2013

Extensions Of S-spaces

Bernd Losert
University of Central Florida

EXTENSIONS OF S-SPACES

by

BERND LOSERT
M.S. Florida Atlantic University, 2010

A dissertation submitted in partial fulfilment of the requirements
for the degree of Doctor of Philosophy
in the Department of Mathematics
in the College of Sciences
at the University of Central Florida
Orlando, Florida

Summer Term
2013

Major Professor: Gary Richardson

ABSTRACT

Given a convergence space X, a continuous action of a convergence semigroup S on X and a compactification Y of X, under what conditions on X and the action on X is it possible to extend the action to a continuous action on Y. Similarly, given a Cauchy space X, a Cauchy continuous action of a Cauchy semigroup S on X and a completion Y of X, under what conditions on X and the action on X is it possible to extend the action to a Cauchy continuous action on Y.

We answer the first question for some particular compactifications like the one-point compactification and the star compactification as well as for the class of regular compactifications. We answer the second question for the class of regular strict completions. Using these results, we give sufficient conditions under which the pseudoquotient of a compactification/completion of a space is the compactification/completion of the pseudoquotient of the given space.

TABLE OF CONTENTS

LIST OF NOMENCLATURE

CHAPTER 1: INTRODUCTION

The study of compactifications and completions already has a long history.[1] Most of these studies have been carried out on a sundry of spaces, metric spaces being the most prominent example. In the context of compactifications, there are two kinds of spaces that have received attention since the 1970s, namely convergence spaces and topological G-spaces[2] and the theory of compactifications of these spaces is now quite mature.[3] There has been some work on generalizing the compactification theory of topological G-spaces to the context of topological S-spaces, where instead of a topological group G we now have a topological semigroup S. This generalization has its roots in the compactification theory of topological semigroups. A recent survey of results in this direction is [18].

Probably the most "natural" direction in which to generalize the compactification theory of convergence spaces and topological S-spaces is to convergence S-spaces. This is the direction we take in this dissertation. It is organized as follows: In Chapter 2, we cover some of the necessary background material needed to make sense out of everything else that follows. In Chapter 3, we define the various notions that are needed to talk about S-spaces and S-extensions, proving some useful facts about them along the way. In Chapter 4, we study the various conditions that are necessary and sufficient for a convergence S-space to have an S-compactification, which is what we call compactifications in this context. We carry out this study by mimicking the compactification theory of convergence spaces as outlined in [13]. Most of the results are concerned with the necessary and sufficient conditions required by a convergence S-space in

[1] See the articles *Hausdorff Compactifications* and *Uniform, Proximal and Nearness Concepts in Topology* in [1] for historical surveys of these notions.

[2] Topological G-spaces are also called topological flows or G-flows, topological transformation groups or topological dynamical systems.

[3] Two of the earliest papers on these spaces are [22] and [6]. See [13] for a summary of results regarding compactifications of convergence spaces and [15] for topological G-spaces.

order for it to have a smallest and largest S-compactification of some type.

Let us now switch from compactifications to completions. The completion theory of uniform spaces is a well-established theory and has had numerous generalizations. The generalization that is of interest to us is the completion theory of Cauchy spaces. Cauchy spaces were first axiomatized in [8] and since then many results have been obtained regarding regular and strict regular completions of Cauchy spaces, e.g. [7, 9, 14]. We generalize many of these results to the setting of "Cauchy S-spaces". This is the topic of Chapter 5.

In the last chapter, Chapter 6, we study the following problem: Suppose X is an S-space and Y is an S-compactification or S-completion of X. Is it possible to find a compactification or completion of the pseudoquotient of X that is equivalent to the pseudoquotient of Y? We answer this question positively in both cases. This line of research was motivated by the papers [4, 5].

CHAPTER 2: BACKGROUND

The point of this chapter is mainly to establish notation, nomenclature and other conventions that will be used throughout this document as some of the definitions and symbolism used here is different from what is seen in the literature. Also, at the end of some sections, we list some results that will be referred to in other chapters.

Filters and Related Notions

We adopt most of the definitions about filters and related notions from Chapter 1 of [3]. Here we review some of the definitions and concepts as well as list some of our own idiosyncratic conventions.

A ***filter*** on a set X is a set $\mathcal{F}$ of subsets of X with the following properties:

1. $\mathcal{F}$ is not empty and does not contain the empty set.
2. $\mathcal{F}$ is closed under finite intersection.
3. $\mathcal{F}$ is upward closed, i.e. if $A \subseteq B \subseteq X$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$.

If $\mathcal{F}$ is a filter on a set X and $\mathcal{B}$ is a subset of $\mathcal{F}$ such that for every $F \in \mathcal{F}$, there is a $B \in \mathcal{B}$ such that $B \subseteq F$, then $\mathcal{B}$ is called a ***basis*** for $\mathcal{F}$.

Given a subset A of a set X, we will write $[A]$ or more specifically $[A]_X$ for the set of all supersets of A in X. When dealing with singleton sets like $\{x\}$, we will write $[x]$ instead of $[\{x\}]$. The set $[x]$ is actually a filter on X called a ***point filter*** . We will encounter point filters frequently throughout this document.

A set $\mathcal{B}$ of subsets of a set X is called a ***filter basis*** on X if

1. $\mathcal{B}$ is not empty and does not contain the empty set, and
2. for every $A, B \in \mathcal{B}$ there is a $C \in \mathcal{B}$ such that $C \subseteq A \cap B$.

Notice how this definition resembles that of a basis for a topology. We will write $[\mathcal{B}]$ or more specifically $[\mathcal{B}]_X$ for the union of all $[B]$ such that $B \in \mathcal{B}$. The set $[\mathcal{B}]$ is in fact a filter on X and $\mathcal{B}$ is a basis for this filter. A filter basis $\mathcal{B}$ is said to ***generate*** a filter $\mathcal{F}$ if $\mathcal{B}$ is a basis for $\mathcal{F}$. The point filter $[x]$ for example is generated by the filter basis $\{\{x\}\}$.

We will denote the set of filters on a set X by $\mathbf{F}(X)$. Let us mention a couple of things about $\mathbf{F}(X)$: The intersection of any family of filters is a filter so $\mathbf{F}(X)$ is closed under intersection. The union of two filters is not necessarily a filter, so $\mathbf{F}(X)$ is not closed under union. The subset relation is a partial order on $\mathbf{F}(X)$: Given $\mathcal{F}$, $\mathcal{G}$ in $\mathbf{F}(X)$, we say that $\mathcal{F}$ is ***finer*** than $\mathcal{G}$ or that $\mathcal{G}$ is ***coarser*** than $\mathcal{F}$ if $\mathcal{G} \subseteq \mathcal{F}$. Not all filters are comparable. There is a coarsest filter on X, namely $[X]$. We will write the ***least upper bound*** of $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X)$ as $\mathcal{F} \vee \mathcal{G}$ whenever it exists. Maximal filters in $\mathbf{F}(X)$ are called ***ultrafilters*** . Ultrafilters come in two varieties: fixed and free. An ultrafilter is called free if the intersection of all its elements is empty; otherwise, it is called fixed. It turns out that a filter is a fixed ultrafilter if and only if it is a point filter. Explicit examples of free ultrafilters do not exist as they are typically constructed using the axiom of choice.

Regarding filter bases, we say that $\mathcal{B}$ is coarser than $\mathcal{B}'$ if and only if $[\mathcal{B}] \subseteq [\mathcal{B}']$. Note that $[\mathcal{B}] \subseteq [\mathcal{B}']$ if and only if for every $B \in \mathcal{B}$ there is a $B' \in \mathcal{B}'$ such that $B' \subseteq B$.

Given $f : X \to Y$, $\mathcal{F} \in \mathbf{F}(X)$ and $\mathcal{G} \in \mathbf{F}(Y)$, we will write $f[\mathcal{F}]$ for the filter generated by $\{f(F) : F \in \mathcal{F}\}$ and we will write $f^{-1}[\mathcal{G}]$ for the filter generated by $\{f^{-1}(G) : G \in \mathcal{G}\}$ whenever the latter does not contain the empty set.

If $X \subseteq Y$ and $\mathcal{F}$ is a filter on X, we will write $[\mathcal{F}]$ or $[\mathcal{F}]_Y$ for the filter $f[\mathcal{F}]$, where f is the natural injection from X into Y. Note that $[\mathcal{F}]_Y$ is just the filter generated on Y by $\mathcal{F}$, considered as a filter basis on Y. Given a filter $\mathcal{G}$ on Y, we will write $\mathcal{G} \cap X$ for the ***trace*** of $\mathcal{G}$ on X, defined to be the set $\{G \cap X \colon G \in \mathcal{G}\}$. Note that $\mathcal{G} \cap X$ is a filter on X if and only if it does not contain $\emptyset$, and in either case it equals to $f^{-1}[\mathcal{G}]$.

Given a family of sets (X_ι) and a family of filters $(\mathcal{F}_\iota)$ where $\mathcal{F}_\iota \in \mathbf{F}(X_\iota)$, the ***product filter*** $\prod \mathcal{F}_\iota$ is the filter on $\prod X_\iota$ generated by the filter basis consisting of sets of the form $\prod F_\iota$, where $F_\iota \in \mathcal{F}_\iota$ and $F_\iota = X_\iota$ for all but finitely many ι.

Useful Results

There are many facts one can prove about filters. Here we list some of the ones that we will use most often. We will take these facts for granted in proofs later on. Some of the proofs of these facts are of the "follow-your-nose" kind, so we will not prove them.

Proposition 2.1. Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the filters $\mathcal{F}$ and $\mathcal{F}'$, respectively, on a set X. Then

(i) $\{B \cup B' \colon B \in \mathcal{B}, B' \in \mathcal{B}'\}$ is a basis for $\mathcal{F} \cap \mathcal{F}'$,

(ii) $\{B \cap B' \colon B \in \mathcal{B}, B' \in \mathcal{B}'\}$ is a filter basis as long as it does not contain the empty set, in which case it generates $\mathcal{F} \vee \mathcal{F}'$.

Proposition 2.2. Let $f \colon X \to Y$ be a function between two sets.

(i) If $\mathcal{B}$ is a basis for a filter $\mathcal{F}$ on X, then $f[\mathcal{B}] = f[\mathcal{F}]$.

(ii) If $\mathcal{B}$ is a basis for a filter $\mathcal{F}$ on Y such that $f^{-1}[\mathcal{F}] \in \mathbf{F}(X)$, then $f^{-1}[\mathcal{B}] = f^{-1}[\mathcal{F}]$.

(iii) For all $\mathcal{F} \in \mathbf{F}(X)$, $f^{-1}[f[\mathcal{F}]] \subseteq \mathcal{F}$ with equality when f is injective.

(iv) $f[\mathcal{F}]$ is an ultrafilter whenever $\mathcal{F} \in \mathbf{F}(X)$ is an ultrafilter.

(v) If $\mathcal{F} \in \mathbf{F}(Y)$ and $f^{-1}[\mathcal{F}] \in \mathbf{F}(X)$, then $\mathcal{F} \subseteq f[f^{-1}[\mathcal{F}]]$ with equality when $\mathcal{F}$ is an ultrafilter or when f is surjective.

(vi) If $f(X) \in \mathcal{F} \in \mathbf{F}(Y)$, then $f^{-1}[\mathcal{F}] \in \mathbf{F}(X)$ and $\mathcal{F} = f[f^{-1}[\mathcal{F}]]$.

(vii) If $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X)$, then

(a) $\mathcal{F} \subseteq \mathcal{G}$ implies $f[\mathcal{F}] \subseteq f[\mathcal{G}]$,

(b) $f[\mathcal{F} \cap \mathcal{G}] = f[\mathcal{F}] \cap f[\mathcal{G}]$,

(c) if $\mathcal{F} \vee \mathcal{G}$ exists, then $f[\mathcal{F}] \vee f[\mathcal{G}]$ exists and $f[\mathcal{F}] \vee f[\mathcal{G}] \subseteq f[\mathcal{F} \vee \mathcal{G}]$,

(d) if f is an injection and $f[\mathcal{F}] \vee f[\mathcal{G}]$ exists, then $f[\mathcal{F}] \vee f[\mathcal{G}] = f[\mathcal{F} \vee \mathcal{G}]$.

(viii) If $\mathcal{F}, \mathcal{G} \in \mathbf{F}(Y)$, then

(a) $\mathcal{F} \subseteq \mathcal{G}$ implies $f^{-1}[\mathcal{F}] \subseteq f^{1}[\mathcal{G}]$,

(b) $f^{-1}[\mathcal{F} \cap \mathcal{G}] = f^{-1}[\mathcal{F}] \cap f^{-1}[\mathcal{G}]$,

(c) if $\mathcal{F} \vee \mathcal{G}$ or $f^{-1}[\mathcal{F}] \vee f^{-1}[\mathcal{G}]$ exists, then $f^{-1}[\mathcal{F}] \vee f^{-1}[\mathcal{G}] = f^{-1}[\mathcal{F} \vee \mathcal{G}]$.

(ix) For all $x \in X$, $f[[x]] = [f(x)]$.

Proposition 2.3. The following propositions about $\mathcal{F} \in \mathbf{F}(X)$ are equivalent.

(i) $\mathcal{F}$ is an ultrafilter.

(ii) For all $A \subseteq X$, either A or its complement is in $\mathcal{F}$.

(iii) $A \cup B \in \mathcal{F}$ implies $A \in \mathcal{F}$ or $B \in \mathcal{F}$.

(iv) If $A \subseteq X$ intersects every set in $\mathcal{F}$, then $A \in \mathcal{F}$.

Also, using Zorn's lemma, there is always an ultrafilter $\mathcal{U} \in \mathbf{F}(X)$ finer than $\mathcal{F}$ and in fact $\mathcal{F}$ is the intersection of all ultrafilters finer than itself.

Proposition 2.4. Let $f \colon X \to Y$ be a function between two sets. Let $\mathcal{F}$ be a filter on X and let $\mathcal{U}$ be an ultrafilter on Y finer than $f[\mathcal{F}]$. Then there exists an ultrafilter $\mathcal{V}$ on X finer than $\mathcal{F}$ such that $f[\mathcal{V}] = \mathcal{U}$.

Proof. Since $f[\mathcal{F}] \subseteq \mathcal{U}$, we have that $f^{-1}[\mathcal{U}]$ is a filter on X. We now prove that $\mathcal{F} \vee f^{-1}[\mathcal{U}]$ exists by proving that $F \cap f^{-1}(U)$ is not empty for arbitrary $F \in \mathcal{F}$ and $U \in \mathcal{U}$: Pick a $y \in f(F) \cap U$. Such a y exists since $f[\mathcal{F}] \subseteq \mathcal{U}$. Since $y \in f(F)$, there exists an $x \in F$ such that $f(x) = y$. Since $f(x) \in U$, we have that $x \in f^{-1}(U)$. Thus, $x \in F \cap f^{-1}(U)$, which means $\mathcal{F} \vee f^{-1}[\mathcal{U}]$ exists as claimed. Now let $\mathcal{V}$ be any ultrafilter finer than $\mathcal{F} \vee f^{-1}[\mathcal{U}]$. Since $f[\mathcal{V}]$ contains every element of $\mathcal{U}$, it follows that $f[\mathcal{V}] = \mathcal{U}$. ∎

Proposition 2.5. Let X and Y be sets, let $\mathcal{F}, \mathcal{F}'$ be filters on X and let $\mathcal{G}, \mathcal{G}'$ be filters on Y. Then $\mathcal{F} \vee \mathcal{F}'$ and $\mathcal{G} \vee \mathcal{G}'$ exist if and only if $(\mathcal{F} \times \mathcal{G}) \vee (\mathcal{F}' \times \mathcal{G}')$ exists as a filter on $X \times Y$.

Proof. The filter $(\mathcal{F} \times \mathcal{G}) \vee (\mathcal{F}' \times \mathcal{G}')$ exists if and only if $\{(F \times G) \cap (F' \times G') \colon F \in \mathcal{F}, F' \in \mathcal{F}', G \in \mathcal{G}, G' \in \mathcal{G}'\} = \{(F \cap F') \times (G \cap G') \colon F \in \mathcal{F}, F' \in \mathcal{F}', G \in \mathcal{G}, G' \in \mathcal{G}'\}$ is a basis if and only if $\mathcal{F} \vee \mathcal{F}'$ and $\mathcal{G} \vee \mathcal{G}'$ exist. ∎

Convergence Spaces

An excellent book about the theory of convergence spaces is [2]. We adopt most of the definitions and results given there except for a few exceptions.

A ***preconvergence structure*** on a set X is a relation between the filters on X and the elements of X, i.e. a collection of pairs of the form $(\mathcal{F}, x)$ where $\mathcal{F}$ is a filter on X and x is an element of X. Given a preconvergence structure p on X, we indicate that the pair $(\mathcal{F}, x)$ belongs to p by writing

$$\mathcal{F} \xrightarrow{p} x \qquad \text{or} \qquad \mathcal{F} \to x \text{ in } (X, p) \qquad \text{or} \qquad \mathcal{F} \to x \text{ in } X \qquad \text{or} \qquad \mathcal{F} \to x$$

and reading it as "$\mathcal{F}$ p-converges to x" or "$\mathcal{F}$ converges to x in (X, p)" or "$\mathcal{F}$ converges to x in X" or "$\mathcal{F}$ converges to x" or "x is a p-limit of $\mathcal{F}$" or "x is a limit of $\mathcal{F}$ in (X, p)" or "x is a limit of $\mathcal{F}$ in X" or "x is a limit of $\mathcal{F}$".

A ***convergence structure*** on a set X is a preconvergence structure on X that satisfies the following “axioms”:

Point filter axiom: Every point filter $[x]$ converges to x.

Subfilter axiom: If $\mathcal{F} \to x$ and $\mathcal{F} \subseteq \mathcal{G}$, then $\mathcal{G} \to x$.

Intersection axiom: If $\mathcal{F} \to x$ and $\mathcal{G} \to x$, then $\mathcal{F} \cap \mathcal{G} \to x$.

The pair (X, p), where X is a set and p is a convergence structure on X, is called a ***convergence space*** . We will usually write X for the convergence space (X, p) unless confusion arises or precision is needed with the notation. The elements of a convergence space will sometimes be called ***points***.

Given two convergence structures p, q on a set X, we say that q is ***finer*** than p or that p is ***coarser*** than q if q-convergence implies p-convergence, i.e. if $\mathcal{F} \xrightarrow{q} x$ implies $\mathcal{F} \xrightarrow{p} x$. The finest convergence structure on X is called the ***discrete convergence structure*** on X. In this convergence structure, $\mathcal{F} \to x$ if and only if $\mathcal{F} = [x]$, i.e. only point filters converge. The coarsest convergence structure on X is called the ***indiscrete or trivial convergence structure*** on X. In this convergence structure, every filter converges to every point. The relations *is finer than* and *is coarser than* are actually partial orders on the set of convergence structures on X that make it a complete lattice.

A function $f\colon X \to Y$ between convergence spaces is called

- ***continuous*** if $\mathcal{F} \to x$ in X implies $f[\mathcal{F}] \to f(x)$ in Y,
- a ***homeomorphism*** if it is a continuous bijection whose inverse is also continuous.

Note that these notions are exactly like those in topology.

The ***closure*** of a subset A of a convergence space (X, p) will be denoted by $\mathrm{cl}\, A$ or $\mathrm{cl}_X A$ or $\mathrm{cl}_p A$ and it consists of all the points in X that are limits of filters containing A. The ***interior*** of A will be denoted by $\operatorname{int} A$ or $\operatorname{int}_X A$ or $\operatorname{int}_p A$ and it consists of all the points x in A with the following property: if $\mathcal{F} \to x$ in X, then $A \in \mathcal{F}$. A point $x \in X$ is said to be an ***adherent point*** (or a cluster point or a limit point) of a filter $\mathcal{F}$ on X if there is a filter $\mathcal{G}$ on X finer than $\mathcal{F}$ that converges to x. The set of adherent points of a filter $\mathcal{F}$ on X will be denoted by $\operatorname{adh}\mathcal{F}$ or $\operatorname{adh}_X \mathcal{F}$ or $\operatorname{adh}_p \mathcal{F}$ and will be called the ***adherence*** of $\mathcal{F}$. We will write $\operatorname{cl}\mathcal{F}$ or $\operatorname{cl}_X \mathcal{F}$ or $\operatorname{cl}_p \mathcal{F}$ for the ***closure filter*** $\{\operatorname{cl} F \colon F \in \mathcal{F}\}$. We will write $\mathcal{U}(x)$ or $\mathcal{U}_X(x)$ or $\mathcal{U}_p(x)$ for the intersection of all filters that converge to x and we will call it the ***neighborhood filter*** of the point x. Other than the fact that the interior/closure of a set is not necessarily open/closed, all these notions are just like those in topology.

Just like there are initial and final topologies, there are initial and final convergence structures: Let (X_ι) be a family of convergence spaces and let X be a set. Given a family (f_ι) of functions the form $f_\iota \colon X \to X_\iota$, the ***initial convergence structure*** on X with respect to the family (f_ι) is the coarsest convergence structure on X making each f_ι continuous. Dually, given a family (f_ι) of functions the form $f_\iota \colon X_\iota \to X$, the ***final convergence structure*** on X with respect to the family (f_ι) is the finest convergence structure on X making each f_ι continuous. Let us briefly look at some of the typical examples of initial and final convergence structures, starting with the subspace convergence structure.

Let Y be a convergence space and let X be a subset of Y. The ***subspace convergence structure*** on X is the coarsest convergence structure on X making the natural injection of X into Y continuous. The set X equipped with the subspace convergence structure makes it into a ***subspace*** of Y and a filter $\mathcal{F}$ on X converges to $x \in X$ if and only if $[\mathcal{F}] \to x$ in Y.

The dual notion of the subspace convergence structure is the quotient convergence structure. It is defined as follows: Let X be a convergence space, let Y be a set and let $f \colon X \to Y$ be a surjection. The ***quotient convergence structure*** on Y is the finest convergence structure on Y

making f continuous. The set Y equipped with the quotient convergence structure is called a ***quotient space*** and a filter $\mathcal{G}$ on Y converges to $y \in Y$ if and only if there exists a filter $\mathcal{F}$ on X and an $x \in X$ such that $\mathcal{F} \to x$ in X, $f[\mathcal{F}] = \mathcal{G}$ and $f(x) = y$.

Let (X_ι) be a family of convergence spaces and let $X = \prod X_\iota$ be their Cartesian product. The ***product convergence structure*** on X is the coarsest convergence structure making each natural projection $\pi_\iota \colon X \to X_\iota$ continuous. The set X equipped with the product convergence structure is called a ***product space*** and a filter $\mathcal{F}$ on X converges to $x \in X$ if and only if $\pi_\iota[\mathcal{F}] \to \pi_\iota(x)$ for each index ι.

A subspace A of a convergence space X is ***dense*** if $\operatorname{cl} A = X$ and ***strictly dense*** if for every $x \in X$ and every filter $\mathcal{F}$ converging to x, there is a filter $\mathcal{G}$ on A such that $[\mathcal{G}] \to x$ and $\operatorname{cl}[\mathcal{G}] \subseteq \mathcal{F}$. Note that being strictly dense implies being dense. This concept of "strictly dense" will prove useful latter on when we define extensions.

A convergence space and its convergence structure are called

- ***Hausdorff*** if no filter has two distinct limits,
- ***regular*** if $\operatorname{cl}\mathcal{F} \to x$ whenever $\mathcal{F} \to x$,
- ***completely regular*** if it is regular and has the same ultrafilter convergence as a completely regular topological space,
- ***compact*** if every ultrafilter converges,
- ***locally compact*** if every convergent ultrafilter has a compact set.

Unlike in topology, compact Hausdorff does not imply regular. This and the definition of complete regularity are really the only topological notions in the theory of convergence spaces that differ from the usual topological ones. Everything else remains the same: Subspaces and products of Hausdorff or regular or completely regular convergence spaces again have this property.

Closed subsets of compact convergence spaces are compact. Compact subsets of Hausdorff convergence spaces are closed. The image of a compact set with respect to a continuous function is compact. Products of compact convergence spaces are compact. Closed subsets of a locally compact convergence space are locally compact and finite products of locally compact convergence spaces are locally compact.

A convergence space is said to be

- ***pseudotopological*** if a filter $\mathcal{F}$ converges to x if and only if every ultrafilter finer than $\mathcal{F}$ converges to x,
- ***pretopological*** if each neighborhood filter $\mathcal{U}(x)$ converges to x,
- ***topological*** if every neighborhood filter has a basis of open sets.

Subspaces and products of pseudotopological, pretopological and topological convergence spaces again have this property. Note that topological convergence spaces are in fact just topological spaces.

If (X,p) is a convergence space, then there is a finest convergence structure σp coarser than p such that $(X,\sigma p)$ is pseudotopological. The space $(X,\sigma p)$ is denoted σX and is called the ***pseudotopological modification*** of X. Similarly, one can define πX, the ***pretopological modification*** of X, and τX, the ***topological modification*** of X.

Useful Results

Proposition 2.6. The following results about a subspace A of a convergence space X are true.

(i) πA is the subspace of πX determined by A.

(ii) If X is compact, regular and Hausdorff, then πA is a Hausdorff topological space.

(iii) A is open (closed) in X if and only if πA is open (closed) in πX.

(iv) If X and πX have the same ultrafilter convergence,[1] then A is a compact (resp. locally compact) subspace of X if and only if πA is a compact (resp. locally compact) subspace of πX.

Proof. The proof of (i) can be found in [2] as the proof of Proposition 1.3.18. The proof of (ii) can be found in [10] as the proof of Proposition 3. The proof of (iii) follows from the fact that $\mathrm{cl}_X = \mathrm{cl}_{\pi X}$ (this is Proposition 1.3.17 in [2]) and the fact that a set is open (resp. closed) if and only if its complement is closed (resp. open). The proof of (iv) goes as follows: A is a compact subspace of X if and only if every ultrafilter on X containing A converges in X to something in A if and only if every ultrafilter on X containing A converges in πX to something in A (using the fact that X and πX have the same ultrafilter convergence) if and only if πA is a compact subspace of πX. The locally compact version follows from the fact that A and πA have the same ultrafilter convergence. ∎

Proposition 2.7. If X is a convergence space, then $\pi X = \tau X$ if and only if cl_X is idempotent (i.e. the closure of any subset of X is always closed).

