# Optimization Problem In Single Period Markets

2013

Tian Jiang
*University of Central Florida*

# OPTIMIZATION PROBLEM IN SINGLE PERIOD MARKETS

by

TIAN JIANG
B.A. Dongbei University of Finance and Economics, 2010

A thesis submitted in partial fulfilment of the requirements
for the degree of Master of Science
in the Department of Mathematics
in the College of Sciences
at the University of Central Florida
Orlando, Florida

Spring Term
2013

Major Professor: Jiongmin Yong

# ABSTRACT

There had been a number of researches that investigated on the security market without transaction costs. The focus of this research is in the area that when the security market with transaction costs is fair and in such fair market how one chooses a suitable portfolio to optimize the financial goal. The research approach adopted in this thesis includes linear algebra and elementary probability. The thesis provides evidence that we can maximize expected utility function to achieve our goal (maximize expected return under certain risk tolerance). The main conclusions drawn from this study are under certain conditions the security market is arbitrage-free, and we can always find an optimal portfolio maximizing certain expected utility function.

**Keywords:** portfolio optimization, arbitrage-free, transaction costs, utility function

This thesis is dedicated to my parents Minglong Jiang and Ying Huang, for giving birth to me and supporting me spiritually throughout my life.

## ACKNOWLEDGMENTS

Foremost, I am taking this opportunity to express my greatest gratitude and regards to my advisor Prof. Jiongmin Yong for his patience, motivation, enthusiasm and immense knowledge. His guidance helped me in all the time of writing my thesis. Without his help, this thesis cannot be completed. I would like to thank the rest of my thesis committee, Prof. Yuanwei Qi, Prof. Zhisheng Shuai, for their helpful suggestions and comments. I am grateful that Prof. Yuanchang Wang and Jingrui Sun gave me a lot of help when I was writing this thesis. In particular, I would like to show my gratitude to Prof. Xin Li for all his help and support.

# TABLE OF CONTENTS

# CHAPTER 1: INTRODUCTION

Suppose a person has an initial wealth, denoted by $V(0)$, a certain dollar amount, at the current moment $t = 0$. At a future time $t = 1$, the initial wealth will become $V(1)$. We now look at how the amount $V(1)$ will be.

(i) Leave the initial wealth $V(0)$ at home. By doing this, the amount will stay the same, i.e.,

$$V(1) = V(0).$$

(ii) Deposit the initial wealth $V(0)$ in a bank. Suppose the interest rate for this period of time is $r$. Then, at the end of this period, we have

$$V(1) = V(0)(1 + r).$$

(iii) Buy certain shares of a stock. Since the stock market is uncertain, we cannot predict the exact price of this stock at $t = 1$. To illustrate, let us assume there are just three possible outcomes: the stock price goes up $a\%$, stays the same and drops down $b\%$. Then the amount $V(1)$ will be:

$$V(1) = \begin{cases} V(0)(1 + a\%), & \text{if the stock price goes up } a\%, \\ V(0), & \text{if the stock price stays the same,} \\ V(0)(1 - b\%), & \text{if the stock price drops down } b\%. \end{cases}$$

(iv) Deposit a portion, $\lambda V(0)$, into a bank and use the rest $(1 - \lambda)V(0)$ to buy a stock. Where

$\lambda \in [0, 1]$. Then the possible V(1) will be

$$V(1) = \begin{cases} \lambda V(0)(1+r) + (1-\lambda)V(0)(1+a\%), & \text{if the stock price goes up } a\%, \\ \lambda V(0)(1+r) + (1-\lambda)V(0), & \text{if the stock price stays the same,} \\ \lambda V(0)(1+r) + (1-\lambda)V(0)(1-b\%), & \text{if the stock price drops down } b\%. \end{cases}$$

In real financial market, there are thousands of stocks, and people can buy or short these stocks. Also, the prices are uncertain in the future. A natural question is if it is possible to choose a portfolio so that the future wealth V(1) will be greater than that if he initial amount $V(0)$ is just deposited in the bank? How one can measure and control the risk?

## Related Literature

A large literature studies portfolio selection in the absence of transaction costs (for example, Pliska (1997)). Constantinides (1986) considered a single risky asset with transaction costs. Later, many treatments have been presented for problems with proportional costs. Optimal portfolio selection given transaction costs is a complex problem. Even with only two assets, solving for optimal strategy in a continuous time model involves a free boundary problem (for example, Davis and Norman (1990) and Liu and Loewenstein (2002)). When there are more securities, the multi-asset continuous time model has been solved only in the extreme case of uncorrelated returns and constant absolute risk aversion (Liu (2004)) or with numerical or heuristic approximations (Leland (2000) or Donohue and Yip (2003)). Most of them applied mean-variance theory which was originated by Markowitz (1952), in which the variance is included in the program objective. Some studies state that maximizing the expected return and at the same time taking care of minimizing the risk can be approximately achieved by maximizing the expected utility for some proper utility function. In Yong's lecture note (2007), the optimal portfolio is determined by using utility function in the market with proportional costs. Under certain constraint (transaction costs are large enough), the existence of optimal solution is proved. In this article, fairness of the market with transaction costs is studied, which was not discussed in the literature. And we study the optimization problems with both proportional and fixed costs in a more comfortable condition, and also try to find out the exact optimal portfolio through maximizing the expected utility function under such comfortable condition.

## A Market Model

Consider two moments: $t = 0, 1$. At $t = 0$, one could expect to have $m > 1$ possible situations at time $t = 1$. We call them events, denoted by $\omega_1, \omega_2, ..., \omega_m$, and define

$$\Omega \doteq \{\omega_1, \omega_2, ..., \omega_m\},$$

which is the set of all possible situations at $t = 1$. At $t = 0$ we cannot tell which event will happen at $t = 1$. A standard approach is to use probability to describe and measure the possibilities of these events. We introduce a function: $P : \Omega \to (0, 1)$, having the following property:

$$\sum_{j=1}^{m} P(\omega_j) = 1.$$

We call $P$ a probability on $\Omega$. Suppose there is a bank account with the interest rate $r$, we denote

$$B(1) = 1 + r,$$

which represents the bank account price at $t = 1$, if the bank account price at $t = 0$ is normalized to $B(0) = 1$. Thus, if $z_0$ dollars is deposited at time $t = 0$, then $z_0 B(1)$ dollars will be received at time $t = 1$. There are two important features of the amount deposited in the bank: (i) The interest rate $r$ is known at $t = 0$, so the amount at $t = 1$ is known at $t = 0$. (ii) The interest rate $r > 0$, so the amount received at $t = 1$ will be greater than the initial amount deposited at $t = 0$. These two can be referred to as deterministic and riskless. Next, suppose there are $n$ stocks in the market, whose prices are certain at current time $t = 0$. We denote

$$S(0) \equiv (S_1(0), S_2(0), ..., S_n(0))^T \in \mathbb{R}^n, \tag{1.1}$$

where $S_i(0)$ is the price of the $i$-th stock at $t = 0$. Viewing at time $t = 0$, the price of $i$-th stock at $t = 1$ is a random variable, and it depends on which event will actually happen. Thus the price of the $i$-th stock at $t = 1$ should be:

$$S_i(1) \equiv (S_i(1, \omega_1), S_i(1, \omega_2), ..., S_i(1, \omega_m))^T \in \mathbb{R}^m, \tag{1.2}$$

where $S_i(1, \omega_j)$ is the price of $i$-th stock at $t = 1$ if event $\omega_j$ happens. We denote:

$$\begin{aligned} S(1) &\equiv (S_1(1), S_2(1), ..., S_n(1)) \\ &\equiv (S(1, \omega_1), S(1, \omega_2), ..., S(1, \omega_m))^T \\ &\equiv \begin{pmatrix} S_1(1, \omega_1) & S_2(1, \omega_1) & \cdots & S_n(1, \omega_1) \\ S_1(1, \omega_2) & S_2(1, \omega_2) & \cdots & S_n(1, \omega_2) \\ \vdots & \vdots & \ddots & \vdots \\ S_1(1, \omega_m) & S_2(1, \omega_m) & \cdots & S_n(1, \omega_m) \end{pmatrix} \in \mathbb{R}^{m \times n}, \end{aligned} \tag{1.3}$$

where $S(1, \omega_j)$ is the prices of these stocks if event $\omega_j$ happens. Now, we suppose an investor who enters the market with $z_0$ dollars deposited in the bank and $z_i$ shares of the $i$-th stock at $t = 0$.

