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Effects Of Polarization And Coherence On The Propagation And The Detection Of Stochastic Electromagnetic Beams

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EFFECTS OF POLARIZATION AND COHERENCE ON THE PROPAGATION AND THE DETECTION OF STOCHASTIC ELECTROMAGNETIC BEAMS

by

MOHAMED SALEM
M.S. Assiut University, Egypt, 2000
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Major Professor: Jannick Rolland
ABSTRACT

Most of the physically realizable optical sources are radiating in a random manner given the random nature of the radiation of a large number of atoms that constitute the source. Besides, a lot of natural and synthetic materials are fluctuating randomly. Hence, the optical fields that one encounters, in most of the applications are fluctuating and must be treated using random or stochastic functions.

Within the framework of the scalar-coherence theory, one can describe changes of the properties of any stochastic field such as the spectral density and the spectral degree of coherence on propagation in any linear medium, deterministic or random. One of the frequently encountered random media is the atmospheric turbulence, where the fluctuating refractive index of such medium severely degrades any signal propagating through it; especially it causes intensity fades of the signal. The usage of stochastic beams at the transmitter instead of deterministic ones has been suggested sometime ago to suppress the severe effects of intensity fluctuations caused by the atmospheric turbulence. In this dissertation, we study the usage of partially coherent beams in long path propagation schemes through turbulent atmosphere such as one frequently encounters in remote sensing, in the use of communication systems, and in guiding. Also the used detection scheme at the receiver is important to quantify the received signal efficiently, hence we compare the performance of incoherent (direct) detection versus coherent (heterodyne) detection upon the use of either one of them at the receiver of the communication system of beams propagating in turbulent atmosphere and namely we evaluate the signal-to-noise-ratio (SNR) for each case.
The scalar-coherence theory ignored the vector nature of stochastic fields, which should be taken into account for some applications such as the ones that depend on the change of the polarization of the field. Recently generalization for the scalar-coherence theory including the vector aspects of the stochastic beams has been formulated and it is well-known as the unified theory of coherence and polarization of stochastic beams. The use of the unified theory of coherence and polarization makes it possible to study both the coherence properties and the polarization properties of stochastic electromagnetic beams on propagation in any linear media. The central quantity in this theory is a $2 \times 2$ matrix that describes the statistical ensemble of any stochastic electromagnetic beam in the space-frequency domain or its Fourier transform in the space-time domain. In this dissertation we derive the conditions that the cross-spectral density matrix of a so-called planar, secondary, electromagnetic Gaussian Schell-model source has to satisfy in order to generate a beam propagating in vacuum. Also based on the unified-theory of coherence and polarization we investigate the subtle relationship between coherence and polarization under general circumstances. Besides we show the effects of turbulent atmosphere on the degree of polarization and the polarization state of a partially coherent electromagnetic beam, which propagates through it and we compare with the propagation in vacuum.

The detection of the optical signals is important; hence it affects the fidelity of the communication system. In this dissertation we present a general analysis for the optical heterodyne detection of stochastic electromagnetic beams. We derive an expression for the SNR when two stochastic electromagnetic beams are mixed coherently on a detector surface in terms of the space-time domain representation of the beams, the beam coherence polarization matrices. We evaluate also the heterodyne efficiency of a heterodyne detection system for stochastic
beams propagating in vacuum and we discuss the dependence of the heterodyne efficiency of the detection process on the changes in the beam parameters as the beam propagates in free space.
To my parents, without their love and support, I would not have been able to get this work done.
ACKNOWLEDGMENTS

All gratitude is due to almighty Allah for His unlimited bounties; He guided and gave me the strength to complete this work.

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<table>
<thead>
<tr>
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<tbody>
<tr>
<td>$\lambda$</td>
<td>Wavelength [m]</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Linear frequency [Hz]</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Angular frequency [rad/sec]</td>
</tr>
<tr>
<td>$k$</td>
<td>Wave number [m$^{-1}$]</td>
</tr>
<tr>
<td>$K$</td>
<td>Wave vector</td>
</tr>
<tr>
<td>$\rho'$</td>
<td>Transverse vector in the source plane</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Transverse vector in the receiver plane</td>
</tr>
<tr>
<td>$r$</td>
<td>Position vector of a point in the Cartesian coordinates</td>
</tr>
<tr>
<td>$s$</td>
<td>Unit vector in specific direction</td>
</tr>
<tr>
<td>$n$</td>
<td>Refractive index of a medium</td>
</tr>
<tr>
<td>$\Delta n$</td>
<td>Fluctuations of the refractive index of random media</td>
</tr>
<tr>
<td>$C_n^2$</td>
<td>Refractive index structure constant [m$^{-2/3}$]</td>
</tr>
<tr>
<td>$l_0$</td>
<td>Inner scale of turbulence [m]</td>
</tr>
<tr>
<td>$L_0$</td>
<td>Outer scale of turbulence [m]</td>
</tr>
<tr>
<td>$R(z)$</td>
<td>Phase front radius of curvature at distance $z$ from the transmitter [m]</td>
</tr>
<tr>
<td>$B_n(\rho)$</td>
<td>Covariance function of refractive index fluctuation in a transverse plane</td>
</tr>
<tr>
<td>$D_n(\rho)$</td>
<td>Structure function of refractive index fluctuations in a transverse plane</td>
</tr>
<tr>
<td>$\Phi_n(\kappa)$</td>
<td>Power spectrum function of refractive index fluctuations in a transverse plane</td>
</tr>
<tr>
<td>$K$</td>
<td>Turbulence spectrum wavenumber [m$^{-1}$]</td>
</tr>
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</table>
\( \Phi_n(K) \)  Three-dimensional refractive index power spectrum function

\( K_m \)  Turbulence inner scale wavenumber [m\(^{-1}\)]

\( K_o \)  Turbulence outer scale wavenumber [m\(^{-1}\)]

\( \rho_o \)  Coherence length of spherical wave propagating in turbulent atmosphere [m]

\( h \)  Planck’s constant

\( \eta_q \)  Quantum efficiency

\( e \)  Electron charge [C]

\( B \)  Filter bandwidth [Hz]
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$$B_{xy} = \frac{1}{2}, \ \sigma_x = \sigma_y = \sigma = 5 \text{ cm}, \ \delta_{xx} = 0.1 \text{ mm}. \ \text{Curves A and D:} \ \delta_{xy} = 0.2 \text{ mm}, \ B \text{ and E:} \ \delta_{xy} = 0.5 \text{ mm}, \ C \text{ and F:} \ \delta_{xy} = 1 \text{ mm}. \ \text{The refractive index structure parameter was chosen as} \ C_n^2 = 10^{-13} \text{ m}^{-2/3}. \ ............................................................... 60$$

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follows; the inner size parameter $l_o = 5 \text{ mm}$ and $C_n^2 = 10^{-17} \text{ m}^{-2/3}$. The propagation distance of the source were assumed as a) $z_s = 0$ and b) $z_s = 1 \text{ km}$.

**Fig. 10.6**: The variation of the normalized SNR ($\text{SNR}^*$) versus the detector radius for both detection techniques. The normalized SNR of the incoherent detection system has been multiplied by factor $10^2$. The source was assumed to be coherent ($\delta_s = \infty$) with wavelength $\lambda = 0.628 \mu\text{m}$ and intensity width $\sigma_s = 5 \text{ mm}$. The propagation distance of the source in the turbulent atmosphere was taken as 3 km. The local oscillator source was assumed to be coherent ($\delta_o = \infty$) with wavelength $\lambda = 0.6328 \mu\text{m}$ and intensity width $\sigma_o = 5 \text{ mm}$. The parameters of the atmospheric turbulence were taken as follows; the inner size parameter $l_o = 5 \text{ mm}$ and a) $C_n^2 = 10^{-15} \text{ m}^{-2/3}$ b) $C_n^2 = 10^{-14} \text{ m}^{-2/3}$.

**Fig. 10.7**: The variation of the normalized SNR ($\text{SNR}^*$) versus the detector radius for both detection techniques. The normalized SNR of the incoherent detection system has been multiplied by factor $10^3$. The propagating source was assumed to be partially coherent one has wavelength $\lambda = 0.6328 \mu\text{m}$ and intensity width $\sigma_s = 5 \text{ mm}$. The propagation distance of the source in the turbulent atmosphere was taken as 1 km. The local oscillator source was assumed to be coherent ($\delta_o = \infty$) with wavelength $\lambda = 0.628 \mu\text{m}$ and intensity width $\sigma_o = 5 \text{ mm}$. The parameters of the atmospheric turbulence were taken as follows; the inner size parameter $l_o = 5 \text{ mm}$ and $C_n^2 = 10^{-17}$.
$m^{-2/3}$. The coherence width of the propagating source was assumed as a) $\delta_s = 5$ mm and b) $\delta_s = 3$ mm.
CHAPTER 1
INTRODUCTORY REMARKS

The majority of the light sources generate optical fields, which fluctuate due to some randomness present in the emission process. Consequently optical fields should be described and treated as stochastic random functions. Generally speaking it is not an easy task to deal with stochastic fields, but within some approximations it is possible to do so. Over the years, fluctuations of optical fields have drawn the attention of many scientists and led to the branch of science known as coherence theory of optical fields. It was found that optical fields may be temporally partially coherent, meaning that there is statistical similarity between the field components generated at different times at some spatial position. Correlation between the field components at different positions in space for the same frequency component or the same instants of time might also exist. They are manifestation of the so-called spatial coherence of optical fields. Most of the times the optical fields are fluctuating both in time and in space but sometime the fields are spectrally pure when the correlation function of the stochastic field can be decomposed into product of its temporal and spatial parts. In recent years the concept of partial coherence has been used in a variety of fields such as in connection with the propagation of electromagnetic fields in the atmosphere, image formation, spectroscopy and interferometry. There are some advantages of using partially coherent beams rather than fully coherent ones in some applications [1] – [4]. In some cases the effects are completely dependent on the coherence properties, for example in optical coherence tomography [5], using so-called coherence gating. Previous observations and the current applications of partially coherent fields indicate the need of taking
the statistical nature of the optical fields into account. In chapter 2 we discuss the key quantity in
the scalar-coherence theory, the second-order correlation function of a stochastic field in the
space-frequency domain and in the space-time domain, as tools for describing the fluctuating
fields and their propagation in various media [6]. The fluctuating nature of scalar wave fields is
usually characterized by second-order correlation functions and their influence is evident, for
example in the interference phenomena and their numerous applications.

