s_i) \exp[i2\pi s_i \cdot x], \quad (3.2)$$

and, from the free-space Faraday's law $\nabla \times E = i\omega \mu_0 H$, the incident magnetic field is given by

$$H_i(x) = \int d^2 s_i \hat{h}_i G(s_i) \exp[i2\pi s_i \cdot x], \quad (3.3)$$

where $\hat{e}_i = \hat{e}_i(s_i)$ and $\hat{h}_i = \hat{h}_i(s_i)$ are the unit direction vectors of the incident fields and they satisfy the relation $\hat{e}_i \times \hat{h}_i = s_i$. Due to the Gaussian shape, the amplitude of the incident electric field is large near the center of the surface and it is negligible near the edge.
If the width of the Gaussian beam is infinity, which is the case of plane wave, the direction of the TE polarized incident electric field can be set to be the y-direction \( \hat{e} = \hat{y} \) without loss of generality. Similarly, it can be assumed that the direction of TM polarized incident electric field lies on the x-z plane. However, for the finite-width Gaussian beam, this intuitive polarization cannot be used because the incident beam is a superposition of many plane waves. Conventionally, for this Gaussian beam, TM polarized incident light is considered to be a superposition of downward plane waves whose electric field does not have a y-component [32].

With the relation between the direction cosine and the electric field \( s_\cdot \hat{e} = 0 \), the unit direction vector of the electric field for TM polarization can be defined by

\[
\hat{e}_{p,i} = \rho^{-1}(\gamma\hat{x} + \alpha\hat{z}),
\]

and, using the relation \( \hat{s} = \hat{e}_p \times s_- \), the direction of the electric field for TE polarization can be expressed by

\[
\hat{e}_{s,i} = \rho^{-1}(\alpha\hat{x} + \rho^2\hat{y} - \beta\hat{z}).
\]

In a similar manner, the unit direction vector of the magnetic field is represented by \( \hat{h}_{s,i} = \hat{e}_{p,i} \) and \( \hat{h}_p = -\hat{e}_{s,i} \). Using Eqs.(3.1) and (3.3), the incident power per unit surface area for the Gaussian beam is given by

\[
P_i = \frac{\eta}{2} \int d^2y |\gamma| |G(s_y)|^2,
\]
for both polarizations. The surface current \( \mathbf{J}_i = \mathbf{J}_i(\mathbf{x}) \) induced by the incident field is calculated by

\[
\mathbf{J}_i|_{\zeta(\mathbf{x})} = \mathbf{n} \times \mathbf{H}_i|_{\zeta(\mathbf{x})},
\]

with the definition of the surface normal vector given by

\[
\mathbf{n} = \mathbf{n}(\mathbf{x}_0) = -\zeta_x \hat{x} - \zeta_y \hat{y} + \hat{z},
\]

where \( \zeta_x = \partial \zeta / \partial x \) and \( \zeta_y = \partial \zeta / \partial y \). In IEM, the finite-size beam is use as incident wave and the finite beam width may affect the result.

### 3.1.2 Surface Field

In this section, the process of converting the integral equation derived in Chapter 2 into a system of linear equations is illustrated in detail. From the Stratton-Chu integral equations, the equations for the magnetic field are given by

\[
\left. \mathbf{H} \right|_{rr_{v1}} = \left. \mathbf{H} \right|_{rr_{v1}} - \int_S da' \left[ \nabla'(g) \times \mathbf{J}' \right]_{r_{ss,v1}},
\]

and the current satisfies the condition \( \mathbf{n} \cdot \mathbf{J} = 0 \). Note that, since there are singularities in the integrands of above equations, they cannot be casted directly into a matrix form. If the observing point \( \mathbf{r} \) is approaching the boundary surface, Eq.(3.9) becomes
\[
\mathbf{H}\big|_{r_0S} = \mathbf{H}\big|_{r_0S} - \lim_{r_0 \to r_0' \pm \varepsilon} \int_S da'[\nabla' g \times \mathbf{J}']_{r_0'S} - P \int_S da'[\nabla' g \times \mathbf{J}']_{r_0'S} \\
0 = \mathbf{H}\big|_{r_0S} - \lim_{r_0 \to r_0' \pm \varepsilon} \int_S da'[\nabla' g \times \mathbf{J}']_{r_0'S} - P \int_S da'[\nabla' g \times \mathbf{J}']_{r_0'S},
\]

(3.10)

where \( P \) denotes the principle value and \( \varepsilon \) is an infinitesimally small number. Since the signs of the derivative of the Green’s function are opposite for the two cases of \( r \to r' \pm \varepsilon \mathbf{\hat{z}} \), the singularity in Eq.(3.10) can be removed by adding the two equations and it leads to

\[
\mathbf{H}\big|_{r_0S} = 2\mathbf{H}\big|_{r_0S} - 2P \int_S da'[\nabla' g \times \mathbf{J}']_{r_0'S}.
\]

(3.11)

Since \( \mathbf{n} \cdot \mathbf{J} = 0 \) and \( \mathbf{J} = \mathbf{\hat{n}} \times \mathbf{H} \), taking the cross product of the unit surface normal \( \mathbf{\hat{n}} \) to both sides of Eq.(3.11) gives

\[
\mathbf{\hat{n}} \times \mathbf{H}\big|_{r_0S} = 2\mathbf{\hat{n}} \times \mathbf{H}\big|_{r_0S} - 2\mathbf{\hat{n}} \times P \int_S da'[\nabla' g \times \mathbf{J}']_{r_0'S}.
\]

(3.12)

Using the notation \( \mathbf{J} = \mathbf{\hat{n}} \times \mathbf{H} \), Eq.(3.12) is rewritten as

\[
\mathbf{J}\big|_{r_0S} = 2\mathbf{J}\big|_{r_0S} - 2P \int_S da'[\mathbf{\hat{n}} \times (\nabla' g \times \mathbf{J}')]_{r_0'S},
\]

(3.13)

where \( \mathbf{\hat{n}} \times \mathbf{H}_i = \mathbf{J}_i \). Note that the integral equation is described in terms of the unknown quantity \( \mathbf{J} \) and the known quantity \( \mathbf{J}_i \). In Cartesian coordinates, Eq. (3.13) is rewritten as

\[
\mathbf{J}(\mathbf{x})\big|_{\mathbf{x} \in S} = 2\mathbf{J}_i(\mathbf{x})\big|_{\mathbf{x} \in S} - 2P \int_S d^2x_i' [\mathbf{\hat{n}} \times (\nabla' g(\mathbf{x}) \times \mathbf{J}(\mathbf{x}'))]_{x_i' \in S}.
\]

(3.14)

After some algebra, the x-component of the surface current \( J_x \) is written as
\[ J_x = 2J_{i,x} + 2P \int d^2 x \left[ -\zeta_x g_x J_x' + \zeta_y g_y J_y' + \zeta_y g_y J_y' + g_z J_z' + g_z J_z' \right], \quad (3.15) \]

and the y-component of the surface current \( J_y \) becomes

\[ J_y = 2J_{i,y} + 2P \int d^2 x \left[ \zeta_x g_x J_x' + \zeta_y g_y J_y' + g_y J_y' - g_y J_y' \right], \quad (3.16) \]

where

\[ g_q = \left[ \frac{\partial}{\partial q} g(x | x') \right]_{\zeta_q(x)} = \left[ (q - q') \left( \frac{i2\pi}{|x - x'|} - \frac{1}{|x - x'|^2} \right) g(x | x') \right]_{\zeta_q(x)}, \quad (3.17) \]

where \( q \) is \( x, y \) or \( z \). The z-component of the surface current satisfies \( J_z = \zeta_x J_x + \zeta_y J_y \).

Equations (3.15) and (3.16) are coupled equations and, using the conventional method of moments’ technique, they can be cast into a matrix form by

\[
\begin{pmatrix}
2J_{i,x} \\
2J_{i,y}
\end{pmatrix} = 
\begin{pmatrix}
J_x \\
J_y
\end{pmatrix} + 2\Delta x \begin{pmatrix}
g_z - \zeta_x g_x - \zeta_y' g_y \\
(\zeta_y - \zeta_y') g_y + g_z - \zeta_y g_y - \zeta_z g_z
\end{pmatrix} \begin{pmatrix}
J_x \\
J_y
\end{pmatrix}, \quad (3.18)
\]

where \( \Delta x \) is the sampling spacing of the surface. The form of Eq.(3.18) is the same as that of \( Ax = b \) and the unknown column vector \( x \) can be obtained by inverting the system matrix \( A \) numerically.

In conclusion, the integral equation is converted to a system of linear equations in Cartesian coordinates. The system of linear equations can be solved by inverting the system matrix and the accuracy of the results depends on the sampling spacing and the inversion algorithm.
3.1.3 Scattered Field

The integral expression of the scattered field obtained in Section 2.4 is used for the IEM simulation in straight forward manner. However, since the incident wave is not a single plane wave, the fractional scattered intensity should be modified.

From the Section 2.4, the total scattered power per unit area per unit solid angle divided by the total incident power per unit area is given by

\[
\frac{dP}{d\Omega} = \frac{\eta}{2} \left| \mathcal{E}_s(s_i) \right|^2 + \frac{\eta}{2} \left| \mathcal{E}_p(s_i) \right|^2, \tag{3.19}
\]

where

\[
\mathcal{E}_s(s_i) = \mathcal{E}(s_i) \cdot \hat{\sigma}_{s+} = \frac{1}{2} \int_S \left[ \ln |d^2 x'| [\mathbf{J}(x')] \cdot \hat{\sigma}_{s+} \right] \exp \left[ -i 2\pi \mathbf{s} \cdot \mathbf{x}' \right] \chi_{\epsilon=\xi}, \tag{3.20}
\]

for TE polarized scattered light and

\[
\mathcal{E}_p(s_i) = \mathcal{E}(s_i) \cdot \hat{\sigma}_{p+} = \frac{1}{2} \int_S \left[ \ln |d^2 x'| [\mathbf{J}(x')] \cdot \hat{\sigma}_{p+} \right] \exp \left[ -i 2\pi \mathbf{s} \cdot \mathbf{x}' \right] \chi_{\epsilon=\xi}, \tag{3.21}
\]

for TM polarized scattered light. Together with the finite size incident beam having the Gaussian spectrum, the fractional scattered intensity is given by

\[
\frac{1}{P_s} \frac{dP}{d\Omega} = \frac{\eta}{2P_s} \left| \mathcal{E}_s(s_i) \right|^2 + \frac{\eta}{2P_s} \left| \mathcal{E}_p(s_i) \right|^2, \tag{3.22}
\]
where $P_i$ is the total incident power given by Eq.(3.6). Basically, Eq.(3.22) can be used to any deterministic surface profile as long as the profile function is a single-valued, continuous and differentiable at least once. The fractional scattered intensity for randomly rough surfaces is obtained by taking ensemble average over many realizations. Since the IEM requires taking ensemble average of individually calculated the fractional scattering intensities, it is time consuming process and it is one of disadvantages of the IEM simulation.

### 3.2 Scattering from One-dimensional Surfaces

Calculating the field scattered by a one-dimensional random rough surface is less computationally intensive, and it has been used for validating various approximate methods. In this section, the numerical approach to solve the two-dimensional scattering problem caused by an ideally conducting one-dimensional random rough surface is described. Similar to the case of three dimensional scattering, a finite-size Gaussian beam is taken as our incident beam. Also, the algorithm of numerically solving the surface integral equations is introduced. The integral equations for two-dimensional scattering can be obtained by applying Dirichlet or Neumann boundary conditions to the scalar Green’s theorem. However, in this section, the equations are derived from the Stratton-Chu equations directly by decomposing its vector quantities. Finally, the expression of the scattered field is illustrated.
3.2.1 Incident Field

Similar to the case of three-dimensional scattering problem, finite size incident beam is adopted to reduce the undesired edge effect instead of the infinite size plane wave. The spectrum of the finite size beam is given by a Gaussian function as [34]

\[
G(\alpha) = \frac{w}{2\sqrt{\pi}} \exp \left[ -\frac{w^2}{4}(2\pi(\alpha + \alpha_i))^2 \right]. \tag{3.23}
\]

where \( w \) is the half-width of the Gaussian spectrum. Using this Gaussian incident beam, the electric field at the boundary surface is obtained as

\[
E_i(x_\perp) = \hat{e} \frac{\omega \mu_0}{k_0} \int d\alpha G(\alpha) \exp\left[ i2\pi s_\perp \cdot x_\perp \right], \tag{3.24}
\]

and, from the free-space Faraday's law \( \nabla \times E = i\omega \mu_0 H \), the magnetic field is given by

\[
H_i(x_\perp) = \hat{h} \int d\alpha G(\alpha) \exp\left[ i2\pi s_\perp \cdot x_\perp \right]. \tag{3.25}
\]

Conventionally, it is assumed that \( \hat{e}_y = \hat{y} \) for TE polarized incident light and that \( \hat{h}_p = \hat{y} \) for TM polarized incident light. The other components are obtained by \( \hat{e} \times \hat{h} = s_\perp \). Since the angular spectrum of the incident wave is Gaussian, the total incident power per unit surface area is given by

\[
P_i = \int d\alpha \langle S_i \cdot \hat{z} \rangle = \frac{\eta}{2} \int d\alpha |G(\alpha)|^2 \gamma. \tag{3.26}
\]
In this section, the finite size beam is introduced as our incident wave. Since the polarization states are intuitive in this two-dimensional scattering problem, the mathematical expressions are much simpler than those of three-dimensional scattering problem.

### 3.2.2 Surface Field

In this section, the algorithm for solving integral equations numerically for the two-dimensional scattering problem caused by a one-dimensional ideally conducting surface is introduced. First, the integral equations are derived by manipulating the integral equations for the three-dimensional scattering problem. For the case of two-dimensional scattering, the polarization state of the scattered field is intuitive and the problem can be divided into TE and TM polarization cases. The integral equations, then, are converted to a system of linear equations for both cases using the method of moments.

Here, for \( \mathbf{r} \in V_1 \), our starting point is the Stratton-Chu formula given by

\[
\begin{align*}
\mathbf{E}_x & = + \int_C ds' \left[ g(i \omega \mu_0 J') + (\nabla' g)(\varepsilon_0^{-1} \rho') \right]_{r \rightarrow C}, \\
\mathbf{H}_x & = - \int_C ds' \left[ (\nabla' g) \times J' \right]_{r \rightarrow C},
\end{align*}
\]

(3.27)

where \( g = g(\mathbf{r} \mid \mathbf{r}') \) is the two-dimensional Green’s function and \( ds' \) is a line segment along the boundary contour line \( C \). If the incident wave is linearly polarized in the y-direction, the Green’s function does not depend on the y-coordinate and the gradient of the Green’s function is decomposed by
\[ \nabla' g = (\hat{n} \cdot \nabla' g) \hat{n} + (\mathbf{i} \cdot \nabla' g) \mathbf{i}. \]

(3.28)

where the surface normal vector is given by \( \mathbf{n} = -\zeta \mathbf{x} + \mathbf{z} \) for one-dimensional surfaces and the tangential vector to the surface is defined by \( \mathbf{t} = \hat{n} \times \hat{y} \). Since the surface current does not have a normal component to the surface, it can be decomposed by

\[ \mathbf{J} = J_y \hat{y} + J_t \hat{t}. \]

(3.29)

where \( J_y = \mathbf{J} \cdot \hat{y} \) and \( J_t = \mathbf{J} \cdot \hat{t} \). Similarly, the electric and magnetic field is decomposed as

\[ \mathbf{E} = E_n \hat{n} + E_y \hat{y} + E_t \hat{t}. \]

(3.30)

\[ \mathbf{H} = H_n \hat{n} + H_y \hat{y} + H_t \hat{t}. \]

Substituting Eqs. (3.28), (3.29) and (3.30) into Stratton-Chu equation, Eq.(3.27) is written by

\[ E_{x,y} = \int_C ds' \left[ g(i\omega \mu_0 J'_s) \right]_{\mathbf{r} = \mathbf{c}} \]

\[ H_{x,y} = \int_C ds' \left[ (-\hat{n} \cdot \nabla' g)J'_y \right]_{\mathbf{r} = \mathbf{c}}, \]

\[ H_{x,n} = \int_C ds' \left[ (\mathbf{i} \cdot \nabla' g)J'_s \right]_{\mathbf{r} = \mathbf{c}} \]

(3.31)

for TE polarized incident wave and

\[ H_{x,y} = \int_C ds' \left[ (\hat{n} \cdot \nabla' g)J'_t \right]_{\mathbf{r} = \mathbf{c}} \]

\[ E_{x,y} = \int_C ds' \left[ (i\omega \mu_0 J'_s) + (\hat{t} \cdot \nabla' g) \rho' \right]_{\mathbf{r} = \mathbf{c}}, \]

\[ E_{x,n} = \int_C ds' \left[ (\hat{c}' \hat{n} \cdot \nabla' g) \rho' \right]_{\mathbf{r} = \mathbf{c}} \]

(3.32)

for TM polarized incident wave.
Let us consider the TE polarization case first. Substituting the integral representation of the scattered field in Eq.(3.31) into the boundary condition, the boundary condition becomes three integral equations given by

\[
0 = E_{x,y} \bigg|_{r = r_c} + \int_{C} ds' \left[ g(x_\perp | x'_\perp) |n| J_y(x') \right]_{x'_\perp, x_\perp \in C},
\]

\[
J_y \bigg|_{r = r_c} = H_{x,z} \bigg|_{r = r_c} + \int_{C} ds' \left[ (\hat{n} \cdot \nabla' g) J_z(x') \right]_{x'_\perp, x_\perp \in C},
\]

\[
0 = H_{x,z} \bigg|_{r = r_c} + \int_{C} ds' \left[ (\hat{i} \cdot \nabla' g) J_y(x') \right]_{x'_\perp, x_\perp \in C}.
\]  

(3.33)

Among the three equations in Eq.(3.33), the first equation is usually considered. Using the relation \( dr' = |n| dx' \), it is written in Cartesian coordinates by

\[
0 = E_{i,y}(x_\perp) \bigg|_{x_\perp \in C} + i \omega \mu_0 \int_{C} dx' \left[ g(x_\perp | x'_\perp) |n| J_y(x') \right]_{x'_\perp, x_\perp \in C}.
\]  

(3.34)

Equation (3.34) is a Fredholm equation of the first kind [84]. Note that applying Dirichlet boundary condition to the scalar Green’s theorem gives the same equation as Eq.(3.30). Since the two-dimensional Green’s function is singular when \( x_\perp = x'_\perp \), the integration cannot be directly converted to a system of linear equations. If we limit the observing point to \( x_\perp \rightarrow x'_\perp \),

\[
 \lim_{x_\perp \rightarrow x'_\perp} \int_{C} dx' \left[ g(x_\perp | x'_\perp) |n(x')| J_y(x') \right]_{x'_\perp, x_\perp \in C} = C_0(x') |n(x')| J_y(x'),
\]  

(3.35)

where \( C_0(x') \) is given by

\[
 \lim_{x' \rightarrow x_{\perp}} \int_{x_{\perp} \rightarrow x'_{\perp}} dx' \left[ g(x_{\perp} | x'_{\perp}) \right]_{x_{\perp}, x'_{\perp} \in C} = C_0(x'),
\]  

(3.36)
and it can be calculated with desired accuracy using the two-dimensional Green’s function. Thus Eq. (3.34) turns into

\[
\frac{i}{\omega \mu_0} E_{i,y}(\mathbf{x}_\perp) \bigg|_{\mathbf{x}_\perp \in \mathcal{C}} = \int_{\mathcal{C}} dx' C_0(x') \delta(x - x') + \int_{\mathcal{C}} \left\{ dx' \left[ g(x_\perp | x'_\perp) |n(x')| J_y(x') \right]_{\mathbf{x}_\perp \in \mathcal{C}} \right\} .
\] (3.37)

Now, using the technique of the method of moments, Eq.(3.37) can be cast into matrix form as

\[
\mathbf{A} \mathbf{x} = \mathbf{b} \quad \text{given by}
\]

\[
\left[ C_0(x_n) \delta_{mn} + (\Delta x) g(x_m, \zeta(x_m) | x_n, \zeta(x_n)) \right] \left[ |n(x_n)| J_y(x_n) \right] = \left[ \frac{i}{\omega \mu_0} E_{i,y}(x_m, \zeta(x_m)) \right],
\] (3.38)

where \( \delta_{mn} \) is the Kronecker delta function which is defined by unity for \( m=n \) and zero for elsewhere. And the \( y \)-component of the surface current \( J_y \) can be obtained by inverting the system matrix \( \mathbf{A} \).

The surface current for the case of TM polarization is obtained in the similar way to the TE polarization case. Substituting the integral representation of the scattered field in Eq.(3.32) into the boundary condition, the boundary condition is given in integral form given by

\[
J_y \big|_{\mathbf{r} \in \mathcal{C}} = H_{i,y} \big|_{\mathbf{r} \in \mathcal{C}} + \int_{\mathcal{C}} ds' \left[ (\hat{\mathbf{n}} \cdot \nabla' g) J_i' \right]_{\mathbf{r}, \mathbf{r}' \in \mathcal{C}}
\]

\[
0 = E_{i,y} \big|_{\mathbf{r} \in \mathcal{C}} + \int_{\mathcal{C}} ds' \left[ i \omega \mu_0 J_i' \right]_{\mathbf{r}, \mathbf{r}' \in \mathcal{C}}
\]

\[
\rho = \varepsilon_0 E_{i,y} \big|_{\mathbf{r} \in \mathcal{C}} + \int_{\mathcal{C}} ds' \left[ (\hat{\mathbf{n}} \cdot \nabla' g) \rho' \right]_{\mathbf{r}, \mathbf{r}' \in \mathcal{C}},
\] (3.39)

Among the above three equations, the first one is usually taken and, in Cartesian coordinates, it is written as
\[ J_i(x_{\perp}) \big|_{x_{\perp} \in C} = H_{i,y}(x_{\perp}) \big|_{x_{\perp} \in C} + \int_{C} dx' \left[ (n \cdot \nabla' g(x_{\perp} | x_{\perp}')) J_i(x') \right]_{x_{\perp} \in C}, \tag{3.40} \]

which is a Fredholm equation of the second kind. Note that applying Neumann boundary condition to the scalar Green’s theorem gives the same result. From the two-dimensional Green’s function, the \( n \cdot \nabla' g \) in the right side of the Eq.(3.40) is given by

\[ n(x') \cdot \nabla' g(x_{\perp} | x_{\perp}') = -\frac{i\pi}{2} \frac{n \cdot (x_{\perp} - x_{\perp}')}{|x_{\perp} - x_{\perp}'|} H_1^{(1)}(2\pi|x_{\perp} - x_{\perp}'|), \tag{3.41} \]

where \( H_1^{(1)} \) is the first order of Hankel function of the first kind. Since the integrand of the integration term in Eq.(3.41) is singular when \( x_{\perp} = x_{\perp}' \), it cannot be cast into the form of a system of linear equations. Similar to the case of the three dimensional scattering problem, if we limit of the observing point by \( x_{\perp} \rightarrow x_{\perp}' \pm \hat{z}, \) it is given by

\[ J_i \big|_{x_{\perp} \in C} = H_{i,y} \big|_{x_{\perp} \in C} + \lim_{x_{\perp} \rightarrow x_{\perp}' \pm \hat{z}} \int_{C} dx' \left[ (n \cdot \nabla' g) J_i \right]_{x_{\perp} \in C} + P \int_{C} dx' \left[ (n \cdot \nabla' g) J_i \right]_{x_{\perp} \in C}, \tag{3.42} \]

and simplified notations \( J_i = J_i(x_{\perp}) \) and \( H_{i,y} = H_{i,y}(x_{\perp}) \) are used. Since the amplitudes of the term \( (n \cdot \nabla' g) \) are the same but its signs are opposite for the two cases of \( x_{\perp} \rightarrow x_{\perp}' \pm \hat{z}, \) adding the above two equations leads to

\[ J_i \big|_{x_{\perp} \in C} = 2H_{i,y} \big|_{x_{\perp} \in C} + 2P \int_{C} dx' \left[ (n \cdot \nabla' g) J_i \right]_{x_{\perp} \in C}. \tag{3.43} \]

Equation (3.43), thus, can be casted into the matrix form as \( Ax = b \) given by
\[
\begin{bmatrix}
\delta_{mn} - 2\Delta x |n(x_n)| g\left[ x_m, \zeta(x_m) | x_n, \zeta(x_n) \right]
\end{bmatrix}
J_t(x_n) = \left[ H_{i,j}(x_m, \zeta(x_m)) \right].
\] (3.44)

The t-component of the surface current \( J_t \) is obtained by inverting the system matrix \( A \).

In conclusion, the integral equations for the surface scatter from an ideally conducting one-dimensional rough surface are derived from the Stratton-Chu formula and, removing their singularities, they are casted into the form of a system of linear equations.

### 3.2.3 Scattered Field

For the case of the two-dimensional scattering problem, if the polarization of the incident light is determined to be either TE or TM, the polarization of the scattered field is also either TE or TM. The scattered electric or magnetic field for the two-dimensional scattering problem is given by

\[
E_{s,1}(x_\perp) = \hat{y} k_0 \int \frac{d\alpha}{\gamma'} |E(\alpha)| \exp[i2\pi s_\perp \cdot x_\perp],
\] (3.45)

for the TE polarization case and

\[
H_{s,1}(x_\perp) = \hat{y} \int \frac{d\alpha}{\gamma'} |E(\alpha)| \exp[i2\pi s_\perp \cdot x_\perp],
\] (3.46)

for TM polarization case and the \( E(\alpha) \) is given by

\[
E(\alpha) = \frac{1}{2} \int dx' |n| J(x') \exp[-i2\pi s_\perp \cdot x_\perp] \big|_{x_\perp=c}. \] (3.47)
Recall that, from the polarization decomposition, \( \mathbf{J} = J_y \hat{y} \) for TE polarization and \( \mathbf{J} = J_x \hat{x} \) for TM polarization. The amplitude of the time-averaged Poynting vector \( \mathbf{S}(\alpha) \) for the scattered light is given by

\[
\langle |\mathbf{S}(\alpha)|^2 \rangle = \frac{\eta}{2} \left| \frac{\mathbf{E}(\alpha)}{\gamma} \right|^2,
\]

for both polarizations. The power per unit surface area for a plane wave is given by the Poynting vector of the plane wave projected to the mean surface normal direction given by

\[
\mathbf{S}(\alpha) \cdot \hat{z} = |\mathbf{S}(\alpha)| \gamma. \]

Thus, the total scattered power per unit surface area \( P_s \) is

\[
P = \frac{\eta}{2} \int d\alpha \frac{1}{\gamma} |\mathbf{E}(\alpha)|^2.
\]

By using the relation \( d\alpha = \gamma d\theta \), the scattered intensity normalized by the incident power can be represented in terms of scattered angle given by

\[
\frac{1}{P_i} \frac{dP}{d\theta} = \frac{\eta}{2P_i} |\mathbf{E}(\theta)|^2,
\]

where \( P_i \) is the incident power per unit surface area for the Gaussian beam described in Section 3.2.2. Equation (3.50) can be applied to any surface profiles as long as the profile function is single-valued, continuous, and differentiable at least once.

In order to apply Eq.(3.50) to randomly rough surfaces, similar to the three dimensional scattering problem, the ensemble average of the scattering intensity is taken and it is given by
\( \frac{\eta}{2P_i} \langle \frac{dP}{d\theta} \rangle = \frac{\eta}{2P_i} \| \mathcal{E}(\theta) \|^2 + \frac{\eta}{2P_i} \left[ \langle |\mathcal{E}(\theta)|^2 \rangle - \langle |\mathcal{E}(\theta)| \rangle^2 \right] . \) \hspace{1cm} (3.51)

The first term on the right side of Eq.(3.51) denotes the coherent specular light, and the second term represents the fluctuation or incoherent scattering light.
Elfouhaily and Guerin conducted an exhaustive survey of approximate surface scatter theories in 2004 [42]. It was claimed that each method has advantages for specific applications under certain situations. Probably, the most widely used methods to predict surface scatter phenomena are the SPM [6, 43-47] and the KA [48-53] method that were established over 40 years ago as the classical Rayleigh-Rice method [6] and the Beckmann-Kirchhoff method [53] respectively. Recently, Krywonos, et al. described a linear systems formulation of surface scatter theory which is referred to as the GHS surface scatter theory [62]. In this section, these three approximate methods are introduced in detail for later discussions. Here, we limited our interest in the perfectly conducting random rough surfaces.

4.1 Small Perturbation Method

The Rayleigh-Rice method is one of the oldest and most popular scattering theories and it is based on the Rayleigh’s hypothesis which is still somewhat controversial [85]. However, the scattering intensity expression predicted by the method turns out to be the same as one obtained by applying the perturbation technique to the rigorous integral equation [47]. In this section, the scattering intensity is derived using small perturbation technique for two-dimensional ideally
conducting surface. We begin with the Stratton-Chu equation for magnetic field obtained in Chapter 2 which is given by

\[
H(r)_{\text{ref} V_1} = H_i(r)_{\text{ref} V_1} - \int_{S} d\alpha \left[ (\nabla' g(r | \mathbf{r}')) \times J(\mathbf{r}') \right]_{r' \in S, \mathbf{r} = \mathbf{r}_V},
\]

\[
0 = H_i(r)_{\text{ref} V_2} - \int_{S} d\alpha \left[ (\nabla' g(r | \mathbf{r}')) \times J(\mathbf{r}') \right]_{r' \in S, \mathbf{r} = \mathbf{r}_V},
\]

(4.1)

where the surface current satisfies the condition,

\[
\mathbf{n} \cdot J(\mathbf{r}) = 0.
\]

(4.2)

The above two equations are solved by the perturbation technique in the following subsections.

