Spectrally Uniform Frames And Spectrally Optimal Dual Frames

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Saliha Pehlivan
University of Central Florida

SPECTRALLY UNIFORM FRAMES

AND

SPECTRALLY OPTIMAL DUAL FRAMES

SALIHA PEHLIVAN
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Major Professors: Deguang Han and Ram Mohapatra

ABSTRACT

Frames have been useful in signal transmission due to the built in redundancy. In recent years, the erasure problem in data transmission has been the focus of considerable research in the case the error estimate is measured by operator (or matrix) norm. Sample results include the characterization of one-erasure optimal Parseval frames, the connection between two-erasure optimal Parseval frames and equiangular frames, and some characterization of optimal dual frames.

If iterations are allowed in the reconstruction process of the signal vector, then spectral radius measurement for the error operators is more appropriate then the operator norm measurement. We obtain a complete characterization of spectrally one-uniform frames (i.e., one-erasure optimal frames with respect to the spectral radius measurement) in terms of the redundancy distribution of the frame. Our characterization relies on the connection between spectrally optimal frames and the linear connectivity property of the frame. We prove that the linear connectivity property is equivalent to the intersection dependence property, and is also closely related to the well-known concept of k-independent set. For spectrally two-uniform frames, it is necessary that the frame must be linearly connected. We conjecture that it is also necessary that a two-uniform frame must be n-independent. We confirmed this conjecture for the case when $N = n+1, n+2$, where N is the number of vectors in a frame for an n-dimensional Hilbert space. Additionally we also establish several necessary and sufficient conditions for the existence of an alternate dual frame to make the

iterated reconstruction to work.

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CHAPTER 1: INTRODUCTION

In the study of Hilbert spaces, an orthonormal basis, possessing some desirable properties, is one of the most important concepts. One such property is that each element in Hilbert space can be written uniquely as a linear combination of the elements in the basis. For instance, in the signal transmission, a signal is thought as a vector in a Hilbert space that is represented as a linear combinations of orthogonal basis vectors. The signal is transmitted to a receiver by transmitting the sequence of coefficients that represents the signal. These coefficients can be computed by taking some inner products. The receiver on the other side reconstructs the signal. However, if one of the coefficients is lost during the transmission, the receiver cannot recovers the signal. The orthogonality property of the basis is restrictive in this sense. This brings us the notion of frame that has redundancy so that if some pieces of information is lost, it is recovered with the other pieces that are received.

A vector in a Hilbert space can be represented by the elements of a frame but not necessarily uniquely as in the case of an orthonormal basis. Thus, frames are considered as a generalization of orthogonal basis. The redundancy property of frames makes it more robust than orthogonal basis in some applications such as signal processing, image processing, coding and sampling. These applications have naturally led to the investigations of optimal frames or dual frames that yield better approximations to the original signals. Typically there are two types of investigations on optimal dual frames: one of them is to find (characterize) and construct optimal frames among a class of frames. Examples of this kind include the known theory established for erasure-optimal Parseval frames (i.e. frames that are erasure optimal in the class of all Parseval frames, (c.f. [7, 10, 12, 13, 18, 22, 34, 35, 36, 50]). The other kind is the investigation of optimal dual frames for a given frame. This case addresses the applications when a particular frame that models the nature of the application is preselected for encoding (decomposition) of the signal. In this case the theory of optimal dual frames (for the purpose of better decoding) all needs to be established (c.f. [38, 41, 42, 44, 45, 46]).

When comes to the terminology of optimal we mean the reconstruction error is minimal with respect to some prescribed measurement. So far most of the investigations use the operator (matrix) norm as the measurement for optimality, and assume the "one-step" (without iterations) reconstruction procedure. However, in real applications, a few steps simple iterations may significantly improve the reconstruction accuracy and in this case the spectral radius (of the error operator) measurement seems to be more natural choice. This idea was first explored by Holmes and Paulsen in [35]. The main focus of this dissertation is on the investigation of spectrally optimal frames. We point out that the spectral radius measurement is the same as the norm measurement if the error operator is positive. For example if we are only interested in Parseval frames and their standard dual frames, then the spectral radius measurement is the same as the norm measurement. Therefore, the novelty of this work is on the spectrally optimal frames that admit an alternate dual frame, not necessarily standard dual frame, which is erasure spectrally optimal dual frame.

The main contribution of this dissertation is to establish the connections between spectrally one-uniform frames and the so-called linear connectivity property of the frame. This leads to the concept of redundancy distribution of a frame. With the help of a characterization of linearly connected frame in terms of intersection dependence property and k-independent property, we are able to completely characterize the spectrally one-uniform frames.More importantly this characterization also provides a method to calculate the minimal-maximal error in terms of the redundancy distribution of the frame, and to construct frames, i.e., spectrally one-uniform frames, that admit minimal-maximal reconstruction errors. As a consequence we obtain that the minimal-maximal error only takes rational values, and in some special cases only linearly connected frames give us the best minimal-maximal reconstruction errors.

Additionally, we provide some partial results on spectrally two-uniform frames. When the number of frame vectors is one more than the dimension of the space, we give the characterization of spectrally two-uniform frames in terms of n-independence property of the frame. Moreover, we give a sufficient condition for a frame to be spectrally two-uniform frame in the case the number of frame vectors is two more than the dimension of the space. Finally, it is shown that there is no spectrally two-uniform frame with four vectors in two dimension.

The rest of the dissertation is organized as follows. In chapter 2 we will review some basics/background related to the frames and dual frames. In chapter 3, we discuss the erasure problem, optimal frames, and optimal dual frames with respect to matrix norm measurement and spectral radius measurement, and we introduce k-erasure spectrally optimal dual frames. In chapter 4, we give some conditions on frames so that their standard dual frames are one-erasure or two-erasure spectrally optimal dual frames. Chapter 5 is devoted to one-uniform frames in which the characterization and construction of one-uniform frames are presented as well as the relation between the linear connectivity property of a frame and one-uniform frame is mentioned. In chapter 6, we examine two-uniform frames for some specific cases. In Chapter 7 and 8, we give some examples on frames and mention future study, respectively.

CHAPTER 2: PRELIMINARIES

In this chapter, we will review some fundemental concepts and results from frame theory in Hilbert space that will be used throughout the thesis. we refer to [20, 31, 32] for more details about the basic theory of frames.

2.1 Frames in Hilbert Spaces

Let H be a Hilbert space with inner product $\langle\, , \rangle$, and norm $\|\cdot\|$. We will begin with the formal definition of frame which is valid in both finite and infinite dimensional Hilbert spaces.

Definition 2.1.1. A collection $\{f_i\}_{i\in\mathbb{N}}$ of elements of a Hilbert space H is called a **frame** for H if there are positive constants $0 < A \leq B < \infty$ such that

$$A||f||^2 \leq \sum_{i\in\mathbb{N}} |\langle f, f_i\rangle|^2 \leq B||f||^2, \quad \text{for all f } \in H. \tag{2.1.1}$$

In the above definition, A and B are called **lower** and **upper frame bounds**, respectively.

A frame is called a **tight frame** if $A = B$, and if $A = B = 1$, it is called **Parseval frame**. If the norm of frame vectors are equal, it is called **uniform frame** and if additionally norm is one, it is called **unit norm frame**.

Now lets see some frame examples on ℓ^2.

Example 2.1.1. i) Standard orthonormal basis is a Parseval frame with $A = 1$.

ii) $\mathcal{F} = \{0, 0, 0, 0, 0, e_1, e_2, e_3, \ldots\}$ is a Parseval frame with $A = 1$.

iii) $\mathcal{F} = \{e_1, e_1, e_1, e_1, e_2, e_3, e_4, \ldots\ldots\}$ is a frame with bounds $A = 1$ and $B = 4$.

iv) $\mathcal{F} = \left\{e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \ldots\ldots\right\}$ is a Parseval frame.

The definition given in (2.1.1) is true for both finite and infinite dimensional Hilbert spaces. However, there is an alternative definiton to frames in finite dimensional Hilbert spaces.

Theorem 2.1.1. A family of elements $\{f_i\}_{i=1}^N$ in a finite dimensional Hilbert space H is a frame for H if and only if $\{f_i\}_{i=1}^N$ spans H; i.e., $\text{span}\{f_i\}_{i=1}^N = H$.

Proof. Assume that $H = \text{span}\{f_i\}_{i=1}^N$. We can find nonzero $h \in H$ with $||h|| = 1$ such that

$$A = \sum_{i=1}^N |\langle h, f_i\rangle|^2 = \min\left\{\sum_{i=1}^N |\langle f, f_i\rangle|^2 \,:\, f \in H\,, ||f|| = 1\right\}, \tag{2.1.2}$$

where $\sum_i |\langle f, f_i\rangle|^2$ is a continuous function of f. We see that $A > 0$ and

$$\sum_{i=1}^N |\langle f, f_i\rangle|^2 = \sum_{i=1}^N |\langle \frac{f}{||f||}, f_i\rangle|^2 ||f||^2 \geq A||f||^2. \tag{2.1.3}$$

Note that by Cauchy-Schwarz' inequality, we have

$$\sum_{i=1}^N |\langle f, f_i\rangle|^2 \leq \sum_{i=1}^N ||f_i||^2 ||f||^2, \tag{2.1.4}$$

and since the sequence of vectors $\{f_i\}_i^N$ is finite, $B = \sum_{i=1}^N ||f_i||^2 < \infty$. Hence, $\{f_i\}_i^N$ is a frame for H.

For the other direction, assume that F is a frame and $\{f_i\}_{i=1}^N$ does not span H. Then there exists a vector $f \in M^\perp$ where M =span$\{f_i\}_{i=1}^N$. Note that f is orthogonal to each f_i. Thus, $\sum_{i=1}^N |\langle f, f_i\rangle| = 0$. This implies that the lower frame bound is 0, which contradicts to the fact that F is a frame. $\square$

Note here that, particularly, this definition implies that every basis for a Hilbert space H is a frame for H. Moreover, a finite collection of vectors $\{f_i\}_i^N$ is a frame for its span, span$\{f_i\}_i^N$.

Proposition 2.1.1. Let $\{f_i\}_{i=1}^N$ be a frame with a lower and upper frame bounds A and B, respectively. Then, $||f_i||^2 \leq B$ for all $i = 1, \ldots, N$. If $||f_i||^2 = B$ for all i, then f_i is orthogonal to every f_j for $j \neq i$. Moreover, if $||f_i||^2 < A$, then $f_i \in$ span$\{f_j\}_{j\neq i}^N$.

Proof. Let $\{f_i\}_{i=1}^N$ be a frame with bounds A and B. Then from the frame definition, for every $j \in \{1, \ldots, N\}$, we have

$$B||f_j||^2 \geq \sum_{i=1}^N |\langle f_j, f_i\rangle|^2 \geq |\langle f_j, f_j\rangle|^2 = ||f_j||^4. \tag{2.1.5}$$

Thus, $||f_i||^2 \leq B$.

For the second part of the proposition assume that $||f_i||^2 = B$, then, from the definition of frame,

$$B||f_j||^2 \geq \sum_{i=1}^N |\langle f_j, f_i\rangle|^2 = |\langle f_j, f_j\rangle|^2 + \sum_{\substack{i=1\\ i\neq j}}^N |\langle f_j, f_i\rangle|^2 = B^2 + \sum_{\substack{i=1\\ i\neq j}}^N |\langle f_j, f_i\rangle|^2, \tag{2.1.6}$$

and, this implies that $\sum_{\substack{i=1\\ i\neq j}}^N |\langle f_j, f_i\rangle|^2 \leq 0$. Therefore, $\langle f_j, f_i\rangle = 0$ for all $i \neq j$.

To show the last part of the proposition, suppose $||f_i||^2 < A$ for all i, and assume for a contradiction that there exist $j \in \{1, \ldots, N\}$ such that f_j is not in the span of $\{f_i\}_{i \neq j}$, in other words, $\langle f_i, f_j \rangle = 0$ for every $i \neq j$. Then, from the definition of frame, we have

$$A||f_j||^2 \leq \sum_{i=1}^{N} |\langle f_j, f_i \rangle|^2 = |\langle f_j, f_j \rangle|^2 + \sum_{\substack{i=1 \\ i \neq j}}^{N} |\langle f_j, f_i \rangle|^2 = ||f_j||^4, \tag{2.1.7}$$

that is, $||f_j||^2 \geq A$. This contradicts with the assumption. Hence, $f_i \in \text{span}\{f_j\}_{j \neq i}^{N}$ for all $i \in \{1, \ldots, N\}$. $\square$

As particular cases of the proposition, we state the following two corollaries:

Corollary 2.1.1. Let $\{f_i\}_{i=1}^{N}$ be a tight frame with frame bound A. Then, $||f_i||^2 \leq A$ for all $i = 1, \ldots, N$, and the inequality holds if and only if f_i is orthogonal to every f_j for $j \neq i$.

Proof. It is enough to show that if f_i is orthogonal to every f_j for $j \neq i$, then $||f_i||^2 = A$, the rest follows from the proof of the above proposition. In fact, assume that f_i is orthogonal to every f_j for $j \neq i$. Then, by the last part of the proposition, we have $||f_i||^2 \geq A$; moreover, we have $||f_i||^2 \leq A$ from the first part of the proposition. Thus, $||f_i||^2 = A$ for all $i \in \{1, \ldots, N\}$. $\square$

Corollary 2.1.2. Let $\{f_i\}_{i=1}^{N}$ be a Parseval frame. Then, $||f_i||^2 \leq 1$ for all $i = 1, \ldots, N$, and the inequality holds if and only if f_i is orthogonal to every f_j for $j \neq i$.

Proposition 2.1.2. If one of the vectors f_j of a Parseval frame $\{f_i\}_{i=1}^{N}$ is removed, then the family of the vectors $\{f_i\}_{i \neq j}$ is either a frame or an incomplete set.

Proof. By Corollary 2.1.2, the norm of vectors of a Parseval frame is either one or less than one. If $||f_j|| = 1$, then f_i is orthogonal to span$\{f_i\}_{i \neq j}$; thus, $\{f_i\}_{i \neq j}$ ceases to be a frame. On the other hand, when $||f_j|| < 1$, $f_j \in \text{span}\{f_i\}_{i \neq j}$. Hence, $\{f_i\}_{i \neq j}$ spans the Hilbert space H, thus, $\{f_i\}_{i \neq j}$ is a frame for H. $\square$

2.2 Frame Operators

In this section, we will try to develop a reconstruction formula for frames similar to the reconstruction formula for orthonormal basis. To find a formula, we first define the analysis and synthesis operators.

Definition 2.2.1. Let $\{f_i\}_{i\in I}$ be a frame for a Hilbert space H and $\{e_i\}_{i\in I}$ be the standard orthonormal basis. The **analysis operator** $\Theta : H \to \ell_2(I)$ is defined to be

$$\Theta(f) = \sum_{i\in I} \langle f, f_i\rangle e_i \quad \text{for all f } \in H. \tag{2.2.1}$$

The adjoint of the analysis operator is called the **synthesis operator** that is given by

$$\Theta^*(e_i) = f_i. \tag{2.2.2}$$

By composing synthesis operator Θ^* with its adjoint operator Θ, we get the **frame operator** S which is given by

$$Sf = \Theta^*\Theta f = \sum_{i\in I} \langle f, f_i\rangle f_i. \tag{2.2.3}$$

Remark 2.2.1. S is self-adjoint and positive operator which follows from

$$S^* = (\Theta^*\Theta)^* = \Theta^*\Theta = S, \tag{2.2.4}$$

and

$$\langle Sf, f\rangle = \left\langle \sum_{i\in I}\langle f, f_i\rangle f_i, f\right\rangle \tag{2.2.5}$$

$$= \sum_{i\in I}\langle f, f_i\rangle\langle f_i, f\rangle \tag{2.2.6}$$

$$= \sum_{i\in I} |\langle f, f_i\rangle|^2, \tag{2.2.7}$$

respectively.

Remark 2.2.2. By the definition (2.1.1) of frame and (2.2.7), we have $A||f||^2 \leq \langle Sf, f\rangle \leq B||f||^2$ for all $f \in H$, or, $AI \leq S \leq BI$. If $\{f_i\}_{i\in I}$ is a tight frame; i.e., $A = B$, then $S = AI$, and if $\{f_i\}_{i\in I}$ is a Parseval frame; i.e., $A = B = 1$, then $S = I$.

Next, we give some properties of analysis operator.

Proposition 2.2.1. Let Θ_{Tf} be an analysis operator for the set of vectors $\{Tf_i\}_{i\in I}$ where $T : H \to H$ is a linear operator. Then, $\Theta_{Tf}h = \Theta_f T^* h$.

Proof. Let $h \in H$. By the definition of analysis operator we have,

$$\Theta_{Tf}h = \sum_{i\in I}\langle h, Tf_i\rangle e_i = \sum_{i\in I}\langle T^*h, f_i\rangle e_i = \Theta_f T^* h. \tag{2.2.8}$$

□

Proposition 2.2.2. Let $\Theta_{\alpha f}$ be an analysis operator of the set of vectors $\{\alpha f_i\}_{i\in I}$ where α is a scalar. Then, $\Theta_{\alpha f} = \bar{\alpha}\Theta_f$.

Proof. Letting $T = \alpha I$ in Proposition 2.2.1, the result follows. □

Following couple propositions show the relationship between frames and its corresponding analysis and frame operators, respectively. In other words, frames can be characterized by analysis and frame operators.

Proposition 2.2.3. Let H be a finite dimensional Hilbert space. Then, $\{f_i\}_{i=1}^N$ is a frame for H if and only if the analysis operator Θ is one-to-one.

Proof. First, suppose that $\{f_i\}_{i=1}^N$ is a frame for H. And assume that $\Theta f = 0$ for some $f \in H$. Then, $\sum_{i=1}^N \langle f, f_i \rangle e_i = 0$, which means that $\langle f, f_i \rangle = 0$ for all $i = 1, \ldots, N$ because $\{e_i\}_{i=1}^N$ is the standard orthonormal basis. On the other hand, since $\{f_i\}_{i=1}^N$ is a frame, we can write every $f \in H$ as a linear combination of frame vectors such that $f = \sum_{i=1}^N c_i f_i$ for some constants c_i. Then

$$\langle f, f \rangle = \left\langle f, \sum_{i=1}^N c_i f_i \right\rangle = \sum_{i=1}^N \bar{c_i} \langle f, f_i \rangle = 0 \tag{2.2.9}$$

Hence $f = 0$, and f is one-to-one.

Now, suppose that Θ is one-to-one, and assume for a contradiction that $\{f_i\}_{i=1}^N$ is not a frame for H; i.e., $\{f_i\}_{i=1}^N$ does not span H. Then there exist nonzero $f \in H$ such that $\langle f, f_i \rangle = 0$ for all $i = 1, \ldots, N$. Thus, we have $\Theta f = \sum_{i=1}^N \langle f, f_i \rangle e_i = 0$. This contradicts with Θ being one-to-one. Hence, $\{f_i\}_{i=1}^N$ is a frame for H. $\square$

Proposition 2.2.4. Let H be a finite dimensional Hilbert space. Then, $\{f_i\}_{i=1}^N$ is a frame for H if and only if the frame operator S is invertible.

Proof. First assume that $\{f_i\}_{i=1}^N$ is a frame for H. To show that S is one-to-one, assume further that $Sf = 0$. Then by (2.2.7), we have $\sum_{i=1}^N |\langle f, f_i \rangle| = 0$. This implies that $||f|| = 0$ by the definition of frame. Hence, f is one-to-one. Now, to show that S is onto, assume that there exist nonzero element f in the orthogonal complement of the range of S. Then $\langle Sg, f\rangle = 0$ for all $g \in H$. Thus, $\langle Sf, f\rangle = 0$. Again, from (2.2.7) and the definition of frame, $f = 0$. Therefore, range of S is the entire space H.

To show the opposite direction, assume that S is invertible with the inverse operator S^{-1}. Then, for each $f \in H$

$$f = SS^{-1}f = \sum_{i=1}^{N}\langle S^{-1}f, f_i\rangle f_i = \sum_{i=1}^{N}\langle f, S^{-1}f_i\rangle f_i. \tag{2.2.10}$$

This shows that $\{f_i\}_{i=1}^N$ spans H and, therefore, $\{f_i\}_{i=1}^N$ is a frame. □

If the inverse S^{-1} of frame operator is applied to the frame vectors f_i for $i = 1, \ldots, N$, then the new collection of vectors $\{S^{-1}f_i\}_{i=1}^N$ is a frame and its frame bounds are characterized by the frame bounds of $\{f_i\}_{i=1}^N$.

Proposition 2.2.5. If $\{f_i\}_{i=1}^N$ is a frame for a finite dimensional H with corresponding frame operator S and frame bounds A and B, then $\{S^{-1}f_i\}_{i=1}^N$ is also a frame for H with lower and upper frame bounds B^{-1} and A^{-1}, respectively. Moreover, the frame operator for $\{S^{-1}f_i\}_{i=1}^N$ is S^{-1}.

Proof. Recall from Remark 2.2.2 that $AI \leq S \leq BI$. Now, applying S^{-1} to each side, we have

$$S^{-1}A \leq S^{-1}S = I \quad \Rightarrow S^{-1} \leq A^{-1}I, \tag{2.2.11}$$

$$I = S^{-1}S \leq S^{-1}B \quad \Rightarrow S^{-1} \geq B^{-1}I, \tag{2.2.12}$$

which is

$$B^{-1}I \leq S^{-1} \leq A^{-1}I, \tag{2.2.13}$$

or,

$$B^{-1}||f||^2 = \langle B^{-1}f, f\rangle \le \langle S^{-1}f, f\rangle \le \langle A^{-1}f, f\rangle = A^{-1}||f||^2 \quad \text{for all f } \in H. \tag{2.2.14}$$

On the other hand,

$$S^{-1}f = S^{-1}SS^{-1}f = S^{-1}\sum_{i=1}^{N}\langle S^{-1}f, f_i\rangle f_i = \sum_{i=1}^{N}\langle f, S^{-1}f_i\rangle S^{-1}f_i. \tag{2.2.15}$$

This implies that

$$\langle S^{-1}f, f\rangle = \Big\langle \sum_{i=1}^{N}\langle f, S^{-1}f_i\rangle S^{-1}f_i, f\Big\rangle = \sum_{i=1}^{N}|\langle f, S^{-1}f_i\rangle|^2. \tag{2.2.16}$$

From (2.2.14) and (2.2.16), we have

$$B^{-1}||f||^2 \le \sum_{i=1}^{N}|\langle f, S^{-1}f_i\rangle|^2 \le A^{-1}||f||^2 \quad \text{for all f } \in H. \tag{2.2.17}$$

Therefore, $\{S^{-1}f_i\}_{i=1}^N$ is a frame with lower and upper frame bounds B^{-1} and A^{-1}, respectively. Note that (2.2.15) shows that S^{-1} is the frame operator for $\{S^{-1}f_i\}_{i=1}^N$. □

Now, we shall show the relationship between frame bounds and the eigenvalues of frame operators.

Proposition 2.2.6. Let $\{f_i\}_{i=1}^N$ be a frame with frame operator S for a finite dimensional H. Then the smallest and largest eigenvalues of S are a lower and an upper frame bounds, respectively, for $\{f_i\}_{i=1}^N$.

Proof. Assume that $\{f_i\}_{i=1}^N$ is a frame for H with frame operator S and n is the dimension of H. For any $f \in H$, we can write $f = \sum_{i=1}^n \langle f, e_i \rangle e_i$, where $\{e_i\}_{i=1}^n$ is the standard orthonormal basis. Then

$$Sf = \sum_{i=1}^n \langle f, e_i \rangle Se_i = \sum_{i=1}^n \lambda_i \langle f, e_i \rangle e_i, \tag{2.2.18}$$

where $\{\lambda_i\}_{i=1}^n$ are the eigenvalues for S corresponding to the eigenvalues $\{e_i\}_{i=1}^n$. And,

$$\langle Sf, f \rangle = \Big\langle \sum_{i=1}^n \lambda_i \langle f, e_i \rangle e_i, f \Big\rangle \tag{2.2.19}$$

$$= \sum_{i=1}^n \lambda_i \langle f, e_i \rangle \langle e_i, f \rangle = \sum_{i=1}^n \lambda_i |\langle f, e_i \rangle|^2. \tag{2.2.20}$$

Note that in (2.2.7), it is shown that $\langle Sf, f \rangle = \sum_{i=1}^N |\langle f, f_i \rangle|^2$, and we also have

$$||f||^2 = \langle f, f \rangle = \Big\langle \sum_{i=1}^n \langle f, e_i \rangle e_i, f \Big\rangle = \sum_{i=1}^n |\langle f, e_i \rangle|^2 \tag{2.2.21}$$

Thus, by (2.2.20) and (2.2.21),

$$\lambda_{\text{min}} ||f||^2 = \lambda_{\text{min}} \sum_{i=1}^n |\langle f, e_i \rangle|^2 \tag{2.2.22}$$

$$\leq \sum_{i=1}^n \lambda_i |\langle f, e_i \rangle|^2 = \sum_{i=1}^N |\langle f, f_i \rangle|^2 \tag{2.2.23}$$

$$\leq \lambda_{\text{max}} \sum_{i=1}^n |\langle f, e_i \rangle|^2 \tag{2.2.24}$$

$$= \lambda_{\text{max}} ||f||^2 \tag{2.2.25}$$

$\square$

2.3 Parseval Frames

In this section, we shall show that Parseval frames have the reconstruction property of orthonormal bases. For the rest of the dissertation, we assume that H is finite dimensional Hilbert space. First, we need to observe the following:

Remark 2.3.1. If the collection of vectors $\{f_i\}_{i=1}^N$ is a Parseval frame then the corresponding analysis operator Θ is an isometry; that is

$$\langle \Theta f, \Theta f\rangle = \langle \Theta^* \Theta f, f\rangle = \langle Sf, f\rangle = \sum_{i=1}^{N} |\langle f, f_i\rangle|^2 = ||f||^2 = \langle f, f\rangle \tag{2.3.1}$$

which follows from (2.2.7) and the definition of Parseval frame ($A = B = 1$). Furthermore, Θ preserves inner products; i.e., $\langle \Theta f, \Theta g\rangle = \langle f, g\rangle$ for every $f, g \in H$.