The vector nature of stochastic fields should be taken into account for some applications
such as the ones that depend on the change of the polarization of the stochastic fields. For
electromagnetic fields when the polarization is not linear and the field is partially coherent, one
needs to use correlation matrices rather than scalar correlation functions to describe such
statistical ensembles. The matrix formulation has been introduced by Wiener [7] a long time ago
inspired by the well-known density matrix formalism in quantum mechanics. Wiener called the
matrix the coherency matrix. The coherency matrix or more appropriately called the polarization
matrix describes the coherence and the polarization properties of an electromagnetic field at one
point. Gori [8] described the matrix formulation for fluctuating electromagnetic fields, which
changes spatially on propagation, and it is known as the beam coherence polarization (BCP)
matrix. Recently a generalized two-point cross spectral density has been introduced, which
indicates an intimate relationship that exists between the coherence and the polarization of a
fluctuating field [9]. The use of the matrix formulation makes it possible to obtain a unified
description of the entire second-order correlation effects in electromagnetic fields such as
coherence, polarization and the spectrum. It has been found that the properties of the random
electromagnetic beams, especially polarization, changes even on propagation in vacuum. They
can be very drastic and introduce many new phenomena [10]. They might have applications in a variety of fields.

In chapter 2 we outline the representation of fluctuating electromagnetic beams in both space-time and space-frequency domains and we give a summary of the newly developed unified theory of coherence and polarization. At the end of chapter 2 we discuss the conditions that the components of the source parameters must satisfy in order for the source to generate an electromagnetic beam.

We consider the propagation of stochastic electromagnetic beams in any linear medium, deterministic or random, such as the atmospheric turbulence. In Chapter 3 we discuss the properties of atmospheric turbulence and we present some models for the atmospheric turbulence. Also we discuss both propagation in free space and turbulent atmosphere. Free space laser beam propagation appears as inexpensive and an easily realizable substitute for fiber optics links whenever the latter is neither feasible nor practical, especially as a possible solution for the so-called last mile problem [11] to increase the bandwidth supplied to the end users of optical communications network. The free space laser link introduces secure, broad band and inexpensive communication link but, on the other hand, it suffers from the effects of atmospheric turbulence through the communication channel. The influence of the atmosphere may be very severe because it causes beam spread, beam wandering and intensity fluctuations at the receiver (scintillation). Several schemes have been suggested to mitigate the effects of the atmospheric turbulence, such as aperture averaging [12], use of adaptive optics [13] and use of partially coherent rather than coherent light on the transmitter [14]. It was found that the use of partially coherent light can improve the performance of the free space link in many cases and mitigates
the effects of turbulence in a reasonable way [15]. It represents a promising approach compared with the other methods. In our current work we investigate some advantages of using partially coherent light instead of coherent light in laser propagation links, as demonstrated in chapter 3.

Moreover we investigate the possibility of using stochastic electromagnetic beams in the transmitter of the system and we discuss the changes of the beam parameters upon propagation in turbulent atmosphere analytically. In chapter 4 we study the effects of turbulent atmosphere on the degree of polarization of a partially coherent electromagnetic beam, which propagates through it. The far-zone behavior of the degree of polarization of stochastic electromagnetic beams propagating in free-space and atmospheric turbulence will be examined. Recently changes in the state of polarization of a class of stochastic electromagnetic beams (so-called electromagnetic Gaussian Schell-model beams), propagating in free space have been investigated [16]. In chapter 5 we extend the analysis to propagation of such beams in homogeneous, isotropic, non-absorbing atmospheric turbulence. We examine the effects of all regimes of turbulence, weak, moderate and strong regimes of atmospheric fluctuations, on the state of polarization of stochastic electromagnetic beams. In several recent papers [17-19] it was shown, both theoretically and experimentally, that the degree of polarization in a Young’s interference pattern depends on the degree of coherence of the light incident on the pinholes. In our current work we further investigate the subtle relationship between coherence and polarization under more general circumstances based on the unified-theory of coherence and polarization. In chapter 6 we show how the degree of coherence of the stochastic electromagnetic beam on the source plane \( z = 0 \) affects the degree of polarization throughout the half-space \( z > 0 \) into which the source radiates.
At the receiver of an optical communications system there are also some issues related to coherence and polarization of the light, where the photo-detectors are generally used. These devices convert light into an electrical current. There exists a variety of detector technologies that provide high performance in the visible and infrared portions of the spectrum for use by free-space laser and communications systems. Two configurations are available to make use of the photo-detectors in the optical receivers, direct detection and coherent detection. It was shown a long time ago that the coherent (heterodyne) detection has many advantages over the direct (incoherent) one [20], but it requires very stringent alignment considerations [21]. In chapter 7 we first present the analysis of heterodyne detection of stochastic electromagnetic beams by analyzing the SNR of the detection system and we demonstrate the effects of polarization and coherence in the averaging process of the detector. We also discuss the effects of the beam parameters on the chosen quality criterion for the detection process. In chapter 8 we study the heterodyne efficiency as another measure of quality for the coherent detection system. We derive a general expression for the heterodyne efficiency of a detection system for stochastic beams at any state of coherence, assuming that the propagation direction for both signals (the received signal and the locally generated one) are slightly different. In chapter 9 we discuss the dependence of the heterodyne efficiency of the coherent detection process on the changes in the parameters of the stochastic beam as the beam propagates in free space. An analytical expression for the heterodyne efficiency in the case when both the received beam and the local oscillator beam belong to a broad class of so-called electromagnetic Gaussian Schell-model beams is given. Finally, in chapter 10 we evaluate the signal-to-noise ratio (SNR) of detection systems for stochastic sources in the transmitter that generate beams propagating in the turbulent atmosphere.
We present a comparison between the performances of incoherent (direct) detection versus coherent (heterodyne) detection upon the use of either one of them at the receiver of such communication system.
CHAPTER 2

POLARIZATION AND COHERENCE PROPERTIES OF
STOCHASTIC ELECTROMAGNETIC BEAMS

In this chapter we discuss polarization and coherence properties of stochastic electromagnetic beams. We begin by introducing some mathematical tools and definitions, which are important for our analysis and we use them first in the context of scalar fluctuating wave fields. We then generalize the analysis to the electromagnetic case and we introduce the matrix representation of stochastic and partially polarized paraxial optical beams. Finally, we derive necessary conditions that the matrix components of the source must satisfy in order to generate electromagnetic optical beams for the case of the so called Gaussian Schell-model source, which is a simple and convenient mathematical model of stochastic electromagnetic beams. The results of this chapter have been published in reference [22].

2.1 Coherence properties of stochastic scalar wave fields

Throughout this section we discuss coherence properties of spatially varying fluctuating wave fields in the space-time domain. Later on we will talk about the corresponding representation in the space-frequency domain. In the following analysis we consider only the electric field; of course one could readily find corresponding results for the magnetic field.

Let us assume that $E^{(r)}(r, t)$ represents a realization of a real fluctuating electric field at position $r$ and time instant $t$. This field is a member of an ensemble $\{E^{(r)}(r, t)\}$. Whereas the electric field is a real function of position and time, it is convenient to consider the associated
complex analytic representation for the field components [23] in the correlations definition. Let us denote the complex analytic component of the electric field for a realization by $E(r, t)$. Hence the electric field is representing a stochastic, random, process. The basic properties of the field are characterized by the second-order correlation function of the field components at different positions and times. The correlation of the field fluctuations are defined by the quantity

$$\Gamma(r_1, r_2; t_1, t_2) = \left\langle E^*(r_1, t_1)E(r_2, t_2) \right\rangle, \quad (2-1)$$

where the asterisk denotes the complex conjugate and the angular brackets denote the ensemble average. Assuming that the process is stationary at least in the wide sense, i.e. the correlation depends only on the difference of $t_1$ and $t_2$, which is the situation that occurs in most of the stochastic processes found in nature. The second-order correlation function then takes the form

$$\Gamma(r_1, r_2, \tau) = \left\langle E^*(r_1, t)E(r_2, t + \tau) \right\rangle, \quad (2-2)$$

where $\tau = t_1 - t_2$. Moreover we shall assume that the process is ergodic i.e. the time average of one realization is equal to the ensemble average. Assuming also that the process is of zero mean and that it is stationary, one can define a Fourier transform relationship, the so called generalized Wiener-Khintchine theorem, expressed by the formula

$$\Gamma(r_1, r_2, \tau) = \int_{-\infty}^{\infty} W(r_1, r_2, \omega)e^{-j\omega\tau}d\omega, \quad (2-3)$$
where \( W(r_1, r_2, \omega) \) denotes the correlation between two members of the ensemble \( \{E(r, \omega)e^{-j\omega t}\} \) composed of the monochromatic realizations of the electric field and hence it could be defined as follows;

\[
W(r_1, r_2, \omega) = \langle E^*(r_1, \omega)E(r_2, \omega) \rangle, \tag{2-4}
\]

where the angular brackets mean averaging over an ensemble of space-frequency realizations.

### 2.2 Coherence and polarization properties of stochastic electromagnetic beams

As mentioned in the previous section for a linearly polarized field, the field fluctuations could be characterized only by one cross correlation function in the space-time or in the space- frequency representations. In the case of a randomly fluctuating vector (electromagnetic) field one needs nine correlation functions to describe the field completely and those nine correlations constitute a 3 × 3 matrix, the so called covariance matrix [24]. If the field has a beam structure then only transverse components of the electric field are presented. With respect to the propagation direction, one could neglect the longitudinal component of the field if not too close to the source and consequently the 3×3 matrix reduces to a 2×2 matrix. In the space time domain the 2×2 matrix has been called the beam coherence polarization (BCP) matrix [25] and it has the following form;

\[

1\text{ Strictly speaking, a stationary random variable has no Fourier transform because it is neither square nor absolute integrable. Nevertheless it can be represented by an ensemble of monochromatic realizations, which however do not have Fourier transform [23].}
\[
\Gamma(r_1, r_2, \tau) = \begin{pmatrix}
\Gamma_{xx}(r_1, r_2, \tau) & \Gamma_{xy}(r_1, r_2, \tau) \\
\Gamma_{yx}(r_1, r_2, \tau) & \Gamma_{yy}(r_1, r_2, \tau)
\end{pmatrix}
\] (2-5)

The elements of the matrix are defined by the expression

\[
\Gamma_{ij}(r_1, r_2, \tau) = \left\langle E_i^*(r_1, t) E_j(r_2, t + \tau) \right\rangle, \quad (i, j = x, y),
\] (2-6)

where the angular brackets denotes time or ensemble average under the assumption that the process is stationary at least in the wide sense and ergodic. For the case of polychromatic stochastic electromagnetic beams it is preferable to use the so-called cross-spectral density matrix [9]

\[
W(r_1, r_2, \omega) = \begin{pmatrix}
W_{xx}(r_1, r_2, \omega) & W_{xy}(r_1, r_2, \omega) \\
W_{yx}(r_1, r_2, \omega) & W_{yy}(r_1, r_2, \omega)
\end{pmatrix}
\] (2-7)

The elements of the matrix are defined by the expression

\[
W_{ij}(r_1, r_2, \omega) = \left\langle E_i^*(r_1, \omega) E_j(r_2, \omega) \right\rangle, \quad (i, j = x, y),
\] (2-8)

where the angular brackets denote the average over all the realization of the monochromatic ensemble. In terms of the elements of the cross-spectral density matrix one can represent the optical spectral intensity, well-known as the spectral density, of the beam at any point \(r\) and at frequency \(\omega\) by the formula

\[
S(r, \omega) = \left\langle E_x^*(r, \omega) E_x(r, \omega) \right\rangle + \left\langle E_y^*(r, \omega) E_y(r, \omega) \right\rangle = Tr(W(r, r, \omega)), \quad (i, j = x, y),
\] (2-9)

where \(Tr\) denotes the trace. The spectral degree of polarization of the beam at any position at frequency \(\omega\) is given by the expression [9]

\[
P(r, \omega) = \sqrt{1 - 4 \frac{Det(W(r, r, \omega))}{(Tr(W(r, r, \omega)))^2}}
\] (2-10)
where $Det$ denotes the determinant of the matrix. Finally, the correlation in the light fluctuating at two points is characterized by the so-called spectral degree of coherence defined as

$$
\eta(r_1, r_2, \omega) = \frac{Tr(W(r_1, r_2, \omega))}{\sqrt{Tr(W(r_1, r_1, \omega)) Tr(W(r_2, r_2, \omega))}}.
$$

(2-11)

It represents the statistical similarity between the fluctuations at any two points across the beam [26]. It is worth mentioning that all the aforementioned quantities have been used within the framework of the recently formulated unified theory of coherence and polarization of stochastic random electromagnetic beams [9]. The use of this theory makes it possible to elucidate the relationship between polarization properties and coherence properties of stochastic electromagnetic beams.