Proof. The statement of this proposition is essentially the same as Proposition 1.3.21 in [2] and the proof is contained therein. ∎

Proposition 2.8. If X is a compact regular Hausdorff convergence space, then $\sigma X = \pi X = \tau X$ and cl_X is idempotent.

[1] Two convergence structures p, q on a set X are said to have the same ultrafilter convergence if for every ultrafilter $\mathcal{U}$ on X and every $x \in X$, $\mathcal{U} \xrightarrow{p} x$ if and only if $\mathcal{U} \xrightarrow{q} x$. In general, we say two convergence spaces have the same ultrafilter convergence if they have the same underlying set and their convergence structures have the same ultrafilter convergence.

Proof. For a proof, see the proof of Theorem 1.4.16 in [2]. For a different proof that cl_X is idempotent, see the proof of Proposition 1 in [10]. ∎

Proposition 2.9. Let X and Y be two convergence spaces. If Y is a locally compact Hausdorff topological space, then $\tau(X \times Y) = \tau X \times \tau Y$.

Proof. For a proof of this, see Theorem 4.2 in [12]. ∎

Proposition 2.10. Let X be a dense subset of Y and let $\mathcal{V}$ be an ultrafilter on Y. Then there exists an ultrafilter $\mathcal{U}$ on X such that $\mathrm{cl}_Y[\mathcal{U}] \subseteq \mathcal{V}$.

Proof. Given $y \in Y - X$, let $\mathcal{U}_y$ be an ultrafilter converging to y (such an ultrafilter exists in X is dense in Y). Given a subset A of X, let $A' = A \cup \{y \in Y - X\colon\ A \in \mathcal{U}_y\}$. Let $\mathcal{U} = \{A \subseteq X\colon\ A' \in \mathcal{V}\}$. We claim that $\mathrm{cl}_Y[\mathcal{U}] \subseteq \mathcal{V}$.

First, we prove that $\mathcal{U}$ is a filter. Since $\emptyset = \emptyset' \notin \mathcal{V}$, $\mathcal{U}$ does not contain $\emptyset$. Let $A, B \subseteq X$ be arbitrary. Since $\{y \in Y - X\colon\ A \cap B \in \mathcal{U}_y\} = \{y \in Y - X\colon\ A \in \mathcal{U}_y\} \cap \{y \in Y - X\colon\ B \in \mathcal{U}_y\}$, it follows that $(A \cap B)' = A' \cap B'$. Thus, if $A, B \in \mathcal{U}$, then $A', B' \in \mathcal{V}$, hence $A' \cap B' = (A \cap B)' \in \mathcal{V}$, hence $A \cap B \in \mathcal{U}$. If $A \subseteq B$, then every $\mathcal{U}_y$ that contains A also contains B, hence $A' \subseteq B'$. Thus, if $A \in \mathcal{U}$ and $A \subseteq B$, then $A' \in \mathcal{V}$, hence $B' \in \mathcal{V}$, hence $B \in \mathcal{U}$.

Now we prove that $\mathcal{U}$ is an ultrafilter. Note that it suffices to prove that $(A \cup B)' = A' \cup B'$ for all $A, B \subseteq X$ because if $A \cup B \in \mathcal{U}$, then $(A \cup B)' = A' \cup B' \in \mathcal{V}$, hence either A' or B' is in $\mathcal{V}$ since $\mathcal{V}$ is an ultrafilter, hence either A or B is in $\mathcal{U}$. The proof that $(A \cup B)' = A' \cup B'$ follows from the fact that $\{y \in Y - X\colon\ A \cup B \in \mathcal{U}_y\} \subseteq \{y \in Y - X\colon\ A \in \mathcal{U}_y\} \cup \{y \in Y - X\colon\ B \in \mathcal{U}_y\}$ since the $\mathcal{U}_y$ are ultrafilters.

Finally, we prove that $\mathrm{cl}_Y[\mathcal{U}] \subseteq \mathcal{V}$. Let $A \in \mathcal{U}$ be arbitrary. Then $A' \in \mathcal{V}$ and since $A' \subseteq \mathrm{cl}_Y A$ by definition of A', it follows that $\mathrm{cl}_Y A \in \mathcal{V}$. Since A is an arbitrary element of $\mathcal{U}$, it follows that

$\mathrm{cl}_Y[\mathcal{U}] \subseteq \mathcal{V}$. ∎

A continuous function $f: X \to Y$ is called a ***proper map*** if for all $y \in Y$ and all and all ultrafilters $\mathcal{U}$ on X such that $f[\mathcal{U}] \to y$, there is an $x \in f^{-1}(\{y\})$ such that $\mathcal{U} \to x$.

Proposition 2.11. Proper maps preserve closure.

Proof. Let $f: X \to Y$ be a proper map between two convergence spaces and let A be a non-empty proper subset of X. The theorem states that $f(\mathrm{cl}_X A) = \mathrm{cl}_Y f(A)$. Since f is continuous, we already have that $f(\mathrm{cl}_X A) \subseteq \mathrm{cl}_Y f(A)$. To prove the reverse inclusion, pick a $y \in \mathrm{cl}_Y f(A)$. Then there is an ultrafilter $\mathcal{V}$ on Y that contains $f(A)$ and converges to y. Since $\mathcal{V}$ contains $f(A)$, it contains $f(X)$, hence $f^{-1}[\mathcal{V}]$ is a filter on X. We claim that $f^{-1}[\mathcal{V}] \vee [A]$ exists: Pick a $V \in \mathcal{V}$ and a $z \in V \cap f(A)$. Then there is an $x \in A$ such that $f(x) = z \in V$, which means $x \in f^{-1}(V) \cap A$. This proves that $f^{-1}(V) \cap A$ is not empty, which means $f^{-1}[\mathcal{V}] \vee [A]$ exists. Now pick an ultrafilter $\mathcal{U}$ on X finer than $f^{-1}[\mathcal{V}] \vee [A]$. Then $f[\mathcal{U}] = f[f^{-1}[\mathcal{V}] \vee [A]] \supseteq f[f^{-1}[\mathcal{V}]] \vee f[[A]] = \mathcal{V} \vee [f(A)] = \mathcal{V}$. Thus $\mathcal{U}$ is an ultrafilter such that $f[\mathcal{U}] = \mathcal{V} \to y$, and since f is proper, there is an $x \in X$ such that $\mathcal{U} \to x$ and $f(x) = y$. Since $\mathcal{U}$ contains A by construction, it follows that $x \in \mathrm{cl}_X A$ and that $f(x) = y \in f(\mathrm{cl}_X A)$. ∎

Proposition 2.12. Let $f: X \to Y$ be a continuous function between two convergence spaces. If X is compact and Y is Hausdorff, then f is proper.

Proof. Pick a $y \in Y$, an ultrafilter $\mathcal{V}$ on Y that converges to Y and an ultrafilter $\mathcal{U}$ on X such that $f[\mathcal{U}] = \mathcal{V}$. Since X is compact, $\mathcal{U}$ converges to some $x \in X$. Since f is continuous $f[\mathcal{U}] \to f(x)$. Since $\mathcal{V} = f[\mathcal{U}] \to y$ and Y is Hausdorff, it follows that $f(x) = y$, so $x \in f^{-1}(\{y\})$. This proves that f is proper. ∎

Cauchy Spaces

To the author's knowledge, there is no standard reference text regarding Cauchy spaces. The majority of the theory is scattered in research articles. In this section, we present the definitions and results about Cauchy spaces that will be useful for our purposes.

A collection $\mathscr{C}$ of filters on a set X is called a ***Cauchy structure*** on X if it satisfies the following "axioms":

Point filter axiom: Every point filter is in $\mathscr{C}$.

Subfilter axiom: If $\mathcal{F} \in \mathscr{C}$ and $\mathcal{F} \subseteq \mathcal{G}$, then $\mathcal{G} \in \mathscr{C}$.

Intersection axiom: If $\mathcal{F}, \mathcal{G} \in \mathscr{C}$ and $\mathcal{F} \vee \mathcal{G}$ exists, then $\mathcal{F} \cap \mathcal{G} \in \mathscr{C}$.

Any pair of the form $(X, \mathscr{C})$, where X is a set and $\mathscr{C}$ is a Cauchy structure on X, is called a ***Cauchy space*** . We will typically write X for the Cauchy space $(X, \mathscr{C})$ and refer to the filters in $\mathscr{C}$ as the ***Cauchy filters*** on X.

If $\mathscr{C}$ and $\mathscr{D}$ are Cauchy structures on a set X, then we say that $\mathscr{C}$ is ***coarser*** than $\mathscr{D}$ or that $\mathscr{D}$ is finer than $\mathscr{C}$ if $\mathscr{D} \subseteq \mathscr{C}$. The finest Cauchy structure on X consists of just the point filters; the coarsest one consists of all the filters.

A function $f\colon X \to Y$ between Cauchy spaces is called ***Cauchy continuous*** if it maps Cauchy filters on X to Cauchy filters on Y. If $f\colon X \to Y$ is a bijective Cauchy continuous function whose inverse is also Cauchy continuous, then f is called a ***Cauchy homeomorphism*** and we say that X and Y are ***Cauchy homeomorphic***.

Every convergence structure $\mathscr{C}$ on a set X induces a convergence structure p on X, where p is defined so that $\mathcal{F} \xrightarrow{p} x$ if and only if $\mathcal{F} \cap [x] \in \mathscr{C}$. Conversely, every Hausdorff convergence structure p on X induces a Cauchy structure $\mathscr{C}$ on X, where $\mathscr{C}$ is defined so that $\mathcal{F} \in \mathscr{C}$ if and only if $\mathcal{F}$ p-converges.

We will sometimes regard a Cauchy space X as a convergence space equipped with the convergence structure induced by the Cauchy strucutre as specified in the prior paragraph. By doing this, we will be able to talk about a subset of a Cauchy space being open or closed or dense or compact or locally compact and so on. Note that with this convention, every convergent filter on a Cauchy space is Cauchy and every Cauchy continuous function is continuous.

A Cauchy space X and its Cauchy structure are called

- ***Hausdorff*** if X as a convergence space is Hausdorff,
- ***regular*** if the closure filter of every Cauchy filter is Cauchy,
- ***complete*** if every Cauchy filter on X converges,
- ***totally bounded*** if every ultrafilter is Cauchy.

Note that if X is a regular Cauchy space, then X is a regular convergence space, but the converse is not true in general. Also note that every complete totally bounded Cauchy space is compact, just like with metric spaces.

Initial and final Cauchy structures are defined in an analogous way as initial and final convergence structures. Here, we review the typical ones that we will be using.

Let X be a subset of a Cauchy space Y. The ***subspace Cauchy structure*** on X is the coarsest Cauchy structure on X making the natural injection from X into Y Cauchy continuous. The set X equipped with the subspace Cauchy structure is called a ***Cauchy subspace*** of Y and a filter $\mathcal{F}$ on X is Cauchy if and only if the filter $[\mathcal{F}]$ on Y is Cauchy. We remark that if X is a Cauchy subspace of Y, then X is a subspace of Y.

Whenever the least upper bound of two Cauchy filters exists, we say that these two Cauchy filters are ***linked*** . If $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ are Cauchy filters and if there is a permutation $j_1, j_2, \ldots, j_n$ of the indices such that $\mathcal{F}_{j_1}, \mathcal{F}_{j_2}$ are linked, $\mathcal{F}_{j_2}, \mathcal{F}_{j_3}$ are linked, $\mathcal{F}_{j_3}, \mathcal{F}_{j_4}$ are linked, etc. we say that $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ are linked. This notion is needed to define the Cauchy quotient structure.

Let X be a Cauchy space, let Y be a set and let $f\colon X \to Y$ be a surjection. The ***quotient Cauchy structure*** on Y is the finest Cauchy structure on Y making f Cauchy continuous. The set Y equipped with the quotient Cauchy structure is called a ***Cauchy quotient space*** and a filter $\mathcal{G}$ on Y is Cauchy if there are Cauchy filters $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n$ on X such that $f[\mathcal{G}_1], f[\mathcal{G}_2], \ldots, f[\mathcal{G}_n]$ are linked and $f[\mathcal{G}_1] \cap f[\mathcal{G}_2] \cap \cdots \cap f[\mathcal{G}_n] \subseteq \mathcal{F}$. It is not necessarily true that a Cauchy quotient space is a convergence quotient space (see Proposition 2.13 for some sufficient conditions).

Products of Cauchy spaces are defined as follows: Given a family of Cauchy spaces (X_ι), the ***product Cauchy structure*** on $X = \prod X_\iota$ is the coarsest Cauchy structure on X making each of the natural projections $\pi_\iota\colon X \to X_\iota$ Cauchy continuous. The set X together with the product Cauchy structure is called a ***Cauchy product space*** and a filter $\mathcal{F}$ on X is Cauchy if and only if $\pi_\iota[\mathcal{F}]$ is Cauchy for every ι. Note that if $X = \prod X_\iota$ is a Cauchy product space, then it is a convergence product space.

Useful results

Proposition 2.13. Let X be a Cauchy space, let Y be a set and let $f\colon X \to Y$ be a surjection. Let p be the quotient convergence structure on Y and let $\mathscr{C}$ be the quotient Cauchy structure on Y.

(i) p is finer than the convergence structure q induced by $\mathscr{C}$.

(ii) If q is Hausdorff, then so is p and consequently the Cauchy structure $\mathscr{D}$ induced by p is finer than $\mathscr{C}$.

(iii) If X is complete and p is Hausdorff, then $\mathscr{C} = \mathscr{D}$, $p = q$ and $\mathscr{C}$ is complete.

Proof.

(i) Suppose $\mathcal{G}$ p-converges to y. By definition of p, there is a filter $\mathcal{F}$ on X and an $x \in X$ such that $\mathcal{F} \to x$ in X, $f[\mathcal{F}] = \mathcal{G}$ and $f(x) = y$. Since $\mathcal{F} \to x$ in X, $\mathcal{F} \cap [x]$ is a Cauchy filter on X, which means that $f[\mathcal{F} \cap [x]] = f[\mathcal{F}] \cap [f(x)] = \mathcal{G} \cap [y] \in \mathscr{C}$ be definition of $\mathscr{C}$ (the filter $f[\mathcal{F} \cap [x]]$ is vacuously linked). Since $\mathcal{G} \cap [y] \in \mathscr{C}$, by definition of q, it follows that $\mathcal{G}$ q-converges to y.

(ii) Suppose $\mathcal{G}$ is a filter on Y that p-converges to y, y'. Since p is finer than q by (i), it follows that $\mathcal{G}$ q-converges to y and y'. Since q is Hausdorff, $y = y'$. This proves that p is Hausdorff. Since p is Hausdorff, $\mathscr{D}$ is well-defined, and if $\mathcal{G} \in \mathscr{D}$, then $\mathcal{G}$ p-converges, which means $\mathcal{G}$ q-converges by (i), which means $\mathcal{G} \in \mathscr{C}$.

(iii) By (ii), we already have that $\mathscr{D} \subseteq \mathscr{C}$. For the reverse inclusion, note that $\mathscr{C}$ is the finest Cauchy structure on Y making f Cauchy continuous. Thus, if $\mathscr{D}$ is a Cauchy structure on Y making f Cauchy continuous, it will follow that $\mathscr{D}$ is coarser than $\mathscr{C}$, i.e. $\mathscr{C} \subseteq \mathscr{D}$. Indeed, given a Cauchy filter $\mathcal{F}$ on X, since X is complete, $\mathcal{F} \to x$ for some $x \in X$, hence $f[\mathcal{F}]$ p-converges to $f(x)$, hence $f[\mathcal{F}] \in \mathscr{D}$ by definition of $\mathscr{D}$.

By (i), we know that p is finer than q, so all we have to do is prove that q is finer than p. If $\mathcal{G}$ q-converges to y, then by definition of $\mathscr{C}$, $\mathcal{G} \cap [y] \in \mathscr{C}$, and since $\mathscr{C} = \mathscr{D}$, it follows that $\mathcal{G} \cap [y] \in \mathscr{D}$, which means $\mathcal{G}$ p-converges to y since the convergence structure induced by $\mathscr{D}$ is exactly p.

Finally, since $\mathscr{C} = \mathscr{D}$ and $\mathscr{D}$ is complete since p is Hausdorff, it follows that $\mathscr{C}$ is complete.

∎

CHAPTER 3: S-SPACES

Basic Notions

We now proceed to generalize two concepts: that of a topological group and that of a continuous action of a topological group on a topological space. The generalizations are straightforward.

A ***convergence semigroup*** is a semigroup equipped with a convergence structure making its binary operation a jointly continuous function. A ***convergence group*** is a group equipped with a convergence structure making its binary operation jointly continuous and the function that maps elements to their inverses continuous.

For notational convenience and unless stated otherwise ***all semigroups will be multiplicative.*** If a semigroup has an identity element, it will be denoted by e. Given subsets A and B of a semigroup, we will write AB for the set $\{ab\colon a \in A, b \in B\}$ and A^{-1} for the set $\{a^{-1}\colon a \in A\}$. Given filters $\mathcal{F}$ and $\mathcal{G}$ of a semigroup, we will write $\mathcal{F}\mathcal{G}$ for the filter generated by $\{FG\colon F \in \mathcal{F}, G \in \mathcal{G}\}$ and $\mathcal{F}^{-1}$ for the filter generated by $\{F^{-1}\colon F \in \mathcal{F}\}$.

A ***left action*** of a semigroup S on a set X is a function $\alpha\colon S \times X \to X$ such that

1. $\alpha(ss', x) = \alpha(s, \alpha(s', x))$ for all $s, s' \in S$ and $x \in X$,
2. if S has an identity element e, then $\alpha(e, x) = x$ for all $x \in X$.

Right actions are defined in an analogous manner. For notational convenience and unless stated otherwise ***all actions will be left actions.***

If α is an action of a semigroup S on a set X and if α is the only action of S on X being considered, we will adopt the following notation: Given $s \in S$ and $x \in X$, we will write $\alpha(s,x)$ as sx. Given $A \subseteq S$ and $B \subseteq X$, we will write $\alpha(A \times B)$ as AB. Given a filter $\mathcal{F}$ on S and a filter $\mathcal{G}$ on X, we will write $\alpha[\mathcal{F} \times \mathcal{G}]$ as $\mathcal{F}\mathcal{G}$.

If S is a semigroup acting on a set X, then X is called an ***S-set*** and a subset A of X is called an ***S-subset*** of X if $SA \subseteq A$. A function $f\colon X \to Y$ between two S-sets is called an ***S-map*** if $f(sx) = sf(x)$ for all $s \in S$ and $x \in X$. A bijection that is also an S-map is called an ***S-isomorphism*** and two S-sets are called ***S-isomorphic*** if there is an S-isomorphism between them.

If a convergence semigroup/group S acts on a convergence space X and if the action of S on X is a jointly continuous function, then S is said to ***act continuously*** on X, the action of S on X is said to be a ***continuous action*** and X is called a ***convergence S-space*** .

Notions like S-subspace, product S-space, quotient S-space, S-homeomorphism, S-embedding, etc. are defined in the obvious manner. By replacing the word “convergence” with “Cauchy” and “continuous” with “Cauchy continuous” in the latter definitions, we obtain the notions of Cauchy semigroup/group , Cauchy continuous action , Cauchy S-space , etc. Note that Cauchy semigroups are convergence semigroups, Cauchy continuous actions are continuous actions, etc. Convergence S-spaces and Cauchy S-spaces will be collectively called ***S-spaces***.

The ***trivial semigroup*** $\{e\}$ is both a convergence semigroup and a Cauchy semigroup in the obvious way and every convergence and Cauchy space X is an $\{e\}$-space with respect to the ***trivial action***, namely the one defined by $ex = x$ for all $x \in X$. Thus, any facts that we prove about convergence and Cauchy S-spaces will hold for convergence and Cauchy spaces in particular.

Any notion with an “S-” prefix has a corresponding notion without an “S-” prefix. To avoid unnecessary repetitiveness, we will omit defining the notions without the “S-” prefix. For example, in the next section we will define what S-extensions are, but we will not define what extensions are since these are implicitly defined: just omit the “S-” prefixes found in the definition of an S-extension to get the definition.

Useful Results

Proposition 3.1. If $\mathcal{F}$ is a filter on a semigroup S and $\mathcal{G}$ is a filter on an S-set X and $\mathcal{U}$ is an ultrafilter finer than $\mathcal{F}\mathcal{G}$, then there exists an ultrafilter $\mathcal{V}$ finer than $\mathcal{G}$ such that $\mathcal{F}\mathcal{V} \subseteq \mathcal{U}$.

Proof. Let $\alpha\colon S \times X \to X$ be the action on X. By Proposition 2.4, there is an ultrafilter $\mathcal{W}$ on $S \times X$ finer than $\mathcal{F} \times \mathcal{G}$ such that $\alpha[\mathcal{W}] = \mathcal{U}$. Let $\mathcal{V}$ be the projection of $\mathcal{W}$ on X. Then $\mathcal{V}$ is an ultrafilter on X, $\mathcal{V}$ is finer than $\mathcal{G}$ and $\mathcal{F} \times \mathcal{G} \subseteq \mathcal{F} \times \mathcal{V} \subseteq \mathcal{W}$, which means $\mathcal{F}\mathcal{V} = \alpha[\mathcal{F} \times \mathcal{V}] \subseteq \alpha[\mathcal{W}] = \mathcal{U}$. ∎

Proposition 3.2. If X is an S-set and $A, B \subseteq S$ and $C, D \subseteq X$, then

(i) $A(C \cap D) \subseteq AC \cap AD$,

(ii) $A(C \cup D) = AC \cup AD$,

(iii) $(A \cap B)C \subseteq AC \cap BC$,

(iv) $(A \cup B)C = AC \cup BC$,

(v) $A \subseteq B$ and $C \subseteq D$ imply $AC \subseteq BD$.

Proof.

(i) If $a \in A$ and $x \in C \cap D$, then $x \in C$ and $x \in D$, so $ax \in AC$ and $ax \in AD$, hence $ax \in AC \cap AD$.

(ii) If $a \in A$ and $x \in C \cup D$, then either $x \in C$ and $x \in D$, so $ax \in AC$ or $ax \in AD$, hence $ax \in AC \cup AD$. If $x \in AC \cup AD$, then either $x = ac$ for some $a \in A$ and $c \in C$ or $x = a'd$ for some $a' \in A$ and $d \in D$, and since $c, d \in C \cup D$, in either case we have that $x \in A(C \cup D)$.

(iii) If $s \in A \cap B$ and $c \in C$, then $s \in A$ and $s \in B$, which means $sc \in AC$ and $sc \in BC$, hence $sc \in AC \cap BC$.

(iv) If $s \in A \cup B$ and $c \in C$, then $s \in A$ or $s \in B$, hence $sc \in AC$ or $sc \in BC$, hence $sc \in AC \cup BC$. If $x \in AC \cup BC$, then either $x = ac$ for some $a \in A$ and $c \in C$ or $x = bc'$ for some $b \in B$ and $c' \in C$, and since $a, b \in A \cup B$, in either case we have that $x \in (A \cup B)C$.

(v) If $s \in A$ and $x \in C$, then $s \in B$ and $x \in D$, hence $sx \in BD$. ∎

Proposition 3.3. If $\mathcal{F}, \mathcal{F}'$ are filters on a semigroup S and $\mathcal{G}, \mathcal{G}'$ are filters on an S-set X, then

(i) $\mathcal{F}(\mathcal{G} \cap \mathcal{G}') = \mathcal{F}\mathcal{G} \cap \mathcal{F}\mathcal{G}'$,

(ii) $(\mathcal{F} \cap \mathcal{F}')\mathcal{G} = \mathcal{F}\mathcal{G} \cap \mathcal{F}'\mathcal{G}$,

(iii) $(\mathcal{F} \cap \mathcal{F}')(\mathcal{G} \cap \mathcal{G}') \subseteq \mathcal{F}\mathcal{G} \cap \mathcal{F}'\mathcal{G}'$,

(iv) $\mathcal{F}(\mathcal{G} \vee \mathcal{G}') \supseteq \mathcal{F}\mathcal{G} \vee \mathcal{F}\mathcal{G}'$ whenever $\mathcal{G} \vee \mathcal{G}'$ exists,

(v) $\mathcal{F} \subseteq \mathcal{F}'$ and $\mathcal{G} \subseteq \mathcal{G}'$ implies $\mathcal{F}\mathcal{G} \subseteq \mathcal{F}'\mathcal{G}'$.

Proof.

(i) Since $\{F(G \cup G') \colon F \in \mathcal{F}, G \in \mathcal{G}, G' \in \mathcal{G}'\}$ is a basis of $\mathcal{F}(\mathcal{G} \cap \mathcal{G}')$ and since $\{FG \cup FG') \colon F \in \mathcal{F}, G \in \mathcal{G}, G' \in \mathcal{G}'\}$ is a basis of $\mathcal{F}\mathcal{G} \cap \mathcal{F}\mathcal{G}'$ and since $F(G \cup G') = FG \cup FG'$ (by Proposition 3.2), it follows that $\mathcal{F}(\mathcal{G} \cap \mathcal{G}') = \mathcal{F}\mathcal{G} \cap \mathcal{F}\mathcal{G}'$.

(ii) The proof is mutatis mutandis the same as (i).

(iii) Pick an $F \in \mathcal{F} \cap \mathcal{F}'$ and a $G \in \mathcal{G} \cap \mathcal{G}'$. Since $F \in \mathcal{F}$ and $G \in \mathcal{G}$, we have that $FG \in \mathcal{F}\mathcal{G}$. Similarly, $FG \in \mathcal{F}'\mathcal{G}'$. The result follows.

(iv) If $\mathcal{G} \vee \mathcal{G}'$ exists, a basis for $\mathcal{F}(\mathcal{G} \vee \mathcal{G}')$ is $\{F(G \cap G') \colon F \in \mathcal{F}, G \in \mathcal{G}, G' \in \mathcal{G}'\}$, and since $F(G \cap G') \subseteq FG \cap FG'$, it follows that $\mathcal{F}\mathcal{G} \vee \mathcal{F}\mathcal{G}'$ exists and is coarser than $\mathcal{F}(\mathcal{G} \vee \mathcal{G}')$.

(v) This is a direct consequence of (v) of Proposition 3.2. ∎

Proposition 3.4. If $f \colon X \to Y$ is an S-map, then

(i) $f(AB) = Af(B)$ for every $A \subseteq S$ and $B \subseteq X$, and

(ii) $f[\mathcal{F}\mathcal{G}] = \mathcal{F}f[\mathcal{G}]$ for every filter $\mathcal{F}$ on S and $\mathcal{G}$ on X.