Theorem 2.3.1. A family of vectors $\{f_i\}_{i=1}^N$ is a Parseval frame if and only if it satisfies the reconstruction property, that is, for every $f \in H$,

$$f = \sum_{i=1}^{N} \langle f, f_i\rangle f_i. \tag{2.3.2}$$

Proof. Assume that $\{f_i\}_{i=1}^N$ is a Parseval frame, and let $\{e_i\}_{i=1}^N$ be the standard orthonormal basis for $\mathbb{C}^N$ and $\{v_i\}_{i=1}^n$ be an orthonormal basis for H. Then, from the reconstruction property of orthonormal basis and Remark 2.3.1, we have

$$
\begin{aligned}
f &= \sum_{i=1}^{n} \langle f, v_i \rangle v_i && (2.3.3)\\
&= \sum_{i=1}^{n} \langle \Theta f, \Theta v_i \rangle v_i && (2.3.4)\\
&= \sum_{i=1}^{n} \Big\langle \sum_{j=1}^{N} \langle f, f_j \rangle e_j \sum_{k=1}^{N} \langle v_i, f_k \rangle e_k \Big\rangle v_i && (2.3.5)\\
&= \sum_{i=1}^{n} \sum_{j=1}^{N} \sum_{k=1}^{N} \langle f, f_j \rangle \overline{\langle v_i, f_k \rangle} \langle e_j, e_k \rangle v_i && (2.3.6)\\
&= \sum_{i=1}^{n} \sum_{j=1}^{N} \langle f, f_j \rangle \overline{\langle v_i, f_j \rangle} v_i && (2.3.7)\\
&= \sum_{j=1}^{N} \langle f, f_j \rangle \sum_{i=1}^{n} \langle f_j, v_i \rangle v_i && (2.3.8)\\
&= \sum_{j=1}^{N} \langle f, f_j \rangle f_j. && (2.3.9)
\end{aligned}
$$

Thus, $\{f_i\}_{i=1}^{N}$ satisfies reconstruction property in (2.3.2).

For the converse, assume that (2.3.2) holds true for the family of vectors $\{f_i\}_{i=1}^{N}$. Then

$$
\begin{aligned}
||f||^2 = \langle f, f \rangle &= \Big\langle f, \sum_{i=1}^{N} \langle f, f_i \rangle f_i \Big\rangle && (2.3.10)\\
&= \sum_{i=1}^{N} \overline{\langle f, f_i \rangle} \langle f, f_i \rangle && (2.3.11)\\
&= \sum_{i=1}^{N} |\langle f, f_i \rangle|^2. && (2.3.12)
\end{aligned}
$$

Therefore, $\{f_i\}_{i=1}^N$ is a Parseval frame. $\square$

Proposition 2.3.1. If the collection of vectors $\{f_i\}_{i=1}^N$ in H is a frame for H with frame operator S, then $\{S^{-\frac{1}{2}}f_i\}_{i=1}^N$ is a Parseval frame for H.

Note 2.3.1. The frame operator S being a positive invertible operator has a positive square root operator $S^{\frac{1}{2}}$. Similarly, since S^{-1} is positive operator, there is a corresponding positive square root operator $S^{-\frac{1}{2}}$. Both $S^{\frac{1}{2}}$ and $S^{-\frac{1}{2}}$ are self-adjoint operators.

Proof. Let $\{f_i\}_{i=1}^N$ be a frame for H with frame operator S. Then, from Note 2.3.1, we have

$$f = S^{-\frac{1}{2}}SS^{-\frac{1}{2}}f = S^{-\frac{1}{2}}\sum_{i=1}^{N}\left\langle S^{-\frac{1}{2}}f, f_i\right\rangle f_i \tag{2.3.13}$$

$$= \sum_{i=1}^{N}\left\langle S^{-\frac{1}{2}}f, f_i\right\rangle S^{-\frac{1}{2}}f_i \tag{2.3.14}$$

$$= \sum_{i=1}^{N}\left\langle f, S^{-\frac{1}{2}}f_i\right\rangle S^{-\frac{1}{2}}f_i, \tag{2.3.15}$$

which means that $\left\{S^{-\frac{1}{2}}f_i\right\}_{i=1}^N$ satisfies reconstruction formula; hence, $\left\{S^{-\frac{1}{2}}f_i\right\}_{i=1}^N$ is a Parseval frame for H.

$\square$

2.4 Dual Frames

For every frame, we have a general reconstruction formula similar to the reconstruction formula (2.3.2) for Parseval frames. To define reconstruction formula, we need a new set of vectors called dual frames.

Definition 2.4.1. Let $\{f_i\}_{i=1}^N$ be a frame for a Hilbert space H. A set of vectors $\{g_i\}_{i=1}^N$ which satisfies the following formula

$$f = \sum_{i=1}^{N} \langle f, g_i \rangle f_i = \sum_{i=1}^{N} \langle f, f_i \rangle g_i, \quad \text{for all } f \in H \tag{2.4.1}$$

is called a **dual frame** for $\{f_i\}_{i=1}^N$. The set of vectors $\{S^{-1}f_i\}_{i=1}^N$ is a dual frame (will be shown later) for $\{f_i\}_{i=1}^N$, and is called **standard** or **canonical dual frame**. If $\{g_i\}_{i=1}^N$ is not a standard dual, it is called an **alternate dual frame**.

Proposition 2.4.1. Let $F = \{f_i\}_{i=1}^N$ be a frame. Then $\{S^{-1}f_i\}_{i=1}^N$ is a dual frame for F.

Proof. Recall that the frame operator S for a frame $\{f_i\}_{i=1}^N$ is given by

$$Sf = \sum_{i=1}^{N} \langle f, f_i \rangle f_i, \quad \text{for all } f \in H. \tag{2.4.2}$$

Since S is a positive and invertible operator, we can substitute S^{-1} for f in Equation (2.4.2), and we get the reconstruction formula

$$\begin{aligned} f = S(S^{-1}f) &= \sum_{i=1}^{N} \langle S^{-1}f, f_i \rangle f_i & (2.4.3)\\ &= \sum_{i=1}^{N} \langle f, S^{-1}f_i \rangle f_i. & (2.4.4) \end{aligned}$$

using the fact that S^{-1} is self-adjoint. Similarly, if we apply S^{-1} to both sides of Equation (2.4.2), we obtain the dual of reconstruction formula

$$f = S^{-1}(Sf) = S^{-1}\Big(\sum_{i=1}^{N}\langle f, f_i\rangle f_i\Big) \tag{2.4.5}$$

$$= \sum_{i=1}^{N}\langle f, f_i\rangle S^{-1}f_i. \tag{2.4.6}$$

Thus, by (2.4.4) and (2.4.6), we conclude that $\{S^{-1}f_i\}_{i=1}^N$ is a dual frame for F. $\square$

Remark 2.4.1. Standard dual of a tight frame F is $A^{-1}F$. Indeed, using the fact that $S = AI$, the inverse of frame operator is $A^{-1}I$; thus, $S^{-1}F = A^{-1}F$. In particular, the standard dual of a Parseval frame F is itself because $S = I$ in Parseval case.

Remark 2.4.2. Standard dual of the frame $\{S^{-1}f_i\}_{i=1}^N$ is $\{f_i\}_{i=1}^N$ because of the fact that the frame operator for the frame $\{S^{-1}f_i\}_{i=1}^N$ is S^{-1}.

Definition 2.4.2. Let $F = \{f_i\}_{i=1}^N$ and $G = \{g_i\}_{i=1}^N$ be sequences in a Hilbert space H, and let Θ_F and Θ_G be the corresponding analysis operators for F and G, respectively. Then, if $\Theta_F \perp \Theta_G$, F and G are called **orthogonal sequences**. If these sequences F and G are frames, they are called **orthogonal frames**.

Proposition 2.4.2. Let $F = \{f_i\}_{i=1}^N$ and $G = \{g_i\}_{i=1}^N$ be sequences in a Hilbert space H. Then F and G are orthogonal if and only if $\Theta_F^*\Theta_G = 0$, where Θ_F and Θ_G are the corresponding analysis operators for F and G, respectively.

Proof. Let F and G be sequences in H with analysis operators Θ_F and Θ_G, respectively. Then

$$\Theta_F^*\Theta_G = 0 \Leftrightarrow \langle \Theta_F f, \Theta_G g\rangle = \langle f, \Theta_F^*\Theta_G g\rangle = 0, \quad \text{for all } f, g \in H \tag{2.4.7}$$

$$\Leftrightarrow \Theta_F \perp \Theta_G. \tag{2.4.8}$$

□

Now we shall show the relationship between standard dual and alternative dual by giving the characterization of duals.

Proposition 2.4.3. Let $F = \{f_i\}_{i=1}^N$ be a frame with frame operator S. Then, $G = \{g_i\}_{i=1}^N$ is a dual frame of F if and only if there exists a sequence $H = \{h_i\}_{i=1}^N$ such that $\Theta_H^*\Theta_F = 0$ and $\{g_i\}_{i=1}^N = \{S^{-1}f_i + h_i\}_{i=1}^N$, where Θ_F and Θ_H are the corresponding analysis operators for F and H.

Proof. Assume that $G = \{g_i\}_{i=1}^N$ is a dual of $F = \{f_i\}_{i=1}^N$, and let $h_i = g_i - S^{-1}f_i$. Then

$$\sum_{i=1}^N \langle f, f_i\rangle h_i = \sum_{i=1}^N \langle f, f_i\rangle g_i - \sum_{i=1}^N \langle f, f_i\rangle S^{-1} f_i \tag{2.4.9}$$

$$= f - f = 0 \tag{2.4.10}$$

This implies that, for every $f, h \in H$

$$\Big\langle \sum_{i=1}^N \langle f, f_i\rangle h_i, h \Big\rangle = \sum_{i=1}^N \langle f, f_i\rangle \langle h_i, h\rangle = \langle \Theta_F f, \Theta_H h\rangle \tag{2.4.11}$$

$$= \langle \Theta_H^*\Theta_F f, h\rangle = 0. \tag{2.4.12}$$

Therefore, $\Theta_H^* \Theta_F = 0$.

Conversely, assume that there exist a sequence $\{h_i\}_{i=1}^N$ such that $\{g_i\}_{i=1}^N = \{S^{-1} f_i + h_i\}_{i=1}^N$ with $\Theta_H^* \Theta_F = 0$. Then, for all $f, h \in H$

$$\langle \Theta_H^* \Theta_F f, h \rangle = \langle \Theta_F f, \Theta_H h \rangle \tag{2.4.13}$$

$$= \sum_{i=1}^{N} \langle f, f_i \rangle \langle h_i, h \rangle = 0. \tag{2.4.14}$$

This implies that $\sum_{i=1}^N \langle f, f_i \rangle h_i = 0$ for all f in H. Thus,

$$\sum_{i=1}^{N} \langle f, f_i \rangle g_i = \sum_{i=1}^{N} \langle f, f_i \rangle S^{-1} f_i + \sum_{i=1}^{N} \langle f, f_i \rangle h_i \tag{2.4.15}$$

$$= f + 0 = f, \tag{2.4.16}$$

which implies that G is a dual of F.

$\square$

2.5 Traces of Frame Operators

2.5.1 Traces of Operators

Theorem 2.5.1. Let T be a linear operator on a Hilbert Space H, and n be the dimension of H. Assume that $k \geq n$ and $N \geq n$. If $\{v_i\}_{i=1}^k$ and $\{f_i\}_{i=1}^N$ are frames for H with corresponding dual frames $\{w_i\}_{i=1}^k$ and $\{g_i\}_{i=1}^N$, then

$$\sum_{i=1}^{k} \langle T v_i, w_i \rangle = \sum_{i=1}^{N} \langle T f_i, g_i \rangle. \tag{2.5.1}$$

Proof.

$$\sum_{i=1}^{k}\langle Tv_i, w_i\rangle = \sum_{i=1}^{k}\Big\langle \sum_{j=1}^{N}\langle Tv_i, g_j\rangle f_j, w_i\Big\rangle \tag{2.5.2}$$

$$= \sum_{i=1}^{k}\sum_{j=1}^{N}\langle Tv_i, g_j\rangle\langle f_j, w_i\rangle \tag{2.5.3}$$

$$= \sum_{j=1}^{N}\sum_{i=1}^{k}\langle f_j, w_i\rangle\langle Tv_i, g_j\rangle \tag{2.5.4}$$

$$= \sum_{j=1}^{N}\Big\langle \sum_{i=1}^{k}\langle f_j, w_i\rangle v_i, T^* g_j\Big\rangle \tag{2.5.5}$$

$$= \sum_{j=1}^{N}\langle f_j, T^* g_j\rangle \tag{2.5.6}$$

$$= \sum_{j=1}^{N}\langle Tf_j, g_j\rangle. \tag{2.5.7}$$

□

Corollary 2.5.1. Let T be a linear operator and $\{f_i\}_{i=1}^N$ be a frame for H with dual frame $\{g_i\}_{i=1}^N$. Then

$$tr(T) = \sum_{i=1}^{N}\langle Tf_i, g_i\rangle. \tag{2.5.8}$$

Proof. In Theorem 2.5.1, let $k = n$ and $\{v_i\}_{i=1}^n$ be the standard orthonormal basis; i.e., $\{e_i\}_{i=1}^n$. Since $tr(T) = \sum_{i=1}^{n}\langle Te_i, e_i\rangle$, the result follows from the Theorem. □

Corollary 2.5.2. Let $\{f_i\}_{i=1}^N$ be a frame of H with dual frame $\{g_i\}_{i=1}^N$. Then

$$n = \sum_{i=1}^{N}\langle f_i, g_i\rangle. \tag{2.5.9}$$

Proof. In Corollary 2.5.1, let T be an Identity operator I_n. Then, the result is immediate. $\square$

Remark 2.5.1. As a special case of the above Corollary, for Parseval frames $\{f_i\}_{i=1}^N$, we have

$$n = \sum_{i=1}^{N} \langle f_i, f_i \rangle = \sum_{i=1}^{N} ||f_i||^2, \tag{2.5.10}$$

that is, the dimension of Hilbert space H is the sum of the squares of the lengths of frame vectors.

2.5.2 Uniform Parseval Frames

Proposition 2.5.1. If $\{f_i\}_{i=1}^N$ is a uniform Parseval frame, then

$$||f_i|| = \sqrt{\frac{n}{N}} \quad \text{for all } i, \tag{2.5.11}$$

where n is the dimension of H.

Proof. Since the norm of vectors is uniform , for any $j \in \{1, \ldots, N\}$, we have

$$||f_j||^2 = \frac{1}{N} \sum_{i=1}^{N} ||f_i||^2 = \frac{n}{N}, \tag{2.5.12}$$

where the last equality follows from Remark 2.5.1. $\square$

Proposition 2.5.2. Let H be a Hilbert space with dimension n, and let $n < N$. Then, if one of the vectors f_j of a uniform Parseval frame $\{f_i\}_{i=1}^N$ for H is removed, then $\{f_i\}_{i \neq j}$ is a frame for H.

Proof. Assume that $\{f_i\}_{i=1}^N$ is a uniform Parseval frame. Then, by Cauchy-Schwarz inequality, for a nonzero $f \in H$, we have

$$||f||^2 = |\langle f, f_j \rangle|^2 + \sum_{\substack{i=1 \\ i \neq j}}^{N} |\langle f, f_i \rangle|^2 \tag{2.5.13}$$

$$\leq ||f||^2 ||f_j||^2 + \sum_{\substack{i=1 \\ i \neq j}}^{N} |\langle f, f_i \rangle|^2. \tag{2.5.14}$$

This implies that

$$||f||^2 \big(1 - ||f_j||^2\big) \leq \sum_{\substack{i=1 \\ i \neq j}}^{N} |\langle f, f_i \rangle|^2. \tag{2.5.15}$$

We need to have $1 - ||f_j||^2 > 0$ so that $\{f_i\}_{i \neq j}$ is a frame. Indeed, $1 - ||f_j||^2 > 0$ because $\{f_i\}_{i=1}^N$ is a uniform Parseval frame where $||f_j||^2 = \dfrac{n}{N}$ by Proposition 11, and $n < N$ by assumption.

□

2.6 Group Representation Frames

Special structured frames have significance in applications and in theory. One of these frames has a group structure, and is gained by applying a unitary group representations to a fixed vector in a Hibert space.

To define group representation frames, first we define unitary representations which are related with unitary operators.

Definition 2.6.1. Let H and K be two Hilbert spaces, and $T : H \to K$ be a linear operator. Then T is called a **unitary operator** if it is an isometry; i.e. $\|Tf\| = \|f\|$ for every $f \in H$, and is surjective.

Proposition 2.6.1. Let $T : H \to K$ be a linear operator. Then the following are equivalent:

(i) T is an unitary.

(ii) T preserves the inner product; i.e., $\langle Tx, Ty\rangle = \langle x, y\rangle$ for x, y in H, and T is surjective.

(iii) $T^* = T^{-1}$.

Let G and $\tilde{G}$ be two groups. a mapping from G to $\tilde{G}$ is called a group homomorphism if $\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$ and $\pi(g^{-1}) = (\pi(g))^{-1}$ for all g, g_1, g_2 in G.

Definition 2.6.2. Let G be a group. A group homomorphism π from G into the group of all the unitary operators on a Hilbert space H is called a **unitary representation**. This means that $\pi(g)$, $\pi(h)$ are unitary operators on H with $\pi(gh) = \pi(g)\pi(h)$ and $(\pi(g))^{-1} = \pi(g^{-1})$ for all g, h in G.

Proposition 2.6.2. For any unitary representation π, we have

(i) $\pi(e) = I$.

(ii) $\pi(g)^* = \pi(g^{-1})$ for all $g \in G$.

(iii) $\|\pi(g)\phi\| = \|\phi\|$ for all $g \in G$.

Proof. (i) $\pi(e) = \pi(gg^{-1}) = \pi(g)\pi(g^{-1}) = \pi(g)(\pi(g))^{-1} = I$, because of the fact that π is a homomorphism.

(ii) $\pi(g)^* = (\pi(g))^{-1} = \pi(g^{-1})$ because π is a unitary operator and group homomorphism.

(iii) Since π is a unitary operator, we have

$$\|\pi(g)\phi\|^2 = \langle \pi(g)\phi, \pi(g)\phi\rangle = \langle \phi, \phi(g)^*\pi(g)\phi\rangle = \langle \phi, \phi\rangle = \|\phi\|^2. \tag{2.6.1}$$

□

Definition 2.6.3. A unitary representation π of a group G on H is called a **frame representation** if there exist a vector $\phi \in H$ such that $\{\pi(g)\phi\}_{g\in G}$ is a frame for H. In this situation, it is said that $\{\pi(g)\phi\}_{g\in G}$ is a **group frame** and ϕ is a **frame vector** for π.

Note that any group frame is a uniform frame since $\|\pi(g)\phi\| = \|\phi\|$; in other words, every vector in the group frame has the same norm that is the norm of the frame vector ϕ.

The following proposition tells us that the canonical dual frame is a group frame.

Proposition 2.6.3. Let $\{\pi(g)\phi\}_{g\in G}$ be a group frame for H. Then the canonical dual of $\{\pi(g)\phi\}_{g\in G}$ is of the following form: $\{\pi(g)\xi\}_{g\in G}$ for some $\xi \in H$.

Proof. Let S be the frame operator for $\{\pi(g)\phi\}_{g\in G}$. Then by the definition of frame operator S, we have

$$S\pi(g)f = \sum_{g'\in G}\langle \pi(g)f, \pi(g')\phi\rangle\pi(g')\phi \tag{2.6.2}$$

$$= \pi(g)\pi(g)^{-1}\sum_{g'\in G}\langle f, \pi(g)^{-1}\pi(g')\phi\rangle\pi(g')\phi \tag{2.6.3}$$

$$= \pi(g)\sum_{g'\in G}\langle f, \pi(g^{-1}g')\phi\rangle\pi(g^{-1}g'\phi \tag{2.6.4}$$

$$= \pi(g)\sum_{g'\in G}\langle f, \pi(g'')\phi\rangle\pi(g'')\phi \tag{2.6.5}$$

$$= \pi(g)Sf \tag{2.6.6}$$

for all $f \in H$ and $g, g', g'' \in G$. Therefore, $S\pi(g) = \pi(g)S$ for all $g \in G$. This implies that

$S^{-1}\pi(g) = \pi(g)S^{-1}$ since

$$S\pi(g) = \pi(g)S \Leftrightarrow S\pi(g)S^{-1} = \pi(g)SS^{-1} \Leftrightarrow S\pi(g)S^{-1} = \pi(g) \tag{2.6.7}$$

$$\Leftrightarrow S^{-1}S\pi(g)S^{-1} = S^{-1}\pi(g) \Leftrightarrow \pi(g)S^{-1} = S^{-1}\pi(g). \tag{2.6.8}$$

Let $\xi = S^{-1}\phi$. Then for every $g \in G$, we have $S^{-1}\pi(g)\phi = \pi(g)\xi$. Thus, the canonical dual of $\{\pi(g)\phi\}_{g\in G}$ is a group frame of the form $\{s^{-1}\pi(g)\phi\}_{g\in G} = \{\pi(g)\xi\}_{g\in G}$. $\square$

CHAPTER 3: ERASURES

3.1 The Erasure Problem

The property of frames that the number of vectors, N, greater than or equal to the dimension, n, of the Hilbert space has a great significance in applications. For instance, in coding theory, the information of a vector f is transmitted, or encoded, by the analysis operator Θf, that is, $\Theta f = \{\langle f, f_i\rangle\}_{i=1}^N$, where $\{f\}_{i=1}^N$ is a frame for a Hilbert space H. On the other side, the receiver reconstructs, or decodes, the vector f, by the help of synthesis operator, $\Theta^*\Theta f$. If there is no erasure, the receiver is able to reconstruct f completely. If there is loss of data or any erasure, however, the receiver still may be able to reconstruct f perfectly with the help of redundancy property of frames, which is the quantity $\frac{N}{n}$.

To deal with the erasures, maximum errors for erasures are to be minimized. To minimize the maximal errors for erasures, two approaches are provided in [35] and [44]. One approach provided by Holmes and Paulsen in [35] is to select an optimal frame for erasures. On the other hand, second approach provided by Lopez and Han in [44] is to select optimal dual frames for erasures for a given frame. Second approach is motivated mainly, because of the limitations on optimal frames, to give more freedom to frames that are to be used in coding. To find optimal frame means to find a best frame that minimizes the error on reconstructed vectors; however, to find an optimal dual frame for a given frame is to find a best dual frame that minimizes the error on reconstructed vectors.

To make the notion of optimal frames and optimal dual frames precise, let us first define the error operator E_Λ for erasures. Let D be an $N \times N$ diagonal matrix with m ones and $n - m$ zeros, and $\mathcal{D}_m$ be the set of all such diagonal matrices, D. For any frame pairs $F = \{f_i\}_{i=1}^N$ and $G = \{g_i\}_{i=1}^N$, where G is the dual frame of F, and Θ_F and Θ_G are the respective analysis operators for F and G, the error operator for m-erasure where $\Lambda = \{i_1, \ldots, i_m\}$ is defined by

$$E_\Lambda(f) = f - \sum_{i \notin \Lambda} \langle f, f_i \rangle g_i = \sum_{i \in \Lambda} \langle f, f_i \rangle g_i = \Theta_G^* D \Theta_F f, \tag{3.1.1}$$

and the maximum error when m-erasures occur is defined by

$$\max\{||\Theta_G^* D \Theta_F|| \, : \, D \in \mathcal{D}_m\}, \tag{3.1.2}$$

where $|| \cdot ||$ is a measurement for the error operator (it could be the usual matrix norm, Hilbert-Schmidt norm or some other measurement). The goal is either to characterize the dual frame G that minimizes the maximum error if a frame F is preselected, or to characterize Parseval frames F such that $\max\{||\Theta_G^* D \Theta_F|| \, : \, D \in \mathcal{D}_m\}$ is minimal among all the Parseval frames. The similar setup can be used for other applications (e.g. optimal for sparsity, noise control). In the following sections, we will give precise definitions for optimal frames and optimal dual frames, and give some results.

3.2 Optimal Frames for Erasures

From now on, for a frame $F = \{f_i\}_{i=1}^N$ for a Hilbert space H of dimension n, we will call F an (N, n) frame, and we will let $\|\cdot\|$ be a matrix norm. Throughout this section, we let F be a Parseval frame.