2.3 Beam conditions for radiation generated by an electromagnetic Gaussian Schell-model source

It has been recently shown that the changes in the propagation of the spectral density, the spectral degree of coherence and the spectral degree of polarization of a fluctuating beam of electromagnetic radiation may be determined from the knowledge of the $2 \times 2$ electric correlation matrix, known as the cross-spectral density matrix of the electromagnetic beam [9], [10]. Uses of this matrix have already been made in the studies of the effects of the turbulent atmosphere on a beam propagating through it [27], [28] and in investigations of the effects of liquid crystals and spatial light modulators on the degree of polarization of a fluctuating electromagnetic beam [29].

In the following we derive the conditions which the cross-spectral density matrix of a so-called planar, secondary, electromagnetic, Gaussian Schell-model source has to satisfy in order
that it generates a beam. The results are generalizations of the conditions derived previously (Ref. [23], Sect. 5.6.4) for the generation of such beams, within the framework of the scalar theory of coherence. These conditions are likely to be of importance in connection with the design of diffusers that generate highly directional beams for various applications, such as guiding, aiming, and communications.

Consider a planar, secondary, fluctuating electromagnetic source, located in the plane \( z = 0 \) and radiating into the half-space \( z > 0 \). We assume that the fluctuations are represented by a statistical ensemble that is stationary, at least in the wide sense. We may characterize the second-order correlation properties of the source by the \( 2 \times 2 \) cross-spectral density matrix [28], [30] defined by the expression

\[
W_{ij} \left( \mathbf{p}_1, \mathbf{p}_2, 0; \omega \right) = \begin{pmatrix}
W_{xx} \left( \mathbf{p}_1, \mathbf{p}_2, 0; \omega \right) & W_{xy} \left( \mathbf{p}_1, \mathbf{p}_2, 0; \omega \right) \\
W_{yx} \left( \mathbf{p}_1, \mathbf{p}_2, 0; \omega \right) & W_{yy} \left( \mathbf{p}_1, \mathbf{p}_2, 0; \omega \right)
\end{pmatrix}.
\]  

(2-12)

Here \( \mathbf{p}_1 \) and \( \mathbf{p}_2 \) are the two-dimensional position vectors of two points \( Q_1 \) and \( Q_2 \) in the source plane, \( \omega \) denotes the frequency and

\[
W_{ij} \left( \mathbf{p}_1, \mathbf{p}_2, 0; \omega \right) = \left\langle E_i^* \left( \mathbf{p}_1, 0; \omega \right) E_j \left( \mathbf{p}_2, 0; \omega \right) \right\rangle, \quad (i = x, y; \quad j = x, y).
\]

(2-13)

where \( E_i \) and \( E_j \) denote the components of the electric field in two mutually orthogonal \( x \) and \( y \) directions, perpendicular to the \( z \)-axis and the sharp brackets denote an average, taken over an ensemble of realizations of the electric field, in the sense of coherence theory in the space-frequency domain (Ref. [23], Sec. 4.7.1). We will only consider beams generated by
electromagnetic Gaussian Schell-model sources [28] and [31]. For such sources the elements of
the cross-spectral density matrix have the form

$$ W_{ij}(\mathbf{p}_1, \mathbf{p}_2, 0; \omega) = \sqrt{S_i(\mathbf{p}_1, 0; \omega)} \sqrt{S_j(\mathbf{p}_2, 0; \omega)} \mu_{ij}(\mathbf{p}_2 - \mathbf{p}_1, 0; \omega), \quad (i = x, y; \quad j = x, y). $$

(2-14)

In this formula $S_i$ and $S_j$ denote the spectral densities (proportional to the average electric
energy densities) of the $i$ and $j$ components of the electric field in the plane $z = 0$ and $\mu_{ij}$
denotes the degree of correlation between the two components. These quantities are given by the
expressions [28]

$$ S_j(\mathbf{p}, 0; \omega) = A_j^2 \exp\left[-\frac{\rho_j^2}{2\sigma_j^2}\right], \quad (j = x, y), $$

(2-15)

$$ \mu_{ij}(\mathbf{p}_2 - \mathbf{p}_1, 0; \omega) = B_{ij} \exp\left[-\frac{(\mathbf{p}_2 - \mathbf{p}_1)^2}{2\delta_{ij}^2}\right], \quad (i = x, y; \quad j = x, y). $$

(2-16)

The parameters $A_j$, $B_{ij}$, $\sigma$ and $\delta_{ij}$ are independent of position but, in general, depend on the
frequency [32].

Let us now determine the electric cross-spectral density matrix of the radiated electric
field in the far zone (denoted by the superscript $\infty$), at points $P_1$ and $P_2$, specified by position
vectors $\mathbf{r}_1 = \eta \mathbf{s}_1$ and $\mathbf{r}_2 = r_2 \mathbf{s}_2$ ($s_1^2 = s_2^2 = 1$). Following a similar argument as used in the scalar
theory (Ref. 23, Sec 5.4-1) we readily find that

$$ W^{(\infty)}_{ij}(\mathbf{r}_1, \mathbf{s}_1, r_2 \mathbf{s}_2, \omega) = (2\pi k)^2 \cos\theta_1 \cos\theta_2 \tilde{W}^{(0)}_{ij}(-k\mathbf{s}_{1z}, k\mathbf{s}_{2z}, \omega) \frac{\exp[ik(r_2 - r_1)]}{r_1 r_2}, \quad (i = x, y; \quad j = x, y), $$

(2-17)
where
\[
\tilde{W}_{ij}^{(0)}(\mathbf{f}_1, \mathbf{f}_2, \omega) = \frac{1}{(2\pi)^{4}} \iint W_{ij}(\mathbf{p}_1', \mathbf{p}_2', 0, \omega) \exp \left[ -i \left( \mathbf{f}_1 \cdot \mathbf{p}_1' + \mathbf{f}_2 \cdot \mathbf{p}_2' \right) \right] d^2 \rho_1' d^2 \rho_2',
\]
\[(i = x, y; \quad j = x, y), \quad (2-18)\]
is the four-dimensional Fourier transform of \( W_{ij}(\mathbf{p}_1', \mathbf{p}_2', 0, \omega) \). The quantities \( \mathbf{s}_{1\perp} \) and \( \mathbf{s}_{2\perp} \) are projections, considered as two-dimensional vectors, of the unit vectors \( \mathbf{s}_1 \) and \( \mathbf{s}_2 \) onto the source plane \( z = 0 \) and \( \theta_1, \theta_2 \) are the angles which the unit vectors \( \mathbf{s}_1 \) and \( \mathbf{s}_2 \) make with a positive \( z \)-direction (see Fig. 2.1).

Fig. 2.1: Illustrating the notation relating to propagation of an electromagnetic beam.
We see from Eq. (2-17) that in order to determine \( W^{(o)}_{ij} \) we must first calculate the four-dimensional Fourier transform of the matrix elements \( W^{(0)}_{ij} \). One finds from Eq. (2-18) and (2-14)-(2-16) that

\[
\tilde{W}^{(0)}_{ij}(f_1, f_2, \omega) = \frac{A_i A_j B_{ij}}{(2\pi)^4} \left[ \exp \left( -\frac{\mathbf{p}_1^2 + \mathbf{p}_2^2}{4\sigma^2} \right) \exp \left( -\frac{\mathbf{p}_2^2 - \mathbf{p}_1^2}{2\delta^2} \right) \right]
\]

\[
\times \exp \left[ -i (f_1 \cdot \mathbf{p}_1 + f_2 \cdot \mathbf{p}_2) \right] d^2 \mathbf{p}_1 d^2 \mathbf{p}_2
\]

\[
= \frac{A_i A_j B_{ij}}{(2\pi)^4} \left[ \exp \left( -\frac{a_{ij}\mathbf{p}_1^2 + a_{ij}\mathbf{p}_2^2 - 2b_{ij}\mathbf{p}_1 \cdot \mathbf{p}_2}{2\delta^2} \right) \right]
\]

\[
\times \exp \left[ -i (f_1 \cdot \mathbf{p}_1 + f_2 \cdot \mathbf{p}_2) \right] d^2 \mathbf{p}_1 d^2 \mathbf{p}_2,
\]

\[(i = x, y; \quad j = x, y), \quad (2-19)\]

where

\[
a_{ij} = \frac{1}{2} \left( \frac{1}{2\sigma^2} + \frac{1}{\delta_{ij}^2} \right), \quad b_{ij} = \frac{1}{2\delta_{ij}^2}. \quad (2-20)\]

The Fourier transform on the right hand side of Eq. (2-19) may be readily evaluated and one finds that

\[
\tilde{W}^{(0)}_{ij}(f_1, f_2, \omega) = \frac{A_i A_j B_{ij}}{(2\pi)^4 (a_{ij}^2 - b_{ij}^2)} \exp \left[ \left( \alpha_{ij} f_1^2 + \alpha_{ij} f_2^2 + 2\beta_{ij} f_1 \cdot f_2 \right) \right],
\]

\[(i = x, y; \quad j = x, y), \quad (2-21)\]

where

\[
\alpha_{ij} = \frac{a_{ij}}{4(a_{ij}^2 - b_{ij}^2)}, \quad \beta_{ij} = \frac{b_{ij}}{4(a_{ij}^2 - b_{ij}^2)}. \quad (2-22)\]
On substituting from Eq. (2-21) into Eq. (2-17) we obtain the following expression for the elements of the cross-spectral density matrix of the electric field in the far zone generated by an electromagnetic Gaussian Schell-model source:

$$W_{ij}^{(\omega)}(r_1, s_1, r_2, s_2, \omega) = k^2 \cos \theta_i \cos \theta_j \frac{A_x A_y B_{ij}}{(a_y^2 - b_y^2)} \exp \left[ -2k^2 (\alpha_{ij} - \beta_{ij} s_1 \cdot s_2) \right] \frac{\exp[i k (r_2 - r_1)]}{r_1 r_2}$$

$$(i = x, y; \quad j = x, y), \quad (2-23)$$

The spectral density (more precisely, the average electric energy density) at a point $r$ ($r = r_1 = r_2$) is given by the expression

$$S(\omega)(r) = \langle E_x^*(r, \omega) E_x(r, \omega) \rangle + \langle E_y^*(r, \omega) E_y(r, \omega) \rangle$$