### 4.1.1 Perturbation of Integral Equations

First, together with \( \mathbf{H} = H_i + H_s \), the integral equation given in Eq.(4.1) is represented in Cartesian coordinate as

\[
\begin{align*}
H_i(x)_{\text{ref} V_1} &= -\int_{S} |\mathbf{n}| d^2 x' \left[ (\nabla' g(x | \mathbf{x}')) \times J(x') \right]_{x' \in S, \mathbf{x} = \mathbf{x}_V}, \\
H_s(x)_{\text{ref} V_1} &= +\int_{S} |\mathbf{n}| d^2 x' \left[ (\nabla' g(x | \mathbf{x}')) \times J(x') \right]_{x' \in S, \mathbf{x} = \mathbf{x}_V}.
\end{align*}
\]

(4.3)

By taking gradient of the Fourier representation of the Green’s function, it is obtained

\[
(\nabla' g(x | \mathbf{x}') = \int d^2 \mathbf{s}_1 \frac{1}{2\gamma} \mathbf{s}_1 \exp[i2\pi \mathbf{s}_1 \cdot (\mathbf{x} - \mathbf{x}')],
\]

(4.4)
where $s_+$ is for the vacuum region and $s_-$ is for the conducting region. The scattered magnetic field in the vacuum region can be decomposed by the upward plane waves as

$$\mathbf{H}_s(\mathbf{x}) = \int d^2 s \left[ \frac{1}{y} s_+ \times \mathbf{E}(s^i) \right] \exp[i2\pi s_+ \cdot \mathbf{x}]. \quad (4.5)$$

Similarly, the incident downward plane wave in conducting region can be decomposed by the downward plane waves as

$$\mathbf{H}_i(\mathbf{x}) = \hat{\mathbf{h}}_i \int d^2 s \delta(s_i + s_{h,i}) \exp[i2\pi s_- \cdot \mathbf{x}]. \quad (4.6)$$

Substituting Eq.(4.4), (4.5) and (4.6) into Eq.(4.3), the integral equations become

$$s_+ \times \mathbf{E}(s_i) = s_+ \times \frac{1}{2} \int_{s^i} |\mathbf{n}'| d^2 x' \mathbf{J}(x'_i) \exp[-i2\pi s_+ \cdot \mathbf{x}']_{x < s}, \quad (4.7)$$

for the vacuum region and

$$\hat{\mathbf{h}}_i \delta(s_i + s_{h,i}) = s_- \times \frac{1}{2\gamma} \int_{s^i} |\mathbf{n}'| d^2 x' \mathbf{J}(x'_i) \exp[i2\pi s_- \cdot \mathbf{x}']_{x \geq s}, \quad (4.8)$$

for the conducting region. The surface current $\mathbf{J}(x'_i)$ is solved by using Eq.(4.8) and the condition for the surface current described in Eq.(4.2) in perturbation manner. And the calculated surface current is used to obtain scattered field by using Eq.(4.7).

Assuming that the surface is very smooth, the exponential term in Eqs.(4.7) and (4.8) is expanded as
\[ \exp \left[ \pm i 2 \pi \zeta \right] \approx 1 \pm i 2 \pi \zeta, \]  
\tag{4.9} \]

where \( \lambda^{(m)} \) denotes the m-th order of perturbation. Also, the surface current is expanded by

\[ |n'| \mathbf{J}(x_i) = \mathbf{J}^{(0)}(x_i) \lambda^{(0)} + \mathbf{J}^{(1)}(x_i) \lambda^{(1)}. \]  
\tag{4.10} \]

Note that the amplitude of the surface normal in the left side of Eq.(4.10) is inserted for later convenience. Substituting Eqs.(4.9) and (4.10) into Eqs.(4.7) and (4.8), and rearranging terms in the perturbation order, the two Eqs.(4.7) and (4.8) becomes

\[ s_+ \times \mathcal{E}(s_i) = s_+ \times \frac{1}{2} \mathcal{F} \left[ J^{r(0)} \lambda^{(0)} + \left( J^{r(1)} - i 2 \pi \zeta' J^{r(0)} \right) \lambda^{(1)} \right], \]  
\tag{4.11} \]

for the vacuum region and

\[ \hat{n}_i \delta(s_i + s_{i,i}) = s_- \times \frac{1}{2} \mathcal{F} \left[ J^{r(0)} \lambda^{(0)} + \left( J^{r(1)} + i 2 \pi \zeta' J^{r(0)} \right) \lambda^{(1)} \right], \]  
\tag{4.12} \]

for the conducting region. The simplified notations \( J^{r(m)} = J^{(m)}(x_i) \) and \( \zeta' = \zeta(x_i) \) are used and the second order term is ignored. Utilizing the convolution theorem, it is obtained that

\[ \mathcal{F} \left[ Z(x') J^{(m)}(x') \right] = \int d^2 s_i Z(s_i) J^{(m)}(s_i) = Z(s_i) \otimes j^{(m)}(s_i), \]  
\tag{4.13} \]

where \( \otimes \) is the convolution operator and \( j^{(m)} = j^{(m)}(s_i) \) is given by

\[ j^{(m)}(s_i) = \mathcal{F} \left[ J^{(m)}(x_i) \right], \]  
\tag{4.14} \]

and \( Z = Z(s_i) \) is given by

62
\[
Z(s_{\|}) = \mathcal{F}[\zeta(x_{\|})].
\] (4.15)

Together with the identity in Eq.(4.13) and the definition in Eq.(4.14), the Eqs.(4.11) and (4.12) are simplified to

\[
s_{+} \times \mathcal{E}(s_{\|}) = \frac{1}{2} \left( s_{+} \times j^{(0)} \right) \lambda^{(0)} + \frac{1}{2} \left[ s_{+} \times j^{(1)} - i2\pi s_{+} \times (Z \otimes j^{(0)}) \right] \lambda^{(1)},
\] (4.16)

for the vacuum region and

\[
\hat{h}, \partial(s_{\|} + s_{h}) = \frac{1}{2\gamma} \left( s_{-} \times j^{(0)} \right) \lambda^{(0)} + \frac{1}{2\gamma} \left[ s_{-} \times j^{(1)} + i2\pi s_{-} \times (Z \otimes j^{(0)}) \right] \lambda^{(1)},
\] (4.17)

for the conducting region.

On the other hand, the condition for surface current given in Eq.(4.2) is represented in Cartesian coordinate as

\[
(-\zeta, \hat{x} - \zeta, \hat{y} + \hat{z}) \cdot \mathbf{J} = 0.
\] (4.18)

Substituting the expanded surface current shown in Eq.(4.10) into Eq.(4.17), and balancing it in the perturbation order, it is obtained that

\[
J_{z}^{(0)} = 0
\]
\[
J_{z}^{(1)} = \zeta, J_{x}^{(0)} + \zeta, J_{y}^{(0)}.
\] (4.19)
Taking the Fourier transform on the both sides on the first equation in Eq.(4.19), it is easily attained that attained that \( j_z^{(0)} = 0 \), which is \( \mathbf{j}^{(0)} = j_{x}^{(0)} \mathbf{x} + j_{y}^{(0)} \mathbf{y} \). Taking Fourier transform of the both sides of the second equation in Eq. (4.19), it turns out

\[
j_z^{(1)} = \mathcal{F}\left[ \zeta_x J_x^{(0)} + \zeta_y J_y^{(0)} \right] = i2\pi \int d^2 s' \left\{ \left[ (s_i - s'_i) Z(s_i - s'_i) \right], \mathbf{j}^{(0)}(s'_i) \right\},
\]

which is a convolution integral. The Eqs.(4.17), (4.19) and (4.20) are used for calculating surface current up to the first order and the results will be put to Eq.(4.16).

**4.1.2 Surface Field**

Balancing terms in the perturbation order, Eq.(4.17) is divided into

\[
\hat{\mathbf{h}}.\delta(s_i + s'_i) = \frac{1}{2\gamma} \mathbf{s} \times \mathbf{j}^{(0)},
\]

for the zeroth order and

\[
0 = \mathbf{s} \times \mathbf{j}^{(1)} + i2\pi \mathbf{s} \times \left( Z \otimes \mathbf{j}^{(0)} \right),
\]

for the first order.

The direction of the incident wave is conveniently described in the orthonormal coordinate system \((\mathbf{\hat{h}}_{x,}, \mathbf{\hat{h}}_{p,}, \mathbf{s}_{-})\) introduced in Section 2.4. The direction of the electric field for a TE polarized incident plane wave is represented by \( \mathbf{\hat{e}}_{x,i} = \mathbf{\hat{\sigma}}_{p,}(-\mathbf{s}_i) \) and the direction of the electric
field is written as \( \hat{e}_{pi} = \hat{\sigma}_{p} (-s_i) \) for TM polarized incident plane wave. Therefore, the corresponding magnetic field of the incident plane wave is set to

\[
\hat{h}_i = h_p \hat{\sigma}_{s,i} + h_e \hat{\sigma}_{p,i},
\]  

(4.23)

where \( h_s \) and \( h_p \) are the amplitude of the magnetic fields for TE and TM polarization respectively, and the simplified notations, \( \hat{\sigma}_{s,i} = \hat{\sigma}_{s}(-s_{i}), \) and \( \hat{\sigma}_{p,i} = \hat{\sigma}_{p}(-s_{i}), \) are used. The zeroth order surface current can be decomposed by using the new coordinate system given by

\[
\hat{j}^{(0)} = j_s^{(0)} \hat{\sigma}_{s_+} + j_p^{(0)} \hat{\sigma}_{p_+} + j_k^{(0)} s_-, \quad \text{and Eq. (4.21) is rewritten by}
\]

\[
(h_p \hat{\sigma}_{s,i} + h_e \hat{\sigma}_{p,i}) \delta(s_l + s_{l,i}) = \frac{1}{2\gamma} s_- \times \left( j_s^{(0)} \hat{\sigma}_{s_+} + j_p^{(0)} \hat{\sigma}_{p_+} + j_k^{(0)} s_-. \right).
\]  

(4.24)

Since \( \hat{h}_i \) must be perpendicular to the direction cosine vector \( s_- \), the equation holds only for the case \( s_- = -s_i \). Thus, the two components perpendicular to \( s_- \) are calculated by

\[
j_s^{(0)} = -2\gamma h_s \delta(s_l + s_{l,i})
\]

\[
j_p^{(0)} = +2\gamma h_p \delta(s_l + s_{l,i}).
\]  

(4.25)

The other component which is parallel to \( s_- \) is calculated by substituting Eq.(4.25) into the Eq.(4.20) and given by

\[
j_k^{(0)} = -2\rho h_p \delta(s_l + s_{l,i}).
\]  

(4.26)

All together, the zero order surface field vector is expressed as
\[ \mathbf{j}^{(0)} = 2\delta (\mathbf{s}_i + \mathbf{s}_{i,i'}) \tilde{\mathbf{j}}^{(0)}, \]  

(4.27)

where

\[ \tilde{\mathbf{j}}^{(0)} = -\gamma_i h_i \tilde{\mathbf{a}}_{s,i} + \gamma_i h_i \tilde{\mathbf{a}}_{p,i} - \rho_i h_i \mathbf{a}_i. \]  

(4.28)

The zero order surface current is the same as the surface current due to the flat surface and, therefore, the zero order surface current \( \mathbf{j}^{(0)} \) do not satisfy the condition \( \mathbf{n} \cdot \mathbf{J} = 0 \). However, the components \( j_i^{(0)} \), \( j_p^{(0)} \) and \( j_z^{(1)} \) satisfy the condition and it results from the fact that the perturbation order of the \( z \)-component is one order higher than the others in Eq.(4.18).

Let us move on the first order surface current. The \( z \)-component of the first order current is calculated by combining Eqs.(4.20), (4.28) and (4.29), and given by

\[ j_z^{(1)} = i2\pi \int d^2 s_i' \left\{ \left[ (\mathbf{s}_i - \mathbf{s}_i') Z(\mathbf{s}_i - \mathbf{s}_i') \right] \cdot 2\delta (\mathbf{s}_i + \mathbf{s}_{i,i'}) \tilde{\mathbf{j}}^{(0)} \right\} = -q_0 \gamma^{-1} \tilde{\mathbf{j}}^{(0)} \cdot (\mathbf{s}_i + \mathbf{s}_{i,i'}), \]  

(4.29)

where \( q_0 = i4\pi Z(\mathbf{s}_i - \mathbf{s}_i') \). Substituting the zero order surface current obtained in Eqs.(4.28) and (4.29) into the Eq.(4.22), the equation is simply expressed as

\[ 0 = \mathbf{s} \times \mathbf{j}^{(1)} + q_0 \mathbf{s} \times \tilde{\mathbf{j}}^{(0)}, \]  

(4.30)

and it is used that

\[ \int d^2 s_i' Z(\mathbf{s}_i - \mathbf{s}_i') \delta (\mathbf{s}_i + \mathbf{s}_{i,i'}) = Z(\mathbf{s}_i + \mathbf{s}_{i,i'}). \]  

(4.31)
Using the new orthogonal coordinate system, the first order surface current is represented as 
\[ \mathbf{j}^{(1)} = j_s^{(1)} \hat{\mathbf{s}}_s + j_p^{(1)} \hat{\mathbf{s}}_p + j_k^{(1)} \mathbf{s}. \] 
Equation (4.31) implies that the amplitudes of the projections of the \( \mathbf{j}^{(1)} \) and \( \mathbf{\tilde{j}}^{(0)} \) onto the \( \hat{\mathbf{s}}_s \) direction must be equal to each other and their signs must be opposite. The similar argument is hold for the projections onto the \( \hat{\mathbf{s}}_p \) direction. Therefore, Eq.(4.30) determines \( j_s^{(1)} \) and \( j_p^{(1)} \) components and they are calculated by

\[
\begin{align*}
j_s^{(1)} &= -q_0 \mathbf{\tilde{j}}^{(0)} \cdot \hat{\mathbf{s}}_s \\
j_p^{(1)} &= -q_0 \mathbf{\tilde{j}}^{(0)} \cdot \hat{\mathbf{s}}_p \end{align*}
\]

The third component \( j_k^{(1)} \) can be calculated from the two components and the z-component but it is unnecessary to obtain it if only first order scattered field is of interest.

### 4.1.3 Scattered Field

Using the zero and first order surface current, the first order scattered field will be calculated in this section. For upward plane waves, it is convenient for describing polarization of upward plane waves to use the orthonormal coordinate system \((\hat{\mathbf{s}}_s, \hat{\mathbf{s}}_p, \mathbf{s})\) introduced in Section 2.4. Under the new coordinate system, for upward plane waves, the direction of the electric field is \( \hat{\mathbf{e}}_s = \hat{\mathbf{s}}_s \) for a TE polarization and it is \( \hat{\mathbf{e}}_p = \hat{\mathbf{s}}_p \) for TM polarization. Combining zero order surface current and the Eq.(4.16), the first order scattered field is obtained by

\[
2 \mathbf{s} \times \mathbf{\mathcal{E}}(\mathbf{s}_s) = \mathbf{s} \times \mathbf{j}^{(1)} - q_0 \mathbf{s} \times \mathbf{j}^{(0)}. 
\]
Using the expression of the first order surface current in Eq.(4.33), the \( \mathbf{\hat{s}}_{x} \) component of \( \mathbf{E}(\mathbf{s}) \) is calculated by

\[
2\mathbf{E}^{(1)} \cdot \mathbf{\hat{s}}_{x} = \mathbf{j}^{(1)} \cdot \mathbf{\hat{s}}_{x} - q_{0} \mathbf{j}^{(0)} \cdot \mathbf{\hat{s}}_{x} = \mathbf{j}^{(1)} \cdot \mathbf{\hat{s}}_{x} - q_{0} \mathbf{\hat{j}}^{(0)} \cdot \mathbf{\hat{s}}_{x} = 2j_{x}^{(1)}
\]

\[= -2q_{0} \mathbf{j}^{(0)} \cdot \mathbf{\hat{s}}_{x} \quad \text{(4.34)}\]

Similarly, the \( \mathbf{\hat{s}}_{p} \) component of \( \mathbf{E}(\mathbf{s}) \) is calculated by

\[
2\mathbf{E}^{(1)} \cdot \mathbf{\hat{s}}_{p} = \mathbf{j}^{(1)} \cdot \mathbf{\hat{s}}_{p} - q_{0} \mathbf{\hat{j}}^{(0)} \cdot \mathbf{\hat{s}}_{p} = -\mathbf{j}^{(1)} \cdot \mathbf{\hat{s}}_{p} - 2\rho \mathbf{j}^{(1)} \cdot \mathbf{\hat{z}} + q_{0} \mathbf{\hat{j}}^{(0)} \cdot \mathbf{\hat{s}}_{p} = -2j_{p}^{(1)} - 2\rho \mathbf{j}^{(1)}
\]

\[= 2q_{0} \mathbf{j}^{(0)} \cdot \mathbf{\hat{s}}_{p} + 2q_{0} \gamma \rho \mathbf{j}^{(0)} \cdot (\mathbf{s} + \mathbf{s}_1) \quad \text{(4.35)}\]

The first order scattered field is represented by the zero order surface current. After some algebra, the vector inner products are calculated by

\[
\mathbf{\hat{s}}_{x} \cdot \mathbf{\hat{s}}_{x} = q_{1} \quad \mathbf{\hat{s}}_{x} \cdot \mathbf{\hat{s}}_{p} = q_{2} \gamma \\
\mathbf{\hat{s}}_{p} \cdot \mathbf{\hat{s}}_{x} = -q_{2} \gamma, \quad \mathbf{\hat{s}}_{p} \cdot \mathbf{\hat{s}}_{p} = q_{1} \gamma \gamma + \rho \rho \,
\]

\[\text{(4.36)}\]

and

\[
\mathbf{\hat{s}}_{x} \cdot \mathbf{s} = -q_{2} \rho \quad \mathbf{\hat{s}}_{x} \cdot \mathbf{s}_1 = 0 \\
\mathbf{\hat{s}}_{p} \cdot \mathbf{s} = -q_{1} \rho \gamma, \quad \mathbf{\hat{s}}_{p} \cdot \mathbf{s}_1 = -\rho \gamma \\
\mathbf{s} \cdot \mathbf{s}_1 = q_{1} \rho \rho, \quad \mathbf{s}_1 \cdot \mathbf{s}_1 = \rho \rho
\]

\[\text{(4.37)}\]

where \( q_{1} = \mathbf{\hat{s}}_1 \cdot \mathbf{\hat{s}}_1 \) and \( q_{2} = -\mathbf{(\hat{s}_1 \times \hat{s}_1)} \cdot \mathbf{\hat{z}} \). Using these vector inner products and Eq.(4.29), the first order surface current becomes
\[
\mathbf{\mathcal{E}}^{(i)} \cdot \hat{s}_{+} = -\gamma_i q_0 q_1 h_s - q_0 q_2 h_p \\
\mathbf{\mathcal{E}}^{(i)} \cdot \hat{s}_{p+} = \gamma^{-1} \gamma_i q_0 q_2 h_s - \gamma^{-1} (q_i + p p) q_0 h_p .
\] (4.38)

If incident wave is TE polarized, the amplitudes of the magnetic field are \( h_s = 1 \) and \( h_p = 0 \). On the other hand, if the polarization of the incident wave is TM, they become \( h_s = 0 \) and \( h_p = 1 \).

The fractional scattered intensity is given by

\[
\frac{1}{P_i} dP = \frac{1}{\gamma_i} \left[ |\mathbf{\mathcal{E}}^{(i)}(s_{||}) \cdot \hat{s}_{s+}|^2 + |\mathbf{\mathcal{E}}^{(i)}(s_{||}) \cdot \hat{s}_{p+}|^2 \right].
\] (4.39)

Equation (4.39) can be applied to any surface profile as long as the profile function is single valued, continuous, and differentiable at least once. For random surface surfaces whose statistical properties are characterized in terms of surface PSD function, \( W(s_{||}) = \langle Z(s_{||}) Z^*(s_{||}) \rangle \), the first order scattered intensity is obtained as

\[
\left\langle \frac{1}{P_i} dP \right\rangle = \frac{1}{\gamma_i} \left\langle \left| \mathbf{s}_{s+} \times \mathbf{\mathcal{E}}^{(i)}(s_{||}) \right|^2 \right\rangle = 4 \cdot (2\pi)^2 q_{ss} W(s_{||} + s_{||}) ,
\] (4.40)

where

\[
q_{ss} = \gamma_i |\hat{s}_{s+} \times \hat{s}_{s+}|^2 ,
\] (4.41)

for TE polarized incident and TE polarized scattered light, and

\[
q_{ss} = \gamma_i |\hat{s}_{p+} \times \hat{s}_{p+}|^2 ,
\] (4.42)
for TE polarized incident and TM polarized scattered light, and

\[ q_{ps} = \frac{\gamma}{\gamma_i} |\hat{s}_{i\perp} \times \hat{s}_i|^2, \]  

(4.43)

for TM polarized incident and TE polarized scattered light, and

\[ q_{pp} = \frac{1}{\gamma_i \gamma} \left| |s_{i\parallel}| - \hat{s}_{i\parallel}, \hat{s}_i|^2, \right. \]  

(4.44)

for TM polarized incident and TM polarized scattered light. Note that, for TE polarized scattered light, scattering intensity depends on the square of cosine function, which leads its value of bidirectional reflectance distribution function at both ±90 degrees to zeros regardless of surface power spectral density function. And the scattering intensity, for TM polarized incident light, diverges to infinity when incident angle approach to ±90 degrees. Some researchers have reported that these perturbation results fail to predict scattering distribution for certain situation even though the surfaces roughness is quite smooth [86, 87].

### 4.2 Kirchhoff Approximation Method

The Kirchhoff approximation (KA) method is also referred to Physical Optics method. In KA, instead of solving the boundary condition given by integral equation, the tangential plane approximation which is consistent with the Kirchhoff boundary condition is used. In the following subsections, the surface field is obtained using the approximation, and the expression
of scattered field is further derived. Finally, the ensemble average for a random rough surface exhibiting Gaussian statistics is derived.

4.2.1 Scattered Field

A scalar field \( U = U(\mathbf{r}) \) satisfies the homogeneous Helmholtz equation, and the Green’s function \( g = g(\mathbf{r} | \mathbf{r'}) \) satisfies the inhomogeneous Helmholtz equation. From the Green’s second theorem, it is given by

\[
\int_{V} d\mathbf{r} [U'(\nabla^2 g) - g(\nabla^2 U')] = \int_{S} d\mathbf{a} \left[ \frac{\partial g}{\partial n'} - g \frac{\partial U'}{\partial n'} \right]_{\mathbf{r} \in S} ,
\]

(4.45)

where the simplified notations \( U' = U(\mathbf{r'}) \) and \( n' = n'(\mathbf{r'}) \) are used. Using the homogeneous and inhomogeneous Helmholtz equations, the left side of the equation reduces to

\[
(LS) = \int_{V} d\mathbf{r} [U'(-k_0^2 g + \delta(\mathbf{r} - \mathbf{r'})) - g(-k_0^2 U')] = U .
\]

(4.46)

Together with the identity \( \frac{\partial g}{\partial n} = \mathbf{n} \cdot \nabla g \), Eq.(4.45) is rewritten by

\[
U = \int_{S} d\mathbf{a}' \left[ U'(\hat{n} \cdot \nabla g) - g(\hat{n}' \cdot \nabla U') \right]_{\mathbf{r} \in S} .
\]

(4.47)

In Cartesian coordinates, using the Fourier space representation of the Green’s function and its derivatives, the scattered field is written by
\[ U(x) = -\int d^2 s' \frac{1}{\gamma} \mathcal{U}(s') \exp[2\pi s \cdot x], \quad (4.48) \]

where

\[ \mathcal{U}(s) = \frac{1}{2} \int_S d^2 x' \left[ \left( n' \cdot s \right) U(x') + \frac{i}{2\pi} n' \cdot \left( \nabla' U(x') \right) \right] \exp[-i2\pi s \cdot x'], \quad (4.49) \]

Here the scattered field is represented by the superposition of upward plane waves. To calculate the scattered field, the information about both the field value, \( U(x')\big|_{x \in S} \), and its derivative value, \( \nabla U(x')\big|_{x \in S} \), at the boundary surface are required. Here, the Kirchhoff approximation is applied. First, for ideally conducting surface, the surface field at the boundary is assumed to be the same as the twice of the incident field given by

\[ U_s(x)\big|_{x \in S} = 2U_i(x)\big|_{x \in S} = 2\exp[-i2\pi s_i \cdot x]\big|_{x \in S}, \quad (4.50) \]

and the normal derivative of the surface field vanishes

\[ n \cdot \nabla U(x)\big|_{x \in S} = 0, \quad (4.51) \]

which is called the tangential plane approximation for an ideal conductor. With these assumptions, Eq.(4.49) is simplified to

\[ \mathcal{U}(s) = \int_S d^2 x' (n' \cdot s) \exp[-i2\pi (s_i + s) \cdot x']\big|_{x \in S}, \quad (4.52) \]

The term \((n' \cdot s)\) is called an obliquity factor for rough surfaces.
Let us introduce an observing hemisphere which is denoted by \( O \) and it is located far away from the boundary surface \( S \). Applying stationary phase method \([88]\) to Eq.(4.48), the field on the observing hemisphere is calculated by

\[
U(x)|_{x \in O} = i \mathcal{U}(s) \frac{\exp[i 2\pi |x|]}{|x|}.
\]

At the observing hemisphere, the infinitesimal radiated power illuminating small area is

\[
dP = \frac{dP}{d\Omega} = \left| U(x)|_{x \in O} \right|^2 d^2 x_i.
\]

Using the identity \( \gamma d^2 x_i = |x|^2 d\Omega \), the infinitesimal power contained in unit solid angle is

\[
\frac{dP}{d\Omega} = \left[ \left| U(x)|_{x \in O} \right|^2 \frac{|x|^2}{\gamma} \right] = \frac{1}{\gamma} \left| \mathcal{U}(s) \right|^2.
\]

The total incident power is given by \( P_i = \gamma_i A \), where \( A \) is the total mean surface area and the power per solid angle divided by the total incident power is written by

\[
\frac{1}{P_i} \frac{dP}{d\Omega} = \frac{1}{\gamma_i A} \int_S \left| \mathcal{U}(s) \right|^2 = \frac{1}{\gamma_i A} \int_S d^2 x_i (\mathbf{n} \cdot \mathbf{s}) \exp\left[-i 2\pi (\mathbf{s} + \mathbf{s'}) \cdot \mathbf{x}|_{x \in S} \right]^2.
\]

Note that the surface normal in Eq.(4.56) depends on the variable \( \mathbf{n} = \mathbf{n}(x) \) and, if the surface is perfectly smooth, it is obtained \( \mathbf{n}' \cdot \mathbf{s} = \gamma \).
4.2.2 Geometrical Factor

In this section, Eq.(4.56) will be further simplified. Let us introduce variables

\[ du_q = \zeta_q \exp[-i2\pi(\gamma_i + \gamma)\zeta'(x')] dq', \]  

(4.57)

and

\[ v_q = \exp[-i2\pi(\alpha_i + \alpha)q'], \]  

(4.58)

where \( q \) is \( x, y \) or \( z \). Recalling that \( \mathbf{n} \cdot \mathbf{s} = -\zeta_x \alpha - \zeta_y \beta + \gamma \), the function \( \mathcal{U}(\mathbf{s}_\parallel) \) is rewritten as

\[ \mathcal{U}(\mathbf{s}_\parallel) = \alpha \int dy' v_y \int v_i du_i + \beta \int dx' v_x \int v_i du_i - \gamma \int dx' v_y v_i \int du_i. \]  

(4.59)

Since \( u_i \) and \( dv_i \) is given by

\[ u_i = -\frac{1}{i2\pi(\gamma_i + \gamma)} \exp[-i2\pi(\gamma_i + \gamma)\zeta'(x')], \]  

(4.60)

\[ dv_i = -i2\pi(\alpha_i + \alpha) \exp[-i2\pi(\alpha_i + \alpha)\zeta(x')] dx', \]

the first term in Eq.(4.59) is calculated as

\[ \alpha \int dy' v_y \int v_i du_i = \alpha \int dy' v_y \left( -\int u_i dv + u_i v_i \right)_\infty = \frac{\alpha(\alpha_i + \alpha)}{(\gamma_i + \gamma)} \int d^2 x' \exp[-i2\pi(s_i + s) \cdot x'] \big|_{s=s}, \]  

(4.61)

where the edge term is neglected and the other terms can be obtained similarly. Combining all the three terms, the function \( \mathcal{U}(\mathbf{s}_\parallel) \) becomes
The integration term is now independent from the surface normal vector. Thus, the fractional scattered intensity is rewritten by

\[
\mathcal{U}(s_i) = \frac{s \cdot (s + s_i)}{\gamma' + \gamma} \int d^2x_i \exp \left[ -i 2\pi (s_i + s) \cdot \mathbf{x} \right] \chi_{x \in S}. 
\]  

(4.62)

where the geometrical factor \( F(s_i, s) \) is given by

\[
F(s_i, s) = \frac{s \cdot (s + s_i)}{\gamma(\gamma' + \gamma)}. 
\]  

(4.64)

Note that the geometrical factor is different from that in Reference [53], but it is consistent with Reference [89]. Equation (4.64) can be applied to any surface profile as long as the surface profile function is continuous, differentiable at least once and single-valued function. In the next section, the intensity function is further developed to a random surface profile assuming Gaussian statistics.