A Parseval frame F' is called optimal frame for 1-erasure if it satisfies

$$\delta^1_{F'} = \min_F \max\{||\Theta^*_F D\Theta_F|| \,:\, D \in \mathcal{D}_1\}, \tag{3.2.1}$$

and a Parseval frame F' is called optimal frame for any m-erasure if it is optimal for $(m-1)-$erasure and

$$\delta^m_{F'} = \min_F \max\{||\Theta^*_F D\Theta_F|| \,:\, D \in \mathcal{D}_m\}. \tag{3.2.2}$$

In other words, a Parseval frame that is optimal for $m-$erasures is optimal for m or less erasures.

One erasure optimal Parseval frames are characterized in [35].

Proposition 3.2.1. An (N, n) Parseval frame is 1-erasure optimal if and only if it is uniform. Moreover, minimum error, $\delta^1_{F'}$, is n/N.

Proof. Let $F = \{f_i\}_{i=1}^N$ be an (N, n) Parseval frame. Note that for one erasure case, we have that

$$\|\Theta^*_F D\Theta_F\| = \|\langle f_i, f_i\rangle\| = \|f_i\|^2, \tag{3.2.3}$$

where $D \in \mathcal{D}_1$. Therefore,

$$\max\{||\Theta_F^* D \Theta_F|| \,:\, D \in \mathcal{D}_1\} = \max\{\|f_i\|^2 \,:\, i = 1, \dots, N\}. \tag{3.2.4}$$

Moreover, we note that

$$tr(\Theta\Theta^*) = \sum_{i=1}^{N} \langle f_i, f_i \rangle = \sum_{i=1}^{N} \|f_i\|^2 = n, \tag{3.2.5}$$

which implies that $\|f_i\|^2 \geq n/N$ for some i. Thus,

$$\delta_{F'}^1 = \min_F \max\{\|f_i\|^2 \,:\, i = 1, \dots, N\} = n/N. \tag{3.2.6}$$

This means that if F is 1-erasure optimal Parseval frame, then by 3.2.5 we have $\|f_i\|^2 = n/N$ for all $i = 1, \dots, N$, i.e., F is a uniform frame. By the same arguments, if F is a uniform frame, then it is 1-erasure optimal Parseval frame. $\square$

Definition 3.2.1. If F is an (N, n) uniform Parseval frame and $\|\Theta^* D \Theta\|$ is a constant for all D where D is a diagonal matrix with 2 ones and $N - 2$ zeros on the diagonal, and Θ^* and Θ are synthesis operator and analysis operator of F, respectively, then F is called **2-uniform Parseval frame**.

The following Theorem provides an alternative definition for a 2-uniform Parseval frame that is given in [35].

Theorem 3.2.1. Assume that F is a uniform (N, n) Parseval frame. Then, F is 2-uniform if and only if $|\langle f_i, f_j\rangle| = c$ is constant for all $i \neq j$ where

$$c = \sqrt{\frac{n(N-n)}{N^2(N-1)}}. \tag{3.2.7}$$

Proof. Let F be a uniform Parseval frame, then $\|f_i\|^2 = \frac{n}{N}$. Assume that ith and jth coefficients are erased. And let D be a diagonal matrix with 2 ones and $N-2$ zeros on the diagonal. Because of the fact that $D^2 = D = D^*$, we have

$$\|\Theta^* D\Theta\| = \|\Theta^* D^* D\Theta\| = \|D\Theta(D\Theta)^*\| = \|D\Theta\Theta^* D\| \tag{3.2.8}$$

$$= \left\| \begin{bmatrix} \langle f_i, f_i\rangle & \langle f_j, f_i\rangle \\ \langle f_i, f_j\rangle & \langle f_j, f_j\rangle \end{bmatrix} \right\| = \left\| \begin{bmatrix} n/N & \langle f_j, f_i\rangle \\ \langle f_i, f_j\rangle & n/N \end{bmatrix} \right\|. \tag{3.2.9}$$

The norm of the matrix above is the spectral radius of the matrix because it is positive definite. Then, the spectral radius of the matrix is

$$\max\left\{ \Big| n/N + |\langle f_i, f_j\rangle| \Big|, \Big| n/N - |\langle f_i, f_j\rangle| \Big| \right\} = n/N + |\langle f_i, f_j\rangle| \tag{3.2.10}$$

Therefore, F is a 2-uniform Parseval frame if and only if $|\langle f_i, f_j\rangle| = c$ is a constant for all $i \neq j$.

To find the exact value of c, note the followings

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\langle f_i, f_j\rangle\langle f_j, f_i\rangle = \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\langle f_i, f_j\rangle\langle f_j, f_i\rangle + \sum_{i=1}^{N}\langle f_i, f_i\rangle\langle f_i, f_i\rangle \tag{3.2.11}$$

$$= (N^2 - N)c + \sum_{i=1}^{N}|\langle f_i, f_i\rangle|^2 \tag{3.2.12}$$

$$= (N^2 - N)c + N\frac{n^2}{N^2}, \tag{3.2.13}$$

and

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\langle f_i, f_j\rangle\langle f_j, f_i\rangle = \sum_{i=1}^{N}\langle f_i, f_i\rangle = n. \tag{3.2.14}$$

From (3.2.13) and (3.2.14), it follows that

$$c = \sqrt{\frac{n(N-n)}{N^2(N-1)}}. \tag{3.2.15}$$

□

The proof of the theorem implies that 2-uniform Parseval frames are 2-erasure optimal Parseval frames.

For a 2-uniform (N, n) Parseval frame, $\Theta\Theta^*$ can be written in the following way:

$$\Theta\Theta^* = \begin{bmatrix} \langle f_1, f_1\rangle & \langle f_2, f_1\rangle & \dots & \langle f_N, f_1\rangle \\ \langle f_1, f_2\rangle & \langle f_2, f_2\rangle & \dots & \langle f_N, f_2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle f_1, f_N\rangle & \langle f_2, f_N\rangle & \dots & \langle f_N, f_N\rangle \end{bmatrix} = \begin{bmatrix} n/N & \pm c & \dots & \pm c \\ \pm c & n/N & \dots & \pm c \\ \vdots & \vdots & \ddots & \vdots \\ \pm c & \pm c & \dots & n/N \end{bmatrix}. \tag{3.2.16}$$

In other words, $\Theta\Theta^* = \frac{n}{N}I + cQ$ where $Q = (q_{ij})$ is a self adjoint matrix with $q_{ii} = 0$ for all i and $|q_{ij}| = 1$ for all $i \neq j$.

Definition 3.2.2. Let F be a 2-uniform (N, n) Parseval frame. Then, the $(N \times N)$ matrix Q derived above is called **signature matrix** of F.

In [35], the characterization of 2-uniform Parseval frames is given in the following way:

Proposition 3.2.2. Let Q be a signature matrix of a 2-uniform (N, n) Parseval frame F. Then

$$Q^2 = (N-1)I + \mu Q, \tag{3.2.17}$$

where

$$\mu = (N-2n)\sqrt{\frac{N-1}{n(N-n)}}. \tag{3.2.18}$$

Conversely, let Q be a signature matrix of the form

$$Q^2 = (N-1)I + \mu Q, \quad \mu^2 \neq -4(N-1). \tag{3.2.19}$$

Then, Q is a signature matrix of a 2-uniform (N, n) Parseval frame with $n = \dfrac{N}{2} - \dfrac{\mu N}{2\sqrt{4(N-1)+\mu^2}}$ and $\Theta\Theta^* = \frac{n}{N}I + cQ$.

Proof. Let F be a uniform (N, n) Parseval frame with analysis Θ and synthesis Θ^* operators. Suppose Q is the signature matrix of F. Note first that $\left(\Theta\Theta^*\right)^2 = \Theta\Theta^*\Theta\Theta^* = \Theta\Theta^*$, since $\Theta^*\Theta = I_N$. Using the definition of signature matrix and the note above, we have

$$\left(\Theta\Theta^*\right)^2 = \left(\frac{n}{N}I + cQ\right)^2 = \frac{n^2}{N^2}I + 2c\frac{n}{N}Q + c^2Q^2 = \frac{n}{N}I + cQ. \tag{3.2.20}$$

This implies that

$$Q^2 = (N-1)I + (N-2n)Q\sqrt{\frac{N-1}{n(N-n)}}. \tag{3.2.21}$$

For the second part of the proposition, let P be a matrix of the form $P = aI + cQ$ where

$$a = \frac{1}{2} - \frac{\mu}{2\sqrt{4(N-1)+\mu^2}} \quad \text{and} \quad c^2 = \frac{a-a^2}{N-1} = \frac{1}{4(N-1)+\mu^2} \tag{3.2.22}$$

Note that P is self-adjoint matrix because Q is so. For Q to be a signature matrix of a frame F, it is sufficient that P satisfy $P^2 = P$ so that P can be factored as $\Theta\Theta^*$ and we obtain a frame. In fact,

$$P^2 = (aI + cQ)^2 = a^2I + 2acQ + c^2Q^2 \tag{3.2.23}$$

$$= a^2I + 2acQ + c^2\big((N-1)I + \mu Q\big) \tag{3.2.24}$$

$$= (a^2 + Nc^2 - c^2)I + (2ac + \mu c^2)Q \tag{3.2.25}$$

$$= aI + cQ = P. \tag{3.2.26}$$

where the last equality follows from

$$a^2 + Nc^2 - c^2 = a^2 + c^2(N-1) = a^2 + \frac{a - a^2}{N-1}(N-1) = a \tag{3.2.27}$$

and

$$2ac + \mu c^2 = c(2a + \mu c) = c\left(1 - \frac{\mu}{\sqrt{4(N-1)+\mu^2}} + \mu\frac{1}{\sqrt{4(N-1)+\mu^2}}\right) = c \tag{3.2.28}$$

by 3.2.22. □

3.3 Optimal Dual Frames

Given a frame F, we search for a dual frame G of F, which makes the error of erasures minimum. Now, in the following subsections, we will look at the optimal dual frames with respect to matrix norm measurement and spectral radius measurement.

3.3.1 Optimality with respect to Matrix Norm Measurement

Let a frame F be given. Then a dual frame G' for F is called optimal dual frame of F for 1-erasure if

$$\delta^{(1)}_{F,G'} = \min_{G} \max\{||\Theta_G^* D \Theta_F|| \, : \, D \in \mathcal{D}_m\}, \tag{3.3.1}$$

and a dual frame G' for F is called optimal dual frame of F for any m-erasure if it is optimal for $(m-1)-$erasure and

$$\delta_{F,G'}^{(m)} = \min_{G} \max\{||\Theta_G^* D\Theta_F|| \,:\, D \in \mathcal{D}_m\}. \tag{3.3.2}$$

In [44], the condition in which the standard dual of a frame is the unique optimal dual frame for $m-$erasures is given.

Theorem 3.3.1. Let $F = \{f_i\}_{i=1}^N$ be an (N, n) frame for a Hilbert space H. If $\|S^{-1}f_i\| \cdot \|f_i\|$ is constant for all i, then the standard dual is the unique optimal dual frame for $m-$erasure.

In particular, the standard dual of a uniform tight frame is the optimal dual frame for $m-$erasures. In fact, because the frame operator S of a tight frame is of the form $S = AI$, where A is the frame bound, $\|S^{-1}f_i\| = \frac{1}{A}\|f_i\|$ for all i. Using the uniformness of the frame, we obtain the conditions of the Theorem.

The necessary and sufficient condition for the standard dual of a frame to be the 1-erasure optimal dual frame is given in [41]. Let F be an (N, n) frame and $c = \max\big\{\|S^{-1}f_i\| \cdot \|f_i\| \,:\, i \in \{1, \ldots, N\}\big\}$. Define $H_i = \text{span}\{f_i \,:\, i \in \Lambda_j\}$ for $j = 1, 2$, where $\Lambda_1 = \{i \,:\, |S^{-1}f_i\| \cdot \|f_i\| = c\}$, and $\Lambda_2 = \{1, \ldots, N\}\backslash\Lambda_1$.

Theorem 3.3.2. The standard dual is the unique 1-erasure optimal dual if and only if $H_1 \cap H_2 = \{0\}$ and $\{f_i\}_{i\in\Lambda_2}$ is linearly independent set.

Proposition 3.3.1. For an (N, n) Parseval frame $F = \{f_i\}_{i=1}^N$ for H, the standard dual is the unique optimal dual frame for $m-$erasure if and only if $\|f_i\|$ is constant for all i.

3.3.2 Optimality with respect to Spectral Radius Measurement

Most of the research so far (c.f [41, 35, 44]) have focused on measuring the error of the reconstructed vector by operator norm. For example, it is known that a Parseval frame is one-erasure optimal if and only if it is uniform, and it is 2-erasure optimal if it is equiangular (c.f. [35]). For the case when a frame F is preselected, optimal dual problems for erasures were studied for example in [41, 42, 44], optimal dual frame for sparsity was investigated in [38], and some other optimality was also studied for different purposes (c.f. [16, 17, 30, 40, 45, 46]).

Now consider the case when iterations are applied in the reconstruction process: Let $F = \{f_i\}_{i=1}^N$ be a frame and $G = \{g_i\}_{i=1}^N$ be a dual frame of F in a Hilbert space H with dimension n. For any $f \in H$, we have

$$f = \sum_{i=1}^{N} \langle f, g_i \rangle f_i = \sum_{i=1}^{N} a_i f_i, \tag{3.3.3}$$

where $\langle f, g_i \rangle = a_i$. Let $\Lambda = \{i \,:\, a_i \text{ is lost or erased}\}$ and $\Lambda^c = \{1, \ldots, N\} \backslash \Lambda$. Now, we can rewrite the reconstruction formula for f in the following way;

$$f = \sum_{i \in \Lambda} \langle f, g_i \rangle f_i + \sum_{i \in \Lambda^c} \langle f, g_i \rangle f_i, \tag{3.3.4}$$

or equivalently,

$$f = E_\Lambda f + R_\Lambda f, \tag{3.3.5}$$

where $E_\Lambda f = \sum_{i\in\Lambda} \langle f, g_i\rangle f_i$ and $R_\Lambda f = \sum_{i\in\Lambda^c} \langle f, g_i\rangle f_i$. Note that $E_\Lambda + R_\Lambda = I$. This implies that the receiver knows both operators E_Λ and R_Λ. The first step approximation of f is given by $f^{(1)} = R_\Lambda f$. However, we can achieve higher approximation accuracy by employing the following iterations:

$$
\begin{aligned}
f^{(1)} &= R_\Lambda f && (3.3.6)\\
f^{(2)} &= E_\Lambda f^{(1)} + R_\Lambda f && (3.3.7)\\
f^{(3)} &= E_\Lambda f^{(2)} + R_\Lambda f && (3.3.8)\\
\vdots \quad & \quad \vdots && (3.3.9)\\
f^{(n)} &= E_\Lambda f^{(n-1)} + R_\Lambda f. && (3.3.10)
\end{aligned}
$$

Then, the error of the reconstruction is

$$
\begin{aligned}
f - f^{(n)} &= E_\Lambda f - E_\Lambda f^{(n-1)} = E_\Lambda (f - f^{(n-1)}) && (3.3.11)\\
&= E_\Lambda (E_\Lambda f - E_\Lambda f^{(n-2)}) = E_\Lambda^2 (f - f^{(n-2)}) && (3.3.12)\\
&= E_\Lambda^{n-1}(f^{(1)} - f) = E_\Lambda^n f. && (3.3.13)
\end{aligned}
$$

Thus, we have

$$
\|f - f^{(n)}\| = \|E_\Lambda^n f\| \le \|E_\Lambda^n\| \|f\| \ . \tag{3.3.14}
$$

To measure the error, we need to look at the norm of E_Λ^n, $\|E_\Lambda^n\|$. It can be estimated by the spectral radius of E_Λ. Recall that

$$r(E_\Lambda) \leq \|E_\Lambda\|. \tag{3.3.15}$$

In the case that E_Λ is positive or normal, $E_\Lambda^* E_\Lambda = E_\Lambda E_\Lambda^*$, we have $\|E_\Lambda\| = r(E_\Lambda)$, thus,

$$\|E_\Lambda\| = \max_i |\lambda_i|, \tag{3.3.16}$$

where λ_i is an eigenvalue of E_Λ. But, it could happen that $r(E_\Lambda) << \|E_\Lambda\|$. In this case,

$$\lim_{n\to\infty} \|E_\Lambda^n\|^{1/n} = r(E_\Lambda), \tag{3.3.17}$$

where $r(E_\Lambda)$ is the spectral radius of E_Λ. Therefore, the spectral radius $r(E_\Lambda)$ of E_Λ satisfies

$$r(E_\Lambda) = \max\left\{|\lambda| \ : \ \lambda \in \sigma(E_\Lambda)\right\} = \lim_{k\to\infty} \|E_\Lambda^n\|^{1/n}. \tag{3.3.18}$$

Definition 3.3.1. Let F be a frame and G be a dual frame of F. For each k, let

$$r_{F,G}^{(k)} = \max\{r(E_\Lambda) \ : \ |\Lambda| = k\} \tag{3.3.19}$$

and

$$r_F^{(k)} = \min\{r_{F,G}^{(k)} : G \text{ is a dual frame of } F\}, \tag{3.3.20}$$

where $|\Lambda|$ denotes the cardinality of Λ. A dual frame G of F is called 1**-erasure spectrally optimal** if $r_{F,G}^{(1)} = r_F^{(1)}$. We say that G is k**-erasure spectrally optimal** if it is $(k-1)$-erasure spectrally optimal and $r^{(k)}_{F,G} = r^{(k)}_F$

Clearly we have $r^{(k)}_{F,G} \leq \delta^{(k)}_{F,G}$. In the iterated reconstruction introduced in this section, the reconstruction error of a signal f is dominated by $\|E^n_\Lambda\| \cdot ||f||$. Therefore in order to completely recover f as $n \to \infty$, we need the necessary condition that $r^{(k)}_{F,G} < 1$ (or a more stronger $\delta^{(k)}_{F,G} < 1$). In this section we present two conditions to ensure this inequality. The first one is a necessary and sufficient condition on the frame F such that this happens for one of the dual frames G. The second one is a necessary and sufficient condition on the triple (N, n, k) such that there exists a dual frame pair (F, G) for H with the property that $r^{(k)}_{F,G} < 1$. Both results involve the standard dual frames. So at the end of this section we give a sufficient condition under which the standard dual frame is 1-erasure spectrally optimal.

Proposition 3.3.1. Let $F = \{f_i\}_{i=1}^N$ be a frame for a Hilbert space H of dimension n. Assume that k represents the number of erased coefficients in the frame expansion, and S is the frame operator of F. Then the following are equivalent:

(i) Every $(N-k)$ vectors span the Hilbert space H,

(ii) $\delta^{(k)}_{S^{-1/2}F, S^{-1/2}F} < 1$,

(iii) $r^{(k)}_{F, S^{-1}F} < 1$,

(iv) There exists a dual frame G of F such that $r^{(k)}_{F,G} < 1$.

Proof. Let Λ and Λ^c be the set of indices associated with erased coefficients and received coefficients respectively with $|\Lambda| = k$ and $|\Lambda^c| = N - k$.

"$(iii) \Rightarrow (iv)$" is obvious.

"$(i) \Rightarrow (ii)$:" Let $E_\Lambda = \Theta^*_{S^{-1/2}F} D_\Lambda \Theta_{S^{-1/2}F}$ and $R_{\Lambda^c} = \Theta^*_{S^{-1/2}F} D_{\Lambda^c} \Theta_{S^{-1/2}F}$. Assume that

$\delta^{(k)}_{S^{-1/2}F,S^{-1/2}F} = 1$. Then, there exists a Λ_j with $|\Lambda_j| = k$ such that $||E_{\Lambda_j}|| = 1$. Because E_{Λ_j} is a positive definite matrix, $||E_{\Lambda_j}|| = r(E_{\Lambda_j})$. This implies that 1 is in the spectrum of E_{Λ_j}. Thus, there exists $0 \neq f \in H$ such that $E_{\Lambda_j} f = f$. Since $E_{\Lambda_j} + R_{\Lambda_j^c} = I$, we get that $R_{\Lambda_j^c} f = 0$. Thus,

$$
\begin{aligned}
0 &= \left\langle f, R_{\Lambda_j^c} f \right\rangle && (3.3.21)\\
&= \left\langle f, \sum_{i \in \Lambda_j^c} \left\langle f, S^{-1/2} f_i \right\rangle S^{-1/2} f_i \right\rangle && (3.3.22)\\
&= \sum_{i \in \Lambda_j^c} \overline{\left\langle f, S^{-1/2} f_i \right\rangle} \left\langle f, S^{-1/2} f_i \right\rangle && (3.3.23)\\
&= \sum_{i \in \Lambda_j^c} \left| \left\langle f, S^{-1/2} f_i \right\rangle \right|^2. && (3.3.24)
\end{aligned}
$$

This implies that

$$f \perp S^{-1/2} f_i \text{ for all } i \in \Lambda_j^c. \qquad (3.3.25)$$

Therefore, we can find $(N - k)$ vectors $\{f_i \,:\, i \in \Lambda_j^c\}$ that do not span H. Hence
$\delta^{(k)}_{S^{-1/2}F,S^{-1/2}F} < 1$.

"$(ii) \Rightarrow (iii)$:" Assume that $\delta^{(k)}_{S^{-1/2}F,S^{-1/2}F} < 1$. Then

$$\begin{aligned}
\sigma(B_\Lambda) &= \sigma(\Theta^*_{S^{-1}F} D_\Lambda \Theta_F) &(3.3.26)\\
&= \sigma(S^{-1/2}\Theta^*_{S^{-1/2}F} D_\Lambda \Theta_F) &(3.3.27)\\
&= \sigma(\Theta^*_{S^{-1/2}F} D_\Lambda \Theta_F S^{-1/2}) &(3.3.28)\\
&= \sigma(\Theta^*_{S^{-1/2}F} D_\Lambda \Theta_{S^{-1/2}F}) &(3.3.29)\\
&= \sigma(A_\Lambda). &(3.3.30)
\end{aligned}$$

Because A_Λ is a positive definite matrix, we have $||A_\Lambda|| = r(A_\Lambda)$. From (3.3.30) we have,

$$r(B_\Lambda) = ||A_\Lambda|| < 1. \tag{3.3.31}$$

So, $r^{(k)}_{F,S^{-1}F} < 1$.

"$(iv) \Rightarrow (i)$:" Assume that there exist $(N-k)$ vectors $\{f_i : i \in \Lambda^c\}$ that do not span H. Then there exist $0 \neq f$ in H such that

$$f \perp f_i \text{ for all } i \in \Lambda^c, \tag{3.3.32}$$

which implies that $R_{\Lambda^c} f = 0$ where $R_{\Lambda^c} = \Theta^*_G D_{\Lambda^c} \Theta_F$. Since $R_\Lambda + E_\Lambda = I$, for $E_\Lambda = \Theta^*_G D_\Lambda \Theta_F$, we have $f = E_\Lambda f$. Thus, $1 \in \sigma(E_\Lambda)$, which implies that $r^{(k)}_{F,G} \geq 1$. $\square$

Proposition 3.3.2. Let n be the dimension of H. Then the following are equivalent:

(i) $N - k \geq n$,

(ii) There exists a frame F such that $\delta^{(k)}_{S^{-1/2}F,S^{-1/2}F} < 1$,

(iii) There exists a frame F such that $r^{(k)}_{F,S^{-1}F} < 1$,

(iv) There exists dual pair (F, G) such that $r^{(k)}_{F,G} < 1$.

Proof. The equivalence of (ii), (iii) and (iv) has been established in the proof of Proposition 3.3.1. Hence, it is enough to show that $(i) \Rightarrow (ii)$ and $(iv) \Rightarrow (i)$.

"$(i) \Rightarrow (ii)$:" Assume that $N - k \geq n$. Let $x_1, x_2, \ldots, x_N$ be distinct nonzero real numbers. Construct an $n \times N$ matrix A in the following way:

$$A = \begin{pmatrix} 1 & 1 & \cdots & \cdots & 1 \\ x_1^1 & x_2^1 & \cdots & \cdots & x_N^1 \\ x_1^2 & x_2^2 & \cdots & \cdots & x_N^2 \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & \cdots & x_N^{n-1} \end{pmatrix} \tag{3.3.33}$$

Any $n \times n$ matrix, say V, consisting of any n columns of A, is a Vandermonde matrix. Thus,

$$|V| = \prod_{\substack{i,j=i_1,i_2,\ldots,i_n \\ i>j}} (x_i - x_j) \neq 0. \tag{3.3.34}$$

This means that every n column vectors of A are linearly independent, i.e., every n column vectors of A span H. Let F be the frame consisting of all the columns of A. Then every $N - k$ vectors from F span H. Therefore, by Proposition 3.3.1, we get $\delta^{(k)}_{S^{-1/2}F,S^{-1/2}F} < 1$.

"$(iv) \Rightarrow (i)$" : Let (F, G) be a dual frame pair with the property that $r^{(k)}_{F,G} < 1$. Then, by

Proposition $3.3.1$, every $N - k$ vectors in F span H. Thus, $N - k \geq n$. $\square$

CHAPTER 4: SPECTRALLY OPTIMAL STANDARD DUAL FRAMES

In this chapter, we provide some conditions on frames so that their standard dual frames are one-erasure or two-erasure spectrally optimal dual frames.