(iii) If f is a bijection, then f^{-1} is also an S-map.

Proof. (i) Since f is an S-map, $f(AB) = \{f(sx) : s \in A, x \in B\} = \{sf(x) \colon s \in A, x \in B\} = Af(B)$. (ii) A basis for $f[\mathcal{F}\mathcal{G}]$ is $\{f(FG) \colon F \in \mathcal{F}, G \in \mathcal{G}\} = \{Ff(G) \colon F \in \mathcal{F}, G \in \mathcal{G}\}$, and since this latter set is a basis for $\mathcal{F}f[\mathcal{G}]$, it follows that $f[\mathcal{F}\mathcal{G}] = \mathcal{F}f[\mathcal{G}]$. (iii) If f is a bijection, then for given $s \in S$ and $y \in Y$, we have that $f(sf^{-1}(y)) = sf(f^{-1}(y)) = sy = f(f^{-1}(sy))$, and since f is injective, it follows that $sf^{-1}(y) = f^{-1}(sy)$. ∎

Proposition 3.5. If S is a semigroup, then $[s][s'] = [ss']$ for all $s, s' \in S$.

Proof. A basis for $[s][s']$ is $\{\{s\}\{s'\}\} = \{\{ss'\}\}$ and this latter set is a basis for $[ss']$, hence $[s][s'] = [ss']$. ∎

Proposition 3.6. If X is an S-set, then $[s]\mathcal{U}$ is an ultrafilter on X for every $s \in S$ and every ultrafilter $\mathcal{U}$ on X.

Proof. Pick an $s \in S$ and define $f \colon X \to X$ by $f(x) = sx$. If $\mathcal{U}$ is an ultrafilter on X, then $f[\mathcal{U}]$ is an ultrafilter on X. Now note that $f[\mathcal{U}] = [s][\mathcal{U}$. ∎

Proposition 3.7. Let $f \colon X \to Y$ be a continuous function between two convergence S-spaces. If $A \subseteq S$ and $B \subseteq X$, then $f(A \operatorname{cl}_X B) \subseteq \operatorname{cl}_Y f(AB)$. Consequently, if $\mathcal{F}$ is a filter on S and $\mathcal{G}$ is a filter on X, then $\operatorname{cl}_Y f[\mathcal{F}\mathcal{G}] \subseteq f[\mathcal{F} \operatorname{cl}_X \mathcal{G}]$.

Proof. Let $s \in A$ and $x \in \operatorname{cl}_X B$. Since $x \in \operatorname{cl}_X B$, there is a filter $\mathcal{F}$ on X that contains B and converges to x, and since $AB \in [s]\mathcal{F} \to sx$, it follows that $sx \in \operatorname{cl}_X(AB)$. Since f is continuous, $f(sx) \in f(\operatorname{cl}_X(AB)) \subseteq \operatorname{cl}_Y f(AB)$. ∎

Proposition 3.8. Let X be an S-subset of an S-set Y. If $\mathcal{F}$ be a filter on S and $\mathcal{G}$ be a filter on Y such that $\mathcal{G} \cap X \in \mathbf{F}(X)$, then $\mathcal{F}\mathcal{G} \cap X \subseteq \mathcal{F}(\mathcal{G} \cap X)$.

Proof. Follows from the fact that $F(G \cap X) \subseteq FG \cap FX \subseteq FG \cap X$ for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$. ∎

S-Extensions

An ***S-extension*** of a convergence S-space X is a convergence S-space Y together with a dense S-embedding f of X in Y; and if, in addition, $f(X)$ is strictly dense in Y, we say that Y is a ***strict S-extension*** of X. S-extensions of Cauchy S-spaces are defined in an analogous manner. For simplicity and notational convenience, ***we will normally treat an S-extension of an S-space X as if it were a superset of X***.

A compact Hausdorff S-extension is called an ***S-compactification*** . A complete Hausdorff S-extension is called an ***S-completion*** . Notice that these notions are required to be Hausdorff, primarily because we want to take advantage of the theorems below and these theorems require the extensions to be Hausdorff.

Let Y, Z be two S-extensions of a convergence S-space X. We say that Y is ***larger*** or ***greater*** than Z if there is a continuous surjective S-map $f\colon Y \to Z$ whose restriction on X is the identity function on X. We say that Y is ***equivalent*** to Z if there is an S-homeomorphism $f\colon Y \to Z$ whose restriction on X is the identity function on X. These two relations, adjusted accordingly, also apply to S-extensions of Cauchy S-spaces.

Theorem 3.9. If Y, Z are Hausdorff S-extensions of an S-space X, then Y is equivalent to Z if and only if Y is larger than Z and Z is larger than Y.

Proof. The forward implication is easy, so we focus on proving the reverse implication. Suppose $f\colon Y \to Z$ and $g\colon Z \to Y$ are continuous or Cauchy continuous surjective S-maps such that $f(x) = g(x) = x$ for all $x \in X$. We need to show that f and g are inverses of each other, so we need to prove that $f \circ g$ is the identity function on Z and $g \circ f$ is the identity function on Y. Consider $f \circ g$. Since X is dense in Y and Y is Hausdorff and since the identity function on Z and $f \circ g$ are continuous extensions of the identity function on X, these two extensions must be the same. A similar argument applies to $g \circ f$, thus finishing the proof. ∎

The relation "is larger than" is a preorder on the collection of Hausdorff S-extensions of a given S-space. The above theorem essentially says that we can think of this preorder as a partial order if we agree not to distinguish between equivalent S-extensions. Since equivalent S-extensions are S-homeomorphic, it makes sense not to distinguish them.

Theorem 3.10. Let Y, Z be two S-extensions of a convergence S-space X and let $f\colon Y \to Z$ be a continuous function such that $f(x) = x$ for all $x \in X$. If Z is Hausdorff, then

(i) f is an S-map and

(ii) $f(Y - X) \subseteq f(Z - X)$,

and if in addition Y is compact, then

(iii) f is onto and consequently Y is greater than Z,

(iv) $f(Y - X) = Z - X$, and

(v) X is open in Y if and only if X is open in Z.

Proof.

(i) Since f is the identity function on X, $f(sx) = sx = sf(x)$ for all $s \in S$ and $x \in X$. Let $s \in S$ and $y \in Y$ be arbitrary. Since X is dense in Y, there is a filter $\mathcal{F}$ on X such that $[\mathcal{F}]_Y \to y$ in Y, hence $[s]\mathcal{F} \to sy$ in Y, and since f is continuous, we have that $f[[s]\mathcal{F}] \to f(sy)$ in Z, $f[\mathcal{F}] \to f(y)$ in Z and $[s]f[\mathcal{F}] \to sf(y)$ in Z. Since $\{sf(F)\colon F \in \mathcal{F}\}$ is a basis for $[s]f[\mathcal{F}]$ and since $\{f(sF)\colon F \in \mathcal{F}\}$ is a basis for $f[[s]\mathcal{F}]$ and since $sf(F) = f(sF)$ for all $F \in \mathcal{F}$ (because f

is an S-map on X), it follows that $[s]f[\mathcal{F}] = f[[s]\mathcal{F}]$. Since Z is Hausdorff and since $f(sy)$ and $sf(y)$ are limits of $[s]f[\mathcal{F}] = f[[s]\mathcal{F}]$, it follows that $f(sy) = sf(y)$.

(ii) Let $y \in Y - X$. Suppose $f(y) = x$ for some $x \in X$. Since $\mathrm{cl}_Y X = Y$, there is an ultrafilter $\mathcal{U}$ on X such that $[\mathcal{U}]_Y \to y$ in Y. Since f is continuous, $f[\mathcal{U}] \to f(y) = x$ in Z. Since f is the identity on X, the trace of $f[\mathcal{U}]$ on X is just $\mathcal{U}$, and since $f[\mathcal{U}] \to x$ in Z, it follows that $\mathcal{U} \to x$ in X. But this implies that $[\mathcal{U}]_Y \to x$ in Y, which means that $[\mathcal{U}]$ converges to two distinct limits in Y, namely x and y. But this is a contradiction since Y is Hausdorff. Ergo, $f(y) \in Z - X$, and since y is an arbitrary point in $Y - X$, it follows that $f(Y - X) \subseteq Z - X$.

(iii) Since $X = f(X) \subseteq f(Y)$ and X is dense in Z, it follows that $\mathrm{cl}_Z f(Y) = Z$. Since f is continuous and Y is compact, $f(Y)$ is compact, and since Z is Hausdorff, $f(Y)$ is closed in Z. Thus, $f(Y) = Z$.

(iv) Since f is onto, $f(Y) = Z$, and since $f(X) = X$ and $f(Y - X) \subseteq Z - X$, it follows that $f(Y - X) = Z - X$.

(v) Assume that X is open in Y. Then $Y - X$ is closed and therefore compact since Y is compact. Since $f(Y - X) = Z - X$ and f is continuous, $Z - X$ is compact and therefore closed since Z is Hausdorff. It follows that X is open in Z. Now suppose X is not open in Y. Then $Y - X$ is not closed, which means there is a $x \in X$ and a filter $\mathcal{F}$ on Y that converges to x and contains $Y - X$. By the continuity of f, $f[\mathcal{F}] \to f(x) = x$ in Z, and by (ii), $f[\mathcal{F}]$ contains $f(Y - X) = Z - X$. This means that $Z - X$ is not closed and that X is not open in Z. ∎

Theorem 3.11. Let Y be a regular Hausdorff strict extension of a convergence S-space X that has the following property:

(P) For every convergent filter $\mathcal{F}$ on S and every filter $\mathcal{G}$ on X such that $[\mathcal{G}]$ converges in Y, the filter $[\mathcal{F}\mathcal{G}]$ converges in Y.

Then there is a continuous action of S on Y making Y into a regular strict S-extension of X.

Proof. If $X = Y$, there is nothing to do. Otherwise, extend the action of S from X to Y as follows:

(A) Given $s \in S$ and $y \in Y - X$, let sy be the limit in Y of the filter $[\mathcal{F}\mathcal{G}]$, where $\mathcal{F}$ is any filter on S converging to s in S and $\mathcal{G}$ is any filter on X such that $[\mathcal{G}]$ converges to y in Y.

Let us check that this is well-defined. Let $s \in S$, let $y \in Y - X$ and let $\mathcal{F}$ be any filter on S converging to s (e.g. $[s]$). Since X is dense in Y, there is a filter $\mathcal{G}$ on X such that $[\mathcal{G}] \to y$ in Y. By (P), $[\mathcal{F}\mathcal{G}]$ converges in Y and by (A) sy is the limit of $[\mathcal{F}\mathcal{G}]$. Now suppose $\mathcal{F}'$ is another filter on S converging in S to s and $\mathcal{G}'$ is another filter on X such that $[\mathcal{G}'] \to y$ in Y. Then $\mathcal{F} \cap \mathcal{F}' \to s$ in S and $[\mathcal{G}] \cap [\mathcal{G}'] = [\mathcal{G} \cap \mathcal{G}'] \to y$ in Y. By (P), $[(\mathcal{F} \cap \mathcal{F}')(\mathcal{G} \cap \mathcal{G}')] \to sy$ in Y. By Proposition 3.3, $(\mathcal{F} \cap \mathcal{F}')(\mathcal{G} \cap \mathcal{G}') \subseteq \mathcal{F}\mathcal{G} \cap \mathcal{F}'\mathcal{G}'$, hence $[\mathcal{F}\mathcal{G} \cap \mathcal{F}'\mathcal{G}'] \to sy$ in Y. Since Y is Hausdorff and $[\mathcal{F}\mathcal{G} \cap \mathcal{F}'\mathcal{G}']$ is a subset of both $[\mathcal{F}\mathcal{G}]$ and $[\mathcal{F}'\mathcal{G}']$, both $[\mathcal{F}\mathcal{G}]$ and $[\mathcal{F}'\mathcal{G}']$ must converge in Y to the same limit, namely sy.

We now check that the given action of S on Y is indeed an action: Let $s, s' \in S$, let $y \in Y - X$, and let $\mathcal{G}$ be a filter on X such that $[\mathcal{G}] \to y$ in Y. By (A), $[[s]\mathcal{G}] \to sy$. Since $[s'][s] = [s's]$ and since $[s's] \to s's$ in S, by (A) we have that $[[s's]\mathcal{G}] \to (s's)y$ in Y. To show that $(s's)y = s'(sy)$, we consider two cases. (i) Suppose $sy = x$ for some $x \in X$. Then since X is a subspace of Y and $[[s]\mathcal{G}] \to sy = x$ in Y, we have that $[s]\mathcal{G} \to x$ in X, hence $[s']([s]\mathcal{G}) = [s's]\mathcal{G} \to s'x = s'(sy)$ in X, hence $[[s's]\mathcal{G}] \to s'(sy)$ in Y, hence $s'(sy) = (s's)y$ since Y is Hausdorff and $[[s's]\mathcal{G}] \to (s's)y$. (ii) Suppose $sy = y'$ for some $y' \in Y - X$. By (A), $[[s'][s]\mathcal{G}] \to s'y' = s'(sy)$, and since $[[s'][s]\mathcal{G}] = [[s's]\mathcal{G}] \to (s's)y$ and Y is Hausdorff, it follows that $s'(sy) = (s's)y$. Lastly, if $e \in S$, then by (A) $[[e]\mathcal{G}] \to ey$, but since $[[e]\mathcal{G}] = [\mathcal{G}] \to y$ and Y is Hausdorff, it follows that $ey = y$.

Before we prove S acts continuously on Y, we will need the following result:

Lemma. For every $A \subseteq S$ and every $B \subseteq X$, $A \operatorname{cl}_Y B \subseteq \operatorname{cl}_Y AB$. Consequently, for every filter $\mathcal{F}$ on S and every filter $\mathcal{G}$ on X, $\operatorname{cl}_Y[\mathcal{F}\mathcal{G}] \subseteq \mathcal{F} \operatorname{cl}_Y[\mathcal{G}]$.

Proof. Let $s \in A$ and $y \in \operatorname{cl}_Y B$. Suppose $y = x$ for some $x \in X$. Then $x \in \operatorname{cl}_X B$. Since $A \operatorname{cl}_X B \subseteq \operatorname{cl}_X(AB)$ by Proposition 3.7 and since $\operatorname{cl}_X(AB) \subseteq \operatorname{cl}_Y(AB)$, it follows that $sy \in \operatorname{cl}_Y(AB)$. Now suppose $y \in Y - X$. Since $y \in \operatorname{cl}_Y B$, there is a filter $\mathcal{F}$ on Y that contains B and converges to y in Y. Since $\mathcal{F}$ contains B, it contains X, hence the trace $\mathcal{G}$ of $\mathcal{F}$ on X is a filter on X such that $[\mathcal{G}] \to y$ in Y. By (A), $[[s]\mathcal{G}] \to sy$ in Y. Since $[[s]\mathcal{G}] = [s][\mathcal{G}] = [s]\mathcal{F}$ and $AB \in [s]\mathcal{F} \to sy$, it follows that $sy \in \operatorname{cl}_Y(AB)$.

We are now ready to prove that S acts continuously on Y. Let $\mathcal{F} \to s$ in S and let $\mathcal{G} \to y$ in Y. Since Y is a strict extension of X and $\mathcal{G} \to y$ in Y, there exists a filter $\mathcal{H}$ on X such that $[\mathcal{H}] \to y$ in Y and $\operatorname{cl}_Y[\mathcal{H}] \subseteq \mathcal{G}$. We consider two cases. (i) Suppose $y = x$ for some x in X. Since X is a subspace of Y and $[\mathcal{H}] \to x$ in Y, we have that $\mathcal{H} \to x$ in X, hence $\mathcal{F}\mathcal{H} \to sx$ in X since X is a convergence S-space, hence $[\mathcal{F}\mathcal{H}] \to sx$ in Y. Since Y is regular, $\operatorname{cl}_Y[\mathcal{F}\mathcal{H}] \to sx$ in Y. Using the lemma above and the fact that $\operatorname{cl}_Y[\mathcal{H}] \subseteq \mathcal{G}$, we have that $\operatorname{cl}_Y[\mathcal{F}\mathcal{H}] \subseteq \mathcal{F}\operatorname{cl}_Y[\mathcal{H}] \subseteq \mathcal{F}\mathcal{G}$, hence $\mathcal{F}\mathcal{G} \to sx$ in Y. (ii) Suppose $y \in Y - X$. By (A), $[\mathcal{F}\mathcal{H}] \to sy$ in Y, and since Y is regular, $\operatorname{cl}_Y[\mathcal{F}\mathcal{H}] \to sy$ in Y. Using the lemma above and the fact that $\operatorname{cl}_Y[\mathcal{H}] \subseteq \mathcal{G}$ we have that $\operatorname{cl}_Y[\mathcal{F}\mathcal{H}] \subseteq \mathcal{F}\operatorname{cl}_Y[\mathcal{H}] \subseteq \mathcal{F}\mathcal{G}$, hence $\mathcal{F}\mathcal{G} \to sy$ in Y.

Since S acts continuously on Y and Y is a regular strict extension of X, it follows that Y is a regular strict S-extenions of X. ∎

Remainder-Invariant S-Extensions

If Y is an S-extension of an S-space X, then the complement $Y - X$ is called the ***remainder*** of the extension. An S-extension with a one-point remainder is called a ***one-point extension*** .

An S-extension Y of an S-space X is called ***remainder invariant*** whenever the remainder is an S-subset of Y.

If Y is an S-extension of an S-space X and S is a group, then Y is remainder invariant: if for some $s \in S$ and $y \in Y - X$ we have that $sy \in X$, then $s^{-1}sy = y \in X$, which is a contradiction. This observation implies that any result that we prove about remainder-invariant S-extensions will hold for S-extensions where S is a convergence group.

The following theorem essentially says that the collection of Hausdorff remainder-invariant S-extensions of a given S-space is partially ordered if we agree not to distinguish between equivalent S-extensions.

Theorem 3.12. If Y and Z are S-extensions of an S-space X and Y is larger than Z, then Y is remainder invariant if and only if Z is remainder invariant.

Proof. Since Y is larger than Z, there exists a continuous or Cauchy continuous surjective S-map $f\colon Y \to Z$ such that $f(x) = x$ for all $x \in X$. By Theorem 3.10, $f(Y - X) = Z - X$. Suppose Y is remainder invariant and $z \in Z - X$. Since $f(Y - X) = Z - X$, there exists a $y \in Y - X$ such that $f(y) = z$. Since Y is remainder invariant, $sy \in Y - X$ for all $s \in S$, and since $f(Y - X) = Z - X$, it follows that $f(sy) = sf(y) = sz \in Z - X$ for all $s \in S$. Thus, Z is remainder invariant. Now suppose Z is remainder invariant and $y \in Y - X$. Since $f(Y - X) = Z - X$ and Z is remainder invariant, $f(sy) = sf(y) \in Z - X$ for all $s \in S$. Since $f(X) = X$, for any $s \in S$, it is impossible for sy to be in X, hence sy must be in $Y - X$. This proves that Y is remainder invariant. ∎

Adherence-Restrictive S-Spaces

A convergence S-space X is called ***adherence restrictive*** if for all convergent filters $\mathcal{F}$ on S and all filters $\mathcal{G}$ on X with empty adherence, the filter $\mathcal{F}\mathcal{G}$ has empty adherence. A Cauchy S-space X is called adherence restrictive if for every convergent filter $\mathcal{F}$ on S and every Cauchy filter $\mathcal{G}$ on X with empty adherence, the filter $\mathcal{F}\mathcal{G}$ has empty adherence. Note that compact convergence S-spaces and complete Cauchy S-spaces are vacuously adherence restrictive.

The theorem below shows that adherence-restrictive S-spaces are a generalization of those spaces that have convergence or Cauchy groups acting on them.

Theorem 3.13. If X is an S-space and S is a convergence or a Cauchy group, then X is adherence restrictive.

Proof. Let $\mathcal{F} \to s$ in S and let $\mathcal{G}$ be a filter or a Cauchy filter on X with empty adherence. By way of contradiction, suppose x is an adherent point of $\mathcal{F}\mathcal{G}$. Let $\mathcal{U}$ be an ultrafilter finer than $\mathcal{F}\mathcal{G}$ that converges to x (afforded to us by the fact that $x \in \operatorname{adh} \mathcal{F}\mathcal{G}$) and let $\mathcal{V}$ be an ultrafilter finer than $\mathcal{G}$ such that $\mathcal{F}\mathcal{V} \subseteq \mathcal{U}$ (afforded to us by Proposition 3.1). Since S is a convergence/Cauchy group, $\mathcal{F}^{-1} \to s^{-1}$ and consequently $\mathcal{F}^{-1}\mathcal{U} \to s^{-1}x$. We now prove that $\mathcal{F}^{-1}\mathcal{U} \vee \mathcal{V}$ exists. This will lead to a contradiction since it implies that $\mathcal{V} \subseteq \mathcal{F}^{-1}\mathcal{U} \vee \mathcal{V} \to s^{-1}x$, contradicting that $\operatorname{adh} \mathcal{G} = \emptyset$. Let $F^{-1}U \cap V$ be an arbitrary basis element of $\mathcal{F}^{-1}\mathcal{U} \vee \mathcal{V}$. Since $\mathcal{F}\mathcal{V} \subseteq \mathcal{U}$, the set $U \cap FV$ is not empty and contains a point x. Since $x = u$ for some $u \in U$ and $x = sv$ for some $s \in F$ and $v \in V$, we have that $u = sv$, and since S is a group, $s^{-1}u = v$, and consequently, $s^{-1}u = v \in F^{-1}U \cap V$. ∎

Theorem 3.14. If Y is a compact or complete Hausdorff S-extension of an S-space X, then Y is remainder invariant if and only if X is adherence restrictive.

Proof. Suppose Y is a compact Hausdorff S-extension of a convergence S-space X. If $Y = X$, then Y is vacuously remainder-invariant and X is vacuously adherence-restricted, so we assume that $Y \neq X$. Suppose Y is remainder invariant. By way of contradiction, suppose X is not adherence restrictive, i.e. suppose there is a convergent filter $\mathcal{F}$ on S and a filter $\mathcal{G}$ on X such that $\operatorname{adh}\mathcal{G} = \emptyset$ and $\mathcal{F}\mathcal{G}$ has an adherent point x. Let $\mathcal{U}$ be an ultrafilter finer than $\mathcal{F}\mathcal{G}$ that converges to x in X and let s be a limit of $\mathcal{F}$. Since $\mathcal{F}\mathcal{G} \subseteq \mathcal{U}$ and $\mathcal{U}$ is an ultrafilter, by Proposition 3.1 there is an ultrafilter $\mathcal{V}$ on X finer than $\mathcal{G}$ such that $\mathcal{F}\mathcal{V} \subseteq \mathcal{U}$. Since $\operatorname{adh}\mathcal{G} = \emptyset$, the filter $\mathcal{V}$ does not converge in X, hence $[\mathcal{V}]$ converges in Y to some $y \in Y - X$ (using the fact that $X \subset Y$ and that Y is compact). This means that $\mathcal{F}[\mathcal{V}] \to sy$ in Y, and since $\mathcal{F}\mathcal{V} \subseteq \mathcal{U}$, $[\mathcal{U}] \to sy$ in Y. Since Y is remainder invariant, $sy \in Y - X$, but this means x and sy are distinct limits of $[\mathcal{U}]$, contradicting that Y is Hausdorff.

Now suppose Y is not remainder invariant. Then for some $x \in X$, $y \in Y - X$ and $s \in S$, $sy = x$. Since X is dense in Y, y is a point of closure of X, which means that there is some ultrafilter $\mathcal{V}$ on Y containing X that converges to y. The trace $\mathcal{U}$ of $\mathcal{V}$ on X is therefore an ultrafilter on X that does not converge in X, for otherwise $\mathcal{V}$ would have two limits in Y, contradicting that Y is Hausdorff. Using Proposition 3.4, we have that $[[s]\mathcal{U}] = [s][\mathcal{U}] = [s]\mathcal{V} \to sy = x$ in Y, hence $[s]\mathcal{U} \to x$ in X. Since $\mathcal{U}$ is a filter on X with empty adherence, it follows that X is not adherence restrictive.

The proof when Y is a complete Hausdorff S-extension of a Cauchy S-space X is basically the same: just use completeness wherever compactness is used and use Cauchy filters instead of filters where appropriate. ∎

Theorem 3.15. If a convergence S-space X has a compact Hausdorff S-extension, then X is adherence restritive if and only if for every non-convergent ultrafilter $\mathcal{U}$ on X and every $s \in S$, $[s]\mathcal{U}$ does not converge.

Proof. The forward implication follows from the definition of being adherence restrictive, so we focus on proving the reverse implication, which we will prove by proving the contrapositive statement. Suppose X is not adherence restrictive. Then there exists a convergent filter $\mathcal{F}$ on S and a filter $\mathcal{G}$ on X with empty adherence such that $\operatorname{adh}\mathcal{F}\mathcal{G} \neq \emptyset$. Pick an $x \in \operatorname{adh}\mathcal{F}\mathcal{G}$. Then there is an ultrafilter $\mathcal{U}$ finer than $\mathcal{F}\mathcal{G}$ that converges to x and by Proposition 3.1 there exists an ultrafilter $\mathcal{V}$ on X finer than $\mathcal{G}$ such that $\mathcal{F}\mathcal{V} \subseteq \mathcal{U}$. Let Y be a compact Hausdorff S-extension of X. Then $[\mathcal{U}] \to x$ in Y and $[\mathcal{V}]$ converges to some y in Y. Let s be a limit of $\mathcal{F}$ in X. Then $\mathcal{F}[\mathcal{V}] \to sy$ in Y and since $\mathcal{F}[\mathcal{V}] \subseteq [\mathcal{U}]$ and Y is Hausdorff, it follows that $sy = x$. By Proposition 3.6, $[s]\mathcal{V}$ is an ultrafilter on X and it either converges in X or it doesn't. We claim that it does, for otherwise, $[s][\mathcal{V}]$ would converge to something in $Y - X$, contradicting that $[s][\mathcal{V}] \to sy = x$. This completes the proof. ∎

CHAPTER 4: S-COMPACTIFICATIONS

Compactifications of Convergence Spaces - A Review

We now review some results about compactifications in the theory of convergence spaces. We will generalize some of these results in subsequent sections. These results are taken from [13].