### 4.2.3 Scattered Field for Random Rough Surface

If the surface is a random rough surface and its statistical properties are homogeneous and isotropic over the whole its area, the scattering intensity is described by in terms of its statistical properties. For later convenience, let us define a quantity \( \mathcal{U}_s(s_i) \) given by
\( \Phi_s(s_i) = A^{-1/2} \int d^2 x' \exp\left[ -i2\pi(s_i + s) \cdot x' \right] \chi_{\text{inc}}. \) (4.65)

Using this quantity, the ensemble average of the incoherent fractional scattered intensity is written by

\[ \left\langle \frac{1}{P_i} \frac{\partial P}{\partial \Omega} \right\rangle_s = \frac{\gamma}{\gamma_i} F^2 \left\langle |\Phi_s(s_i)|^2 \right\rangle - \frac{\gamma}{\gamma_i} F^2 \left| \left\langle \Phi_s(s_i) \right\rangle \right|^2. \] (4.66)

where a simplified notation \( F^2 = F^2(s_i, s) \) is used. Utilizing the autocorrelation theorem of Fourier transfer theory, the second term in Eq. (4.66) is written as

\[ \left\langle |\Phi_s(s_i)|^2 \right\rangle = \int d^2 x' \left\langle |\Phi_s(x' | s_i)|^2 \right\rangle \exp\left[ -i2\pi(s_{k_i} + s_i) \cdot x' \right]. \] (4.67)

where

\( \Phi_s(x' | s_i) = \exp\left[ -i2\pi(\gamma_i + \gamma)\zeta(x') \right]. \) (4.68)

Now, let us assume that, as discussed in Section 2.1.4, the surface height distribution is statistically stationary, ergodic and normally distributed having Gaussian probability density function \( p(\zeta) \) given by

\[ p(\zeta) = \frac{1}{\sigma_{\text{tot}} \sqrt{2\pi}} \exp\left[ -\frac{\zeta^2}{2\sigma_{\text{tot}}^2} \right]. \] (4.69)

where \( \sigma_{\text{tot}} \) is root mean square (rms) roughness value. With this probability density function, the mean value is calculated by
\[
\langle \mathcal{U}_i(\mathbf{x}_i | \mathbf{s}_i) \rangle = \int d\zeta \, p(\zeta) \exp[-i2\pi(\gamma_i + \gamma)\zeta] = \exp\left\{ -\frac{1}{2} [2\pi \sigma_{tot}(\gamma_i + \gamma)]^2 \right\},
\]

(4.70)

which is a constant with respect to the spatial variables. On the other hand, the term \( |\mathcal{U}_s(\mathbf{s}_i)|^2 \) is written as

\[
|\mathcal{U}_s(\mathbf{s}_i)|^2 = \frac{1}{A} \int d^2x' \int d^2x'' \exp[-i2\pi(\mathbf{s}_i + \mathbf{s}) \cdot (\mathbf{x}' - \mathbf{x}'')]_{\zeta = \zeta(\mathbf{s}_i)}.
\]

(4.71)

With the change of variables, \( \mathbf{x}_i' - \mathbf{x}_i'' = \mathbf{x}_i \), Eq. (4.71) becomes

\[
|\mathcal{U}_s(\mathbf{s}_i)|^2 = \int d^2x_i \mathcal{U}_s(\mathbf{x}_i | \mathbf{s}_i) \exp[-i2\pi(\mathbf{s}_i + \mathbf{s}_i) \cdot \mathbf{x}_i],
\]

(4.72)

where

\[
\mathcal{U}_s(\mathbf{x}_i | \mathbf{s}_i) = \frac{1}{A} \int d^2x_i \exp[-i2\pi(\gamma_i + \gamma) \cdot (\zeta(\mathbf{x}_i') - \zeta(\mathbf{x}_i' - \mathbf{x}_i))].
\]

(4.73)

The ensemble average of \( |\mathcal{U}_s(\mathbf{s}_i)|^2 \) is obtained by calculating ensemble average of \( \mathcal{U}_s(\mathbf{x}_i | \mathbf{s}_i) \). If the two random variables, \( \zeta_1 = \zeta(\mathbf{x}_i') \) and \( \zeta_2 = \zeta(\mathbf{x}_i' - \mathbf{x}_i) \), are jointly normal, the joint probability density function \( p_j(\zeta_1, \zeta_2) \) is given by

\[
p_j(\zeta_1, \zeta_2) = \frac{1}{2\pi\sigma_{tot}^2 \sqrt{1 - C^2(\mathbf{x}_i)}} \exp\left[ -\frac{\zeta_1^2 - 2C(\mathbf{x}_i)\zeta_1\zeta_2 + \zeta_2^2}{2\sigma_{tot}^2(1 - C^2(\mathbf{x}_i))} \right],
\]

(4.74)

where \( C(\mathbf{x}_i) \) is their correlation and it depends only the spatial separation between the two variables because it is assumed that the surface is statistically stationary. Since the surface is
assumed to be ergodic, the expectation value of $\mathcal{U}_s(x_i | s_i)$ is the same as ensemble average of $\mathcal{U}_s(x_i | s_i)$ and it is given by

$$\langle \mathcal{U}_s(x_i | s_i) \rangle = \int d\zeta_1 d\zeta_2 p_r(\zeta_1, \zeta_2) \exp\left[-i2\pi(\gamma_i + \gamma) \cdot (\zeta_1 - \zeta_2)\right].$$

Thus, the second term in Eq.(4.66) becomes

$$\left\langle \left| \mathcal{U}_s(s_i) \right|^2 \right\rangle = \int d^2 x_i \left\langle \mathcal{U}_s(x_i | s_i) \right\rangle \exp\left[-i2\pi(s_{ki} + s_i) \cdot x_i\right].$$

Together with Eqs.(4.67) and (4.72), the fractional incoherent scattered intensity becomes

$$\left\langle \frac{1}{P} \frac{\partial P}{\partial \Omega} \right\rangle_s = \frac{\gamma}{\gamma_i} F^2 \int d^2 x_i \left[ \left\langle \mathcal{U}_s(x_i | s_i) \right\rangle - \left\langle \mathcal{U}_s(x_i | s_i) \right\rangle \right] \exp\left[-i2\pi(s_{ki} + s_i) \cdot x_i\right],$$

where

$$\langle \mathcal{U}_s(x_i | s_i) \rangle - \langle \mathcal{U}_s(x_i | s_i) \rangle = \exp\left\{- \left[2\pi\sigma_{tot}(\gamma_i + \gamma)\right]^2 \right\} \left( \exp\left\{- \left[2\pi\sigma_{tot}(\gamma_i + \gamma)\right]^2 \right\} - 1 \right).$$

The scattering intensity from randomly rough surfaces is represented by using their correlation function. If the correlation function is given by a Gaussian function, the asymptotic behaviors of the scattering intensity distribution when surface roughness is very small and very large can be founded in other literatures [53]. Since the KA model is based on the tangential plane approximation, it is expected that the results will be valid when the surface slope is small. It is also expected that the model does not give accurate results in the situation where there are multiple scattering and/or shadowing effects.
4.3 Generalized Harvey-Shack Method

In this section, the Generalized Harvey-Shack (GHS) method is introduced. The GHS method is based on scalar diffraction theory and, similar to the KA method, the Kirchhoff boundary condition. The method originates from a linear systems formulation of surface scatter phenomena (so-called original Harvey-Shack surface scatter theory [70]), but removes its paraxial assumption. The surface transfer function which characterizes the scattering behavior is first introduced, then a necessary renormalization process is discussed, and finally the angle spread function (or scattered radiance function) is described.

4.3.1 Angle Spread Function

If a scalar field, \( U = U(x) \), satisfies the homogeneous Helmholtz equation, applying Green’s second theorem in Cartesian coordinate gives

\[
\int_{\gamma'} dy' \left[ U' \left( \nabla'^2 g_- \right) - g_- \left( \nabla'^2 U' \right) \right] = \int_{S} da \left[ U' \frac{\partial g_-}{\partial n'} - g_- \frac{\partial U'}{\partial n'} \right]_{x'=S}. \tag{4.79}
\]

where we set the function \( g_- = g_-(x | x') \) as

\[
g_-(x | x') = g(x | x') - g(x'' | x'), \tag{4.80}
\]
where \( \mathbf{x}'' = \mathbf{x}'(\mathbf{x}) = x\hat{x} + y\hat{y} - z\hat{z} \), which is the mirror image of the position \( \mathbf{x} \) to the x-y plane, and 

\[ g(\mathbf{x} | \mathbf{x}') \]

is the Green’s function. And the function \( g_{-}(\mathbf{x} | \mathbf{x}') \) satisfies the equation

\[
\nabla^2 g_{-}(\mathbf{x} | \mathbf{x}') + k_0^2 g_{-}(\mathbf{x} | \mathbf{x}') = -\delta(\mathbf{x} - \mathbf{x}') + \delta(\mathbf{x}'' - \mathbf{x}').
\]

(4.81)

For \( \mathbf{x} \in V_1 \), since \( \mathbf{x}'' \) is outside the space, the left side of the Eq.(4.79) reduces to

\[
\int dV'[U(\mathbf{x}')(-\delta(\mathbf{x} - \mathbf{x}') + \delta(\mathbf{x}'' - \mathbf{x}'))] = U(\mathbf{x}).
\]

(4.82)

On the other hand, the term \( \partial g / \partial n = \mathbf{n} \cdot \nabla g_{-} \) in the right side of Eq.(4.79) reduces to

\[
\mathbf{n} \cdot \nabla g_{-}(\mathbf{x} | \mathbf{x}') = \mathbf{n} \cdot \nabla g(\mathbf{x} | \mathbf{x}') - \mathbf{n} \cdot \nabla g(\mathbf{x}'' | \mathbf{x}') = 2\mathbf{n} \cdot \nabla g(\mathbf{x} | \mathbf{x}').
\]

(4.83)

Together with Eqs.(4.82) and (4.83), Eq.(4.79) becomes

\[
U(\mathbf{x}) = 2\int_{S} d^2 s' \left[ U(\mathbf{x}')(\mathbf{n}' \cdot \nabla' g(\mathbf{x} | \mathbf{x}')) \right]_{s' \in S}.
\]

(4.84)

which is the first Rayleigh-Sommerfeld solution [90].

Here, we set the surface as a flat plane at \( z = 0 \). Manipulating Eq.(4.84) using a Fourier space representation of the Green’s function, Eq.(4.84) can be written as

\[
U(\mathbf{x}) = -\int d^2 s_{\parallel} \frac{1}{\gamma} \mathcal{U}(\mathbf{s}_{\parallel}) \exp \left[ i2\pi\mathbf{s} \cdot \mathbf{x} \right],
\]

(4.85)

where

\[
\mathcal{U}(\mathbf{s}_{\parallel}) = \gamma \mathcal{F}[U_0(x'_{\parallel})],
\]

(4.86)
where \( \gamma = \hat{z} \cdot \mathbf{s} = \mathbf{n}' \cdot \mathbf{s} \) and simplified notation \( U_0(x'_i) = U(x'_i) \big|_{x'=0} \) is used. Equation (4.86) states that the Fourier transform of the field distribution at the boundary plane becomes \( \mathcal{F}(\mathbf{s}_i)/\gamma \).

Again, let us to return to Eq.(4.84). Recalling that if observing point is located on the observing hemisphere denoted by \( \mathbf{x} \in \mathcal{O} \), the term \( \hat{z} \cdot \nabla g \) in the equation is approximately

\[
\hat{z} \cdot \nabla g(x\mid x') \big|_{x=0} \approx \frac{1}{2} i \gamma \exp[-i 2 \pi \mathbf{s}_i \cdot \mathbf{x}_i] \frac{\exp[i 2 \pi |x|]}{|x|} . \tag{4.87}
\]

Using this asymptotic expression, Eq.(4.84) can be written as

\[
U(x) \big|_{x=0} = i \gamma \mathcal{F}[U_0(x_i')] \frac{\exp[i 2 \pi |x|]}{|x|} . \tag{4.88}
\]

Since infinitesimal radiated power irradiated from a small area can be expressed by

\[
dP = \frac{dP}{d\Omega} d\Omega = |U(x)|^2 \big|_{x=0} d^2 x_i , \tag{4.89}
\]

the infinitesimal power contained in unit solid angle is obtained

\[
\frac{dP}{d\Omega} = \left[ |U(x)|^2 \frac{|\mathbf{x}^2|}{\gamma} \big|_{x=0} \right] = \gamma \mathcal{F}[U_0(x'_i)]^2 = \gamma \left| \mathcal{U}(\mathbf{s}_i) \right|^2 , \tag{4.90}
\]

where the identity \( \gamma d^2 x_i = |\mathbf{x}^2|_{x=0} d\Omega \) is used. By integrating both sides of Eq.(4.90), the total power is calculated by
\[ P = \int d^2 s \| D_1(s) \| \mathcal{U}(s) \|^2, \]  \hspace{1cm} (4.91)

where \( D_1(s) \) is the unit disk function given by \( D_1(s) = 1 \) for \( |s| \leq 1 \) and \( D_1(s) = 0 \) elsewhere. If the function \( \mathcal{U}(s) \) has zero value for the domain \( |s| > 1 \), that is \( D_1(s) \mathcal{U}(s) = \mathcal{U}(s) \), applying Parseval’s theorem gives

\[ P = \int d^2 s \left| \mathcal{U}(s) \right|^2 = \int d^2 x |U_0(x')|^2 = P_t. \]  \hspace{1cm} (4.92)

Equation (4.92) states that the total incident power should be equal to the total radiated power.

Now, let us introduce a quantity called as angel spread function (ASF) defined by

\[ \text{ASF}(s) = \frac{dL}{dE} = \frac{dP}{P d^2 s} = \frac{1}{\gamma P \partial \Omega}, \]  \hspace{1cm} (4.93)

which is a scattered radian distribution function (rather than the more conventional scattered intensity distribution function). Using Eq.(4.90) or Eq.(4.91), the ASF becomes

\[ \text{ASF}(s) = \mathcal{F}[\mathcal{H}(x)]. \]  \hspace{1cm} (4.94)

Using the autocorrelation theorem of Fourier transform theory, the function \( \mathcal{H}(x) \) is given by

\[ \mathcal{H}(x) = \frac{1}{P_t} \int d^2 x' \left[ U_0(x') U_0^*(x' - x) \right], \]  \hspace{1cm} (4.95)
The *angle spread function* is thus represented by the Fourier transform of the *surface transfer function*, $\mathcal{H}(x_i)$. The relationships described by Eqs.(4.94) and (4.95) form an exact analogy with the definitions and relationship between the *point spread function* and the *optical transfer function* of modern image formation theory. This linear systems formulation has gone a long way towards bringing complicated surface scatter phenomena out of the theoretical realm into a practical engineering realm that can readily be applied to image degradation problems by optical engineers and image analysts.

### 4.3.2 Renormalization Constant

Let us assume that an upward plane wave is traveling through the volume and it has direction cosine vector $s_0$, and its irradiance is unity. The irradiance by the incident wave at the flat boundary plane is

$$U(x)|_{z=0} = \sqrt{\gamma_0} \exp[i2\pi s_0 \cdot x] |_{z=0}.$$  \hspace{1cm} (4.96)

In this case, function $\mathcal{H}(x_i)$ becomes

$$\mathcal{H}(x_i) = \gamma_0 \exp[i2\pi s_{i,0} \cdot x_i],$$  \hspace{1cm} (4.97)

and the ASF is calculated by

$$ASF(s_i) = \mathcal{F}\{\exp[i2\pi s_{i,0} \cdot x_i]\} = \delta(s_i - s_{i,0}).$$  \hspace{1cm} (4.98)
Equation (4.99) states that the angle spread function is shift invariant relative to variations in the incident angle, which is trivial but important. If, for any reason, the field at the flat boundary surface has phase (and possibly amplitude) variation, the field on the flat surface can be written as

\[ U(x_i) = \sqrt{\gamma_0} P_0(x_i) \exp\left[i2\pi s_{\parallel,0} \cdot x_i\right]. \]  

(4.99)

where \( P_0(x_i) \) is a complex the pupil function. The function \( \mathcal{H}(x_i) \) is written by

\[ \mathcal{H}(x_i) = \gamma_0 \exp\left[i2\pi s_{\parallel,0} \cdot x_i\right] \mathcal{H}'(x_i), \]  

(4.100)

where \( \mathcal{H}'(x_i) \) is given by

\[ \mathcal{H}'(x_i) = \int d^2 x'_i \frac{P_0(x'_i - x_i) P'_0(x'_i)}{d^2 x'_i P_0(x'_i) P'_0(x_i)}. \]  

(4.101)

Especially, if the pupil function does not depends on the direction cosines, the ASF becomes

\[ \text{ASF}(s_i) = \delta(s_i - s_{\parallel,0}) \otimes \mathcal{F}[\mathcal{H}(x_i)], \]  

(4.102)

which means, again, the angle spread function is shift invariant relative to variations in the direction cosines \( s_{\parallel,0} \). However, if the pupil function depends on the direction cosines, the ASF cannot be written in the form of convolution, which means it is not shift invariant to the direction cosines.
This formalism can be applied to many obstacles such as apertures, lenses, and gratings as long as we can obtain the effective pupil function at the flat plane. One easy and intuitive way to obtain the pupil function is applying the Kirchhoff boundary condition. By counting optical path difference (OPD) and putting adequate amplitude change, the pupil function can be described. However, as pointed out in references [62,65,67], sometimes ASFs obtained using this method have non-zero values for the domain $|s_0| > 1$. It then violates the condition, $D_i(s_0)\mathcal{U}(s_0) = \mathcal{U}(s_0)$, therefore it violates the law of energy conservation. However, instead of discarding the Kirchhoff boundary condition, which has been successfully used for describing many optical phenomena, simply a constant is introduced as

$$K \int d^2s_1 D_i(s_0) \left|\frac{\mathcal{U}(s_0)}{\gamma}\right|^2 = \int d^2s_1 \left|\frac{\mathcal{U}(s_0)}{\gamma}\right|^2 = \int d^2x_0 \left|U_0(x_0)\right|^2 = P. \quad (4.103)$$

or equivalently,

$$K = \frac{P/P_i}{\int d^2s_1 D_i(s_0) \text{ASF}(s_0)}. \quad (4.104)$$

where $P$ is the total scattered power and the constant $K$ is called a renormalization constant. The validity and limitation of this method in some situations can be found in other literatures [66,67].

85
4.3.3 Surface Transfer Function

In the view of the previous section, the surface scattering problem reduces to obtaining the effective pupil function in the presence of a rough surface.

Let’s assume that the incident wave is given as a downward plane wave with direction cosine vector $s_{i}$, and that its irradiance is given by unity. If the surface is perfectly smooth, we cannot distinguish the reflected upward wave by the surface from an upward wave having direction cosine $s_{0} = -\alpha \hat{x} - \beta \hat{y} + \gamma \hat{z}$ without the reflection.

However, if the surface is not perfectly smooth, due to the Kirchhoff boundary condition, the amplitude and phase of the field at the virtual flat plane $z = 0$ is changed compared to the field when the surface is perfectly smooth. The phase difference is obtained by measuring the optical path difference between the rough and flat surface, and the differences are changed along the observation position as well as the incident angle. Thus, the optical path difference has two parameters; one is related to the observing position and the other is related to the incident angle given by

$$OPD(x_{i} \parallel s_{i}) = (\gamma_{i} + \gamma)\zeta(x_{i}). \tag{4.105}$$

Since the phase variation is severe, the amplitude variation is ignored for perfectly conducting surfaces. Using this assumption, the pupil function is written as

$$P_{0}(x_{i} \parallel s_{i}) = \exp[i2\pi(\gamma_{i} + \gamma)\zeta(x_{i})]. \tag{4.106}$$
The function $\mathcal{H}(x_j)$ is given by

$$\mathcal{H}(x_{||} | s_{||}) = \gamma_0 \exp[i2\pi s_{||} \cdot x_{||}] \mathcal{H}_{s}(x_{||} | s_{||}), \quad (4.107)$$

where $\mathcal{H}_{s}(x_{||} | s_{||})$ is

$$\mathcal{H}_{s}(x_{||} | s_{||}) = \frac{1}{A} \int d^3x' \exp[-i2\pi(\gamma + \gamma')(\zeta(x_{||}) - \zeta(x'_{||} - x_{||}))]. \quad (4.108)$$

where $A$ is the area of the virtual flat surface and the term $\mathcal{H}_{s}(x_{||})$ is called the surface transfer function. Then, the ASF becomes

$$ASF(s_{||}) = \mathcal{F}[\mathcal{H}(x_{||} | s_{||})] = \int d^3x' \mathcal{H}_{s}(x_{||} | s_{||}) \exp[i2\pi(s_{||} - s_{||}) \cdot x_{||}]. \quad (4.109)$$

where

$$\mathcal{H}_{s}(x_{||} | s_{||}) = \frac{1}{A} \int d^3x' \exp[-i2\pi(\gamma + \gamma')(\zeta(x'_{||}) - \zeta(x'_{||} - x_{||}))] = \mathcal{H}_0(x_{||} | s_{||}) \quad (4.110)$$

The equations (4.109) and (4.110) are exactly identical to equations (4.74) and (4.75) in the Kirchhoff approximation discussed in Section 4.2.3, if we use $s_{||,0} = -s_{||,i}$. Note that, since the surface transfer function has dependency on the direction cosines, the ASF cannot be written in the form of a convolution. It indicates that the ASF is not shift invariant to the incident angle, however, in small incident and scattered angle, the direction cosines, $\gamma_i$ and $\gamma$, can be assumed to be unity, then the ASF becomes shift invariant to the incident angle.

87
4.3.4 Scattering Intensity

With the definition of ASF and Eqs.(4.78) and (4.110), the ensemble average of ASF is calculated by

$$\langle ASF(s_l) \rangle = \mathcal{F}(\langle \mathcal{H}(x_l | s_l) \rangle \exp[i2\pi s_{l0} \cdot x_l],$$  \hspace{1cm} (4.111)$$

where

$$\langle \mathcal{H}(x_l | s_l) \rangle = \exp \left[2\pi \sigma_{\text{tot}} (\gamma_l + \gamma) [1 - C(x_l)] \right].$$ \hspace{1cm} (4.112)$$

Here, one approximation is added. Since geometrical structure having less scale than the wavelength of the incident light does not contribute to diffraction (or negligible if any) phenomena at far field, the contribution by such structures of the surface to the scattering can be removed. Or the surface profile can be smoothed so that it does not have higher frequencies than the reciprocal of the incident wavelength. Then, the total rms roughness $\sigma_{\text{tot}}$ is replaced to relevant rms roughness $\sigma_{\text{rel}}$ defined by [62,91]

$$\hat{\sigma}_{\text{rel}} = \int d^2 s_l D_l(s_l - s_{l0})W(s_l).$$ \hspace{1cm} (4.113)$$

Together with the relevant rms roughness and using Eq.(4.68), the coherent part of the ensemble averaged ASF is obtained as

$$\langle ASF(s_l) \rangle_p = \exp \left[-(4\pi \sigma_{\text{rel}} \gamma_l)^2 \right] \delta(s_l - s_{l0}).$$ \hspace{1cm} (4.114)$$
which corresponds to the specular light and the incoherent part of the ensemble averaged ASF is

\[
\langle SF(s_i) \rangle_S = \mathcal{F} \left[ \exp \left\{ -\left[ 2\pi \sigma_{rel}(\gamma_i + \gamma) \right]^2 \left[ \exp \left\{ \left[ 2\pi \sigma_{rel}(s_i - s_{||}) \right]^2 C(x_i) \right] - 1 \right] \right\} \right].
\] (4.115)

Equation (4.115) states that the total energy in the specular ray depends on the direction of the incident wave and the relevant rms roughness of the surface. Also the equation states that the fractional radiant power contained in the scattering part of ASF is

\[
B = 1 - \exp \left[ -\left( 4\pi \sigma_{rel}\gamma_i \right)^2 \right].
\] (4.116)

This is the modern version of the classical expression for the total integrated scatter (TIS) [91]. Applying the renormalization method, the renormalized scattering part of the ensemble averaged ASF becomes

\[
\langle SF(s_i) \rangle_{S,R} = K \langle SF(s_i) \rangle_S,
\] (4.117)

where the renormalization constant is given by

\[
K = \frac{B}{\int d^2s_i D(s_i) \langle SF(s_i) \rangle_S}.
\] (4.118)

Lastly, by the definition of the angle spread function, the scattering intensity is easily obtained

\[
\left\langle \frac{1}{P_i} \frac{\partial P_i}{\partial \Omega} \right\rangle = \gamma \langle SF(s_i) \rangle_{S,R}.
\] (4.119)
The GHS method is based on the scalar diffraction theory and, similar to the KA method, Kirchhoff approximation. By introducing the surface transfer function, it linearizes the surface scattering formulation and categorizes it as diffraction phenomena. The surface transfer function is given by the two parameters, incident and scattered direction cosines, and the mathematical form of the scattering intensity is very similar to one of the KA except the geometrical factor, rms roughness and the renormalization constant.
CHAPTER 5: NUMERICAL COMPARISON FOR ONE DIMENSIONAL RANDOM ROUGH SURFACES

Several previous researchers have reported upon the valid domain of the SPM, KA and other methods by comparing their scattering predictions to that predicted by the IEM [17-30]. Almost all of their work has been done for one-dimensional random rough surfaces with Gaussian statistics because the IEM provides fairly stable and reliable predictions for that case with current computing technology. In this chapter, we restrict our attention to perfectly conducting one-dimensional random surfaces. Firstly we systematically investigate the region of validity of the GHS method for perfectly conducting one-dimensional random rough surfaces with Gaussian statistics and compare it to the region of validity of the SPM and KA. Next, we extend the work to surfaces having fractal-like structure which has been less intensively investigated but which more closely characterizes conventional ground and polished optical surfaces. Although we have used the wavelength normalized coordinates, which means the unit of length is the wavelength, the wavelength dependence is explicitly expressed when values of surface statistical parameters are specified throughout this chapter in order to make it easy to compare the results in this chapter to other literature.
5.1 Surface with Gaussian Statistics

Surfaces having Gaussian statistics can be characterized by rms roughness $\sigma_{tot}$ and the auto-covariance length $l_c$. In wavelength scaled coordinates, the auto-covariance (ACV) function for one-dimensional surfaces is given by

$$ACV(x) = \sigma_{tot}^2 \exp\left[-\left(x/l_c\right)^2\right], \quad (5.1)$$

and the corresponding one-dimensional surface PSD function is given by

$$W(\alpha) = \sqrt{\pi}l_c \sigma_{tot}^2 \exp\left[-(\pi l_c \alpha)^2\right]. \quad (5.2)$$

The rms roughness, the surface auto-covariance length and the incident angle are the three parameters that determine the behavior of surface scattering for ideally conducting surfaces having a Gaussian auto-covariance function. In the following Sections, we limited our attention to the practical range of surface parameters defined by $0<\sigma_{tot}<\lambda$ and $0<l_c<3\lambda$ for our numerical analysis. And predictions for TE polarization of the three approximate methods described in the previous chapters are numerically compared to the rigorous IEM prediction.

5.1.1 Scattering Intensity Distribution

Figure 5-1 shows the scattering intensity as a function of scattering angle for the IEM (black asterisk), SPM (blue dashed line), KA (green dotted line), and GHS (red solid line) for two different sets of parameters. In Figure 5-1(a), the case of $\sigma_{tot}=0.01\lambda$, $l_c=0.2\lambda$, $\theta_i=30^\circ$, which is a
smooth surface with a short correlation length, is considered. Both predictions of the SPM and GHS agree well with the IEM prediction, and the KA result agrees near the specular direction but fails in the wide angle regime. In particular, it does not converge to zero at ±90° scattering angles, which is a non-physical situation. The case of \( \sigma_{tot}=0.2\lambda, l_c=2\lambda, \theta_i=0^\circ \), which is categorized as in moderately rough with a long correlation length, is illustrated in Figure 5-1(b), and both intensity profiles from the KA and GHS methods show good agreement to the IEM prediction; however, as expected, the SPM fails for this moderately rough surface.

![Figure 5-1](image1.png)

**Figure 5-1:** Scattered intensity of the IEM (asterisk), SPM (dashed line), KA (dotted line), and GHS (solid line) for (a) \( \sigma_{tot}=0.01\lambda, l_c=0.2\lambda, \theta_i=30^\circ \), (b) \( \sigma_{tot}=0.2\lambda, l_c=2\lambda, \theta_i=0^\circ \).

Figures 5-2(a) and 5-2(b) illustrate that the GHS has reasonable accuracy for moderately rough surfaces, but the other two fail to predict an accurate intensity distribution for certain situations. However they fail for different reasons; the SPM fails because the smooth-surface approximation
is violated and the KA fails because the small correlation lengths produce large angle scattering that violates the paraxial limitation.

Figure 5-2: Scattering intensity for (a) $\sigma_{tot}=0.1\lambda$, $l_c=0.25\lambda$, $\theta_i=0^\circ$, (b) $\sigma_{tot}=0.3\lambda$, $l_c=0.8\lambda$, $\theta_i=0^\circ$, (c) $\sigma_{tot}=0.4\lambda$, $l_c=2.5\lambda$, $\theta_i=50^\circ$ and (d) $\sigma_{tot}=\lambda$, $l_c=2\lambda$, $\theta_i=0^\circ$. The SPM is not plotted in (c) and (d).
Figures 5-2(c) and 5-2(d) are for even rougher surfaces, thus the SPM predictions have been omitted. Figure 5-2(c) shows the predicted intensity distribution by the KA and GHS for the case of \( \sigma_{\text{tot}} = 0.4\lambda, l_c = 2.5\lambda, \theta_i = 50^\circ \).