4.1 1-Erasure Spectrally Optimal Standard Dual Frames

The following Theorem provides a large class of frames for which the standard dual is 1-erasure spectrally optimal.

Theorem 4.1.1. Let $F = \{f_i\}_{i=1}^N$ be a frame with the frame operator S in a Hilbert space of dimension n. Then, if $\|S^{-1/2}f_i\|^2 = \frac{n}{N}$, standard dual $S^{-1}F$ of F is one erasure spectrally optimal dual.

Proof. Recall that for a frame F and any dual G of F we have

$$\sum_{i=1}^{N} \langle f_i, g_i \rangle = n. \tag{4.1.1}$$

Then,

$$
\begin{aligned}
|\langle f_i, g_i\rangle| &= |\langle f_i, S^{-1}f_i + h_i\rangle| && (4.1.2)\\
&= |\langle f_i, S^{-1}f_i\rangle + \langle f_i, h_i\rangle| && (4.1.3)\\
&= |\langle S^{-1/2}f_i, S^{-1/2}f_i\rangle + \langle f_i, h_i\rangle| && (4.1.4)\\
&= |\|S^{-1/2}\|^2 + \langle f_i, h_i\rangle| && (4.1.5)\\
&= \left|\frac{n}{N} + \langle f_i, h_i\rangle\right| && (4.1.6)
\end{aligned}
$$

From Equation (4.1.1), we have

$$\max_{i\in\{1,\ldots,N\}} \left\{|\langle f_i, g_i\rangle|\right\} \geq \frac{n}{N}. \tag{4.1.7}$$

The minimum of (4.1.7) among all dual frames G of F is

$$\min_{G} \max_{i\in\{1,\ldots,N\}} \left\{|\langle f_i, g_i\rangle|\right\} = \frac{n}{N}. \tag{4.1.8}$$

This together with (4.1.6) implies that $\langle f_i, h_i\rangle = 0$ for all i, i.e., $|\langle f_i, g_i\rangle| = n/N$ for all i. If not there would be a j such that $\langle f_i, h_i\rangle < 0$, i.e., $|\langle f_i, g_i\rangle| < n/N$ which contradicts with the fact in 4.1.1. Thus, $S^{-1}F$ is an one erasure spectrally optimal dual frame.

□

We have derived a sufficient condition for standard dual to be 1-erasure spectrally optimal in the following proposition; however, we are still missing a necessary condition.

Let $F = \{f_i\}_{i=1}^N$ be an (N, n)-frame, and $c = \max\{\|S^{-1/2}f_i\| : i = 1, \ldots, N\}$. Set $\Lambda_1 = \{i :$

$\|S^{-1/2}f_i\| = c\}$ and $\Lambda_2 = \{1, \ldots, N\} \backslash \Lambda_1$. Let $H_j =$span$\{x_i : i \in \Lambda_j\}$ for $j = 1, 2$.

Proposition 4.1.1. Let $F = \{f_i\}_{i=1}^N$ be a (N, n) frame for H. If $H_1 \cap H_2 = \{0\}$, then $S^{-1}F$ is 1-erasure spectrally optimal dual frame.

Proof. Assume that $G = \{g_i\}_{i=1}^N$ is a 1-erasure spectrally optimal dual of F. Write $g_i = S^{-1}f_i + h_i$. Then

$$\sum_{i=1}^{N} \langle f, h_i \rangle f_i = 0. \tag{4.1.9}$$

Hence, we have

$$\sum_{i \in \Lambda_1} \langle f, h_i \rangle f_i + \sum_{i \in \Lambda_2} \langle f, h_i \rangle f_i = 0. \tag{4.1.10}$$

By assumption, this implies that

$$\sum_{i \in \Lambda_1} \langle f, h_i \rangle f_i = 0 \quad \text{and} \quad \sum_{i \in \Lambda_2} \langle f, h_i \rangle f_i = 0. \tag{4.1.11}$$

Then, we have

$$\sum_{i \in \Lambda_1} \langle h_i, f_i \rangle = 0 \quad \text{and} \quad \sum_{i \in \Lambda_2} \langle h_i, f_i \rangle = 0. \tag{4.1.12}$$

Note that

$$|\langle S^{-1}f_i + h_i, f_i \rangle| = |\langle S^{-1}f_i, f_i \rangle + \langle h_i, f_i \rangle| = |\|S^{-1/2}f_i\|^2 + \langle h_i, f_i \rangle| \tag{4.1.13}$$

Because G is a 1-erasure spectrally optimal dual frame we have

$$\text{For } i \in \Lambda_1 \, : \, |\langle S^{-1}f_i + h_i, f_i\rangle| = |c^2 + \langle h_i, f_i\rangle| \leq |\langle S^{-1}f_i, f_i\rangle| = c^2 \tag{4.1.14}$$

$$\text{For } i \in \Lambda_2 \, : \, |\langle S^{-1}f_i + h_i, f_i\rangle| = |\|S^{-1/2}f_i\|^2 + \langle h_i, f_i\rangle| < c^2 \tag{4.1.15}$$

This implies that $|c^2 + Re\langle h_i, f_i\rangle| \leq c^2$ and $|c^2 + Im\langle h_i, f_i\rangle| \leq c^2$ for $i \in \Lambda_1$. Thus $Re\langle h_i, f_i\rangle \leq 0$ and $Im\langle h_i, f_i\rangle \leq 0$ for all $i \in \Lambda_1$. We claim that $Re\langle h_i, f_i\rangle = 0$ and $Im\langle h_i, f_i\rangle = 0$ for all $i \in \Lambda_1$. In fact, if $Re\langle h_i, f_i\rangle < 0$ for some $i \in \Lambda_1$, then there must be some $j \in \Lambda_1$ such that $Re\langle h_j, f_j\rangle > 0$ because $\sum_{i\in\Lambda_1}\langle h_i, f_i\rangle = 0$ ($\sum_{i\in\Lambda_1} Re\langle h_i, f_i\rangle = 0$). Thus, $Re\langle h_i, f_i\rangle = 0$ for all $i \in \Lambda_1$. Similarly, we show that $Im\langle h_i, f_i\rangle = 0$ for all $i \in \Lambda_1$. Thus, $\langle h_i, f_i\rangle = 0$ for all $i \in \Lambda_1$. Moreover, $\|S^{-1/2}f_i\| < c$ for all $i \in \Lambda_2$. Therefore, $r^{(1)}_{F,S^{-1}F} = c \leq r^{(1)}_{F,G} = r^{(1)}_F$, and consequently $r^{(1)}_{F,S^{-1}F} = r^{(1)}_F$. Thus $S^{-1}F$ is a 1-erasure spectrally optimal dual frame. □

Proposition 4.1.2. If $S^{-1}F$ is 1-erasure spectrally optimal, then there exist f_i ($i \in \Lambda_1$) such that $f_i \notin H_2$.

Proof. Assume the contrary that $f_i \in H_2$ for all $i \in \Lambda_1$. Then, for $\Lambda_1 = \{i_1, \ldots, i_k\}$ and $\Lambda_2 = \{i_{k+1}, \ldots, i_N\}$, we have

$$f_{i_p} = \sum_{j=k+1}^{N} c^{(p)}_{i_j} f_{i_j} \quad \text{or} \tag{4.1.16}$$

$$f_{i_p} - \sum_{j=k+1}^{N} c^{(p)}_{i_j} f_{i_j} = 0 \tag{4.1.17}$$

where $p = 1, \ldots, k$. Note that the set $\{1, -c^{(p)}_{i_{k+1}}, \ldots, -c^{(p)}_{i_N}\}$ is orthogonal to the set $\{f_{i_p}, f_{i_{k+1}}, \ldots, f_{i_N}\}$

for all $p = 1, \dots, k$. Then, there exist sets such that

$$\begin{aligned} \{h_{i_1}^{(1)}, 0, \dots, 0, h_{i_{k+1}}^{(1)}, \dots, h_{i_N}^{(1)}\} \quad &\perp \quad \{f_{i_1}, \dots, f_{i_k}, f_{i_{k+1}}, \dots, f_{i_N}\} \\ \vdots \qquad\qquad\qquad & \qquad\qquad\qquad \vdots \\ \{0, \dots, 0, h_{i_k}^{(k)}, h_{i_{k+1}}^{(k)}, \dots, h_{i_N}^{(k)}\} \quad &\perp \quad \{f_{i_1}, \dots, f_{i_k}, f_{i_{k+1}}, \dots, f_{i_N}\} \end{aligned} \tag{4.1.18}$$

Let $v = [v_1, \dots, v_k, v_{k+1}, \dots, v_N]$ be a vector such that

$$v = \Big[h_{i_1}^{(1)}, \dots, h_{i_k}^{(k)}, \sum_{p=1}^{k} h_{i_{k+1}}^{(p)}, \dots, \sum_{p=1}^{k} h_{i_N}^{(p)}\Big]. \tag{4.1.19}$$

Then, $\sum_{i=1}^{N} \langle f_i, v_i \rangle = 0$. We can rescale the vector v by t, sufficiently small, so that we have $\langle f_i, v_i \rangle < 0$ for all $i = 1, \dots, N$. Thus,

$$|\langle f_i, S^{-1} f_i \rangle + \langle f_i, t v_i \rangle| < |\langle f_i, S^{-1} f_i \rangle| \leq \max\{\|S^{-1/2} f_i\|^2 \,:\, i = 1, \dots, N\}. \tag{4.1.20}$$

This implies that there exist 1-erasure spectrally optimal dual of F which is not $S^{-1}F$. Therefore, if the standard dual $S^{-1}F$ of F is 1-erasure spectrally optimal dual frame then there exists at least one $j \in \Lambda_1$ such that $f_j \notin H_2$ $\square$

Throughout the thesis, we will be using the following property of linearly independent sets in some of the proofs.

Proposition 4.1.3. Assume that $\{f_1, \dots, f_k\}$ is a linearly independent set in a Hilbert space H. Then, there exists a vector $v \in H$ such that $\langle f_i, v \rangle < 0$ for all $i = 1, \dots, k$.

Proof. Let $\{e_1, \dots, e_k\}$ be the standard basis for a $k-$ dimensional Hilbert space and T be the linear transformation such that $Te_i = f_i$ for $i = 1, \ldots, k$. Note first that there exists a vector $h \in H$ such that $\langle e_i, h \rangle < 0$ for all $i = 1, \ldots, k$. Then, we have

$$\langle f_i, v \rangle = \langle Te_i, (T^*)^{-1}h \rangle = \langle e_i, T^*(T^*)^{-1}h \rangle = \langle e_i, h \rangle < 0 \tag{4.1.21}$$

This implies that $\langle f_i, v \rangle < 0$, where $v = (T^*)^{-1}h$. $\square$

4.2 2-Erasure Spectrally Optimal Standard Dual Frames

The following proposition provides a sufficient condition for a standard dual to be a 2-erasure spectrally optimal dual frame.

Proposition 4.2.1. Let $F = \{f_i\}_{i=1}^N$ be a frame in H with frame operator S. Assume that $\|S^{-1/2}f_i\|^2 = \frac{n}{N}$. Then if $\langle f_i, S^{-1}f_j \rangle = \sqrt{\frac{Nn-n^2}{N^2(N-1)}}$ for $i \neq j$, then $S^{-1}F = \{S^{-1}f_i\}_{i=1}^N$ is 2-erasure spectrally optimal dual frame.

Proof. Let F be a frame with $\|S^{-1/2}f_i\|^2 = \frac{n}{N}$ and $\langle f_i, S^{-1}f_j \rangle = \sqrt{\frac{Nn-n^2}{N^2(N-1)}}$ for $i \neq j$. Assume that $S^{-1}F$ is not a 2-erasure spectrally optimal dual frame. Consider the error operator E_Λ for $|\Lambda| = 2$. Then the spectral radius of error operator is

$$r(E_\Lambda) = r(\Theta_{S^{-1}F}^* D_\Lambda \Theta_F) = r(\Theta_{S^{-1}F}^* D_\Lambda^* D_\Lambda \Theta_F) = r(D_\Lambda \Theta_F \Theta_{S^{-1}F}^* D_\Lambda^*). \tag{4.2.1}$$

where Θ_F and $\Theta_{S^{-1}F}$ are analysis operators for F and G, respectively, and D_Λ is an N by N diagonal matrix with $d_{ii} = 1$ for $i \in \Lambda$ and zero otherwise. For the spectral radius of E_Λ, we consider the characteristic function of $A_{i,j}$;

$$A_{i,j} = \begin{pmatrix} \langle S^{-1}f_i, f_i\rangle & \langle S^{-1}f_j, f_i\rangle \\ \langle S^{-1}f_i, f_j\rangle & \langle S^{-1}f_j, f_j\rangle \end{pmatrix} = \begin{pmatrix} \frac{n}{N} & \sqrt{\frac{Nn-n^2}{N^2(N-1)}} \\ \sqrt{\frac{Nn-n^2}{N^2(N-1)}} & \frac{n}{N} \end{pmatrix}, \tag{4.2.2}$$

for $i \neq j$ and $i, j \in 1, \ldots, n$. The characteristic function is

$$(\frac{n}{N} - \lambda)(\frac{n}{N} - \lambda) - \langle S^{-1}f_j, f_i\rangle\langle S^{-1}f_i, f_j\rangle = (\frac{n}{N} - \lambda)^2 - \frac{Nn - n^2}{N^2(N-1)} = 0, \tag{4.2.3}$$

which leads to

$$\lambda = \frac{n}{N} \pm \sqrt{\langle S^{-1}f_j, f_i\rangle\langle S^{-1}f_i, f_j\rangle} = \frac{n}{N} \pm \sqrt{\frac{Nn - n^2}{N^2(N-1)}}. \tag{4.2.4}$$

Then

$$r^{(2)}_{F,S^{-1}F} = \frac{n}{N} + \sqrt{\frac{Nn - n^2}{N^2(N-1)}}. \tag{4.2.5}$$

Assume that G is a 2-erasure spectrally optimal dual frame. Thus,

$$\frac{n}{N} + \sqrt{\langle g_j, f_i\rangle\langle g_i, f_j\rangle} = r^{(2)}_{F,G} < r^{(2)}_{F,S^{-1}F} = \frac{n}{N} + \sqrt{\frac{Nn - n^2}{N^2(N-1)}}. \tag{4.2.6}$$

This implies that

$$\langle g_j, f_i\rangle\langle g_i, f_j\rangle < \frac{Nn - n^2}{N^2(N-1)} \quad \text{for all } i \neq j. \tag{4.2.7}$$

Note that

$$\sum_{j=1}^{N}\sum_{i=1}^{N}\langle g_j, f_i\rangle\langle g_i, f_j\rangle = \sum_{j=1}^{N}\sum_{i\neq j}\langle g_j, f_i\rangle\langle g_i, f_j\rangle + \sum_{j=1}^{N}|\langle g_j, f_j\rangle|^2 \tag{4.2.8}$$

$$< N(N-1)\frac{Nn-n^2}{N^2(N-1)} + N\frac{n^2}{N^2} = n \tag{4.2.9}$$

However, this contradicts to the fact that

$$\sum_{j=1}^{N}\sum_{i=1}^{N}\langle g_j, f_i\rangle\langle g_i, f_j\rangle = \sum_{j=1}^{N}\langle g_j, f_j\rangle = n. \tag{4.2.10}$$

Therefore, $S^{-1}F$ is a 2-erasure spectrally optimal dual frame of F. $\square$

CHAPTER 5: SPECTRALLY ONE-UNIFORM FRAMES

In this chapter, we define and investigate spectrally one-uniform frames that admit one-erasure spectrally optimal dual frames. Our investigation is based on the characterization and construction of such frames. Doing so we show that spectrally one-uniform frames are closely related to the linear connectivity property of a frame. By the linear connectivity property of frames, we are able to define what we call uniform redundancy distribution of frames. With the help of redundancy distribution of a frame, we show that we can construct a spectrally one-uniform frame.

5.1 Spectrally One-Uniform Frames

In this section, we define spectrally one-uniform frames and show the relationship between spectrally one-uniform frames and one-erasure spectrally optimal dual frames.

Recall that for the 1-erasure case, spectral radius of the error operator satisfies $r_{F,G}^{(1)} = \max\{|\langle g_i, f_i\rangle| : 1 \leq i \leq N\}$. Therefore, for one erasure spectrally optimal dual frame we have $r_F^{(1)} \geq n/N$ since $\sum_{i=1}^{N}\langle g_i, f_i\rangle = n$. This leads to the question of characterizing all the frames F such that $r_F^{(1)} = n/N$, and the questions of how to compute $r_F^{(1)}$ and how to construct frames F and their duals G with prescribed maximal error $r_{F,G}^{(k)}$. It turns out that the answers to all these questions rely on an interesting connectivity property for finite sequences (or subset) of nonzero vectors in H. From application point of view we are only interested in frames consisting of nonzero vectors.

So we will assume this property throughout the rest of the dissertation.

Definition 5.1.1. Let F be an (N, n) frame. Then F is called **spectrally one-uniform frame** if there exists a dual frame G of F such that $\langle g_i, f_i \rangle = c$ for all $i = 1, \ldots, N$ where $c = n/N$.

Theorem 5.1.1. Let F be an (N, n) frame. Then F is spectrally one-uniform frame if and only if there exists a dual G such that $r_F^{(1)} = r_{F,G}^{(1)} = n/N$.

Proof. If we assume that F is spectrally one-uniform frame then there exists a dual frame G such that $\langle g_i, f_i \rangle = n/N$ for all $i = 1, \ldots, N$ by definition. This implies that $r_{F,G}^{(1)} = n/N$. Note that $r_F^{(1)} = \min_{G'}\{|\langle g_i, f_i \rangle| : i = 1, \ldots, N\} = n/N$, where G' is a dual frame of F since we have the fact that $\sum_{i=1}^{N} \langle g_i, f_i \rangle = n$, i.e., $\max\{|\langle g_i, f_i \rangle| : i = 1, \ldots, N\} \geq n/N$. Thus, $r_F^{(1)} = r_{F,G}^{(1)}$.

Now assume that $r_F^{(1)} = n/N$. Then there exists a dual frame G such that $r_{F,G}^{(1)} = n/N$ which implies that $\max\{|\langle g_i, f_i \rangle| : i = 1, \ldots, N\} = n/N$. Assume that there exists j such that $\langle g_j, f_j \rangle < n/N$. Then there must exists a j' such that $\langle g_{j'}, f_{j'} \rangle > n/N$ since $\sum_{i=1}^{N} \langle g_i, f_i \rangle = n$. However, this contradicts to $\max\{|\langle g_i, f_i \rangle| : i = 1, \ldots, N\} = n/N$. Thus, for all $i = 1, \ldots, N$, $\langle g_i, f_i \rangle = n/N$. Therefore, F is spectrally one-uniform frame. $\square$

5.2 Linearly Connected Sequences

In this section, we define three properties of frames; linear connectivity, intersection dependence and k-independence properties, on which the characterization of spectrally one-uniform frames rely. We prove that the linear connectivity property is equivalent to the intersection dependence property, and is also closely related to the well-known concept of k-independent set.

We say that two vectors f and g in a sequence F of vectors are *linearly F-connected (or simply, connected)* if there exist vectors $\{u_1, ..., u_\ell\}$ from F such that $\{g, u_1, ..., u_\ell\}$ are linearly independent and $f = cg + \sum_{i=1}^{\ell} c_i u_i$ with c, c_i all nonzero. Clearly connectivity is reflexive and symmetric. We will show that it is also transitive which turns to be a key property needed to prove our main results. We will use the notation $f \overset{F}{\leftrightarrow} g$ if f and g are F-connected.

Definition 5.2.1. Let $F = \{f_i\}_{i=1}^N$ be a finite sequence of nonzero vectors in H. We say that F

(i) is **linearly connected** if every two vectors in F are F-connected.

(ii) has the **intersection dependent property** if $H_\Lambda \cap H_{\Lambda^c} \neq \{0\}$ holds for every proper subset Λ of $\{1, ..., N\}$, where H_Λ is the subspace spanned by $\{f_i : i \in \Lambda\}$.

(ii) is k**-independent** if every k vectors in F are linearly independent.

The following theorem 5.2.4 states that all these three properties are closely related. While this result is needed as one of the main ingredients in characterizing and constructing spectrally one-uniform frames, it is also an independently interesting property that may have applications in other area of research.

To prove Theorem 5.2.4 we need the following definition of support of a vector f and some of its properties.

Let $\mathcal{B} = \{f_1, \dots, f_n\}$ be a basis for a Hilbert space H and $f = \sum_{i=1}^n c_i f_i \in H$. We define the *support* of f with respect to the basis $\mathcal{B}$ by

$$\mathrm{supp}_{\mathcal{B}}(f) = \{i \,:\, c_i \neq 0\}. \tag{5.2.1}$$

We first need to prove the transitivity property for F-connected vectors.

Theorem 5.2.1. Let F be a finite sequence in H. Assume that f_1, f_2 and f_3 are vectors in F such that $f_1 \overset{F}{\leftrightarrow} f_3$ and $f_2 \overset{F}{\leftrightarrow} f_3$. Then $f_1 \overset{F}{\leftrightarrow} f_2$.

Proof. By the definition of connected vectors, we have

$$f_1 = \alpha f_3 + \sum_{i \in \Lambda_1} \alpha_i f_i, \quad \text{with } \alpha, \alpha_i \text{ all nonzero and} \tag{5.2.2}$$

$$f_2 = \beta f_3 + \sum_{i \in \Lambda_2} \beta_i f_i, \quad \text{with } \beta, \beta_i \text{ all nonzero,} \tag{5.2.3}$$

and $\{f_3, f_i : i \in \Lambda_1\}$ and $\{f_3, f_i : i \in \Lambda_2\}$ are linearly independent subsets of F. Solving the equation in (5.2.3) for f_3 and then substituting it to the equation (5.2.2), we obtain

$$f_1 = \frac{\alpha}{\beta} f_2 - \frac{\alpha}{\beta} \sum_{i \in \Lambda_2} \beta_i f_i + \sum_{i \in \Lambda_1} \alpha_i f_i. \tag{5.2.4}$$

Set $M = \operatorname{span}\{f_i : i \in \Lambda_1 \cup \Lambda_2\}$. Then there exists a subset Λ_2' of Λ_2 such that $\mathcal{B} = \{f_i : i \in \Lambda = \Lambda_1 \cup \Lambda_2'\}$ is a basis for M. Thus the equation in (5.2.4) can be rewritten as

$$f_1 = \frac{\alpha}{\beta} f_2 + \sum_{i \in \Lambda} \gamma_i f_i. \tag{5.2.5}$$

To show that f_1 is F-connected to f_2, we consider two cases depending on whether f_2 is in M or not.

Case 1: Assume that $f_2 \notin M$. Since $\{f_i : i \in \Lambda\}$ is linearly independent, we get that $\{f_2, f_i : i \in \Lambda\}$ is linearly independent. Moreover, α, β, γ_i all are nonzero by the assumptions in (5.2.2) and (5.2.3). Hence, by the definition of connected vectors, we have $f_1 \overset{F}{\leftrightarrow} f_2$.

Case 2: Assume that $f_2 \in M$. In this case, we take the relations between the supports of the vectors f_1 and f_2 into account. Let $\Omega_i = \mathrm{supp}_{\mathcal{B}}(f_i)$ for $i = 1, 2, 3$.

(i) Assume that $\Omega_1 \cap \Omega_2 \neq \emptyset$. Then there exist $i_k \in \Omega_1 \cap \Omega_2$. By the definition of supports of f_1 and f_2, we have

$$f_1 = a_{i_k} f_{i_k} + \sum_{\substack{i\in\Omega_1 \\ i\neq i_k}} a_i f_i \quad \text{and} \tag{5.2.6}$$

$$f_2 = b_{i_k} f_{i_k} + \sum_{\substack{i\in\Omega_2 \\ i\neq i_k}} b_i f_i. \tag{5.2.7}$$

Solving the equation in (5.2.7) for f_{i_k} and substituting it to (5.2.6), we obtain

$$f_1 = a_2' f_2 + \sum_{\substack{i\in\Omega_1\cup\Omega_2 \\ i\neq i_k}} a_i' f_i. \tag{5.2.8}$$

We note here that $f_2 \notin \mathrm{span}\{f_i \,:\, i \in \Omega_1 \cup \Omega_2,\, i \neq i_k\}$ and $a_2' = \frac{a_{i_k}}{b_{i_k}}$ is nonzero. Thus, by Case 1, the result follows.