Theorem 4.1. Let X be a non-compact Hausdorff convergence space. Let $\omega X = X \cup \{\omega\}$, where $\omega \notin X$, and define a preconvergence structure on ωX as follows:

(a) A filter $\mathcal{F}$ on ωX converges to a point x of X if there is a filter $\mathcal{G}$ on X that converges to x and $[\mathcal{G}] \subseteq \mathcal{F}$.

(b) A filter $\mathcal{F}$ on ωX converges to ω if there is a filter $\mathcal{G}$ on X with empty adherence such that $[\mathcal{G}] \cap [\omega] \subseteq \mathcal{F}$.

Then ωX equipped with this preconvergence structure is a compactification of X called the ***Rao compactification*** of X and it is equal to the Alexandrov one-point compactification when X is locally compact and topological.

Theorem 4.2. Let X be a non-compact Hausdorff convergence. For $A \subseteq X$, let A' denote the set of non-convergent ultrafilters on X containing A and let $A^* = A \cup A'$. Given a filter $\mathcal{F}$ on X, let $\mathcal{F}^*$ denote the filter on X^* generated by $\{F^* \subseteq X^* : F \in \mathcal{F}\}$. Define a preconvergence structure on X^* as follows:

(a) A filter $\mathcal{F}$ on X^* converges to a point x of X if there is a filter $\mathcal{G}$ on X that converges to x such that $\mathcal{G}^* \subseteq \mathcal{F}$.

(b) A filter $\mathcal{F}$ on X^* converges to an ultrafilter $\mathcal{U} \in X'$ if $\mathcal{U}^* \subseteq \mathcal{F}$.

Then X^* with the above preconvergence structure is a compactification of X, which we will call the ***star compactification*** X.

Theorem 4.3. The following statements about be a Hausdorff convergence space X are equivalent.

(i) X has finitely many non-convergent ultrafilters.

(ii) X is open in each of its compactifications.

(iii) X has a smallest compactification (equivalent to ωX when X is not compact)

(iv) X has a largest compactification (equivalent to X^* when X is not compact).

Theorem 4.4. If Y is a regular compactification of a convergence space X, then πY is a regular topological compactification πX.

Proof. Every ultrafilter on Y converges in Y and hence converges in πY, which means πY is compact. Since $Y = \mathrm{cl}_Y X = \mathrm{cl}_{\pi Y} X$ and since πX is a subspace of πY and πY is Hausdorff and topological (by Proposition 2.6), it follows that πY is a topological compactification of πX. Now recall that compactifications of topological spaces are always regular. ∎

Theorem 4.5. The following statements about a Hausdorff convergence space X are equivalent.

(i) X is completely regular.

(ii) X and πX agree on ultrafilter convergence and πX is a completely regular topological space.

(iii) X has a regular compactification.

(iv) X has a largest regular compactification.

The largest regular compactification of a completely regular Hausdorff convergence space X is called the ***Stone-Čech regular compactification*** and it is denoted βX. The smallest regular compactification of a non-compact completely regular Hausdorff convergence space X is ωX. Both of these compactifications, when they exist in this context, are strict.

Theorem 4.6. The follows statements about a completely regular Hausdorff convergence space X are equivalent.

(i) X has a smallest regular compactification (equivalent to ωX when X is not compact).

(ii) πX is a locally compact topological space.

(iii) X is a locally compact convergence space.

(iv) X is open in each of its regular compactifications.

From Compactifications to S-Compactifications

This section contains some results that generalize those corresponding results from the previous section. We start with the one-point compactification ωX.

The Rao S-Compactification

Theorem 4.7. Let X be a non-compact Hausdorff convergence S-space and let ωX be as in Theorem 4.1. Extend the action of S to ωX by letting $s\omega = \omega$ for all $s \in S$. Then ωX is a one-point remainder-invariant S-compactification of X if and only if X is adherence restrictive.

Proof. Some algebra shows that the action of S on ωX is well-defined. We now prove that the action is continuous. Let $\mathcal{F} \to s$ in S and let $\mathcal{G}$ be a filter on ωX. We consider two cases. (i) If $\mathcal{G}$ converges to a point x of X, there is a filter $\mathcal{H}$ on X such that $\mathcal{H} \to x$ and $\mathcal{H} \subseteq \mathcal{G}$. This means that $\mathcal{F}\mathcal{H} \subset \mathcal{F}\mathcal{G}$ and since $\mathcal{F}\mathcal{H} \to sx$, it follows that $\mathcal{F}\mathcal{G} \to sx$. (ii) If $\mathcal{G} \to \omega$, there is a filter $\mathcal{H}$ on X such that $\mathrm{adh}\,\mathcal{H} = \emptyset$ and $[\mathcal{H}] \cap [\omega] \subseteq \mathcal{G}$. Since X is adherence restrictive, $\mathrm{adh}\,\mathcal{F}\mathcal{H} = \emptyset$. By

Proposition 3.3, $\mathcal{F}([\mathcal{H}] \cap [\omega]) = \mathcal{F}[\mathcal{H}] \cap \mathcal{F}[\omega]$. Since $\{F\{\omega\}\colon F \in \mathcal{F}\}$ is a basis for $\mathcal{F}[\omega]$ and since $F\{\omega\} = \{s\omega\colon s \in F\} = \{\omega\}$, we have that $\mathcal{F}[\omega] = [\omega]$. By Proposition 3.4 we have that $\mathcal{F}[\mathcal{H}] = [\mathcal{F}\mathcal{H}]$. Since $\mathcal{G} \supseteq [\mathcal{H}] \cap [\omega]$, we have that $\mathcal{F}\mathcal{G} \supseteq \mathcal{F}([\mathcal{H}] \cap [\omega]) = [\mathcal{F}\mathcal{H}] \cap [\omega]$, and since $\operatorname{adh} \mathcal{F}\mathcal{H} = \emptyset$, it follows that $\mathcal{F}\mathcal{G} \to \omega$. We have thus shown that if $\mathcal{G}$ converges to some arbitrary point y of ωX, then $\mathcal{F}\mathcal{G} \to sy$, which means the action is continuous and that ωX is an S-space. By construction, ωX is remainder-invariant S-extension of X. This fact together with Theorem 4.1 prove that ωX is a one-point remainder-invariant S-compactification of X.

Conversely, if ωX is a one-point remainder-invariant S-compactification of X, then by Theorem 3.14, X is adherence restrictive. ∎

Henceforth, the default action on ωX ***is the one specified in the aforementioned theorem.***

Theorem 4.8. A non-compact Hausdorff convergence S-space X has a smallest remainder-invariant S-compactification if and only if it is adherence restrictive and open in each of its remainder-invariant S-compactifications. Moreover, if X has a smallest remainder-invariant S-compactification, it is equivalent to ωX.

Proof. Suppose X has a smallest remainder-invariant S-compactification Y. By Theorem 3.14, X is adherence restrictive, hence by Theorem 4.7, ωX as is a one-point remainder-invariant S-compactification of X. Since ωX is larger than Y and X is open in ωX, by Theorem 3.10, Y has a one-point remainder, X is open in Y and therefore X is open in each of its remainder-invariant S-compactifications.

Now suppose X is adherence restrictive and open in each of its remainder-invariant S-compactifications. Then by Theorem 4.7, ωX is a one-point remainder-invariant S-compactification of X. We now prove that ωX is the smallest remainder-invariant S-compactification of X: Let Y be any remainder-invariant S-compactification of X and define $f\colon Y \to \omega X$ so that f is the

identity on X and everything in the remainder of Y gets mapped to ω. First, we prove that f is continuous: Let $\mathcal{F}$ be a filter on Y. We consider two cases: (i) Suppose $\mathcal{F}$ converges to a point x of X. Since X is open in Y and $\mathcal{F} \to x$, it follows that $X \in \mathcal{F}$. This implies that $\mathcal{F}$ has a trace $\mathcal{G}$ on X which converges to x and that $\mathcal{G}$ is a basis on Y for $\mathcal{F}$. Since $\mathcal{G} \to x$ in X, $[\mathcal{G}]_{\omega X} \to x$ in ωX. Since $\mathcal{G}$ is a basis for $\mathcal{F}$, $f[\mathcal{F}] = f[\mathcal{G}]$, and since $\mathcal{G}$ is a filter on X and f is the identity on X, $f[\mathcal{G}] = [\mathcal{G}]_{\omega X}$. Thus, $f[\mathcal{F}] \to x = f(x)$ in ωX. (ii) Now suppose $\mathcal{F}$ converges to some point y of $Y - X$. If $Y - X \in \mathcal{F}$, then $f[\mathcal{F}] = [\omega] \to \omega = f(y)$. Otherwise, the intersection of any two sets in $\mathcal{F}$ must contain elements of X, which means that $\mathcal{F}$ has a trace $\mathcal{G}$ on X. We claim that $\mathcal{G}$ has empty adherence: Suppose not, i.e. suppose there is an ultrafilter $\mathcal{U}$ on X finer than $\mathcal{G}$ that converges to some point x of X. Since $\mathcal{F} \subseteq [\mathcal{G}]_Y \subseteq [\mathcal{U}]_Y$ and since $[\mathcal{U}]_Y \to x$ and $\mathcal{F} \to y$, $[\mathcal{U}]_Y$ has two distinct limits, namely x and y, contradicting that Y is Hausdorff. We now claim that $[\mathcal{G}]_{\omega X} \cap [\omega] \subseteq f[\mathcal{F}]$, proving that $f[\mathcal{F}] \to \omega = f(y)$: Pick an arbitrary basis element $f(F)$ of $f[\mathcal{F}]$. Since $f(F) = f((F \cap X) \cup (F - X)) = f(F \cap X) \cup f(F - X) \subseteq (F \cap X) \cup \{\omega\}$ and since $(F \cap X) \cup \{\omega\}$ is an arbitrary basis element of $[\mathcal{G}]_{\omega X} \cap [\omega]$, it follows that $[\mathcal{G}]_{\omega X} \cap [\omega] \subseteq f[\mathcal{F}]$ as claimed. This completes the proof that f is continuous. All that is left to prove is that f is an S-map: Pick an $s \in S$, an $x \in X$ and a $y \in Y - X$. Since f is the identity function on X, $f(sx) = sx = sf(x)$. Since Y is remainder-invariant, $sy \in Y - X$, and since $f(Y - X) = \{\omega\}$, we have that $f(sy) = \omega = s\omega = sf(y)$.
∎

Example 4.9. Let $X = [0,1)$ and $S = (0,1]$ be equipped with their usual topologies, let the operation on S be multiplication and let S act on X via multiplication. With this setup, X is an S-space. Let $Y = [0,1]$ be equipped with its usual topology and let the action of S on X be multiplication. In this way, Y is also an S-space, and in fact it is a one-point S-compactification of X. However, Y is not remainder invaraint: if $s = \frac{1}{2}$ and $y = 1$, then $sy = \frac{1}{2} \in X$ even though $y \in Y - X$. Letting $\omega = 1$, let us now consider ωX. It is homeomorphic to Y but it is not S-homeomorphic to Y because it is not even an S-space since the action of S on ωX is not continuous at (s, ω)

for every $s \neq 1$. For example, if $\mathcal{F}$ is the elementary filter on ωX corresponding to the sequence whose nth term is $\frac{n}{n+1}$, then $[\frac{1}{2}]\mathcal{F} \to \frac{1}{2}$, but if the action of S on ωX were continuous, we should have $[\frac{1}{2}]\mathcal{F} \to 1$. It follows from Theorem 4.7 that X is not adherence restrictive.

Example 4.10. Let $X = [0,\infty)$ and let S be the multiplicative group $(0,\infty)$. Let S act on X via multiplication and equip X and S with their usual topologies. Then X is an S-space and $\omega X = [0,\infty]$ with its usual topology is an S-compactification X. Note that since S is a group, X is adherence restrictive and ωX is remainder invariant.

The Star S-Compactification

Theorem 4.11. Let X be a non-compact Hausdorff S-space and suppose that the convergence structure on S is discrete and that X is adherence restrictive. Then there is an action of S on the star compactification X^* of X making it a remainder-invariant S-compactification of X.

Proof. Recall that $X^* = X \cup X'$, where X' consists of all the non-convergent ultrafilters on X. Extend the action of S to X^* by letting $s\mathcal{U} = [s]\mathcal{U}$ for each $\mathcal{U} \in X'$ and each $s \in S$. Let us check that this is well-defined. Fix an $s \in S$ and a $\mathcal{U} \in X'$. Since X is adherence restrictive, $[s]\mathcal{U}$ does not converge in X. By Proposition 3.6, $[s]\mathcal{U}$ is a ultrafilter. Thus, $[s]\mathcal{U} \in X'$, which means that $s\mathcal{U} \in X^*$. A little algebra shows that this action is a valid action on X^* making it remainder invariant.

Let us now prove that this action is continuous. Let $\mathcal{F} \to s$ in S and let $\mathcal{G}$ be a filter on X^*. Since S has the discrete convergence structure, $\mathcal{F} = [s]$. Suppose $\mathcal{G} \to x \in X$. Then there is a filter $\mathcal{H}$ on X that converges to x such that $\mathcal{H}^* \subseteq \mathcal{G}$. We now claim that $([s]\mathcal{H})^* \subseteq [s]\mathcal{H}^*$. This follows from the following lemma.

Lemma. Let $s \in S$ and $A \subseteq X$ be arbitrary. Then $sA' \subseteq (sA)'$ and consequently $sA^* \subseteq (sA)^*$.

Proof. Let $\mathcal{U} \in A'$. Then $A \in \mathcal{U}$ and $\mathcal{U}$ is non-convergent. Since X is adherence restrictive, $[s]\mathcal{U}$ is non-convergent. By Proposition 3.6, $[s]\mathcal{U}$ is a ultrafilter. Since $sA \in [s]\mathcal{U}$, it follows that $[s]\mathcal{U} \in (sA)'$.

Since $[s]\mathcal{H} \to sx$ in X and $([s]\mathcal{H})^* \subseteq [s]\mathcal{H}^* \subseteq [s]\mathcal{G}$, it follows that $[s]\mathcal{G} \to x$ in X^*. Now suppose $\mathcal{G} \to \mathcal{U} \in X'$. Then $\mathcal{U}^* \subseteq \mathcal{G}$. Since $[s]\mathcal{U} \in X'$ and $([s]\mathcal{U})^* \subseteq [s]\mathcal{U}^* \subseteq [s]\mathcal{G}$, it follows that $[s]\mathcal{G} \to [s]\mathcal{U} = s\mathcal{U}$ in X^*. This concludes the proof that S acts continuously on X^*.

We have thus proved that X^* is an S-space and since it is a compactification of X, it follows that it is an S-compactification of X. ∎

Henceforth, the action on X^* will be the one given in the prior theorem. Note that this action only makes sense when X is adherence restrictive. Using Theorem 4.3, we obtain the following result.

Theorem 4.12. Let X be a non-compact Hausdorff S-space and suppose that the convergence structure on S is discrete and that X is adherence restrictive. If X' is finite, then X^* is the largest S-compactification of X.

Regular S-Compactifications

Theorem 4.13. Every convergence S-space with a regular S-compactification has a strict regular S-compactification.

Proof. Let Y be a regular S-compactification of a convergence S-space X. Let ρY denote the modification of Y with the following preconvergence structure: $\mathcal{F} \to y$ in ρY if and only if there is a filter $\mathcal{G}$ on X such that $[\mathcal{G}] \to y$ in Y and $\mathrm{cl}_Y[\mathcal{G}] \subseteq \mathcal{F}$. We will prove that ρY is a strict extension of X.

We first prove that ρY is a convergence space. Pick an $x \in X$. Since $[[x]_X]_Y = [x]_Y \to x$ in Y and since $\mathrm{cl}_Y[x]_Y \subseteq [x]_Y$, it follows that $[x]_Y \to x$ in ρY. Pick a $y \in Y - X$. Since X is dense in Y there exists a filter $\mathcal{G}$ on X such that $[\mathcal{G}] \to y$ in Y. Since $[\mathcal{G}] \to y$, we have that $y \in \mathrm{cl}_Y G$ for all $G \in \mathcal{G}$, hence $\mathrm{cl}_Y[\mathcal{G}] \subseteq [y]$, hence $[y] \to y$ in ρY. If $\mathcal{F} \to y$ in ρY and $\mathcal{F} \subseteq \mathcal{G}$, then there is a filter $\mathcal{H}$ on X such that $[\mathcal{H}] \to y$ in Y and $\mathrm{cl}_Y[\mathcal{H}] \subseteq \mathcal{F} \subseteq \mathcal{G}$, hence $\mathcal{G} \to y$ in ρY. If $\mathcal{F}$ and $\mathcal{F}'$ are filters on Y that converge to y in ρX, then there are filtes $\mathcal{G}$ and $\mathcal{G}'$ on X such that $[\mathcal{G}]$ and $[\mathcal{G}']$ converge to y in Y, $\mathrm{cl}_Y[\mathcal{G}] \subseteq \mathcal{F}$ and $\mathrm{cl}_Y[\mathcal{G}'] \subseteq \mathcal{F}'$, and since $[\mathcal{G} \cap \mathcal{G}'] = [\mathcal{G}] \cap [\mathcal{G}'] \to y$ in Y and $\mathrm{cl}_Y[\mathcal{G} \cap \mathcal{G}'] = \mathrm{cl}_Y[\mathcal{G}] \cap \mathrm{cl}_Y[\mathcal{G}'] \subseteq \mathcal{F} \cap \mathcal{F}'$, it follows that $\mathcal{F} \cap \mathcal{F}' \to y$ in ρY. This concludes the proof that ρY is a convergence space.

Next, note that convergence in ρY implies convergence in Y: If $\mathcal{F} \to y$ in ρY, then there is a filter $\mathcal{G}$ on X such that $[\mathcal{G}] \to y$ in Y and $\mathrm{cl}_Y[\mathcal{G}] \subseteq \mathcal{F}$, and since Y is regular, $\mathrm{cl}_Y[\mathcal{G}] \to y$ in Y and consequently $\mathcal{F} \to y$ in Y.

Now let us prove that Y and ρY agree on ultrafilter convergence. Suppose $\mathcal{V}$ is an ultrafilter on Y that converges to y in Y. By Proposition 2.10, $\mathrm{cl}_Y[\mathcal{U}] \subseteq \mathcal{V}$ for some ultrafilter $\mathcal{U}$ on X. Since Y is compact and $[\mathcal{U}]$ is an ultrafilter on Y, $[\mathcal{U}]$ converges to some $y' \in Y$. Since Y is regular, $\mathrm{cl}_Y[\mathcal{U}] \to y'$, and since $\mathrm{cl}_Y[\mathcal{U}] \subseteq \mathcal{V}$ and Y is Hausdorff, $y' = y$. Thus, $\mathcal{U}$ is a filter on X such that $[\mathcal{U}] \to y$ in Y and $\mathrm{cl}_Y[\mathcal{U}] \subseteq \mathcal{V}$, which means $\mathcal{V} \to y$ in ρY. Now suppose that $\mathcal{V} \to y$ in ρY. Since convergence in ρY implies convergence in Y, it follows that $\mathcal{V} \to y$ in Y.

Since Y and ρY agree on ultrafilter convergence, it follows that

(i) ρY is compact Hausdorff since Y Hausdorff compact,

(ii) $\mathrm{cl}_Y = \mathrm{cl}_{\rho Y}$ because for any $A \subseteq Y$ we have that $y \in \mathrm{cl}_Y A$ if and only if $A \in \mathcal{V} \to y$ in Y for some ultrafilter $\mathcal{V}$ on Y if and only if $A \in \mathcal{V} \to y$ in ρY for some ultrafilter $\mathcal{V}$ on Y if and only if $y \in \mathrm{cl}_{\rho Y} A$,

(iii) X is a strictly dense subspace of ρY by the definition of ρY and the fact that X is a subspace of Y, and

(iv) ρY is regular because if $\mathcal{F} \to y$ in ρy, then there is a filter $\mathcal{G}$ on X such that $[\mathcal{G}] \to y$ in Y and $\mathrm{cl}_Y[\mathcal{G}] \subseteq \mathcal{F}$, hence $\mathrm{cl}_Y[\mathcal{G}] = \mathrm{cl}_{\rho Y}[\mathcal{G}] \to y$ in ρY. By Proposition 2.8, $\mathrm{cl}_Y = \mathrm{cl}_{\rho Y}$ is idempotent. Since $\mathrm{cl}_{\rho Y}[\mathcal{G}] \subseteq \mathcal{F}$ and $\mathrm{cl}_{\rho Y}$ is idempotent, $\mathrm{cl}_{\rho Y} \mathrm{cl}_{\rho Y}[\mathcal{G}] = \mathrm{cl}_{\rho Y}[\mathcal{G}] \subseteq \mathrm{cl}_{\rho Y} \mathcal{F}$, hence $\mathrm{cl}_{\rho Y} \mathcal{F} \to y$ in ρY.

We have thus proved that ρY is a regular strict compactification of X. We now prove that S acts continuously on ρY. This will prove that ρY is an S-space and thus a regular strict S-compactification of X. Let $\mathcal{F} \to s$ in S and $\mathcal{G} \to y$ in ρY. Since $\mathcal{G} \to y$ in ρY, there is a filter $\mathcal{H}$ on X such that $[\mathcal{H}] \to y$ in Y and $\mathrm{cl}_Y[\mathcal{H}] \subseteq \mathcal{G}$. Since the S acts continuously on Y, $[\mathcal{F}\mathcal{H}] = \mathcal{F}[\mathcal{H}] \to sy$ in Y. By Proposition 3.7, $\mathrm{cl}_Y \mathcal{F}[\mathcal{H}] \subseteq \mathcal{F} \mathrm{cl}_Y[\mathcal{H}] \subseteq \mathcal{F}\mathcal{G}$. Thus, $\mathcal{F}\mathcal{H}$ is a filter on X such that $[\mathcal{F}\mathcal{H}] \to sy$ and $\mathrm{cl}_Y[\mathcal{F}\mathcal{H}] \subseteq \mathcal{F}\mathcal{G}$, which means $\mathcal{F}\mathcal{G} \to sy$ in ρY. ∎

Theorem 4.14. Let S be a Hausdorff convergence semigroup acting continuously on a Hausdorff convergence space X and let Y be a regular strict compactification of X. Declare a filter on S to be Cauchy if it converges and declare a filter $\mathcal{F}$ on X to be Cauchy if $[\mathcal{F}]$ converges in Y. Then there is an action of S on Y making it a regular strict S-compactification of X if and only if S acts Cauchy-continuously on X. Moreover, if $Y = \beta X$ and S acts Cauchy-continuously on X, then there is an action of S on βX making it the largest regular S-compactification of X.

Proof. First note that the collections of Cauchy filters on S and on X are valid Cauchy structures: Since $[s] \to s$ for every $s \in S$, $[s]$ is Cauchy. For every $x \in X$, $[x]_X \to x$ in X, hence $[x]_Y = [[x]_X]_Y \to x$ in Y since Y is an extension of X, hence $[x]_X$ is Cauchy. Let $\mathcal{F}$ be a Cauchy filter on S and let $\mathcal{G}$

be a Cauchy filter on X. Then $\mathcal{F} \to s$ for some $s \in S$ and $[\mathcal{G}] \to y$ in Y for some $y \in Y$. If $\mathcal{F}'$ is finer than $\mathcal{F}$ and $\mathcal{G}'$ is finer than $\mathcal{G}$, then $\mathcal{F}' \to s$ and $[\mathcal{G}'] \to y$ in Y, so $\mathcal{F}'$ and $\mathcal{G}'$ are Cauchy. If $\mathcal{F}, \mathcal{F}'$ are Cauchy filters on S such that $\mathcal{F} \vee \mathcal{F}'$ exists, then $\mathcal{F}$ and $\mathcal{F}'$ converge, and since $\mathcal{F} \vee \mathcal{F}'$ is finer than $\mathcal{F}$ and $\mathcal{F}'$, $\mathcal{F} \vee \mathcal{F}'$ converges to both the limit of $\mathcal{F}$ and the limit of $\mathcal{F}'$, and since Y is Hausdorff, the limits of $\mathcal{F}$, $\mathcal{F}'$ and $\mathcal{F} \vee \mathcal{F}'$ must be the same, hence $\mathcal{F} \cap \mathcal{F}'$ converges to this limit, which means that $\mathcal{F} \cap \mathcal{F}'$ is Cauchy. If $\mathcal{G}, \mathcal{G}'$ are Cauchy filters on X such that $\mathcal{G} \vee \mathcal{G}'$ exists, then $[\mathcal{G}]$ and $[\mathcal{G}']$ converge in Y, and since $[\mathcal{G} \vee \mathcal{G}'] = [\mathcal{G}] \vee [\mathcal{G}']$ is finer than $[\mathcal{G}]$ and $[\mathcal{G}']$, $[\mathcal{G}] \vee [\mathcal{G}']$ converges to both the limit of $[\mathcal{G}]$ and the limit of $[\mathcal{G}']$, and since Y is Hausdorff, the limits of $[\mathcal{G}]$, $[\mathcal{G}']$ and $[\mathcal{G}] \vee [\mathcal{G}']$ must be the same, hence $[\mathcal{G}] \cap [\mathcal{G}'] = [\mathcal{G} \cap \mathcal{G}']$ converges to this limit, which means that $\mathcal{G} \cap \mathcal{G}'$ is Cauchy.

Suppose there is an action of S on Y making it a regular strict S-compactification of X. Let $\mathcal{F}$ be a Cauchy filter on S and let $\mathcal{G}$ be a Cauchy filter on X. Then $\mathcal{F} \to s$ in S for some $s \in S$ and $[\mathcal{G}] \to y$ in Y for some $y \in Y$. Since S acts continuously on Y, $\mathcal{F}[\mathcal{G}] = [\mathcal{F}\mathcal{G}] \to sy$ in Y, which means $\mathcal{F}\mathcal{G}$ is a Cauchy filter on X by definition. This proves that S acts Cauchy-continuously on X.