The GHS predictions agree well with the rigorous IEM predictions over the entire range of scattered angles; however, the KA prediction exhibits significant error near the peak and in the forwarding scattering direction due to the large (non-paraxial) incident angle. In Figure 5-2(d), a very rough surface (\( \sigma_{\text{tot}} = \lambda \)) is considered, and the overall angular behavior predicted by the GHS theory is quite close to that predicted by the rigorous IEM except near the specular direction where the IEM predictions exhibit irregularities due to considerable multiple scattering [92].

There is, of course, no specular beam for such a rough surface; i.e. the total integrated scatter (TIS) is approximately unity. The KA method fails badly due to the large surface roughness. Extensive empirical scatter predictions with the KA method have indicated that it remains quite accurate, at this correlation width, for \( \sigma_{\text{tot}} = 0.5\lambda \). This statement will be validated later in Figure 5-3(b).

### 5.1.2 Error Map for Region of Validity

To quantitatively illustrate the relative accuracy of the three approximate surface scatter methods over the entire practical range of surface parameters, we define the mean error in the predicted scattered radiant power from -90° to 90° normalized by the total scattered radiant power.
\[ \varepsilon = \frac{1}{\pi} \sum_{m} \left| I(\theta_m) \Delta \theta_m - I_{IEM}(\theta_m) \Delta \theta_m \right| \sum_{m} I_{IEM}(\theta_m) \Delta \theta_m \]  

Figure 5-3 illustrates contour maps of this error over the two-dimensional surface parameter space for each of the approximate theories for normal incidence. Note that, for our simulation of scattered intensity with the IEM, one thousand realizations were used and the averaged intensity of the IEM predictions still exhibits minor fluctuations. The SPM has high accuracy over the entire range of correlation widths; however, the domain is severely restricted to very smooth surfaces as shown in Figure 5-3(a).

![Figure 5-3](image-url)

(a) (b) (c)

Figure 5-3: Error contour maps for normal incidence; (a) the SPM, (b) the KA and (c) the GHS.

The KA can be considered to have a high degree of accuracy over a much larger fraction of the two-dimensional surface parameter space, increasing almost linearly to moderately rough surfaces for increasing correlation length as shown in Figure 5-3(b). However, the KA prediction
is quite inaccurate \( (\varepsilon > 0.2) \) over a significant fraction of the domain represented by the upper left corner of Figure 5-3(b). Meanwhile, the GHS theory exhibits slightly less accuracy than the SPM and KA in some regions of their domain of validity, but it has a much broader valid domain in general. For example the entire illustrated domain is valid if \( \varepsilon < 0.2 \) is considered as the criterion as illustrated in Figure 5-3(c).

It is widely believed that the validity of the KA method depends on the curvature of the surface irregularities making up the surface profile [17,34]. Due to the tangential plane approximation, the KA is considered to be accurate when the curvature is less than the wavelength of interest. In Figure 5-4(a), the numerically calculated contour lines of the rms slope of the surface are plotted on top of the error map previously shown in Figure 5-3(b).

Figure 5-4: Contour lines of (a) rms slope and (b) rms curvature superposed on top of the error map for the KA method with normally incident light.
In Figure 5-4(b), contour lines of rms curvature are plotted on top of the same error map. From the two figures, it is clear that the validity of the KA method is more strongly correlated to the rms slope than to the rms curvature of the surface.

![Figure 5-4(a)](image1)

![Figure 5-4(b)](image2)

**Figure 5-5:** Contour lines of rms slope superposed on top of the error map for the GHS for (a) $\theta_i=0^\circ$ and (b) $\theta_i=40^\circ$.

Likewise, in Figure 5-5(a), the numerically calculated contour lines of the rms slope of the surface are plotted on top of the error profile map shown in Figure 5-3(c). For normally incident light, the GHS method is relatively accurate when the rms slope is less than the wavelength of the scattered light. Figure 5-5(b) indicates that for a 40 degree incident angle, the value of the error increases for a given value of rms slope. This degradation in accuracy with increasing incident angle may be caused by the fact that the GHS does not account for shadowing effects and multiple scattering.
Figure 5-6: Error contour maps for the SPM with (a) $\theta_i=20^\circ$ (d) $\theta_i=40^\circ$ (g) $\theta_i=60^\circ$, for the KA with (b) $\theta_i=20^\circ$ (e) $\theta_i=40^\circ$ (h) $\theta_i=60^\circ$, and for the GHS with (c) $\theta_i=20^\circ$ (f) $\theta_i=40^\circ$ (i) $\theta_i=60^\circ$. 
In order to more clearly show the incident angle dependency of the region of validity for the three approximate theories, error maps for the incident angles of 20, 40 and 60 degrees are provided in Figure 5-6. The shifting of the error contour lines with increasing incident angle can be studied; however, if an error value $\varepsilon < 0.2$ is maintained as a criterion for validity, the entire domain of two-dimensional surface parameter space illustrated remains valid for the GHS method at $\theta_i = 20^\circ$ and at $\theta_i = 40^\circ$. Not until $\theta_i = 60^\circ$ does a portion of this domain (upper central) exhibit an error value greater than $\varepsilon = 0.2$. However, within the small slope regime (lower right), it still shows a small error of value, $\varepsilon < 0.05$, for very large incident angles. It can be seen in Figure 5-6 that, if a different criterion for validity is chosen, or if a particular region in the domain is of interest, the KA method is actually be more accurate than the GHS method.

5.1.3 Total Integrated Scatter

For some applications the total amount of scattered light is more important than its angular distribution. Hence, it is necessary that a scattering theory accurately predict the TIS. The TIS is defined as the diffusely reflected (scattered) radiant power divided by the total reflected radiant power. It has historically been analytically approximated by the following expression [91]

$$TIS = \frac{\text{Diffuse Reflectance}}{\text{Total Reflectance}} \approx 1 - \exp\left[-(4 \pi \gamma, \hat{\sigma})^2\right]. \quad (5.4)$$

It has recently been emphasized that Eq.(5.4) is ambiguous unless the relevant spatial frequency band-limits on the surface roughness $\sigma$ are specified [91].
The GHS uses $\sigma_{rel}$ as defined in previous chapter and described in more detail in [62,91]; however, the KA methods have traditionally used the total rms roughness, $\sigma_{tot}$. Since the rigorous IEM method is known to conserve energy, we will define the error in the TIS predictions by

$$
\varepsilon = \frac{|TIS - TIS_{IEM}|}{TIS_{IEM}}.
$$

(5.5)

Figure 5-7(a) illustrates the error map over the two-dimensional surface parameter space when $\sigma_{tot}$ is used in Eq.(5.4), and Figure 5-7(b) illustrates the error map when $\sigma_{rel}$ is used in Eq.(5.4). Figure 5-7(a) illustrates that, for normal incidence, Eq.(5.4) is very accurate for Gaussian surfaces over the whole domain of interest except for very smooth surfaces with small correlation lengths, even when using $\sigma_{tot}$. Using $\sigma_{rel}$ in Eq.(5.4) provides even better accuracy as shown by Figure 5-7(b).

Figure 5-7: Error contour map for TIS predicted by Eq.(15) for $\theta_i=0^\circ$ with (a) $\sigma_{tot}$, and (b) $\sigma_{rel}$. 

101
By definition, the TIS can be obtained by integrating the scattering function over the entire hemisphere. Figure 5-8 illustrates the accuracy of TIS values obtained by integrating the scattering function predicted by the SPM, KA and GHS surface scatter theories for normally incident light.

![Figure 5-8](image)

Figure 5-8: Error contour maps of the TIS value predicted by numerically integrating the scattering function predicted by (a) the SPM, (b) the KA and (c) the GHS for normal incidence.

The SPM is again restricted to smooth surfaces, and does not conserve energy for even moderately rough surfaces as shown in Figure 5-8(a). The TIS value predicted by the KA can again be considered to have a high degree of accuracy over a much larger fraction of the two-dimensional surface parameter space, increasing almost linearly to moderately rough surfaces for increasing correlation length as shown in Figure 5-8(b). Comparing Figure 5-8 with Figure 5-3 suggests that the accuracy of the scattered intensity maps predicted by the SPM and
the KA is highly correlated to the corresponding accuracy of the TIS; i.e., to the fact that the SPM and the KA is highly correlated to the corresponding accuracy of the TIS; i.e., to the fact that the SPM and the KA approximate theories do not conserve energy over the full domain of relevant surface parameters. Meanwhile, the corresponding TIS error map of the GHS is accurate almost over the entire two-dimensional surface parameter domain as shown by Figure 5-8(c). In fact, it is identical to Figure 5-7(b) because the scattering function in the GHS surface scatter theory is re-normalized to have the TIS value predicted by Eq.(5.4) with the appropriate band-limited roughness given by $\sigma_{rel}$.

5.1.4 Simulation Parameters

Throughout the previous sections in this Chapter, predictions by the three approximate methods were compared to predictions by the IEM. The formalism of the IEM is described in Chapter 3, but the parameters used for the calculations must be discussed. To show the behavior of the error maps in the domain of interest, 55 cases were selected for rms roughness from 0.002$\lambda$ to 1$\lambda$. For every roughness, 60 cases of the correlation length from 0.05$\lambda$ to 3$\lambda$ were calculated. Note that all the lengths are wavelength scaled. The total length of the surface was chosen to be 20$\lambda$ and 1000 realizations were carried out. The Gaussian beam discussed in Section 3.2.1 is used as incident beam and the width of the Gaussian spectrum of it is chosen to be one-fourth of the surface length as used by other researchers [34]. The total number of sampling points was carefully chosen depending on the correlation length and roughness. In our experience, the total number of sampling points which is required to achieve energy conservation depends more on
the rms slope than the correlation length. Sixteen sampling points per wavelength were used for surfaces having rms slopes smaller than unity, and a denser sampling ratio is used for the case of steeper rms slopes.

However, due to the restriction of computer memory and calculation time, no more than 2000 total sampling points were used. Thus, in the regime of short correlation length and very steep mean slope, the reliability of the IEM is possibly reduced. Figure 5-9(a) shows the total fractional reflected energy predicted by the IEM for the normal incidence case and Figure 5-9(b) is for a 40 degree incident angle case. In both figures, the energy conservation condition is achieved with less than 0.5% error over almost the entire domain of interest. However, it must be mentioned that, although energy conservation is one of the necessary conditions, the fulfillment of the requirement does not always guarantee the accuracy of the method.

Figure 5-9: Contour map of total fractional reflected energy as a function of the rms roughness and the surface correlation length for the IEM: (a) $\theta_i=0^\circ$ (b) $\theta_i=40^\circ$.  

104
5.2 Surface with Fractal-like Structure

In this section, we focus on the validity of the three approximate methods for fractal-like surfaces. Since it is known that well-polished optical surfaces have fractal-like structure [93-98], the domain of the validity for fractal surfaces would be useful when analyzing surface scatter from optical elements. The surface PSD of fractal surfaces follow the inverse power law given by

\[ W(\alpha) = C_0 |\alpha|^{-c}, \]

where \( C_0 \) is some constant and \( c \) is the slope of the PSD function in log-log scale. Unlike the surfaces having Gaussian PSD, the fractal surfaces are not characterized by rms roughness and correlation length [99,100]. Therefore, it is needed to obtain the region of validity in terms of the parameters, \( C_0 \) and \( c \), which characterize fractal surfaces rather than rms roughness and correlation length. There are a few authors that have discussed surface scatter from fractal surfaces [28,31,34,101], but unlike these previous researchers, we obtain the valid domain by directly comparing scattering predictions by the approximate methods and the IEM. And the valid domain is represented in terms of the parameters that characterize the fractal surfaces.

In the Section 5.2.1, the adequate surface modeling and simulation parameters are discussed. Using the modeling and parameters, scattering prediction by the IEM is obtained with reasonable accuracy and the region of validity is obtained by comparing it to the scattering prediction of the three approximate methods in the followed sections.
5.2.1 Surface Modeling and Simulation Parameters

Due to the limitation of the computer memory and calculation speed, the surface length and the sampling spacing must be finite when a surface profile is numerically generated for the integral method. However, it leads to unavoidable finite frequency band limit and this limit results in inaccurate predictions because a fractal surface has the surface power spectrum following an unbounded inverse power law. This is a serious problem because the IEM cannot act as a reference to examine the validity of an approximate method anymore. However, since this problem comes from the finite size of the sampled surface length and sampling spacing, one can guess that larger surface length and smaller sampling spacing gives a closer prediction to the exact answer for the integral method.

Based on this idea, the relation between scattered intensity and the frequency limits is first investigated and, from the relation, a reasonable high frequency limit for the IEM prediction is obtained. Also, by introducing abc-function as the PSD function, the problem caused by the low frequency limit is reasonably avoided. Thus, even though our IEM result is not an exact prediction for a fractal surface, it should be much more accurate than other approximate methods.

In Section 5.2.1.1, two surface generating models are discussed to reveal the characteristics of surface scatter from fractal surfaces. Based on those characteristics, proper sampling parameters for the IEM calculation are obtained in Section 5.2.1.2.
5.2.1.1 Surface Modeling

One of the famous functions for generating fractal surface profiles is the Weierstrass-Mandelbrot (WM) function [31,34]. It generates a surface profile for which the PSD function exactly follows the inverse power law for a given band-limited frequency region in numerical simulation. However, due to its frequency band, the edge effect must be considered when approximate methods are used to calculate the scattering distribution from the surface. On the other hand, the abc-function is frequently used as a surface PSD function for evaluating surface scatter from the surfaces with fractal structure in approximate methods [62,97,102] but it is rarely used in the IEM simulation. Since, unlike the WM function, it does not have frequency band limitation, the approximate methods can easily calculate the scattering intensity from the surface characterized by the function. However, the abc-function does not produce exact fractal structure. Throughout this subsection, the characteristics of the surface scatter from the surface generated by the two surface generating models are investigated by using the IEM calculation. And from those characteristics, a proper surface generating method is selected.

First, as a surface generating model, the WM function is considered. It characterizes surfaces by the Hurst exponent \( H \), low frequency limit, high frequency limit and the band-limited rms roughness \( h_{\text{rms}} \). The one-dimensional surface profile is generated by

\[
\zeta(x) = Ch_{\text{rms}} \sum_{n=0}^{N-1} B^{-nH} \sin(k_n B^n x + \phi_n),
\]

(5.7)
where $N_t$ is the number of tones, $k_0$ is the fundamental frequency, $\phi_n$ is uniformly distributed random phase of the $n$-th harmonic, and $C, B$ are some functions of the other parameters (See [31] for details). Note that the Hurst exponent $H$ is related to the slope parameter $c$ in Eq. (5.6) by $c = 1 + 2H$, and the high frequency and the low frequency limit becomes $1/2\Delta x$ and $1/L$ respectively if the Nyquist sampling is applied.

As other authors mentioned [28, 31], there is an issue that the generated surface whose frequency is truncated at a certain frequency is not an exact fractal because an ideal fractal surface does not exhibit such a high frequency limit. Moreover, the profile function of the ideal fractal surface is not differentiable due to its unbounded spatial frequency. However, this truncation is justified by the fact that an ideal fractal surface model is an approximation to the real surface profile whose fractal structure must be broken, at least, in the atomic scale. Thus, it is reasonable to set a high frequency limit where the fractal structure is broken and this frequency truncation makes the surface smoother and differentiable.

However, it is not clear that at which frequency the fractal structure is broken. Moreover, even though it is known, for example if its scale is the inverse of the atomic scale, numerical calculation of the intensity prediction using the IEM is almost impossible because the sampling spacing must be the atomic scale which requires a very large number of sampling points which is not acceptable by the computer memory size.

Thus, a less accurate but more numerically efficient method is taken to determine the sampling spacing as being motivated by the fact that the surface structure much smaller than the wavelength do not significantly contribute to the scattering mechanism [28,31]. When comparing
scattering intensity distributions as increasing high frequency limit (or the frequency to be truncated as upper limit) of sampled surface, if the intensity distributions remain almost the same when high frequency limit is further increased, the lowest high frequency limit which produces the same intensity profile can be selected as the highest frequency (or the frequency to be truncated as upper limit) for our simulation. Since increasing high frequency limit corresponds to narrowing the sampling spacing, determining high frequency limit is the process of selecting sampling spacing.

Figure 5-10 illustrates this method for the surface of which Hurst exponent is 0.5 and surface length is $20\lambda$.

Figure 5-10: Five incoherent intensity distributions of which sampling spacing are $\Delta x=\lambda/4$, $\lambda/8$, $\lambda/16$, $\lambda/32$ and $\lambda/64$ respectively. The surface length is $20\lambda$ and the $H=0.5$. The bandlimited rms roughness for the surface with $\Delta x=\lambda/64$ is $0.5\lambda$. The number of realizations is 1000.
The five intensity distributions predicted by the IEM from the five different surfaces of which sampling spacing is \( \Delta x = \lambda/4, \lambda/8, \lambda/16, \lambda/32 \) and \( \lambda/64 \) respectively, and all the other parameters are the same. In Figure 5-10, the intensity profile looks almost unchanged when the sampling spacing is smaller than \( \lambda/8 \). Thus, with the assumption that the intensity distribution will not be significantly changed when the sampling spacing becomes even smaller than \( \lambda/64 \), the sampling spacing \( \Delta x = \lambda/8 \) can be used for numerically generating surface profile.

In comparison to those who take great care to set the highest frequency, there are only a few researchers who have set the lowest frequency [28]. Since it is intuitively determined by the inverse of the surface length, the influence of the finite surface length to the scattering intensity has been rarely investigated. Regarding the relation between the scattering light and the low frequency of the surface, according to the SPM, lower spatial frequencies contribute to the scatter behavior nearer to the specular direction in the smooth surface regime. However, this simple rule may not be valid when the surface roughness is moderately or quite rough.

Figure 5-11(a) shows an interesting scattering behavior which is opposite to this common expectation. Fixing all the other parameters, the five scattering intensity curves are obtained from the surface length of \( L = 10\lambda, 20\lambda, 40\lambda, 80\lambda \) and \( 160\lambda \), and their periodograms are shown in Figure 5-11(b). From their surface length, the lowest frequencies are given by \( 1/10\lambda, 1/20\lambda, 1/40\lambda, 1/80\lambda \) and \( 1/160\lambda \) respectively. First, let us consider the two cases of \( L = 160\lambda \) and \( L = 10\lambda \). For the case of \( L = 160\lambda \), the bandlimited rms roughness is 0.35\( \lambda \) and the total integrated scatter value is 0.998, which means the surface is quite rough. For the case of \( L = 10\lambda \), the bandlimited rms roughness is 0.08\( \lambda \) and the total integrated scatter value is 0.639 resulting in moderately rough surface.
Figure 5-11: (a) Five incoherent intensity distributions of which surface length are \( L=10\lambda, 20\lambda, 40\lambda, 80\lambda \) and \( 160\lambda \) respectively and (b) their averaged periodogram. The sampling spacing is \( \Delta x = \lambda/8 \) and \( H=0.5 \) for all the cases. The bandlimited rms roughness for the surface with \( L=160\lambda \) is \( 0.35\lambda \). The number of realizations is 1000.

The difference in the low frequency limit of the two surfaces induces different scattering intensity behavior in the vicinity of the specular direction and ±90 degrees of scattering angle. However, their scattering behavior is almost identical between the scattering angles of 20 to 80 degrees. Again, let us consider another scattering intensity by the surface with \( L=20\lambda \) in the same figure. Its scattering curve between angles of 20 to 80 degrees is almost the same as the previous two cases. Also, the curve between angles of 10 to 20 degrees is similar to those by the surface with \( L=160\lambda \). From this observation, it is possible to obtain a meaningful scattering intensity profile with a relatively small number of sampling points if we accept some error near the specular direction and 90 degrees of scattering angle.
Figure 5-12(a) shows the detailed scattering intensity distribution near the specular direction and all the parameters are the same to those cases of Figure 5-11(a). From the figure, the scattering intensity distribution seems to have no singularity in specular direction and it appears that the shape of the scattering distribution will not be changed as the surface length is increased beyond $40\lambda$.

![Scattering Intensity vs Scattering Angle](image)

Figure 5-12: Angular behavior between -20 to 20 degrees of scattering angles of five incoherent intensity distributions for the surfaces whose length are $L=10\lambda$, $20\lambda$, $40\lambda$, $80\lambda$ and $160\lambda$ respectively. The bandlimited rms surface roughness for the case of $L=160\lambda$ is (a) $0.35\lambda$ (b) $0.07\lambda$. The sampling spacing is $\Delta x=\lambda/8$ and the $H=0.5$ for all the cases. The number of realizations is 1000.

But this is not always true. In Figure 5-12(b), the scattering intensity distributions are shown for the surface of which the Hurst exponent is still 0.5 but the rms roughness is smaller than the previous one. The bandlimited rms roughness of the surface with $L=160\lambda$ and $L=10\lambda$ are $0.07\lambda$.
and $0.01\lambda$ respectively and the total integrated scatter values for those two cases are 0.549 and 0.04 respectively. From the Figure 5-12(b), the scattering intensity distribution looks that it has a singularity at the specular direction and larger surface length produces stronger peak at the singular point. Thus, even though the behaviors of the scattering intensity distributions are similar at the most of the scattering angles, the insufficiently large surface length misleads the scattering intensity profile largely near the specular direction.

In principle, WM function can be characterized by the four parameters which are highest frequency, lowest frequency, Hurst exponent and slope parameter. However, recalling that WM function is used to generate fractal surfaces profile which does not has highest frequency limit, the highest frequency is not actual parameter but it is something to be carefully obtained. Regarding the lowest frequency limit, different lowest frequency truncation produces different scattering distribution. Nevertheless, from the Figure 5-11 and 5-12, the difference looks limited to large and near specular angle behavior of scattering distribution in logarithmic scale. Thus, if the priority must be set among the parameters, the Hurst exponent and slopes parameter are the most important parameters characterizing fractal surfaces when WM model is used for generating surface profile.

Now, let us move to another surface generating method referred to as spectral method. The spectral method also can be used to generate fractal surface profile [103,104]. One advantage of the method is that it can be used to generate surface profile with any given power spectrum function. Here, we employ an abc-function as our surface PSD function given by
\[ W(\alpha) = \frac{a}{\left(1 + b^2 \alpha^2\right)^{1/2}}, \] (5.8)

where \(a, b\) is some constant given by the surface statistical properties. Due to the shape of the function, \(f_0 = 1/b\lambda\) is called as shoulder frequency. Since the function converges to a certain value when the frequency approaches to zeros, the total rms roughness \(\sigma_{tot}\) is given by

\[ \sigma_{tot} = \sqrt{2} \cdot \left[ \frac{a\sqrt{\pi}}{b(c-1)} \frac{\Gamma((c+1)/2)}{\Gamma(c/2)} \right]^{1/2}, \] (5.9)

when the slope parameter \(c\) is larger than unity, and \(\Gamma\) is the gamma function. Its one-dimensional Fourier transform gives analytic auto-covariance function given by

\[ ACV(x) = 2 \cdot \sqrt{2\pi} \cdot \frac{a}{b} \cdot \frac{2^{-c/2}}{\Gamma(c/2)} \left( \frac{2\pi|x|}{b} \right)^{(c-1)/2} K_{(c-1)/2} \left( \frac{2\pi|x|}{b} \right), \] (5.10)

where \(K\) is the modified Bessel function of the second kind. Since the function behavior follows the inverse power law where the frequency is larger than the shoulder frequency, the abc-function has been frequently used to modeling surface PSD of fractal surfaces [62,97,102].

Unlike the band-limited WM model, abc-function has neither lowest nor highest frequency limit, which could be a problem when the surface profile is numerically generated for integral method because there must be unavoidable lowest and highest frequency limits in numerical approach. Regarding the high frequency limit, an artificial high frequency limit can be adopted as discussed in previous section. Meanwhile, the issue that there is no lowest frequency limit requires little more attention because it means that the abc-function model is characterizing infinitely large size
surfaces. As recalling the IEM simulation for the surfaces whose PSD function is Gaussian, the infinite size of the surface length is not a serious problem if the surface length is large enough compared to the wavelength and/or correlation length. In other word, if the PSD function converges to a certain constant value when frequency approaches to zero, the scattering intensity profile is not significantly changed as increasing surface length beyond a certain sufficiently large surface length. With this observation, it seems that it is possible to set proper lowest frequency to be truncated when surface profiles are numerically generated.

Still there is a questionable issue. Unlike the surface generated by using WM function, the surface generated by using abc-function does not have real fractal structure due to the low frequency behavior of the abc-function because the fractal-like structure is broken where the surface frequencies are lower than the shoulder frequency. However, the scattering distributions by the surface generated by WM function and abc-function are closely related.

To illustrate this point, let us consider the two surface profiles whose PSD function is shown in Figure 5-13(a). The surface A is generated by the abc-function and the surface B is generated using WM function. Both of them have the same slope in logarithmically spaced frequency domain, specifically the $c=2$ for the surface A and the Hurst exponent $H=0.5$ for the surface B. The total rms roughness of the surface A is $\sigma_{\text{tot}}=0.30\lambda$ and the band-limited rms roughness $h_{\text{rms}}=0.30\lambda$ for the surface B. Also, the maximum frequency is chosen by $f_{\text{max}}=4/\lambda$ for both surfaces. For the surface A, the surface length is chosen to be $L_A=160\lambda$ in order to make the lowest frequency is small enough compared to its shoulder frequency $f_0=1/20\lambda$. For surface B, its surface length is selected to be $L_B=32\lambda$ in order to make its band-limited rms roughness to be
almost the same as the total rms roughness of the surface A. The PSD function and the surface lengths for the two surfaces are different, but the scattering intensity profiles (asterisk for surface A and solid line for surface B) looks very similar to each other as shown in Figure 5-13(b) over the most of the scattering angles.

Figure 5-13: (a) PSD function for surface A and B. (b) The scattering intensity for surface A (asterisk) and B (solid line). The number of realization is 1000.

But, as shown in Figure 5-12(b), the scattering distribution from fractal surface sometimes has singularity at the specular direction. Figure 5-14 shows the intensity distributions of such a case by using the two surface generating models having the similar relation to the case of Figure 5-13. The surface A is generated by the abc-function whose parameters are $a=0.016$, $b=20$ and $c=2$. The surface B is generated by the WM function whose Hurst exponent $H=0.5$, bandlimited rms roughness is $h_{rms}=0.050\lambda$ and the lowest frequency is $f_{min}=1/32\lambda$. In Figure 5-13, there is huge
discrepancy in the vicinity of the specular direction between the two scattering distributions. However, at the most of the scattering angles in logarithmic scale, the two intensity curves share many characteristics like curvature and slope which are quite different from the intensity curve shown in Figure 5-13(b).

Figure 5-14: The scattering intensity for surface A (dotted line) and B (solid line). The total rms roughness of surface A is $\sigma_{\text{tot}} = 0.05\lambda$ and the bandlimited rms roughness of surface B is $h_{\text{rms}} = 0.05\lambda$. The slope parameter is $c = 2$ for surface A and Hurst exponent is $H = 0.5$ for surface B. The number of realizations is 1000.

This similarity comes from the behavior of the abc-function which follows inverse power law at the frequency larger than its shoulder frequency. The difference near the specular direction comes from its low frequency behavior. Even though abc-function does not generate exact fractal surface profile, the scattering intensity is very similar to those from the fractal surface. Moreover, with proper sampling parameters, the abc-function produces quite close results to that by WM
function. Regarding scattering prediction using approximate methods, if abc-function is employed as the surface PSD function, the scattering calculation becomes much easier because the ACV function is analytically given and the edge effect can be ignored. On the other hand, if we adapt the WM function as our surface generating model for numerical integral method, for a fair comparison, we are forced to use expressions of approximate methods for deterministic surfaces to obtain scattering intensity and repeat calculation and take ensemble average of them. Since, the analytic formulation of ensemble average is an important part for the KA and GHS methods, using the WM function has a disadvantage for comparison purposes.

It is thus reasonable to employ the abc-function as our PSD function for comparing the IEM results with other approximate methods. And the validity obtained by using the abc-function could be directly connected to the validity for the real fractal surfaces.

5.2.1.2 Sampling Parameters for Numerical Simulation

Throughout the previous section, the advantages of employing an abc-function as the surface PSD model were discussed. Since our final goal is estimating the validity of other approximate methods by comparing their prediction to the IEM prediction, it is important to obtain IEM predictions accurately. Since the accuracy of the IEM results relies on the proper selection of the sampled surface length and the sampling spacing, choosing those sampling parameters properly for the abc-function is required. In this Section, the proper sampled surface length and the sampling spacing are obtained for different values of the total rms roughness and the $c$ parameter of the abc-function.
Generally, the surface generated by the abc-function can be parameterized by the three parameters of $a$, $b$ and $c$. However, here we fixed the parameter $b=20\lambda$ for our simulation because, from the previous section, it can be deduced that the variation of $b$ value does not change the scattering intensity distribution significantly except near specular direction and near $\pm90$ degrees. The parameter $a$, determines the total rms roughness for the abc-function with fixed $b$ and $c$ parameters. Since the total rms roughness is a more familiar parameter for analyzing scattering phenomena than the parameter $a$, the total rms roughness $\sigma_{tot}$ is used instead of $a$. Thus, in our analysis, the parameters characterizing random rough surfaces are reduced to $\sigma_{tot}$ and the slope $c$ value with the fixed value of $b=20\lambda$.

Typically, using smaller sampling spacing and larger surface length produces a more accurate prediction, but it takes a longer calculation time and requires more memory size. Thus a trade-off between accuracy and the calculation load is made. In other words, finding the proper sampling parameters is the process of finding a minimum surface length and a maximum sampling spacing with which the scattering intensity distribution is almost same as that from the surface generated by using much larger surface length and much narrower sampling spacing. Thus, to select proper sampling parameters, the scattering intensity comparison is needed. In order to make it easy to compare two scattering intensity distributions, let us introduce an error value by

$$
\varepsilon_{12} \equiv \frac{1}{m} \sum_{m} \left| \log I_1(\theta_m) - \log I_2(\theta_m) \right|.
$$

Since we are interested in the scattered intensity shape in a logarithmic scale, we take the log of the scattering intensity profile. The error value can be interpreted by the averaged scattering
intensity difference in logarithmic scale over all sampled scattering angles, thus a smaller error value means the two intensity distributions are similar to each other.