(2-24)

and is readily evaluated by substituting from Eq. (2-23) into this formula. One then finds that

$$S(\omega)(r, \omega) = k^2 \cos^2 \theta \frac{A_x^2}{(a_{xx}^2 - b_{xx}^2)} \exp \left[ -2k^2 (\alpha_{xx} - \beta_{xx}) \right] + \frac{A_y^2}{(a_{yy}^2 - b_{yy}^2)} \exp \left[ -2k^2 (\alpha_{yy} - \beta_{yy}) \right],$$

(2-25)

if $\theta_1 = \theta_2 = \theta$. In order that the cross-spectral density matrix gives rise to a beam propagating close to the $z$-axis, the spectral density $S(\omega)(r, \omega)$ has to be negligible except when the unit vector $s$ lies in a narrow solid angle about the $z$-axis, and then $\cos \theta$ may be approximated by unity. We see from Eq. (2-25) that this will be the case when

$$\frac{1}{2(a_{xx} - \beta_{xx})} \ll k^2 \quad \text{and} \quad \frac{1}{2(a_{yy} - \beta_{yy})} \ll k^2,$$

(2-26)

or, in terms of source parameters,
\[
\frac{1}{4\sigma^2} + \frac{1}{\delta_{xx}^2} \ll \frac{2\pi^2}{\lambda^2} \quad \text{and} \quad \frac{1}{4\sigma^2} + \frac{1}{\delta_{yy}^2} \ll \frac{2\pi^2}{\lambda^2}.
\]

These conditions are analogous to the beam conditions that were derived previously for scalar Gaussian Schell-model beams [23, Sect. 5.6.4]. It is to be noted that the conditions (2-27) do not contain any information about the off-diagonal elements of the cross-spectral density matrix of the source and, consequently, they are independent of the state of the polarization of the source.

2.4 Conclusions

Through this chapter we discussed the polarization and the coherence properties of stochastic electromagnetic beams. We introduced some mathematical tools and definitions, which are important for our analysis and we used them first in the context of scalar fluctuating wave fields. We then generalized the analysis to the electromagnetic case and we introduced the matrix representation of stochastic and partially polarized paraxial optical beams. Finally, we have derived constraints on the elements of the $2 \times 2$ electric cross-spectral density matrix of an electromagnetic Gaussian Schell-model source in order that the source generates a beam. These conditions are generalizations of conditions for generation of scalar Gaussian Schell-model beams.
CHAPTER 3
PROPAGATION OF PARTIALLY COHERENT BEAMS IN TURBULENT ATMOSPHERE

In this chapter we outline a method to describe the interaction between stochastic electromagnetic beams and the turbulent atmosphere. The method makes it possible to characterize the changes of the parameters of a propagating electromagnetic beam with any state of coherence at any distance from the generating source. We begin by discussing some statistical properties of the atmospheric turbulence and the dependence of the refractive index of the atmosphere on different meteorological parameters. We will also discuss different models, which have been suggested for the spectrum of the refractive index variations. We will then describe the propagation of optical beams in atmospheric turbulence by using an analysis based on Maxwell’s equations and perturbation theory. Finally, we will compare beam spread width of beams of different states of coherence. Two simple theorems are established, which indicate some advantages that are gained from the use of partially coherent sources beams over fully coherent ones in long-distance propagation in the turbulent atmosphere. The results of this chapter have been published in reference [33].

3.1 Introduction

Free-space communication is a growing field, which is replacing traditional fiber optics communication in certain applications such as tracking, aiming and guiding and to overcome
some problems, which are currently degrading the optical communication system, for example
the so-called current last mile problem [34]. In the free-space optical communication one uses
propagating electromagnetic wave as signal carrier; hence one must take into account the optical
properties of the medium and their effects on propagation of the electromagnetic wave. Free-
space communication uses frequencies at which the medium is transparent, so that the absorption
and the scattering effects of the particles and the molecules in the atmosphere are neglected,
whereas the used wavelengths are comparable in size to the atmospheric particles. On the other
hand in a free space, the major drawback that is degrading the system performance is the
fluctuation of the refractive index of the atmosphere, causing optical turbulence. The influence of
optical turbulence, which causes in general deterioration of the free-space communication
system, includes beam wandering and beam broadening beyond the diffraction limits. The most
severe known effect of the atmospheric turbulence is the intensity fluctuations of the beam,
known also as scintillation; it affects the bit error rate (BER) and hence the fidelity of the
communication system. The fluctuations in the refractive index are usually very small,
approximately of the order of $10^{-6}$, but its accumulated effect is usually significant, especially for
long distance of propagation. The changes of the refractive index are mainly due to temperature
variation in the atmosphere, while some other factors such as the humidity and the atmospheric
pressure also contribute, but usually they are less significant.

In the following we give a brief account of the nature of turbulence and some of its
effects. In the next section we introduce some statistical concepts, which we will use later such
as the covariance function and the structure function to represent the fluctuations of a stochastic
variable in the spatial domain and the power spectrum of the fluctuations to represent them in the
conjugate (inverse) domain. In section 3.3 we will outline the essence of Kolmogorov theory of turbulence and some widely used power spectrum models of the turbulent atmosphere. In section 3.4 we will present one of the classic methods that have been used to study the propagation of an electromagnetic field in turbulent atmosphere. The major advantage of the method is that its results give deep insight into the physics of the problem. Finally, in section 3.5 we will use a previously derived analytical expression for the beam spread width of partially coherent beams propagating in the turbulent atmosphere to demonstrate the advantages of using them in some applications of free-space communication rather than using coherent beams.

3.2 Statistical concepts of random fields

In this section we consider the properties of the spatially varying stochastic fields and the corresponding properties in the conjugate domain (the spatial frequency domain) will also be presented. Generally speaking stochastic fields exhibit random spatial variations of specific parameters and also may fluctuate in time; usually one describes the fluctuations in terms of higher order moments of the fluctuating field.

Let us begin by assuming the spatially varying stochastic quantity \( n(R) \) to be the refractive index, \( R \) being a position vector of a typical point in a medium. For this random field we define a spatial covariance function by the formula [35]:

\[
B_n(R_1, R_2) = \langle [n(R_1) - \langle n(R_1) \rangle][n(R_2) - \langle n(R_2) \rangle]\rangle,
\]

(3-1)

where \( R_1 \) and \( R_2 \) are position vector of two points in the random field. For homogeneous random fields the covariance function is invariant with respect to translation i.e. it depends only
on the difference of the two points not on their separate values. Assuming homogeneous random field one can express the covariance function of the refractive index variation as [35];

\[ B_n(\mathbf{R}_1, \mathbf{R}_2) = B_n(\mathbf{R}) = \langle n(\mathbf{R}_1)n(\mathbf{R}_1 + \mathbf{R}) \rangle - < n >^2, \]  

(3-2)

where \( \mathbf{R} = \mathbf{R}_2 - \mathbf{R}_1 \) and \( < n > \) denotes the mean of \( n(\mathbf{R}) \). For isotropic random fields the statistical moments are also rotationally invariant and the covariance function then depends on the magnitude of the difference in the positions only i.e. \( B_n(R) = B_n(|\mathbf{R}|) \).

In some cases it is more convenient to work in the inverse domain. There exists a spectral representation for the covariance function for some random fields. For homogeneous random fields the three-dimensional Fourier transform relation between the covariance function and the power spectrum \( \Phi_n(\mathbf{K}) \) is [35]

\[ \Phi_n(\mathbf{K}) = \left( \frac{1}{2\pi} \right)^3 \iiint_{-\infty}^{\infty} B_n(\mathbf{R}) e^{i\mathbf{K} \cdot \mathbf{R}} d\mathbf{R}. \]  

(3-3)

where \( \mathbf{K} \) is the vector wave number. The corresponding inverse relation is readily obtained as [35]

\[ B_n(\mathbf{R}) = \iiint_{-\infty}^{\infty} \Phi_n(\mathbf{K}) e^{-i\mathbf{K} \cdot \mathbf{R}} d\mathbf{K}. \]  

(3-4)

Both the covariance and the power spectrum are real and even functions of the arguments. Hence, Eqs. (3-3) and (3-4) could be written in terms of cosine transforms [35];

\[ \Phi_n(\mathbf{K}) = \left( \frac{1}{2\pi} \right)^3 \iiint_{-\infty}^{\infty} B_n(\mathbf{R}) \cos(\mathbf{K} \cdot \mathbf{R}) d\mathbf{R}, \]  

(3-5)

\[ B_n(\mathbf{R}) = \iiint_{-\infty}^{\infty} \Phi_n(\mathbf{K}) \cos(\mathbf{K} \cdot \mathbf{R}) d\mathbf{K}. \]  

(3-6)
If the random field is also assumed to be isotropic, by integrating over the angular spherical variables, one obtains the following formulas [35]

$$\Phi_n(K) = \frac{1}{2\pi^2 K} \left[ B_n(R) \sin(K.R) RdR, \right.$$  

$$\left. B_n(R) = \frac{4\pi}{R} \int_0^\infty K\Phi_n(K) \sin(K.R)dK. \right. \quad (3-7)$$

One is usually interested in the covariance function in the transverse plane or in the two-dimensional expression for the covariance function. For homogeneous and isotropic random field one then finds that [36]

$$B_n(\rho) = 2\pi \int_0^\infty \Phi_n(\kappa, 0) J_0(\kappa \rho) \kappa d\kappa, \quad (3-9)$$

where $\kappa$ is the transverse magnitude of the wave number, $J_0$ is the Bessel function of the first kind and zero order and $\rho$ is the magnitude of the transverse projection of the position vector.