Now suppose S acts Cauchy-continuously on X. Then for every $\mathcal{F} \to s$ in S and every filter $\mathcal{G}$ on X such that $[\mathcal{G}] \to y$ in Y, the filter $[\mathcal{F}\mathcal{G}]$ converges in Y since $\mathcal{F}$ and $\mathcal{G}$ are Cauchy filters by definition and $\mathcal{F}\mathcal{G}$ is a Cauchy filter since the action on X is Cauchy continuous. Thus, Y satisfies property (P) of Theorem 3.11, hence there is an action of S on Y making it a regular strict S-compactification of X. If $Y = \beta X$, then since βX is a regular strict compactification of X, it follows the action of S on βX given by (A) in the proof of Theorem 3.11 makes βX into a regular strict S-compactification of X. We claim that it is the largest regular S-compactification of X. Let Z be a regular S-compactification of X. Since βX is the Stone-Čech regular compactification of X and Z is compact regular Hausdorff space, the identity function $\mathrm{id}_X \colon X \to Z$ has a continuous extension $f \colon \beta X \to Z$. We claim that f is an S-map. Clearly, it is an S-map on X since it is the identity function on X. Let $s \in S$ and $y \in Y - X$ be arbitrary. Since X is dense in y, there is a filter $\mathcal{G}$ on X such that $[\mathcal{G}] \to y$ in Y. Since f is continuous, $f[\mathcal{G}] \to f(y)$ in Z. By (A), $[[s]\mathcal{G}] = [s][\mathcal{G}] \to sy$, and since f is continuous, $f[[s]\mathcal{G}] \to f(sy)$. Since S acts continuously on Z, $[s]f[\mathcal{G}] \to sf(y)$. Since f is the identity function on X, $f[\mathcal{G}] = [\mathcal{G}]_Z$ and so $[s]f[\mathcal{G}] = [s][\mathcal{G}]_Z = [[s]\mathcal{G}]_Z = f[[s]\mathcal{G}]$, and since Z is Hausdorff, $f(sy) = sf(y)$. We have proved that f is a continuous S-map from βX to Z such that $f(x) = x$ for all $x \in X$. By Theorem 3.10, it follows that βX is larger than Z. Since Z is an arbitrary regular S-compactification of X, it follows that βX is the largest regular S-compactification of X. ∎

Theorem 4.15. If X be a completely regular Hausdorff convergence S-space, then statements (i) – (iii) are equivalent and (iii) implies (iv).

(i) ωX is a one-point remainder-invariant regular S-compactification of X.

(ii) X is locally compact, non-compact and adherence restrictive.

(iii) X has a remainder-invariant regular S-compactification and it is open in each of its remainder-invariant regular S-compactification.

(iv) X has a smallest remainder-invariant regular S-compactification.

Proof. (i) $\Longrightarrow$ (ii): Suppose ωX is a one-point remainder-invariant regular S-compactification of X. Then X is non-compact and by Theorem 3.14 X adherence restrictive. By Theorem 4.4, $\pi(\omega X)$ is a one-point topological compactification of πX, which means πX is locally compact. Since X is completely regular, by Theorem 4.5, πX and X agree on ultrafilter convergence. Since πX is locally compact, it follows by Theorem 2.6 that X is locally compact.

(ii) $\Longrightarrow$ (i) and (iii): Assume X is locally compact, non-compact and adherence restrictive. Since X is locally compact, by Theorem 4.6, ωX is the smallest regular compactification of X. Since X is adherence restrictive, by Theorem 4.7, ωX is a one-point remainder-invariant regular S-compactification of X. Again, by Theorem 4.6, X is open in each of its regular compactifications, including its remainder-invariant regular S-compactifications.

(iii) $\implies$ (ii) and (iv). Assume X has a remainder-invariant regular S-compactification and it is open in each of its remainder-invariant regular S-compactification. Let Y be an arbitrary remainder-invariant regular S-compactification of X. By Theorem 4.4, πY is a topological compactification of πX. Since X is open in Y, by part (iii) of Proposition 2.6, πX is open in πY. Since a dense subset of a compact Hausdorff topological space is locally compact if and only if it is open, and since πX is an open dense subset of πY and πY is a compact Hausdorff topological space, it follows that πX is locally compact. Since X is completely regular, by (ii) of Theorem 4.5, πX and X agree on ultrafilter convergence, and so by part (iv) of Theorem 2.6, X is locally compact. By Theorem 3.14, X is adherence restrictive, so by Theorem 4.7, ωX is a one-point remainder-invariant S-compactification of X. Since X is locally compact, by Theorem 4.6, ωX is the smallest regular compactification of X and in particular it smallest remainder-invariant regular S-compactification of X. ∎

It is an open question whether the statement (iv) of the above theorem is equivalent to any of the other statements. In other words, it is an open question whether ωX is always the smallest remainder-invariant regular S-compactification.

Theorem 4.16. If a convergence S-space X has a regular S-compactification, then it has a largest regular S-compactification.

Proof. Let (Y_ι) be a family consisting of all the regular S-compactifications of X, let Y denote the product convergence space $\prod_\iota Y_\iota$ and for each index ι let π_ι be projection map of Y onto Y_ι. Define an action of S on Y so that for $s \in S$ and $y \in Y$, sy satisfies $\pi_\iota(sy) = s\pi_\iota(y)$ for each index ι. Let us prove that this action is a valid action on Y: Let $s, s' \in S$ and $y \in Y$ be arbitrary. Then $\pi_\iota(s(s'y)) = s\pi_\iota(s'y) = ss'\pi_\iota(y) = \pi_\iota((ss')y)$ for every index ι, hence $s(s'y) = (ss')y$. Finally, if $e \in S$, then $\pi_\iota(ey) = e\pi_\iota(y) = \pi_\iota(y)$ for every index ι, hence $ey = y$.

We now prove that the action on Y is continuous: Let $\mathcal{F} \to s$ in S and $\mathcal{G} \to y$ in Y. Then $\pi_\iota[\mathcal{G}] \to \pi_\iota(y)$ and since the action on Y_ι is continuous, it follows that $\mathcal{F}\pi_\iota[\mathcal{G}] \to s\pi_\iota(y)$. Since $\mathcal{F}\pi_\iota[\mathcal{G}] = \pi_\iota[\mathcal{F}\mathcal{G}]$ by Proposition 3.4 and since $s\pi_\iota(y) = \pi_\iota(sy)$, we have that $\pi_\iota[\mathcal{F}\mathcal{G}] \to \pi_\iota(sy)$. Since this is true for every $\iota \in A$, it follows that $\mathcal{F}\mathcal{G} \to sy$.

Since the given action on Y is continuous and Y is the product of compact regular Hausdorff convergence spaces, it follows that Y is a compact regular Hausdorff convergence S-space.

Define $f\colon X \to Y$ so that $\pi_\iota(f(x)) = x$ for all $x \in X$ and all indices ι. This is well defined since X is a subspace of each Y_ι. We claim that f is an S-embedding of X in Y. By definition, f is one-to-one. To see that it is an S-map, note that for every $s \in S$ and $x \in X$ and every index ι, $\pi_\iota(f(sx)) = sx = s\pi_\iota(f(x)) = \pi_\iota(sf(x))$, where this last equality follows from the definition of the action on Y. Suppose that $\mathcal{F} \to x$ in X. Since $\pi_\iota[f[\mathcal{F}]] = [\mathcal{F}]_{Y_\iota} \to x$ in Y_ι for every index ι, it follows that $f[\mathcal{F}] \to f(x)$ in Y. Now suppose that $f[\mathcal{F}] \to f(x)$ in Y for some filter $\mathcal{F}$ on X and some $x \in X$. Then $\pi_\iota[f[\mathcal{F}]] = [\mathcal{F}]_{Y_\iota} \to \pi_\iota(f(x)) = x$ in Y_ι for each ι, and since X a subspace of each Y_ι, it follows that $\mathcal{F} \to x$ in X. This concludes the proof that f is an S-embedding of X in Y.

We now prove that $\operatorname{cl} f(X)$ is a regular S-compactification of X. First note that $\operatorname{cl} f(X)$ is an S-subspace of Y since $S \operatorname{cl} f(X) \subseteq \operatorname{cl}(Sf(X))$ by Proposition 3.7 and $\operatorname{cl}(Sf(X)) = \operatorname{cl}(f(SX)) \subseteq \operatorname{cl} f(X)$. Since Y is compact, regular and Hausdorff, the closure operator on Y is idempotent by Proposition 2.8, which means $\operatorname{cl} f(X)$ is closed, which means $\operatorname{cl} f(X)$ is compact (and of course regular and Hausdorff). Moreover, since $f(X)$ is automatically dense in $\operatorname{cl} f(X)$, it follows that $\operatorname{cl} f(X)$ is a regular S-compactification of X.

Finally, since the restriction of π_ι to $\operatorname{cl} f(X)$ is a continuous S-map from $\operatorname{cl} f(X)$ to Y_ι such that $\pi_\iota(f(x)) = x$ for all $x \in X$, it follows from Theorem 3.10 that $\operatorname{cl} f(X)$ is larger than Y_ι. Since this is true for every Y_ι, and since (Y_ι) consists of all the regular S-compactifications of X, it follows that $\operatorname{cl} f(X)$ is the largest regular S-compactification of X. ∎

CHAPTER 5: S-COMPLETIONS

Completions of Cauchy Spaces - A Review

In this section, we outline some of the main results regarding completions of Cauchy spaces, which we will generalize in the next section to the context of Cauchy S-spaces.

Let X be a Cauchy space with Cauchy structure $\mathscr{C}$. Define a relation $\sim$ on $\mathscr{C}$ so that $\mathcal{F} \sim \mathcal{G}$ if and only if $\mathcal{F} \cap \mathcal{G} \in \mathscr{C}$. The relation $\sim$ is an equivalence relation on $\mathscr{C}$ and two filters $\mathcal{F}, \mathcal{G} \in \mathscr{C}$ are called ***equivalent*** if $\mathcal{F} \sim \mathcal{G}$. (In terms of sequences, the relation $\sim$ can be thought of as an equivalence relation between Cauchy sequences such that two Cauchy sequences are equivalent if when you interleave them you get a Cauchy sequence.) Given $\mathcal{F} \in \mathscr{C}$, we will write $\langle \mathcal{F} \rangle$ for the equivalence class containing $\mathcal{F}$. Note that the intersection of any two filters in $\langle \mathcal{F} \rangle$ is again in $\langle \mathcal{F} \rangle$, and if $\mathcal{G} \in \langle \mathcal{F} \rangle$, then every filter finer than $\mathcal{G}$ is also in $\langle \mathcal{F} \rangle$. We will use these facts repeatedly and implicitly in many of the proofs later on. We will use X^* to denote the set of equivalence classes of equivalent Cauchy filters on X. Note that X is "contained" in X^* via the correspondence $\phi\colon X \to X^*$ given by $\phi(x) = \langle [x] \rangle$.

A completion Y of Cauchy space X is in ***standard form*** if $Y = X^*$, ϕ is the dense embedding and $\phi[\mathcal{F}] \to \langle \mathcal{F} \rangle$ for every $\mathcal{F} \in \mathscr{C}$. Basically, a completion is in standard form if its points consist of equivalence classes of equivalent Cauchy filters.

Theorem 5.1 (Theorem 5 in [21])**.** Every completion of a Cauchy space is equivalent to one in standard form.

We know that in the setting of metric spaces and more generally uniform spaces, every Hausdorff space has a completion. The same is true in this setting. In fact, there are many possible completions of a Hausdorff Cauchy space (cf. Theorem 3 of [21]).

Hausdorff plus complete does not imply regular in the setting of Cauchy spaces, so it makes sense to talk about regular completions. In [11], the authors introduce the so-called Σ operator in order to construct regular completions. It is defined as follows:

For every subset A of a Cauchy space X, define ΣA to be the set of equivalence classes $\langle \mathcal{F} \rangle \in X^*$ that contain a Cauchy filter $\mathcal{G}$ that contains A, and for every $\mathcal{F} \in \mathscr{C}$, let $\Sigma\mathcal{F}$ be the filter on X^* generated by $\{\Sigma F \colon F \in \mathcal{F}\}$ (see Proposition 5.4 for a proof that this is indeed a basis). Let $\Sigma\mathscr{C}$ denote the collection of filters on X^* defined so that $\mathcal{H} \in \Sigma\mathscr{C}$ if and only if $\mathcal{H}$ is finer than $\Sigma\mathcal{F}$ for some $\mathcal{F} \in \mathscr{C}$ and let $\Sigma X = (X^*, \Sigma\mathscr{C})$. In general, ΣX fails to be a Cauchy space. However, we have the following theorem.

Theorem 5.2 (Corollary 1.6 in [11])**.** If a Cauchy space has a regular strict completion in standard form, then the completion must be equal to ΣX.

There is another kind of completion available for Cauchy spaces that satisfies a property that is stronger than Hausdorff. The completions are constructed via the use of the Γ operator . It is related to the Σ operator and it works as follows: Given a subset A of a Cauchy space X, let $\Gamma A = \phi(A) \cup (\Sigma A - \phi(X))$. Note that $\langle \mathcal{F} \rangle \in \Gamma A$ if and only if either $\mathcal{F} \to x$ for some $x \in A$ or $\mathcal{F} \sim \mathcal{G}$ for some non-convergent Cauchy filter $\mathcal{G}$ on X that contains A. We now prove a sundry of facts regarding the Σ and Γ operators.

Proposition 5.3. The following statements are true about any two subsets A and B of a Cauchy space X.

(i) $\phi(A) \subseteq \Gamma A \subseteq \Sigma A$ with equality when A is a singleton set.

(ii) $A \subseteq B$ implies $\Sigma A \subseteq \Sigma B$ and $\Gamma A \subseteq \Gamma B$.

(iii) $\Sigma(A \cap B) \subseteq \Sigma A \cap \Sigma B$ and $\Gamma(A \cap B) \subseteq \Gamma A \cap \Gamma B$.

(iv) $\Sigma(A \cup B) = \Sigma A \cup \Sigma B$ and $\Gamma(A \cup B) = \Gamma A \cup \Gamma B$.

(v) $\Sigma A \cap \phi(X) = \phi(\operatorname{cl} A)$ and $\phi(\operatorname{cl} A) \cup \Gamma A = \Sigma A$.

(vi) $\phi^{-1}(\Gamma A) = A$ and $\phi^{-1}(\Sigma A) = \operatorname{cl} A$.

Proof.

(i) The first inclusion follows from the definition. The second inclusion follows from the fact that $\phi(A) \subseteq \Sigma A$: Since $\phi(A)$ consists of all those equivalence classes $\langle [x] \rangle$ such that $x \in A$, it follows that $A \in [x]$ for every $\langle [x] \rangle \in \phi(A)$, hence $\langle [x] \rangle \in \Sigma A$. Now suppose $A = \{x\}$. Then $\langle \mathcal{F} \rangle \in \Sigma\{x\}$ if and only if there is a Cauchy filter $\mathcal{G}$ on X such that $\{x\} \in \mathcal{G} \sim \mathcal{F}$ if and only if $[x] \sim \mathcal{F}$ if and only if $\langle \mathcal{F} \rangle = \langle [x] \rangle = \phi(x)$. This proves that $\phi(\{x\}) = \Sigma\{x\}$ and since $\phi(\{x\}) \subseteq \Gamma\{x\} \subseteq \Sigma\{x\}$, it follows that $\phi(\{x\}) = \Gamma\{x\} = \Sigma\{x\}$.

(ii) If $\langle \mathcal{F} \rangle \in \Sigma A$, then there is a $\mathcal{G} \sim \mathcal{F}$ such that $A \in \mathcal{G}$, and since $A \subseteq B$, it follows that $B \in \mathcal{G}$ and that $\langle \mathcal{F} \rangle \in \Sigma B$. Also, since $A \subseteq B$, $\phi(A) \subseteq \phi(B)$ and $\Sigma A - \phi(X) \subseteq \Sigma B - \phi(X)$ by what we have just shown, hence $\Gamma A \subseteq \Gamma B$.

(iii) Follows from (ii).

(iv) The reverse inclusions, $\Sigma(A \cup B) \supseteq \Sigma A \cup \Sigma B$ and $\Gamma(A \cup B) \supseteq \Gamma A \cup \Gamma B$, follow from (ii). Suppose $\langle \mathcal{F} \rangle \in \Sigma(A \cup B)$. Then there is a $\mathcal{G} \sim \mathcal{F}$ such that $A \cup B \in \mathcal{G}$, Let $\mathcal{U}$ be an ultrafilter finer than $\mathcal{G}$. Then $\mathcal{G} \sim \mathcal{U}$ and so $\mathcal{F} \sim \mathcal{U}$ since $\sim$ is transitive. Furthermore, $\mathcal{U}$ contains either A or B, which means that $\langle \mathcal{F} \rangle$ belongs to either ΣA or ΣB and hence to $\Sigma A \cup \Sigma B$. Now suppose $\langle \mathcal{F} \rangle \in \Gamma(A \cup B)$. If $\mathcal{F} \to x$ in X for some $x \in A \cup B$, then $\langle \mathcal{F} \rangle$ belongs to either ΓA or ΓB and hence to $\Gamma A \cup \Gamma B$. Otherwise, $\mathcal{F} \sim \mathcal{G}$ for some non-convergent Cauchy filter $\mathcal{G}$ on X that contains $A \cup B$. Let $\mathcal{U}$ be an ultrafilter finer than $\mathcal{G}$. Then $\mathcal{G} \sim \mathcal{U}$ and so $\mathcal{F} \sim \mathcal{U}$ since $\sim$ is transitive. Furthermore, $\mathcal{U}$ contains either A or B, which means that $\langle \mathcal{F} \rangle$ belongs to either ΓA or ΓB and hence to $\Gamma A \cup \Gamma B$.

(v) The proof that $\Sigma A \cap \phi(X) = \phi(\operatorname{cl} A)$ goes as follows: $\langle [x] \rangle \in \Sigma A \cap \phi(X)$ if and only if there is a Cauchy filter $\mathcal{G}$ on X such that $A \in \mathcal{G} \to x$ if and only if $x \in \operatorname{cl} A$ if and only if $\phi(x) = \langle [x] \rangle \in \phi(\operatorname{cl} A)$. Now since $\Sigma A \cap \phi(X) = \phi(\operatorname{cl} A)$, it follows that $\phi(\operatorname{cl} A) \cup \Gamma A = (\Gamma A \cup \Sigma A) \cap (\Gamma A \cup \phi(X))$. Now note that $\Gamma A \cup \Sigma A = \Gamma A \cup \phi(X) = \Sigma A$.

(vi) By definition of ΓA we have that $\phi^{-1}(\Gamma A) = \phi^{-1}(\phi(A)) \cup \phi^{-1}(\Sigma A - \phi(X)) = A \cup \emptyset = A$. Since $\phi(\operatorname{cl} A) \cup \Gamma A = \Sigma A$, we have that $\phi^{-1}(\Sigma A) = \phi^{-1}(\phi(\operatorname{cl} A)) \cup \phi^{-1}(\Gamma A) = \operatorname{cl} A \cup A = \operatorname{cl} A$. ∎

Proposition 5.4. The following are true statements about a Cauchy filter $\mathcal{F}$ on a Cauchy space X.

(i) If $\mathcal{B}$ is basis for $\mathcal{B}$, then $\{\Sigma B \colon B \in \mathcal{B}\}$ and $\{\Gamma B \colon B \in \mathcal{B}\}$ are bases for filters on X^* denoted $\Sigma\mathcal{F}$ and $\Gamma\mathcal{F}$, respectively.

(ii) $\Sigma\mathcal{F} \subseteq \Gamma\mathcal{F} \subseteq \phi[\mathcal{F}]$ with equality when $\mathcal{F}$ is a point filter.

(iii) $\mathcal{F} = \phi^{-1}[\Gamma\mathcal{F}]$ and $\operatorname{cl}\mathcal{F} = \phi^{-1}[\Sigma\mathcal{F}]$ and $\phi[\operatorname{cl}\mathcal{F}] \cap \Gamma\mathcal{F} = \Sigma\mathcal{F}$ and $\Sigma\mathcal{F} \subseteq [\langle\mathcal{F}\rangle]$.

(iv) If $\mathcal{F} \to x$, then $\Gamma(\mathcal{F} \cap [x]) \subseteq [\langle\mathcal{F}\rangle] = [\phi(x)]$. If $\mathcal{F}$ does not converge, then $\Gamma\mathcal{F} \subseteq [\langle\mathcal{F}\rangle]$.

Proof.

(i) Both $\{\Sigma B \colon B \in \mathcal{B}\}$ and $\{\Gamma B \colon B \in \mathcal{B}\}$ are not empty since B is not empty. Both $\{\Sigma B \colon B \in \mathcal{B}\}$ and $\{\Gamma B \colon B \in \mathcal{B}\}$ do not contain the empty set since $\phi(B)$ is not empty for all $B \in \mathcal{B}$ and $\phi(B) \subseteq \Gamma B \subseteq \Sigma B$ by (i) of Proposition 5.3. And by (iii) of that same proposition, it follows that both $\{\Sigma B \colon B \in \mathcal{B}\}$ and $\{\Gamma B \colon B \in \mathcal{B}\}$ are bases.

(ii) Follows from (i) of Proposition 5.3.

(iii) By (vi) of Proposition 5.3, we immediately have that $\mathcal{F} = \phi^{-1}[\Gamma\mathcal{F}]$ and that $\phi^{-1}[\Sigma\mathcal{F}] = \operatorname{cl}\mathcal{F}$; also $\phi[\operatorname{cl}\mathcal{F}] \cap \Gamma\mathcal{F} = \Sigma\mathcal{F}$ follows automatically from (v) of that same proposition. Since $F \in \mathcal{F} \sim \mathcal{F}$ for every $F \in \mathcal{F}$, it follows that $\langle\mathcal{F}\rangle \in \Sigma F$ for every $F \in \mathcal{F}$, hence $\Sigma\mathcal{F} \subseteq [\langle\mathcal{F}\rangle]$.

(iv) If $\mathcal{F} \to x$, then $\mathcal{F} \cap [x]$ is Cauchy, which means $\mathcal{F} \sim [x]$, which means $\langle\mathcal{F}\rangle = \langle [x] \rangle = \phi(x)$, and since $\phi(x) = \langle [x] \rangle \in \phi(F \cup \{x\}) \subseteq \Gamma(F \cup \{x\})$ for every $F \in \mathcal{F}$, it follows that $\Gamma(\mathcal{F} \cap [x]) \subseteq [\phi(x)] =$

$[\langle\mathcal{F}\rangle]$. If $\mathcal{F}$ does not convergence, then for every $F \in \mathcal{F}$ we have that $\langle\mathcal{F}\rangle \in \Sigma F - \phi(X)$, which means $\langle\mathcal{F}\rangle$ belongs to every element of $\Gamma\mathcal{F}$, which means $\Gamma\mathcal{F} \subseteq [\langle\mathcal{F}\rangle]$. ∎

Given a Cauchy space X with Cauchy structure $\mathscr{C}$, we will write $\Gamma\mathscr{C}$ for the set of filters on X^* defined so that $\mathcal{F} \in \Gamma\mathscr{C}$ if and only if $\Gamma\mathcal{G} \subseteq \mathcal{F}$ for some $\mathcal{G} \in \mathscr{C}$. We will also write ΓX for the pair $(X^*, \Gamma\mathscr{C})$.

We say that a Cauchy space, resp. its Cauchy structure, is ***separated*** if for every two Cauchy filters $\mathcal{F}$, $\mathcal{G}$, if $\Gamma\mathcal{F} \vee \Gamma\mathcal{G}$ exists, then $\mathcal{F} \cap \mathcal{G}$ is Cauchy (which means $\mathcal{F} \sim \mathcal{G}$ and $\langle\mathcal{F}\rangle = \langle\mathcal{G}\rangle$).

Theorem 5.5. The following statements are true about a separated Cauchy space X.

(i) ΓX is a complete Hausdorff Cauchy space.
(ii) ΓX is a completion of X in standard form.
(iii) ΓX is totally bounded whenever X is totally bounded.
(iv) If X is regular, then $\Sigma X = \Gamma X$.

Proof.

(i) Let $\mathscr{C}$ be the Cauchy structure on X. By part (iv) of Proposition 5.4, $\Gamma\mathscr{C}$ contains all the point filters on X^*. By definition of $\Gamma\mathscr{C}$, if $\mathcal{H} \in \Gamma\mathscr{C}$ and $\mathcal{H}'$ is finer than $\mathcal{H}$, then $\mathcal{H}' \in \Gamma\mathscr{C}$. Now suppose that $\mathcal{H}$ and $\mathcal{H}'$ are two filters in $\Gamma\mathscr{C}$ such that $\mathcal{H} \vee \mathcal{H}'$ exists. By definition, there are two filters $\mathcal{F}$ and $\mathcal{F}'$ in $\mathscr{C}$ such that $\Gamma\mathcal{F} \subseteq \mathcal{H}$ and $\Gamma\mathcal{F}' \subseteq \mathcal{H}'$. Since $\mathcal{H} \vee \mathcal{H}'$ exists, so does $\Gamma\mathcal{F} \vee \Gamma\mathcal{F}'$. Since X is separated, $\mathcal{F} \cap \mathcal{F}' \in \mathscr{C}$, which means $\Gamma(\mathcal{F} \cap \mathcal{F}') \in \Gamma\mathscr{C}$. By (iv) of Proposition 5.3, $\Gamma(\mathcal{F} \cap \mathcal{F}') = \Gamma\mathcal{F} \cap \Gamma\mathcal{F}'$ and since $\Gamma\mathcal{F} \cap \Gamma\mathcal{F}' \subseteq \mathcal{H} \cap \mathcal{H}'$, it follows that $\mathcal{H} \cap \mathcal{H}' \in \Gamma\mathscr{C}$. This completes the proof that $\Gamma\mathscr{C}$ is a Cauchy structure on X^*, hence ΓX is a Cauchy space.

Before we prove that ΓX is Hausdorff, we will need the following three lemmas.

Lemma 1. If $\mathcal{H}$ is a filter on ΓX that converges to $\langle \mathcal{F}_1 \rangle$ and $\langle \mathcal{F}_2 \rangle$, then there is a Cauchy filter $\mathcal{G}$ on X such that $\Gamma\mathcal{G} \subseteq \mathcal{H} \cap [\langle \mathcal{F}_1 \rangle] \cap [\langle \mathcal{F}_2 \rangle]$.