First, let us consider the sampled surface length. Figure 5-15(a) shows the scattering intensity profiles generated by surfaces with a total rms roughness of $\sigma_{\text{tot}}=0.1\lambda$ and a slope parameter of $c=1.2$, with the surface length varying from $L=10\lambda$ to $200\lambda$ with a step size of $10\lambda$.

![Scattering profiles](image)

**Figure 5-15:** (a) The scattering intensity distributions with the various surface lengths from $L=10\lambda$ to $200\lambda$. (b) The error curves. All the calculation is done for the case of $\sigma_{\text{tot}}=0.1\lambda$, $c=1.2$, $\Delta x=\lambda/2$.

The scattering profiles are changing as increasing the surface length, but the difference becomes smaller when the surface length is large enough. This tendency is verified by Figure 5-15(b) where the calculated error values from the scattering distributions are shown. The error values are calculated between the two scattering intensity distributions from a surface and the other
surface whose length is smaller by $10\lambda$. The difference between the two surfaces becomes smaller when the surface length becomes larger. Also, when the surface lengths are larger than $L=40\lambda-60\lambda$, it looks that the error curve is saturated. This is expected because the abc-function converges at the frequency smaller than its shoulder frequency $f_0=1/20\lambda$.

Regarding the two error curves in Figure 5-15(b), the tendency of the two curves is the same but the solid red curve converges to a value around $\varepsilon=0.04$ and the dotted blue is saturated near a value around $\varepsilon=0.06$. This difference can be explained by the limited number of realizations. Since the IEM uses Monte-Carlo technique when generating surface, there must be some fluctuation in intensity distributions even though the distributions are averaged over 1000 realizations. It is expected that a smaller number of realizations induces a higher level of residual error, which is the reason that the dotted blue error curve (400 realizations) converges to the higher value than the value the solid red error curve (1000 realizations) does to. Thus, in our simulation, the difference in the scattering intensity distributions smaller than a certain level of error value cannot be distinguished due to the residual error cause by the limited number of realizations.

Figure 5-16 shows four error curves which are obtained in the similar way to the Figure 5-16(b) but the rms roughness and the slope parameter are different. The surface A has $\sigma_{\text{tot}}=0.1\lambda$ and $c=1.2$, the surface B has $\sigma_{\text{tot}}=0.1\lambda$ and $c=2.8$, the surface C has $\sigma_{\text{tot}}=0.9\lambda$ and $c=1.2$, and the surface D has $\sigma_{\text{tot}}=0.9\lambda$ and $c=2.8$. In the figure, regardless of rms roughness and the slope parameter, the four curves are saturated around $L=40\lambda$ to $60\lambda$. Thus, it can be deduced that the proper surface lengths are not sensitive to those sampling parameters and they can be chosen as a
value between $L=40\lambda$ to $60\lambda$ at least the surface whose rms roughness is between $0.1\lambda < \sigma_{tot} < 0.9\lambda$ and whose parameter $c$ is between $1.2 < c < 2.8$ with the fixed parameter $b=20\lambda$. Note that $\Delta x = \lambda/8$ of sampling spacing is used for surface A, B and D, but $\Delta x = \lambda/31$ is used for surface C in order to satisfy the condition of energy conservation. Since the sampling spacing is so narrow that many sampling points are required for the surface C, only 6 cases of the surface length are calculated and shown in the figure.

![Error curves for the four cases of surface parameter sets.](image)

**Figure 5-16:** The error curves for the four cases of surface parameter sets. For all the cases, 1000 realizations are used.

When it comes to selecting sampling spacing, the methodology is the same as selecting sampled surface length. But, in addition to that, the energy conservation must be considered. Figure 5-17(a) shows the scattering intensity distribution generated by the surfaces whose rms roughness is $\sigma_{tot}=0.9\lambda$ and the slope parameter is $c=1.2$ with the surface length of $L=40\lambda$, but the
sampling spacing are different from $\Delta x=\lambda/2$ to $\Delta x=\lambda/36$. The scattering intensity distributions are changing as the sampling spacing is decreased, but the difference becomes smaller when the sampling spacing is small enough. This tendency can be verified in the Figure 5-17(b) where the calculated error values from the scattering distributions in Figure 5-17(a) are shown. The error values are calculated between the two scattering intensity distributions from a surface and the other surface whose number of sampling per wavelength ($\lambda/\Delta x$) is smaller by 2.

Figure 5-17: (a) Scattering intensity distribution with the various number of sampling points per wavelength from $\Delta x=\lambda/2$ to $\Delta x=\lambda/36$. (b) Error curve (black solid) and total reflected energy (red solid). All the calculation is done for the case of $\sigma_{tot}=0.1\lambda$, $c=1.2$, $L=40\lambda$, and 1000 realizations are used.

The y-axis on the right-side of the Figure 5-17(b) represents the total energy of the reflected light. Considering both energy conservation and the error value, there is a reasonable minimum value of $\lambda/\Delta x$ which produces relatively small error and satisfies the condition of energy conservation.
Figure 5-17(a) shows the main restriction is the condition of energy conservation rather than the condition of low error value.

Figure 5-18(a) shows the error curves for the four different surface parameter sets of the rms roughness and the slope parameter. Surface A has the parameter set of $\sigma_{tot}=0.1\lambda$ and $c=1.2$, surface B has $\sigma_{tot}=0.6\lambda$ and $c=2$, surface C has $\sigma_{tot}=0.9\lambda$ and $c=1.2$, and surface D has $\sigma_{tot}=0.9\lambda$ and $c=2.8$.

Contrasting to the case of the surface length, the sampling spacing must be different for different surface parameter sets because the surfaces having larger r.m.s roughness or smaller slope
parameter require more dense sampling spacing to satisfy the condition of energy conservation as shown in Figure 5-18(b).

In Figure 5-19, we choose 9 cases in the domain of $0.1\lambda < \sigma_{tot} < 0.9\lambda$ and 9 cases in the domain of $1.2 < c < 2.8$, and consider all the combinations of the surface parameter sets. The figure shows the selected number of the sampling points per wavelength for all the 81 cases of surface parameter sets.

![Figure 5-19: Selected minimum number of the sampling points per wavelength for the 9x9 cases of the surface parameter sets (c, \sigma_{tot}).](image)

The value of $\lambda/\Delta x$ is chosen by trading off between the error value, the condition of energy conservation and the calculation load. The rule of thumb is that the error value must go down to a value around $\varepsilon=0.04$ when 1000 realizations are considered, and that the value of the total reflected energy must be at least 0.98, and that the total number of sampling must be smaller than
4000. Each number is obtained by comparing the scattering intensity distribution as the number of the sampling points per wavelength is increased. From Figure 5-19, it is deduced that a larger value of $\lambda/\Delta x$ is required for smaller slope parameter and larger value of rms roughness.

In Figure 5-20, we choose 36 cases in the domain $0.1 < \sigma_{\text{tot}} < 0.9$ and another 36 cases in the domain $1.2 < c < 2.8$ and all the combinations of them are considered. Figure 5-20(a) and 5-20(b) show the total reflected energy for the 36x36 cases using the IEM. In order to calculate scattering intensity for a given surface parameter set, instead of calculating $\lambda/\Delta x$ for every 36x36 cases, we pick up the value from the nearest parameter set in Figure 5-19 in the surface parameter space.

Figure 5-20: Total reflected energy using the IEM for 36x36 cases of the surface parameter sets $(c, \sigma_{\text{tot}})$ for (a) $\theta_i=0^\circ$ and (b) $\theta_i=60^\circ$. The surface length is chosen by $L=40\lambda - 60\lambda$, and $\lambda/\Delta x$ value is picked up from the Figure 5-19.
Figure 5-20(a) is the case of normal incident angle and Figure 5-20(b) is the case of $\theta_i=60^\circ$. The total reflected energy is almost unity for most of the cases, but it drops when the slope parameter is small and rms roughness is large. Even though the $\lambda/\Delta x$ is obtained by comparing intensity distribution of the normal incident case, the energy conservation condition is satisfied for the non-normal incident angle case using the same surface parameters as shown in Figure 5-20(b). Also, recalling that the condition of energy conservation is a stronger constraint than the condition for the error value as shown in Figure 5-17, the $\lambda/\Delta x$ in Figure 5-20 can be used for calculating non-normal angle of incidence cases.

By comparing intensity distributions while changing the two sampling parameters, sampled surface length and sampling spacing, the sampling parameters are carefully selected for each surface parameter set, rms roughness and the slope parameter. The surface length is not sensitive, if it is carefully chosen, to the surface parameters, but the proper sampling spacing is dramatically changed when the surface parameters are different. Those sampling parameters are used to calculate the IEM result in the next section.

5.2.2 Scattering Intensity Distribution

In this Section, scattering predictions by the SPM, KA and GHS are compared to the IEM predictions for the TE polarization case. To reveal the tendency of the validity of those approximate methods according to rms roughness and slope parameter, the comparison is done for $36\times36$ cases of parameter sets with the fixed $b=20\lambda$. In calculating the scattering intensity
using the IEM, we use the Gaussian beam whose spectrum is a Gaussian function as our incident wave as discussed in Section 3.2.1 and the width of the Gaussian beam is chosen to be one-sixth of surface length as did other researchers [34]. Also, the induced field by the incident wave at the surface is calculated in the Fourier domain numerically and the scattering intensity is obtained by averaging 1000 realizations. Similar to the previous sections, an error value is introduced by

\[ \varepsilon \equiv \frac{1}{m} \sum_{m} \left| \log I(\theta_m) - \log I_{\text{IEM}}(\theta_m) \right|, \]

which is the mean difference between the log of predicted scattering intensity distributions by an approximate method and the IEM. Since the log value of the intensity is taken, the error value would be useful for comparing intensity distributions when it is plotted logarithmically and the error value is referred to as log error value. Similarly, another error value is introduced as

\[ \varepsilon^* \equiv \frac{\sum_{m} \left| I(\theta_m) \Delta \theta_m - I_{\text{IEM}}(\theta_m) \Delta \theta_m \right|}{\sum_{m} I_{\text{IEM}}(\theta_m) \Delta \theta_m}, \]

which is the mean difference between predicted intensity distributions by an approximate method and the IEM. This error value would be useful for comparing intensity distributions when it is plotted linearly, and the error value is referred to as linear error value.

Figure 5-21 shows scattering intensity distributions predicted by the SPM (blue dashed line), KA (green dotted line), GHS (red solid line) and IEM (black asterisk) for the case of (a) \( \sigma_{\text{tot}}=0.025\lambda, c=1.4, \theta_i=40^\circ \) and (b) \( \sigma_{\text{tot}}=0.6\lambda, c=2.8, \theta_i=0^\circ \). In Figure 5-21(a), since the rms roughness is small, both the SPM and IEM results agree to each other as we expected. The log error value of the
SPM is $\varepsilon=0.06$ which is quite smaller than those of the KA and GHS. In Figure 5-21(b), the roughness is quite rough so that the roughness is out of valid domain of the SPM, but the KA prediction shows good behavior near the specular direction. However, the KA prediction fails at the large scattering angles leading its log error value $\varepsilon=0.46$. In both cases, the GHS prediction show quite good agreement not only near specular direction and but also over the whole scattering angles.

Figure 5-21: Scattering intensity distributions by the IEM, SPM, KA and GHS for the case of (a) $\sigma_{tot}=0.025\lambda$, $c=1.4$, $\theta_i=40^\circ$ (b) $\sigma_{tot}=0.6\lambda$, $c=2.8$, $\theta_i=0^\circ$.

Figure 5-22 shows the predicted scattering intensity for the three different surface parameter sets. In Figure 5-22(a), the scattering distributions for the case of $\sigma_{tot}=0.1\lambda$, $c=1.4$, $\theta_i=40^\circ$ are plotted in logarithmic scale and the SPM prediction shows good agreement with the IEM prediction except near specular direction leading to an error of $\varepsilon=0.16$. It is quite interesting that the SPM
prediction agrees well at the large scattering angles even the rms roughness is not quite small. Regarding the GHS, its prediction is superior near the specular direction than the one by the SPM but it predicts a little stronger scattering at large scattering angles leading to $\varepsilon=0.20$.

Figure 5-22: Scattering intensity distributions by the IEM, SPM, KA and GHS for the case of (a) $\sigma_{\text{tot}}=0.1\lambda$, $c=1.4$, $\theta_i=40^\circ$, (b) the same to (a) but in linear scale, (c) $\sigma_{\text{tot}}=0.3\lambda$, $c=2$, $\theta_i=0^\circ$, and (d) $\sigma_{\text{tot}}=0.3\lambda$, $c=2.8$, $\theta_i=60^\circ$. 

130
If the distribution is plotted in linear scale as shown in the Figure 5-22(b), it is obvious that the SPM overestimates scattering near specular direction, but the GHS and KA predict the scattering behavior quite accurately. Figure 5-22(c) shows the scattering predictions for the case of $\sigma_{tot}=0.3\lambda$, $c=2$, $\theta_i=0^\circ$, which is a moderately rough surface. The SPM does not predict the scattering properly and the KA fails to predict accurately both near the specular direction and at large scattering angles. In Figure 5-22(d), the surface roughness is the same as in Figure 5-22(c) but the slope of the PSD is larger and the incident angle is 60 degrees. Again, the SPM overestimates scattering near the specular direction, and the KA prediction deviates at the large scattering angles. However, in both cases shown in Figure 5-22(c) and (d), predictions from the GHS theory are quite accurate over the entire scattering angle domain.

Figure 5-23 shows the predicted scattered intensity by the SPM, KA, GHS and the rigorous IEM theories for three different incident angles and two surface parameter sets. In Figure 5-23(a), the scattering distributions for incident angles if $0^\circ$, $30^\circ$ and $60^\circ$ and surface parameters of $\sigma_{tot}=0.15\lambda$, $c=2$ are plotted. The SPM predicts overly strong scattering near the specular direction and the amount of incorrectness is reduced as angle of incidence becomes larger. The KA predictions show good agreement to the IEM result near the specular direction for all the incident angles but their large angle behavior is inaccurate in all cases. Predictions from the GHS theory exhibit good accuracy over all scattering angles and for all incident angles illustrated in the figure. Figure 5-23(b) illustrates the scattering predictions from a surface whose roughness and the slope parameter are larger than the case of Figure 5-23(a).
Figure 5-23: Scattering intensity distributions by the IEM, SPM, KA and GHS for the case of (a) $\sigma_{_{\text{tot}=0.15\lambda, c=2, \theta_i=0^\circ, 30^\circ, 60^\circ}}$ and (b) $\sigma_{_{\text{tot}=0.3\lambda, c=2.4, \theta_i=0^\circ, 30^\circ, 60^\circ}}$.

The overall shape of the scattering distribution is less sharply peaked in the specular direction, but, at the case of the $60^\circ$ incident angle, the shape becomes similar to that in Figure 5-23(a). Again the KA predictions are valid only near the specular angle, but the GHS theory remains quite accurate not only in the specular direction but also at large scattering angles. Note that in Figure 5-23(b), the SPM results are not plotted because the rms roughness is obviously out of its valid domain.

5.2.3 Error Map for Region of Validity

To reveal the surface parameter dependency of the SPM validity, both log- and linear-scale error values for the SPM are calculated for the 36×36 surface parameter cases and they are illustrated.
in Figure 5-24(a) and Figure 5-24(b) respectively. In logarithmic scale, as shown in Figure 5-24(a), the scattering intensity distributions at large angles predicted by the SPM agree very well to the IEM predictions not only for the surface whose rms roughness is very small, but also rms roughness is moderately large. However, the valid region of the SPM shrinks largely in Figure 5-24(b) where the linear-scale error values are potted because the SPM overestimates scattering at the near specular direction. Thus the region of validity of the SPM looks similar to one obtained for surfaces having Gaussian PSD function and it is largely restricted by the smooth surface approximation.

![Figure 5-24](image)

(a) (b)

**Figure 5-24:** Map of error values for SPM (a) in logarithmic scale and (b) in linear scale for normal incident angle.

Figure 5-25 shows error maps for the KA method where the error map is calculated using (a) logarithmically plotted intensity and (b) linearly plotted intensity. The KA predictions show relatively good agreement near specular direction for the cases of large slope parameter and...
small to large rms roughness, which is leading the smooth variation of the error value over the entire surface parameter space in Figure 5-25(b). However, in Figure 5.25(a) where the error values are calculated using logarithmically plotted intensity, the KA predictions almost always fail over the space. This result comes from the disagreement at large angle behavior of the KA scattering predictions as shown in Figures 5-21, 5-22 and 5-23.

Figure 5-25: Map of error values for the KA (a) in logarithmic scale (b) in linear scale for normal incident angle

The log- and linear-scale error values for the GHS are calculated and illustrated in Figure 5-26. Comparing Figure 5-26 and Figures 5-24 and 5-25, the GHS has much smaller error value than the SPM and KA over the large area of the surface parameter space for both log- and linear-scale. Moreover, except the cases of large rms roughness and small slope parameter, the variation of error values are quite small.
In order to compare the incident angle dependency of the error values for the SPM, KA and GHS, the log-scale error map for the three methods in the case of $\theta_i=20^\circ$, $40^\circ$ and $60^\circ$ are shown in Figure 5-27. The region of validity of the SPM looks that it does not affected by the change of the incident angle. Meanwhile, for both the GHS and KA, the valid area in the domain reduces as the incident angle is increased. But, in the three incident angle cases, the error maps show that the GHS has wider range of validity for large incident angles than the KA method.
Figure 5-27: Map of error values in logarithmic scale for the SPM in the case of (a) $\theta_i=20^\circ$ (d) $\theta_i=40^\circ$ (g) $\theta_i=60^\circ$, for the KA in the case of (b) $\theta_i=20^\circ$ (e) $\theta_i=40^\circ$ (h) $\theta_i=60^\circ$ and for the GHS in the case of (c) $\theta_i=20^\circ$ (f) $\theta_i=40^\circ$ (i) $\theta_i=60^\circ$. 
5.2.4 Total Integrated Scatter

Since most optical surfaces have fractal-like structure, estimating TIS value for those surfaces is important. Figure 5-28 illustrates the accuracy of TIS values obtained by integrating the scattering function predicted by the SPM, KA and GHS surface scatter theories at normal incidence. The predicted TIS by the SPM shows good agreement in smooth surface regime regardless of c parameter values but the region of validity is largely confined to the regime as shown in Figure 5-28(a). The TIS values predicted by the KA are quite accurate when the ratio of rms roughness to c parameter is small, in the Figure 5-28(b), but the predictions are not reliable when the ratio is increased. Figure 5-28(c) shows that the GHS predictions are very accurate over almost entire surface parameter space.

![Figure 5-28: Map of error values in logarithmic scale at normal incidence for (a) the SPM, (b) the KA and (c) the GHS.](image-url)
Figure 5-29: Map of error values in logarithmic scale for the SPM in the case of (a) $\theta_i=20^\circ$, (d) $\theta_i=40^\circ$, (g) $\theta_i=60^\circ$, the KA in the case of (b) $\theta_i=20^\circ$, (e) $\theta_i=40^\circ$, (h) $\theta_i=60^\circ$, and the GHS in the case of (c) $\theta_i=20^\circ$, (f) $\theta_i=40^\circ$, (i) $\theta_i=60^\circ$. 
Figure 5-29 shows the region of validity of the TIS prediction by the three approximate methods at 20, 40, 60 degrees of incident angle. For these all incident angles, SPM is largely restricted to smooth surface limitation. One interesting characteristic in SPM’s TIS prediction is that its valid domain is increased when angle of incidence is increased and it is explained that the surface looks smoother for the incident light when the incident angle is larger. The valid domain of KA’s TIS predictions is quite similar until the incident angle is increased up to 40 degrees. However, if the incident angle becomes larger, quite complicated error map is obtained because, for this large incident angle case, its geometrical factor plays a critical role. Meanwhile, the GHS’s TIS predictions show good agreement almost entire surface parameter space for up to 60 degrees of incident angle.
5.2.5 Scattering Intensity for Different Shoulder Frequency

In previous sections, the value of the parameter $b$ is fixed as 20. However, from the observations in Section 5.2.1, it is expected that the scattering behavior from the surface having a larger $b$ value which corresponds to smaller shoulder frequency would be almost the same except near the specular direction.

Figure 5-30: (a) PSD functions for surface A, B and C with the same parameter $c=2$. (b) The scattering intensity distributions predicted by the IEM for the three surfaces in log-log scale for normal incidence.

Figure 5-30(a) shows PSD function of the three surfaces. The surface A has $0.02\lambda$ of rms roughness with $b=20\lambda$, the surface B has $0.045\lambda$ of rms roughness with $b=100\lambda$, and the surface C has $0.14\lambda$ of rms roughness with $b=500\lambda$. All the three surfaces have the same slope parameter value of 2. Figure 5-30(b) shows the IEM scattering predictions in log-log scale for the normal
incident angle case and it shows that a different shoulder frequency with the same slope parameter changes the scattering behavior only in the near specular directions.

Figure 5-31: Predicted intensity distributions by the SPM, KA, GHS and IEM in the case of $\theta_i=0^\circ$ and $\theta_i=60^\circ$ (a) for the surface A and (b) for the surface B.

Figure 5-31(a) shows the predicted scattering distribution for the surface A in the case of $\theta_i=0^\circ$ and $\theta_i=60^\circ$ by the four different scattering methods. Since the rms roughness is small, the SPM prediction is quite accurate, the GHS has reasonable accuracy for entire scattering and incident angles, and the KA has good agreement near the specular direction, but fails to predict accurately at large scattering angles. This trend holds for the surface B as shown in Figure 5-31(b).

Figure 5-32(a) shows the predicted scattering intensity for the surface C whose shoulder frequency is much smaller than that of surface A. The characteristic of the smooth-surface scattering intensity is that it has strong singular-like behavior near the specular direction. If the
smooth-surface approximation is not strictly satisfied, the SPM fails to accurately predict scattering in the near specular direction, but the GHS and KA show good agreement as shown in Figure 5-32(b). But unlike the KA, the GHS predicts reasonably accurate prediction at large scattering angles as well.

Figure 5-32: Predicted intensity distributions by the SPM, KA, GHS and IEM in the case of (a) $\theta_i=0^\circ$ and $\theta_i=60^\circ$ in linear-log scale and (b) $\theta_i=0^\circ$ in log-log scale.

Due to the limitation of the computer memory, the simulation for the case with much smaller shoulder frequency case is not calculated. However, as shown in this section, it is strongly possible that the trend of the region of validity obtained in previous sections still holds for the cases with much smaller shoulder frequency.
In this chapter, the scattering distributions from ideally conducting two-dimensional random rough surfaces are calculated using the IEM, SPM, KA and GHS methods. Their full angular behavior in direction cosine space are shown and compared to each other. Since the scattering calculation from two-dimensional random surfaces is computationally intensive especially for the numerical method, the comparison is performed for a few surface parameter sets instead of obtaining the region of validity over a large surface parameter space. This comparison work helps to reveal the connection between the regions of validity for one-dimensional surfaces and those for two-dimensional ones.

In Section 6.1, the predicted scattering intensities from the two-dimensional surfaces with the Gaussian PSD by the four scattering models are compared to each other. Also the results are compared to those obtained from the one-dimensional surfaces having the same surface parameter sets. In Section 6.2, the simulation parameters used for calculating IEM prediction are presented and the method of calculating an integration which is important for computing approximate methods is discussed.
6.1 Surfaces with Gaussian Statistics

Let us consider a two-dimensional random rough surface with a Gaussian auto-correlation function that has an rms roughness of $\sigma_{\text{tot}}$ and an correlation length of $l_c$. Its auto-covariance function is given by

$$ACV(r) = \sigma_{\text{tot}}^2 \exp \left[ - \left( \frac{r}{l_c} \right)^2 \right], \quad (6.1)$$

where $r = (x^2 + y^2)^{1/2}$ and its corresponding surface PSD function is

$$PSD(\rho) = \pi \sigma_{\text{tot}}^2 l_c^2 \exp \left[ - (\pi l_c \rho)^2 \right]. \quad (6.2)$$

where $\rho = (\alpha^2 + \beta^2)^{1/2}$. With these auto-correlation and surface PSD functions, the two-dimensional randomly rough surfaces are generated and the scattering intensity distributions are calculated using the IEM. Also, the predictions by the SPM, KA, and GHs are computed and their results are compared to those by IEM.

Figure 6-1 shows the full angular scattered intensity distributions in direction cosine space predicted by the IEM and SPM for the case of $\sigma_{\text{tot}}=0.001\lambda$, $l_c=0.5\lambda$, $\theta_i=0^\circ$. for TE polarized incident wave. In Figure 6-1(a) and (b), the scattering predictions by the IEM and SPM for TE polarized incident and TE polarized scattered wave (referred to as ss-polarization) are presented and, in (c) and (d), the predictions by the two methods for TE incident and TM scattered wave (referred to as sp-polarization) are shown.
Figure 6-1: Full angular scattering distribution predicted by the IEM for (a) ss, (c) sp, (e) ps, (g) pp and by the SPM for (b) ss, (d)sp, (f) ps (h) pp for the case of $\sigma_{tot}=0.001\lambda$, $l_c=0.5\lambda$, $\theta_i=0^\circ$.

In (e) and (f), the predictions by the methods for TM incident and TE scattered wave (referred to as ps-polarization) are illustrated and, lastly in (g) and (h), the predictions by the methods for TM incident and TE scattered wave (referred to as pp-polarization) are shown. The four components predicted by the two methods are virtually identical as we expected. For the IEM simulation, 5,000 realizations were carried out and the total reflected fractional energy calculated by the IEM is 0.9998.

In Figure 6-2(a), the unpolarized incoherent scattering intensity distributions from the two-dimensional surface with $\sigma_{tot}=0.001\lambda$, $l_c=0.5\lambda$ at normal incidence (the same as the case of Figure 6-1) along the x-axis predicted by the four methods are plotted.
Figure 6-2: The scattering distribution predicted by the four methods (a) for the unpolarized light from the two-dimensional surface having $\sigma_{tot}=0.001\lambda$, $l_c=0.5\lambda$ and (b) for the one-dimensional surface (TE for the SPM and IEM) having the same surface parameters at normal incidence.

In Figure 6-2(b), the TE polarized scattered intensities from the one-dimensional randomly rough surface having the same rms roughness and correlation length predicted by the IEM and SPM are superimposed by the predictions of the GHS and KA for the same surface. Comparing (a) and (b), the scattering intensity distributions from the two-dimensional surfaces are different from those from the one-dimensional surface. However, it looks that the characteristics of the three approximate methods remain the same if the surface parameters are the same. The SPM and KA predictions show good agreement to the predictions by the IEM. And the GHS prediction has little deviation from the ones by other methods.

In Figure 6-3, the full angular intensity distributions predicted by the (a) IEM, (b) GHS, (c) KA, (d) SPM for unpolarized scattered light from the surface having $\sigma_{tot}=0.2\lambda$, $l_c=2\lambda$ at normal incidence are plotted. Since the surface is moderately rough the total number of realizations is chosen to be 10,000 for the IEM simulation and the total fractional energy calculated by the IEM is 0.9991. The SPM predicts a much higher intensity distribution as indicated by its colorbar, but the other two approximate methods predict similar distributions to the prediction by the IEM. Figure 6-4(a) shows the predicted scattering intensity profiles along the x-axis by the four methods for the same surface parameters used in Figure 6-3. In Figure 6-4(b), the scattered intensities from the one-dimensional surface having the same rms roughness and correlation length predicted by the methods are superimposed.
In both Figure 6-4(a) and (b), the KA predictions agree well to the predictions by the IEM. Regarding the GHS, it predicts, for the two-dimensional surface, little stronger intensity near the specular direction than the other two methods and this characteristic can be also found in one-dimensional result shown in (b). In both one- and two-dimensional cases, it is obvious that the SPM is not valid for this moderately rough surface.
Figure 6-4: The scattering distribution predicted by the four methods (a) for the unpolarized light from the two-dimensional surface with $\sigma_{\text{tot}}=0.2\lambda$, $l_c=2\lambda$ and (b) from the one-dimensional surface (TE for the SPM and IEM) having the same surface parameters at normal incidence.

Figure 6-5 shows the full angular intensity distributions predicted by the (a) IEM, (b) GHS and (c) KA for the unpolarized scattered light from the surface having $\sigma_{\text{tot}}=0.6\lambda$, $l_c=2\lambda$ at normal incidence are plotted. The total number of realizations is chosen to be 10,000 for the IEM simulation and the total fractional energy calculated by the IEM is 0.9865. Since the surface is quite rough, the SPM prediction is not shown in Figure 6-5. In the figure, the IEM calculation shows little fluctuation near the specular direction and it is caused by the Monte-Carlo technique. The GHS predicts little stronger scattering near the specular direction, and the KA does little weaker scattering. These characteristics are revealed more clearly in Figure 6-6(a) where the scattering intensity distributions are plotted along the x-axis.
Figure 6-5: Full angular distributions predicted by the (a) IEM, (b) GHS and (c) KA for unpolarized scattering intensity from the surface having $\sigma_{tot}=0.6\lambda$, $l_c=2\lambda$ at normal incidence.

Figure 6-6: The scattering distribution predicted by the IEM, GHS and KA (a) for the unpolarized light from the two-dimensional surface (b) from the one-dimensional surface (TE for the IEM) having the same surface parameters.