Some random fields are not strictly homogeneous and isotropic but may consider being locally homogeneous or locally isotropic. In such case the random variable should be defined as [35];

$$n(R) = \langle n(R) \rangle + n'(R), \quad (3-10)$$

where $\langle n(R) \rangle$ is a slowly varying mean field and $n'(R)$ is the fluctuating part of the random field of zero mean. For any random fields one could use the definition of the structure function, which provides the usual framework for analyzing such fields. The structure function of the random field is defined as [35]:

$$D_s(R_1, R_2) = \langle [n(R_1) - n(R_2)]^2 \rangle. \quad (3-11)$$
where $\mathbf{R}_1$ and $\mathbf{R}_2$ are position vectors of two points in the random field. Furthermore, if the random field is locally homogeneous and also locally isotropic, the structure function can be simplified as follows [35];

$$D_n(R) = \langle [n(R) - n(\mathbf{R}_1 + R)]^2 \rangle.$$  \hfill (3-12)

There are Fourier transforms relationships connecting the structure function and the power spectrum of the refractive index. For locally homogeneous and locally isotropic random fields the transform relations are given by the formulas [35];

$$\int_{-\infty}^{\infty} \Phi(K) K^2 dK = 2 \pi \int_{0}^{\infty} \sin(KR) dR \int_{0}^{\infty} \left[ 1 - \frac{\sin(KR)}{KR} \right] K^2 dK.$$  \hfill (3-13)

$$\Phi_n(K) = \frac{1}{4\pi^2 K^2} \int_{0}^{\infty} \sin(KR) \frac{d}{dR} \left[ R^2 \frac{d}{dR} D_n(R) \right] dR.$$  \hfill (3-14)

An expression for the structure function in the transverse plane or a two-dimensional expression can be readily derived. For locally homogeneous and locally isotropic random fields the expression is [37]

$$D_n(\rho) = 4\pi \int_{0}^{\infty} \left[ 1 - J_0(\kappa \rho) \right] \Phi_n(\kappa, 0) d\kappa.$$  \hfill (3-15)

where $\kappa$ is the transverse magnitude of the wave number, $J_0$ is the Bessel function of the first kind and zero order and $\rho$ is the magnitude of the transverse projection of the position vector.

For random fields that are both homogeneous and isotropic, one can relate the covariance function to the structure function. These functions will possess Fourier transforms with the same power spectrum in this case. When the random field is locally homogeneous and locally isotropic
the Fourier transform relation between the covariance function and the power spectrum is no longer valid and one has to use the structure function to characterize the random field.

### 3.3 Modeling of the turbulent atmosphere

In this section we will give a brief overview of the mechanism that produces the atmospheric turbulence. Also we will review the classical models of the power spectrum of the refractive index fluctuations. We begin by discussing the Kolmogorov theory of turbulence.

The classical studies of turbulence are concerned with the turbulent motion of the velocity in viscous fluid. When the viscosity, characterized by the Reynolds number, exceeds certain critical value, the motion of the wind changes from laminar flow (uniform changes) to turbulent flow. The wind velocity is found to be fluctuating around a certain mean value, at least locally, and it could be represented as a random variable. Kolmogorov [38] used dimensional analysis to describe the turbulent motion of the air flow and he assumed that the velocity fluctuations can be represented by a locally homogeneous and locally isotropic random field for the so called inertial range. In the framework of his derivation Kolmogorov used the energy cascade theory of turbulence illustrated in Fig. 3.1. The main hypothesis of this theory is that the energy cascades from larger to smaller scales i.e. turbulent eddies break down into smaller and smaller structures provided that the air is not compressible, the source of the energy at large scale ($L_0$) is the convection or wind shear and the energy is dissipated at small scale $l_0$ by viscosity. The range of the scales from $L_0$ to $l_0$ is called the inertial range, which has sizes of eddies (called optical turbules) with different velocity fluctuations.
Fig. 3.1: Kolmogorov cascade theory of turbulence.

The eddies have been found to share certain statistical similarity. This enabled Kolmogorov to build a statistical model for their structure function and it was found that they possess essentially local homogeneity and local isotropy in this range. The major source of the turbulent motion is the fluctuation in the wind velocity. Using dimensional analysis, by equating the heat flux in and the heat flux out, Kolmogorov found the well known $2/3$ power law for the structure function of the wind velocity with longitudinal distance within the inertial range;

$$D_v(R) = C_v^2 R^{2/3},$$  \hspace{1cm} (3-16)
where \( C_v^2 \) is called the wind velocity structure constant. It depends on the amount of energy in the turbulence. The corresponding power spectrum for the above structure function can be found by using Eq. (3-14) and is found to be

\[
\Phi_v(K) = 0.033 C_v^2 K^{-11/3}, \quad \frac{1}{L_0} << K << \frac{1}{l_0}.
\]  

(3-17)

The structure function of the refractive index does not depend directly on the wind velocity but it has been found that the refractive index fluctuations in the atmosphere are given by the expression [39];

\[
n'(R) = \frac{77.6 \times 10^{-6} P(R)}{T(R)} \left[ 1 + \frac{7.52 \times 10^{-3}}{\lambda^2} \right].
\]  

(3-18)

Here \( P(R) \) is the atmospheric pressure in millibars, \( T(R) \) is the atmospheric temperature in Kelvin and \( \lambda \) is the wavelength in microns. In the turbulent motion the meteorological conditions such as the temperature gradient, the atmospheric pressure and the specific moisture are known to be passive parameters. The turbulent motion of the winds leads to the mix of the passive parameters in a random way and, consequently, some irregularities in the refractive index of the atmosphere will arise. They are known as eddies (optical turbules). The refractive index structure function in the atmosphere has been found to obey the same 2/3 power law of the velocity fluctuations [40]

\[
D_n(R) = C_n^2 R^{2/3}.
\]  

(3-19)

Here \( C_n^2 \) is the refractive index structure constant and it is a measure of the strength of the fluctuations in the refractive index. Values of \( C_n^2 \) vary from \( 10^{-17} \) or less, which represents weak turbulence condition to \( 10^{-13} \) or more, which represents strong turbulence. \( C_n^2 \) varies with the
temperature and also with the height of the propagation path. The corresponding power spectrum is the well-known Kolmogorov power spectrum. In the inertial range it is;

\[
\Phi_n(K) = 0.033C_n^2K^{-11/3}, \quad \frac{1}{L_0} \ll K \ll \frac{1}{l_0}.
\] (3-20)

From Eq. (3-20) it is clear that the Kolmogorov spectrum has a singularity at \( K = 0 \). To use this expression in some calculation one has to justify that the large scale eddies have very large values and the small scale eddies are negligible, which is not justified in all situations. To overcome the difficulties several modifications have been made. Tatarskii [41] suggested the use of a Gaussian function to truncate the spectrum in the range \( K > \frac{1}{l_0} \) and he introduced the spectrum; known as Tatarskii spectrum

\[
\Phi_n(K) = 0.033C_n^2K^{-11/3}\exp(-K^2/K_m^2), \quad K >> \frac{1}{L_0},
\] (3-21)

where \( K_m^2 = 5.92/l_0 \). To remove the singularity at \( K = 0 \), one used the so-called von Kármán power spectrum and the modified von Kármán power spectra given as [35] respectively;

\[
\Phi_n(K) = 0.033C_n^2(K^2 + K_0^2)^{-11/6}, \quad 0 \leq K < \infty,
\] (3-22a)

\[
\Phi_n(K) = 0.033C_n^2\frac{\exp(-K^2/K_m^2)}{(K^2 + K_0^2)^{11/6}}, \quad 0 \leq K < \infty,
\] (3-22b)

where \( K_0^2 = 2\pi/L_0 \). Measurements of the structure function of the atmospheric temperature showed the existence of a bump at high spatial frequencies [42]. Consequently the refractive index variation obeys nearly the same power spectrum. Andrews [43] proposed a modified
atmospheric spectrum which included the effect of the bump at the high spatial frequencies. It is
given by the formula
\[ \Phi_n(K) = 0.033 C_n^2 \left[ 1 + 1.802(K / K_l) - 0.254(K / K_l)^{7/6} \right] \exp\left(-\frac{K^2 / K_l^2}{(K^2 + K_0^2)^{11/6}}\right), \quad 0 \leq K < \infty, \quad (3-23) \]
where \( K_l^2 = 3.3/l_0 \) and \( K_0^2 = 2\pi/L_0 \).

3.4 Classical theory of monochromatic beams propagation in turbulent atmosphere

The classical approach of studying the propagation of electromagnetic waves in turbulent
atmosphere begins with Maxwell’s equation. One can obtain an analytical formula for the
propagating beam parameters at any transverse plane. This provides more physical insights into
the beam changes upon propagation.

Let us assume a monochromatic beam
\[ E(\rho, z, t) = E(\rho, z)e^{-j\omega t} \quad (3-24) \]
propagating in the turbulent atmosphere. Here \( E(\rho, z) = U(\rho, z)e^{jkz} \), \( U(\rho, z) \) represents the
complex amplitude of the electric field component variation with the transverse position vector \( \rho \)
at any plane \( z \), \( k = \omega/c \) is the wave number and \( c \) is the speed of light in vacuum. In a medium, \( E \)
satisfies the inhomogeneous Helmholtz equation [44];
\[ \nabla^2 E(\rho, z) + k^2 n^2(\rho)E(\rho, z) + 2\nabla(E(\rho, z)\nabla \ln n(\rho)) = 0 \quad (3-25) \]
The last term on the left hand side of Eq. (3-25) represents depolarization. For the propagation at
optical wavelength in turbulent atmosphere where the size of the smallest eddy is much larger
than the wavelength, the scattering is mostly in the forward direction contained in a narrow cone close to beam axis. We may then neglect this term [36] and Eq. (3-25) becomes;

$$\nabla^2 \mathbf{E}(\mathbf{p}, z) + k^2 n^2(\mathbf{p}) \mathbf{E}(\mathbf{p}, z) = 0$$  \hspace{1cm} (3-26)

The above equation is a stochastic equation because the refractive index is a random variable. Several approaches may be applied to deduce approximate solution for this equation and we will obtain the solution using perturbation expansion. In any transverse plane the index of refraction in the turbulent atmosphere can be written as;

$$n(\mathbf{p}) = \langle n(\mathbf{p}) \rangle + n'(\mathbf{p})$$  \hspace{1cm} (3-27)

where $\langle n(\mathbf{p}) \rangle$ is approximately equals to unity and $\langle n(\mathbf{p}) \rangle \gg n'(\mathbf{p})$. Let us express the electric field as a coherent summation of the incident field and all multiple scattered terms:

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1 + .......$$ \hspace{1cm} (3-28)

where $\mathbf{E}_0$ represents the incident wave and $\mathbf{E}_n$ is a contribution of the order of $(n'(\mathbf{p}))^n$. For very small fluctuations, using the perturbation theory one has [36];

$$\nabla^2 \mathbf{E}_0 + k^2 \mathbf{E}_0 = 0$$ \hspace{1cm} (3-29)

$$\nabla^2 \mathbf{E}_1(\mathbf{p}, z) + k^2 \mathbf{E}_1(\mathbf{p}, z) + 2k^2 n'(\mathbf{p}, z) \mathbf{E}_1(\mathbf{p}, z) = 0$$ \hspace{1cm} (3-30)

An analytical solution of the last equation can be found in certain circumstances and one could obtain an expression for the electric field in any transverse plane $z$. Assuming that the paraxial approximation is valid i.e. assuming that the electric field varies slowly along the $z$-direction, one obtains the Green’s function solution of the Helmholtz equation (3-30) as [45];

$$G(\mathbf{p}', \mathbf{p}, z) = -\frac{jk}{2\pi z} \exp[jkz + \frac{jk}{2z}|\mathbf{p} - \mathbf{p}'|^2 + \psi(\mathbf{p}', \mathbf{p})]$$ \hspace{1cm} (3-31)
where \( \mathbf{\rho} \) and \( \mathbf{\rho}' \) are transverse position vectors in the source plane and in the observation plane respectively and \( \psi(\mathbf{\rho}', \mathbf{\rho}) \) represents the complex phase arising from the random medium [45].