Then he has a initial wealth:

$$V(0) = V_0 = z_0 + \sum_{i=1}^{n} z_i S_i(0) = z_0 + S(0)^T z, \tag{1.4}$$

where $z = (z_1, z_2, ..., z_n)^T \in \mathbb{R}^n$. We call $Z \equiv (z_0, z_1, z_2, ..., z_n)^T \in \mathbb{R}^{n+1}$ a portfolio. Suppose this investor holds such a portfolio between $t = 0$ and $t = 1$. Then, viewing at $t = 0$, the total wealth at $t = 1$ will be:

$$V(1, \omega, z) \stackrel{\Delta}{=} z_0 B(1) + \sum_{i=1}^{n} z_i S_i(1, \omega), \quad \omega \in \Omega, \tag{1.5}$$

which is a random variable. Also, using (1.4), (1.5) can be rewritten as:

$$V(1, \omega, V_0, z) = [V_0 - S(0)^T z] B(1) + \sum_{i=1}^{n} z_i S_i(1, \omega). \tag{1.6}$$

Using (1.1)–(1.3) we may write (1.6) as follows:

$$\begin{aligned} V(1, V_0, z) &= \begin{pmatrix} V(1, \omega_1, V_0, z) \\ \vdots \\ V(1, \omega_m, V_0, z) \end{pmatrix} \\ &= [V_0 - S(0)^T z] B(1) \mathbf{1} + S(1) z \\ &= V_0 B(1) \mathbf{1} + [S(1) - B(1) \mathbf{1} S(0)^T] z, \end{aligned} \tag{1.7}$$

where $\mathbf{1} = (1, 1, ..., 1)^T \in \mathbb{R}^m$. In what follows, we call $\{(\Omega, P), S(0), S(1), r\}$ a market without transaction costs, denoted by $\mathbf{M_0}$.

## Fairness of the Market Without Transaction Costs

Intuitively, a market is not fair if someone could make a risk-free profit with zero initial wealth. In another word, if some one comes to the market with nothing at hand, by borrowing some money at $t = 0$ and investing it, at $t = 1$, he/she could get some positive amount after paying back the borrowed amount together with the interest, no matter what event happens, then the market is not fair. Therefore, a market is fair, if the above described thing does not exist. To rigorously study the fairness of the market, we now introduce some relevant notations.

**Definition 1.1.** *Let $M_0 \equiv \{(\Omega, P), S(0), S(1), r\}$ be given.*

*(i) The law of one price holds for the market $\mathbf{M_0}$ if any pairs $(V_0, z), (\overline{V_0}, \overline{z}) \in \mathbb{R} \times \mathbb{R}^n$ of initial wealth and portfolio satisfying:*

$$V(1, \omega, V_0, z) = V(1, \omega, \overline{V_0}, \overline{z}), \quad \forall \omega \in \Omega, \tag{1.8}$$

*must lead to $V_0 = \overline{V_0}$.*

*(ii) Portfolio $\overline{z} \in \mathbb{R}^n$ is said to be dominant over portfolio $z \in \mathbb{R}^n$ if for some initial wealth $V_0 \in \mathbb{R}$,*

$$V(1, \omega, V_0, \overline{z}) > V(1, \omega, V_0, z), \quad \forall \omega \in \Omega. \tag{1.9}$$

*When the above $\overline{z}$ and $z$ exist, we say that the market $\mathbf{M_0}$ admits dominant strategies. Otherwise, we say that the market has no dominant strategies.*

*(iii) Portfolio $z \in \mathbb{R}^n$ is called an arbitrage opportunity in $\mathbf{M_0}$ if the following holds:*

$$\begin{cases} V(1, \omega, 0, z) \geq 0, & \forall \omega \in \Omega, \\ V(1, \omega_j, 0, z) > 0, & \textit{for some } \omega_j \in \Omega. \end{cases} \tag{1.10}$$

*When the above $z$ exists, we say that $\mathbf{M_0}$ admits arbitrage opportunities. Otherwise, we say that the market is arbitrage-free, or has no arbitrage.*

We also have some criteria helping us to determine whether the market has law of one price, dominant trading strategy or arbitrage opportunities. (The theorem below is cited from [1]).

**Theorem 1.2.** *Let market $\mathbf{M_0}$ be given.*

*(i) Law of one price holds for $\mathbf{M_0}$ if and only if there exists a vector $\mu \in \mathbb{R}^m$ such that:*

$$\mathbf{1}^T \mu = 1, \tag{1.11}$$

*and*

$$B(1)S(0) = [S(1)]^T \mu. \tag{1.12}$$

*(ii) Market $\mathbf{M_0}$ has no dominant strategies if and only if there exists a $\mu \in \overline{\mathbb{R}^m_+}$, such that:* (1.11)–(1.12) *hold, where*

$$\overline{\mathbb{R}^m_+} \triangleq \{x \in \mathbb{R}^m | x \equiv (x_1, x_2, ..., x_m)^T, x_i \geq 0, 1 \leq i \leq m\}.$$

.

*(iii) Market $\mathbf{M_0}$ is arbitrage-free if and only if there exists a $\mu \in \mathbb{R}^m_+$, such that:* (1.11)–(1.12) *hold, where*

$$\mathbb{R}^m_+ \triangleq \{x \in \mathbb{R}^m | x \equiv (x_1, x_2, ..., x_m)^T, x_i > 0, 1 \leq i \leq m\}.$$

It is not hard to see that any $\mu \in \overline{\mathbb{R}^m_+}$ satisfying (1.11) defines a probability measure. Let us denote $\mu(\cdot)$ by the following:

$$\mu(\omega_j) = \mu_j \tag{1.13}$$

Note that for any $\mu \in \mathbb{R}^m_+$ satisfying (1.11) the induced probability $\mu(\cdot)$ defined by (1.13) is equivalent to $P(\cdot)$, in the sense that $\mu(\omega_j) = 0$ if and only if $P(\omega_j) = 0$.

From the above theorem, we can easily get the following results which gives the relationship among the law of one price, dominant trading strategies and arbitrage opportunities: (The proposition below is cited from [1]).

**Proposition 1.3.** *For market* $\mathbf{M_0}$*, the following implications hold:*

*Market* $\mathbf{M_0}$ *is arbitrage-free* $\Rightarrow$ *market* $\mathbf{M_0}$ *has no dominant strategies* $\Rightarrow$ *the law of one price holds for* $\mathbf{M_0}$.

Here, we should notice that the converse of the above proposition is not true. The following are some examples show that the converse does not hold:

**Example 1.4.** Let

$$\begin{cases} m = 2, n = 1, r = 1, \\ S_1(0) = 10, S_1(1, \omega_1) = 12, S_1(1, \omega_2) = 10. \end{cases}$$

$B(1)S(0) = 20$, and $[S(1)]^T = (12, 10)$. Then from (1.11) and (1.12), we get:

$$\begin{cases} \mu_1 + \mu_2 = 1 \\ 12\mu_1 + 10\mu_2 = 20 \end{cases}$$

It is clear that $\mu = (5, -4)$, and $\mu \in \mathbb{R}^2$ but $\mu \notin \overline{\mathbb{R}^2_+}$. According to Theorem 1.2, we know that in this market law of one price holds, but it has dominant strategies. In fact, we note that (by (1.7)):

$$\begin{aligned} V(1,\omega,V_0,z_1) &= V_0B(1)+z_1[S_1(1,\omega)-S_1(0)B(1)] \\ &= 2V_0+Z_1[S_1(1,\omega)-20]=\begin{cases} 2V_0-8z_1, & \omega=\omega_1 \\ 2V_0-10z_1, & \omega=\omega_2 \end{cases} \end{aligned}$$

Therefore, if we choose portfolio $z_1<0$, $z_1$ is always a dominant strategy over portfolio 0. From the above, we can have a market where law of one price holds, admitting dominant strategies.

**Example 1.5.** Let

$$\begin{cases} m=2, n=1, r=1, \\ S_1(0)=10, S_1(1,\omega_1)=20, S_1(1,\omega_2)=10. \end{cases}$$

Then from (1.11) and (1.12), we get:

$$\begin{cases} \mu_1+\mu_2=1 \\ 20\mu_1+10\mu_2=20 \end{cases}$$

It is clear that $\mu=(1,0)$, and $\mu\in\overline{\mathbb{R}^2_+}$ but $\mu\notin\mathbb{R}^2_+$. According to theorem 1.2, we know that in the market has no dominant trading strategies, but has arbitrage opportunities. In fact, we note that (by (1.7)):

$$\begin{aligned} V(1,\omega,0,z_1) &= z_1[S_1(1,\omega)-S_1(0)B(1)] \\ &= z_1[S_1(1,\omega)-20]=\begin{cases} 0, & \omega=\omega_1 \\ -10z_1, & \omega=\omega_2 \end{cases} \end{aligned}$$

Hence, any $z_1<0$ is an arbitrage opportunity, but there are not dominant trading strategies.

To be complete, we present the following example which shows that the law of one price might fail in $\mathbf{M_0}$.