Figure 6-6(b) illustrates the predicted scattered intensities from the one-dimensional surface having the same rms roughness and correlation length. The characteristics can be found in
one-dimensional results although the shape and the height of the scattering distributions are
different.

Figure 6-7 shows the full angular intensity distributions predicted by the (a) IEM, (b) GHS and
(c) KA for the unpolarized scattered light from the surface having $\sigma_{tot}=0.8\lambda$, $l_c=2\lambda$ at normal
incidence are plotted. The total number of realizations is chosen to be 10,000 for the IEM
simulation and the total fractional energy calculated by the IEM is 0.9753.

![Figure 6-7: Full angular distributions predicted by the (a) IEM, (b) GHS and (c) KA for the
unpolarized scattering intensity from the surface having $\sigma_{tot}=0.8\lambda$, $l_c=2\lambda$ at normal
incidence.](image)

Since the surface is very rough, the SPM prediction is not shown and the GHS prediction shows
weaker scattering prediction than the IEM prediction near the specular direction and the KA
predicts much weaker scattering. In the Figure 6-8(a), the scattering intensity distributions by the
three methods are plotted along the x-axis and Figure 6-8(b) shows the scattered intensities from
the one-dimensional one with the same rms roughenss and correlation length. The scattering
distributions are different, but the characteristic of the scattered intensity from the two-dimensional surfaces can be found in the results calculated from the one-dimensional surface having the same surface parameters.

Figure 6-8: The scattering distribution predicted by the IEM, GHS and KA (a) for the unpolarized light from the two-dimensional surface (b) from the one-dimensional surface (TE for the IEM) having the same surface parameters.

Throughout this section, the scattered intensity predictions from the two-dimensional random rough surfaces are calculated using the IEM, SPM, KA and GHS method. The results are compared to those obtained from the one-dimensional surfaces with the same surface parameters. This comparison work shows that the two intensity predictions are, obviously, different but it indicates that there is strong possibility that the region of validity obtained from the one-dimensional surfaces can be applied to the scattering problem from the two-dimensional surfaces.
6.2 Numerical Consideration

Numerical simulation for two-dimensional random rough surfaces is computationally intensive for both analytic and numerical methods. Especially, it is extremely computer memory and computation time demanding work for the numerical method.

When it comes to the IEM simulation, the most severe limiting factor is the computer memory as noted by other researchers [32], but the computing time is also a crucial factor. By trading off between the computer memory and calculation time, with currently available computers for us, the practical limitation of the total number of sampling of the sampled surface is 112×112.

In our simulation, the number of samples per wavelength is chosen as 7λ for the surfaces with a Gaussian PSD function. The sampling ratio is directly connected to the reliability of the simulation because the energy conservation condition can be achieved when a large enough number is used, especially for rough surfaces. In our simulation, with the sampling ratio, energy conservation condition is checked for every result and they are specified when the IEM results are illustrated. Achieving energy conservation does not guarantee the accuracy of the simulation but it is not trivial condition when solving 2×112×112 unknowns.

Due to the limitation of the total number of sampling, the surface size is chosen by 16λ×16λ for the surfaces with Gaussian PSD function. And the width of the Gaussian incident beam is chosen by 4λ. The total number of sampling for the direction cosine space of the scattered light is selected as 101×101 in linear scale as like other researchers [32]. The total number of realizations is chosen by 10,000 to get smooth enough scattered light distribution from randomly
rough surfaces. However, even with this large number of realizations, some fluctuations are found in the IEM results.

Regarding analytic methods, an integration term requires great care when it is numerically evaluated. In the analytic expressions of the KA and GHS, there is a common integration given by

\[
I(s_i) = \int d^2x_i \exp \left\{ \left[ 2\pi \sigma (\gamma_i + \gamma_i(s_i)) \right]^2 C(x_i) \right\} \exp \left\{ -i 2\pi (s_i + s_i) \cdot x_i \right\}. \tag{6.3}
\]

where \(\sigma = \sigma_{\text{tot}}\) for the KA and \(\sigma = \sigma_{\text{rel}}\) for the GHS. Replacing the variables as \(s'_i = s_{k,i} + s_i\), the integration term is rewritten by

\[
I(s'_i) = \int d^2x_i \exp \left\{ \left[ 2\pi \sigma (\gamma_i + \gamma'_i(s'_i)) \right]^2 C(x_i) \right\} \exp \left\{ -i 2\pi s'_i \cdot x_i \right\}, \tag{6.4}
\]

where

\[
\gamma'_i(s'_i) = \left[ 1 - (\alpha' - \alpha_i)^2 - (\beta' - \beta_i)^2 \right]^{1/2}. \tag{6.5}
\]

The auto-correlation function we have considered throughout this chapter depends on the radial distance only and the integration is, in polar coordinates, rewritten by

\[
I(s'_r) = 2\pi \int rdr \exp \left\{ \left[ 2\pi \sigma (\gamma_i + \gamma'_r) \right]^2 C(r) \right\} J_0(2\pi s'_r), \tag{6.6}
\]

where \(s'_r = (\alpha'^2 + \beta'^2)^{1/2}\). Unfortunately, when ACV function is given by the Gaussian function, there is no analytic closed form and it forces us to calculate the integration numerically. First, the integrand can be expanded as
\[ \exp[gC(r)] = \sum_{n=0}^{\infty} \frac{g^n}{n!} C^n(r), \]  

(6.7)

where \( g = g(\gamma') = [2\pi\sigma(\gamma + \gamma')]^2 \). Using this expansion, the integration is written by

\[
I(s') = 2\pi \sum_{n=0}^{\infty} \left[ \frac{g^n}{n!} \int rdrC^n(r)J_0(2\pi s') \right],
\]

(6.8)

which is the summation of Hankel transforms. If the correlation function is given by the Gaussian function, the analytic form of the integrand the in above equation is given so that the integration can be calculated by summing all terms (up to ‘infinite’ order). However, if the correlation function is given by K-correlation function which is not used in this chapter but important for typical optical surfaces, since the analytic form of the Hankel transform of \( C^n(r) \) is not known, each expansion term is calculated numerically for each scattering direction cosine.

There is another numerical method for directly calculating the integration. However, the correlation function varies in logarithmic scale so that the numerical integration requires tight sampling spacing with large domain size. Talman suggested an algorithm called FFTLog which is taking the Hankel transform of an arbitrary function numerically, according to Hamilton [105], which varies smoothly in logarithmic space. The fact that we know the behavior of Bessel function in logarithmic space enables us to calculate Hankel transform in logarithmic space with relatively small number of sampling points. The integration in Eq.(6.9) is not a Hankel transform because the integrand depends on the direction cosine but we still can use the FFTLog algorithm.
by picking up a value from the result of the transformation and repeating to taking the transformation. Details can be found in other literature [106].
Image degradation due to conventional aberrations has become well-understood over the last century, and surface scatter phenomena has been investigated extensively for the last half century. However, the two image degradation mechanisms are usually treated separately even though they are essentially the same phenomenon; i.e., the deviation of the light from the ideal direction as it propagates through an imaging system.

Recently, Peterson suggested a way of analyzing image degradation due to surface scatter in a multi-element system [107,108], and subsequently, Harvey, et al. extended Peterson’s method to the more general case of moderately rough surfaces [109]. However, their analyses are still restricted to paraxial, aberration free optical systems. In this chapter, image degradation, as characterized by the PSF, is obtained in explicit form, for systems in which both aberrations and surface scatter are dominant image degradation mechanisms.

This explicit form, expressed as the convolution of the geometrical PSF\(_G\) with a scattering PSF\(_S\) is convenient; however, the accuracy depends upon the validity of the approximations and assumptions made in the mathematical development. Goodman has shown that a similar linear systems approach to image analysis as degraded by diffraction and geometrical aberrations is not strictly true [90], and Harvey and Krywonos have quantitatively evaluated the assumption for different amounts of defocus [110]. They then proceed to successfully apply the linear system formulation of image quality to the case of an X-ray telescope [111,112].
This chapter confirms that the explicit expression for the system PSF is not rigorously true; however, it discusses the assumptions and approximations made in the mathematical formulation, explains why the explicit expression should be sufficiently accurate for most optical engineering applications.

For short wavelength EUV applications, the effects of aberrations exceed the Rayleigh’s diffraction limit by a substantial factor and diffraction effects become relatively insignificant [71]. Also at these short wavelengths, scattering effects are drastically increased for a given residual surface roughness. The size of the Airy disk is thus small compared to both the geometrical ‘spot size’ and ‘scattering effects’. We have thus ignored diffraction effects in the analysis presented in this chapter and discuss image degradation due only to scatter effects in the presence of aberrations.

After introducing notations and definitions used in this chapter in Section 7.1, the fundamental formalism is derived in Section 7.2 and it will be validated in Section 7.3 before extending this analysis to the case of image degradation due to conventional aberrations and surface scatter in Section 7.4. In the following sections, absorption and vignetting effects are ignored.

7.1 Notations

Figure 7-1 illustrates an optical system consisting of a series of coaxial optical surfaces [113]. The Gaussian image of a point object formed by the first optical surface acts as an object for the
second optical surface, and so on. Thus, the image plane of the \( j \)th optical surface (referred to as \( j \)th image plane) is the same as the object plane for the \((j+1)\)th optical surface (referred to as \((j+1)\)th object plane). The first object plane is the same as the object plane of the entire optical system, and last image plane is the same as image plane of the entire optical system.

Figure 7-1: Schematic layout of an optical imaging system consisting of a series of coaxial optical surfaces.

Unlike previous chapters the position vector \( \mathbf{x}_j = x_j \hat{x} + y_j \hat{y} \) denote two-dimensional vector on the x-y plane and the vector is lying on the \( j \)th image plane and \( \mathbf{x}_0 \) denotes the position where a ray intersects the object plane. \( z_j \) and \( z'_j \) are defined by the conjugate distances of the \( j \)th optical
surface divided by the index of refraction of the corresponding space, and the paraxial transverse magnification of the $j$th optical surface is given by $m_j = z_j' / z_j$.

Each optical surface has its own entrance pupil and exit pupil. The exit pupil of the $j$th optical surface (referred to as $j$th exit pupil) coincides with the entrance pupil of the $(j+1)$th optical surface (referred to as $(j+1)$th entrance pupil). The first entrance pupil is the same as the entrance pupil of the entire optical system, and the last exit pupil is the same as the exit pupil of the entire optical system. $\xi_j = \xi_j' \hat{x} + \eta_j \hat{y}$ denotes the ray ‘intersection’ position at the $j$th entrance pupil plane and $\xi'_j = \xi'_j \hat{x} + \eta'_j \hat{y}$ denotes the ray ‘intersection’ position at the $j$th exit pupil plane. The entrance pupil and exit pupil have one-to-one correspondence and the relation is given by $\xi'_j = m_{pj} \xi_j$, where $m_{pj}$ is the magnification between the two pupils and it is not necessarily a constant, but it turns out to be a constant under the fourth order (wavefront error) approximation. $r_j$ is defined by the distance from the center of the $j$th exit pupil to the $j$th image plane divided by the index of refraction of the corresponding space.

Each optical surface may have aberrations, and the primary wave aberration (fourth order aberrations) function [88] of the $j$th optical surface (referred to as $j$th wave aberration function) is defined at the $j$th exit pupil and given by

$$W_j(\xi_j' | x_{j-1}) = W_{040}p_j^2 + W_{134}p_j c_j + W_{222}c_j^2 + W_{220}p_j q_j + W_{311}q_j c_j,$$

(7.1)

Where $p_j = |\xi_j'|^2 / |\xi_{j,max}'|^2$, $q_{j+1} = |x_j| / |x_{j,max}|^2$, and $c_j = \xi_j' \cdot x_{j-1} / |\xi_{j,max}'| |x_{j,max}|$ where $|\xi_{j,max}'|$ is the maximum pupil height in the $j$th exit pupil, and $|x_{j,max}|$ is the maximum object height in the
jth object plane. Note that \(|x_{j, \text{max}}| = m_{m} \cdots m_{j-1} |x_{0, \text{max}}|\), where \(|x_{0, \text{max}}|\) is the maximum object height in the object plane. If an optical system consists of \(n\)-elements, the primary aberration function of the total system \(W\) is obtained by

\[
W(\xi_{o} | x_{0}) = \sum_{j=1}^{n} W_{j}(\xi'_{j} | x_{j-1}) = \sum_{j=1}^{n} W_{j}(\xi'_{j} / (\xi'_{j} / \xi_{j}')) m_{m} \cdots m_{j-1} x_{0}.
\]  

(7.2)

The third order transverse ray aberration function \(\varepsilon_{j}(\xi'_{j} | x_{j-1})\) is introduced, and defined by

\[
\varepsilon_{j}(\xi'_{j} | x_{j-1}) = -r_{j} \nabla_{j} W_{j}(\xi'_{j} | x_{j-1}),
\]

(7.3)

which is a ray displacement with respect to the Gaussian image position in the \(j\)th image plane, and \(\nabla_{j} = (\partial / \partial \xi'_{j}) \hat{x} + (\partial / \partial \eta'_{j}) \hat{y}\) is the gradient operator. The Lagrange invariant has been used to obtain the third order transverse ray aberration function of the total \(n\)-element optical system

\[
\varepsilon(\xi'_{o} | x_{0}) = -r_{n} \nabla_{n} W(\xi'_{o} | x_{0}) = -r_{n} \nabla_{n} \left[ \sum_{j=1}^{n} W_{j}(\xi'_{j} | x_{j-1}) \right] = - \sum_{j=1}^{n} [m_{j+1} \cdots m_{n} r_{j} \nabla_{j} W_{j}(\xi'_{j} | x_{j-1})],
\]

(7.4)

\[
= \sum_{j=1}^{n} [m_{j+1} \cdots m_{n} \varepsilon_{j}(\xi'_{j} | x_{j-1})]
\]

Note that the last exit pupil coincides with the exit pupil of the total optical system.
7.2 Integral Expression for Geometrical PSF

In this section, the geometrical PSF is represented in an integral form. First, geometrical PSF for a reduced optical system is described and then it is extended to an optical system consisting of a series of coaxial optical surfaces in the followed section.

7.2.1 Geometrical PSF for a Reduced Optical System

The concept of geometrical PSF is simple and straightforward but an analytic expression of it is rarely discussed. In this section, one of analytic expressions of the geometrical PSF is proposed for further discussion. To avoid any order related issues, let’s assume that the relation between the transverse ray aberration $\varepsilon(\xi')$ and the wave aberration $W(\xi')$ is given by

$$\varepsilon(\xi') = \mathcal{F}[W(\xi')].$$  \hfill (7.5)

It is assumed that the operator $\mathcal{F}$ gives exact ray aberration, and it reduces to $-r_j \nabla_j$ under the third order approximation.

Figure 7-2(a) shows the pupil plane and 7-2(b) shows the corresponding image plane. The shadowed infinitesimal area denoted as $da_p$ in Figure 7-2(a) is mapped into the shadowed infinitesimal square of which area is $da_i$ which is centered on $x_c$ in Figure 7-2(b) by the operator. Welford [114] states that the irradiance value at a given point in image plane is proportional to the ratio of $da_p$ to $da_i$, and Mahajan [113] shows that the explicit relation between them given by
where \( E_p \) is the irradiance at pupil plane and it is not necessarily constant in Mahajan’s expression, but we restrict our attention on the case of constant \( E_p \) for simplicity.

\[
E(x_c) = E_p \frac{da_p}{da_i},
\]

(7.6)

Meanwhile, Smith [71] uses a ray-tracing technique to describe the geometrical PSF. According to him, the geometrical PSF is obtained through three steps. The first step is discretizing the pupil plane uniformly, and the second step is obtaining the ray intersection position in the image plane by tracing each ray passing through the center of each discretized area, and the last step is counting the number of rays fallen on the given small area in image plane.

In order to show the equivalence of the two approaches, consider a small area \( \Delta a_p \) in the pupil plane which is mapped into small square of which area is \( \Delta a_i \) centered on \( x_c \) in the image plane.
by the operator \( \mathcal{F} \). Smith’s statement can be expressed mathematically in the following steps. First, represent the position of discrete sampling point in pupil plane as \( \xi'_m \), where \( m \) represents the \( m \)th sampling point out of total \( N \) sampling points in the pupil. Second, trace each ray and obtain the ray intersection position in the image plane, which is represented by using the operator \( \mathcal{F}[W(\xi'_m)] \). Third, introduce a function \( E \) which is the combination of delta functions centered on the ray intersection positions in the image plane by

\[
E(x) = E_p \frac{a_p}{N} \sum_{m=1}^{N} \delta(x - \mathcal{F}[W(\xi'_m)]),
\]  

where \( a_p \) is the total area of the pupil and \( E_p a_p / N \) is the power which a sampled ray is carrying. Lastly, take the average of the function \( E(x) \) over the small square \( \Delta a_i \) in image plane which is expressed by

\[
\mathcal{E}(x_c) = \frac{1}{\Delta a_i} \int d^2x E(x) \\
= E_p \frac{a_p}{N} \frac{1}{\Delta a_i} \int d^2x \sum_{m=1}^{N} \delta(x - \mathcal{F}[W(\xi'_m)]) = E_p \frac{a_p}{N} \frac{1}{\Delta a_i} \int d^2x \sum_{m=1}^{M} \delta(x),
\]  

where \( M \) is the number of sampling points inside the \( \Delta a_p \) in the pupil plane. Equation (7.8) could be one of mathematical illustration of Smith’s method. If we take the limit of \( N \to \infty \), the ratio of \( M / N \) approaches \( da_p / a_p \), and, in addition, if we take the limit of \( \Delta a_i \to 0 \), the ratio of

164
the two small areas becomes \( \Delta a_p / \Delta a_i = da_p / da_i \). Thus, the value of \( \mathcal{E} \) in those limits is given by

\[
\lim_{\Delta a \to 0} \left( \lim_{N \to \infty} \mathcal{E} \right) = \lim_{\Delta a \to 0} \left( E_p \frac{\Delta a_p}{\Delta a_i} \right) = E_p \frac{da_p}{da_i},
\]

which is the identical expression to Eq.(7.6) with constant \( E_p \) and mathematical ground that the number of samples in the pupil plane must be large.

In order to extend the mathematical illustration from the case of discrete \( \xi' \) to the case of continuous variable \( \xi' \), let us to define a hexahedron function \( \tau(x) \) by

\[
\tau(x) = \lim_{\Delta a \to 0} \frac{1}{\Delta a_i}, \quad \text{for} \quad -\frac{\Delta x}{2} < x < \frac{\Delta x}{2} , \quad -\frac{\Delta y}{2} < y < \frac{\Delta y}{2} , \quad \text{elsewhere}
\]

where \( \Delta a_i = (\Delta x)^2 \), and the physical dimension of this hexahedron function is the inverse of area.

Using the hexahedron function, the function \( E(x) \) becomes simply

\[
E(x) = E_p \int d^2\xi' \tau(x - \mathcal{S}[W(\xi')]) \equiv PSF_G(x).
\]

Let us consider the case of \( x = x_c \). If the ray intersection position given by the mapping operator \( \mathcal{S}[W(\xi')] \) is fallen inside of the shadowed square in Figure 7-2(b), the integrand in Eq.(7.11) becomes a constant value of \( 1/\Delta a_i \). If the ray intersection position given by the operator
\( J[W(\xi')] \) is fallen outside of the square, the integrand becomes zero. Equation (7.11) is, thus, reduced to

\[
E(x_e) = \lim_{\Delta n \to 0} \, \lim_{\Delta a_p \to 0} E_p \int d^2 \xi \, \frac{1}{\Delta a_i} = \lim_{\Delta n \to 0} \, \lim_{\Delta a_p \to 0} \frac{\Delta a_p}{\Delta a_i} = E_p \frac{da_p}{da_i},
\]

which is identical to Eq.(7.9). The physical dimension of \( E(x) \) in Eq.(7.11) is irradiance, and, by definition, it can be considered as a geometrical PSF. In the following sections, a delta function is used instead of \( \tau(x) \). The delta function is usually considered as the limiting form of a Gaussian function, but it is used instead of \( \tau(x) \) for mathematical simplicity. In conclusion, if only the third order ray aberration (or fourth order wave aberrations) is concerned, the geometrical PSF is expressed by

\[
PSF_G(x) = E_p \int d^2 \xi' \delta(x + r_j \nabla_j W(\xi')),
\]

and the above expression is referred to as the geometrical aberration in this Chapter.

### 7.2.2 Geometrical PSF of a Series of Coaxial Optical Surfaces

Figure 7-3 illustrates an imaging system consisting of a single optical surface. A ray bundle leaves from the ray intersection position \( x_0 \) with the direction cosine \( s_0 \) in the first object plane toward the position \( \xi_1 \) in entrance pupil. The ray bundle strikes the optical surface at the point \( p_1 \) and it is refracted or reflected. After the refraction or reflection, its direction cosine is changed to
s\_1 and the ray bundle intersects the image plane at x\_1. The distance between x\_0 and \( \xi\_1 \) is denoted by \( l = l(x\_0, \xi) \) and the distance between \( \xi' \) and x is denoted by \( l' = l'(x, \xi') \).

Figure 7-3: Schematic layout of a single surface optical imaging system.

Assume that a point source is located at x\_obj in object space with its intensity \( I_0(s\_0) \). Then, its ‘radiance’ can be represented by

\[
L(s\_0, x\_0) = I_0(s\_0)\delta(x\_0 - x\_obj).
\]  

(7.14)

The total power collected by the entrance pupil is

\[
P_t = \int da\_0 \int d\Omega L(s\_0, x) = \int da\_0 E\_obj,
\]

(7.15)

where \( da\_0 \) is infinitesimal areas in the object plane and \( E\_obj \) is the irradiance of the point source at the object given by
\[ E_{\text{obj}} = \int d\Omega I_0(s_0) \delta(x_0 - x_{\text{obj}}) = P \delta(x_0 - x_{\text{obj}}). \quad (7.16) \]

The solid angle of an infinitesimal area in the entrance pupil \( da_{p1} \) from the point source position is given by

\[ d\Omega = \frac{(s_0 \cdot \hat{z}) da_{p1}}{l_i^2}. \quad (7.17) \]

Using the solid angle, the total power collected by the entrance pupil is rewritten by

\[ P = \int da_{p1} \left( \frac{s_0 \cdot \hat{z}}{l_i^2} \right) I_0(s_0) = \int da_{p1} E_{p1}, \quad (7.18) \]

where the irradiance of the entrance pupil plane is

\[ E_{p1} = \frac{(s_0 \cdot \hat{z})}{l_i^2} I_0(s_0). \quad (7.19) \]

Combining the equations (7.16) and (7.18), the expression of the irradiance of the point source is, again, given by

\[ E_{\text{obj}} = P \delta(x_0 - x_{\text{obj}}) = \int da_{p} E_{p1} \delta(x_0 - x_{\text{obj}}). \quad (7.20) \]

The infinitesimal power emitted from the infinitesimal object plane area is \( dP = da_0 E_{\text{obj}} \) so that the infinitesimal power collected by an infinitesimal entrance pupil area emitted from the infinitesimal object plane area is

\[ d^2 P_{i} = da_0 da_{p1} E_{p1} \delta(x_0 - x_{\text{obj}}). \quad (7.21) \]
Figure 7-4: Schematic layout of the entrance pupil and exit pupil for a single optical surface. Vignetting is ignored.

As illustrated in Figure 7-4, for an observer located in the object plane, it looks that all the ray bundles are entering the entrance pupil. Since the entrance pupil and exit pupil have one-to-one correspondence given by ξ' = mξ, the infinitesimal power carried by a ray bundle strikes entrance pupil at ξ (or the infinitesimal power illuminates the exit pupil near the position ξ') is

\[ dP = E_p da_p = E_p m^2 da'_{p1} = E'_{p1} da'_{p1}, \quad (7.22) \]

where \( da'_{p1} \) is the infinitesimal area at the exit pupil and \( E'_{p1} \) is the exit pupil irradiance. Thus, the power, emitted from the infinitesimal object plane area \( da_0 \), illuminating the infinitesimal exit pupil area is

\[ d^2 P_1 = da_0 da'_{p1} E'_{p1} \delta(x_0 - x_{obj}). \quad (7.23) \]
For an observer located in the image plane, although the ray bundle is refracted or reflected at the optical surface, it looks that all the ray bundles are emitted from the exit pupil. Since it looks that the exit pupil emits the light, we can define the BRDF of the exit pupil which is defined by

\[
\frac{dL}{dE} = \frac{1}{(s'_i \cdot \hat{z})} \frac{1}{P_i} \frac{dP}{d\Omega} = BSDF_i(s'_i, s_0 \mid \xi'_i).
\] (7.24)

where \(s'_i\) is the direction cosines of a ray bundle emitted \textit{from the exit pupil} at the ray intersection position \(\xi'_i\), and \(s_0\) is the direction cosines of a ray bundle striking the entrance pupil. Since the exit pupil BSDF may not be homogeneous over the exit pupil, it has dependency on the pupil position. Using the exit pupil BSDF, the radiant power carried by a ray bundle emitted from \(\xi'_i\) in the exit pupil and intersects the image plane at \(x_1\) is represented by

\[
dP = P_i BSDF_i(s'_i, s_0 \mid \xi'_i)(s'_i \cdot \hat{z})\frac{da_1}{l^2},
\] (7.25)

where \(d\Omega = (s'_i \cdot \hat{z})da_i/l^2\) is used. Combining the equations (7.23) and (7.25), the power carried by the ray bundle leaving the object plane at \(x_0\), passing through \(\xi_i, \xi'_i\), and intersecting the object plane at \(x_1\) is given by

\[
d^3P = E^i_{\rho l} \frac{BSDF_i(s'_i, s_0 \mid \xi'_i)}{l^2} \delta(x_0 - x_{obj})(s'_i \cdot \hat{z})^2 da'_i da_0 da_1.
\] (7.26)

Now, let us move to the situation illustrated in Figure 7-5. If there is another refracting or reflecting optical surfaces followed by the first optical surface, the ray bundle having direction
cosine \( s_1 \) strikes the point \( p_2 \) on the second optical surface and, again, it is reflected or reflected. Its direction cosine is changed to \( s_2 \) and the ray bundle intersects the second image plane at the position \( x_2 \).

![Diagram of two-surface optical imaging system](image)

**Figure 7-5:** Schematic layout of a two-surface optical imaging system. If \( x_1 \) is equal to \( x_{01} \), the system becomes conventional optical imaging system.

For an observer located in the second object plane, it looks that the ray bundle enters the second entrance pupil at \( \xi_2 \). Since the second entrance pupil is the same as the first exit pupil, \( \xi_2 = \xi'_1 \), the infinitesimal power carried by the ray bundle intersects the entrance pupil at \( \xi_2 \) is just \( d^1 P_1 \).

Since the entrance pupil has one-to-one correspondence to exit pupil, \( \xi'_2 = m_{p2} \xi_2 \), the infinitesimal power *illuminates* the second exit pupil is given by

171
\[
d^3 P_{p2} = E'_{p2} \frac{BSDF_1(s'_1, s_0 | \xi'_1)}{l_1^2} \delta(x_0 - x_{obj}) (s'_1 \cdot \hat{z})^2 da'_p da_0 da_1.
\]

(7.27)

where \( E'_{p2} \) is the irradiance of the second exit pupil and \( da'_p \) is the infinitesimal area in the second exit pupil. For an observer who is lying on the second image plane, it looks that the ray bundle having direction cosines \( s'_2 \) emitted from the second exit pupil at \( \xi'_2 \). By introducing the BSDF of the second exit pupil, the infinitesimal power carried by a ray bundle emerges from the second exit pupil is

\[
d^4 P_2 = (s'_2 \cdot \hat{z}) (d^3 P_{p2}) BSDF_2(s'_2, s'_1 | \xi'_2) (s'_2 \cdot \hat{z}) da_2 \frac{1}{l_2^2}.
\]

(7.28)

where \( d\Omega = (s'_2 \cdot \hat{z}) da_2 / l_2^2 \) is used. Thus, the infinitesimal power carried by a ray bundle originates from \( x_0 \), enters the first entrance pupil at \( \xi_1 \) and emerges from the second exit pupil \( \xi'_2 \) is given by

\[
d^4 P_2 = E'_{p2} \frac{BSDF_1(s'_1, s_0 | \xi'_1)}{l_1^2} \frac{BSDF_2(s'_2, s'_1 | \xi'_2)}{l_2^2} \delta(x_0 - x_{obj}) (s'_1 \cdot \hat{z})^2 (s'_2 \cdot \hat{z})^2 da'_p da_0 da_1 da_2.
\]

(7.29)

Note that the second exit pupil is the exit pupil of the entire optical system in a series of two-coaxial surfaces system.