With this Green’s function one obtains a solution for the electric field distribution at the observation plane \( z = \text{const} > 0 \), the expression [46];

\[
U(\mathbf{\rho}, z) = \frac{-jk}{2\pi z} e^{jkr} \iint U(\mathbf{\rho}', 0) \exp\left[jkz + \frac{jk}{2z} |\mathbf{\rho} - \mathbf{\rho}'|^2 + \psi(\mathbf{\rho}', \mathbf{\rho})\right] d^2 \mathbf{\rho}'
\]  

(3-32)

This formula expresses the so-called extended Huygens-Fresnel principle. It is a generalization of the usual Huygens-Fresnel principle: from propagation in free space to propagation in a random medium [47].

### 3.5 Long-distance propagation of partially coherent beams through atmospheric turbulence

There exists considerable amount of literature dealing with propagation of coherent scalar beams through the turbulent atmosphere [48]. Recently propagation of partially coherent beams through turbulence began to be investigated and some of this work has indicated that under certain circumstances partially coherent beams may be less affected by turbulence than fully coherent ones [2], [15], [49], [50] and [51]. In this Section we show, on the basis of a recently derived formula for the r.m.s. width of beams generated by so-called Gaussian Schell-model sources and propagating through a turbulent medium that two simple, interesting, and potentially useful results can be deduced. One of these indicates that the state of coherence of the source does not affect the r.m.s. width of the beam, after the beam has traveled a sufficiently long distance through turbulence. The other predicts that sources of lower degree of spatial coherence generate
beams, which are more stable\textsuperscript{2} than those produced by sources of high degree of spatial coherence. These results are independent of any particular model for the atmospheric fluctuations. We illustrate the results by some computed examples.

### 3.5.1 Two theorems

Consider a beam generated by a planar, secondary, Gaussian Schell-model source [23, Sect. 5.4.2], located in the plane \( z = 0 \) and propagating into the half-space \( z > 0 \) containing a random, statistically homogeneous, isotropic medium. Such a source is characterized by the intensity distribution

\[
I^{(0)}(\mathbf{p}, \omega) = A \exp \left( -\frac{\|\mathbf{p}\|^2}{\sigma_j^2} \right)
\]  

(3-33)

and by the spectral degree of coherence

\[
\mu^{(0)}(\mathbf{p}_1, \mathbf{p}_2, \omega) = \exp \left( -\frac{\|\mathbf{p}_2 - \mathbf{p}_1\|^2}{2\sigma_\mu^2} \right).
\]  

(3-34)

In these formulas \( \mathbf{p}, \mathbf{p}_1, \) and \( \mathbf{p}_2 \) are two-dimensional vectors transverse to the beam axis, which coincides with the \( z \)-axis (see Fig. 3.2) and \( A, \sigma_j, \) and \( \sigma_\mu \) are positive constants, which may depend on the frequency \( \omega \). In order that such a source generates a beam, the condition

\[
\frac{1}{\sigma_j^2} + \frac{2}{\sigma_\mu^2} \ll k^2
\]  

(3-35)

must be satisfied, where \( k = 2\pi/\lambda = \omega/c, \lambda \) being the wavelength and \( c \) the speed of light in vacuum ([23, Sect. 5.6.4]).

\textsuperscript{2} In the sense that, they are less affected by atmospheric changes.
It has been shown in [50] that the r.m.s. width of the beam at a distance $z$ from the source is given by the expression

$$\overline{\rho^2}(z) = \sigma_i^2 \left( 1 + \frac{z^2}{z_R^2} + \frac{F_z \rho^2}{\sigma_i^2} \right). \quad (3-36)$$

In this expression $z_R$ denotes the Rayleigh range, defined as the distance at which the cross-sectional area of the beam doubles on propagation in free space. It was shown in [52] that the Rayleigh range $z_R$ of a Gaussian Schell-model beam in free space is given by the expression

$$z_R = \frac{k\sigma_i}{\sqrt{(1/\sigma_i^2) + (2/\sigma_p^2)}}. \quad (3-37)$$

The constant $F_z$ in the formula (3-36) is defined as
\[ F_2 = \frac{4}{3} \pi^2 \int_0^\infty K^3 \Phi_n(K) \, dK, \quad (3-38) \]

where \( \Phi_n(K) \) represents the spatial power spectrum of the refractive index fluctuations of the random medium. The factor \( F_2 \) represents the effect of the atmospheric turbulence, while \( z_R \) depends on the spectral degree of coherence of the light across the source (through \( \sigma_I \)) and on the size of the source (through \( \sigma_I \)). It is to be noted that Eq. (3-36) applies irrespective of the particular model which might be used to represent the fluctuations of the refractive index of the medium.

From Eq. (3-36) the following result follows:

Theorem I. The r.m.s. width of a beam generated by a Gaussian Schell-model source, after it traveled sufficiently long distance \( z \), is given by the expression

\[ \sqrt{\rho^2(z)} \sim \sqrt{F_2 z^{3/2}}. \quad (3-39) \]

By sufficiently long is meant here that \( z \gg \max(z_1, z_2) \) where \( z_1 = (\sigma_I^2 / F_2)^{1/3} \) and \( z_2 = \sigma_I^2 / F_2 z_R^2 \), where \( z_R \) (the Rayleigh range) is given by Eq. (3-37).

The expression (3-39) is seen to be independent of the state of coherence of the source and of the r.m.s. width of the intensity distribution across it. It depends only on the spatial power spectrum \( \Phi_n(K) \) of the refractive index fluctuations. Examples of the values of the distances \( z_1 \) and \( z_2 \) are given in the following for propagation in the atmosphere with fluctuations characterized by the Tatarskii spectrum.

One could see that the first and the second terms in the right-hand side of Eq. (3-36) may be neglected if
\[ z \gg \left( \frac{\sigma_i^2}{F_2} \right)^{1/3} \equiv z_1 \text{ and } z \gg \frac{\sigma_i^2}{F_2^2 \sigma_R^2} \equiv z_2. \] (3-40)

If the Tatarskii spectrum of atmospheric turbulence is used, i.e. if

\[ \Phi_n(K) = 0.0333 C_n^2 K^{-11/3} \exp \left( -\frac{K^2}{K_m^2} \right), \] (3-41)

where \( C_n^2 \) is the structure constant of the refractive index fluctuations of the turbulence and \( K_m = 5.92/l_0 \), \( l_0 \) denoting the inner scale of turbulence, one has

\[ F_2 \equiv \frac{4\pi^2}{3} \int_0^{\infty} K^2 \Phi_n(K) \, dK = 2.186 C_n^2 l_0^{-1/3}. \] (3-42)

On substituting from Eqs.(3-37) and (3-42) into Eq.(3-40), we find that

\[ z_1 = \left( \frac{\sigma_i^2 l_0^{1/3}}{2.186 C_n^2} \right)^{1/3}, \] (3-43)

and

\[ z_2 = \frac{l_0^{1/3}}{1.093 C_n^2 k^2} \left( \frac{1}{2 \sigma_i^2} + \frac{1}{\sigma_\mu^2} \right). \] (3-44)

It is to be noted that the relative magnitude of \( z_1 \) and \( z_2 \) may greatly differ, depending on the values of the parameters such as \( C_n^2 \), \( \sigma_i \) and \( \sigma_\mu \). With \( C_n^2 = 10^{-14} \text{ m}^{-2/3}, l_0 = 10 \text{ mm}, k = 10^7 \text{ m}^{-1} \) (\( \lambda = 628 \text{ nm} \), \( \sigma_i = 10 \text{ mm}, \sigma_\mu = 2 \text{ mm} \), for example, the inequalities (3-40) with \( z_1 \) and \( z_2 \) given by Eqs.(3-43) and (3-44) imply that Theorem I will apply if

\[ z \gg z_1 \sim 995 \text{ m and } z \gg z_2 \sim 50.3 \text{ km}, \] (3-45)

i.e., when the beam traveled a distance much greater than about 50 km. This Theorem is in agreement with some results presented in Ref. [53]
Next we will examine the ratio of the r.m.s. widths of the beam propagating through the atmosphere to that in free space (subscript FS). For propagation in free space $F_2 = 0$, and consequently, according to Eq.(3-36),

$$[\bar{\rho^2}(z)]_{FS} = \sigma^2_f \left( 1 + \frac{z^2}{z^2_R} \right). \quad (3-46)$$

It then follows from Eqs.(3-36) and (3-46) that

$$G(z) \equiv \sqrt{\bar{\rho^2}(z)} \sqrt{[\bar{\rho^2}(z)]_{FS}} = \left[ 1 + \left( \frac{z}{z_R} \right)^2 + \left( \frac{F_2}{\sigma^2_f} \right) \left( \frac{z^3}{z^2_R} \right) \right]^{1/2}. \quad (3-47)$$

For a propagation distance $z >> z_R$, Eq.(3-47) takes the simpler form

$$G(z) \approx \left[ 1 + F_2 \left( \frac{z}{\sigma^2_f} \right)^2 \right]^{1/2}. \quad (3-48)$$

On substituting for $z_R$ from Eq.(3-37), this formula becomes

$$G(z) \approx \left[ 1 + \frac{k^2 F_2}{1/\sigma^2_f + (2/\sigma^2_\mu)} z \right]^{1/2}. \quad (3-49)$$

If $\sigma_\mu$ decreases, with the other parameters being kept fixed, the second term in the bracket on the right-hand side (rhs) of Eq.(3-49) also decreases. In the limit as $\sigma_\mu \to 0$ (i.e. incoherent limit), the r.h.s of Eq.(3-49) may be approximated by unity, and hence

$$G(z) \to 1, \quad \text{as } \sigma_\mu \to 0. \quad (3-50)$$

It is not difficult to show from Eqs.(3-47) and (3-37) that the behavior indicated by Eq.(3-50) actually holds for any propagation distance, not restricted for $z >> z_R$. These results imply the following theorem.
Theorem II: The ratio, $G(z)$, of the r.m.s. width of a beam generated by a scalar Gaussian Schell-model source, propagating through the turbulent atmosphere, to the r.m.s. width of the beam generated by the same Gaussian Schell-model source in free space decreases as the spatial coherence of the source decreases (i.e., as $\sigma_\mu$ decreases). As the source becomes nearly incoherent (very small $\sigma_\mu$ but with the beam condition (3) satisfied), the ratio $G(z)$ approaches unity, i.e. the beam width is hardly affected by propagation through atmospheric turbulence. This theorem applies for propagation over any distance.