**Example 1.6.** Let

$$\begin{cases} m = 2, n = 1, r = 1, \\ S_1(0) = 10, S_1(1, \omega_1) = S_1(1, \omega_2) = 12. \end{cases}$$

Then for initial wealth $V_0 \in \mathbb{R}$ and $z_1 \in \mathbb{R}$, we have:

$$\begin{aligned} V(1, \omega, V_0, z_1) = & \quad B(1)[V_0 - S_1(0)z_1] + S_1(1, \omega)z_1 \\ = & \quad 2V_0 + [12 - 20]z_1 = 2V_0 - 8z_1. \end{aligned}$$

Consequently, by choosing

$$z_1 = \frac{2V_0 - \lambda}{8}$$

for any $\lambda \in \mathbb{R}$ one has

$$V(1, \omega, V_0, z_1) = \lambda,$$

which is independent of $V_0$. This means that in this market, the law of one price fails.

# CHAPTER 2: FAIRNESS OF THE MARKET WITH TRANSACTION COSTS

In reality, expenses, called transaction costs, incur when people buy or sell securities. Because the transaction costs are always positive, they diminish returns. If we do not take into account the transaction costs in real life, a seemingly profitable portfolio would cause serious loss when the transaction costs are high or we have a very big trading volume. So it is necessary to discuss a market with transaction costs. Here, we assume that a market just only have brokers commissions which include proportional transaction cost and fixed transaction cost (or entrance fee).

## Market Model with Transaction costs

Let a market $\mathbf{M}$ be given. Suppose an investor has an initial position $(z_0, z) \in \mathbb{R} \times \mathbb{R}^n$. The corresponding market value of the position is

$$V_0(z_0, z) = z_0 + S(0)^T z, \tag{2.1}$$

where $z_0$ is the amount deposited in the bank, and $z = (z_1, z_2, ...z_n)^T$ with $z_i$ being the share number of the $i$-th stock. Viewing at $t = 0$, this portfolio in the future moment $t = 1$ will have market value

$$V(1, z_0, z) = z_0 B(1)\mathbf{1} + S(1)z \tag{2.2}$$

If at $t = 0$, a transaction is made, so that the position becomes $(\overline{z}_0, z+y)$, where $y = (y_1, y_2, ..., y_n)^T$ with $y_i$ being the transacted share number of the $i$-th stock, then under self-financing condition [1],

[1] A portfolio is self-financing if there is no exogenous infusion or withdrawal of money; the purchase of a new asset must be financed by the sale of an old one.

it is necessary that

$$\overline{z}_0 = z_0 - S(0)^T y - \sum_{i=1}^{n} (\lambda_i S_i(0)\,|y_i| + c_i \mathbf{1}_{\{y_i \neq 0\}}), \tag{2.3}$$

where $\lambda_i \in [0,1)$ is the transaction cost rate for the $i$-th stock and $c_i \in [0,\infty)$ is the fixed cost for the $i$-th stock when we have a transaction on this stock. We can rewrite (2.3):

$$\overline{z}_0 = z_0 - S(0)^T y - \|y\|_\lambda - \left\langle c, \mathbf{1}_{\{y\}} \right\rangle, \tag{2.4}$$

where

$$\|y\|_\lambda = \sum_{i=1}^{n} \lambda_i S_i(0)\,|y_i|, \quad c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \quad \mathbf{1}_{\{y\}} = \begin{pmatrix} \mathbf{1}_{\{0\}^c}(y_1) \\ \vdots \\ \mathbf{1}_{\{0\}^c}(y_1) \end{pmatrix}, \quad \{0\}^c = \mathbb{R}\backslash\{0\}.$$

Then the market value of this new portfolio at $t = 1$ becomes:

$$\begin{aligned} V(1, \overline{z}_0, z + y) &= \overline{z}_0 B(1)\mathbf{1} + S(1)(z + y) \\ &= [z_0 - S(0)^T y - \|y\|_\lambda - \left\langle c, \mathbf{1}_{\{y\}} \right\rangle] B(1)\mathbf{1} + S(1)(z + y) \qquad (2.5) \\ &\equiv V(1, z_0, z, y). \qquad (2.6) \end{aligned}$$

Componentwise, we have

$$\begin{aligned} V(1, \omega, \overline{z}_0, z + y) &= [z_0 - S(0)^T y - \|y\|_\lambda - \left\langle c, \mathbf{1}_{\{y\}} \right\rangle] B(1) + S(1, \omega)(z + y) \qquad (2.7) \\ &\equiv V(1, \omega, z_0, z, y), \qquad \forall \omega \in \Omega. \qquad (2.8) \end{aligned}$$

Hereafter, we denote

$$\begin{aligned}
\mathbf{M}(\lambda, c) &= \{\mathbf{M_0}, \lambda, c\} \\
&\equiv \{(\Omega, P), S(0), S(1), r, \lambda, c\} \qquad (2.9)
\end{aligned}$$

to be the market with transaction costs. Thus, by identifying $\mathbf{M}_0 = \{\mathbf{M}_0, 0, 0\}$, we have $\mathbf{M}(0, 0) = \mathbf{M}_0$.

Fairness of the Market with Transaction Costs

The following proposition is from [1].

**Proposition 2.1.** *In the market* $\mathrm{M}_0$*, the following are equivalent:*

*(i) Market* $\mathrm{M}_0$ *admits dominant strategies.*

*(ii) There exists a portfolio* $z \in \mathbb{R}^n$*, such that*

$$V(1, \omega, 0, z) \equiv (-S(0)^T z)B(1) + \sum_{i=1}^{n} z_i S_i(1, \omega) > 0, \quad \forall \omega \in \Omega. \qquad (2.10)$$

The above proposition tells us that if there exists a portfolio $z \in \mathbb{R}$, such that

$$\begin{aligned}
[\Delta S(\omega_j)]z &\equiv [S(1, \omega_j) - S(0)^T B(1)]z \\
&= V(1, \omega_j, 0, z) > 0, \quad \text{for} \quad 1 \leq j \leq m. \qquad (2.11)
\end{aligned}$$

where $\Delta S(\omega_j) = S(1, \omega_j) - S(0)^T B(1)$, then the market $\mathrm{M}_0$ admits a dominant trading strategy. So if a market has dominant trading strategies, one, starting with zero initial wealth, can earn a positive profit by choosing this dominant strategy no matter which event will happen in the future.

This is risk-less.

Similarly, in the market $\mathbf{M}(\lambda, c)$, if one, starting with zero initial wealth, can always earn a positive profit in the future without taking any risk, we say the market $\mathbf{M}(\lambda.c)$ admits dominant trading strategies. Mathematically, we have:

**Definition 2.2.** *A portfolio $y \in \mathbb{R}^n$ is said to be a dominant trading strategy in $\mathbf{M}(\lambda, c)$, if*

$$V(1, \omega_j, 0, 0, y) > 0, \quad \textit{for all} \quad 1 \leq j \leq m. \tag{2.12}$$

*When the above $y$ exists, we say that the market $\mathbf{M}(\lambda, c)$ admits dominant trading strategies. Otherwise, we say there is no dominant trading strategies in this market.*

From (2.5), it is not hard to see (2.12) is equivalent to:

$$[\Delta S(\omega_j)]y > [\|y\|_\lambda + \langle c, \mathbf{1}_{\{y\}} \rangle]B(1) \quad \text{for} \quad 1 \leq j \leq m. \tag{2.13}$$

From Definition 1.1, we see that when the market $\mathbf{M}_0$ has arbitrage opportunities, one, starting with zero initial wealth, will not have any risk of losing money, and when some events happen, he/she will have a positive profit by choosing the arbitrage opportunity as his/her portfolio. Here, in the market $\mathbf{M}(\lambda, c)$ , we have the modified definition of arbitrage opportunity.

**Definition 2.3.** *A portfolio $y \in \mathbb{R}^n$ is said to be an arbitrage opportunity in $\mathbf{M}(\lambda, c)$, if*

$$\begin{cases} V(1, \omega, 0, 0, y) \geq 0, & \forall \omega \in \Omega, \\ V(1, \omega_j, 0, 0, y) > 0, & \textit{for some } \omega_j \in \Omega. \end{cases} \tag{2.14}$$

*When the above $y$ exists, we say that the market $\mathbf{M}(\lambda, c)$ admits arbitrage opportunities. Otherwise, we say the market is arbitrage-free.*

From (2.5), it is not hard to see that (2.14) is equivalent to:

$$\begin{cases} [\Delta S(\omega)]y \geq [\|y\|_\lambda + \langle c, \mathbf{1}_{\{y\}} \rangle]B(1) \quad , & \forall \omega \in \Omega, \\ [\Delta S(\omega_j)]y > [\|y\|_\lambda + \langle c, \mathbf{1}_{\{y\}} \rangle]B(1) \quad , & \text{for some } \omega_j \in \Omega. \end{cases} \tag{2.15}$$

We have the following result:

**Proposition 2.4.** *Suppose* $\mathbf{M}(\lambda, c)$ *has no dominant trading strategies (or is arbitrage-free), then the property remains in* $\mathbf{M_0}$.