Using the similar manner, the power carried by the ray bundle leaving the object plane at \( x_0 \), enters the first entrance pupil at \( \xi_1 \) and emerges from the \( n \)th exit pupil \( \xi'_n \) is obtained by
\[ d^{n+2} P_n = E'_{p2} \left[ \prod_{j=1}^{n} \frac{BSDF_j (s'_j, s'_{j-1} | \xi'_j) (s'_j \cdot \hat{z})^2 \delta(x_0 - x_{obj})}{l_j^2} da_j \right] \delta(x_0 - x_{obj}) da_{p0} da_0, \]  

(7.30)

where the notation \( s'_0 = s_0 \) is used for simplicity. The irradiance distribution at the \( n \)th image plane is, therefore, calculated by integrating both sides of above equation as

\[ E_n(x_0) = \frac{dP}{da_n} = \int da_p \int da_0 \cdots da_{n-1} \left[ \prod_{j=1}^{n} \frac{BSDF_j (s'_j, s'_{j-1} | \xi'_j) (s'_j \cdot \hat{z})}{l_j^2} \right] E_0(x_0), \]  

(7.31)

where simplified notations \( da_p = da'_{pa} \) and \( E_0(x_0) = E_p \delta(x_0 - x_{obj}) \) where \( E_p = E'_{pa} \) are used. Eq.(7.31) can be considered as the geometrical PSF for a series of coaxial optical surfaces. In the equation, the refracting or reflecting phenomena are described in terms of the BSDFs of the exit pupils. Equation (7.31) can be simplified under the small angle approximations given by

\[ s'_j \cdot \hat{z} \cong 1, \]  

(7.32)

and

\[ l'_j \cong r'_j. \]  

(7.33)

In Cartesian coordinates, under this small angle condition, the Eq.(7.31) is written as

\[ E_n(x_a) = \int d^2 x'_a dx_0 \cdots dx_{n-1} \left[ \prod_{j=1}^{n} \frac{1}{l_j^2} BSDF_j (s'_j, s'_{j-1} | \xi'_j) \right] E_0(x_0). \]  

(7.34)

Equation (7.34) will be referred to as the irradiance distribution function in the \( n \)th image plane, and it will be calculated for special cases in the followed sections.
7.3 Geometrical PSF in the Presence of Aberration

In Section 7.3.1, the irradiance function at the $n$th object plane is calculated under the assumption that there is no aberration in the optical system. Then, the method will be extended to the system where optical elements have aberrations. Regarding aberrations, we confine our interest in only up to third order aberrations.

7.3.1 Aberration Free Optical System

First, let us consider a single optical surface having no aberration. The irradiance function at the first image plane is given by

$$E_i(x_i) = \int d^2\xi d^2x_0 \left\{ \frac{1}{\rho^2} BSDF_i(s'_i, s_0 | \xi'_i) \right\} E_0(x_0).$$  \hspace{1cm} (7.35)

To calculate the integration, we need to evaluate the BSDF of the exit pupil for the optical surface. Since there is no aberration, the ray intersection position at the first image plane of a ray bundle emerging from a position $\xi'$ at the exit pupil is easily expressed by

$$x_{01} = m_i x_0,$$ \hspace{1cm} (7.36)

where the ray intersection position does not depend on the exit pupil position because the surface is assumed to be aberration free. However, its direction cosines has exit pupil position dependence given by
where the subscript $\theta$ in the symbol $s_{01}$ denotes the specular direction and recall that we assumed that the angles are small so that the $z$-component of the direction cosine vector is always unity. The direction cosines of a ray bundle intersecting an arbitrary position $x_i$ at image plane can be represented as

$$s'_i = \frac{x_i - \xi'_i}{n'_i},$$

(7.38)

and the ray bundle does not carry any power except when its direction cosine is the same as $s'_i = s_{01}$. Therefore, the BSDF of the exit pupil for an aberration free optical surface can be written by

$$BSDF_i(s'_i, s_0 \mid \xi'_i) = \delta(s'_i - s_{01}) = \delta\left(\frac{x_i}{n'_i} - \frac{m_i x_0}{n'_i}\right).$$

(7.39)

Note that, BSDF of the exit pupil for an aberration free optical surface does not depend on the exit pupil position. Using the identity,

$$\delta(ax) = a^2\delta(x),$$

(7.40)

The exit pupil BSDF is rewritten as

$$BSDF_i(s'_i, s_0 \mid \xi'_i) = \frac{n'_i}{z'_i} \delta\left(\frac{x_2}{z'_i} - \frac{m_i x_0}{z'_i}\right).$$

(7.41)
Substituting the above expression into the Eq. (7.35), the irradiance distribution function at the first image plane for a single aberration free optical surface is obtained by

\[
E(x_1) = \int d^2 \xi_1 d^2 x_0 \frac{1}{z_1^2} \delta \left( \frac{x_1}{z_1} - \frac{x_0}{z_1} \right) E_0(x_1) = P \delta(x_1 - m_0 x_{obj}),
\]

(7.42)

and the trivial solution is reproduced.

Let us extend this method to an \( n \)-element aberration free optical system. The irradiance function in \( n \)th image plane is given by

\[
E_n(x_n) = \int d^2 \xi_n d^2 x_0 \cdots d^2 x_{n-1} \left[ \prod_{j=1}^{n} \frac{1}{\xi_j'} BSDF_j(s_j', s_{j-1}' | \xi_j') \right] E_0(x_0).
\]

(7.43)

Using the BSDF of the exit pupil for an aberration free optical surface, the above equation becomes

\[
E_n(x_n) = \int d^2 \xi_n d^2 x_0 \cdots d^2 x_{n-1} \left[ \prod_{j=1}^{n} \frac{1}{\xi_j'} \delta \left( \frac{x_j}{\xi_j'} - \frac{x_{j-1}}{\xi_j'} \right) \right] E_0(x_0).
\]

(7.44)

After some algebra, the irradiance function in \( n \)th image plane for a coaxial aberration free \( n \)-optical-surfaces system is obtained by

\[
E_n(x_n) = P \delta(x_n - m x_{obj}).
\]

(7.45)
where \( m = m_1 \cdots m_n \) is the total magnification of the entire optical system. Again, the trivial solution is reproduced. This formalism is extended to the case where the optical surfaces have aberrations in the next section.

### 7.3.2 Aberrated Optical System

Consider a situation where there is only one optical surface but it has aberrations. The irradiance function at the first object plane is given by

\[
E_i(x_i) = \int d^2 \xi d^2 x_0 \left[ \frac{1}{\eta^2} \text{BSDF}(s', s_0 | \xi') \right] E_0(x_0). 
\]  

Due to the aberrations, up to the fourth order, the ray intersection position at the first image plane is expressed by

\[
x_i = m_i x_0 \lambda^{(0)} + \varepsilon_i(\xi', x_0) \lambda^{(4)},
\]

where \( \lambda^{(0)} \) denotes the order of aberrations. The direction cosines of the ray bundle emerging from the exit pupil intersection position \( \xi' \), and intersecting the first image plane at the position \( x_1 \) are given by

\[
s'_{01} = \left( m_i x_0 \lambda^{(0)} + \varepsilon_i(\xi', x_0) \lambda^{(4)} \right) - \frac{\xi'}{\eta'}.
\]

Thus, the BSDF of the exit pupil at the exit pupil position \( \xi' \) is given by
\[ BSDF_i(s'_i, s_0 | \xi'_i) = \delta(s'_i - s_{0i}) = \delta \left( \frac{x_2}{n'} - \frac{m_i x_0 \lambda^{(0)} + E_i(\xi'_i | x_0) \lambda^{(4)}}{n'} \right). \]  

(7.49)

Note that, due to the aberration, the BSDF of the exit pupil does depend on the exit pupil position. Using the delta function property, the function is rewritten as

\[ BSDF_i(s'_i, s_0 | \xi'_i) = \frac{n'^2}{z'^2} \delta \left( \frac{x_2}{z'_2} - \frac{m_i x_0 \lambda^{(0)} - E_i(\xi'_i | x_0)}{z'_2} \right). \]

(7.50)

Substituting the exit pupil BSDF in the presence of aberrations to Eq.(7.46), the irradiance function in the first image plane for the aberrated single optical surface is given by

\[ E(x_i) = \int d^2 x_0 \int d^2 \xi'_i \frac{1}{z'_2} \delta \left( \frac{x_1}{z'_1} - \frac{x_0}{z_1} - \frac{E_i(\xi'_i | x_0)}{z'_1} \right) E_0(x_0). \]

(7.51)

Using the identity

\[ \delta \left( \frac{x}{a} - b + c \right) = \int d^2 x' \delta \left( \frac{x'}{a} - b \right) \delta(x - x' + ac), \]

(7.52)

Eq.(7.51) turns into

\[ E(x_i) = \int d^2 \xi'_i \left[ \frac{1}{z'^2} \delta \left( \frac{x'_i}{z'_1} - \frac{x_{obj}}{z_1} \right) \right] \left[ \int d^2 \xi'_i \delta \left( \frac{x_i - x'_i - E_i(\xi'_i | x_{obj})}{z'_1} \right) \right]. \]

(7.53)

The variable \( x'_i \) in Eq.(7.53) can be interpreted as the Gaussian image position of the point object and the term in the second square brackets is the geometrical PSF shown in Section 7.2.1.
Thus, the irradiance function in the first image plane for the aberrated single optical surface is simply rewritten by

$$E(x_i) = \delta(x_i - m_{i,0}) \otimes PSF_G(x_1 \mid x_{obj}). \quad (7.54)$$

In conclusion, for an aberrated single optical surface, the irradiance in the image plane is represented by the convolution of the geometrical PSF and the delta function located in the Gauss image point of the point source.

Let us move to another situation where there are two coaxial optical surfaces having aberrations respectively. The irradiance distribution function at the second image plane is given by

$$E_2(x_2) = \int d^2\xi_2 d^2x_0 d^2x_1 \left[ \frac{1}{r_1^2} BSDF_1(s_1', s_0 \mid \xi_1') \frac{1}{r_2^2} BSDF_2(s_2', s_1' \mid \xi_2') \right] E_0(x_0). \quad (7.55)$$

Using the exit pupil BSDF for an aberrated optical surface, the above equation is written by

$$E_2(x_2) = \int d^2\xi_2 d^2x_0 \frac{1}{z_1^2 z_2^2} \int d^2x_1 \left[ \delta\left( x_i - \frac{x_0\lambda^{(0)}}{z_1} - \frac{x_1(\xi_1', x_0)\lambda^{(4)}}{z_1'} \right) \times \right.$$

$$\left. \delta\left( x_i - \frac{x_2\lambda^{(0)}}{z_2} - \frac{x_2(\xi_2', x_1)\lambda^{(4)}}{z_2'} \right) \right] E_0(x_0). \quad (7.56)$$

However, under the fourth order approximation, the square brackets term in Eq.(7.56) should be modified because the integration in the square brackets contains higher order aberration term than the fourth order one. If we calculate the integration, the ray intersection position at the second image plane is
\[ x_2 = m_2 \left( m_1 x_0 \lambda^{(0)} + \xi_1 (x_0 | x_1) \lambda^{(4)} \right) + \xi_2 \left[ m_1 x_0 \lambda^{(0)} - \xi_1 (x_0 | x_1) \lambda^{(4)} \right] \lambda^{(4)}. \]  

(7.57)

Since we should ignore higher order aberrations than the fourth order one, the second term in the above equation is simplified to

\[ \xi_2 \left[ m_1 x_0 \lambda^{(0)} - \xi_1 (x_0 | x_1) \lambda^{(4)} \right] \lambda^{(4)} = \xi_2 \left( \xi_2^2 | m_1 x_0 \right) \lambda^{(4)}. \]  

(7.58)

Now, the \( x_2 \) can be expressed by

\[ x_2 = m_2 m_1 x_0 + m_2 \xi_1 (x_0 | x_1) + \xi_2 \left( \xi_2 | m_1 x_0 \right). \]  

(7.59)

Or equivalently,

\[ \delta(x - x_2) = \int d^2 x_1 \delta(x_1 - m_1 x_0) \delta(x - m_2 x_1 - m_2 \xi_1 | x_0) - \xi_2 (x_0 | m_1 x_0), \]  

(7.60)

which does not contain higher order term. Therefore, the square brackets term in Eq.(7.44) is replaced by

\[ \text{[bracket]} = \int d^2 x_1 \delta \left( \frac{x_1}{z_1} - \frac{x_0}{z_1} \right) \delta \left( \frac{x_2}{z_2} - \frac{x_1}{z_2} - \frac{\xi_1 (x_0 | x_1)}{z_2} \right). \]  

(7.61)

where

\[ \xi (\xi_2 | x_0) = m_2 \xi_1 (\xi_1 | x_0) + \xi_2 (\xi_2 | m_1 x_0). \]  

(7.62)
Recall that $\xi'_1$ is the function of $\xi'_2$ with simple pupil magnification. Using Eq.(7.61), Eq.(7.56) becomes

$$
E(x_2) = \int d^2x_0 d^2x_2 \int d^2x_1 \left[ \frac{1}{z_1^2} \delta \left( \frac{x_1}{z_1} - \frac{x_{obj}}{z_1} \right) \frac{1}{z_2^2} \delta \left( \frac{x'_2}{z_2} - \frac{x_1}{z_2} \right) \right] \times
$$

$$
\left[ \int d^2\xi_2' \delta \left( x_2 - x'_2 - \mathbf{e}(\xi'_2 | x_{obj}) \right) \right] E_0(x_0) \quad \text{(7.63)}
$$

If we replace the dummy variables $x_2$ and $x'_2$ to each other, the Eq.(7.63) is written by

$$
E(x'_2) = \int d^2x_0 d^2x_2 T(x_2 | x_0) PSF_G(x'_2 - x_2 | x_0) \delta(x_0 - x_{obj}). \quad \text{(7.64)}
$$

where the function $T(x_2 | x_0)$ is

$$
T(x_2 | x_0) = \int d^2x_1 \left[ \prod_{n=1}^{2} \frac{1}{z_n^2} \delta \left( \frac{x_n}{z_n} - \frac{x_0}{z_n} \right) \right]. \quad \text{(7.65)}
$$

and the geometrical PSF is given by

$$
PSF_G(x'_2 - x_2 | x_0) = E_p \int d^2\xi_2' \delta \left( x'_2 - x_2 - \mathbf{e}(\xi'_2 | x_0) \right) \quad \text{(7.66)}
$$

Finally, the irradiance distribution function at the second image plane of the two coaxial aberrated optical surfaces becomes

$$
E(x_2) = \delta \left( x_2 - m_2 m_{obj} \right) \otimes PSF_G(x_2 | x_{obj}), \quad \text{(7.67)}
$$

which is, again, a simple convolution of the geometrical PSF with a delta function centered on the Gaussian image position of the point source.
Let us now consider an optical imaging system with \( n \) aberrated optical surfaces. The irradiance distribution function in the \( n \)th image plane is given by

\[
E_n(x_n) = \int \frac{d^2 \xi_n d^2 x_0}{r_j^2} \prod_{j=1}^{n} \frac{1}{z_j^2} BSDF_j(s_j', s_j' \mid \xi_j') E_0(x_0).
\]  

(7.68)

Using the exit pupil BSDF for an aberrated single optical surface, the above equation becomes

\[
E(x_n) = \int \frac{d^2 \xi_n d^2 x_0}{r_j^2} \prod_{j=1}^{n-1} \frac{1}{z_j^2} \delta \left( \frac{x_j - x_0}{z_j} - \frac{x_{j-1} - \xi_j}{z_{j-1}} \right) E_0(x_0),
\]  

(7.69)

where a simplified notation \( \xi_j = \xi_j(\xi_j' \mid x_{j-1}) \) is used. Counting only up to fourth order aberrations, the square brackets term is simplified to

\[
[\text{bracket}] = \int d^2 x_1 \cdots d^2 x_{n-1} \prod_{j=1}^{n-1} \frac{1}{z_j^2} \delta \left( \frac{x_j - x_0}{z_j} \right) \delta \left( \frac{x_{j-1} - \xi_j}{z_{j-1}} \right) \delta \left( \frac{x_{n-1} - \xi_j}{z_{n-1}} \right).
\]  

(7.70)

where

\[
\xi = \xi(\xi_n' \mid x_0) = \sum_{j=1}^{n} m_{j+1} \cdots m_n \xi_j(\xi_j' \mid m_1 \cdots m_{j-1} x_0),
\]  

(7.71)

which is the identical equation to Eq.(7.4). Using Eq.(7.70), Eq.(7.69) turns into

\[
E(x_n) = \int \frac{d^2 \xi_n d^2 x_0}{r_j^2} \prod_{j=1}^{n-1} \frac{1}{z_j^2} \delta \left( \frac{x_j - x_0}{z_j} \right) \times \delta \left( \frac{x_{n-1} - \xi_j}{z_{n-1}} \right) \delta \left( \frac{x_n - \xi_j}{z_n} \right) E_0(x_0)
\]  

(7.72)

(\text{continues})
After some manipulation, the above equation becomes

\[ E(x_n') = \int d^2 x_0 d^2 x_n T(x_n | x_0) PSF_G(x_n' - x_n | x_0) \delta(x_0 - x_{obj}). \]  \(7.73\)

where the function \( T(x_n | x_0) \) is given by

\[ T(x_n | x_0) = \int d^2 x_1 \cdots d^2 x_{n-1} \prod_{j=1}^{n} \frac{1}{z_j'} \delta\left( \frac{x_j}{z_j'} - \frac{x_{j-1}}{z_{j-1}} \right) = \delta(x_n - mx_0). \]  \(7.74\)

and the geometrical PSF is

\[ PSF_G(x_n' - x_n | x_0) = E_p \int d^2 x_n' \delta(x_n' - x_n - e(x_n' | x_0)). \]  \(7.75\)

Finally, the irradiance distribution function in the \( n \)th image plane of the system with \( n \) coaxial aberrated optical surfaces is obtained as

\[ E(x_n) = \delta(x_n - mx_{obj}) \otimes PSF_G(x_n | x_{obj}), \]  \(7.76\)

which is a simple convolution of the geometrical PSF with a delta function centered on the Gaussian image position of the point source.

### 7.4 Geometrical PSF in the Presence of both Aberration and Surface Scattering

In the previous sections, we assumed there was no surface scattering from the optical surfaces. All the ray trajectories, therefore, were determined by the Snell’ law and it causes the delta
function in the \textit{BSDF of the exit pupil}. However, if there is surface scatter from optical elements, the ray looks that it does not obey Snell’s law macroscopically and the BSDF of the exit pupil is not such a simple delta function.

Assuming that the optical surfaces have isotropic and homogeneous roughness over the whole of its area, and that only small incident and small scattered angles are considered, the ASF of a surface can be considered as shift invariant relative to the incident angle as discussed in Chapter 4.3. Under this assumption, the ASF can be represented by

\[
ASF(s_x, s_0) = ASF(s_x - s_0),
\]

where \(s_0\) is the direction cosines of the incident wave or ray bundle and \(s_x\) is that of scattered wave or ray bundle.

A single scattered ray can be understood as a ray deviated from the specular direction, carrying the energy dictated by the BSDF of the exit pupil. From the point of view of wavefront analysis, the angle of deviation from a certain reference direction can be expressed in terms of a wavefront error which can be expressed as

\[
W'(x_0, \xi', s') = W(x_0, \xi') + W_s(s', \xi'),
\]

where \(W_s\) is the wavefront error of a scattered ray relative to the specular ray. Under the assumption of shift invariance of the BSDF of the exit pupil to the incident angle, the wavefront error of a scattered ray is a linear function of the pupil coordinates. Specifically, its mathematical form is the same as a wavefront tilt error, which is
\[ W''(x_0, \xi', s'_s) = W(x_0, \xi') - n s'_s \cdot \xi', \quad (7.79) \]

where \( n \) is the index of refraction of the space after the refraction or reflection. Using Eq. (7.79), the formalism developed in previous sections will be used in a straight-forward manner to analyze image degradation due to scattered light in the absence of aberrations. In the following subsection, the formalism will be extended to compute image degradation due to surface scatter in the presence of aberrations.

### 7.4.1 Aberration Free Optical System

Figure 7-6 shows the transverse ray aberration in the \( j \)th image plane due to the surface scatter in the \( j \)th optical surface.

![Figure 7-6: Ray aberration due to scattering in a single optical surface.](image)

185
The ray aberration, \( n_j z_j' s_{sj} \), is caused by the scattered ray bundle from the optical surface with the scattered direction cosines \( s_{sj} \) relative to the specular direction and the power carried by the ray bundle is given by surface ASF. However, for an observer located in the image plane, it looks that the ray aberration caused by the scattered ray from the exit pupil with the scattered direction cosines \( s'_{sj} \) relative to the specular direction and the power carried by the ray bundle is given by the exit pupil BSDF. From the geometry, these two direction cosines have the relation \( z_j' s_{sj} = r_j s'_{sj} \) and the fractional infinitesimal power carried by this ray bundle is

\[
\frac{dP}{P} = ASF_j(s_j) d^2 s_j = \frac{r_j^2}{z_j'^2} \ ASF_j \left( \frac{r_j}{z_j'} s_j \right) d^2 s_j' .
\]  

(7.80)

Thus, the BSDF of the \( j \)th exit pupil is written as

\[
BSDF_j(s'_j, s'_{j-1} | \xi'_j) = \frac{r_j'^2}{z_j'^2} \ ASF_j \left( \frac{r_j'}{z_j'} s'_j, s'_{j-1} | \xi'_j \right) .
\]  

(7.81)

Note that the surface normal is assumed to be \( \hat{z} \) under the small angle approximation. If the specular ray bundle has direction cosines \( s_{0j} \) and it emerges from the position \( \xi'_j \) in the \( j \)th exit pupil, the direction cosine of the specular ray is

\[
s'_{0j} = \frac{m_j x_{j-1} - \xi_j}{r'_j}.
\]  

(7.82)

On the other hand, if the scattered ray bundle intersects the \( j \)th image plane at the position \( x_j \), its direction cosines is given by
\[ s'_j = s'_{0j} + s'_{cj} = \frac{x_j - \xi_j}{r'_j}. \] (7.83)

Since the surface ASF is assumed to be shift invariant with respect to incident direction cosine, the BSDF of the \( j \)th exit pupil is obtained by

\[
BSDF_j(s'_j, s_{j-1} | \xi'_j) = \frac{r'^2_j}{z'^2_j} ASF_j\left(\frac{r'_j}{z'_j} (s'_j - s'_{0j}) | \xi'_j \right) = \frac{r'^2_j}{z'^2_j} ASF_j\left(\frac{x_j - m_j x_{j-1}}{z'_j} \right). \] (7.84)

Let us now consider an optical imaging system with \( n \)-surfaces. The irradiance distribution function in the \( n \)th image plane for an \( n \)-surface optical imaging system of a point source is

\[
E_n(x_n) = \int d^2 \xi'_n d^2 x_0 \cdots d^2 x_{n-1} \left[ \prod_{j=1}^{n} \frac{1}{r'_j} BSDF_j(s'_j, s'_{j-1} | \xi'_n) \right] E_0(x_0), \] (7.85)

Using the BSDFs of the exit pupils in the presence of the scattering from optical surfaces, the above equation becomes

\[
E(x_n) = \int d^2 \xi'_n d^2 x_0 \left[ \int d^2 x_1 \cdots d^2 x_{n-1} \prod_{j=1}^{n} \frac{1}{z'^2_j} ASF_j\left(\frac{x_j - x_{j-1}}{z'_j} \right) \right] E_0(x_0). \] (7.86)

After painstaking algebra, mainly due to the magnifications of the optical elements, Eq.(7.86) reduces to simpler form of

\[
E(x_n) = PSF(x_n) \otimes \delta(x_n - m x_{obj}), \] (7.87)
where $PSF_s$ denotes the scattering PSF given by

$$PSF_s(x_n) = \frac{1}{d_n^2}ASF_n \left( \frac{x_n}{d_n} \right) \otimes \cdots \otimes \frac{1}{d_1^2}ASF_1 \left( \frac{x_n}{d_1} \right), \quad (7.88)$$

where the magnified conjugate distance is given by $d_j = z'h_j/h$, where $h_j$ is the marginal ray height at the $j$th optical surface, $h$ is the exit pupil height of the total optical system, and $z'$ is the image-side conjugate distance of the total optical system. If it is assumed that only the $j$th surface is rough, and all the other surfaces are perfectly smooth, Eq.(7.88) reduces to

$$PSF_s(x_n) = P_{inc} \frac{1}{d_j^2}ASF_j \left( \frac{x_n}{d_j} \right), \quad (7.89)$$

which is identical to Peterson’s result [107,108].

### 7.4.2 Aberrated Optical System

Let us now move to the optical imaging system of a series of $n$ coaxial aberrated rough optical surfaces. A ray bundle which is scattered at $j$th optical surface with the scattering direction cosines $s_{ij}$ relative to the specular direction intersects the $j$th image plane at the position $x_j$. The corresponding specular ray having direction cosine $s_{0j}$ emerges from the position $\xi_j'$ in the $j$th exit pupil. Since the ray intersection position of the specular ray in the $j$th image plane is given by $m_jx_{j-1} + \varepsilon_j$, the direction cosine of the specular ray is
\[ s'_{0j} = \frac{(m_j x_{j-1} + e_j) - \xi_j}{r'_j}. \]  
(7.90)

On the other hand, the direction cosine of the scattered ray bundle from the exit pupil is

\[ s'_j = s'_{0j} + s'_{sj} = \frac{x_j - \xi_j}{r'_j}. \]  
(7.91)

Since the ASF is assumed to be shift invariant with respect to the incident direction cosine, the BSDF of the $j$th exit pupil is obtained by

\[ BSDF_j(s'_j, s'_{j-1} | \xi'_j) = \frac{r'^2_j}{z'^2_j} \text{ASF}_j \left( \frac{r'_j}{z'_j} (s'_j - s'_{0j}) | \xi'_j \right) \]

\[ = \frac{r'^2_j}{z'^2_j} \text{ASF}_j \left( \frac{r'_j}{z'_j} \left( \frac{x_j}{r'_j} - \frac{m_j x_{j-1}}{r'_j} - \frac{e_j}{r'_j} \right) \right). \]  
(7.92)

With some algebra, the function is rewritten by

\[ BSDF_j(s'_j, s'_{j-1} | \xi'_j) = \frac{r'^2_j}{z'^2_j} \int d^2 s_j \text{ASF}_j(s_j) \delta \left( \frac{x_j}{z'_j} - \frac{x_{j-1}}{z_j} - s_j - \frac{e_j}{z'_j} \right). \]  
(7.93)

The irradiance distribution function in the $n$th image plane for $n$-surfaces optical imaging system is given by

\[ E_n(x_n) = \int d^2 \xi'_{n-1} d^2 x_0 \cdots d^2 x_{n-1} \left[ \prod_{j=1}^n \frac{1}{r'^2_j} BSDF_j(s'_j, s'_{j-1} | \xi'_j) \right] E_0(x_0), \]  
(7.94)

Using the calculated exit pupil BRDFs in the presence of surface scatter and aberration, the above equation is written by
\[
E(x_n) = \int d^2 x_n d^2 x_0 \left[ \int d^2 s_1 \cdots d^2 s_j \prod_{j=1}^{n} ASF_j(s_j) \right] 	imes \\
\left[ \int d^2 x_1 \cdots d^2 x_{n-1} \prod_{j=1}^{n} \frac{1}{z_j'} \delta \left( \frac{x_j - x_{j-1}}{z_j} - s_j - \frac{e_j}{z_j'} \right) \right] E_0(x_0) .
\] (7.95)

The first integration in the second square brackets term in the above equation is simplified to
\[
\int d^2 x_1 \prod_{j=1}^{2} \frac{1}{z_j'} \delta \left( \frac{x_j - x_{j-1}}{z_j} - s_j - \frac{e_j}{z_j'} \right) = \int d^2 x_1 \delta \left( x_1 - m_1 x_0 - z'_1 s_1 - e_1 \right) \delta \left( x_2 - m_2 x_1 - z'_2 s_2 - e_2 \right) .
\] (7.96)

The Eq.(7.96) implies that the ray intersect position at the second image plane is
\[
x_2 = m_2 \left( m_1 x_0 + z'_1 s_1 + e_1 (\xi'_1 \mid x_0) \right) + z'_2 s_2 + e_2 \left[ \xi'_2 \mid m_1 x_0 + z'_1 s_1 + e_1 (\xi'_1 \mid x_0) \right] .
\] (7.97)

In the square brackets term in the above equation, if we count up to fourth order aberration and ignore the fourth order aberration of scattered light, the square brackets term becomes
\[
e_2 \left[ \xi'_2 \mid m_1 x_0 + z'_1 s_1 + e_1 (\xi'_1 \mid x_0) \right] = e_2 (\xi'_2 \mid m_1 x_0) .
\] (7.98)

Therefore, the ray intersection position at the second image plane is given by
\[
x_2 = m_2 \left( m_1 x_0 + z'_1 s_1 \right) + z'_2 s_2 + m_2 e_1 (\xi'_1 \mid x_0) + e_2 (\xi'_2 \mid m_1 x_0) ,
\] (7.99)

or equivalently
\[
\delta (x - x_2) = \int d^2 x_1 \delta \left( x_1 - \frac{z'_1}{z_1} x_0 - z'_1 s_1 \right) \delta \left( x - \frac{z'_2}{z_2} x_1 - z'_2 s_2 - \frac{z'_2}{z_2} e_1 (\xi'_1 \mid x_0) - e_2 (\xi'_2 \mid m_1 x_0) \right) .
\] (7.100)

Thus the first integration in the second square brackets term in Eq.(7.95) becomes
\[ \int d^2x_i \delta \left( \frac{x_i - x_0}{z_i} - s_i \right) \delta \left[ \frac{x - x_0}{z_2} - s_2 - 1 \left( m_2 \epsilon_1 (\xi' \mid x_0) + \epsilon_2 (\xi' \mid m_2 x_0) \right) \right] \], \quad (7.101) \]

and, in similar manner, the second square brackets term in the Eq.(7.95) becomes

\[ \left[ \right] = \int d^2x_1 \cdots d^2x_{n-1} d^2x_n' \left[ \prod_{j=1}^{n-1} \frac{1}{z_j^2} \delta \left( \frac{x_j - x_{j-1}}{z_j} - s_j \right) \right] \delta \left( \frac{x_n' - x_{n-1}}{z_n} - s_j \right) \delta (x_n - x_n' - \epsilon), \quad (7.102) \]

where

\[ \epsilon (\xi' \mid x_0) = \sum_{j=0}^{n} m_{j+1} \cdots m_n \epsilon_j (\xi' \mid m_{j+1} \cdots m_1 x_0). \quad (7.103) \]

Substituting Eq.(7.102) into Eq.(7.95), the Eq.(7.95) turns into

\[ E(x_n') = \int d^2x_0 d^2x_n \left[ \int d^2x_1 \cdots d^2x_{n-1} \prod_{j=1}^{n-1} \frac{1}{z_j^2} \text{ASF}_j \left( \frac{x_j - x_{j-1}}{z_j} \right) \right] \times \left[ \int d^2x_n' \delta \left( x_n - x_n' - \epsilon (\xi' \mid x_0) \right) \right] E_0(x_0) \], \quad (7.104) \]

Finally, the irradiance distribution function in the nth image plane of an imaging system with n coaxial optical surfaces in the presence of both aberration and scattering is obtained by

\[ E(x_n) = \text{PSF}_s(x_n) \otimes \text{PSF}_G(x_n \mid x_{\text{obj}}) \otimes \delta (x_n - m x_{\text{obj}}). \quad (7.105) \]

Equation (7.105) is a simple convolution of the geometrical PSF with the scattering PSF, which is the main conclusion of this chapter. The scattering ‘effect’ and geometrical aberration ‘effect’
can be separately calculated, then their combined effect can be calculated approximately by their convolution.