### 3.5.2 Numerical results

In the following the two theorems are illustrated by some computed examples. Figs. 3.3 and 3.4 are illustrating Theorem I. In Fig. 3.3 we show the variation of the r.m.s. width $\sqrt{\rho^2(z)}$ of a partially coherent beam (PCB) generated by a Gaussian Schell-model source and a Gaussian laser beam when both of them are propagating through the turbulent atmosphere and also we considered the free-space propagation of the beams for comparison. The PCB beam has 628 nm wavelength, the intensity width of the beam was chosen as $\sigma_I = 10$ mm and the spectral degree of coherence was chosen as $\sigma_\mu = 2$ mm. The Gaussian laser beam has 628 nm wavelength, the intensity width of the beam was chosen as $\sigma_I = 10$ mm and the spectral degree of coherence was chosen as $\sigma_\mu \to \infty$. The turbulent atmosphere was modeled by Tatarskii spectrum with $C_n^2 = 10^{-14}$ m$^{-2/3}$, $l_0 = 10$ mm, and $k = 10^7$ m$^{-1}$ ($\lambda = 628$ nm), $C_n^2 = 0$ was used for free-space propagation. In Fig. 3.3a we compare between the spread widths of the two beams in different media, for short-propagation distances (does not satisfy the limits of validity for Theorem I). As is shown in Fig. 3.3a the PCB is spreading more than the Gaussian laser beam even in free space.
In Fig. 3.3b we compare between the spread widths of the two beams in different media, for long-propagation distances (satisfy the limits of validity for Theorem I). Two observations might be dragged from Fig. 3.3b; first the spread widths of the two beams when propagating in the turbulent atmosphere at long-distances are the same as assisted by Theorem I and the second observation is that the beam spread widths of the two beams are different when the two beams are propagating in free space.

To check the effect of the atmospheric turbulence model on the validity of Theorem I, we repeated our simulation for Fig. 3.3 for the modified von Karman model. The atmospheric parameters were chosen as $C_n^2 = 10^{-14}$ m$^{-2/3}$, $l_0 = 10$ mm, $L_o = 10$ m and $k = 10^7$ m$^{-1}$ ($\lambda = 628$ nm), $C_n^2 = 0$ was used for free-space propagation. As shown in Fig. 3.4, the same observations that we have deduced from Fig. 3.3 are clear also in this case and we can conclude that Theorem I is applied regardless the choice of the model of atmospheric turbulence.

In Fig. 3.5 we illustrate Theorem II for two different models of the spatial power spectrum of atmospheric fluctuations. In Fig. 3.5a we show the variation of the ratio $G(z)$ versus the propagation distance for four beams generated by Gaussian Schell-model sources. The four beams have 628 nm wavelength, the intensity widths of the beams were chosen as $\sigma_I = 10$ mm and the spectral degrees of coherence of the beams were chosen as $\sigma_\mu = 0.2$ mm, $\sigma_\mu = 2$ mm, $\sigma_\mu = 20$ mm and $\sigma_\mu \to \infty$. The turbulent atmosphere was modeled by Tatarskii spectrum with $C_n^2 = 10^{-14}$ m$^{-2/3}$, $l_0 = 10$ mm, and $k = 10^7$ m$^{-1}$ ($\lambda = 628$ nm), $C_n^2 = 0$ was used for free-space propagation. As one can see from Fig. 3.5a the ratio $G(z)$ is getting smaller as the degree of coherence of the beam decreases as stated by Theorem II. We repeated our simulation for Fig. 3.5a for the modified von Karman model. The atmospheric parameters were chosen as
\( C_n^2 = 10^{-14} \text{ m}^{-2/3}, l_0 = 10 \text{ mm}, L_o = 10 \text{ m} \) and \( k = 10^7 \text{ m}^{-1} (\lambda = 628 \text{ nm}) \), \( C_n^2 = 0 \) was used for free-space propagation. As is shown in Fig. 3.5b the trend of the variation of the ratio \( G(z) \) is similar to its trend that observed in Fig. 3.5a, hence one can conclude that Theorem II still applied regardless the chosen model of atmospheric turbulence.
Fig. 3.3: The r.m.s. width $\sqrt{\rho^2(z)}$ of a beam generated by a Gaussian Schell-model source, on propagating through the atmosphere, modeled by Tatarskii spectrum, for ranges of propagation (a) $0 \leq z \leq 5\text{ km}$ and (b) $0 \leq z \leq 500\text{ km}$. The following values of the parameters were used: $C_n^2 = 10^{-14}$ m$^{-2/3}$ ($C_n^2 = 0$ for free-space propagation), $l_0 = 10$ mm, $k = 10^7$ m$^{-1}$ ($\lambda = 628$ nm), and $\sigma_I = 10$ mm. For a laser, $\sigma_\mu \to \infty$. 

\[
\sqrt{\rho^2(z)} \text{[m]} \quad (a)
\]

\[
\sqrt{\rho^2(z)} \text{[m]} \quad (b)
\]
Fig. 3.4: The r.m.s. width $\sqrt{\rho^2(z)}$ of a beam generated by a Gaussian Schell-model source, on propagating through the atmosphere, modeled by a modified von Karman spectrum $[\Phi_n(K) = 0.033C_n^2(K^2 + K_0^2)^{-11/6} \exp(-K^2 / K_m^2)$ with $K_0 = 2\pi/L_0$ and $K_m = 5.92/L_0$], for ranges of propagation (a) $0 \leq z \leq 5$ km and (b) $0 \leq z \leq 500$ km. The values of the parameters used here are the same as in Fig. 3.3, but in this case the outer scale $L_0$ of turbulence has to be included. We chose $L_0 = 10$ m.
Fig. 3.5: The ratio $G(z)$ of the r.m.s. width of a beam in atmospheric turbulence to the r.m.s. width in free space, generated by a Gaussian Schell-model source, for two models of the turbulence: (a) Tatarskii spectrum and (b) modified von Karman spectrum. The same values of the parameters were used as in connection with Figs. 3.3 and 3.4.
3.6 Conclusions

Through this chapter we outlined a method to describe the interaction between electromagnetic beams and the turbulent atmosphere. First, we discussed some statistical properties of the atmospheric turbulence and the dependence of the refractive index of the atmosphere on different meteorological parameters. We then described the propagation in atmospheric turbulence by using an analysis based on Maxwell’s equations and on small perturbation theory. Finally, we compared beam spread widths of beams of different states of coherence.

Two simple theorems have been established, which indicated some advantages that can be gained from the use of partially coherent source beams over fully coherent ones in a long-distance propagation in the turbulent atmosphere. Some practical consequences can follow from the results of our two theorems. Because, according to Theorem I, the spreading of a beam generated by a Gaussian Schell-model source, after it has traveled over a sufficiently long distance through a turbulent atmosphere is independent of the coherence properties of the source, there is no need to use high quality lasers for pointing, tracking and guiding through the atmosphere over long enough distances. A poor quality laser or a “mosaic” of independent lasers would do just as well. Of course considerations of the power output would also have to be taken into account. Theorem II indicates that beams produced by sources of lower spatial degree of coherence are more stable in a well defined sense, namely that they are less affected by changes in the atmospheric conditions.

These results make it clear that partially coherent beams should be seriously considered for use in any long path propagation scheme such as one frequently encounters in remote sensing, in the use of communication systems, and in guiding, for example.
CHAPTER 4
POLARIZATION CHANGES IN PARTIALLY COHERENT ELECTROMAGNETIC BEAMS PROPAGATING THROUGH TURBULENT ATMOSPHERE

In this chapter we study the effects of the turbulent atmosphere on the degree of polarization of a partially coherent electromagnetic beam, which propagates through it. The beam is described by a $2 \times 2$ cross-spectral density matrix and is assumed to be generated by a planar, secondary, electromagnetic Gaussian Schell-model source. The analysis is based on the recently formulated unified theory of coherence and polarization mentioned earlier and on the extended Huygens-Fresnel principle. We begin by studying the behavior of the degree of the polarization in the intermediate zone, i.e. in the region of space where the coherence properties of the beam and the atmospheric turbulence are competing. We also show analytically that the degree of polarization of a beam generated by a partially polarized Gaussian-Schell model source which propagates through atmospheric turbulence tends to the same as it has at the source plane, with increasing distance of propagation. This result is independent of the spectral degree of coherence of the source and of the strength of atmospheric turbulence. We illustrate the analysis by numerical examples. The results of this chapter have been published in references [28] and [54].

4.1 Introduction

It is generally believed that the changes in the degree of polarization of a random electromagnetic beam propagating through the turbulent atmosphere are negligible [36], [55] and
However, this opinion is based on the assumption that the beam is monochromatic. In recent years it was found that when the beam is partially coherent and hence not strictly monochromatic, the degree of polarization generally changes as the beam propagates, even in free space, as James first showed in a seminal paper [57]. Somewhat similar results were later obtained by Gori et al.[32]. These investigations showed that such changes arise from correlation properties of the field in the source plane.

The effects of atmospheric turbulence on partially coherent beams have been studied up to now only within the framework of the scalar theory and, consequently, such treatments cannot provide any information about polarization properties of the beam. These early studies were mainly concerned with uses of partially coherent beams for free-space optical (FSO) communications ([15], [58]). Later, investigations have been carried out, which elucidate some aspects of the propagation of random electromagnetic beams through the turbulent atmosphere ([27], [28]).

The goal of this chapter is to study such changes in the polarization of stochastic beams, at finite distances from the source, on propagation in free space and as well as in the turbulent atmosphere. As already mentioned our analysis is based on the unified theory of coherence and polarization [9], which makes it possible to determine how the degree of polarization of a light beam changes on propagation in any linear medium, deterministic or random [10]. We show also that after propagating a sufficiently long distance the degree of polarization, whilst changing with the distance of propagation, returns to its initial value (its value in the source plane), irrespective of the atmospheric conditions. The results are illustrated by a number of computed curves.
4.2 Propagation of the electric cross-spectral density matrix of an electromagnetic beam in the atmosphere

We begin by first considering the propagation of an electromagnetic wave in a linear medium. It follows from Maxwell equations that the space-dependent part \( \mathbf{E}(\mathbf{r}; \omega) \) of a monochromatic electric field vector \( \mathbf{E}(\mathbf{r}; \omega) \exp(-i\omega t) \) propagating in a dielectric medium satisfies the equation ([44], Sec 14-1.1)

\[
\nabla^2 \mathbf{E}(\mathbf{r}; \omega) + k^2 n^2(\mathbf{r}; \omega) \mathbf{E}(\mathbf{r}; \omega) + \nabla \left[ \mathbf{E}(\mathbf{r}; \omega) \nabla \ln n^2(\mathbf{r}; \omega) \right] = 0 ,
\]

where \( k = 2\pi/\lambda = \omega/c \), \( \lambda \) being the wavelength, \( \omega \) is the angular frequency, \( c \) the speed of light in vacuum, and \( n(\mathbf{r}; \omega) \) is the refractive index of the medium. We note that only the third term in Eq. (4-1) couples the Cartesian components of the electric field. Elementary arguments show that this coupling term may be neglected if the refractive index varies slowly with position; more precisely if the fractional change \( |\Delta n/n| \) is much smaller than unity in distances of the order of the wavelength. Under these circumstances we may replace Eq. (4-1) by the equation

\[
\nabla^2 \mathbf{E}(\mathbf{r}; \omega) + k^2 n^2(\mathbf{r}; \omega) \mathbf{E}(\mathbf{r}; \omega) = 0 .
\]