(ii) Assume that $\Omega_1 \cap \Omega_2 = \emptyset$. Let $f_3 = \sum_{i\in\Omega_3} c_i f_i$. Then, if we replace f_1, f_2 and f_3 in equations (5.2.2) and (5.2.3) with the sums associated with the supports of the vectors, we have the following two equations:

$$\sum_{i\in\Omega_1} a_i f_i - \alpha \sum_{i\in\Omega_3} c_i f_i = \sum_{i\in\Lambda_1} \alpha_i f_i \tag{5.2.9}$$

$$\sum_{i\in\Omega_2} b_i f_i - \beta \sum_{i\in\Omega_3} c_i f_i = \sum_{i\in\Lambda_2} \beta_i f_i. \tag{5.2.10}$$

These imply that $\Lambda_1 \subseteq \Omega_1 \cup \Omega_3$ and $\Lambda_2 \subseteq \Omega_2 \cup \Omega_3$. We claim that neither $\Omega_1 \cap \Omega_3$ nor $\Omega_2 \cap \Omega_3$ is an empty set. In fact, if $\Omega_1 \cap \Omega_3 = \emptyset$, then $\Lambda_1 = \Omega_1 \cup \Omega_3$ which implies that $\Omega_3 \subseteq \Lambda_1$. Thus, $f_3 \in \operatorname{span}\{f_i : i \in \Lambda_1\}$. This contradicts to the assumption that $\{f_3, f_i : i \in \Lambda_1\}$ is linearly independent. Therefore, $\Omega_1 \cap \Omega_3 \neq \emptyset$. Similarly, it is shown that $\Omega_2 \cap \Omega_3 \neq \emptyset$.

Now, let $i_1 \in \Omega_1 \cap \Omega_3$ and $i_2 \in \Omega_2 \cap \Omega_3$. Consider, $\tilde{\Lambda} = \big(\Lambda \backslash \{i_1, i_2\}\big) \cup \{1, 2\}$. We want to show that $\{f_i : i \in \tilde{\Lambda}\}$ is linearly independent; thus, a basis for M. Assume that

$$d_1 f_1 + d_2 f_2 + \sum_{\substack{i \in \Lambda \\ i \neq i_1, i_2}} d_i f_i = 0. \tag{5.2.11}$$

Substituting f_1 and f_2 with their associated sums, we have

$$d_1 \sum_{i \in \Omega_1} a_i f_i + d_2 \sum_{i \in \Omega_2} b_i f_i + \sum_{\substack{i \in \Lambda \\ i \neq i_1, i_2}} d_i f_i = 0. \tag{5.2.12}$$

After pulling out f_{i_1} and f_{i_2} from the summations, we obtain

$$d_1 a_{i_1} f_{i_1} + d_2 b_{i_2} f_{i_2} + d_1 \sum_{\substack{i \in \Omega_1 \\ i \neq i_1}} a_i f_i + d_2 \sum_{\substack{i \in \Omega_2 \\ i \neq i_2}} b_i f_i + \sum_{\substack{i \in \Lambda \\ i \neq i_1, i_2}} d_i f_i = 0. \tag{5.2.13}$$

Since Ω_1 and Ω_2 are subsets of Λ, the above equation can be reduced to the following;

$$d_1 a_{i_1} f_{i_1} + d_2 b_{i_2} f_{i_2} + \sum_{\substack{i \in \Lambda \\ i \neq i_1, i_2}} d'_i f_i = 0. \tag{5.2.14}$$

This implies that $d_1 a_{i_1} = d_2 b_{i_2} = 0$ because $\{f_i : i \in \Lambda\}$ is linearly independent. We note that a_{i_1} and b_{i_2} are nonzero since they are associated with the supports of f_1 and f_2. Thus, $d_1 = d_2 = 0$.

Therefore, the equation in (5.2.11) is reduced to

$$\sum_{\substack{i\in\Lambda \\ i\neq i_1,i_2}} d_i f_i = 0. \tag{5.2.15}$$

Since $\{f_i : i \in \Lambda, i \neq i_1, i_2\}$ is linearly independent, we get that $d_i = 0$ for all $i \in \Lambda\backslash\{i_1, i_2\}$. Thus, $\tilde{\mathcal{B}} = \{f_i : i \in \tilde{\Lambda}\}$ is linearly independent, and hence it is a basis for M. So, f_3 can be written in terms of the vectors in $\tilde{\mathcal{B}}$ as

$$f_3 = c'_1 f_1 + c'_2 f_2 + \sum_{\substack{i\in\Omega_3 \\ i\neq i_1,i_2}} c'_i f_i. \tag{5.2.16}$$

We claim that both c'_1 and c'_2 are nonzero. Indeed, if $c'_1 = 0$, then $c'_2 \neq 0$ because otherwise, $\Omega_3 \subseteq \Omega_3\backslash\{i_1, i_2\}$ which is not possible. Recall that the support of f_3 with respect to the basis $\mathcal{B}$ is Ω_3. We have the representation: $f_3 = c_{i_1} f_{i_1} + c_{i_2} f_{i_2} + \sum_{\substack{i\in\Omega_3 \\ i\neq i_1,i_2}} c_i f_i$, where all the scalars are nonzero. Combining this with (5.2.16), we obtain

$$f_3 = c'_2 f_2 + \sum_{\substack{i\in\Omega_3 \\ i\neq i_1,i_2}} c'_i f_i = c_{i_1} f_{i_1} + c_{i_2} f_{i_2} + \sum_{\substack{i\in\Omega_3 \\ i\neq i_1,i_2}} c_i f_i. \tag{5.2.17}$$

This implies that

$$f_2 = \frac{c'_{i_1}}{c'_2} f_{i_1} + \frac{c'_{i_2}}{c'_2} f_{i_2} + \frac{1}{c'_2} \sum_{\substack{i\in\Omega_3 \\ i\neq i_1,i_2}} (c_i - c'_i) f_i. \tag{5.2.18}$$

Since c'_{i_1}, c'_{i_2} and c'_2 are nonzero, we get that $i_1, i_2 \in \Omega_2$. Recall that $i_1 \in \Omega_1$. So we have $i_1 \in \Omega_1 \cap \Omega_2$. This contradicts to the assumption that $\Omega_1 \cap \Omega_2 = \emptyset$. Hence, $c'_1 \neq 0$. Similarly, it can be shown that $c'_2 \neq 0$.

By the equation in (5.2.16) for f_1, we get

$$f_1 = \frac{1}{c'_1} f_3 - \frac{c'_2}{c'_1} f_2 - \frac{1}{c'_1} \sum_{\substack{i \in \Omega_3 \\ i \neq i_1, i_2}} c'_i f_i. \tag{5.2.19}$$

To show that f_1 and f_2 are F connected it is sufficient to show that $\{f_3, f_2, f_i : i \in \Omega_3, i \neq i_1, i_2\}$ is linearly independent since we already know that c'_1, c'_2, c'_i for $i \in \Omega_3 \backslash \{i_1, i_2\}$ all are nonzero. Note that $\{2, i : i \in \Omega_3, i \neq i_1, i_2\} \subset \tilde{\mathcal{B}}$, so it is enough to show that $f_3 \notin \operatorname{span}\{f_2, f_i : i \in \Omega_3, i \neq i_1, i_2\}$. Indeed, if $f_3 \in \operatorname{span}\{f_2, f_i : i \in \Omega_3, i \neq i_1, i_2\}$, then by (5.2.16), $\operatorname{supp}_{\tilde{\mathrm{B}}}(f_3) \subseteq \operatorname{supp}_{\tilde{\mathrm{B}}}(f_3) \backslash \{1\}$ which gives a contradiction. Hence, $\{f_3, f_2, f_i : i \in \Omega_3, i \neq i_1, i_2\}$ is linearly independent, and thus $f_1 \overset{F}{\leftrightarrow} f_2$. $\square$

As a result of the transitivity property of connected vectors, we have the following consequence:

Corollary 5.2.2. Let $F = \{f_1, \dots, f_N\}$ be a connected sequence of H. Then for any nonzero vector $f \in \operatorname{span}\{f_1, \dots, f_N\}$, the sequence $\{f_1, \dots, f_N, f\}$ is connected.

Proof. Let $f \in \operatorname{span}\{f_1, \dots, f_N\}$. To show that $\{f_1, \dots, f_N, f\}$ is connected, we show that $f \overset{F}{\leftrightarrow} f_j$ for all $j \in \Lambda = 1, \dots, N$. Write $f = \sum_{i \in \Lambda'} \alpha_i f_i$ with α_i nonzero for all $i \in \Lambda'$, and $\{f_i : i \in \Lambda' \subseteq \Lambda\}$ is linearly independent. This implies that $f \overset{F}{\leftrightarrow} f_i$ for any fixed $i \in \Lambda'$. Because $\{f_1, \dots, f_N\}$ is connected, $f_i \overset{F}{\leftrightarrow} f_j$ for any $i \in \Lambda', j \in \Lambda$. Therefore, by transitivity property of connected vectors, $f \overset{F}{\leftrightarrow} f_j$ for all $j \in \Lambda$. $\square$

Let $F = \{f_i\}_{i=1}^N$ be a frame of H and let $\emptyset \neq \Lambda \subset \{1, \dots, N\}$ and $\Lambda^{\mathsf{c}} = \{1, \dots, N\} \backslash \Lambda$. Define $H_\Lambda = \operatorname{span}\{f_i : i \in \Lambda\}$ and $H_{\Lambda^{\mathsf{c}}} = \operatorname{span}\{f_i : i \in \Lambda^{\mathsf{c}}\}$.

Proposition 5.2.3. If $H_\Lambda \cap H_{\Lambda^c} \neq \{0\}$, then there exists $i_1 \in \Lambda$, $i_2 \in \Lambda^c$ such that $f_{i_1} \overset{F}{\leftrightarrow} f_{i_2}$.

Proof. Since $H_\Lambda \cap H_{\Lambda^c} \neq \{0\}$, there exist $0 \neq f \in H_\Lambda \cap H_{\Lambda^c}$. So, there exist $\emptyset \neq \Lambda_1 \subseteq \Lambda$ and $\emptyset \neq \Lambda_2 \subseteq \Lambda^c$ such that

$$f = \sum_{i \in \Lambda_1} \alpha_i f_i = \sum_{j \in \Lambda_2} \beta_j f_j \tag{5.2.20}$$

with α_i and β_j all are nonzero, and $\{f_i : i \in \Lambda_1\}$ and $\{f_j : j \in \Lambda_2\}$ are linearly independent sets. Then for a fixed $i_1 \in \Lambda_1$, we have

$$f_{i_1} = -\frac{1}{\alpha_{i_1}} \sum_{\substack{i \in \Lambda_1 \\ i \neq i_1}} \alpha_i f_i + \frac{1}{\alpha_{i_1}} \sum_{j \in \Lambda_2} \beta_j f_j, \tag{5.2.21}$$

which implies that $f_{i_1} \in \operatorname{span}\{f_i : i \in \Lambda_1 \cup \Lambda_2,\ i \neq i_1\}$. Thus there exists $\Lambda_3 \subseteq (\Lambda_1 \cup \Lambda_2) \backslash \{i_1\}$ such that $f_{i_1} = \sum_{i \in \Lambda_3} \gamma_i f_i$ with $\gamma_i \neq 0$ ($i \in \Lambda_3$) and $\{f_i : i \in \Lambda_3\}$ is linearly independent. Since $f_{i_1} \notin \operatorname{span}\{f_i : i \in \Lambda_1,\ i \neq i_1\}$, we have $\Lambda_2 \cap \Lambda_3 \neq \emptyset$. Let $i_2 \in \Lambda_2 \cap \Lambda_3 \subset \Lambda^c$. Then, f_{i_1} and f_{i_2} are F-connected. $\square$

Now, we are ready to prove the first main Theorem of this section that gives the relationship among the three concepts; connectedness, IDP and $\ell-$independence of sets.

Theorem 5.2.4. Let $F = \{f_i\}_{i=1}^N$ be a sequence of H and let ℓ =dim span$\{f_i : 1 \leq i \leq N\}$. Then the following are equivalent:

(i) F is linearly connected.

(ii) F has the intersection dependent property.

(iii) F contains an $\ell-$independent subset of cardinality of at least $\ell+1$.

Proof. We can assume that F is a frame for H and dim H= ℓ.

"$(i) \Rightarrow (ii)$:" Assume that F is connected. Let $\emptyset \neq \Lambda \subset \{1,\dots,N\}$, and $\Lambda^{\mathrm{c}} = \{1,\dots,N\}\backslash\Lambda$. By assumption, for any fixed $j \in \Lambda$ and $j' \in \Lambda^c$, $f_j \overset{F}{\leftrightarrow} f_{j'}$. In other words, there exist a sequence of vectors $\{f_i \,:\, i \in \Lambda_1 \cup \Lambda_2,\, \Lambda_1 \subset \Lambda,\, \Lambda_2 \subset \Lambda^c\}$ such that $\{f_{j'}, f_i \,:\, i \in \Lambda_1 \cup \Lambda_2\}$ is linearly independent and

$$f_j = \alpha f_{j'} + \sum_{i\in\Lambda_1} \alpha_i f_i + \sum_{i\in\Lambda_2} \beta_i f_i \tag{5.2.22}$$

with α, α_i and β_i all are nonzero. If we move the vectors associated with Λ_1 to one side and keep the rest in the other side, we have

$$f_j - \sum_{i\in\Lambda_1} \alpha_i f_i = \alpha f_{j'} + \sum_{i\in\Lambda_2} \beta_i f_i. \tag{5.2.23}$$

Note that the right hand side of the equation in (5.2.23) is nonzero because $\{f_{j'}, f_i \,:\, i \in \Lambda_2\}$ is linearly independent and α and β_i are nonzero for $i \in \Lambda_2$. Moreover, the right side of (5.2.23) is in H_{Λ^c} while the left side is in H_Λ. Therefore, $H_\Lambda \cap H_{\Lambda^c} \neq \{0\}$

"$(ii) \Rightarrow (iii)$:" Let $N = \ell + k$ and we prove the statement by induction on k. For $k = 1$, we show that every ℓ members in F are linearly independent. Without losing the generality it suffices to show that $\{f_1, ..., f_\ell\}$ are linearly independent. Assume to the contrary that $\{f_1, ..., f_\ell\}$ are linearly dependent. Then $\dim H_\Lambda \leq \ell - 1$, where $\Lambda = \{1, ..., \ell\}$. This implies that $f_{\ell+1} \notin H_\Lambda$

since $\dim \operatorname{span}\{f_i : 1 \leq i \leq \ell+1\} = \ell$. Therefore $H_\Lambda \cap H_{\Lambda^c} = \{0\}$, which contradicts to the assumption that F has the intersection dependence property.

To complete the rest of the proof, we only need to show that if F has the intersection dependence property but does not have the ℓ-independent property, then there exists a proper subset Ω of $\{1, ..., N\}$ such that has $F' = \{f_i : i \in \Omega\}$ has the intersection dependence property and $\dim \operatorname{span} F' = \ell$.

Since F is not ℓ-independent, there exists ℓ members from F that are linearly dependent. Without losing the generality we can assume that $f_1 = \sum_{i\in\Lambda_0} \alpha_i f_i$, where Λ_0 is a subset of $\{2, ..., N\}$ of cardinality less than ℓ, and $\alpha_i \neq 0$. Let $F' = \{f_2, ..., f_N\}$. We show that F' still has the intersection dependence property. Assume to the contrary that F' does not have the intersection dependence property. Then there is a nonempty proper subset Λ of $\{2, ..., N\}$ such that $H_\Lambda \cap H_{\Lambda^c} = \{0\}$, where $\Lambda^c = \{2, ..., N\} \setminus \Lambda$. This implies that both $\Lambda_1 := \Lambda \cap \Lambda_0 \neq \emptyset$ and $\Lambda_2 := \Lambda^c \cap \Lambda_0 \neq \emptyset$. If fact, for example, if $\Lambda_2 := \Lambda^c \cap \Lambda_0 = \emptyset$, then $\Lambda_0 \subseteq \Lambda$. So $f_1 \in H_\Lambda$ and therefore we have $H_{\Lambda\cup\{1\}} \cap H_{(\Lambda\cup\{1\})^c} = H_\Lambda \cap H_{(\Lambda)^c} = \{0\}$. This leads to a contradiction since F has the intersection dependence property.

Write $\Lambda_1 = \{i_1, ..., i_k\}$ and $\Lambda_2 = \{j_1, ..., i_m\}$. Then $k + m = |\Lambda_0| < \ell$. This implies that either $\dim H_\Lambda > k$ or $\dim H_{\Lambda^c} > m$ since $\dim H_\Lambda + \dim H_{\Lambda^c} = \ell$. We can assume that $\dim H_{\Lambda^c} > m$. Extend Λ_2 to a basis $\{j_1, ..., j_m, j_{m+1}, ..., j_M\}$ ($M > m$) for H_{Λ^c} and Λ_1 to a basis $\{i_1, ..., i_k, ..., i_K\}$ ($K \geq k$) for H_Λ. Then $\mathcal{B} = \{f_i : i \in B\}$ is a basis for H, where $B = \{j_1, ..., j_m, j_{m+1}, ..., j_M\} \cup \{i_1, ..., i_k, ..., i_K\}$. Let $A = \{i : f_i \in \operatorname{span}\{f_{j_{m+1}}, ..., f_{j_M}\}\}$. Note that $\operatorname{supp}_{\mathcal{B}}(f_1) = \Lambda_0$, and Λ_0 and $\{j_{m+1}, ..., j_M\}$ are disjoint subsets of B. Moreover, $\{f_{j_1}, ..., f_{j_m}\} \cup \{f_{i_1}, ..., f_{i_k}, ..., f_{i_K}\}$ is a basis for H_{A^c}. (Here we use the assumption that $f_1 \in \operatorname{span}\{f_i : i \in \Lambda_0\}$ and Λ_0 is a subset of A^c).

By the intersection dependence property we obtain that $H_A \cap H_{A^c} \neq \{0\}$. Therefore there exist vectors $g_1 \in \text{span}\{f_{j_{m+1}}, ..., f_{j_M}\}$, $g_2 \in \text{span}\{f_{j_1}, ..., f_{j_m}\}$ and $g_3 \in \text{span}\{f_{i_1}, ..., f_{i_K}\}$ such that $g_1 = g_2 + g_3 \neq 0$. This is impossible since

$$\{f_{j_{m+1}}, ..., f_{j_M}\} \cup \{f_{j_1}, ..., f_{j_m}\} \cup \{f_{i_1}, ..., f_{i_K}\} \tag{5.2.24}$$

are linearly independent.

"$(iii) \Rightarrow (i)$:" Let S be an ℓ independent subsequence of F. Without losing the generality, we assume that $S = \{f_i : i \in \Lambda\}$ for $\Lambda = \{i : 1 \leq i \leq \ell + 1\}$. First we show that S is connected. Then the rest follows from Corollary 5.2.2 since $f_{i'} \in \text{span}S$ for any $f_{i'} \in F$ with $i' \geq \ell + 2$. Assume that S is not a connected subsequence. Then there exist f_j and f_k in S such that f_j is not F connected to f_k. Note that $f_j \in \text{span}\{f_i : i \in \Lambda, i \neq j\}$ because, otherwise, if $f_j \notin \text{span}\{f_i : i \in \Lambda, i \neq j\}$ then $\dim S = \ell + 1$ which contradicts to the assumption that $\ell = \dim \text{span}\{f_i : i = 1, \ldots, N\}$. Write

$$f_j = \sum_{i \in \Lambda'} \alpha_i f_i \tag{5.2.25}$$

with $\alpha_i \neq 0$ and $\Lambda' \subseteq \Lambda \backslash \{j, k\}$. Then $|\Lambda' \cup \{j\}| \leq \ell$. This, by (5.2.25), implies that $\big\{f_i : i \in \Lambda' \cup \{j\}\big\}$ is linearly dependent set, thus, gives a contradiction to ℓ-independence of S. Therefore, S is a connected subsequence. $\square$

As a consequence of Theorem 5.2.4 we obtain the following partition of frames:

Corollary 5.2.5. Let $F = \{f_i\}_{i=1}^N$ be a sequence of H. Then there exists a (unique up to permutations) partition $\{\Lambda_j\}_{j=1}^J$ of $\{1, 2, ..., N\}$ such that each $\{f_i\}_{i \in \Lambda_j}$ is linearly connected, and H is the direct sum of the subspaces H_j =span$\{f_i \,:\, i \in \Lambda_j\}$.

Proof. If F is not connected, then by Theorem 5.2.4, there exists a nonempty proper subset Λ of $\{1, \ldots, N\}$ such that $H_\Lambda \cap H_{\Lambda^c} = \{0\}$. If both $\{f_i\}_{i\in\Lambda}$ and $\{f_i\}_{i\in\Lambda^c}$ are connected, then the result follows. If not, by the same argument we can decompose the unconnected part until we obtain connected subsets. By this decomposition, we obtain a partition $\{\Lambda_j\}_{j=1}^J$ of $\{1, \ldots, N\}$ such that the direct sum of the subspaces H_j $(1 \leq j \leq J)$ is the original space H. $\square$

5.3 Redundancy Distribution of a Frame

In this section, we introduce redundancy distribution of frames that helps us to characterize and construct spectrally one-uniform frames, and to compute maximum erasure errors, $r^{(1)}_{F,G}$. Using Corollary 5.2.5, we introduce the following definition:

Definition 5.3.1. Let $F = \{f_i\}_{i=1}^N$ be a frame for H, and let H_j, Λ_j be as in Corollary 5.2.5. Then the **redundancy distribution** of F is defined to be $\left\{\frac{\dim H_j}{|\Lambda_j|}\right\}_{1\leq j\leq J}$. We say that F has the **uniform redundancy distribution** if $\frac{\dim H_j}{|\Lambda_j|}$ is a constant for all j.

Let $G = \{g_1, \ldots, g_N\}$ be a dual frame of a frame $F = \{f_1, \ldots, f_N\}$. Define $\Lambda_G = \big\{i : \langle g_i, f_i\rangle = n/N\big\}$ and $\Lambda_G^{\mathsf{c}} = \{1, 2, \ldots, N\}\backslash\Lambda_G$.

Lemma 5.3.1. Let $|\Lambda_G^{\mathsf{c}}| \geq 1$ and i_1, $i_2 \in \Lambda_G^{\mathsf{c}}$. If f_{i_1} and f_{i_2} are F-connected, then there exists a dual G' such that $|\Lambda_{G'}| > |\Lambda_G|$.

Proof. Assume that f_{i_1} and f_{i_2} are F-connected. Then there exist a basis $\mathcal{B} \subset \{f_1, \ldots, f_N\}$ such that $f_{i_2} \in \mathcal{B}$ and $i_2 \in \operatorname{supp}_{\mathcal{B}}(f_{i_1})$,

$$f_{i_1} = c_2 f_{i_2} + \sum_{\substack{i_k \in \operatorname{supp}(f_{i_1}) \\ k \neq 2}} c_k f_{i_k} \quad (c_2 \neq 0). \tag{5.3.1}$$

We can select G' in such a way that either $\langle g'_{i_1}, f_{i_1} \rangle = \frac{n}{N}$ or $\langle g'_{i_2}, f_{i_2} \rangle = \frac{n}{N}$. Indeed, since $\{f_{i_k} : i_k \in \operatorname{supp}_{\mathcal{B}}(f_{i_1})\}$ is linearly independent, there exists $u \in H$ such that $\langle u, c_k f_{i_k} \rangle = \langle \bar{c}_k u, f_{i_k} \rangle = 0$ for $i_k \in \operatorname{supp}(f_{i_1}) \backslash \{i_2\}$ and $\langle u, c_2 f_{i_2} \rangle + \langle g_{i_2}, f_{i_2} \rangle = \langle \bar{c}_2 u, f_{i_2} \rangle + \langle g_{i_2}, f_{i_2} \rangle = \frac{n}{N}$.

Now, let $u_{i_k} = \bar{c}_k u$ for $i_k \in \operatorname{supp}(f_{i_1})$, $u_{i_1} = -u$ and $u_{i_k} = 0$ otherwise. Then

$$\sum_{k=1}^{N} \langle f, u_{i_k} \rangle f_{i_k} = \sum_{i_k \in \operatorname{supp}(f_{i_1})} \langle f, \bar{c}_k u \rangle f_{i_k} + \langle f, -u \rangle f_{i_1} \tag{5.3.2}$$

$$= \sum_{i_k \in \operatorname{supp}(f_{i_1})} \langle f, u \rangle c_k f_{i_k} - \langle f, u \rangle f_{i_1} \tag{5.3.3}$$

$$= \langle f, u \rangle \left(\sum_{i_k \in \operatorname{supp}(f_{i_1})} c_k f_{i_k} - f_{i_1} \right) = 0. \tag{5.3.4}$$

Note that $\langle g'_{i_k}, f_{i_k} \rangle = \langle g_{i_k}, f_{i_k} \rangle$ for all $i_k \neq i_1, i_2$ and $\langle g'_{i_2}, f_{i_2} \rangle = \frac{n}{N}$. Thus, $\{g'_{i_k}\} = \{g_{i_k} + u_{i_k}\}$ is a dual frame of F such that $|\Lambda_{G'}| \geq |\Lambda_G| + 1$.

$\square$

Remark 5.3.1. If $|\Lambda_G^{\mathsf{c}}| = 1$, then for all $i = 1, \ldots, N$ $\langle g_i, f_i \rangle = \frac{n}{N}$. Indeed, by assumption $N-1$ vectors, say $f_1, \ldots, f_{N-1}$, have the property $\langle g_i, f_i \rangle = \frac{n}{N}$. Because $\sum_{i=1}^{N} \langle g_i, f_i \rangle = n$, we also have $\langle g_N, f_N \rangle = \frac{n}{N}$.

Corollary 5.3.2. If F is a connected frame, then there exists a dual $G' = \{g'_i\}_{i=1}^{N}$ such that $\langle g'_i, f_i \rangle = n/N$ for all i.