7.5 Approximations and Assumptions

To obtain Eq.(7.105), on top of the small angle approximation, mainly an assumption and an approximation were used. The assumption is that the ASF is shift invariant with respect to incident angle for both small incident and scattering angles. This is not true in general as discussed in Chapter 4, however, for slightly rough surfaces, the ASF has been shown to be directly proportional to surface power spectrum density (PSD) function for small scattering and incident angles [62,69,70]. The above assumption is thus totally valid for smooth surfaces when the approximation \( \cos \theta_i \equiv \cos \theta_s \equiv 1 \) is valid. For moderately rough surfaces, which are our main interest, the assumption is not strictly true. Nevertheless, as discussed in Section 4.3, approximately the ASF can be considered as shift-invariant if both incident and scattered angle are small. Thus perhaps the departure from shift-invariant behavior could be calculated numerically to determine the range of parameters over which the shift-invariant behavior is approximately valid. Experimental validation may is another possible alternative.

The main approximation is specified in Eq.(7.98). In the equation, the aberration of the light is ignored if the light is once scattered. Thus, the ray trajectory predicted by the equation, if \( n_j z_j' s_j >\mathbf{e}_j \) (referred to as case 1), may not be considered to be valid. However, if \( n_j z_j' s_j <\mathbf{e}_j \)
or the order of the amount of $z''s_j$ is similar to the order of $\varepsilon_j$ (referred as case 2), the approximation gives close results. Thus, some ray intersection position at the $(j+1)$th image plane predicted by Eq.(7.98) corresponding to case 1 would not be valid, but some others corresponding to case 2 could be considered to be an acceptable approximation. From experimental observations, well-polished optical surfaces have exponentially decaying BSDFs (also ASFs) relative to the specular ray direction [93-98], and rays deviated far from the specular ray (case 1) carry an extremely small amount of radiant power compared to the power carried by a scattered ray near the specular direction (case 2). Thus, a ray whose trajectory is predicted with a relatively large amount of error carries very small amount of energy and a ray whose trajectory is predicted with a relatively small amount of error carries very large amount of energy. Thus, we consider the approximation (7.98) to be sufficiently valid for performing engineering calculations for state-of-the-art mirror surfaces.
In this chapter, image quality for the SUVI telescope is estimated for different wavelengths, square sizes of detector and incident field angles. The BRDFs of the mirror surfaces are obtained by using the GHS surface scatter theory introduced in Chapter 4 exhibiting excellent agreement to the rigorous method for fractal surfaces as shown in Chapter 5 and 6, but having a computable analytic formula which the rigorous IEM method does not for two-dimensional fractal surfaces. With the predicted BRDFs, the composite PSF in the presence of both surface scatter and aberration is calculated by applying the result of Chapter 7. Finally, image quality of the EUV two-mirror telescope is evaluated in terms of the fractional ensquared energy which is commonly used as an image quality requirement on many NASA astronomy programs.

In the following section, the image quality estimation of a Cassegrain type two mirror telescope which is very similar to the SUVI telescope is performed first. Using predicted BRDFs, the analytic method is applied and its results are compared to the irradiance distributions separately calculated by using non-sequential ray-tracing techniques supported by well-known optical simulation software. This comparison provides numerical validation of the analytic method. Then, in the followed sections, image quality predictions of the SUVI telescope are carried out with the same methodology.
8.1 Predicting Image Quality of a Cassegrain Type Telescope

Most commercially-available image analysis codes have the ability to calculate scattering behavior [72-75]. The software package named OSAC claims that it calculates image degradation from parameterized surface roughness data and its ability is shown through a research publication [115], but the functional form of the input surface PSD is limited to a few special cases. Typically, optical analysis software are able to calculate the irradiance distribution in the focal plane in the presence of surface scattering with given BRDF by tracing rays non-sequentially. However, performing non-sequential ray-tracing is a time-consuming process and it is difficult to analyze and gain insight concerning the dominant image degradation mechanisms. Thus, in this section, the image degradation of a Cassegrain type telescope is predicted using the simple analytic formula introduced in Chapter 7 and the results are compared to the ones obtained using commercial software with the same BRDFs. This validation enables us to apply the simple formula to the analysis of the SUVI telescope.

First, the optical prescription of a telescope is specified and the geometrical PSF is calculated using conventional ray tracing techniques. Then, from the measured surface metrology data of a mirror surface, the BRDF of the surface is predicted using the GHS surface scatter theory. The scattering PSF is calculated through numerical convolution of the predicted BRDFs, and the composite PSF for the telescope suffering surface scatter and conventional aberrations is predicted using the convolution of the geometrical and scattering PSFs as discussed in Chapter 7. Finally, the predicted PSF is compared to the Zemax prediction.
8.1.1 Geometrical PSF of the Telescope

Figure 8-1 illustrates the schematic layout of a Cassegrain type two-mirror telescope operating at EUV wavelengths [109].

![Schematic layout of a Cassegrain type two-mirror telescope](image)

Figure 8-1  Schematic layout of a Cassegrain type two-mirror telescope

Its effective focal length is $f=1750\,\text{mm}$ and entrance pupil diameter is $h=190\,\text{mm}$ with an obscuration ratio of $\epsilon=0.47$, and the stop is located at the first mirror. A mosaic detector array with $21\,\mu\text{m} \times 21\,\mu\text{m}$ pixel size is placed in its image plane. This telescope is intended to provide full solar disc images, requiring a field-of-view of $\pm0.5^\circ$. The telescope design is optimized to achieve similar geometrical spot sizes throughout the field of view as provided in Figure 8-2. The coefficient of the wavefront defocus error is $W_{020}=-30.35\lambda$ and all the five Seidel aberration coefficients are given by $W_{040}=0.42\lambda$, $W_{131}=-7.52\lambda$, $W_{222}=47.63\lambda$, $W_{220}=38.23\lambda$ and $W_{311}=2.61\lambda$ for the wavelength of $\lambda=9.4\,\text{nm}$. 
Figure 8-2: Geometrical spot diagram of the telescope for five different incident angles.

Figure 8-3: Calculated geometrical PSF for the Cassegrain type two-mirror telescope for 4×4 detector size (a) for normal incidence and (b) for 0.5 degrees of incident angle. The irradiance is plotted in logarithmic scale.
The geometrical PSF is expressed by a combination of delta functions but, for practical reasons, the sequential ray-tracing technique with the discrete sampling approach described in Chapter 7 is used to calculate geometrical PSF [71]. The calculated geometrical PSFs for the Cassegrain type two-mirror telescope for $4 \times 4$ detector pixel size image plane are plotted in Figure 8-3(a) at normal incidence and (b) at 0.5 degrees of incident angle. In the figure, the irradiance is plotted in logarithmic scale.

8.1.2 Predicted BSDF of a Mirror Surface

Regarding the roughness of the mirror surfaces, it is assumed that the statistical properties of the roughness of the two mirror surfaces are isotropic and homogeneous. Figure 8-4 shows the measured surface PSD of a mirror surface plotted in log-log scale. Four different measuring techniques are used because each technique has inherent spatial frequency limitations. A full aperture interferometer is used for measuring the surface PSD in the low spatial frequency region and a micro phase-measuring interferometer is used to measure the surface PSD in the mid spatial frequency region. Finally, to attain surface PSD values in the high spatial frequency domain, atomic force microscopy is used. The composite surface PSD is obtained by combining the results of these different sets of metrology data.

Since it is well known that most optical surfaces fabricated by conventional abrasive grinding and polishing techniques on ordinary amorphous glassy materials exhibit an inverse power-law surface PSD [93-98], the measured PSD data are fitted to an abc-function as illustrated in Figure
The three parameter values of this abc-function SUVI Spec PSD function are given by by
\(a=610\,\text{Å}\text{mm}\), \(b=120\,\text{mm}^{-1}\) and \(c=1.08\).

Figure 8-4: Measured surface PSD of a mirror surface of the Cassegrain type two mirror telescope. The PSD is measured from four different metrology instruments for different spatial frequency bands. An abc-function has been fitted to the experimental data to characterize the surface.

In Chapter 4, the analytic expression of the scattering BRDF is provided by GHS surface scatter theory and its special form for the two-dimensional surfaces having rotationally symmetric surface PSF is given by Chapter 6. Using the parameter values fitted to the measured surface PSD data, the two-dimensional BRDF is predicted by the GHS for the wavelength of \(\lambda=9.4\,\text{nm}\) at normal incidence and shown in Figure 8-5(a) in direction cosine space. For this normally incident light, the relevant rms roughness is calculated as \(0.0726\,\lambda\) and TIS value is predicted as
0.57, which means the surface is moderately rough at the wavelength. In addition, the renormalization constant is calculated as 1.47.

Figure 8-5: Predicted BRDF plotted in logarithmic scale at the wavelength of $\lambda=9.4nm$ for (a) normal incidence, (b) $\theta_i=20^\circ$, (c) $\theta_i=40^\circ$ and (d) $\theta_i=60^\circ$. The number on the color bar represents the log of the BRDF values.
In Figure 8-4(b), (c) and (d), the predicted BRDFs are illustrated, in direction cosine space, for the three different incident angles, 20°, 40° and 60°. The calculated relevant rms roughness values are 0.0725λ, 0.0723λ and 0.0718λ for those incident angles respectively. Compared to the normal incidence case, the rms roughness is not significantly changed due to the exponentially decaying surface PSD. The predicted TIS values at the three non-normal incidence cases are 0.51, 0.38 and 0.18 respectively and the total amount of scattered radiant power is predicted to decrease with increasing incident angle. In addition, the renormalization constants are calculated as 1.45, 1.40 and 1.35 for those three cases respectively. For the case of 20 degrees of the incident angle, although the BRDF is not symmetric about the y-axis, its behavior near specular direction looks quite symmetric about its specular direction even though the surface is moderately rough. However, this symmetric-like behavior breaks severely when the angle of incidence is further increased to 40 and 60 degrees.

8.1.3 Scattering PSF of the Telescope

The scattering PSF for the Cassegrain type two-mirror telescope is calculated using analytic method introduced in Chapter 7. Since the predicted TIS value, at λ=9.4nm and normal incidence for a single surface, is 0.56, it is expected that about 19% of the radiant energy resides in the specular beam and over 81% of the radiant energy is scattered for the two-mirror telescope [109]. At this wavelength, surface scatter is thus a very dominant image degradation mechanism, which is the situation where the scattering PSF plays an important role.
The object, which is the Sun, is located at an infinite distance from the telescope and the stop is located at the first mirror surface, then, the magnified conjugate distances are given by
\[ d_1 = f, \]
\[ d_2 = f \varepsilon, \]
where \( f \) is the effective focal length of the entire system, \( h \) is the entrance pupil diameter, and \( \varepsilon \) is the obscuration ratio. Using the analytic method described in Chapter 7, the scattering PSF is given by

\[
PSF_5(x) = \frac{1}{f^2 \varepsilon^2} BRDF_2 \left( \frac{x}{f \varepsilon} \right) \otimes \frac{1}{f^2} BRDF_1 \left( \frac{x}{f} \right). \tag{8.1}
\]

Figure 8-6: Calculated scattering PSF for the Cassegrain type two-mirror telescope for (a) 4x4 detector size and (b) 16x16 detector size at the wavelength of \( \lambda=9.4nm \). The irradiance is plotted in logarithmic scale.

With the assumption that the surface PSDs of the two mirrors of the telescope are identical, the scattering PSF for the two mirror telescope at the wavelength of \( \lambda=9.4nm \) is calculated and
shown in Figure 8-6(a) for 4×4 and (b) for 16×16 detector size image plane. In the figures, the irradiance values are scaled logarithmically.

The scattering PSF is rotationally symmetric and the function is also described in terms of a radial variable in the telescope focal plane as shown in Figure 8-7. The BRDFs are given by the summation of delta function corresponding to the coherent specular light and the scattering function corresponding incoherent scattered light. Thus Eq.(8.1) is represented as the sum of four components; specular-specular (PP), scatter-specular (SP), specular-scatter (PS) and scatter-scatter (SS) [110].

Figure 8-7: Rotationally symmetrical scattering PSD function at the wavelength of $\lambda=9.4\text{nm}$. The specular-specular component is omitted in this figure.

At the wavelength of $\lambda=9.4\text{nm}$, the fractional radiant powers contained in the four components are $P_{pp}=0.1892$, $P_{ps}=0.2458$, $P_{sp}=0.2458$ and $P_{ss}=0.3192$, where $P_{pp}$ denotes the fractional
radiant power contained in the PP component and so on. It is worth noting that less than 20% of the total energy reaching the focal plane will reside in the specular beam, or image core. Furthermore, almost 32% of the energy will reside in the scattered-scattered component. From the fact that the irradiance distribution of the Airy function decreases as an inverse power law with a slope of -3, whereas other components obey an inverse power law with a slope of approximately -2, one can expect that the contribution of the direct-direct component to the irradiance distribution is weak. Therefore, the scattered-scattered light is indeed the dominant component of the irradiance distribution at this very short wavelength. Clearly there is a strong need to be able to perform accurate predictions of image quality as degraded by surface scatter effects from real metrology data throughout the optical fabrication process.

8.1.4 Composite PSF of the Telescope

In this section, the composite PSF in the presence of aberrations and surface scatter for the Cassegrain type two-mirror telescope is calculated and compared to the irradiance distribution using non-sequential ray tracing method provided by well-known commercial software.

First, the situation where there is no aberration is considered. For the aberration free optical system, the composite PSF becomes simply the scattering PSF. Figure 8-8 shows the scattering PSF obtained using numerical convolution and the irradiance obtained from the commercially available ZEMAX [74] and ASAP [72] optical analysis codes.
Figure 8-8: Irradiance distribution predicted by three different methods: analytic approximation method (red asterisk), non-sequential ray tracing provided by Zemax (blue solid line) and ASAP (green dotted line). The irradiance values are normalized to the incident power collected to the entrance pupil.

The same design for the aberration-free two-mirror telescope and the same predicted BSDF data to both the primary and the secondary mirror are applied to the three methods. The three approaches are in excellent agreement for this application which does not satisfy the smooth-surface approximation.

When optical surfaces have aberrations, in Chapter 7, it is shown that the PSF is given by the convolution of the geometrical PSF and the scattering PSF. The logarithmically scaled composite PSF is calculated and shown in Figure 8-9(a) in the image plane with size 4×4 detector pixels for a field angle of 0.5°. In Figure 8-9(b), a logarithmically scaled irradiance distribution calculated using non-sequential ray tracing technique provided by Zemax is shown. Since the software does not provide a BRDF prediction, the two-dimensional predicted BRDFs by the GHS for several different angles of incidence are used as an input. To obtain reliable results, one billion rays are
non-sequentially traced with a Monte Carlo technique and the results are normalized to the total power arrived at the image plane when there is no scattering.

![Image](a)

![Image](b)

![Image](c)

![Image](d)

Figure 8-9: Point spread function by scattering and aberration for 0.5° incident field angle by (a) the convolution method, (b) Zemax, (c) contour map of (a), and (d) contour map of (b).
These two irradiance distributions are virtually indistinguishable by visual observations. In Figure 8-9(c) and (d), the corresponding contour maps of the irradiance distributions (or PSFs) are shown.

![Figure 8-10: Contour map of point spread function by scattering and aberration for 0.5° incident field angle by (a) the convolution method, (b) Zemax, (c) contour map of (a), and (d) contour map of (b).]
Again the two computational techniques produce virtually identical results, although the contour lines are somewhat smeared due to the Monte Carlo technique used in the non-sequential tracing of discrete rays. Note that the diameter of the Airy disk for full aperture at 9.4nm wavelength is about 0.21μm and the full width of a detector pixel is 21μm, thus diffraction effects are quite insignificant.

In Figure 8-10, the logarithmically scaled irradiance distributions obtained using the two methods and their corresponding contour maps in the image plane with the size of 16×16 detector pixel for the same field angle are presented. Again, the two computational techniques continue to provide almost indistinguishable results.

For estimating optical system performance, the fractional encircled energy is often used. However, in the case of an aberrated image, the PSF is no longer rotationally symmetric and square detector pixels are used, the fractional energy contained in a square (referred to as ensquared energy) is chosen for evaluating system performance. The ensquared energy plots centered on the Gaussian image of the point source for 0° and 0.5° of incident field angle are shown in Figure 8-11, and the ensquared energy for the aberration-free case is also plotted (dotted line) as a reference. Since the dynamic range of the plot is over four decades and the number of sampling points is restricted, five ensquared energy plots for different sizes of the image plane are superimposed. Figure 8-11 shows excellent agreement between the convolution result (solid line) and Zemax result (asterisks). Furthermore, the ensquared energies for the two aberrated cases approach the ensquared energy of the aberration-free case for positions far from the Gaussian image.
Figure 8-11: The ensquared energy for 0° and 0.5° incident field angle of the convolution method (solid line), Zemax (asterisk), and aberration free case (dotted line) centered on the Gaussian image of the point source.

Roughly speaking, aberrations are caused by macroscopic features (surface height deviations) from the ideal reference surface figure, and scattering is caused by microscopic features from the mean surface. Macroscopic (low spatial frequency) roughness contributes small angle ray deviations from the specular direction, and microscopic (high spatial frequency) roughness contributes large angle scattering from the specular direction. Thus, the existence of aberrations does not change the high-frequency scattering behavior as shown above in Figure 8-11.
8.2 Image Degradation Prediction for the SUVI Telescope

In this section, the image quality estimation for the Solar Ultra Violet Imager telescope is performed [116]. The SUVI telescope has the same geometrical figures to the Cassegrain type telescope described in previous sections. However it has two filters located in front of its two mirror surfaces respectively in order to observe the object with six different wavelengths. The schematic shape of the filter is shown in Figure 8-12.

![Figure 8-12: Schematic shape of the filter.](image)

The six wavelengths are given by 93.9Å, 131.2Å, 171.1Å, 195.1Å, 284.2Å, and 303.8Å and image quality prediction is needed for all these wavelengths of light. In this chapter, these six wavelengths are referred to as SUVI wavelengths. Due to the asymmetric shape of the filter, the aperture shape is not rotationally symmetric. With the defocus error and Seidel aberration
coefficients, the calculated geometrical PSFs for the different incident field angles are shown in Figure 8-13 for this special aperture.

![Direction cosine for incident light](image1)

(a) Direction cosine for incident light

(b) (c)

(d) (e) (f)

Figure 8-13: (a) Direction cosines of incident light and calculated geometrical PSFs for the SUVI telescope for 2×2 detector pixel size at (b) normal incidence, (c) \( \theta_i=0.5^\circ, \phi_i=90^\circ \) (d) \( \theta_i=0.5^\circ, \phi_i=45^\circ \), (e) \( \theta_i=0.5^\circ, \phi_i=0^\circ \) and (f) \( \theta_i=0.5^\circ, \phi_i=-45^\circ \). The irradiance is plotted in logarithmic scale.

The two measured surface metrology data sets characterizing each state-of-the-art EUV telescope mirrors are provided and they are shown in Figure 8-14(a) for the primary mirror.
surface and (b) for the secondary mirror surface. To remove unrealistic behavior of the measured PSD data, they are fitted to the combination of five abc-functions for the primary and secondary mirror surfaces respectively.

![Graph](Image)

**Figure 8-14:** The measured surface metrology data characterizing state-of-the-art EUV telescope mirror for (a) the primary and (b) the secondary mirror surface.

The $a$ parameters are $1.9 \times 10^2$, $2.0 \times 10^{-3}$, $1.6 \times 10^{-3}$, $2 \times 10^{-3}$, $3.0 \times 10^{-6}$, the $b$ parameters are $11$, $5.0 \times 10^{-3}$, $7.0 \times 10^4$, $7.0 \times 10^5$, $8.0 \times 10^{-6}$; and the $c$ parameters are $1.56$, $1.57$, $2.3$, $6$ and $5.5$ for the five abc-functions respectively for the primary mirror surface. And the $a$ parameters are $65$, $1.6$, $4.2 \times 10^{-3}$, $1.8 \times 10^{-3}$, $2.4 \times 10^4$, and the $b$ parameters are $1.4$, $9.3 \times 10^{-1}$, $7.0 \times 10^{-3}$, $6.0 \times 10^{-4}$, $6.0 \times 10^{-5}$ and the $c$ parameters are $7.6$, $1.37$, $2.2$ $2.5$ and $5.7$ for the five abc-functions respectively for the secondary mirror surface. The unit of $a$ parameter is given by [$A\cdot mm$], the unit of $b$ parameter is [$mm^{-1}$] and the $c$ parameter is a dimensionless quantity.
Figure 8-15: Predicted BRDFs plotted in logarithmic scale at (a) $\lambda=93.9\,\text{Å}$ (b) $303.8\,\text{Å}$ for the primary mirror surface and at (c) $\lambda=93.9\,\text{Å}$ and (d) $303.8\,\text{Å}$ for the secondary mirror surface at normal incidence. The number on the color bar represents the log of the BRDF values.

The BRDFs of the two mirror surfaces as predicted by the GHS surface scatter theory, for the six SUVI wavelengths and the four predicted two-dimensional BRDF distributions, for the shortest and longest SUVI wavelength and for primary and secondary mirror surfaces, are shown in
Figure 8-15 at normal incidence. In Figure 8-16(a), the six BRDFs of the primary mirror surface for the six SUVI wavelengths are superimposed and the six BRDFs of the secondary mirror surface are presented in Figure 8-16(b) at normal incidence.

Figure 8-16: Predicted BRDFs of (a) the primary mirror surface and (b) the secondary mirror surface for the six SUVI wavelengths at normal incidence.

For this normally incident light, the relevant rms roughness is calculated as $0.073\lambda$ and the TIS value is predicted as 0.57, which means the surface is quite rough at the shortest SUVI wavelength, and the relevant rms roughness is calculated as $0.023\lambda$ and TIS value is predicted as 0.08, which means the surface is moderately rough at the longest SUVI wavelength for the primary mirror surface. In addition, the relevant rms surface roughness values are calculated as 0.052, 0.040, 0.035 and 0.024 for the other four SUVI wavelengths and the corresponding TIS values are computed by 0.35, 0.22, 0.17 and 0.08 for the primary mirror surface. Thus the
surface cannot be considered a smooth surface. Similarly, for the second mirror surface, the relevant rms roughness values are calculated as 0.064, 0.046, 0.035, 0.031, 0.021 and 0.020 for the six SUVI wavelengths respectively and the corresponding TIS values are computed by 0.48, 0.28, 0.18, 0.14, 0.07 and 0.06. These TIS values reveal that the secondary surface is also moderately rough at the SUVI wavelengths.

The scattering PSFs are calculated with the predicted BRDFs at the six SUVI wavelengths. In Figure 8-17, the scattering PSFs at the shortest and the longest SUVI wavelengths are plotted in the image plane with $4 \times 4$ detector pixel size for the normal incidence and the incident field angle of 0.5° with $\phi = 0°$. At the shortest SUVI wavelength, it is predicted that about 22% of the radiant power resides in the specular beam and about 78% of the radiant power is scattered.

Figure 8-17: Calculated scattering PSF for the SUVI telescope (a) at $\lambda = 93.9\text{Å}$ and (b) at $303.8\text{Å}$ for $4 \times 4$ detector pixel size image plane. The irradiance is plotted in logarithmic scale.
Meanwhile, at the longest SUVI wavelength, it is predicted that about 86% of the radiant energy contained in the specular beam and about 14% of the radiant energy is scattered for the SUVI telescope.

Figure 8-18: Point spread function by surface scatter and aberrations for (a) normal incidence, (b) $\theta_i=0.5^\circ$ with $\phi_i=90^\circ$ at $\lambda=93.9\,\text{Å}$ and for (c) normal incidence and (d) $\theta_i=0.5^\circ$ with $\phi_i=90^\circ$ at $\lambda=303.8\,\text{Å}$. The values in color bar denote log of the irradiance.
Using the geometrical PSF and scattering PSF, the composite PSF in the presence of surface scatter and conventional aberrations for the SUVI telescope is calculated at the six SUVI wavelengths and for different field angles. In Figure 8-18, the logarithmically scaled composite PSFs in the image plane with the 4×4 detector pixels size for the normal incidence and the field angle 0.5° with \( \phi_i = 90° \) at the shortest and longest SUVI wavelengths are plotted.

![Composite PSFs](image)

Figure 8-19: Predicted fractional ensquared energy at normal incidence for the six SUVI wavelengths. The dotted lines denote required value.

The fractional ensquared energy plots centered on the Gaussian image of the point source at normal incidence for the six SUVI wavelengths are shown in Figure 8-19. The ensquared energy values are normalized to the total power collected by the full size of the detector for each wavelength and the size of the square are converted to arc-second which is the common unit in astronomy. The dotted curves denote the required ensquared energy values for the six different
image plane sizes. Comparing the curves for the shortest SUVI wavelength to one for the largest wavelength, the image quality is severely degraded by the surface scatter. The ensquared energy values are dramatically changed near the 1 arcsec because of the aberrations.

Figure 8-20: Predicted fractional ensquared energy for the six SUVI wavelengths at (a) $\theta_i=0.5^\circ$ $\phi_i=90^\circ$ (b) $\theta_i=0.5^\circ$ $\phi_i=45^\circ$ (c) $\theta_i=0.5^\circ$ $\phi_i=0^\circ$ and (d) $\theta_i=0.5^\circ$ $\phi_i=-45^\circ$. 

218
In Figure 8-20, the predicted fractional ensquared energy curves for the six SUVI wavelengths for different field angles are presented. Due to the aberration, the behavior near the Gaussian image points is quite different. But it is predicted that their behavior at the region far from the Gaussian image point is similar to each other.

In conclusion, the image quality of the SUVI telescope suffering both surface scatter and conventional aberrations is estimated in terms of PSF and ensquared energy for different wavelengths and different incident field angles. The conventional aberrations are a dominant image degrading factor at the longest SUVI wavelength, but the surface scatter is also curial factor at the shortest SUVI wavelength.
CHAPTER 9: CONCLUSION

At the beginning of this dissertation, we started from a situation where surface scatter is an important issue. A Cassegrain type telescope is to be launched to observe the Sun, and the main wavelengths of interest are EUV wavelengths (9~30nm). Due to the limitation of grinding and polishing technology, the two mirrors produce a large amount of scattered light which degrades the image quality severely at these wavelengths.

To analyze this image degradation due to surface scatter, we have reviewed some surface scattering models in detail. The Stratton-Chu formula is rigorously derived from the Maxwell’s equation and it gives exact answer to the surface scatter problem. However, the formula is not readily calculable except in very limited special cases because the formula is given by an integral equation. Therefore, to calculate the integral equation for two-dimensional random rough surfaces, we first turned to a rigorous numerical method called, in this dissertation, the integral equation method (IEM). But, with currently available computational technologies, it is practically impossible to obtain the scattering predictions accurately.

Analytic approximations were our alternative choice. Although they have their own limitations, they provide quite reliable scattering predictions within their valid domain. We have reviewed the three analytic approximations, the SPM, KA, and GHS, and we have investigated their region of validity to choose one of them for predicting image blurring due to surface scatter for the SUVI telescope.
We obtained the region of validity of the three approximate methods respectively by numerically comparing their predictions to the rigorous IEM predictions. First we performed the comparison work for one-dimensional random rough surfaces having Gaussian statistics and then we extended our comparison work to surfaces having fractal-like structure. In our numerical comparison, the GHS has a broader valid domain than the other two approximate methods for surfaces with fractal-like structure. Then, we performed the comparison, although it is done for limited cases, for two-dimensional random rough surfaces and found that the region of validity obtained using two-dimensional scattering problem can be applicable to three dimensional surface scatter phenomena.

Based on our comparison, the surface scatter prediction from the measured metrology data for the SUVI telescope mirrors have been calculated using the GHS surface scatter theory. However, none of surface scattering models provide scattering prediction with curved mean surface which is the actual shape of the SUVI mirror surface. Thus, the image quality, in terms of the PSF, should be separately evaluated. Some commercial software provides non-sequential ray tracing capabilities and it can be used for computing the PSF directly. However, we have analyzed the image degradation mechanism due to surface scatter and aberrations in an optical imaging system, and obtained a simple formula for the PSF for systems suffering from both surface scatter effects and conventional aberrations. Also we have demonstrated that the simple formula provides accurate results but takes much less calculation time compared to the time-consuming non-sequential ray tracing method.
Finally, with the GHS surface scatter theory and the simple formula of the PSF in the presence of surface scatter and aberrations, the PSFs of the SUVI telescope have been calculated for different wavelengths and field angles. The calculated PSFs are converted to ensquared energy curves which are commonly used as an image quality requirement on many NASA astronomy programs.
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