Proof. Since $\mathcal{H}$ converges to $\langle \mathcal{F}_i \rangle$, $\mathcal{H} \cap [\langle \mathcal{F}_i \rangle] \in \Gamma\mathscr{C}$, which means there is some $\mathcal{G}_i \in \mathscr{C}$ such that $\Gamma\mathcal{G}_i \subseteq \mathcal{H} \cap [\langle \mathcal{F}_i \rangle]$. Since $\mathcal{H}$ is finer than each $\Gamma\mathcal{G}_i$, it follows that $\Gamma\mathcal{G}_1 \vee \Gamma\mathcal{G}_2$ exists, and since X is separated, $\mathcal{G}_1 \cap \mathcal{G}_2 \in \mathscr{C}$. By (iv) of Proposition 5.3 we have that $\Gamma(\mathcal{G}_1 \cap \mathcal{G}_2) = \Gamma\mathcal{G}_1 \cap \Gamma\mathcal{G}_2 \subseteq \mathcal{H} \cap [\langle \mathcal{F}_1 \rangle] \cap [\langle \mathcal{F}_2 \rangle]$. Thus, $\mathcal{G} = \mathcal{G}_1 \cap \mathcal{G}_2$ is Cauchy filter on X such that $\Gamma\mathcal{G} \subseteq \mathcal{H} \cap [\langle \mathcal{F}_1 \rangle] \cap [\langle \mathcal{F}_2 \rangle]$.

Lemma 2. For every Cauchy filter $\mathcal{G}$ on X and every $x \in X$, if $\Gamma\mathcal{G} \subseteq [\phi(x)]$, then $\langle \mathcal{G} \rangle = \phi(x)$.

Proof. Since $\Gamma\mathcal{G} \subseteq [\phi(x)]$, for every $G \in \mathcal{G}$, we have that $\phi(x) \in \Gamma G$, and since this implies that $x \in G$ by definition of ΓG, we conlude that $\mathcal{G} \subseteq [x]$, which means $\mathcal{G} \cap [x]$ is Cauchy, which means $\langle \mathcal{G} \rangle = \phi(x)$.

We are now ready prove that ΓX is Hausdorff: Let $\mathcal{H}$ be a filter on ΓX. We consider three cases. (i) Suppose $\mathcal{H}$ converges to $\phi(x)$ and $\phi(y)$. By Lemma 1, there is a Cauchy filter $\mathcal{G}$ on X such that $\Gamma\mathcal{G} \subseteq \mathcal{H} \cap [\phi(x)] \cap [\phi(y)]$. Since $\Gamma\mathcal{G}$ is coarser than both $[\phi(x)]$ and $[\phi(y)]$, by Lemma 2 it follows that $\langle \mathcal{G} \rangle = \phi(x) = \phi(y)$. (ii) Suppose $\mathcal{H}$ converges to $\phi(x)$ and to $\langle \mathcal{F} \rangle$, where $\mathcal{F} \in \mathscr{C}$ is a non-convergent. By Lemma 1, there is a Cauchy filter $\mathcal{G}$ on X such that $\Gamma\mathcal{G} \subseteq \mathcal{H} \cap [\phi(x)] \cap [\langle \mathcal{F} \rangle]$. By Lemma 2, $\langle \mathcal{G} \rangle = \phi(x)$. By part (iv) of Proposition 5.4, $\Gamma\mathcal{F} \subseteq [\langle \mathcal{F} \rangle]$, and since $\Gamma\mathcal{G} \subseteq [\langle \mathcal{F} \rangle]$, it follows that $\Gamma\mathcal{F} \vee \Gamma\mathcal{G}$ exists, which means $\langle \mathcal{G} \rangle = \langle \mathcal{F} \rangle$ since X is separated. Thus, $\langle \mathcal{G} \rangle = \phi(x) = \langle \mathcal{F} \rangle$. (iii) Suppose $\mathcal{H}$ converges to $\langle \mathcal{F}_1 \rangle$ and $\langle \mathcal{F}_2 \rangle$, where each $\mathcal{F}_i \in \mathscr{C}$ is non-convergent. By Lemma 1, there is a Cauchy filter $\mathcal{G}$ on X such that $\Gamma\mathcal{G} \subseteq \mathcal{H} \cap [\langle \mathcal{F}_1 \rangle] \cap [\langle \mathcal{F}_2 \rangle]$. By part (iv) of Proposition 5.4, $\Gamma\mathcal{F}_i \subseteq [\langle \mathcal{F}_i \rangle]$, and since $\Gamma\mathcal{G} \subseteq [\langle \mathcal{F}_i \rangle]$, it follows that $\Gamma\mathcal{F}_i \vee \Gamma\mathcal{G}$ exists, which means $\langle \mathcal{G} \rangle = \langle \mathcal{F}_i \rangle$ since X is separated. Thus, $\langle \mathcal{G} \rangle = \langle \mathcal{F}_1 \rangle = \langle \mathcal{F}_2 \rangle$.

Finally, we prove that ΓX is complete. The proof follows from the following lemma since every Cauchy filter on ΓX is finer than some $\Gamma\mathcal{G}$.

Lemma 3. For every $\mathcal{G} \in \mathscr{C}$, $\Gamma\mathcal{G} \to \langle\mathcal{G}\rangle$ in ΓX.

Proof. We want to show that $\Gamma\mathcal{G} \cap [\langle\mathcal{G}\rangle] \in \Gamma\mathscr{C}$. If $\mathcal{G}$ is non-convergent, then by part (iv) of Proposition 5.4, $\Gamma\mathcal{G} \subseteq [\langle\mathcal{G}\rangle]$ so $\Gamma\mathcal{G} \cap [\langle\mathcal{G}\rangle] \in \Gamma\mathscr{C}$. If $\mathcal{G} \to x$ for some $x \in X$, then by part (iv) of Proposition 5.4, $\Gamma(\mathcal{G} \cap [x]) \subseteq [\phi(x)]$, which means $\Gamma(\mathcal{G} \cap [x]) \cap [\langle\mathcal{G}\rangle] \in \Gamma\mathscr{C}$, which means $\Gamma\mathcal{G} \cap [\langle\mathcal{G}\rangle] \in \Gamma\mathscr{C}$.

(ii) Let us first prove that ϕ is an Cauchy embedding. If $\mathcal{F} \in \mathscr{C}$, then $\Gamma\mathcal{F} \in \Gamma\mathscr{C}$ by definition of $\Gamma\mathscr{C}$, and since $\Gamma\mathcal{F} \subseteq \phi[\mathcal{F}]$ by (ii) of Proposition 5.4, it follows that $\phi[\mathcal{F}] \in \Gamma\mathscr{C}$. Now suppose $\phi[\mathcal{F}] \in \Gamma\mathscr{C}$ for some $\mathcal{F} \in \mathbf{F}(X)$. Then there is a $\mathcal{G} \in \mathscr{C}$ such that $\Gamma\mathcal{G} \subseteq \phi[\mathcal{F}]$, hence $\mathcal{F} = \phi^{-1}[\phi[\mathcal{F}]] \supseteq \phi^{-1}[\Gamma\mathcal{G}] = \mathcal{G}$, where the last equality follows from (iii) of Proposition 5.4. Now we prove that $\phi(X)$ is dense in ΓX and conclude that ΓX is in standard form: For every Cauchy filter $\mathcal{F}$ on X, $\Gamma\mathcal{F} \to \langle\mathcal{F}\rangle$ in ΓX by Lemma 3. Since $\Gamma\mathcal{F} \subseteq \phi[\mathcal{F}]$ by (ii) of Proposition 5.4, it follows that $\phi[\mathcal{F}] \to \langle\mathcal{F}\rangle$ in ΓX.

(iii) Suppose X is totally bounded and let $\mathcal{V}$ be an ultrafilter on X^*. For each $\langle\mathcal{F}\rangle \in X^* - \phi(X)$, pick an ultrafilter $\mathcal{U}_{\langle\mathcal{F}\rangle} \in \langle\mathcal{F}\rangle$. Given $A \subseteq X$, let $\Psi A = \phi(A) \cup \{\langle\mathcal{F}\rangle : A \in \mathcal{U}_{\langle\mathcal{F}\rangle}\}$.

(a) $\Psi(A \cap B) = \Psi A \cap \Psi B$ for all $A, B \subseteq X$: Follows from the fact that $\phi(A \cap B) = \phi(A) \cap \phi(B)$ (since ϕ is one-to-one) and the fact that $\{\langle\mathcal{F}\rangle : A \cap B \in \mathcal{U}_{\langle\mathcal{F}\rangle}\} = \{\langle\mathcal{F}\rangle : A \in \mathcal{U}_{\langle\mathcal{F}\rangle}\} \cap \{\langle\mathcal{F}\rangle : B \in \mathcal{U}_{\langle\mathcal{F}\rangle}\}$.

(b) $\Psi(A \cup B) = \Psi A \cup \Psi B$ for all $A, B \subseteq X$: Follows from the fact that $\phi(A \cup B) = \phi(A) \cup \phi(B)$ and the fact that $\{\langle\mathcal{F}\rangle : A \cup B \in \mathcal{U}_{\langle\mathcal{F}\rangle}\} = \{\langle\mathcal{F}\rangle : A \in \mathcal{U}_{\langle\mathcal{F}\rangle}\} \cup \{\langle\mathcal{F}\rangle : B \in \mathcal{U}_{\langle\mathcal{F}\rangle}\}$ since the $\mathcal{U}_{\langle\mathcal{F}\rangle}$ are ultrafilters.

(c) $\Psi A \subseteq \Gamma A$ for all $A \subseteq X$: This follows from the fact that $\{\langle\mathcal{F}\rangle : A \in \mathcal{U}_{\langle\mathcal{F}\rangle}\} \subseteq \Sigma A - \phi(X)$.

(d) $\mathcal{U} := \{A \subseteq X : \Psi A \in \mathcal{V}\}$ is an ultrafilter on X: Since $\Psi X = X^* \in \mathcal{V}$, $\mathcal{U}$ is not empty. Since $\Psi\emptyset = \emptyset$, $\mathcal{U}$ does not contain the empty set. If $A \subseteq B \subseteq X$, then $\Psi A \subseteq \Psi B$ by (b), hence $A \in \mathcal{U}$ implies $B \in \mathcal{U}$. If $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$ by (a). If $A \cup B \in \mathcal{U}$, then by (b) either

$A \in \mathcal{U}$ or $B \in \mathcal{U}$.

(e) $\{\Psi A\colon A \in \mathcal{U}\}$ is a basis for a filter on X^*, which we denote $\Psi\mathcal{U}$: Since $\mathcal{U}$ contains X but not the empty set, $\{\Psi A\colon A \in \mathcal{U}\}$ is not empty and does not contain the empty set. By (a), it follows that $\{\Psi A\colon A \in \mathcal{U}\}$ is a basis.

(f) $\Gamma\mathcal{U} \subseteq \Psi\mathcal{U} \subseteq \mathcal{V}$: First note that $\{\Psi A\colon A \in \mathcal{U}\} = \{\Psi A\colon \Psi A \in \mathcal{V}\} \subseteq \mathcal{V}$, hence $\Psi\mathcal{U} \subseteq \mathcal{V}$. Since $\{\Gamma A\colon A \in \mathcal{U}\} = \{\Gamma A\colon \Psi A \in \mathcal{V}\}$ and $\{\Psi A\colon \Psi A \in \mathcal{V}\}$ is finer than $\{\Gamma A\colon A \in \mathcal{U}\}$ by (c), it follows that $\Gamma\mathcal{U} \subseteq \Psi\mathcal{U}$.

Since X is totally bounded and $\mathcal{U}$ is an ultrafilter on X, $\mathcal{U}$ is Cauchy, hence $\Gamma\mathcal{U}$ is Cauchy, and since $\Gamma\mathcal{U} \subseteq \mathcal{V}$, it follows that $\mathcal{V}$ is Cauchy.

(iv) Let $\mathscr{C}$ be the Cauchy structure on X and let $\mathcal{H} \in \Sigma\mathscr{C}$. Then $\mathcal{H}$ is finer than $\Sigma\mathcal{F}$ for some Cauchy filter $\mathcal{F} \in \mathscr{C}$. Since X is regular, $\mathrm{cl}\,\mathcal{F} \in \mathscr{C}$, so $\phi[\mathrm{cl}\,\mathcal{F}] \in \Gamma\mathscr{C}$. Since $\phi[\mathrm{cl}\,\mathcal{F}] \subseteq \phi[\mathcal{F}]$ and $\Gamma\mathcal{F} \subseteq \phi[\mathcal{F}]$ by (ii) of Proposition 5.4, it follows that $\phi[\mathrm{cl}\,\mathcal{F}] \vee \Gamma\mathcal{F}$ exists, which means $\phi[\mathrm{cl}\,\mathcal{F}] \cap \Gamma\mathcal{F} \in \Gamma\mathscr{C}$. By (iii) of Proposition 5.4, $\phi[\mathrm{cl}\,\mathcal{F}] \cap \Gamma\mathcal{F} = \Sigma\mathcal{F}$, hence $\Sigma\mathcal{F} \in \Gamma\mathscr{C}$, hence $\mathcal{H} \in \Gamma\mathscr{C}$. This proves that $\Sigma\mathscr{C} \subseteq \Gamma\mathscr{C}$. For the reverse inclusion, if $\mathcal{H} \in \Gamma\mathscr{C}$, there is a $\mathcal{G} \in \mathscr{C}$ such that $\Gamma\mathcal{G} \subseteq \mathcal{H}$ and since $\Sigma\mathcal{G} \subseteq \Gamma\mathcal{G}$ by (iii) of Proposition 5.4, it follows that $\mathcal{H} \in \Sigma\mathscr{C}$. Thus, $\Sigma\mathscr{C} = \Gamma\mathscr{C}$ and consequently $\Sigma X = \Gamma X$. ∎

Part (i) of the above theorem implies that separated Cauchy spaces are Hausdorff and by part (iv) we see being separated and regular is slightly stronger than being Hausdorff and regular.

We will say that a Cauchy space, resp. its Cauchy structure, is ***precompact*** if it is totally bounded and has a regular completion.

Theorem 5.6. If X is a precompact Cauchy space, then ΣX is a regular strict completion of X.

Proof. Since X is precompact, it has a regular completion, which means that X is regular and Hausdorff. We now prove that ΣX is a completion. First, we prove that ΣX is a Cauchy space. Let

$\mathscr{C}$ be the Cauchy structure on X. By (iii) of Proposition 5.4, $\Sigma\mathscr{C}$ contains every point filter on X^* so it satisfies the point filter axiom. By definition of $\Sigma\mathscr{C}$, it automatically satisfies the subfilter axiom. Before we prove that $\Sigma\mathscr{C}$ satisfies the intersection axiom, we will need the following lemma.

Lemma 1. If $\mathcal{F}, \mathcal{G} \in \mathscr{C}$ and $\Sigma\mathcal{F} \vee \Sigma\mathcal{G}$ exists, then $\mathcal{F} \cap \mathcal{G} \in \mathscr{C}$ and consequently $\Sigma\mathcal{F} \cap \Sigma\mathcal{G} \in \Sigma\mathscr{C}$.

Proof. Since X is precompact, it has a regular completion Y, and by Theorem 5.1, we assume that Y is in standard form. By Proposition 1.5 in [11], $\Sigma A = \mathrm{cl}_Y\, \phi(A)$ for all $A \subseteq X$. Thus, since $\Sigma\mathcal{F} \vee \Sigma\mathcal{G} = \mathrm{cl}_Y\, \phi[\mathcal{F}] \vee \mathrm{cl}_Y\, \phi[\mathcal{G}]$ exists, it follows that $\mathrm{cl}_Y\, \phi[\mathcal{F}] \cap \mathrm{cl}_Y\, \phi[\mathcal{G}] = \mathrm{cl}_Y\, \phi[\mathcal{F} \cap \mathcal{G}]$ is a Cauchy filter on Y, which means $\phi[\mathcal{F} \cap \mathcal{G}]$ is a Cauchy filter on Y, which means $\mathcal{F} \cap \mathcal{G}$ is a Cauchy filter on X since Y is in standard form. By (iv) of Proposition 5.3, $\Sigma(\mathcal{F} \cap \mathcal{G}) \subseteq \Sigma\mathcal{F} \cap \Sigma\mathcal{G}$ and since $\Sigma(\mathcal{F} \cap \mathcal{G}) \in \Sigma\mathscr{C}$, it follows that $\Sigma\mathcal{F} \cap \Sigma\mathcal{G} \in \Sigma\mathscr{C}$.

Now we are ready to prove that $\Sigma\mathscr{C}$ satisfies the intersection axiom: Suppose $\mathcal{H}, \mathcal{H}' \in \Sigma\mathscr{C}$ and $\mathcal{H} \vee \mathcal{H}'$ exists. By definition of $\Sigma\mathscr{C}$, there are $\mathcal{G}, \mathcal{G}' \in \mathscr{C}$ such that $\Sigma\mathcal{G} \subseteq \mathcal{H}$ and $\Sigma\mathcal{G}' \subseteq \mathcal{H}'$. Since $\mathcal{H} \vee \mathcal{H}'$ exists, so does $\Sigma\mathcal{G} \vee \Sigma\mathcal{G}'$. By Lemma 1, $\Sigma\mathcal{G} \cap \Sigma\mathcal{G}' \in \Sigma\mathscr{C}$ and since $\Sigma\mathcal{G} \cap \Sigma\mathcal{G}' \subset \mathcal{H} \cap \mathcal{H}'$, it follows that $\mathcal{H} \cap \mathcal{H}' \in \Sigma\mathscr{C}$ since $\Sigma\mathscr{C}$ satisfies the subfilter axiom. This concludes the proof that $\Sigma\mathscr{C}$ is Cauchy structure and that ΣX is a Cauchy space.

Now we prove that ΣX is complete: Let $\mathcal{H} \in \Sigma\mathscr{C}$ be arbitrary. Pick a $\mathcal{G} \in \mathscr{C}$ such that $\Sigma\mathcal{G} \subseteq \mathcal{H}$. We claim that $\mathcal{H} \to \langle\mathcal{G}\rangle$ in ΣX, i.e. $\mathcal{H} \cap [\langle\mathcal{G}\rangle] \in \Sigma\mathscr{C}$. This follows immediately from (iii) of Proposition 5.4, for it says that $\Sigma\mathcal{G} \subseteq [\langle\mathcal{G}\rangle]$, hence $\Sigma\mathcal{G} \subseteq \mathcal{H} \cap [\langle\mathcal{G}\rangle]$ and consequently $\mathcal{H} \cap [\langle\mathcal{G}\rangle] \in \Sigma\mathscr{C}$ by definition of $\Sigma\mathscr{C}$.

Next, we prove that $\phi\colon X \to \Sigma X$ is a Cauchy embedding: Suppose $\mathcal{F} \in \mathscr{C}$. By (ii) of Proposition

5.4, $\Sigma\mathcal{F} \subseteq \phi[\mathcal{F}]$, hence $\phi[\mathcal{F}] \in \Sigma\mathscr{C}$. Now suppose $\phi[\mathcal{F}] \in \Sigma\mathscr{C}$ for some $\mathcal{F} \in \mathbf{F}(X)$. Since ΣX is complete, $\phi[\mathcal{F}] \to \langle\mathcal{G}\rangle$ in ΣX for some $\mathcal{G} \in \mathscr{C}$. This means that $\phi[\mathcal{F}] \cap [\langle\mathcal{G}\rangle] \in \Sigma\mathscr{C}$, which means there is an $\mathcal{H} \in \mathscr{C}$ such that $\Sigma\mathcal{H} \subseteq \phi[\mathcal{F}] \cap [\langle\mathcal{G}\rangle]$. Since $\Sigma\mathcal{H} \subseteq \phi[\mathcal{F}]$, by (iii) of Proposition 5.4, $\mathrm{cl}\,\mathcal{H} = \phi^{-1}[\Sigma\mathcal{H}] \subseteq \mathcal{F}$. Since X is regular, $\mathrm{cl}\,\mathcal{H}$ and consequently $\mathcal{F}$ are in $\mathscr{C}$.

Next, we prove that $\phi[\mathcal{F}] \to \langle\mathcal{F}\rangle$ in ΣX for all $\mathcal{F} \in \mathscr{C}$. Since $\phi[\mathcal{F}] \in \Sigma\mathscr{C}$, there is a $\mathcal{G} \in \mathscr{C}$ such that $\Sigma\mathcal{G} \subseteq \phi[\mathcal{F}]$. By (iii) of Proposition 5.4, $\Sigma\mathcal{G} \subseteq [\langle\mathcal{G}\rangle]$, hence $\Sigma\mathcal{G} \subseteq \phi[\mathcal{F}] \cap [\langle\mathcal{G}\rangle]$, hence $\phi[\mathcal{F}] \cap [\langle\mathcal{G}\rangle] \in \Sigma\mathscr{C}$ since $\Sigma\mathcal{G} \in \Sigma\mathscr{C}$. This means that $\phi[\mathcal{F}] \to \langle\mathcal{G}\rangle$ in ΣX. Since $\phi[\mathcal{F}]$ is an upper bound of both $\Sigma\mathcal{F}$ and $\Sigma\mathcal{G}$, it follows that $\Sigma\mathcal{F} \vee \Sigma\mathcal{G}$ exists, which by Lemma 1 means that $\mathcal{F} \cap \mathcal{G} \in \mathscr{C}$, i.e. $\langle\mathcal{F}\rangle = \langle\mathcal{G}\rangle$. Thus, $\phi[\mathcal{F}] \to \langle\mathcal{F}\rangle$ in ΣX.

Before we continue proving stuff about ΣX, we will need the following lemma.

> **Lemma 2.** For all $A \subseteq X$, $\mathrm{cl}_{\Sigma X}\,\phi(A) = \Sigma A$. Consequently, for all $\mathcal{F} \in \mathscr{C}$, $\mathrm{cl}_{\Sigma X}\,\phi[\mathcal{F}] = \Sigma\mathcal{F}$.
>
> *Proof.* If $\langle\mathcal{F}\rangle \in \Sigma A$, then $A \in \mathcal{G} \sim \mathcal{F}$ for some $\mathcal{G} \in \mathscr{C}$, and since $\phi(A) \in \phi[\mathcal{G}] \to \langle\mathcal{G}\rangle$ in ΣX, it follows that $\langle\mathcal{F}\rangle = \langle\mathcal{G}\rangle \in \mathrm{cl}_{\Sigma X}\,\phi(A)$. This proves that $\Sigma A \subseteq \mathrm{cl}_{\Sigma X}\,\phi(A)$. For the reverse inclusion, let $\langle\mathcal{F}\rangle \in \mathrm{cl}_{\Sigma X}\,\phi(A)$. Then there is an $\mathcal{H} \in \Sigma\mathscr{C}$ such that $\phi(A) \in \mathcal{H} \to \langle\mathcal{F}\rangle$ in ΣX. Since $\phi(A) \in \mathcal{H}$, we have that $\phi(X) \in \mathcal{H}$, which means that $\mathcal{G} := \phi^{-1}[\mathcal{H}] \in \mathbf{F}(X)$. Since $\phi[\mathcal{G}] = \mathcal{H} \to \langle\mathcal{F}\rangle$ in ΣX and $\phi[\mathcal{F}] \to \langle\mathcal{F}\rangle$ in ΣX, it follows that $\phi[\mathcal{G}] \cap \phi[\mathcal{F}] = \phi[\mathcal{G} \cap \mathcal{F}] \to \langle\mathcal{F}\rangle$ in ΣX. Thus, $\phi[\mathcal{G} \cap \mathcal{F}] \in \Sigma\mathscr{C}$, which means $\mathcal{G} \cap \mathcal{F} \in \mathscr{C}$ since ϕ is a Cauchy embedding, which means $\mathcal{G} \sim \mathcal{F}$. Since $A \in \mathcal{G} \sim \mathcal{F}$, it follows that $\langle\mathcal{F}\rangle \in \Sigma A$.

Now we can prove that $\phi(X)$ is strictly dense: Suppose $\mathcal{H} \to \langle\mathcal{F}\rangle$ in ΣX. Pick a $\mathcal{G} \in \mathscr{C}$ such that $\Sigma\mathcal{G} \subseteq \mathcal{H} \cap [\langle\mathcal{F}\rangle]$. By Lemma 2, $\Sigma\mathcal{G} = \mathrm{cl}_{\Sigma X}\,\phi[\mathcal{G}] \subseteq \mathcal{H} \cap [\langle\mathcal{F}\rangle]$, hence $\mathrm{cl}_{\Sigma X}\,\phi[\mathcal{G}] \subseteq \mathcal{H}$. Since $\Sigma\mathcal{G} \subseteq [\langle\mathcal{F}\rangle]$, it follows that $\Sigma\mathcal{G} \cap [\langle\mathcal{F}\rangle] \in \Sigma\mathscr{C}$, hence $\Sigma\mathcal{G} \to \langle\mathcal{F}\rangle$ in ΣX. Since $\Sigma\mathcal{G} = \mathrm{cl}_{\Sigma X}\,\phi[\mathcal{G}] \subseteq \phi[\mathcal{G}]$, we also have that $\phi[\mathcal{G}] \to \langle\mathcal{F}\rangle$ in ΣX. This concludes the proof that $\phi(X)$ is strictly dense and that ΣX is a strict completion of X.

The next order of business is to prove that ΣX is Hausdorff. Before we do so, we make the following observation: Since X is precompact, it has a regular completion Y, and by Theorem 5.1, we assume that Y is in standard form. Let $\mathscr{C}_Y$ be the Cauchy structure on Y. By Proposition 1.5 in [11] $\mathrm{cl}_Y\,\phi(A) = \Sigma A$ for all $A \subseteq X$. By Lemma 2, it follows that $\mathrm{cl}_Y\,\phi[\mathcal{F}] = \mathrm{cl}_{\Sigma X}\,\phi[\mathcal{F}] = \Sigma\mathcal{F}$ for all $\mathcal{F} \in \mathscr{C}$. Since Y is a regular completion, $\mathrm{cl}_Y\,\phi[\mathcal{F}] \in \mathscr{C}_Y$ for every $\mathcal{F} \in \mathscr{C}$, hence $\Sigma\mathcal{F} \in \mathscr{C}_Y$ for every $\mathcal{F} \in \mathscr{C}$, hence $\Sigma\mathscr{C} \subseteq \mathscr{C}_Y$. Now suppose that $\mathcal{H} \to \langle\mathcal{F}_i\rangle$ in ΣX for some $\mathcal{F}_i \in \mathscr{C}$, $i = 1,2$. Then $\mathcal{H} \cap [\langle\mathcal{F}_i\rangle] \in \Sigma\mathscr{C} \subseteq \mathscr{C}_Y$, which means $\mathcal{H} \to \langle\mathcal{F}_i\rangle$ in Y since Y is complete. Since Y is Hausdorff, it follows that $\langle\mathcal{F}_1\rangle = \langle\mathcal{F}_2\rangle$.