Proof. Since F is connected, we apply Lemma 5.3.1 successively to any dual frame G that is constructed as a result of Lemma 5.3.1 until we obtain a dual frame G' with $\langle g_i', f_i \rangle = n/N$ for all i. $\square$

Example 5.3.1. The converse of Corollary 5.3.2 is not true. Consider the frame $F = \{e_1, e_1, e_1\} \cup \{e_2, e_2, e_2\}$ in $\mathbb{C}^2$. It has a dual $G = \{e_1/3, e_1/3, e_1/3\} \cup \{e_2/3, e_2/3, e_2/3\}$ with $\langle g_i, f_i \rangle = 1/3$ for $i = 1, 2, 3$. However, $f_3 = e_1$ and $f_4 = e_2$ are not F-connected.

Let $F = \{f_i\}_{i=1}^N = \cup_{j=1}^J F_j$ be a frame with a partition $\{\Lambda_j\}_{j=1}^J$ of $\{1, \ldots, N\}$ and $F_j = \{f_i : i \in \Lambda_j\}$ is linearly connected. Let $n_j = \dim H_j = \dim \operatorname{span}\{f_i : i \in \Lambda_j\}$, $N_j = |\Lambda_j|$ and $d_j = \frac{n_j}{N_j}$.

Proposition 5.3.3. There exist a dual frame $G_j = \{g_i\}_{i \in \Lambda_j}$ of F_j with $\langle g_i, f_i \rangle = d_j$ for all $i \in \Lambda_j$. Moreover, there exist a dual frame G' of F such that $\langle g_i', f_i \rangle = d_j$ for all $i \in \Lambda_j$.

Proof. Since for a fixed $j = 1, \ldots, J$, F_j is a connected frame for H_j, by Corollary 5.3.2, there exist a dual $G_j = \{g_i\}_{i \in \Lambda_j}$ of F_j such that $\langle g_i, f_i \rangle = \frac{n_j}{N_j} = d_j$ for all $i \in \Lambda_j$. This completes the first part of the proof. For the second part, let $h \in H$. Then h has a unique representation:

$$h = h_1 + \cdots + h_j + \cdots + h_J, \tag{5.3.5}$$

where $h_j \in H_j$. For a fixed $j \in \{1, \ldots, J\}$, define a projection

$$P_j : H \to H_j \quad \text{such that } P_j h = h_j. \tag{5.3.6}$$

Now set $g_i' = P_j^* g_i$ for $i \in \Lambda_j$, where P_j^* is the adjoint operator of P_j. Since $P_j f_i = f_i$ for all

$i \in \Lambda_j$, we get that

$$\langle g_i', f_i \rangle = \langle P_j^* g_i, f_i \rangle = \langle g_i, P_j f_i \rangle = \langle g_i, f_i \rangle \quad \text{for all } i \in \Lambda_j. \tag{5.3.7}$$

Thus, it is enough to show that $\cup_{j=1}^{J} G_j'$ is a dual for F. For any $h = h_1 + \cdots + h_j + \cdots + h_J \in H$, we have

$$\sum_{i=1}^{N} \langle h, g_i' \rangle f_i = \sum_{i \in \Lambda_1} \langle h, g_i' \rangle f_i + \cdots + \sum_{i \in \Lambda_j} \langle h, g_i' \rangle f_i + \cdots + \sum_{i \in \Lambda_J} \langle h, g_i' \rangle f_i \tag{5.3.8}$$

$$= \sum_{i \in \Lambda_1} \langle h, P_1^* g_i \rangle f_i + \cdots + \sum_{i \in \Lambda_j} \langle h, P_j^* g_i \rangle f_i + \cdots + \sum_{i \in \Lambda_J} \langle h, P_J^* g_i \rangle f_i \tag{5.3.9}$$

$$= \sum_{i \in \Lambda_1} \langle P_1 h, g_i \rangle f_i + \cdots + \sum_{i \in \Lambda_j} \langle P_j h, g_i \rangle f_i + \cdots + \sum_{i \in \Lambda_J} \langle P_J h, g_i \rangle f_i \tag{5.3.10}$$

$$= \sum_{i \in \Lambda_1} \langle h_1, g_i \rangle f_i + \cdots + \sum_{i \in \Lambda_j} \langle h_j, g_i \rangle f_i + \cdots + \sum_{i \in \Lambda_J} \langle h_J, g_i \rangle f_i \tag{5.3.11}$$

$$= h_1 + \cdots + h_j + \cdots + h_J = h. \tag{5.3.12}$$

Therefore, $\cup_{j=1}^{J} G_j'$ is a dual of F with $\langle g_i', f_i \rangle = d_j$ for all $j \in \Lambda_j$. □

The following lemma gives us the precise value of $r_F^{(1)}$ for any given frame F.

Lemma 5.3.4. Let F be a frame and $\{d_j\}_{j=1}^{J}$ be its redundancy distribution. Then $r_F^{(1)} = \max\{d_j : j = 1, ..., J\}$. In particular, $r_F^{(1)}$ only takes rational values.

Proof. Let G be a dual of F with $r_{F,G}^{(1)} = r_F^{(1)}$. By Proposition 5.3.3, there exists a dual $\{g_i'\}_{i=1}^{N}$ of

F with $\langle g_i', f_i\rangle = d_j$ for all $i \in \Lambda_j$ and $1 \leq j \leq J$. Then

$$r_F^{(1)} \leq \max_{1\leq i\leq N} |\langle g_i', f_i\rangle| = \max_{1\leq j\leq J} d_j. \tag{5.3.13}$$

For the other side of the inequality, let Q_j be the orthogonal projection from H onto H_j, where $H_j = \mathrm{span}\{f_i : i \in \Lambda_j\}$. Then we show that for any dual $G = \{g_i\}_{i=1}^N$ of F, $\{Q_j g_i\}_{i\in\Lambda_j}$ is a dual of $\{f_i\}_{i\in\Lambda_j}$. Let $h' \in H_j$. Then we have that

$$\begin{aligned} h' &= \sum_{i=1}^N \langle h', g_i\rangle f_i = \sum_{i=1}^N \langle Q_j h', g_i\rangle f_i = \sum_{i=1}^N \langle h', Q_j g_i\rangle f_i && (5.3.14)\\ &= \sum_{i\in\Lambda_1} \langle h', Q_j g_i\rangle f_i + \cdots + \sum_{i\in\Lambda_j} \langle h', Q_j g_i\rangle f_i + \cdots + \sum_{i\in\Lambda_J} \langle h', Q_j g_i\rangle && (5.3.15)\\ &= \sum_{i\in\Lambda_j} \langle h', Q_j g_i\rangle f_i, && (5.3.16) \end{aligned}$$

where in the last equality we use the fact that H is the direct sum of H_j's. Thus, $\{Q_j g_i\}_{i\in\Lambda_j}$ is a dual of $\{f_i\}_{i\in\Lambda_j}$, as claimed.

Since $\max_{i\in\Lambda_j} |\langle Q_j g_i, f_i\rangle| \geq \frac{\dim H_j}{|\Lambda_j|} = d_j$ and $\langle Q_j g_i, f_i\rangle = \langle g_i, Q_j f_i\rangle = \langle g_i, f_i\rangle$ for all $i \in \Lambda_j$, for each j we get that $\max_{i\in\Lambda_j} |\langle g_i, f_i\rangle| \geq d_j$. This implies that

$$\max_{1\leq i\leq N} |\langle g_i, f_i\rangle| \geq \max_{1\leq j\leq J} d_j; \tag{5.3.17}$$

hence, $r_F^{(1)} \geq \max\{d_j : 1 \leq j \leq J\}$. Therefore we obtain $r_F^{(1)} = \max\{d_j : 1 \leq j \leq J\}$. $\square$

5.4 Characterization and Construction of Spectrally One Uniform Frames

In this section, we give the characterization and construction of spectrally one-uniform frames.

In the following theorem, with the help of Theorem 5.2.4, we are able to characterize all the frames that admit dual frames so that the maximal one-erasure reconstruction error is minimal.

Theorem 5.4.1. Let $F = \{f_i\}_{i=1}^N$ be a frame for H. Then the following are equivalent:

(i) F is spectrally one uniform frame;

(ii) F has the uniform redundancy distribution;

(iii) There exists a dual frame $G = \{g_i\}_{i=1}^N$ of F such that $\langle g_i, f_i \rangle = n/N$ for all i;

(iv) There exists a dual frame $G = \{g_i\}_{i=1}^N$ of F such that $|\langle g_i, f_i \rangle| = n/N$ for all i.

Proof. Clearly we have $(iii) \Rightarrow (iv)$ and $(iv) \Rightarrow (i)$.

"$(i) \Rightarrow (ii)$:" Let $r_F^{(1)} = \frac{n}{N}$. Assume for a contradiction that F does not have the uniform redundancy distribution. Then, there exists $j_0 \in \{1, \ldots, J\}$ such that $d_{j_0} < \frac{n}{N}$ since $d_j \leq \frac{n}{N}$ for all j by Lemma 5.3.4. Then, for $n_j = \dim H_j$ and $N_j = |\Lambda_j|$, we have

$$n = \sum_{j=1}^{J} n_j = \sum_{j=1}^{J} N_j d_j = \sum_{j \neq j_0} N_j d_j + N_{j_0} d_{j_0} \tag{5.4.1}$$

$$< (N - N_{j_0}) \frac{n}{N} + N_{j_0} \frac{n}{N} = n, \tag{5.4.2}$$

which is a contradiction. Thus, F has the uniform redundancy distribution.

"$(ii) \Rightarrow (iii)$:" Assume that F has the uniform redundancy distribution. Then $d_j = d$ for each $j = 1, \ldots, J$. Since G_j is a dual of F_j with $\langle g_i, f_i \rangle = d_j = d$ for all $i \in \Lambda_j$, $\langle g_i, f_i \rangle = d$ for all

$i \in \{1, \ldots, N\}$. The fact that $n = \sum_{i=1}^{N} \langle g_i, f_i \rangle$ implies $d = n/N$. Thus $\langle g_i, f_i \rangle = n/N$ for all i. $\square$

Our third main result tells us that frames with any prescribed redundancy distributions can be easily constructed.

Theorem 5.4.2. Let $d_j = \frac{n_j}{N_j} \in (0, 1)$ with the property that $\sum_{j=1}^{J} n_j = n$ and $\sum_{j=1}^{J} N_j = N$. Then there exists a frame $F = \{f_i\}_{i=1}^{N}$ such that its redundancy distribution is $\{d_j\}_{j=1}^{J}$. Moreover, such a frame F can be explicitly constructed out of any given basis of H.

Proof. Let $d_j = \frac{n_j}{N_j} \in (0, 1)$. Let $F = \{f_i\}_{i=1}^{n}$ be a basis for an n-dimensional Hilbert space H. Then we can decompose F into J subsets $F_j = \{f_i\}_{i \in \Lambda_j}$ with $\Lambda_j = \{n_{j-1} + 1, \ldots, n_{j-1} + n_j\}$ so that $\sum_{j=1}^{J} n_j = n$. Clearly, F_j is a basis for $H_j = \mathrm{span}\{f_i : i \in \Lambda_j\}$. For any j, pick any $N_j - n_j$ vectors, say $h_{j,k}$ for $1 \leq k \leq N_j - n_j$ with $h_{j,1} = \sum_{i=n_{j-1}+1}^{n_{j-1}+n_j} f_i$, from H_j. Note that the set $\{h_{j,1}, f_i : i \in \Lambda_j\}$ is connected. Indeed, since $h_{j,1}$ is connected with every f_i, $i \in \Lambda_j$, by the way it is chosen, f_i is connected with f_k for all $i \neq k$, $i, k \in \Lambda_j$ by the transitivity of connected vectors. Then, by Corollary 5.2.2, $F'_j = \{f_i : i \in \Lambda_j\} \cup \{h_{j,k} : 1 \leq k \leq N_j - n_j\}$ is a connected frame for H_j with cardinality N_j. Hence, $\frac{\dim H_j}{|\Lambda'_j|} = \frac{n_j}{N_j} = d_j$. By the construction of F'_j, for any fixed j none of the vectors in F'_j is connected with any of the vectors in F'_ℓ with $\ell \neq j$; thus, we have a frame $F' = \cup_{j=1}^{J} F'_j$ for H with redundancy distribution $\{d_j\}_{j=1}^{J}$. $\square$

Let $\mathcal{G}$ be a group and π is a unitary representation of $\mathcal{G}$ on a Hilbert space H. Then a *group representation frame* is a frame of the form $\{\pi(g)\phi\}_{g \in \mathcal{G}}$, where ϕ is a fixed vector in H. This type of frames have played important roles in establishing the connections of frame theory with representation theory, operator algebra theory, operator valued measures and applications (c.f.

[1, 23, 24, 25, 27, 28, 32, 33, 37]).

Corollary 5.4.3. (i) If F is a uniform Parseval frame, then it has the uniform redundancy distribution.

(ii) If F is a group representation frame, then it has the uniform redundancy distribution.

(iii) Assume that F has the uniform redundancy distribution and N and n are co-prime to each other. Then F is connected.

Proof. (i) Let $F = \{f_i\}_{i=1}^N$ be a uniform Parseval frame and S be the frame operator of F. Then, for all i,

$$|\langle S^{-1} f_i, f_i \rangle| = |\langle f_i, f_i \rangle| = \|f_i\|^2 = \frac{n}{N}. \tag{5.4.3}$$

By Theorem 5.4.1, F has the uniform redundancy distribution.

(ii) For a fixed vector $\phi \in H$, let $F = \{\pi(g)\phi\}_{g \in \mathcal{G}}$ be a group representation frame with associated frame operator S where π is a group representation of a finite group $\mathcal{G}$ on H. Then, since $\{S^{-1}\pi(g)\phi\}_{g \in \mathcal{G}} = \{\pi(g)S^{-1}\phi\}_{g \in \mathcal{G}}$ is a canonical dual of F, we have that

$$\langle \pi(g) S^{-1}\phi, \pi(g)\phi \rangle = \langle S^{-1}\phi, \phi \rangle = \langle S^{-1/2}\phi, S^{-1/2}\phi \rangle = \|S^{-1/2}\phi\| \tag{5.4.4}$$

is a constant for all $g \in \mathcal{G}$. Hence, by Theorem 5.4.1, F has the uniform redundancy distribution.

(iii) Let n and N be coprime and F be a frame with the uniform redundancy distribution. For

$n_j = \dim H_j$ and $N_j = |\Lambda_j|$, by Theorem 5.4.1, we have that

$$\frac{n}{N} = \frac{n_j}{N_j} \quad \text{for } 1 \le j \le J \tag{5.4.5}$$

with $n = \sum_{j=1}^{J} n_j$ and $N = \sum_{j=1}^{J} N_j$. Then $n_j N = n N_j$ for any fixed $j \in \{1, \ldots, J\}$. By assumption n does not divide N; so n divides n_j. Since $n \ge n_j$, we get $n = n_j$. This implies that $J = 1$. Thus F is a connected frame. $\square$

CHAPTER 6: SPECTRALLY TWO-UNIFORM FRAMES

The main aim of this chapter is to characterize spectrally two-uniform frames that have 2-optimal dual frames with respect to spectral radius measurement. At this moment, we do not have a complete characterization of two erasure spectrally optimal dual frames but we will present some partial results.

6.1 Spectrally Two Uniform Frames

In this section, we define spectrally two-uniform frames and give their relationship with two-erasure spectrally optimal dual frames.

Definition 6.1.1. Let F be an (N, n) frame. Then F is called **spectrally two-uniform frame** if there exists a dual frame G of F such that $\langle g_j, f_i\rangle\langle g_i, f_j\rangle = c$ is constant where $c = \frac{Nn-n^2}{N^2(N-1)}$.

Theorem 6.1.1. Let F be an (N, n) spectrally one-uniform frame. Then F is spectrally two uniform frame if and only if there exists a dual G such that $r_F^{(2)} = r_{F,G}^{(2)} = \frac{n}{N} + \sqrt{\frac{Nn-n^2}{N^2(N-1)}}$. In other words, F is spectrally two uniform frame if and only if F admits 2-erasure spectrally optimal dual frame.

Proof. Let F be a spectrally two uniform frame and G be a one-erasure spectrally optimal dual frame of F with $\langle g_j, f_i\rangle\langle g_i, f_j\rangle = \frac{Nn-n^2}{N^2(N-1)}$ for all $i \neq j$. Consider the error operator E_Λ for

$|\Lambda| = 2$. Then the spectral radius of error operator is

$$r(E_\Lambda) = r(\Theta_G^* D_\Lambda \Theta_F) = r(\Theta_G^* D_\Lambda^* D_\Lambda \Theta_F) = r(D_\Lambda \Theta_F \Theta_G^* D_\Lambda^*). \tag{6.1.1}$$

where Θ_F and Θ_G are analysis operators for F and G, respectively, and D_Λ is an N by N diagonal matrix with $d_{ii} = 1$ for $i \in \Lambda$ and zero otherwise. For the spectral radius of E_Λ, we consider the characteristic function of $A_{i,j}$;

$$A_{i,j} = \begin{pmatrix} \langle g_i, f_i \rangle & \langle g_j, f_i \rangle \\ \langle g_i, f_j \rangle & \langle g_j, f_j \rangle \end{pmatrix} \tag{6.1.2}$$

for $i \neq j$ and $i, j \in 1, \ldots, n$. The characteristic function is

$$(\langle g_i, f_i \rangle - \lambda)(\langle g_j, f_j \rangle - \lambda) - \langle g_j, f_i \rangle \langle g_i, f_j \rangle = 0. \tag{6.1.3}$$

Since G is 1-erasure spectrally optimal dual frame, i.e., $\langle g_i, f_i \rangle = n/N$, the characteristic function becomes

$$\left(\frac{n}{N} - \lambda\right)^2 - \langle g_j, f_i \rangle \langle g_i, f_j \rangle = 0, \tag{6.1.4}$$

which leads to

$$\lambda = \frac{n}{N} \pm \sqrt{\langle g_j, f_i \rangle \langle g_i, f_j \rangle}. \tag{6.1.5}$$

Since F is spectrally two uniform frame, $\langle g_j, f_i\rangle\langle g_i, f_j\rangle = \frac{nN-n^2}{N^2(N-1)}$ for all $i \neq j$. Therefore,

$$r^{(2)}_{F,G} = \frac{n}{N} + \sqrt{\frac{nN-n^2}{N^2(N-1)}}. \tag{6.1.6}$$

Assume that $c_{ji} = \langle g_j, f_i\rangle\langle g_i, f_j\rangle$ is not constant for all $j \neq i$. Then

$$\sum_{j,i} |c_{ji}| \geq \sum_{j,i} c_{ji} = \frac{nN-n^2}{N}. \tag{6.1.7}$$

This implies that

$$\max_{i,j} |c_{ji}| \geq \frac{nN-n^2}{N^2(N-1)} = c. \tag{6.1.8}$$

Thus, there exist j_0, i_0 such that $|c_{j_0 i_0}| \geq c$. Then

$$\begin{aligned}
\big|\frac{n}{N} \pm (c_{j_0i_0})^{1/2}\big| &= \big|\frac{n}{N} \pm |c_{j_0i_0}|^{1/2} e^{i\theta/2}\big| && (6.1.9)\\
&= \big|\frac{n}{N} \pm |c_{j_0i_0}|^{1/2}\cos\frac{\theta}{2} \pm i|c_{j_0i_0}|^{1/2}\sin\frac{\theta}{2}\big| && (6.1.10)\\
&= \Big(\big(\frac{n}{N} \pm |c_{j_0i_0}|^{1/2}\cos\frac{\theta}{2}\big)^2 + |c_{j_0i_0}|\sin^2\frac{\theta}{2}\Big)^{1/2} && (6.1.11)\\
&= \Big(\frac{n^2}{N^2} + |c_{j_0i_0}|\cos^2\frac{\theta}{2} \pm 2\frac{n}{N}|c_{j_0i_0}|^{1/2}\cos\frac{\theta}{2} + |c_{j_0i_0}|\sin^2\frac{\theta}{2}\Big)^{1/2} && (6.1.12)\\
&= \Big(\frac{n^2}{N^2} + |c_{j_0i_0}| \pm 2\frac{n}{N}|c_{j_0i_0}|^{1/2}\cos\frac{\theta}{2}\Big)^{1/2}. && (6.1.13)
\end{aligned}$$

Hence

$$\max\left|\frac{n}{N}+(c_{j_0i_0})^{1/2}\right| \tag{6.1.14}$$

$$= \max\left\{\left(\frac{n^2}{N^2}+|c_{j_0i_0}|+2\frac{n}{N}|c_{j_0i_0}|^{1/2}\cos\frac{\theta}{2}\right)^{1/2},\ \left(\frac{n^2}{N^2}+|c_{j_0i_0}|-2\frac{n}{N}|c_{j_0i_0}|^{1/2}\cos\frac{\theta}{2}\right)^{1/2}\right\} \tag{6.1.15}$$

$$\geq \left(\frac{n^2}{N^2}+c+\frac{2n}{N}c^{1/2}\right)^{1/2} = \frac{n}{N}+c^{1/2}. \tag{6.1.16}$$

This implies that

$$r_F^{(2)} = \min_G \max\{r(A_{i,j})\} = \frac{n}{N}+c^{1/2} = \frac{n}{N}+\sqrt{\frac{nN-n^2}{N^2(N-1)}}. \tag{6.1.17}$$

Hence, $r_F^{(2)} = r_{F,G}^{(2)}$.

For the other side of the proof, assume that F has a one-erasure spectrally optimal dual G such that $r_{F,G}^{(2)} = \frac{n}{N}+\sqrt{\frac{nN-n^2}{N^2(N-1)}}$. Assume that there exist a pair of indices (i_0,j_0) such that $\langle g_{j_0},f_{i_0}\rangle\langle g_{i_0},f_{j_0}\rangle < \frac{nN-n^2}{N^2(N-1)}$. Because of the fact

$$\sum_{j=1}^{N}\sum_{i=1}^{N}\langle g_j,f_i\rangle\langle g_i,f_j\rangle = \sum_{j=1}^{N}\langle g_j,f_j\rangle = n. \tag{6.1.18}$$

there exist a pair of indices (i_1,j_1) such that $\langle g_{j_1},f_{i_1}\rangle\langle g_{i_1},f_{j_1}\rangle > \frac{nN-n^2}{N^2(N-1)}$. This implies that

$$r_{F,G}^{(2)} > \frac{n}{N}+\sqrt{\frac{nN-n^2}{N^2(N-1)}}. \tag{6.1.19}$$

However, this contradicts to the assumption. Therefore, $\langle g_j,f_i\rangle\langle g_i,f_j\rangle$ has to be constant for all

$i, j, i \neq j$. This implies that F is spectrally two uniform frame. $\square$

6.2 Spectrally Two-uniform Frames for $N = n + 1$ and $N = n + 2$

In this section, we give the characterization of spectrally two uniform frames when $N = n+1$ and give a necessary condition on frames for $N = n + 2$.

Proposition 6.2.1. Let $F = \{f_i\}_{i=1}^N$ be a spectrally one-uniform frame for an n-dimensional H. If $N = n + 1$, then F is a spectrally 2-uniform frame if and only if F is n-independent.

Proof. First of all, we assume that F is an n-independent frame and we show that F is a spectrally 2-uniform frame. Let $F = \left\{e_1, \dots, e_n, \sum_{i=1}^n a_i e_i\right\}$ be a frame where $\{e_i\}_{i=1}^n$ is the standard orthonormal basis for an n-dimensional Hilbert space. Because F is spectrally one-uniform frame, it has a 1-erasure spectrally optimal dual, say G. Then by the definition of frames we have

$$e_i = \sum_{i=1}^{n+1} \langle e_i, f_i \rangle g_i = g_i + a_i g_{n+1}, \qquad \text{for } i = 1, \dots, n. \tag{6.2.1}$$

Since G is 1-erasure spectrally optimal,

$$\langle f_i, g_i \rangle = \langle e_i, g_i \rangle = \frac{n}{n+1} \qquad \text{for } i = 1, \dots, n, \tag{6.2.2}$$

$$\langle f_{n+1}, g_{n+1} \rangle = \sum_{i=1}^n a_i \langle e_i, g_{n+1} \rangle = \frac{n}{n+1}. \tag{6.2.3}$$

Moreover, by 6.2.1 we have ,

$$\langle e_i, e_i \rangle = \langle e_i, g_i \rangle + a_i \langle e_i, g_{n+1} \rangle = 1 \qquad \text{for } i = 1, \ldots, n, \tag{6.2.4}$$

$$\langle e_j, e_i \rangle = \langle e_j, g_i \rangle + a_i \langle e_j, g_{n+1} \rangle = 0 \qquad \text{for } j \neq i. \tag{6.2.5}$$

Therefore, for $k \neq \ell$, $k, \ell = 1, \ldots, n$,

$$\begin{aligned}
\langle f_k, g_\ell \rangle \langle f_\ell, g_k \rangle &= \langle e_k, g_\ell \rangle \langle e_\ell, g_k \rangle && (6.2.6)\\
&= -a_\ell \langle e_k, g_{n+1} \rangle \cdot -a_k \langle e_\ell, g_{n+1} \rangle && (6.2.7)\\
&= a_\ell \frac{1 - \langle e_k, g_k \rangle}{a_k} \cdot a_k \frac{1 - \langle e_\ell, g_\ell \rangle}{a_\ell} && (6.2.8)\\
&= \Big(1 - \frac{n}{n+1}\Big)\Big(1 - \frac{n}{n+1}\Big) = \frac{1}{(n+1)^2}, && (6.2.9)
\end{aligned}$$

which is equivalent to $\frac{nN-n^2}{N^2(N-1)}$ where in this case $N = n+1$. Furthermore, for $k = 1, \ldots, n$ we have

$$\langle f_k, g_{n+1}\rangle\langle f_{n+1}, g_k\rangle = \langle e_k, g_{n+1}\rangle\langle\sum_{i=1}^{n} a_i e_i, g_k\rangle \tag{6.2.10}$$

$$= \langle e_k, g_{n+1}\rangle\Bigg(\sum_{i=1}^{n} a_i\langle e_i, g_k\rangle\Bigg) \tag{6.2.11}$$

$$= \frac{1-\langle e_k, g_k\rangle}{a_k}\Bigg(\sum_{i\neq k} a_i\langle e_i, g_k\rangle + a_k\langle e_k, g_k\rangle\Bigg) \tag{6.2.12}$$

$$= \frac{1-\langle e_k, g_k\rangle}{a_k}\Bigg(\sum_{i\neq k} -a_k a_i\langle e_i, g_{n+1}\rangle + a_k(1-a_k\langle e_k, g_{n+1}\rangle)\Bigg) \tag{6.2.13}$$

$$= \frac{1-\langle e_k, g_k\rangle}{a_k}a_k\Bigg(1-\sum_{i=1}^{n} a_i\langle e_i, g_{n+1}\rangle\Bigg) \tag{6.2.14}$$

$$\frac{1}{a_k(n+1)}a_k\Big(1-\frac{n}{n+1}\Big) = \frac{1}{(n+1)^2} \tag{6.2.15}$$

since $\langle e_k, g_k\rangle = n/(n+1)$ and $\sum_{i=1}^{n} a_i\langle e_i, g_{n+1}\rangle = n/(n+1)$.