*Proof.* Suppose a market $\mathbf{M}(\lambda, c)$ admits a dominant trading strategy $y \in \mathbb{R}^n$. Then (2.13) holds, which is the same as:

$$[\Delta S(\omega_j)]y > [\|y\|_\lambda + \langle c, \mathbf{1}_{\{y\}} \rangle]B(1) > 0, \quad \text{for} \quad 1 \leq j \leq m. \tag{2.16}$$

Clearly $y$ is also a dominant trading strategy in $\mathbf{M_0}$. The proof for the case of arbitrage-free is similar. □

Various Conditions for the Market being Arbitrage-free

**Proposition 2.5.** *In the market* $\mathbf{M}(\lambda, c)$*, suppose*

$$|E[\Delta S_i]| \leq \lambda_i S_i(0) B(1), \quad \forall 1 \leq i \leq n. \tag{2.17}$$

*Then* $\mathbf{M}(\lambda, c, )$ *is arbitrage-free.*

*Proof.* Suppose (2.17) holds. If $y \in \mathbb{R}^n$ is an arbitrage opportunity in $\mathbf{M}(\lambda, c)$, then for each

$1 \leq j \leq m$,

$$\begin{aligned} V(1, \omega_j, 0, 0, y) &= \Delta S(\omega_j) y - \sum_{i=1}^{n} (\lambda_i S_i(0)|y_i| + c_i \mathbf{1}_{\{y_i \neq 0\}}) B(1) \\ &= \sum_{i=1}^{n} (\Delta S_i(\omega_j) y_i - \lambda_i S_i(0) B(1)|y_i| - c_i \mathbf{1}_{\{y_i \neq 0\}} B(1)) \geq 0, \end{aligned}$$

and the strict inequality holds for some $j_0$. Thus,

$$\begin{aligned} 0 &< E[V(1, 0, 0, y)] \\ &= \sum_{i=1}^{n} \{E[\Delta S_i] y_i - \lambda_i S_i(0) B(1)|y_i| - c_i \mathbf{1}_{\{y_i \neq 0\}} B(1)\} \\ &\leq \sum_{i=1}^{n} \{|E[\Delta S_i]||y_i| - \lambda_i S_i(0) B(1)|y_i| - c_i \mathbf{1}_{\{y_i \neq 0\}} B(1)\} \\ &= \sum_{i=1}^{n} \{[|E[\Delta S_i]| - \lambda_i S_i(0) B(1)]|y_i| - c_i \mathbf{1}_{\{y_i \neq 0\}} B(1)\} < 0, \end{aligned}$$

which is a contradiction. Thus, $\mathbf{M}(\lambda, c)$ is arbitrage-free. $\square$

**Proposition 2.6.** *Let $c \in \mathbb{R}_+^n$, suppose there exists a $\overline{\omega} \in \Omega$ such that*

$$|\Delta S_i(\overline{\omega})| \leq \lambda_i S_i(0) B(1), \quad \textit{for each } i = 1, 2, ..., n. \tag{2.18}$$

*Then the market $\mathbf{M}(\lambda, c)$ is arbitrage-free.*

*Proof.* Suppose (2.18) holds for some $\overline{\omega} \in \Omega$. Then for any $y \in \mathbb{R}^n$ we have:

$$\begin{aligned} V(1,\overline{\omega},0,0,y) &= \sum_{i=1}^{n} \{\Delta S_i(\overline{\omega}) y_i - \lambda_i S_i(0) B(1) |y_i| - c_i \mathbf{1}_{\{y_i \neq 0\}} B(1)\} \\ &\leq \sum_{i=1}^{n} \{|\Delta S_i(\overline{\omega})||y_i| - \lambda_i S_i(0) B(1) |y_i| - c_i \mathbf{1}_{\{y_i \neq 0\}} B(1)\} \\ &= \sum_{i=1}^{n} \{[|\Delta S_i(\overline{\omega})| - \lambda_i S_i(0) B(1)] |y_i| - c_i \mathbf{1}_{\{y_i \neq 0\}} B(1)\} < 0. \end{aligned}$$

Thus there does not exist a $y \in \mathbb{R}^n$, that satisfies (2.16). So the market $\mathbf{M}(\lambda, c)$ is arbitrage-free.

□

**Proposition 2.7.** *In the market $\mathbf{M}(\lambda, c)$, if for every $(\epsilon_1, \epsilon_2, ..., \epsilon_n) \in \{-1, 1\}^n$, there exists an $\overline{\omega}(\epsilon_1, \epsilon_2, ..., \epsilon_n) \in \Omega$, such that:*

$$\max\{\epsilon_1 \Delta S_1(\overline{\omega}) - \lambda_1 S_1(0) B(1), ..., \epsilon_n \Delta S_n(\overline{\omega}) - \lambda_n S_n(0) B(1)\} < 0 \tag{2.19}$$

*Then the market $\mathbf{M}(\lambda, c)$ is arbitrage-free.*

*Proof.* For any $y \in \mathbb{R}^n$, by (2.19), there exists an $\overline{\omega} \in \Omega$, such that

$$\begin{aligned} V(1,\overline{\omega},0,0,y) &= \sum_{i=1}^{n} \{\Delta S_i(\overline{\omega}) y_i - \lambda_i S_i(0) B(1) |y_i| - c_i \mathbf{1}_{\{y_i \neq 0\}} B(1)\} \\ &= \sum_{y_i > 0} \{\Delta S_i(\overline{\omega}) y_i - \lambda_i S_i(0) B(1) y_i - c_i \mathbf{1}_{\{y_i \neq 0\}} B(1)\} \\ &\quad + \sum_{y_i < 0} \{\Delta S_i(\overline{\omega}) y_i + \lambda_i S_i(0) B(1) y_i - c_i \mathbf{1}_{\{y_i \neq 0\}} B(1)\} \\ &< \sum_{y_i \neq 0} [|\Delta S_i(\overline{\omega})| - \lambda_i S_i(0) B(1)] |y_i| \leq 0. \end{aligned}$$

Thus the market $\mathbf{M}(\lambda, c)$ is arbitrage-free. □

**Proposition 2.8.** *If the market* $\mathbf{M}(\lambda, c)$ *is arbitrage-free, then for each* $i = 1, 2, ..., n$*, there at least exists a* $j = 1, 2, ..., m$ *depending on* $i$*, such that:*

$$\Delta S_i(\omega_j) < \lambda_i S_i(0)B(1) + c_i B(1). \tag{2.20}$$

*Proof.* Suppose there exists a stock $i$, such that:

$$\Delta S_i(\omega_j) \geq \lambda_i S_i(0)B(1) + c_i B(1), \quad \text{for} \quad 1 \leq j \leq m. \tag{2.21}$$

Then taking $y = (0, 0, ..., y_i, 0, ..., 0) \in \mathbb{R}^n$, where $y_i > 0$. Because we have:

$$\begin{aligned} V(1, \omega_j, 0, 0, y) &= \sum_{i=1}^{n} \{\Delta S_i(\omega_j) y_i - \lambda_i S_i(0) B(1) y_i - c_i \mathbf{1}_{\{y_i \neq 0\}} B(1)\} \\ &= \Delta S_i(\omega_j) y_i - \lambda_i S_i(0) B(1) y_i - c_i B(1) \\ &> \Delta S_i(\omega_j) y_i - \lambda_i S_i(0) B(1) y_i - c_i y_i B(1) > 0. \end{aligned}$$

Thus such $y$ is an arbitrage opportunity, a contradiction. □

**Theorem 2.9.** *Let* $\lambda_i > 0$*,* $c_i > 0$ *for each* $i = 1, 2, ..., n$*. The market* $\mathbf{M}(\lambda, c)$ *is arbitrage-free if and only if*

$$\sup_{\|\eta\|=1} [\min_{1 \leq j \leq m} \Delta S(\omega_j)\eta - \widehat{\lambda}^T \eta^+ B(1)] \leq 0, \tag{2.22}$$

*where*

$$\widehat{\lambda} = (\lambda_1 S_1(0), \lambda_2 S_2(0), ..., \lambda_n S_n(0))^T, \tag{2.23}$$

*and*

$$\eta^+ = (|\eta_1|, |\eta_2|, ..., |\eta_n|)^T, \quad \forall \eta = (\eta_1, \eta_2, ..., \eta_n) \in \mathbb{R}^n. \tag{2.24}$$

*Proof.* First of all, the market is arbitrage-free if and only if for any $y \in \mathbb{R}^n \setminus \{0\}$, there exists an

$j \in \{1, 2, ..., m\}$, depending on $y$ such that

$$[\Delta S(\omega_j)]y < \sum_{i=1}^{n}(\lambda_i S_i(0)\,|y_i| + c_i \mathbf{1}_{\{y\neq 0\}})B(1). \tag{2.25}$$

Now for any $y \in \mathbb{R}^n \setminus \{0\}$, $\frac{y}{\|y\|} \in \partial B_1(0) \equiv \{\eta \in \mathbb{R}^n |\, \|\eta\| = 1\}$. Thus by (2.20), we have some $1 \leq j \leq m$, such that:

$$\begin{aligned} [\Delta S(\omega_j)]y = \|y\|\,[\Delta S(\omega_j)]\frac{y}{\|y\|} &\leq \|y\|\,\lambda^T(\frac{y}{\|y\|})^+ B(1) \\ &= \|y\|_\lambda B(1) \\ &< (\|y\|_\lambda + \left\langle c, \mathbf{1}_{\{y\}} \right\rangle)B(1) \\ &\equiv \sum_{i=1}^{n}(\lambda_i S_i(0)\,|y_i| + c_i \mathbf{1}_{\{y\neq 0\}})B(1). \end{aligned}$$

Thus, the market is arbitrage-free.