Using (3) of Theorem 2.2 in [11], ΣX is regular. This completes the proof that ΣX is a regular strict completion of X. ∎

A well-known result from the theory of uniform spaces is that there is a one-to-one correspondence between the set of totally bounded uniformities on a completely regular Hausdorff topological space X and the set of Hausdorff compactifications of X. Theorem 5.7 below is a generalization of this result. To make sense of the statement of the theorem, we will need the following notions:

Given a convergence space X, we say that a Cauchy space Y ***induces*** X if Y regarded as a convergence space is *equal* to X (i.e. Y and X are the same set with the same convergence strucutre). The set of all Cauchy spaces that induce a given convergence space X can be ordered in the following way: If X_1 and X_2 are Cauchy spaces that induce X, we will say that X_1 ***is greater than*** X_2 if and only if the Cauchy structure on X_1 is finer than that of X_2. Note that if X_1 is greater than X_2 and X_2 is greater than X_1, then $X_1 = X_2$.

Theorem 5.7 (Theorem 4.2 in [11]). If X is a completely regular Hausdorff convergence space, then there is an isomorphism between the ordered set of precompact Cauchy structures on X that induce X and the ordered set of regular strict compactifications of X.

S-Completion Theory

We now proceed to generalize some of the results from the previous section. Let X be a Cauchy S-space and let S act on X^* as follows: Given $s \in S$ and $\langle \mathcal{F} \rangle \in X^*$, let $s\langle \mathcal{F} \rangle = \langle [s]\mathcal{F} \rangle$. This action on X^* is well-defined: Suppose $\langle \mathcal{F} \rangle = \langle \mathcal{G} \rangle$ and let $s \in S$ be arbitrary. Then $s\langle \mathcal{F} \rangle = \langle [s]\mathcal{F} \rangle$ and $s\langle \mathcal{G} \rangle = \langle [s]\mathcal{G} \rangle$. Since $\mathcal{F} \cap \mathcal{G}$ is Cauchy, so is $[s](\mathcal{F} \cap \mathcal{G}) = [s]\mathcal{F} \cap [s]\mathcal{G}$, hence $\langle [s]\mathcal{F} \rangle = \langle [s]\mathcal{G} \rangle$. It is also a valid action on X^*: Let $s, s \in S$ and $\langle \mathcal{F} \rangle \in X^*$ be arbitrary. Then $(ss')\langle \mathcal{F} \rangle = \langle [ss']\mathcal{F} \rangle = \langle [s][s']\mathcal{F} \rangle = s\langle [s']\mathcal{F} \rangle = s(s'\langle \mathcal{F} \rangle)$. Also, if $e \in S$, then $e\langle \mathcal{F} \rangle = \langle [e]\mathcal{F} \rangle = \langle \mathcal{F} \rangle$.

Henceforth, ***the default action on X^* will be the one given above.*** We now ask ourselves under what conditions on X is this action above a Cauchy continuous action on ΣX and ΓX. To answer this question, we first make some observations about how the action behaves with respect to the injection ϕ, the sigma operator Σ and the gamma operator Γ. This is the subject of the following lemma.

Lemma 5.8. If X is a Cauchy S-space, then

(i) ϕ is an S-map,

(ii) $A(\Sigma B) \subseteq \Sigma(AB)$ for all $A \subseteq S$ and $B \subseteq X$,

(iii) if X is adherence restrictive, then $A(\Gamma B) \subseteq \Gamma(AB)$ for all $A \subseteq S$ and $B \subseteq X$, and

Proof.

(i) Given $s \in S$ and $x \in X$, $\phi(sx) = \langle [sx] \rangle$. Since $[sx] = [s][x]$, it follows that $\langle [sx] \rangle = \langle [s][x] \rangle = s\langle [x] \rangle = s\phi(x)$ where the second-to-last equality follows from the definition of the action on X^*.

(ii) Let $s \in A$ and $\langle \mathcal{F} \rangle \in \Sigma B$ be arbitrary. Then there is a $\mathcal{G}$ in $\langle \mathcal{F} \rangle$ that contains B, hence $sB \in [s]\mathcal{G}$. Since $[s]\mathcal{G} \in \langle [s]\mathcal{F} \rangle$ and $[s]\mathcal{G}$ contains sB, it follows that $\langle [s]\mathcal{F} \rangle \in \Sigma(sB) \subseteq \Sigma(AB)$.

(iii) Let $s \in A$ and $\langle \mathcal{F} \rangle \in \Gamma B$ be arbitrary. We consider two cases. (i) Suppose $\mathcal{F} \to x$ in X for some $x \in B$. Then $\langle \mathcal{F} \rangle = \phi(x) \in \phi(B)$ and consequently $s\langle \mathcal{F} \rangle = \langle [s]\mathcal{F} \rangle = s\phi(x) = \phi(sx) \in \phi(AB) \subseteq \Gamma(AB)$. (ii) Suppose there is a non-convergent Cauchy filter $\mathcal{G}$ on X such that $B \in \mathcal{G} \sim \mathcal{F}$. Since X is adherence-restritive, $[s]\mathcal{G}$ does not converge. Since $\mathcal{F} \cap \mathcal{G}$ is Cauchy, so is $[s](\mathcal{F} \cap \mathcal{G}) = [s]\mathcal{F} \cap [s]\mathcal{G}$, hence $\langle [s]\mathcal{F} \rangle = \langle [s]\mathcal{G} \rangle$. Thus, $[s]\mathcal{G}$ is non-convergent Cauchy filter on X such that $AB \in [s]\mathcal{G} \sim [s]\mathcal{F}$, which means $s\langle \mathcal{F} \rangle = \langle [s]\mathcal{F} \rangle \in \Gamma(AB)$. ∎

Theorem 5.9. If X is a Cauchy S-space, then

(i) if ΣX is a Cauchy completion of X, then it is also an S-completion of X,

(ii) if X is adherence restrictive and ΓX is a Cauchy completion of X, then ΓX is also an S-completion of X.

Proof. By (i) of Lemma 5.8, ϕ will be a dense Cauchy S-embedding, so all we have to do is verify that the action on ΣX and ΓX is Cauchy continuous.

(i) Let $\mathcal{F}$ be a Cauchy filter on S and let $\mathcal{H}$ be a Cauchy filter on ΣX. Since $\mathcal{H}$ is Cauchy, there is a Cauchy filter $\mathcal{G}$ on X such that $\Sigma\mathcal{G} \subseteq \mathcal{H}$. By (ii) of Lemma 5.8, $\Sigma(\mathcal{F}\mathcal{G}) \subseteq \mathcal{F}(\Sigma\mathcal{G}) \subseteq \mathcal{F}\mathcal{H}$, and since $\Sigma(\mathcal{F}\mathcal{G})$ is Cauchy, it follows that $\mathcal{F}\mathcal{H}$ is Cauchy.

(ii) Let $\mathcal{F}$ be a Cauchy filter on S and let $\mathcal{H}$ be a Cauchy filter on ΓX. Since $\mathcal{H}$ is Cauchy, there is a Cauchy filter $\mathcal{G}$ on X such that $\Gamma\mathcal{G} \subseteq \mathcal{H}$. By (iii) of Lemma 5.8, $\Gamma(\mathcal{F}\mathcal{G}) \subseteq \mathcal{F}(\Gamma\mathcal{G}) \subseteq \mathcal{F}\mathcal{H}$, and since $\Gamma(\mathcal{F}\mathcal{G})$ is Cauchy, it follows that $\mathcal{F}\mathcal{H}$ is Cauchy. ∎

The following theorem generalizes the result of Theorem 5.7. Note: when a Cauchy S-space Y induces a convergence S-space X, it also means that the actions of S on X and Y are the same.

Theorem 5.10. Let X be a completely regular Hausdorff convergence space and let S be a complete Cauchy semigroup. Suppose that S acts continuously on X. Then there is an isomorphism between the ordered set of precompact Cauchy S-space that induce X and the ordered set of regular strict S-compactifications of X.

Proof. Let $\mathfrak{X}$ be the set of all precompact Cauchy S-spaces that induce X and let $\mathfrak{K}$ denote the set of regular strict S-compactifications of X, ordered in the usual way. Note that we will not distinguish between isomorphic objects in $\mathfrak{X}$ and $\mathfrak{K}$.

Let $X_1 \in \mathfrak{X}$. By Theorems 5.6 and 5.9, ΣX_1 is a regular strict S-completion of X. We claim that ΣX_1 is a regular strict S-compactification of X_1: Since ΣX_1 is regular as a Cauchy space, it is regular as a convergence space. Since ΣX_1 is a Hausdorff strict S-extension in the Cauchy sense, it is automatically a Hausdorff strict S-extension in the convergence space sense. All we must show now is that ΣX_1 is compact: Pick an ultrafilter $\mathcal{V}$ on ΣX_1 and let ϕ_1 be the dense Cauchy S-embedding of X_1 in ΣX_1. By Proposition 2.10, there is an ultrafilter $\mathcal{U}$ on X such that $\operatorname{cl}\phi_1[\mathcal{U}] \subseteq \mathcal{V}$. Since X_1 is precompact, $\mathcal{U}$ is Cauchy, hence $\phi_1[\mathcal{U}]$ is Cauchy, and since ΣX_1 is regular, $\operatorname{cl}\phi_1[\mathcal{U}]$ is Cauchy and consequently $\mathcal{V}$ is Cauchy. We have thus shown that $\Sigma X_1 \in \mathfrak{K}$ and with this in mind, we claim that the operator $\Sigma\colon X_1 \mapsto \Sigma X_1$ is the claimed isomorphism between $\mathfrak{X}$ and $\mathfrak{K}$.

First, we prove that Σ is injective. Let $X_1, X_2 \in \mathfrak{X}$ be arbitrary. For $i = 1,2$, let $\langle \mathcal{F} \rangle_i$ denote the equivalence class of Cauchy filters on X_i containing $\mathcal{F}$ and let ϕ_i be the embeddding of X_i into ΣX_i. Suppose ΣX_1 and ΣX_2 are equivalent S-compactifications of X. We want to show that X_1 and X_2 have the same Cauchy filters. Since ΣX_1 and ΣX_2 are equivalent, there is an S-homeomorphism $f\colon \Sigma X_1 \to \Sigma X_2$ such that $f \circ \phi_1 = \phi_2$. If $\mathcal{F}$ is a Cauchy filter on X_1, then $\phi_1[\mathcal{F}] \to \langle \mathcal{F} \rangle_1$ in ΣX_1, hence $f[\phi_1[\mathcal{F}]] = \phi_2[\mathcal{F}] \to f(\langle \mathcal{F} \rangle_1)$ in ΣX_2, hence $\phi_2[\mathcal{F}]$ is a Cauchy filter on ΣX_2, which means $\mathcal{F}$ is a Cauchy filter on X_2. This proves that every Cauchy filter on X_1 is a Cauchy filter on X_2. Since f is a homeomorphism, the same argument with the roles of X_1 and

X_2 swapped proves that every Cauchy filter on X_2 is a Cauchy filter on X_1.

Next, we prove that Σ is surjective. Let $Y \in \mathfrak{K}$, let $\mathscr{C}_1$ be the Cauchy structure on X defined so that $\mathcal{F} \in \mathscr{C}_1$ if and only if $[\mathcal{F}]$ converges in Y and let $X_1 = (X, \mathscr{C}_1)$. Since Y is Hausdorff and S is complete, we can and will regard Y as a complete Cauchy S-space. Moreover, by definition of X_1, Y is a regular strict completion of X_1. We now proceed to prove that $X_1 \in \mathfrak{X}$. First, we prove that the action of S on X_1 is Cauchy continuous: Let $\mathcal{F}$ be a Cauchy filter on S and let $\mathcal{G}$ be a Cauchy filter on X_1. Since S and Y are complete, there is an $s \in S$ and a $y \in Y$ such that $\mathcal{F} \to s$ in S and $[\mathcal{G}] \to y$ in Y. Since $[s][\mathcal{G}] = [[s]\mathcal{G}] \to sy$ in Y, it follows that $[s]\mathcal{G}$ is a Cauchy filter on X_1. This proves that the action of S on X_1 is Cauchy continuous and that X_1 is a Cauchy S-space. Since Y is a compactification, every ultrafilter on X_1 is automatically Cauchy by definition of X_1 and since Y regular completion of X_1, it follows that X_1 is precompact. Lastly, we prove that X_1 induces X: Given a Cauchy filter $\mathcal{G}$ on X_1 and a point $x \in X_1$, then $\mathcal{G} \cap [x]_X$ is Cauchy if and only if $[\mathcal{G} \cap [x]_X] = [\mathcal{G}] \cap [x]_Y$ converges in Y if and only if $[\mathcal{G}] \to x$ in Y if and only if $\mathcal{G} \to x$ in X. This concludes the proof that $X_1 \in \mathfrak{X}$. To finish the proof that Σ is surjective, just note that by Theorems 5.1 and 5.2, ΣX_1 with $\Sigma\mathscr{C}_1$ as the Cauchy structure is equivalent to Y.

Finally, we prove that Σ is order preserving. Pick $X_1, X_2 \in \mathfrak{X}$ and suppose X_1 is greater than X_2, i.e. suppose every Cauchy filter on X_1 is a Cauchy filter on X_2. We want to prove that ΣX_1 is greater than ΣX_2, i.e. we want to prove that there is a continuous surjective S-map $h\colon \Sigma X_1 \to \Sigma X_2$ such that $h \circ \phi_1 = \phi_2$. Actually, in light of Theorem 3.10, we only need to prove there is a continuous function $h\colon \Sigma X_1 \to \Sigma X_2$ such that $h \circ \phi_1 = \phi_2$. With this in mind, define $h\colon \Sigma X_1 \to \Sigma X_2$ by $h(\langle \mathcal{F} \rangle_1) = \langle \mathcal{F} \rangle_2$ and let Σ_i denote the Σ operator on ΣX_i, $i = 1, 2$. First, we prove that h is well-defined: Suppose $\mathcal{F}$ and $\mathcal{G}$ are Cauchy filters on X_1 and that $\langle \mathcal{F} \rangle_1 = \langle \mathcal{G} \rangle_1$. We want to check that $h(\langle \mathcal{F} \rangle_1) = h(\langle \mathcal{G} \rangle_1)$ or equivalently that $\langle \mathcal{F} \rangle_2 = \langle \mathcal{G} \rangle_2$. Since $\langle \mathcal{F} \rangle_1 = \langle \mathcal{G} \rangle_1$, it follows that $\mathcal{F} \cap \mathcal{G}$ is a Cauchy filter on X_1, and since every Cauchy filter on X_1 is a Cauchy filter on X_2, it follows that $\mathcal{F} \cap \mathcal{G}$ is a Cauchy filter on X_2, which means that $\langle \mathcal{F} \rangle_2 = \langle \mathcal{G} \rangle_2$. Next we check that $h \circ \phi_1 = \phi_2$:

Pick an $x \in X$. Then $h(\phi_1(x)) = h(\langle [x] \rangle_1) = \langle [x] \rangle_2 = \phi_2(x)$. To prove that h is continuous and we prove that h is Cauchy continuous since Cauchy continuity implies continuity. We will need the following result: for every $A \subseteq X$, we have that $h(\Sigma_1 A) \subseteq \Sigma_2 A$. The proof of this result goes like this: Pick an $\langle \mathcal{F} \rangle_1 \in \Sigma_1 A$. Then there is a $\mathcal{G} \in \langle \mathcal{F} \rangle_1$ such that $A \in G$. Since $\langle \mathcal{G} \rangle_1 = \langle \mathcal{F} \rangle_1$, it follows that $h(\langle \mathcal{F} \rangle_1) = h(\langle \mathcal{G} \rangle_1)$, i.e. $\langle \mathcal{G} \rangle_2 = \langle \mathcal{F} \rangle_2$. Thus, $A \in \mathcal{G} \in \langle \mathcal{F} \rangle_2$, which means $\langle \mathcal{F} \rangle_2 \in \Sigma_2 A$. This ends the proof of the result and using it we can now prove that h is Cauchy continuous: Pick a Cauchy filter $\mathcal{H}$ on ΣX_1. Then by definition of ΣX_1, there is a Cauchy filter $\mathcal{G}$ on X_1 such that $\Sigma_1 \mathcal{G} \subseteq \mathcal{H}$. By the result we just proved, $\Sigma_2 \mathcal{G} \subseteq h[\Sigma_1 \mathcal{G}] \subseteq h[\mathcal{H}]$, which means $h[\mathcal{H}]$ is a Cauchy filter on ΣX_2.

∎

CHAPTER 6: APPLICATIONS TO PSEUDOQUOTIENTS

Pseudoquotients, also known as generalized quotients, are abstractions of the quotient field construction of an integral domain. It is a construction that arose and is primarily used in the area of generalized functions. For a review of the history and application of pseudoquotients, the reader is referred to [19, 20]. In this chapter, we answer the following question in many particular instances: When is the pseudoquotient of an S-compactification or S-completion of an S-space X equivalent to the S-compactification or S-completion, respectively, of the pseudoquotient of X? This chapter can be seen as a continuation of the papers [4, 5].

Preliminaries

Pseudoquotients are defined in the following manner:

Let S be a commutative monoid acting on a set X from the right. Suppose that that the action on X is right-cancellative, i.e. suppose that $xs = ys$ implies $x = y$ for all $s \in S$ and $x, y \in X$. Define a binary relation $\sim$ on $X \times S$ so that $(x, s) \sim (y, t)$ if and only if $xt = ys$. Then $\sim$ is an equivalence relation on $X \times S$ and the collection of equivalence classes of $\sim$ is called the ***pseudoquotient*** of X with respect to (the action of) S. The pseudoquotient is typically denoted $B(X,S)$ in the literature, but we will simplify this and write the pseudoquotient as $Q(X)$. We will write the equivalence class in $Q(X)$ containing the element (x, s) as x/s. There is an obvious right action of S on $Q(X)$, namely the one defined by $(x/s)t = xt/s$.

Henceforth, S ***will denote a commutative monoid and every action of*** S ***will be a right-cancellative action.***

For notational convenience, we adopt the following notations: Given $A \subset S$ and $B \subset X$, we will write B/A for the set $\{x/s \colon x \in B, s \in A\}$. Given filters $\mathcal{F}$ on S and $\mathcal{G}$ on X, we will write $\mathcal{G}/\mathcal{F}$ for the filter generated by $\{G/F \colon G \in \mathcal{G}, F \in \mathcal{F}\}$.

Let us make an observation about $Q(X)$ that will be useful later on. Given $A \subseteq S$ and $B \subseteq X$, then $x/s \in B/A$ if and only if there are elements $s' \in A$ and $x' \in B$ such that $x'/s' = x/s$. Given $A \subseteq S$ and $B, C \subseteq X$, then $(B \cup C)/A = B/A \cup C/A$ and $(B \cap C)/A \subseteq B/A \cap C/A$.

If X is an S-subset of an S-set Y, then it is not necessarily true that $Q(X) \subseteq Q(Y)$ since for $x \in X$ and $s \in S$, the equivalence class x/s in $Q(X)$ will generally have fewer elements than x/s in $Q(Y)$.

Proposition 6.1. If X is an S-subset of an S-set Y, then $Y - X$ is an S-subset of Y if and only if $Q(X) \subseteq Q(Y)$.

Proof. Suppose $Q(X)$ is not a subset of $Q(Y)$. Then for some $x \in X$ and $s \in S$, the equivalence class x/s in $Q(Y)$ must contain an element of the form (y, t) where $t \in S$ and $y \in Y - X$. This means that $x/s = y/t$, hence $xt = ys \in X$, hence $Y - X$ is not an S-subset of Y. Conversely, if $Y - X$ is not an S-subset of Y, then for some $y \in Y - X$ and $t \in S$, we have that $yt \in X$. Now fix an $s \in S$ and let $x = yt$. Then $xs = yts = yst$ and so $x/t = ys/s$ in $Q(Y)$, which means x/t in $Q(Y)$ contains (ys, s), which means that x/t in $Q(X)$ is a proper subset of x/t in $Q(Y)$, which means $Q(X)$ is not a subset of $Q(Y)$. ∎

Pseudoquotient S-Spaces

If X is a convergence space, then we will treat $Q(X)$ as a quotient convergence space where the convergence structure is defined as follows: A filter $\mathcal{H}$ on $Q(X)$ converges to a point x/s of $Q(X)$ if and only if there are filters $\mathcal{F}$ on S and $\mathcal{G}$ on X and points $s' \in S$ and $x' \in X$ such that $\mathcal{F} \to s'$

in S, $\mathcal{G} \to x'$ in X, $x'/s' = x/s$ and $\mathcal{G}/\mathcal{F} \subseteq \mathcal{H}$. If X is a Cauchy space, then we will treat $Q(X)$ as a Cauchy quotient space where the Cauchy structure is defined as follows: A filter $\mathcal{G}$ on $Q(X)$ is Cauchy if and only if there are Cauchy filters $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ on S and Cauchy filters $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n$ on X such that the filters in $\{\mathcal{G}_i/\mathcal{F}_i : i = 1, 2, \ldots, n\}$ are linked and $\bigcap_{i=1}^{n} \mathcal{G}_i/\mathcal{F}_i \subseteq \mathcal{G}$.

Theorem 6.2. If X is an S-space, then

(i) $Q(X)$ is an S-space,

(ii) if X is a Hausdorff convergence S-space, then so is $Q(X)$, and

(iii) if X is a complete Hausdorff Cauchy S-space and S is complete, then $Q(X)$ is a complete Hausdorff Cauchy S-space and convergence in $Q(X)$ is given by the quotient convergence structure.

Proof.

(i) Suppose X is a convergence S-space. Let $\mathcal{F} \to t$ in S and let $\mathcal{G} \to x/s$ in $Q(X)$. We want to prove that $\mathcal{G}\mathcal{F} \to xt/s$. Since $\mathcal{G} \to x/s$ in $Q(X)$, there are filters $\mathcal{F}'$ on S and $\mathcal{G}'$ on X such that $\mathcal{F}' \to s'$ for some $s' \in S$, $\mathcal{G}' \to x'$ for some $x' \in X$, $x'/s' = x/s$ and $\mathcal{G}'/\mathcal{F}' \subseteq \mathcal{G}$. Since $\mathcal{G}'\mathcal{F} \to x't$ and $\mathcal{F}' \to s'$ and $x't/s' = xt/s$, it follows that $\mathcal{G}'\mathcal{F}/\mathcal{F}' \to xt/s$. Finally, since $\mathcal{G}'\mathcal{F}/\mathcal{F}' = (\mathcal{G}'/\mathcal{F}')\mathcal{F} \subseteq \mathcal{G}\mathcal{F}$, we get that $\mathcal{G}\mathcal{F} \to xt/s$ in $Q(X)$.

Now suppose X is a Cauchy S-space. Let $\mathcal{G}$ be a Cauchy filter on $Q(X)$ and let $\mathcal{F}$ be a Cauchy filter on S. We want to prove that $\mathcal{G}\mathcal{F}$ is Cauchy. Since $\mathcal{G}$ is Cauchy, there are Cauchy filters $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ on S and Cauchy filters $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n$ on X such that the filters in $\{\mathcal{G}_i/\mathcal{F}_i : i = 1, 2, \ldots, n\}$ are linked and $\bigcap_{i=1}^{n} \mathcal{G}_i/\mathcal{F}_i \subseteq \mathcal{G}$. By part (iv) of Proposition 3.3, the filters in $\{\mathcal{G}_i\mathcal{F}/\mathcal{F}_i : i = 1, 2, \ldots, n\}$ are linked, and since $\bigcap_{i=1}^{n} \mathcal{G}_i\mathcal{F}/\mathcal{F}_i \subseteq \mathcal{G}\mathcal{F}$, it follows that $\mathcal{G}\mathcal{F}$ is Cauchy.

(ii) Suppose X is a Hausdorff convergence S-space. Let $\mathcal{H}$ be a filter on $Q(X)$ and suppose it has two limits: x_1/s_1 and x_2/s_2. Then for $i = 1,2$ there are filters $\mathcal{F}_i$ on S and $\mathcal{G}_i$ on X and elements $s'_i \in S$ and $x'_i \in X$ such that $\mathcal{F}_i \to s'_i$ in S, $\mathcal{G}_i \to x'_i$ in X, $x'_i/s'_i = x_i/s_i$ and $\mathcal{G}_i/\mathcal{F}_i \subseteq \mathcal{H}$. If $x'_1/s'_1 = x'_2/s'_2$, then $x_1/s_1 = x_2/s_2$ and we are done, so suppose $x'_1/s'_1 \neq x'_2/s'_2$. Since $\mathcal{H}$ is an upper bound of $\mathcal{G}_i/\mathcal{F}_i$ for $i = 1,2$, it follows that $\mathcal{G}_1/\mathcal{F}_1 \vee \mathcal{G}_2/\mathcal{F}_2$ exists. Since $\mathcal{G}_1 \to x'_1$ in X and $\mathcal{F}_2 \to s'_2$ in S, it follows that $\mathcal{G}_1\mathcal{F}_2 \to x'_1 s'_2$ in X. Similarly we have that $\mathcal{G}_2\mathcal{F}_1 \to x'_2 s'_1$ in X. Since $x'_1/s'_1 \neq x'_2/s'_2$, we have that $x'_1 s'_2 \neq x'_2 s'_1$. Since X is Hausdorff, it follows that $\mathcal{G}_1\mathcal{F}_2 \vee \mathcal{G}_2\mathcal{F}_1$ does not exist. This means that for some $F_i \in \mathcal{F}_i$ and $G_i \in \mathcal{G}_i$, $G_1F_2 \cap G_2F_1 = \emptyset$. We now claim that $G_1/F_1 \cap G_2/F_2 = \emptyset$, contradicting that $\mathcal{G}_1/\mathcal{F}_1 \vee \mathcal{G}_2/\mathcal{F}_2$ exists: If $y/t \in G_1/F_1 \cap G_2/F_2$, then there are $t_i \in F_i$ and $y_i \in G_i$ such that $y/t = y_i/t_i$, $i = 1,2$. Since $y_1/t_1 = y_2/t_2$, it follows that $y_1 t_2 = y_2 t_1 \in G_1F_2 \cap G_2F_1$, which is a contradiction. Thus, $G_1/F_1 \cap G_2/F_2 = \emptyset$ as claimed and the resulting contradiction implies that $x'_1/s'_1 \neq x'_2/s'_2$ is false.