This proves that if F is n-independent then F is spectrally 2-uniform frame.

For the other side of the proof, assume that F is not n-independent such that $F = \{e_1, \ldots, e_n, \sum_{i=1}^{s} a_i e_i\}$ for $s < n$. Let G be a 1-erasure spectrally optimal dual frame. Then by the definition of frame we have

$$e_i = g_i + a_i g_{n+1} \qquad \text{for } i = 1, \ldots, s \tag{6.2.16}$$

$$e_i = g_i \qquad \text{for } i = s+1, \ldots, n, \tag{6.2.17}$$

and

$$\langle e_i, e_i\rangle = \langle e_i, g_i\rangle + a_i\langle e_i, g_{n+1}\rangle = 1 \qquad \text{for } i = 1, \ldots, s, \tag{6.2.18}$$

$$\langle e_j, e_i\rangle = \langle e_j, g_i\rangle + a_i\langle e_j, g_{n+1}\rangle = 0 \qquad \text{for } i = 1, \ldots, s,\ j \neq i, \tag{6.2.19}$$

$$\langle e_i, e_i\rangle = \langle e_i, g_i\rangle = 1 \qquad \text{for } i = s+1, \ldots, n, \tag{6.2.20}$$

$$\langle e_j, e_i\rangle = \langle e_j, g_i\rangle = 0 \qquad \text{for } i = s+1, \ldots, n,\ j \neq i. \tag{6.2.21}$$

Since G is 1-erasure spectrally optimal dual of F, we have $\langle f_i, g_i\rangle = \langle e_i, g_i\rangle = n/(n+1)$ for $i = 1, \ldots, n+1$ and

$$\langle f_{n+1}, g_{n+1}\rangle = \sum_{i=1}^{s} a_i\langle e_i, g_{n+1}\rangle = \frac{n}{n+1}. \tag{6.2.22}$$

Note that since $\langle e_j, g_n\rangle = 0$ for $j \neq n$ by 6.2.21 we have

$$\langle f_{n+1}, g_n\rangle = \sum_{i=1}^{s} a_i\langle e_i, g_n\rangle = 0 \tag{6.2.23}$$

Thus, $\langle f_n, g_{n+1}\rangle\langle f_{n+1}, g_n\rangle = 0$. Therefore, F is not spectrally 2-uniform frame. $\square$

Proposition 6.2.2. Let F be a spectrally one uniform frame. For $N = n+2$, if F is a spectrally 2-uniform frame, then F is n-independent.

Proof. Assume that F is not n-independent. Let $F = \{e_1, \ldots, e_n, \sum_{i=1}^{s} a_i e_i, \sum_{i=1}^{n} b_i e_i\}$ for $s < n$

and G be a 1-erasure spectrally optimal dual of F. Then by the definition of frame we have

$$e_i = g_i + a_i g_{n+1} + b_i g_{n+2} \quad \text{for } i = 1, \ldots, s \tag{6.2.24}$$

$$e_i = g_i + b_i g_{n+2} \qquad \text{for } i = s+1, \ldots, n. \tag{6.2.25}$$

Moreover, we have

$$\langle e_i, e_i \rangle = \langle e_i, g_i \rangle + a_i \langle e_i, g_{n+1} \rangle + b_i \langle e_i, g_{n+2} \rangle = 1 \qquad \text{for } i = 1, \ldots, s, \tag{6.2.26}$$

$$\langle e_j, e_i \rangle = \langle e_j, g_i \rangle + a_i \langle e_j, g_{n+1} \rangle + b_i \langle e_j, g_{n+2} \rangle = 0 \qquad \text{for } i = 1, \ldots, s,\ j \neq i, \tag{6.2.27}$$

$$\langle e_i, e_i \rangle = \langle e_i, g_i \rangle + b_i \langle e_i, g_{n+2} \rangle = 1 \qquad \text{for } i = s+1, \ldots, n, \tag{6.2.28}$$

$$\langle e_j, e_i \rangle = \langle e_j, g_i \rangle + b_i \langle e_j, g_{n+2} \rangle = 0 \qquad \text{for } i = s+1, \ldots, n,\ j \neq i. \tag{6.2.29}$$

Since G is 1-erasure spectrally optimal dual frame, we have

$$\langle f_i, g_i \rangle = \langle e_i, g_i \rangle = \frac{n}{n+2} \quad \text{for } i = 1, \ldots, n, \tag{6.2.30}$$

$$\langle f_{n+1}, g_{n+1} \rangle = \sum_{i=1}^{s} a_i \langle e_i, g_{n+1} \rangle = \frac{n}{n+2}, \tag{6.2.31}$$

$$\langle f_{n+2}, g_{n+2} \rangle = \sum_{i=1}^{n} b_i \langle e_i, g_{n+2} \rangle = \frac{n}{n+2}. \tag{6.2.32}$$

Note that by 6.2.28 we have

$$\langle f_{s+1}, g_{n+2} \rangle = \langle e_{s+1}, g_{n+2} \rangle = \frac{1 - n/(n+2)}{b_{s+1}} = \frac{2}{b_{s+1}(n+2)} \tag{6.2.33}$$

and since $\sum_{i\neq s+1} b_i\langle e_i, g_{n+2}\rangle = n/(n+2) - b_{s+1}\langle e_{s+1}, g_{n+2}\rangle$ by 6.2.32, we have

$$\langle f_{n+2}, g_{s+1}\rangle = \sum_{i=1}^{n} b_i\langle e_i, g_{s+1}\rangle = \sum_{i\neq s+1} b_i\langle e_i, g_{s+1}\rangle + b_{s+1}\langle e_{s+1}, g_{s+1}\rangle \tag{6.2.34}$$

$$= \sum_{i\neq s+1} b_i\langle e_i, g_{s+1}\rangle + b_{s+1}\frac{n}{n+2} \tag{6.2.35}$$

$$= \sum_{i\neq s+1} b_i(-b_{s+1}\langle e_i, g_{n+2}\rangle) + b_{s+1}\frac{n}{n+2} \tag{6.2.36}$$

$$= b_{s+1}\Big(c - \sum_{i\neq s+1} b_i\langle e_i, g_{n+2}\rangle\Big) \tag{6.2.37}$$

$$= b_{s+1}\Big(n/(n+2) - (n/(n+2) - b_{s+1}\langle e_{s+1}, g_{n+2}\rangle)\Big) \tag{6.2.38}$$

$$= b_{s+1}^2\frac{1-n/(n+2)}{b_{s+1}} = b_{s+1}\frac{2}{n+2}. \tag{6.2.39}$$

where the last equality follows from 6.2.28. Thus,

$$\langle f_{s+1}, g_{n+2}\rangle\langle f_{n+2}, g_{k+1}\rangle = \frac{2}{b_{s+1}(n+2)}b_{s+1}\frac{2}{n+2} = \frac{4}{(n+2)^2}. \tag{6.2.40}$$

However, if F were a 2-uniform frame then $\frac{4}{(n+2)^2} = \frac{n(n+2)-n^2}{(n+2)^2(n+1)}$. But this is impossible since $2n \neq 4n+4$. Hence, F is not a 2-uniform frame. $\square$

Let $F = \{e_1, e_2, a_1e_1 + a_2e_2, b_1e_1 + b_2e_2\}$ be a frame and $G = \{g_1, g_2, g_3, g_4\}$ be a 1-erasure spectrally optimal dual frame of F. In the frame definition letting $f = e_1$ and $f = e_2$, respectively, we have

$$e_1 = g_1 + a_1 g_3 + b_1 g_4 \tag{6.2.41}$$

$$e_2 = g_2 + a_2 g_3 + b_2 g_4. \tag{6.2.42}$$

Since G is 1-erasure spectrally optimal dual frame,

$$\langle e_1, g_1 \rangle = \frac{1}{2} = \langle e_2, g_2 \rangle = \langle f_3, g_3 \rangle = \langle f_4, g_4 \rangle. \tag{6.2.43}$$

Then we have

$$\langle e_1, e_1 \rangle = \langle e_1, g_1 \rangle + a_1 \langle e_1, g_3 \rangle + b_1 \langle e_1, g_4 \rangle = 1 \tag{6.2.44}$$

$$\langle e_2, e_2 \rangle = \langle e_2, g_2 \rangle + a_2 \langle e_2, g_3 \rangle + b_2 \langle e_2, g_4 \rangle = 1 \tag{6.2.45}$$

$$\langle e_2, e_1 \rangle = \langle e_2, g_1 \rangle + a_1 \langle e_2, g_3 \rangle + b_1 \langle e_2, g_4 \rangle = 0 \tag{6.2.46}$$

$$\langle e_1, e_2 \rangle = \langle e_1, g_2 \rangle + a_2 \langle e_1, g_3 \rangle + b_2 \langle e_1, g_4 \rangle = 0 \tag{6.2.47}$$

$$\langle f_3, g_3 \rangle = a_1 \langle e_1, g_3 \rangle + a_2 \langle e_2, g_3 \rangle = \frac{1}{2} \tag{6.2.48}$$

$$\langle f_4, g_4 \rangle = b_1 \langle e_1, g_4 \rangle + b_2 \langle e_2, g_4 \rangle = \frac{1}{2}. \tag{6.2.49}$$

Set $\langle e_1, g_4 \rangle = x$ and $\langle e_2, g_4 \rangle = y$. Then by 6.2.44 and 6.2.45, we obtain

$$\langle e_1, g_3 \rangle = \frac{1}{2a_1} - \frac{b_1}{a_1} x \tag{6.2.50}$$

$$\langle e_2, g_3 \rangle = \frac{1}{2a_2} - \frac{b_2}{a_2} y. \tag{6.2.51}$$

By 6.2.48 or 6.2.49, we get a relation between x and y such that

$$x = \frac{1}{2b_1} - \frac{b_2}{b_1}y. \tag{6.2.52}$$

Thus, 6.2.50 becomes

$$\langle e_1, g_3\rangle = \frac{1}{2a_1} - \frac{b_1}{a_1}\big(\frac{1}{2b_1} - \frac{b_2}{b_1}y\big) = \frac{b_2}{a_1}y. \tag{6.2.53}$$

Now we find the pairs by solving the identities in 6.2.46 and 6.2.47 for $\langle e_2, g_1\rangle$ and $\langle e_1, g_2\rangle$ respectively.

$$\begin{aligned}
\langle e_1, g_2\rangle\langle e_2, g_1\rangle =&(-b_2x - a_2\langle e_1, g_3\rangle)(-a_1\langle e_2, g_3\rangle - b_1y) && (6.2.54)\\
=&\bigg(-b_2\big(\frac{1}{2b_1} - \frac{b_2}{b_1}y\big) - a_2\frac{b_2}{a_1}y\bigg)\bigg(-a_1\big(\frac{1}{2a_2} - \frac{b_2}{a_2}y\big) - b_1y\bigg) && (6.2.55)\\
=&\bigg(-\frac{b_2}{2b_1} + \frac{b_2^2}{b_1}y - a_2\frac{b_2}{a_1}y\bigg)\bigg(-\frac{a_1}{2a_2} + \frac{a_1b_2}{a_2}y - b_1y\bigg) && (6.2.56)\\
=&\frac{-a_1b_2 - 2a_2b_1b_2y + 2a_1b_2^2y}{2a_1b_1}\frac{2a_1b_2y - 2a_2b_1y - a_1}{2a_2} && (6.2.57)\\
=&\frac{-2a_1^2b_2^2y + 2a_1a_2b_1b_2y + a_1^2b_2 - 4a_1a_2b_1b_2^2y^2 + 4a_2^2b_1^2b_2y^2}{4a_1a_2b_1} && (6.2.58)\\
&+\frac{2a_1a_2b_1b_2y + 4a_1^2b_2^3y^2 - 4a_1a_2b_1b_2^2y^2 - 2a_1^2b_2^2y}{4a_1a_2b_1} && (6.2.59)\\
=&\frac{(4a_2^2b_1^2b_2 + 4a_1^2b_2^3 - 8a_1a_2b_1b_2^2)y^2 + (4a_1a_2b_1b_2 - 4a_1^2b_2^2)y + a_1^2b_2}{4a_1a_2b_1}. && (6.2.60)
\end{aligned}$$

$$
\begin{aligned}
\langle e_1, g_3\rangle\langle f_3, g_1\rangle =& \langle e_1, g_3\rangle(a_1\langle e_1, g_1\rangle + a_2\langle e_2, g_1\rangle) && (6.2.61)\\
=& \frac{b_2}{a_1}y\bigg(\frac{a_1}{2} + a_2\big(-a_1\langle e_2, g_3\rangle - b_1\langle e_2, g_4\rangle\big)\bigg) && (6.2.62)\\
=& \frac{b_2}{a_1}y\bigg(\frac{a_1}{2} - a_1a_2\big(\frac{1}{2a_2} - \frac{b_2}{a_2}y\big) - a_2b_1y\bigg) && (6.2.63)\\
=& \frac{b_2}{a_1}y(a_1b_2y - a_2b_1y) && (6.2.64)\\
=& \frac{b_2}{a_1}y^2(a_1b_2 - a_2b_1). && (6.2.65)
\end{aligned}
$$

$$
\begin{aligned}
\langle e_1, g_4\rangle\langle f_4, g_1\rangle =& x(b_1\langle e_1, g_1\rangle + b_2\langle e_2, g_1\rangle) && (6.2.66)\\
=& \big(\frac{1}{2b_1} - \frac{b_2}{b_1}y\big)\bigg(\frac{b_1}{2} + b_2\big(-a_1\langle e_2, g_3\rangle - b_1\langle e_2, g_4\rangle\big)\bigg) && (6.2.67)\\
=& \frac{1-2b_2y}{2b_1}\bigg(\frac{b_1}{2} - a_1b_2\big(\frac{1}{2a_2} - \frac{b_2}{a_2}y\big) - b_1b_2y\bigg) && (6.2.68)\\
=& \frac{1-2b_2y}{2b_1}\frac{a_2b_1 - a_1b_2 + 2a_1b_2^2y - 2a_2b_1b_2y}{2a_2} && (6.2.69)\\
=& \frac{a_2b_1 - a_1b_2 + 2a_1b_2^2y - 2a_2b_1b_2y - 2a_2b_1b_2y + 2a_1b_2^2y - 4a_1b_2^3y^2 + 4a_2b_1b_2^2y^2}{4a_2b_1}\\
=& \frac{(4a_2b_1b_2^2 - 4a_1b_2^3)y^2 + (4a_1b_2^2 - 4a_2b_1b_2)y + a_2b_1 - a_1b_2}{4a_2b_1}. && (6.2.70)
\end{aligned}
$$

$$
\begin{aligned}
\langle e_2, g_3\rangle\langle f_3, g_2\rangle =&\langle e_2, g_3\rangle(a_1\langle e_1, g_2\rangle + a_2\langle e_2, g_2\rangle) && (6.2.71)\\
=&\big(\frac{1}{2a_2} - \frac{b_2}{a_2}y\big)\big(a_1(-a_2\langle e_1, g_3\rangle - b_2\langle e_1, g_4\rangle) + \frac{a_2}{2}\big) && (6.2.72)\\
=&\frac{1-2b_2y}{2a_2}\big(- a_1a_2\frac{b_2}{a_1}y - a_1b_2x + \frac{a_2}{2}\big) && (6.2.73)\\
=&\frac{1-2b_2y}{2a_2}\Bigg(- a_2b_2y - a_1b_2\big(\frac{1}{2b_1} - \frac{b_2}{b_1}y\big) + \frac{a_2}{2}\Bigg) && (6.2.74)\\
=&\frac{1-2b_2y}{2a_2}\frac{-2a_2b_1b_2y - a_1b_2 + 2a_1b_2^2y + a_2b_1}{2b_1} && (6.2.75)\\
=&\frac{-2a_2b_1b_2y - a_1b_2 + 2a_1b_2^2y + a_2b_1 + 4a_2b_1b_2^2y^2 + 2a_1b_2^2y - 4a_1b_2^3y^2 - 2a_2b_1b_2y}{4a_2b_1}\\
=&\frac{(4a_2b_1b_2^2 - 4a_1b_2^3)y^2 + (4a_1b_2^2 - 4a_2b_1b_2)y + a_2b_1 - a_1b_2}{4a_2b_1}. && (6.2.76)
\end{aligned}
$$

$$
\begin{aligned}
\langle e_2, g_4\rangle\langle f_4, g_2\rangle =&y(b_1\langle e_1, g_2\rangle + b_2\langle e_2, g_2\rangle) = y\big(b_1(-a_2\langle e_1, g_3\rangle - b_2\langle e_1, g_4\rangle) + \frac{b_2}{2}\big) && (6.2.77)\\
=&y\big(- a_2b_1\frac{b_2}{a_1}y - b_1b_2x + \frac{b_2}{2}\big) && (6.2.78)\\
=&y\Bigg(- \frac{a_2b_1b_2}{a_1}y - b_1b_2\big(\frac{1}{2b_1} - \frac{b_2}{b_1}y\big) + \frac{b_2}{2}\Bigg) && (6.2.79)\\
=&y\big(- \frac{a_2b_1b_2}{a_1}y + b_2^2y\big) && (6.2.80)\\
=&y^2\frac{b_2}{a_1}(a_1b_2 - a_2b_1). && (6.2.81)
\end{aligned}
$$

$$
\begin{aligned}
\langle f_3, g_4\rangle\langle f_4, g_3\rangle =&(a_1\langle e_1, g_4\rangle + a_2\langle e_2, g_4\rangle)(b_1\langle e_1, g_3\rangle + b_2\langle e_2, g_3\rangle) && (6.2.82)\\
=&(a_1x + a_2y)\left(b_1\frac{b_2}{a_1}y + b_2\big(\frac{1}{2a_2} - \frac{b_2}{a_2}y\big)\right) && (6.2.83)\\
=&\big(\frac{a_1}{2b_1} - \frac{a_1b_2}{b_1}y + a_2y\big)\frac{a_1b_2 - 2a_1b_2^2y + 2a_2b_1b_2y}{2a_1a_2} && (6.2.84)\\
=&\frac{a_1 + 2a_2b_1y - 2a_1b_2y}{2b_1}\frac{a_1b_2 - 2a_1b_2^2y + 2a_2b_1b_2y}{2a_1a_2} && (6.2.85)\\
=&\frac{a_1^2b_2 - 2a_1^2b_2^2y + 2a_1a_2b_1b_2y - 4a_1a_2b_1b_2^2y^2 + 2a_1a_2b_1b_2y}{4a_1a_2b_1} && (6.2.86)\\
&+\frac{4a_2^2b_1^2b_2y^2 - 2a_1^2b_2^2y + 4a_1^2b_2^3y^2 - 4a_1a_2b_1b_2^2y^2}{4a_1a_2b_1} && (6.2.87)\\
=&\frac{(4a_2^2b_1^2b_2 + 4a_1^2b_2^3 - 8a_1a_2b_1b_2^2)y^2 + (4a_1a_2b_1b_2 - 4a_1^2b_2^2)y + a_1^2b_2}{4a_1a_2b_1}. && (6.2.88)
\end{aligned}
$$

We observe that equation in 6.2.60 is equal to equation in 6.2.88, equation in 6.2.65 is equal to equation in 6.2.81 and equation in 6.2.70 is equal to equation in 6.2.76. If (F, G) is two uniform frame pair, then all these six equations has to equal to 1/12 since $\frac{nN-n^2}{N^2(N-1)} = \frac{1}{12}$. Now set equation in 6.2.65 to 1/12 and solve for y^2, we get

$$
y^2 = \frac{a_1}{12b_2(a_1b_2 - a_2b_1)}. \tag{6.2.89}
$$

Substituting y^2 to equation in 6.2.60 and setting equation in 6.2.60 to 1/12, we have

$$
\frac{\frac{a_1}{12b_2(a_1b_2-a_2b_1)}4b_2(a_1b_2 - a_2b_1)^2 + 4a_1b_2(a_2b_1 - a_1b_2)y + a_1^2b_2}{4a_1a_2b_1} = \frac{1}{12}. \tag{6.2.90}
$$

After simplification, we get

$$a_1b_2 - a_2b_1 + 12b_2y(a_2b_1 - a_1b_2) + a_1b_2 = a_2b_1. \tag{6.2.91}$$

Solving the equation for y, we get

$$y = \frac{1}{6b_2}. \tag{6.2.92}$$

By 6.2.89 and 6.2.92, we have

$$\frac{a_1}{12b_2(a_1b_2 - a_2b_1)} = \frac{1}{36b_2^2}, \quad \text{i.e.,} \tag{6.2.93}$$

$$a_2b_1 = -2a_1b_2. \tag{6.2.94}$$

Finally, substituting the values of y^2, y and a_2b_1 into the third equation 6.2.70, we get

$$\begin{aligned}
\langle e_1, g_4\rangle\langle f_4, g_1\rangle =& \frac{y^2 = \frac{a_1}{12b_2(a_1b_2-a_2b_1)} 4b_2^2(a_2b_1 - a_1b_2) + 4b_2(a_1b_2 - a_2b_1)\frac{1}{6b_2} + a_2b_1 - a_1b_2}{4a_2b_1} \\
=& -\frac{\frac{a_1b_2}{3} + \frac{2}{3}3a_1b_2 - 3a_1b_2}{-8a_1b_2} && (6.2.95) \\
=& -\frac{1}{6} \neq \frac{1}{12}. && (6.2.96)
\end{aligned}$$

We get a contradiction. Therefore, F is not spectrally two uniform frame.

CHAPTER 7: EXAMPLES

In this chapter, we give some examples on frames. First one is an example on the frame whose standard dual frame is one erasure optimal dual frame with respect to norm measurement but the standard dual frame is not one erasure spectrally optimal. Second example is on a frame that attains a dual frame that is 1-erasure optimal dual frame with respect to norm measurement but not a 1-erasure spectrally optimal dual frame. The last one is an example on how we find a one-erasure spectrally optimal dual frame of a given frame using the partition idea of frames with respect to the redundancy of partitioned frames.

The following frame in two dimension is an example whose standard dual is 1-erasure optimal with respect to norm measure but it is not 1-erasure spectrally optimal dual frame.