Conversely, if the market is arbitrage-free, then for any $\eta \in \mathbb{R}^n$ with $\|\eta\| = 1$, and $\alpha > 0$, we have $j_\alpha \in \{1, 2, ...., m\}$, such that

$$\alpha[\Delta S(\omega_j)]\eta < \alpha\widehat{\lambda}^T\eta^+ B(1) + c_i \mathbf{1}_{\{y\neq 0\}}B(1). \tag{2.26}$$

Thus, dividing by $\alpha$ and sending $\alpha \to \infty$, we may assume that $j_\alpha = j$ (along a sequence, if necessary) and

$$[\Delta S(\omega_{j_\alpha})]\eta \leq \widehat{\lambda}^T\eta^+ B(1). \tag{2.27}$$

Hence,

$$\min_{1\leq j\leq m}[\Delta S(\omega_j)] \leq \widehat{\lambda}^T\eta^+ B(1), \quad \forall \eta \in \partial B_1(0). \tag{2.28}$$

Then, (2.22) follows. $\square$

Let us look at an interesting special case of the above theorem.

**Corollary 2.10.** *When* $n = 1$ *and* $m > 1$, (2.22) *is equivalent to the following:*

$$\min\{\sigma_1\eta, \sigma_2\eta, ..., \sigma_m\eta\} \le \lambda S(0)B(1), \quad \eta = \pm 1, \tag{2.29}$$

*where*

$$\sigma_j = \Delta S(\omega_j), \quad 1 \le j \le m. \tag{2.30}$$

*The above is equivalent to*

$$\min\{\sigma_1, \sigma_2, ..., \sigma_m\} \le \lambda S(0)B(1), \quad \max\{\sigma_1, \sigma_2, ..., \sigma_m\} \ge -\lambda S(0)B(1). \tag{2.31}$$

# CHAPTER 3: OPTIMIZATION PROBLEMS IN SINGLE PERIOD MARKETS

In a market $\mathrm{M}(\lambda, c)$, the return is defined to be the difference between the final wealth and the initial wealth. Since, viewing at current time at $t = 0$, the final wealth in the future at $t = 1$ is uncertain, we need use expected return to measure the uncertain profit. People regard such kind of uncertainty as a risk. Risk is an uncertainty of gain-loss in some future time. Variance of the return rate is one of the common risk measurements.

When a rational person enters a security market, he/she always hopes to maximize the expected return and minimize the risk. We call this behavior risk aversion. Risk aversion is the reluctance of a person to accept a bargain with an uncertain payoff rather than another bargain with a more certain, but possibly lower, expected payoff. For example, a person who is very risk-averse would like to deposit his/her money into the bank with a risk-less but low interest rate, rather than buy some stocks with high expected return, but embedded a chance of losing value. Our goal of choosing portfolio is to maximize the expected return with a risk tolerance level, or minimize the risk for a given expected return level. These two problems are dual each other in some sense.

## Utility Function

We now introduce a function $u(\cdot)$ as follows

(i) strictly increasing:

$$u(x) > u(y), \quad \forall x, y \in \mathcal{D}(u), \quad x > y, \tag{3.1}$$

where, $u : \mathcal{D}(u) \subseteq \mathbb{R} \longrightarrow \mathbb{R}, \mathcal{D}(u)$ is an interval of form $(a, \infty)$, for some $a \in \mathbb{R}$, called the domain of u.

(ii) marginal decrease:

$$u(x+\Delta)-u(x)\le u(y+\Delta)-u(y),\quad \forall x,y\in\mathcal{D}(u),y<x,\Delta>0. \tag{3.2}$$

We can also express such properties in the following way when $u\in\mathcal{C}^2$

$$u'(x)\ge 0,u''(x)\le 0,\qquad \forall x\in[\mathcal{D}(u)]^o\equiv \text{the interior of } \mathcal{D}(u). \tag{3.3}$$

We call such a function a utility function, which can be used to measure people's satisfaction. For example, $u(x)=\log x$ and $u(x)=\sqrt{x}$ are utility functions. Besides these properties, utility function also has some convenient properties for explaining people's risk aversion. Since the shape of utility function is concave, the difference between $u(x)$ and $u(x-y)$ is greater than the difference between $u(x+y)$ and $u(x)$, for $y>0$. Thus, the greater the $y$, the greater the risk or uncertainty, and the more satisfaction is reduced by the difference between $u(x)$ and $u(x-y)$ than satisfaction added by the difference between $u(x+y)$ and $u(x)$. For example, if a people's utility function is $u(x)=\log(x)$, and he has half chance to earn $100+x$ dollars and another half chance to earn $100-x$ dollars, where $x$ is positive and the expected earn $100$ is fixed. Then we want to maximize the expected utility function, e.g.

$$\max_{x\in(0,100)}\{\frac{1}{2}\log(100-x)+\frac{1}{2}\log(100+x)\}.$$

It is not hard to check that when $x=0$ the expected utility function is maximized. And $x=0$ means the risk is minimized. In more general case, If the expected return is fixed, say $W$, and totally we have $m$ events, say $\omega_1,\omega_2,...,\omega_m$. The probability for event $\omega_j$ happens is $P_j$ and the earn when event $\omega_j$ happens is $W + x_j$. Since the expected return is $W$, we have:

$$\begin{aligned} P_1(W+x_1) + P_2(W+x_2) + ... + P_m(W+x_m) &= W \\ P_1x_1 + P_2x_2 + ... + P_mx_m &= 0 \end{aligned} \tag{3.4}$$

Our goal is to maximize the expected utility function $E[u(x)] = E[\log(x)]$, e.g.

$$\begin{cases} \text{max} & P_1 \log(W+x_1) + P_2 \log(W+x_2) + ... + P_m \log(W+x_m) \\ \text{subject to} & P_1x_1 + P_2x_2 + ... + P_mx_m = 0 \end{cases} \tag{3.5}$$

Using Lagrange method:

$$\begin{aligned} Let \quad F = P_1 \log(W+x_1) + P_2 \log(W+x_2) + ... + P_m \log(W+x_m) \\ +\lambda(P_1x_1 + P_2x_2 + ... + P_mx_m) \end{aligned} \tag{3.6}$$

Solve:

$$\begin{cases} F_{x_1} = \frac{P_1}{W+x_1} + \lambda P_1 = 0 \\ ... \\ F_{x_m} = \frac{P_m}{W+x_m} + \lambda P_m = 0 \\ F_\lambda = P_1x_1 + P_2x_2 + ... + P_mx_m = 0 \end{cases} \tag{3.7}$$

Then we have:

$$\begin{cases} x_1 = -\frac{1}{\lambda} - W \\ ... \\ x_m = -\frac{1}{\lambda} - W \\ P_1x_1 + P_2x_2 + ... + P_mx_m = 0 \end{cases} \tag{3.8}$$

Plug $x_1, x_2, .., x_m$ into the last equation, we get:

$$\begin{aligned} -\frac{P_1}{\lambda} - \frac{P_2}{\lambda} - ... - \frac{P_m}{\lambda} &= W \\ -\frac{1}{\lambda} &= W \end{aligned}$$

So $x_1 = x_2 = ... = x_m = 0$, and $(0, 0, ..0)$ is the only critical point. It is not hard to check at this point, the expected utility function has maximum value. When $x_1, x_2, ...x_m$ are all zero, we know the uncertainty is minimized (Variance is zero). It is similar to check $u(x) = \sqrt{x}$.

Thus, maximizing the expected return for certain risk, or minimizing the risk for certain expected return can be achieved by maximizing the expected utility function $\log x$ or $\sqrt{x}$.

## Optimization Problems with Transaction Costs for One Stock

Let us now consider the case with transaction costs, we use the utility function below:

$$\begin{cases} u(x) &= \sqrt{x}, \quad x \geq 0, \\ u(x) &= -\infty, \quad x < 0. \end{cases} \tag{3.9}$$

This utility function can describe a risk-averse people who does not want to take any chance to make him/her in debt. If this happens, his/her utility will become $-\infty$.