(iii) By (ii), $Q(X)$ with the quotient convergence structure is Hausdorff. Since X and S are complete, so is $X \times S$, and since $Q(X)$ with the quotient convergence structure is Hausdorff, by (iii) of Proposition 2.13, $Q(X)$ is complete and convergence in $Q(X)$ with respect to the quotient Cauchy structure is given by the quotient convergence structure. ∎

Proposition 6.3.

(i) If Y is a convergence S-space and X and $Y - X$ are S-subspaces of Y, then

(a) $\mathcal{H} \to x/s$ in $Q(X)$ implies $[\mathcal{H}] \to x/s$ in $Q(Y)$, and

(b) if $\mathcal{F}$ is a filter on S and $\mathcal{G}$ is a filter on X, then $\mathrm{cl}_X\, \mathcal{G}/\mathcal{F} \subseteq \mathrm{cl}_Y [\mathcal{G}]/\mathcal{F} \cap Q(X)$, where $\mathrm{cl}_Y [\mathcal{G}]/\mathcal{F} \cap Q(X)$ is the trace of $\mathrm{cl}_Y [\mathcal{G}]/\mathcal{F}$ on $Q(X)$.

(ii) Let X and Y be Cauchy S-spaces and let $f\colon X \to Y$ be a Cauchy S-embedding of X in Y. Define $g\colon Q(X) \to Q(Y)$ by $g(x/s) = f(x)/s$. Then g is an injective Cauchy continuous S-map.

Proof.

(i) (a) Since $\mathcal{H} \to x/s$ in $Q(X)$, there are filters $\mathcal{F}$ on S and $\mathcal{G}$ on X such that $\mathcal{F} \to s'$ for some $s' \in S$, $\mathcal{G} \to x'$ for some $x' \in X$, $x'/s' = x/s$ and $\mathcal{G}/\mathcal{F} \subseteq \mathcal{H}$. Since $[\mathcal{G}] \to x'$ in Y and $[\mathcal{G}]/\mathcal{F} \subseteq [\mathcal{H}]$, it follows that $[\mathcal{H}] \to x/s$ in $Q(Y)$.

(b) Let $G \in \mathcal{G}$ and $F \in \mathcal{F}$ be arbitrary and let $y_1/s_1 \in \mathrm{cl}_Y\, G/F \cap Q(X)$. Then there exists a $y_2 \in \mathrm{cl}_Y\, G$ and an $s_2 \in F$ such that $y_1/s_1 = y_2/s_2$. Since $y_1/s_1 = y_2/s_2$, it follows that $y_1 s_2 = y_2 s_1$. Since $y_1/s_1 \in Q(X)$, $y_1 = x_1$ for some $x_1 \in X$. Since $x_1 s_2 = y_2 s_1$ and $Y - X$ is an S-subspace of Y, $y_2 = x_2$ for some $x_2 \in X$. Since $x_2 \in \mathrm{cl}_Y\, G$, there is a filter $\mathcal{G}_2$ on Y that contains G and converges to x_2, and since $G \subseteq X$, this filter must also contain X. Thus, $X \in \mathcal{G}_2 \to x_2 \in X$ in Y and since X is a subspace of Y, it follows that $\mathcal{G}_2 \cap X \to x_2$ in X, hence $x_2 \in \mathrm{cl}_X\, G$ and consequently $x_2/s_2 = x_1/s_1 \in \mathrm{cl}_X\, G/F$. This proves that $\mathrm{cl}_Y\, G/F \cap Q(X) \subseteq \mathrm{cl}_X\, G/F$ and thus $\mathrm{cl}_X\, \mathcal{G}/\mathcal{F} \subseteq \mathrm{cl}_Y [\mathcal{G}]/\mathcal{F} \cap Q(X)$ as claimed.

(ii) Let us first prove that g is injective: Suppose $g(x/s) = g(y/t)$. Then $f(x)/s = f(y)/t$, which means $f(x)t = f(y)s$. Since f is an S-map, this latter equality becomes $f(xt) = f(ys)$, and since f is injective, it follows that $xt = ys$ and that $x/s = y/t$.

Now we prove that g is an S-map: $g((x/s)t) = g(xt/s) = f(xt)/s = f(x)t/s = (f(x)/s)t = g(x/s)t$.

Finally, we prove that g is Cauchy continuous: Let $\mathcal{G}$ be a Cauchy filter on $Q(X)$. Then there are Cauchy filters $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ on S and Cauchy filters $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n$ on X such that the filters in $\{\mathcal{G}_i/\mathcal{F}_i \colon i = 1, 2, \ldots, n\}$ are linked and $\bigcap_{i=1}^{n} \mathcal{G}_i/\mathcal{F}_i \subseteq \mathcal{G}$. Since g is injective and $\mathcal{G}_i/\mathcal{F}_i \vee \mathcal{G}_{i+1}/\mathcal{F}_{i+1}$ exists, $g[\mathcal{G}_i/\mathcal{F}_i \vee \mathcal{G}_{i+1}/\mathcal{F}_{i+1}] = g[\mathcal{G}_i/\mathcal{F}_i] \vee g[\mathcal{G}_{i+1}/\mathcal{F}_{i+1}] = f[\mathcal{G}_i]/\mathcal{F}_i \vee f[\mathcal{G}_{i+1}]/\mathcal{F}_{i+1}$ exists. Thus, the filters in $\{f[\mathcal{G}_i]/\mathcal{F}_i \colon i = 1, 2, \ldots, n\}$ are linked and since

$$g\left[\bigcap_{i=1}^{n} \mathcal{G}_i/\mathcal{F}_i\right] = \bigcap_{i=1}^{n} g[\mathcal{G}_i/\mathcal{F}_i] = \bigcap_{i=1}^{n} f[\mathcal{G}_i]/\mathcal{F}_i \subseteq g[\mathcal{G}]$$

it follows that $g[\mathcal{G}]$ is Cauchy. ∎

S-Compactifications

Theorem 6.4. If X is a non-compact Hausdorff adherence-restrictive S-space and S is compact, then $Q(\omega X)$ is a one-point S-compactification of $Q(X)$.

Proof. Since $\omega t = \omega s = \omega$ for all $s, t \in S$ we have that $\omega / s = \omega / t$ for all s, t in S. Since $\{\omega\}$ is an S-subset of ωX, it follows by Proposition 6.1 that $Q(X)$ is an S-subset of $Q(\omega X)$. Thus, $Q(\omega X) - Q(X) = \{\omega / e\}$.

We now prove that $Q(X)$ is a subspace of $Q(\omega X)$. In light of Proposition 6.3, we must only show that if $\mathcal{H}$ is a filter on $Q(X)$ such that $[\mathcal{H}] \to x/s$ in $Q(\omega X)$, then $\mathcal{H} \to x/s$ in $Q(X)$. To this end, suppose $[\mathcal{H}] \to x/s$ in $Q(\omega X)$. Then there are filters $\mathcal{F}$ on S and $\mathcal{G}$ on ωX and there are points $x' \in \omega X$ and $s' \in S$ such that $\mathcal{F} \to s'$ in S, $\mathcal{G} \to x'$ in ωX, $x'/s' = x/s$ and $\mathcal{G}/\mathcal{F} \subseteq [\mathcal{H}]$. Since $x'/s' = x/s$, we have that $x's = xs'$. Since X is adherence restrictive, ωX is remainder invariant, hence $x's \in X$, hence $x' \in X$. Since $\mathcal{G} \to x' \in X$, then there is a filter $\mathcal{G}'$ on X such that $\mathcal{G}' \to x'$ in X and $[\mathcal{G}'] \subseteq \mathcal{G}$. Since $[\mathcal{G}'/\mathcal{F}] = [\mathcal{G}']/\mathcal{F} \subseteq \mathcal{G}/\mathcal{F} \subseteq [\mathcal{H}]$, it follows that $\mathcal{G}'/\mathcal{F} \subseteq \mathcal{H}$. Since $\mathcal{G}' \to x$ and $\mathcal{F} \to s$ and $x'/s' = x/s$, it follows that $\mathcal{H} \to x/s$ in $Q(X)$.

Now we prove that $Q(X)$ is dense in $Q(\omega X)$. Note that it suffices to prove that there is a filter $\mathcal{H}$ on $Q(X)$ such that $[\mathcal{H}] \to \omega/e$ in $Q(\omega X)$. Since X is not compact, there is an ultrafilter $\mathcal{U}$ on X with empty adherence such that $[\mathcal{U}] \to \omega$ in ωX. Since $[\mathcal{U}] \to \omega$ we have that $[\mathcal{U}]/[e] = [\mathcal{U}/[e]] \to \omega/e$ in $Q(\omega X)$. Thus, $\mathcal{H} := \mathcal{U}/[e]$ is a filter on $Q(X)$ such that $[\mathcal{H}] \to \omega/e$.

Finally, since S is compact and ωX is compact, $Q(\omega X)$ is compact. ∎

Theorem 6.5. Let X be an adherence-restrictive S-space, let Y be a regular strict S-compactification of X and suppose that S is compact and regular. Then $Q(Y)$ is a regular S-compactification of $Q(X)$.

Proof. Note that since X is adherence restrictive, Y is remainder invariant, whence $Q(X)$ is an S-subset of $Q(Y)$.

We now prove that $Q(X)$ is a subspace of $Q(Y)$. In light of Proposition 6.3, we must only show that if $\mathcal{H}$ is a filter on $Q(X)$ such that $[\mathcal{H}] \to x/s$ in $Q(Y)$, then $\mathcal{H} \to x/s$ in $Q(X)$. To this end, suppose $[\mathcal{H}] \to x/s$ in $Q(Y)$. Then there are filters $\mathcal{F}$ on S and $\mathcal{G}$ on Y and there are points $y \in Y$ and $t \in S$ such that $\mathcal{F} \to t$ in S, $\mathcal{G} \to y$ in Y, $y/t = x/s$ and $\mathcal{G}/\mathcal{F} \subseteq [\mathcal{H}]$. Since Y is strict, there exists a filter $\mathcal{G}'$ on X such that $[\mathcal{G}'] \to y$ in Y and $\mathrm{cl}_Y[\mathcal{G}'] \subseteq \mathcal{G}$. By Proposition 6.3, $\mathrm{cl}_X \mathcal{G}'/\mathcal{F} \subseteq \mathrm{cl}_Y[\mathcal{G}']/\mathcal{F} \cap Q(X)$, and since $\mathrm{cl}_Y[\mathcal{G}']/\mathcal{F} \cap Q(X) \subseteq \mathcal{G}/\mathcal{F} \cap Q(X) \subseteq \mathcal{H}$, we have that $\mathrm{cl}_X \mathcal{G}'/\mathcal{F} \subseteq \mathcal{H}$. We now prove that $\mathcal{H} \to x/s$, proving that $Q(X)$ is a subspace of $Q(Y)$. Recall that $y/t = x/s$ and that $[\mathcal{G}'] \to y$ in Y. Since Y is remainder invariant and $ys = xt$, it follows that $y = x'$ for some x' in X. Since $[\mathcal{G}'] \to x'$ and X is a subspace of Y, $\mathcal{G}' \to x'$ in X. Since Y is regular, X is regular and so $\mathrm{cl}_X \mathcal{G}' \to x'$ in X. Thus, we have that $\mathrm{cl}_X \mathcal{G}' \to x'$ in X, $\mathcal{F} \to t$ in S, $x'/t = x/s$ and $\mathrm{cl}_X \mathcal{G}'/\mathcal{F} \subseteq \mathcal{H}$, which means $\mathcal{H} \to x/s$ in $Q(X)$.

So far we have proved that $Q(X)$ is an S-subspace of $Q(Y)$. We now need to prove that $Q(Y)$ is compact and that $Q(X)$ is dense in $Q(Y)$. Compactness follows from the fact that S and Y are compact and the quotient map $Y \times S \to Q(Y)$ is continuous. To prove the "dense" part, just note that for a given $y/s \in Q(Y)$, there exists a filter $\mathcal{G}$ on X such that $[\mathcal{G}] \to y$ in Y (since X is dense in Y), and since $[s] \to s$, it follows that $[\mathcal{G}]/[s] \to y/s$ in $Q(Y)$.

Finally, we prove that $Q(Y)$ is regular. Let $\mathcal{H} \to y/s$ in $Q(Y)$. Then there are filters $\mathcal{F}$ on S and $\mathcal{G}$ on Y and points $s' \in S$ and $y' \in Y$ such that $\mathcal{F} \to s'$ in S, $\mathcal{G} \to y'$ in Y, $y'/s' = y/s$ and $\mathcal{G}/\mathcal{F} \subseteq \mathcal{H}$. Since S and Y are compact, $Q(Y)$ is Hausdorff, the quotient map $(y,s) \mapsto y/s$ is proper and preserves closures by Propositions 2.12 and 2.11. This means that $\mathrm{cl}_{Q(Y)}\, \mathcal{G}/\mathcal{F} = \mathrm{cl}_Y\, \mathcal{G}/\,\mathrm{cl}_S\, \mathcal{F}$. Since S and Y are regular, $\mathrm{cl}_S\, \mathcal{F} \to s'$ in S and $\mathrm{cl}_Y\, \mathcal{G} \to y'$ in Y and so $\mathrm{cl}_{Q(Y)}\, \mathcal{G}/\mathcal{F} = \mathrm{cl}_Y\, \mathcal{G}/\,\mathrm{cl}_S\, \mathcal{F} \to y/s$ in $Q(Y)$. Since $\mathcal{G}/\mathcal{F} \subseteq \mathcal{H}$, it follows that $\mathrm{cl}_{Q(Y)}\, \mathcal{G}/\mathcal{F} \subseteq \mathrm{cl}_{Q(Y)}\, \mathcal{H}$, and since $\mathrm{cl}_{Q(Y)}\, \mathcal{G}/\mathcal{F} \to y/s$, it follows that $\mathrm{cl}_{Q(Y)}\, \mathcal{H} \to y/s$. This concludes the proof that $Q(Y)$ is regular. ∎

Corollary 6.6. Let X be an convergence S-space that is not compact, locally compact, completely regular, Hausdorff and adherence restrictive. If S is compact and regular, then $Q(\omega X)$ is a one-point regular S-compactification of $Q(X)$.

Proof. Since X is not compact, locally compact, completely regular and Hausdorff, by Theorem 4.6, ωX is a one-point regular strict compactification of X. By Theorem 6.5, $Q(\omega X)$ is a one-point regular S-compactification of $Q(X)$. ∎

Corollary 6.7. Let X be an adherence-restrictive convergence S-space and suppose that S is a compact Hausdorff topological space. Let Y be a topological S-compactification of X. Then $\tau Q(X)$ and $\tau Q(Y)$ are topological S-spaces and $\tau Q(Y)$ is a topological S-compactification of $\tau Q(X)$.

Proof. Since S is a compact Hausdorff topological space, by Proposition 2.9, both $\tau Q(X)$ and $\tau Q(Y)$ are topological S-spaces and are the pseudoquotients of τX and τY, respectively. Since τY is a topological S-compactification, it is strict and regular, hence by Theorem 6.5, $\tau Q(Y)$ is a regular S-compactification of $\tau Q(X)$. ∎

S-Completions

Theorem 6.8. If X is a Hausdorff Cauchy S-space with pseudoquotient $Q(X)$ and Y is a regular remainder-invariant strict S-completion of X and S is complete, then $Q(Y)$ is an S-completion of $Q(X)$.

Proof. Since Y is a complete Hausdorff S-space, by (iii) of Theorem 6.2, $Q(Y)$ is a complete Hausdorff S-space and convergence in $Q(Y)$ is given by the quotient convergence structure. The burden of the proof now is in showing that $Q(X)$ is a dense Cauchy S-subspace of $Q(Y)$. First we prove that $Q(X)$ is an S-subspace of $Q(Y)$. Since Y is remainder invariant, by Proposition 6.1, $Q(X) \subseteq Q(Y)$. By (ii) of Proposition 6.3, to prove that $Q(X)$ is a Cauchy S-subspace of $Q(Y)$, we must only show that if $\mathcal{H}$ is a filter on $Q(X)$ such that $[\mathcal{H}]$ is Cauchy in $Q(Y)$, then $\mathcal{H}$ is Cauchy in $Q(X)$. Thus, suppose $\mathcal{H}$ is a filter on $Q(X)$ and $[\mathcal{H}]$ is a Cauchy filter on $Q(Y)$. Since $Q(Y)$ is complete, $[\mathcal{H}]$ converges to some $y/s \in Q(Y)$. Thus, there are filters $\mathcal{F}$ on S and $\mathcal{H}'$ on Y and elements $t \in S$ and $y' \in Y$ such that $\mathcal{F} \to t$ in S, $\mathcal{H}' \to y'$ in Y, $y'/t = y/s$ and $\mathcal{H}'/\mathcal{F} \subseteq [\mathcal{H}]$. Since Y is strict and regular, there exists a filter $\mathcal{G}$ on X such that $\mathrm{cl}_Y[\mathcal{G}] \to y'$ and $\mathrm{cl}_Y[\mathcal{G}] \subseteq \mathcal{H}'$. Thus, $\mathrm{cl}_Y[\mathcal{G}]/\mathcal{F} \subseteq [\mathcal{H}]$, and by (b) of (i) of Proposition 6.3, $\mathrm{cl}_X \mathcal{G}/\mathcal{F} \subseteq \mathcal{H}$. Since $\mathrm{cl}_Y[\mathcal{G}] \to y'$ in Y, $\mathrm{cl}_Y[\mathcal{G}]$ and consequently $[\mathcal{G}]$ are Cauchy. Since X is a Cauchy subspace of Y and $[\mathcal{G}]$ is a Cauchy filter on Y, $\mathcal{G}$ is a Cauchy filter on X. Since X is regular (as Y is regular), $\mathrm{cl}_X \mathcal{G}$ is Cauchy. Since $\mathcal{F}$ is a convergent filter on S, $\mathcal{F}$ is Cauchy. Since $\mathrm{cl}_X \mathcal{G}$ and $\mathcal{F}$ are Cauchy, $\mathrm{cl}_X \mathcal{G}/\mathcal{F}$ is Cauchy, and since $\mathrm{cl}_X \mathcal{G}/\mathcal{F} \subseteq \mathcal{H}$, it follows that $\mathcal{H}$ is Cauchy. This concludes the proof that $Q(X)$ is a Cauchy S-subspace of $Q(Y)$.

Now we prove that $Q(X)$ is dense in $Q(Y)$: Given $y/s \in Q(Y)$, there exists a filter $\mathcal{G}$ on X such that $[\mathcal{G}] \to y$ in Y (since X is dense in Y), and since $[s] \to s$ in S, it follows that $[\mathcal{G}]/[s] \to y/s$ in $Q(Y)$. ∎

Theorem 6.9. If X is an adherence-restritive separated Cauchy S-space with pseudoquotient $Q(X)$ and S is complete, then $Q(\Gamma X)$ is an S-completion of $Q(X)$.

Proof. Since X is separated, by (ii) of Theorem 5.5, ΓX is a completion of X in standard form.

Since X is adherence restrictive and ΓX is a completion of X, by (ii) of Theorem 5.9, ΓX is an S-completion of X. Since ΓX is a complete Hausdorff S-space and S is complete, by Theorem 6.2, $Q(\Gamma X)$ is a complete Hausdorff S-space and convergence in $Q(\Gamma X)$ is given by the quotient convergence structure. Define $\varphi\colon Q(X) \to Q(\Gamma X)$ so that $\varphi(x/s) = \phi(x)/s$. We want to show that φ is a dense Cauchy embedding.

By Proposition 6.3, φ is a injective Cauchy continuous S-map. Thus, to prove that φ is an embedding, we must only show that if $\mathcal{H}$ is a filter on $Q(X)$ such that $\varphi[\mathcal{H}]$ is a Cauchy filter on $Q(\Gamma X)$, then $\mathcal{H}$ is a Cauchy filter on $Q(X)$. Thus, suppose $\mathcal{H}$ is a filter on $Q(X)$ and $\varphi[\mathcal{H}]$ is a Cauchy filter on $Q(\Gamma X)$. Since $Q(\Gamma X)$ is complete, $\varphi[\mathcal{H}]$ converges to some $\langle \mathcal{G}\rangle/s \in Q(\Gamma X)$. Thus, there are filters $\mathcal{F}$ on S and $\mathcal{H}'$ on ΓX and elements $t \in S$ and $\langle \mathcal{G}'\rangle \in \Gamma X$ such that $\mathcal{F} \to t$ in S, $\mathcal{H}' \to \langle \mathcal{G}'\rangle$ in ΓX, $\langle \mathcal{G}'\rangle/t = \langle \mathcal{G}\rangle/s$ and $\mathcal{H}'/\mathcal{F} \subseteq \varphi[\mathcal{H}]$. Since $\mathcal{H}' \to \langle \mathcal{G}'\rangle$ in ΓX, $\mathcal{H}' \cap [\langle \mathcal{G}'\rangle]$ is a Cauchy filter on ΓX, which means there is a Cauchy filter $\mathcal{G}_0$ on X such that $\Gamma\mathcal{G}_0 \subseteq \mathcal{H}' \cap [\langle \mathcal{G}'\rangle]$. Thus, $\Gamma\mathcal{G}_0 \subseteq \mathcal{H}'$, which means $\Gamma\mathcal{G}_0/\mathcal{F} \subseteq \mathcal{H}'/\mathcal{F} \subseteq \varphi[\mathcal{H}]$. We claim that $\mathcal{G}_0/\mathcal{F} \subseteq \mathcal{H}$. To see this, pick a $G \in \mathcal{G}_0$ and an $F \in \mathcal{F}$. We want to find an $H \in \mathcal{H}$ such that $H \subseteq G/F$. Since $\Gamma\mathcal{G}_0/\mathcal{F} \subseteq \varphi[\mathcal{H}]$, there is an $H \in \mathcal{H}$ such that $\varphi(H) \subseteq \Gamma G/F$. Since $H \in Q(X)$, it follows that $\varphi(H) \subseteq \phi(X)/S$, and since $\varphi(H) \subseteq \Gamma G/F = (\phi(G) \cup (\Sigma G - \phi(X)))/F = \phi(G)/F \cup (\Sigma G - \phi(X))/F$, it follows that $\varphi(H) \subseteq \phi(G)/F$. Thus, if $x/s \in H$, then $\varphi(x/s) = \phi(x)/s \in \phi(G)/F$, which means there is a $y \in G$ and a $t \in F$ such that $\phi(x)/s = \phi(y)/t$, which means $\phi(x)t = \phi(xt) = \phi(y)s = \phi(ys)$, which means $xt = ys$, which means $x/s \in G/F$. This proves that $H \subseteq G/F$ and that $\mathcal{G}_0/\mathcal{F} \subseteq \mathcal{H}$. Since $\mathcal{G}_0$ and $\mathcal{F}$ are Cauchy, it follows that $\mathcal{H}$ is Cauchy. This concludes the proof that φ is an embedding.

Finally, we proof that $\varphi(Q(X))$ is dense in $Q(\Gamma X)$. Let $\langle \mathcal{G}\rangle/s$ in $Q(\Gamma X)$ be arbitrary. By Lemma 3 in Theorem 5.5, $\Gamma\mathcal{G} \to \langle \mathcal{G}\rangle$ in ΓX. And since $[s] \to s$ in S, it follows that $\Gamma\mathcal{G}/[s] \to \langle \mathcal{G}\rangle/s$ in $Q(\Gamma X)$.

∎

LIST OF REFERENCES

[1] AULL, C. E., AND LOWEN, R. *Handbook of the History of General Topology,* vol. 2. Kluwer Academic Publishers, 1998.

[2] BEATTIE, R., AND BUTZMANN, H. P. *Convergence Structures and Applications to Functional Analysis.* Springer, 2002.

[3] BOURBAKI, N. *General Topology: Chapters 1–4, 18.* Springer, 1998.

[4] BOUSTIQUE, H., MIKUSIŃSKI, P., AND RICHARDSON, G. D. Convergence semigroup actions: generalized quotients. *Applied Gen. Top. 10,* 2 (2009), 173–186.

[5] BOUSTIQUE, H., MIKUSIŃSKI, P., AND RICHARDSON, G. D. Convergence semigroup categories. *Applied Gen. Top. 11,* 2 (2010), 67–88.

[6] DE VRIES, J. On the existence of G-compactifications. *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. 26,* 3 (1978), 275–280.

[7] GAZIK, R., AND KENT, D. C. Regular completions of Cauchy spaces via function algebras. *Bull. Austral. Math. Soc. 11* (1974), 77–88.

[8] KELLER, H. H. Die Limes-Uniformisierbarkeit der Limesräume. *Mathematische Annalen 176,* 4 (1968), 334–341.

[9] KENT, D. C., AND DE EGUINO, K. R. On p-regular completions of Cauchy spaces. *General Topology and Applications 134* (1991), 195.

[10] KENT, D. C., AND RICHARDSON, G. D. Regular compactifications of convergence spaces. *Proc. Amer. Math. Soc. 31* (1972), 571–573.

[11] Kent, D. C., and Richardson, G. D. Regular completions of Cauchy spaces. *Pacific Journal of Mathematics*, 51 (1974), 483–490.

[12] Kent, D. C., and Richardson, G. D. Locally compact convergence spaces. *The Michigan Mathematical Journal 22*, 4 (1976), 353–360.

[13] Kent, D. C., and Richardson, G. D. Compactifications of convergence spaces. *Internat. J. Math. Math. Sci. 2*, 3 (1979), 345–368.

[14] Kent, D. C., and Richardson, G. D. Cauchy spaces with regular completions. *Pacific Journal of Mathematics 111*, 1 (1984), 105–116.

[15] Kozlov, K., and Chatyrko, V. On G-compactifications. *Mathematical Notes 78*, 5 (2005), 649–661.

[16] Losert, B., and Richardson, G. D. Convergence S-compactifications. *submitted*.

[17] Losert, B., and Richardson, G. D. Convergence S-completions. *submitted*.

[18] Megrelishvili, M. Compactifications of semigroups and semigroup actions. *arXiv preprint math/0611786* (2006).

[19] Mikusiński, P. Boehmians and generalized functions. *Acta Mathematica Hungarica 51*, 3 (1988), 271–281.

[20] Mikusiński, P. Generalized quotients with applications in analysis. *Methods and Applications of Analysis 10*, 3 (2003), 377–386.

[21] Reed, E. E. Completions of uniform convergence spaces. *Mathematische Annalen 194*, 2 (1971), 83–108.

[22] Richardson, G. D. A Stone-Čech compactification for limit spaces. *Proc. Amer. Math. Soc 25* (1970), 403–404.