Example 7.2.1. Let F be a $(3, 2)$ frame such that

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} a \\ a \end{bmatrix} \right\}, \tag{7.2.97}$$

where $a^2 = \dfrac{1+\sqrt{3}}{2}$. Then, the frame operator S and its inverse S^{-1} are given by

$$S = \begin{bmatrix} 1+a^2 & a^2 \\ a^2 & 1+a^2 \end{bmatrix} \quad \text{and} \quad S^{-1} = \begin{bmatrix} \frac{1+a^2}{1+2a^2} & -\frac{a^2}{1+2a^2} \\ -\frac{a^2}{1+2a^2} & \frac{1+a^2}{1+2a^2} \end{bmatrix}, \tag{7.2.98}$$

respectively. Thus, the standard dual $S^{-1}F$ of F is

$$\left\{ \begin{bmatrix} \frac{1+a^2}{1+2a^2} \\ -\frac{a^2}{1+2a^2} \end{bmatrix}, \begin{bmatrix} -\frac{a^2}{1+2a^2} \\ \frac{1+a^2}{1+2a^2} \end{bmatrix}, \begin{bmatrix} \frac{a}{1+2a^2} \\ \frac{a}{1+2a^2} \end{bmatrix} \right\}. \tag{7.2.99}$$

Note that

$$\max_{i=1,2,3} \left\{ \|f_i\| \|S^{-1} f_i\| \right\} = \max \left\{ \frac{\sqrt{1+2a^2+2a^4}}{1+2a^2}, \frac{\sqrt{1+2a^2+2a^4}}{1+2a^2}, \frac{2a^2}{1+2a^2} \right\} \tag{7.2.100}$$

$$= \max \left\{ \frac{1+\sqrt{3}}{2+\sqrt{3}}, \frac{1+\sqrt{3}}{2+\sqrt{3}}, \frac{1+\sqrt{3}}{2+\sqrt{3}} \right\} \tag{7.2.101}$$

$$= \frac{1+\sqrt{3}}{2+\sqrt{3}} \approx 0.7321. \tag{7.2.102}$$

Therefore, by Theorem 2.3 in [44], we conclude that $S^{-1}F$ is the optimal dual of F with respect to norm measure. Furthermore,

$$\max_{i=1,2,3} \left\{ |\langle S^{-1} f_i, f_i \rangle| \right\} = \max \left\{ \frac{1+a^2}{1+2a^2}, \frac{1+a^2}{1+2a^2}, \frac{a^2}{1+2a^2} \right\} \tag{7.2.103}$$

$$= \max \left\{ \frac{3+\sqrt{3}}{4+2\sqrt{3}}, \frac{3+\sqrt{3}}{4+2\sqrt{3}}, \frac{1+\sqrt{3}}{2+\sqrt{3}} \right\} \tag{7.2.104}$$

$$= \frac{1+\sqrt{3}}{2+\sqrt{3}} \approx 0.7321. \tag{7.2.105}$$

Thus,

$$\delta_{F,S^{-1}F} = r_{F,S^{-1}F} \approx 0.7321. \tag{7.2.106}$$

On the other hand, F has an optimal dual G that is

$$G = \left\{ \begin{bmatrix} \frac{1+a^2}{1+2a^2} + b \\ -\frac{a^2}{1+2a^2} + b \end{bmatrix}, \begin{bmatrix} -\frac{a^2}{1+2a^2} + b \\ \frac{1+a^2}{1+2a^2} + b \end{bmatrix}, \begin{bmatrix} \frac{a^2}{1+2a^2} - ab \\ \frac{a^2}{1+2a^2} - ab \end{bmatrix} \right\}, \tag{7.2.107}$$

where $b = \dfrac{\sqrt{3}-1}{(2+\sqrt{3})(4+2\sqrt{3})}$. And,

$$\max_{i=1,2,3} \{|\langle g_i, f_i\rangle|\} = \max \left\{ \frac{4+3\sqrt{3}}{7+4\sqrt{3}}, \frac{4+3\sqrt{3}}{7+4\sqrt{3}}, \frac{4+3\sqrt{3}}{7+4\sqrt{3}} \right\} = \frac{4+3\sqrt{3}}{7+4\sqrt{3}} \approx 0.6603. \tag{7.2.108}$$

Hence,

$$r_{F,G} < r_{F,S^{-1}F}. \tag{7.2.109}$$

Therefore, we can find a frame F whose standard dual $S^{-1}F$ is optimal under norm measure and has an optimal dual G under spectral radius measure that is not the standard dual .

Let G be a 1-optimal dual of a frame F with respect to norm measure. This does not imply that G is 1-erasure spectrally optimal dual We can find a dual H, different than G, which is 1-erasure spectrally optimal dual frame. There exists frames whose optimal dual frames with respect to norm and spectral radius measure are different. Lets see the following example.

Example 7.2.2. Let $H = \mathbb{R}^2$. Consider the frame $F = \{f_i\}_{i=1}^{3}$ given in which is

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}. \tag{7.2.110}$$

This is a uniform non Parseval frame. In ([44]) it is given that the standard dual of F is

$$S^{-1}F = \left\{ \begin{bmatrix} \frac{3}{4} \\ -\frac{1}{4} \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} \\ \frac{3}{4} \end{bmatrix}, \begin{bmatrix} \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \end{bmatrix} \right\}. \tag{7.2.111}$$

It is shown that the frame

$$G = \left\{ \begin{bmatrix} \frac{3-\sqrt{3}}{2} \\ \frac{1-\sqrt{3}}{4} \end{bmatrix}, \begin{bmatrix} \frac{1-\sqrt{3}}{2} \\ \frac{3-\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{3}-1}{\sqrt{2}} \\ \frac{\sqrt{3}-1}{\sqrt{2}} \end{bmatrix} \right\} \tag{7.2.112}$$

is the unique optimal dual for F with respect to norm measure. We claim that the frame

$$K = \left\{ \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} \end{bmatrix} \right\} \tag{7.2.113}$$

is an optimal dual frame for F with respect to spectral radius measure.

Proof. First note that to find the spectral radius of frames with 1-erasure, we find the maximum of eigenvalues of the operator $\Theta_G^* D \Theta_F$, where Θ_F and Θ_G^* are the analysis and synthesis operators of the frame F and the dual frame G of F, respectively. In other words, we maximize the eigenvalues

λ_i for $i = 1, 2, 3$ which are of the form

$$\langle x, f_i \rangle g_i = \lambda_i x \text{ for } x \in \mathbb{R}^2. \tag{7.2.114}$$

Let $x = g_i$ in (7.2.114), then $\langle g_i, f_i \rangle g_i = \lambda_i g_i$. If we take the inner product of both side with the frame vector f_i, we get

$$\Big\langle \langle g_i, f_i \rangle g_i, f_i \Big\rangle = \lambda_i \langle g_i, f_i \rangle, \text{ or } \langle g_i, f_i \rangle \langle g_i, f_i \rangle = \lambda_i \langle g_i, f_i \rangle \tag{7.2.115}$$

which implies that $\lambda_i = \langle g_i, f_i \rangle$

An optimal dual frame is the sequence $\{S^{-1} f_i + h_i\}_{i=1}^{3}$ such that

$$\max_i \{ |\langle S^{-1} f_i + h_i, f_i \rangle| \} \tag{7.2.116}$$

is minimal for all $\{h_i\}_{i=1}^{3}$ where $\sum_{i=1}^{3} \langle x, f_i \rangle h_i = 0$ and $x \in \mathbb{R}^2$. Then, all $\{h_i\}_{i=1}^{3}$ must be of the following form ([44]),

$$h_1 = h_2 = \begin{bmatrix} a \\ b \end{bmatrix}, \text{ and } h_3 = \begin{bmatrix} -\sqrt{2}a \\ -\sqrt{2}b \end{bmatrix} \tag{7.2.117}$$

Thus, the function that is to be minimized is

$$L(h) := \max \Big\{ |\langle S^{-1} f_1 + h, f_1 \rangle|, |\langle S^{-1} f_2 + h, f_2 \rangle|, |\langle S^{-1} f_3 - \sqrt{2}h, f_3 \rangle \Big\} \tag{7.2.118}$$

where $h = [a, \ b]^{\mathrm{T}}$.

To simplify the calculations, we first show the existence of an optimal dual frame with respect to spectral radius measure with $a = b$. To show the existence of such a dual frame, we show that $L(\tilde{h}) \le L(h)$, where $h = [a,\, b]^{\mathrm{T}}$ and $\tilde{h} = \left[\frac{a+b}{2},\, \frac{a+b}{2}\right]^{\mathrm{T}}$.

Let $\dagger\, :\, [a,\, b] \mapsto [b,\, a]$. Note that $(S^{-1}f_1)^\dagger = S^{-1}f_2$ and $(S^{-1}f_3)^\dagger = S^{-1}f_3$. Therefore, we obtain

$$
\begin{aligned}
L(\tilde{h}) &= \max\left\{ \left|\left\langle S^{-1}f_1 + \frac{h+h^\dagger}{2}, f_1\right\rangle\right|, \left|\left\langle S^{-1}f_2 + \frac{h+h^\dagger}{2}, f_2\right\rangle\right|,\right. \\
&\qquad \left.\left|\left\langle S^{-1}f_3 + \frac{-\sqrt{2}(h+h^\dagger)}{2}, f_3\right\rangle\right| \right\} && (7.2.119) \\
&= \max\left\{ \frac{1}{2}\big|\langle 2S^{-1}f_1 + h + h^\dagger, f_1\rangle\big|, \frac{1}{2}\big|\langle 2S^{-1}f_2 + h + h^\dagger, f_2\rangle\big|,\right. \\
&\qquad \left.\frac{1}{2}\big|\langle 2S^{-1}f_3 - \sqrt{2}(h+h^\dagger), f_3\rangle\big| \right\} && (7.2.120) \\
&= \max\left\{ \frac{1}{2}\big|\langle S^{-1}f_1 + h, f_1\rangle + \langle S^{-1}f_1 + h^\dagger, f_1\rangle\big|, \frac{1}{2}\big|\langle S^{-1}f_2 + h, f_2\rangle + \langle S^{-1}f_2 + h^\dagger, f_2\rangle\big|,\right. \\
&\qquad \left.\frac{1}{2}\big|\langle S^{-1}f_3 - \sqrt{2}h, f_3\rangle + \langle S^{-1}f_3 - \sqrt{2}h^\dagger, f_3\rangle\big| \right\} && (7.2.121) \\
&\le \max\left\{ \frac{1}{2}\Big(\big|\langle S^{-1}f_1 + h, f_1\rangle\big| + \big|\langle S^{-1}f_1 + h^\dagger, f_1\rangle\big|\Big),\right. \\
&\qquad \frac{1}{2}\Big(\big|\langle S^{-1}f_2 + h, f_2\rangle\big| + \big|\langle S^{-1}f_2 + h^\dagger, f_2\rangle\big|\Big), \\
&\qquad \left.\frac{1}{2}\Big(\big|\langle S^{-1}f_3 - \sqrt{2}h, f_3\rangle\big| + \big|\langle S^{-1}f_3 - \sqrt{2}h^\dagger, f_3\rangle\big|\Big) \right\} && (7.2.122) \\
&= \max\left\{ \frac{1}{2}\Big(\big|\langle S^{-1}f_1 + h, f_1\rangle\big| + \big|\langle (S^{-1}f_2 + h)^\dagger, f_2^\dagger\rangle\big|\Big),\right. \\
&\qquad \frac{1}{2}\Big(\big|\langle S^{-1}f_2 + h, f_2\rangle\big| + \big|\langle (S^{-1}f_1 + h)^\dagger, f_1^\dagger\rangle\big|\Big), \\
&\qquad \frac{1}{2}\Big(\big|\langle S^{-1}f_3 - \sqrt{2}h, f_3\rangle\big| + \big|\langle (S^{-1}f_3 - \sqrt{2}h)^\dagger, f_3^\dagger\rangle\big|\Big) && (7.2.123)
\end{aligned}
$$

$$= \max\Big\{\frac{1}{2}\Big(\big|\langle S^{-1}f_1 + h, f_1\rangle\big| + \big|\langle S^{-1}f_2 + h, f_2\rangle\big|\Big),$$

$$\frac{1}{2}\Big(\big|\langle S^{-1}f_2 + h, f_2\rangle\big| + \big|\langle S^{-1}f_1 + h, f_1\rangle\big|\Big),$$

$$\frac{1}{2}\Big(\big|\langle S^{-1}f_3 - \sqrt{2}h, f_3\rangle\big| + \big|\langle S^{-1}f_3 - \sqrt{2}h, f_3\rangle\big|\Big) \tag{7.2.124}$$

$$\leq \max\Big\{\big|\langle S^{-1}f_1 + h, f_1\rangle\big|, \big|\langle S^{-1}f_2 + h, f_2\rangle\big|, \big|\langle S^{-1}f_3 - \sqrt{2}h, f_3\rangle\big|, \Big\} \tag{7.2.125}$$

$$= L(h), \tag{7.2.126}$$

where the last inequality follows from the fact that $\frac{x+y}{2} \leq \max\{x, y\}$ for $x, y \geq 0$.

Now taking $a = b$, we will find a that minimizes the following function

$$L(a) := \max\Big\{\Big|\frac{3}{4} + a\Big|, \Big|\frac{3}{4} + a\Big|, \Big|\frac{1}{2} - 2a\Big|\Big\}. \tag{7.2.127}$$

We show that for $a = -1/12$, we have

$$L(a) := \max\Big\{\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\Big\} = \frac{2}{3}, \tag{7.2.128}$$

minimal, and therefore,

$$\left\{\begin{bmatrix} \frac{2}{3} \\ \\ -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} -\frac{1}{3} \\ \\ \frac{2}{3} \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{2}}{3} \\ \\ \frac{\sqrt{2}}{3} \end{bmatrix}\right\} \tag{7.2.129}$$

is an optimal dual with respect to spectral radius measure for F. In fact, letting $a = -1/12 + \epsilon$,

$L(a)$ becomes

$$L(a) := \max\left\{\left|\frac{3}{4} - \frac{1}{12} + \epsilon\right|, \left|\frac{3}{4} - \frac{1}{12} + \epsilon\right|, \left|\frac{1}{2} - 2\big(-\frac{1}{12} + \epsilon\big)\right|\right\}. \tag{7.2.130}$$

In order to maximum less than 2/3, $|2/3+\epsilon| < 2/3$ and $|2/3-2\epsilon| < 2/3$. However, $|2/3+\epsilon| < 2/3$ when $-4/3 < \epsilon < 0$ and $|2/3 - 2\epsilon| < 2/3$ when $0 < \epsilon < 2/3$. So, equations never simultaneously less than 2/3.

Hence, we have found an optimal dual K of F with respect to spectral radius measure that is not exactly the optimal dual G with respect to norm measure such that

$$r_{F,K} = \frac{2}{3} < \sqrt{3} - 1 = r_{F,G} = \delta_{F,G}. \tag{7.2.131}$$

$\square$

Now lets see how we can construct a spectrally 1-erasure optimal dual frame for a given frame using redundancy distribution of frame partitions.

Example 7.2.3. Let F be a $(6, 4)$ frame such that

$$\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\2\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right\}. \tag{7.2.132}$$

We can construct a 1-erasure spectrally optimal dual frame G of F in the following way:

First of all, we partition F such as

$$F = F_1 \cup F_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \cup \left\{ \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}. \tag{7.2.133}$$

Then, we find 1-erasure spectrally optimal dual frame G_1 and G_2 for F_1 and F_2, respectively, so that $G = G_1 \cup G_2$. The standard dual frames of F_1 and F_2 are respectively;

$$S^{-1}F_1 = \left\{ \begin{bmatrix} 1/3 \\ -1/3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 1/6 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/3 \\ 5/6 \\ 0 \\ 0 \end{bmatrix} \right\}, \quad S^{-1}F_1 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 4/9 \\ -2/9 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1/9 \\ 4/9 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1/9 \\ 5/9 \end{bmatrix} \right\}. \tag{7.2.134}$$

Then any optimal dual frames G_1 and G_2 of F_1 and F_2 are the sequences $\{S^{-1}f_i^{(1)} + h_i\}_{i=1}^3$ and $\{S^{-1}f_i^{(2)} + k_i\}_{i=1}^3$ with $\sum_{i=1}^3 \langle x, f_i^{(1)} \rangle h_i = 0$ and $\sum_{i=1}^3 \langle x, f_i^{(2)} \rangle k_i = 0$ for $x \in \mathbb{R}^4$. Then, all

$\{h_i\}_{i=1}^3$ and $\{k_i\}_{i=1}^3$ must be of the following form,

$$\{h_1, h_2, h_3\} = \left\{ \begin{bmatrix} -2a \\ -2b \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ b \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -a \\ -b \\ 0 \\ 0 \end{bmatrix} \right\}, \quad \{k_1, k_2, k_3\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ c \\ d \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2c \\ -2d \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2c \\ 2d \end{bmatrix} \right\}. \tag{7.2.135}$$

Thus, G_1 and G_2 are of the form, respectively;

$$G_1 = \left\{ \begin{bmatrix} 1/3 - 2a \\ -1/3 - 2b \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/3 + a \\ 1/6 + b \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/3 - a \\ 5/6 - b \\ 0 \\ 0 \end{bmatrix} \right\}, \tag{7.2.136}$$

$$G_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 4/9 + c \\ -2/9 + d \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1/9 - 2c \\ 4/9 - 2d \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1/9 + 2c \\ 5/9 + 2d \end{bmatrix} \right\}. \tag{7.2.137}$$

For G_1 and G_2 to be 1-erasure spectrally optimal dual for F_1 and F_2, we need $\langle f_i^{(1)}, g_i^{(1)} \rangle = 2/3$ and $\langle f_i^{(2)}, g_i^{(2)} \rangle = 2/3$ for all $i = 1, 2, 3$. This holds true if $a = -1/6, b = 1/6, c = -1/9, d = 1/18$.

Hence,

$$G = \left\{ \begin{bmatrix} 2/3 \\ -2/3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/3 \\ 2/3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1/3 \\ -1/6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 1/6 \\ 1/3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1/3 \\ 1/3 \end{bmatrix} \right\}. \tag{7.2.138}$$

CHAPTER 8: FUTURE STUDY

8.1 Spectrally Two Uniform Frames

Problem 8.1.1. Characterize the frames $F = \{f_i\}_{i=1}^N$ that has a dual $G = \{g_i\}_{i=1}^N$ such that $r_{F,G}^2 = \frac{Nn-n^2}{N^2(N-1)}$, i.e. $\langle f_i, g_j\rangle\langle f_j, g_i\rangle = \frac{Nn-n^2}{N^2(N-1)}$ for all $i \neq j$.

Problem 8.1.2. Let F be a spectrally two-uniform frame and G be a 2-uniform spectrally optimal dual frame of F. Is it true that F is n-independent if and only if G is n-independent? Do they have the same redundancy distribution? (i.e., $r_{F,G} = r_F = r_G$)

Problem 8.1.3. Does the following statement hold true? There exists an (N, n) equiangular frame if and only if there exists spectrally 2-uniform frame.

8.2 Weighted Spectrally Optimal Dual Frames

8.2.1 Weighted 1-Erasure Spectrally Optimal Dual Frames

In erasures, it is not always necessary for each coefficient to have same possibility to be erased. Some coefficients might be more likely to be erased while some have less possibility to be erased. This brings the notion of weighted dual frames and their optimality question as well.

Let ϵ_i be the maximal error allowed for the $i-$th coordinate being erased with $\epsilon_1 \geq \epsilon_2 \geq \ldots \geq$

$\epsilon_N > 0$. Define $\vec{\epsilon} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_N)$. A dual frame $G = \{g_i\}_{i=1}^N$ of $F = \{f_i\}_{i=1}^N$ is called 1-optimal with respect to $\vec{\epsilon}$ if $\langle f_i, g_i \rangle \leq \epsilon_i$ for all i.

Problem 8.2.1. Given $\vec{\epsilon}$ such that

$$\sum_{i=1}^{N} \epsilon_i = n, \tag{8.2.1}$$

characterize the frame F such that there exists a dual G with $\langle f_i, g_i \rangle = \epsilon_i$.

8.2.2 Weighted 2-Erasure Spectrally Optimal Dual Frames

Let $\vec{\epsilon} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{N^2-N})$ be the vector for which each coordinate represents the maximum error allowed for the ijth coordinate being erased such that

$$\epsilon_1 \geq \epsilon_2 \geq \ldots \geq \epsilon_{N^2-N}. \tag{8.2.2}$$

A dual is called 2-optimal with respect to $\vec{\epsilon}$ if $|\langle f_i, g_j \rangle \langle f_j, g_i \rangle| \leq \epsilon_{ij}$ for all $i \neq j$.

Problem 8.2.2. Characterize frames that has a dual such that $\langle f_i, g_j \rangle \langle f_j, g_i \rangle = \epsilon'_{ij}$ for a given $\vec{\epsilon'}$ with $\sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \epsilon'_{ij} = n - \sum_{i=1}^{N} \epsilon_i^2$ where $\epsilon_i = \langle f_i, g_i \rangle$.

8.3 Signal Processing, Quantization and Spectrally Optimal Dual Frames

In application point of view, we are interested in investigating the relationship between k-erasure spectrally optimal dual frames and the quantization of frames in signal processing.

In signal processing, it is of interest to obtain digital representation of the signal. Let the signal be represented by

$$f = \sum_{i \in I} c_i e_i, \tag{8.3.1}$$

where c_i are real or complex numbers. To reduce continuous range of this sequence to a discrete or finite set, we need quantization. A quantizer, say P, is a map such that

$$P : \Gamma \to \Gamma_{\mathcal{A}} \tag{8.3.2}$$

where $\Gamma = \{\sum_{i \in I} c_i e_i \ : \ c_i \in \mathbb{R} \text{ or } c_i \in \mathbb{C}\}$ and $\Gamma_{\mathcal{A}} = \{\sum_{i \in I} q_i e_i \ : \ q_i \in \mathcal{A}\}$, and $\mathcal{A}$ is a discrete/finite set called quantization alphabet. Then

$$\tilde{f} = \sum_{i \in I} q_i e_i \tag{8.3.3}$$

is called quantized expansion where the approximation error is measured by matrix norm, $\|f - \tilde{f}\|$.

In quantizations, the goal is to choose q_i close to c_i. One way of choosing q_i is to choose the closest point in $\mathcal{A}$ to c_i. Quantizers chosen in this form are called pulse code modulation (PCM) algorithms. PCM has some limitations and poor robustness properties. This leaded alternative quantization model that uses frame redundancy. It is shown in [5] that Sigma Delta ($\Sigma\Delta$) quantization outperforms PCM.

The alphabet in $\Sigma\Delta$ quantization is generally the midrise quantization alphabet

$$\mathcal{A}_K^{\delta} = \{(-K + 1/2)\delta, (-K + 3/2)\delta, \ldots, (-1/2)\delta, (1/2)\delta, \ldots, (K - 1/2)\delta\}, \tag{8.3.4}$$

for $K \in \mathcal{N}$ and $\delta > 0$, that consist of $2K$ elements. The 2K-level midrise uniform scalar quantizer with stepsize δ is defined by

$$Q(u) = argmin_{q \in \mathcal{A}_K^\delta} |u - q|. \tag{8.3.5}$$

Let $\{f_i\}_{i=1}^N \subseteq \mathbb{R}^n$ and p be a permutation of $\{1, 2, \ldots, N\}$, called quantization order. The first order $\Sigma\Delta$ quantizer is defined by the following iteration

$$u_i = u_{i-1} + f_{p(i)} - q_i, \tag{8.3.6}$$

$$q_i = Q(u_{i-1} + f_{p(i)}), \tag{8.3.7}$$

where u_0 is prescribed constant. Equation 8.3.6 is simply

$$u_i = u_0 + \sum_{k=1}^{i} (f_{p(k)} - q_k). \tag{8.3.8}$$

In [5], an upper bound for the approximation error in the first order $\Sigma\Delta$ quantization is given by

$$\|f - \tilde{f}\| \leq \|S^{-1}\|_{op} \left(\sigma(F, p) \frac{\delta}{2} + |u_N| + |u_0| \right), \tag{8.3.9}$$

for $|u_0| \leq \delta/2$ and $\|(K - 1/2)\delta\|$, and where S^{-1} is the inverse of frame operator for F and $\sigma(F, p)$ is the frame variation defined by $\sigma(F, p) = \sum_{i=1}^{N-1} \|f_{p(i)} - f_{p(i+1)}\|$. For a (N, n) uniform tight frame F, this bound can be reduced to

$$\|f - \tilde{f}\| \leq \frac{\delta n}{2N} (\sigma(F, p) + 2). \tag{8.3.10}$$

In [6], it is shown that second order $\Sigma\Delta$ outperforms first order $\Sigma\Delta$ quantization and PCM in many settings. And an improved upper bound for approximation error is given.

Under the same assumptions as in the first order $\Sigma\Delta$ quantization, the second order $\Sigma\Delta$ quantizer algorithm is given by

$$u_i = u_{i-1} + f_{p(i)} - q_i, \tag{8.3.11}$$

$$v_i = u_{i-1} + v_{i-1} + f_{p(i)} - q_i, \tag{8.3.12}$$

$$q_i = Q(F(u_{i-1}, v_{i-1}, f_{p(i)})), \tag{8.3.13}$$

where $u_0 = v_0 = 0$, Q is a quantizer and F is a specified quantization rule. For a (N, n) finite unit norm tight frame, an upper bound for the approximation error is

$$\|f - \tilde{f}\| \leq \frac{n}{N}\left(\|v\|_\infty \sigma_2(F, p) + |v_{N-1}|\|\Delta f_{p(N-1)}\| + |u_N|\right), \tag{8.3.14}$$

where $\Sigma_j(F, p) = \sum_{i=1}^{N-j} \|\Delta^j f_{p(n)}\|$, called jth order frame variation of F and Δ^j is the jth order difference operator defined by $\Delta^1 f_i = \Delta f_i = f_i - f_{i+1}$ and $\Delta^j f_i = \Delta^{j-1}\Delta^1 f_i$.

For future research, we are interested in studying on the optimal alternate dual frames to reduce approximation error under both quantization of signal and erasures in signal transmission. Recently, an alternative dual frame is designed for the reconstruction of signals from Sigma-Delta quantized finite frame coefficients [39]. In real settings, beside quantization errors, we expect to have some erasures during transmission. To have better approximations to original signal under both quantization errors and erasures in transmission process, I am planning to examine alternate dual frames, that we reconstruct signal from Sigma-Delta quantization algorithm.

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