In the case of one stock and $m$ events, we denote the expected utility function

$$\begin{aligned} f(y) &= E[u(V(1, z_0, z, y))] \\ &= P_1\sqrt{(z_0 - S(0)y - \lambda S(0)|y| - c\mathbf{1}_{y\neq 0})B(1) + S(1,\omega_1)(z+y)} \\ &\quad +P_2\sqrt{(z_0 - S(0)y - \lambda S(0)|y| - c\mathbf{1}_{y\neq 0})B(1) + S(1,\omega_2)(z+y)} + ... \\ &\quad +P_m\sqrt{(z_0 - S(0)y - \lambda S(0)|y| - c\mathbf{1}_{y\neq 0})B(1) + S(1,\omega_m)(z+y)}, \end{aligned}$$

where $z_0$ is the initial amount deposited in the bank, $z$ is the initial share number of this stock, $y$ is the number of shares transacted at $t = 0$, $S(0)$ is the current price of this stock at $t = 0$ and $S(1, \omega_j)$ is the stock price at $t = 1$ when $j$-th event happens. We can rewrite $f(y)$ in the form of below:

$$\begin{aligned} f(y) &= E[u(V(1, z_0, z, y))] \\ &= P_1\sqrt{(\Delta S(\omega_1)y - \lambda S(0)B(1)|y| + z_0B(1) + S(1, \omega_1)z - c\mathbf{1}_{\{y\neq 0\}}B(1)} \\ &\quad + P_2\sqrt{(\Delta S(\omega_2)y - \lambda S(0)B(1)|y| + z_0B(1) + S(1, \omega_2)z - c\mathbf{1}_{\{y\neq 0\}}B(1)} + ... \\ &\quad + P_m\sqrt{(\Delta S(\omega_m)y - \lambda S(0)B(1)|y| + z_0B(1) + S(1, \omega_m)z - c\mathbf{1}_{\{y\neq 0\}}B(1)}, \end{aligned}$$

where $\Delta S(\omega_j) = S(1, \omega_j) - S(0)B(1)$. In reality, we can safely assume $z_0 > c$, because if a person's tolerance is no debt, then he/she must have a certain amount of initial wealth, at least enough to pay the transaction costs. If we assume that before transaction the investor did not short any stocks ($z > 0$), then under condition $z_0 > \sum_{i=1}^{n} c_i$, one has

$$K_j \equiv z_0B(1) + S(1, \omega_j)z - c\mathbf{1}_{\{y\neq 0\}}B(1) > 0, \quad \forall 1 \leq j \leq m. \tag{3.10}$$

So $f(y)$ becomes

$$\begin{aligned} f(y) &= E[u(V(1, z_0, z, y))] \\ &= P_1\sqrt{\Delta S(\omega_1)y - \lambda S(0)B(1)|y| + K_1} \\ &\quad + P_2\sqrt{\Delta S(\omega_2)y - \lambda S(0)B(1)|y| + K_2} + ... \\ &\quad + P_m\sqrt{\Delta S(\omega_m)y - \lambda S(0)B(1)|y| + K_m} \end{aligned} \tag{3.11}$$

If there exists an event $j \in \{1, 2, ...m\}$, such that

$$\Delta S(\omega_j)y - \lambda S(0)B(1)|y| + K_j < 0. \tag{3.12}$$

The investor's utility will become $-\infty$, and he/she does not hope that happens. So from (3.12), when $y > 0$ we should have:

$$(\Delta S(\omega_j) - \lambda S(0)B(1))y \geq -K_j, \quad \forall 1 \leq j \leq m. \tag{3.13}$$

When $y < 0$ we should have:

$$(\Delta S(\omega_j) + \lambda S(0)B(1))y \geq -K_j, \quad \forall 1 \leq j \leq m. \tag{3.14}$$

**Proposition 3.1.** *In market* $\mathrm{M}(\lambda, c)$*, if there exist* $j_0, \bar{j}_0 \in \{1, 2, ..., m\}$*, such that:*

$$\Delta S(\omega_{j_0}) - \lambda S(0)B(1) < 0, \tag{3.15}$$

$$\Delta S(\omega_{\bar{j}_0}) + \lambda S(0)B(1) > 0. \tag{3.16}$$

*Then, any feasible* $y$ *is bounded.*

*Proof.* Let $j_0, \bar{j}_0 \in \{1, 2, ..., m\}$ such that (3.15)–(3.16) hold, we must have:

$$y \leq \frac{-K_{j_0}}{\Delta S(\omega_{j_0}) - \lambda S(0)B(1)} \tag{3.17}$$

and

$$y \geq \frac{-K_{\bar{j}_0}}{\Delta S(\omega_{\bar{j}_0}) + \lambda S(0)B(1)} \tag{3.18}$$

Define

$$0 < M = \min_{1 \leq j \leq m} \{ \frac{-K_j}{\Delta S(\omega_j) - \lambda S(0)B(1)} | \Delta S(\omega_j) - \lambda S(0)B(1) < 0 \} \tag{3.19}$$

and

$$0 > m = \max_{1 \leq j \leq m} \{ \frac{-K_j}{\Delta S(\omega_j) + \lambda S(0)B(1)} | \Delta S(\omega_j) + \lambda S(0)B(1) > 0 \} \tag{3.20}$$

So, $y$ is bounded by $M$ and $m$. □

**Theorem 3.2.** *In market $\mathbf{M}(\lambda, c)$, if there exist $j_0, \bar{j}_0 \in \{1, 2, ..., m\}$, such that:*

$$\begin{cases} \Delta S(\omega_{j_0}) - \lambda S(0)B(1) < 0, \\ \Delta S(\omega_{\bar{j}_0}) + \lambda S(0)B(1) > 0. \end{cases} \tag{3.21}$$

*Then the market admits an optimal portfolio.*

*Proof.* From Proposition 2.7, we know the market $\mathbf{M}(\lambda, c)$ is arbitrage-free. From (3.11), when $y > 0$,

$$\begin{aligned} f'(y) \quad = \quad & \frac{P_1(\Delta S(\omega_1) - \lambda S(0)B(1))}{2\sqrt{\Delta S(\omega_1)y - \lambda S(0)B(1)y + K_1}} \\ & + \frac{P_2(\Delta S(\omega_2) - \lambda S(0)B(1))}{2\sqrt{\Delta S(\omega_2)y - \lambda S(0)B(1)y + K_2}} + ... \\ & + \frac{P_m(\Delta S(\omega_m) - \lambda S(0)B(1))}{2\sqrt{\Delta S(\omega_m)y - \lambda S(0)B(1)y + K_m}} \end{aligned} \tag{3.22}$$

And it is not hard to see $f''(y) < 0$, since

$$(P_j\sqrt{\Delta S(\omega_j)y - \lambda S(0)B(1)y + K_j})'' < 0, \quad \forall 1 \leq j \leq m. \tag{3.23}$$

If $\lim_{y \to 0^+} f'(y) < 0$, then

$$\max_{y \geq 0} f(y) = f(0) \triangleq \sum_{j=1}^{m} P_j(z_0 B(1) + S(1, \omega_j)z), \tag{3.24}$$

since $f(0) > \lim_{y \to 0^+} f(y)$.

If $\lim_{y \to 0^+} f'(y) > 0$, from Proposition 3.1, we know $y$ is bounded above by $M$. Without loss of generality, we can assume:

$$M = \frac{-K_{j_0}}{\Delta S(\omega_{j_0}) - \lambda S(0)B(1)} \tag{3.25}$$

As $y \to M$,

$$\frac{P_{j_0}(\Delta S(\omega_{j_0}) - \lambda S(0)B(1))}{2\sqrt{\Delta S(\omega_{j_0})y - \lambda S(0)B(1)y + K_{j_0}}} \to -\infty. \tag{3.26}$$

Because, $\lim_{y \to 0^+} f'(y) < \infty$ and (3.25) holds, we must have :

$$\lim_{y \to M} f'(y) \to -\infty. \tag{3.27}$$

And $f'(y)$ is continuous on the interval $(0, M)$, we must have and only have a $\overline{y} \in (0, M)$ such that

$$f'(\overline{y}) = 0. \tag{3.28}$$

So

$$\max_{y \geq 0} f(y) = \max\{f(0), f(\overline{y})\}. \tag{3.29}$$

The proof when $y < 0$ is almost the same.

So, $\max_{m \leq y \leq M} f(y)$ exists, and it is not on its bound $M$ or $m$.

$\square$

From the theorem above, we see in market $\mathbf{M}(\lambda, c)$, the optimal portfolio exists, and we can find out the optimal portfolio, it is either at $\overline{y}$, where $f'(\overline{y}) = 0$, or at $y = 0$. We now present some examples:

**Example 3.3.** Let

$$\begin{cases} m = 3, n = 1, r = 0, \lambda = 0.1, c = 1, z_0 = 10, z = 1, \\ S(0) = 10, S(1,\omega_1) = 10, S(1,\omega_2) = 12, S(1,\omega_3) = 8, \\ P_1 = P_2 = \frac{2}{5}, P_3 = \frac{1}{5}. \end{cases} \tag{3.30}$$

Then by proposition 2.7 the market is arbitrage-free. Now, we use utility function $u(\cdot)$ as follows:

$$\begin{cases} u(x) & = \sqrt{x}, \quad x \geq 0, \\ u(x) & = -\infty, \quad x < 0. \end{cases} \tag{3.31}$$

Hence, we need to maximize the following function:

$$f(y) = \frac{2}{5}\sqrt{-|y| + 20 - \mathbf{1}_{\{y\neq 0\}}} + \frac{2}{5}\sqrt{2y - |y| + 22 - \mathbf{1}_{\{y\neq 0\}}} + \frac{1}{5}\sqrt{-2y - |y| + 18 - \mathbf{1}_{\{y\neq 0\}}} \tag{3.32}$$

Note that:

$$f(y) = \begin{cases} \frac{2}{5}\sqrt{-y+19} + \frac{2}{5}\sqrt{y+21} + \frac{1}{5}\sqrt{-3y+17}, & y > 0, \\ \frac{2}{5}\sqrt{20} + \frac{2}{5}\sqrt{22} + \frac{1}{5}\sqrt{18} \approx 4.51, & y = 0, \\ \frac{2}{5}\sqrt{y+19} + \frac{2}{5}\sqrt{3y+21} + \frac{1}{5}\sqrt{-y+17}, & y < 0. \end{cases} \tag{3.33}$$

When $y > 0$,

$$f'(y) = \frac{1}{10}(-\frac{2}{\sqrt{19-y}} + \frac{2}{\sqrt{21+y}} - \frac{3}{\sqrt{17-3y}}) \tag{3.34}$$

as $y \to 0^+$, we have

$$\lim_{y\to 0^+} f'(y) = \frac{1}{10}(-\frac{2}{\sqrt{19}} + \frac{2}{\sqrt{21}} - \frac{3}{\sqrt{17}}) \approx -0.075 < 0. \tag{3.35}$$

Thus,

$$\max_{y\geq 0} f(y) = f(0) \approx 4.51. \tag{3.36}$$

When $y < 0$,

$$f'(y) = \frac{1}{10}\left(\frac{2}{\sqrt{y+19}} + \frac{6}{\sqrt{3y+21}} - \frac{1}{\sqrt{17-y}}\right) \tag{3.37}$$

as $y \to 0^-$, we have:

$$\lim_{y\to 0^-} f'(y) = \frac{1}{10}\left(\frac{2}{\sqrt{19}} + \frac{6}{\sqrt{21}} - \frac{1}{\sqrt{17}}\right) \approx 0.15 > 0. \tag{3.38}$$

Thus,

$$\max_{y\leq 0} f(y) = f(0) \approx 4.51. \tag{3.39}$$

Hence,

$$\max_{y\in\mathbb{R}} f(y) = f(0) \approx 4.51. \tag{3.40}$$

**Example 3.4.** Let

$$\begin{cases} m = 3, n = 1, r = 0, \lambda = 0.1, c = 1, z_0 = 10, z = 1, \\ S(0) = 10, S(1,\omega_1) = 10, S(1,\omega_2) = 14, S(1,\omega_3) = 8, \\ P_1 = P_2 = \frac{2}{5}, P_3 = \frac{1}{5}. \end{cases} \tag{3.41}$$

We still use utility function as (3.31). Hence we need to maximize the following function:

$$f(y) = \frac{2}{5}\sqrt{-|y| + 20 - \mathbf{1}_{\{y\neq 0\}}} + \frac{2}{5}\sqrt{4y - |y| + 24 - \mathbf{1}_{\{y\neq 0\}}} + \frac{1}{5}\sqrt{-2y - |y| + 18 - \mathbf{1}_{\{y\neq 0\}}} \tag{3.42}$$

Note that:

$$f(y) = \begin{cases} \frac{2}{5}\sqrt{-y+19} + \frac{2}{5}\sqrt{3y+23} + \frac{1}{5}\sqrt{-3y+17}, & y > 0, \\ \frac{2}{5}\sqrt{20} + \frac{2}{5}\sqrt{24} + \frac{1}{5}\sqrt{18} \approx 4.60, & y = 0, \\ \frac{2}{5}\sqrt{y+19} + \frac{2}{5}\sqrt{5y+23} + \frac{1}{5}\sqrt{-y+17}, & y < 0. \end{cases} \tag{3.43}$$

When $y > 0$,

$$f'(y) = \frac{1}{10}(-\frac{2}{\sqrt{19-y}} + \frac{6}{\sqrt{3y+23}} - \frac{3}{\sqrt{17-3y}}) \tag{3.44}$$

as $y \to 0^+$, we have

$$\lim_{y\to 0^+} f'(y) = \frac{1}{10}(-\frac{2}{\sqrt{19}} + \frac{6}{\sqrt{23}} - \frac{3}{\sqrt{17}}) \approx 0.0065 > 0. \tag{3.45}$$

Solve for $y$, when $f'(y) = 0$,

$$0 = f'(y) = \frac{1}{10}(-\frac{2}{\sqrt{19-y}} + \frac{6}{\sqrt{3y+23}} - \frac{3}{\sqrt{17-3y}}). \tag{3.46}$$

We get $y = 0.41$, and $f(0.41) = 4.50$. Thus,

$$\max_{y\geq 0} f(y) = max\{f(0), f(0.41)\} = f(0) = 4.60. \tag{3.47}$$

When $y < 0$,

$$f'(y) = \frac{1}{10}(\frac{2}{\sqrt{y+19}} + \frac{10}{\sqrt{5y+23}} - \frac{1}{\sqrt{17-y}}) \tag{3.48}$$

as $y \to 0^-$, we have:

$$\lim_{y\to 0^-} f'(y) = \frac{1}{10}(\frac{2}{\sqrt{19}} + \frac{10}{\sqrt{23}} - \frac{1}{\sqrt{17}}) \approx 0.23 > 0. \tag{3.49}$$

Thus,

$$\max_{y\leq 0} f(y) = f(0) \approx 4.60. \tag{3.50}$$

Hence,

$$\max_{y\in\mathbb{R}} f(y) = f(0) \approx 4.60. \tag{3.51}$$

**Example 3.5.** Let

$$\begin{cases} m = 3, n = 1, r = 0, \lambda = 0.1, c = 1, z_0 = 10, z = 1, \\ S(0) = 10, S(1,\omega_1) = 10, S(1,\omega_2) = 16, S(1,\omega_3) = 8, \\ P_1 = P_2 = \frac{2}{5}, P_3 = \frac{1}{5}. \end{cases} \tag{3.52}$$

We still use utility function as (3.31). Hence we need to maximize the following function:

$$f(y) = \frac{2}{5}\sqrt{-|y| + 20 - \mathbf{1}_{\{y\neq 0\}}} + \frac{2}{5}\sqrt{6y - |y| + 26 - \mathbf{1}_{\{y\neq 0\}}} + \frac{1}{5}\sqrt{-2y - |y| + 18 - \mathbf{1}_{\{y\neq 0\}}} \tag{3.53}$$

Note that:

$$f(y) = \begin{cases} \frac{2}{5}\sqrt{-y+19} + \frac{2}{5}\sqrt{5y+25} + \frac{1}{5}\sqrt{-3y+17}, & y > 0, \\ \frac{2}{5}\sqrt{20} + \frac{2}{5}\sqrt{26} + \frac{1}{5}\sqrt{18} \approx 4.677, & y = 0, \\ \frac{2}{5}\sqrt{y+19} + \frac{2}{5}\sqrt{7y+25} + \frac{1}{5}\sqrt{-y+17}, & y < 0. \end{cases} \tag{3.54}$$

When $y > 0$,

$$f'(y) = \frac{1}{10}\left(-\frac{2}{\sqrt{19-y}} + \frac{10}{\sqrt{5y+25}} - \frac{3}{\sqrt{17-3y}}\right) \tag{3.55}$$

as $y \to 0^+$, we have

$$\lim_{y\to 0^+} f'(y) = \frac{1}{10}\left(-\frac{2}{\sqrt{19}} + \frac{10}{\sqrt{25}} - \frac{3}{\sqrt{17}}\right) \approx 0.081 > 0. \tag{3.56}$$

Solve for $y$, when $f'(y) = 0$,

$$0 = f'(y) = \frac{1}{10}(-\frac{2}{\sqrt{19-y}} + \frac{10}{\sqrt{5y+25}} - \frac{3}{\sqrt{17-3y}}). \tag{3.57}$$

We get $y = 3.06$, and $f(3.06) = 4.70$. Thus,

$$\max_{y \geq 0} f(y) = max\{f(0), f(3.06)\} = f(3.06) = 4.70. \tag{3.58}$$

When $y < 0$,

$$f'(y) = \frac{1}{10}(\frac{2}{\sqrt{y+19}} + \frac{14}{\sqrt{7y+25}} - \frac{1}{\sqrt{17-y}}) \tag{3.59}$$

as $y \to 0^-$, we have:

$$\lim_{y \to 0^-} f'(y) = \frac{1}{10}(\frac{2}{\sqrt{19}} + \frac{14}{\sqrt{25}} - \frac{1}{\sqrt{17}}) \approx 0.46 > 0. \tag{3.60}$$

Thus,

$$\max_{y \leq 0} f(y) = f(0) \approx 4.677. \tag{3.61}$$

Hence,

$$\max_{y \in \mathbb{R}} f(y) = f(3.06) = 4.70. \tag{3.62}$$

From Theorem 3.2 we can see that when $\lambda$ and $c$ equal to $0$ (the market does not have transaction costs), we have the proposition below:

**Proposition 3.6.** *In market* $\mathrm{M}_\mathbf{0}$*, if there exist* $j_0, \bar{j}_0 \in \{1, 2, ..., m\}$*, such that:*

$$\begin{cases} \Delta S(\omega_{j_0}) < 0, \\ \Delta S(\omega_{\bar{j}_0}) > 0. \end{cases} \tag{3.63}$$

*Then the market admits an optimal portfolio.*

Optimization Problems with $n$ Stocks

In the case of $n$ stocks and $m$ events, we still use the utility function $u(\cdot)$ stated in (3.9). So from (2.5) we denote the expected utility function

$$\begin{aligned} f(y) &= E[V(1, z_0, z, y)] \\ &= \sum_{j=1}^{m} P_j \sqrt{[z_0 - S(0)^T y - \|y\|_\lambda - \langle c, \mathbf{1}_{\{y\}} \rangle] B(1) + S(1, \omega_j)(z + y)}. \end{aligned} \tag{3.64}$$

$f(y)$ can also be written as:

$$f(y) = \sum_{j=1}^{m} P_j \sqrt{\sum_{i=1}^{n} \Delta S_i(1, \omega_j) y_i - \sum_{i=1}^{n} \lambda_i S_i(0) B(1) |y_i| + K_j}, \tag{3.65}$$

where

$$K_j = z_0 B(1) + S(1, \omega_j) z - \left\langle c, \mathbf{1}_{\{y\}} \right\rangle B(1). \tag{3.66}$$

As we discussed in the case of one stock, an investor who enters the security market, must have a certain amount of money that is enough to pay the fixed costs. And suppose the investor did not short stocks before transaction. So we have $K_j > 0$.

**Theorem 3.7.** *In market* $\mathbf{M}(\lambda, c)$*, if the condition in Proposition 2.7 is satisfied, then the market admits optimal solution.*

*Proof.* We just prove the case when $n = 2$, the proof for $n > 2$ is almost the same.

Our goal is to maximize

$$f(y_1, y_2) \triangleq \sum_{j=1}^{m} P_j \sqrt{\alpha_{1j} y_1 + \alpha_{2j} y_2 - \beta_1 |y_1| - \beta_2 |y_2| + K_j} \tag{3.67}$$

where $\alpha_{ij} = \Delta S_i(1, \omega_j)$ , and $\beta_i = \lambda_i S_i(0) B(1)$.

When $y_1 > 0$, $y_2 > 0$, we have

$$f(y_1, y_2) = \sum_{j=1}^{m} P_j \sqrt{(\alpha_{1j} - \beta_1) y_1 + (\alpha_{2j} - \beta_2) y_2 + K_j} \tag{3.68}$$

and since (2.19) holds, there exists a $\omega_{j'} \in \Omega$ such that:

$$\alpha_{1j'} - \beta_1 < 0 \quad , \quad \alpha_{2j'} - \beta_2 < 0.$$

Here, for convenience, denote $\theta_{ij} = \alpha_{ij} - \beta_i$, and we assume $y_2 = \gamma y_1$, where $\gamma > 0$. Then, (3) becomes:

$$f(y_1) = \sum_{j=1}^{m} P_j \sqrt{(\theta_{1j} + \theta_{2j}\gamma) y_1 + K_j} \tag{3.69}$$

Then we have:

$$f'(y_1) = \sum_{j=1}^{m} \frac{P_j(\theta_{1j} + \theta_{2j}\gamma)}{2\sqrt{(\theta_{1j} + \theta_{2j}\gamma) y_1 + K_j}} \tag{3.70}$$

It is not hard to see $f''(y_1) < 0$, since

$$(P_j \sqrt{(\theta_{1j} + \theta_{2j}\gamma) y_1 + K_j})'' < 0, \quad \forall 1 \le j \le m. \tag{3.71}$$

There exists a $\omega_{j'}$, such that $\theta_{1j'} < 0$ and $\theta_{2j'} < 0$. As $y \to \frac{-K_{j'}}{\theta_{1j'} + \theta_{2j'}\gamma}$, $f'(y_1) \to -\infty$.

This tells us that the maximum value, when $y_1, y_2 > 0$, is not on the bound $\theta_{1j'} y_1 + \theta_{2j'} y_2 + K_{j'} = 0$.

If there exists $y_{1_0}, y_{2_0} > 0$, such that:

$$\begin{cases} \frac{\partial f(y_1,y_2)}{\partial y_1} = 0 \\ \frac{\partial f(y_1,y_2)}{\partial y_2} = 0 \end{cases} \tag{3.72}$$

Then $y = (y_{1_0}, y_{2_0})$ is the extreme point when $y_{1_0}, y_{2_0} > 0$.

When $y_1 > 0$, $y_2 = 0$, we have

$$f(y_1, 0) = \sum_{j=1}^{m} P_j \sqrt{\theta_{1j} y_1 + K_j + c_2 B(1)}. \tag{3.73}$$

If there exists $y_{1_1} > 0, y_2 = 0$, such that:

$$f'(y_{1_1}, 0) = 0. \tag{3.74}$$

Then $y = (y_{1_1}, 0)$ is the unique extreme point when $y_{1_1} > 0, y_2 = 0$.

When $y_1 = 0$, $y_2 > 0$, we have

$$f(0, y_2) = \sum_{j=1}^{m} P_j \sqrt{\theta_{2j} y_2 + K_j + c_1 B(1)}. \tag{3.75}$$

If there exists $y_1 = 0, y_{2_1} > 0$, such that:

$$f'(0, y_{2_1}) = 0. \tag{3.76}$$

Then $y = (0, y_{2_1})$ is the unique extreme point when $y_1 = 0, y_{2_1} > 0$. When $y_1 = y_2 = 0$, we have:

$$f(0, 0) = \sum_{j=1}^{m} P_j \sqrt{K_j + c_1 B(1) + c_2 B(1)}. \tag{3.77}$$

Then

$$\max_{y_1 \geq 0, y_2 \geq 0} f(y_1, y_2) = \max\{f(0,0), f(y_{1_1}, 0), f(0, y_{2_1}), f(y_{1_0}, y_{2_0})\}. \tag{3.78}$$

Similarly, we can find the maximum value when $(y_1 \leq 0, y_2 \geq 0)$, $(y_1 \leq 0, y_2 \leq 0)$ and $(y_1 \geq 0, y_2 \leq 0)$. $\square$

From Theorem 3.7 we can see that when $\lambda$ and $c$ equal to $0$ (the market does not have transaction costs), we have the proposition below

**Proposition 3.8.** *In the market* $\mathbf{M_0}$*, if for every* $(\epsilon_1, \epsilon_2, ..., \epsilon_n) \in \{-1, 1\}^n$*, there exists an* $\overline{\omega}(\epsilon_1, \epsilon_2, ..., \epsilon_n) \in \Omega$*, such that:*

$$\max\{\epsilon_1 \Delta S_1(\overline{\omega}), \epsilon_2 \Delta S_2(\overline{\omega}), ..., \epsilon_n \Delta S_n(\overline{\omega})\} < 0. \tag{3.79}$$

*Then the market admits an optimal portfolio.*

# LIST OF REFERENCES


[1] Jiongmin Yong, *Lecture Note: Financial Mathematics*. UCF, 2007.

[2] Stanley R. Pliska, *Introduction to Mathematical Finance*. Blackwell Publishers, 1997.

[3] Jonathan Levin *Choice under Uncertainty*. Stanford University, 2006.

[4] David Autor *Lecture Note: Uncertainty, Expected Utility Theory and the Market for Risk*. MIT, 2010.