This equation shows that the three Cartesian components \( E_x, E_y, E_z \) of the electric field \( \mathbf{E} \) then propagate independently of each other in the sense that each satisfies the equation

\[
\nabla^2 E_j(\mathbf{r}; \omega) + k^2 n^2(\mathbf{r}; \omega) E_j(\mathbf{r}; \omega) = 0 , \quad (j = x, y \text{ or } z).
\]

This does not mean that the three components are necessarily independent of each other because, in general, they will be coupled by boundary conditions.
Suppose that the field is beam-like and it propagates from the plane $z = 0$ into the half-space $z > 0$ close to the $z$-axis, where it encounters turbulent atmosphere. Let $\mathbf{r} = (\rho, z)$ be the position vector at a point in the half-space $z > 0$, $\rho$ is denoting a two-dimensional transverse vector perpendicular to the direction of propagation of the beam (Fig. 4.1). Let $E(\rho',0; \omega)$ represents the electric field at the point $(\rho',0)$ in the plane $z = 0$, which we will refer as the source plane. The field at any point in the half-space $z > 0$ into which the beam is assumed to propagate and which contains the turbulent atmosphere can be expressed by the following well-known expression based on the so-called extended Huygens-Fresnel principle ([48], Sec. 12.2):

$$E_j(\rho, z; \omega) = -\frac{ik \exp(ikz)}{2\pi z} \int E_j(\rho',0; \omega) \exp\left[\frac{ik(\rho - \rho')^2}{2z}\right] \exp(\psi(\rho, \rho', z; \omega)) d^2 \rho',$$

$$(j = x, y). \quad (4-4)$$

In this formula $\psi$ is a random phase factor, which represents the effect of the turbulent atmosphere on a monochromatic spherical wave.

![Fig. 4.1: Illustrating the notation relating to propagation of a beam through a turbulent atmosphere.](image)
Suppose now that the beam is not monochromatic but is polychromatic and spatially partially coherent. It must then be described by a correlation matrix rather than by the field vector. Such a matrix is defined as [9]

$$W(p_1, p_2, z; \omega) \equiv W_{ij}(p_1, p_2, z; \omega) = \left\langle E_i^*(p_1, z; \omega)E_j(p_2, z; \omega) \right\rangle, \quad (i = x, y; \quad j = x, y)$$

(4-5)

where the angular brackets represent the average over an ensemble of realizations of the electric field ([23], Sec. 4.7).

The elements of the cross-spectral density matrix at two points \((p_1, z)\) and \((p_2, z)\) in a transverse plane \(z = \text{const} > 0\) may be obtained on substituting from Eq. (4-4) into Eq. (4-5) and one then finds that

$$W_{ij}(p_1, p_2, z; \omega) = \left( \frac{k}{2\pi z} \right)^2 \int d^2 p_1 \int d^2 p_2 W_{ij}(p_1', \rho_1, 0; \omega) \exp \left[ -ik \frac{(p_1 - p_1')^2 - (p_2 - p_2')^2}{2z} \right] \times \left\langle \exp \left[ \psi^*(p_1, p_1', z, \omega) + \psi(p_2, p_2', z, \omega) \right] \right\rangle_m,$$

(4-6)

where \(\left\langle \ldots \right\rangle_m\) denotes averaging over the ensemble of statistical realizations of the turbulent medium. The elements of the electric cross-spectral density matrix \(W\) at a point \((\rho, z)\) are evidently given by the expressions
where $W_{ij}(\rho_1', \rho_2', 0; \omega), (i = x, y; j = x, y)$ are the elements of the electric cross-spectral density matrix in the source plane $z = 0$. The analytic expression for the components of the matrix can be derived only by approximating the phase structure function, given by the expression in the angular brackets in Eq. (4-7). We will use the quadratic approximation \[45\]

\[
\langle \exp[\psi^*(\rho, \rho_1', z; \omega) + \psi(\rho, \rho_2', z; \omega)] \rangle_m \approx \exp \left[ \frac{-(\rho_1' - \rho_2')^2}{\rho_0^2(z)} \right], \tag{4-8}
\]

where

$$\rho_0(z) = \left(0.55C_n^2k^2z\right)^{3/5} \tag{4-9}$$

is the spatial coherence radius of a spherical wave propagating in turbulence, whose behavior is described by the Kolmogorov model, and $C_n^2$ is the refractive index structure parameter ([48], Sec. 4-2.2), which characterizes the strength of atmospheric turbulence.

Let us assume that the beam is generated by a planar, secondary, electromagnetic Schell-model source located in the same plane $z = 0$, and that it propagates into half space $z > 0$ containing the turbulent medium. The elements of the electric cross-spectral density matrix (4-5) can then be expressed in the form [28]

$$W_{ij}(\rho_1', \rho_2', 0; \omega) = \sqrt{S_i(\rho_1', 0; \omega)} \sqrt{S_j(\rho_2', 0; \omega)} \mu_{ij}(\rho_2' - \rho_1' , 0; \omega), (i = x, y; \ j = x, y) \tag{4-10}$$
In this formula $S_i$ is the spectral density of the component $E_i$ of electric field in the source plane and $\mu_{ij}$ denotes the spectral degree of correlation between the components $E_i$ and $E_j$ in that plane. All these quantities can be determined experimentally [31]. Also it can be shown that $\mu_{ij}$ satisfy the inequalities $|\mu_{ij}| \leq 1$ for all values of their arguments (Appendix A). Let us assume that

$$S_i(\rho',0;\omega) = A_i^2 \exp\left(-\frac{\rho'^2}{2\sigma_i^2}\right), \quad (i = x,y),$$  \hfill (4-11)

$$\mu_{ij}(\rho_{2}^2 - \rho_{1}^2,0;\omega) = B_{ij} \exp\left[-\frac{(\rho_{2}^2 - \rho_{1}^2)^2}{2\chi_{ij}^2}\right], \quad (i = x,y; \quad j = x,y),$$  \hfill (4-12)

where the coefficients $A_i$ and $B_{ij}$ are independent of the position but may depend on the frequency. We assume the same about the variances $\sigma_i$ and $\delta_{ij}$. Moreover, the coefficients $B_{ij}$ satisfy the relations (see [28], Eqs. 2.5 (a)-(c))

$$B_{ij} = 1, \quad i = j$$

$$|B_{ij}| \leq 1, \quad i \neq j$$ \hfill (4-13)

and

$$B_{ji} = B_{ij}^*.$$  

It was also noted in Ref. [32] that if

$$\sigma_x = \sigma_y \equiv \sigma$$  \hfill (4-14)

then the degree of polarization will be the same at all source points.

On substituting from Eqs. (4-11) and (4-12) into Eq. (4-7) and using Eq. (4-14), one obtains the following expressions for the elements of the electric cross-spectral density matrix of the beam in a plane $z = \text{const.} > 0$:
\[ W_{ij}(\rho, \rho, z; \omega) = \frac{A_i A_j B_{ij}}{\Delta^2_{ij}(z)} \exp \left [ -\frac{\rho^2}{2\sigma^2 \Delta^2_{ij}(z)} \right ] , \quad (i = x, y; \quad j = x, y), \quad (4-15) \]

where

\[ \Delta^2_{ij}(z) = 1 + \alpha_{ij} z^2 + 0.98 \left ( \frac{C_n^2}{k} \right )^{2/5} \frac{k^2}{\sigma^2} z^{16/5}, \quad (i = x, y; \quad j = x, y) \quad (4-16) \]

with

\[ \alpha_{ij} = \frac{1}{(k\sigma)^2} \left ( \frac{1}{4\sigma^2} + \frac{1}{\delta^2_{ij}} \right ). \quad (4-17) \]

In the formula (4-15) \( \Delta^2_{ij}(z) \) is the beam-spread factor (called also the expansion coefficient of the beam) determined from the Kolmogorov spectrum model [59]. It indicates how the beam spreads in free space. The third term on the right hand-side of expression (4-16) represents the spread caused by atmospheric turbulence [36].

4.3 The spectral degree of polarization of an electromagnetic Gaussian Schell-model (EGSM) beams propagating in a turbulent atmosphere

The degree of polarization \( P(\rho, z; \omega) \) of a random electromagnetic beam at a point \( (\rho, z) \) is given by the expression [10]

\[ P(\rho, z; \omega) = \sqrt{1 - \frac{4\text{Det} \mathbf{W}(\rho, \rho, z; \omega)}{[\text{Tr} \mathbf{W}(\rho, \rho, z; \omega)]^2},} \quad (4-18) \]

where \( \text{Det} \) and \( \text{Tr} \) denote the determinant and the trace of the matrix \( \mathbf{W}(\rho, \rho, z; \omega) \), respectively. It follows at once from Eqs. (4-18) and Eqs. (4-10) - (4-12) that the degree of polarization in the source plane \( z = 0 \) is given by the expression
\[ P(\rho',0;\omega) = \frac{\sqrt{(A_x^2 - A_y^2)^2 + 4A_x^2 A_y^2 |B_{xy}|^2}}{A_x^2 + A_y^2}. \]  

(4-19)

As already mentioned, the assumption (4-14) ensures that the polarization across the source is uniform. The general expression for the degree of polarization of the EGSM beam at any distance \( z > 0 \) from the source plane can be derived by substituting Eq. (4-15) into Eq. (4-18). One then finds that

\[ P(\rho,z;\omega) = \frac{[F(\rho,z;\omega)]^{1/2}}{G(\rho,z;\omega)}, \]  

(4-20)

where

\[ F(\rho,z;\omega) = \left( \frac{A_x^2}{\Delta_{xx}(z)} \exp \left[ -\frac{\rho^2}{2\sigma^2 \Delta_{xx}(z)} \right] - \frac{A_y^2}{\Delta_{yy}(z)} \exp \left[ -\frac{\rho^2}{2\sigma^2 \Delta_{yy}(z)} \right] \right)^2 - \frac{4A_x^2 A_y^2 |B_{xy}|^2}{\Delta_{xy}(z) \exp \left[ -\frac{\rho^2}{\sigma^2 \Delta_{xy}(z)} \right]}, \]  

(4-21)

\[ G(\rho,z;\omega) = \frac{A_x^2}{\Delta_{xx}(z)} \exp \left[ -\frac{\rho^2}{2\sigma^2 \Delta_{xx}(z)} \right] + \frac{A_y^2}{\Delta_{yy}(z)} \exp \left[ -\frac{\rho^2}{2\sigma^2 \Delta_{yy}(z)} \right]. \]  

(4-22)

At points on the optical axis \( (\rho = 0) \) the expressions (4-21) and (4-22) become

\[ F(0,z;\omega) = \left( \frac{A_x^2}{\Delta_{xx}(z)} - \frac{A_y^2}{\Delta_{yy}(z)} \right)^2 + \frac{4A_x^2 A_y^2 |B_{xy}|^2}{\Delta_{xy}(z)}, \]  

(4-23)

\[ G(0,z;\omega) = \frac{A_x^2}{\Delta_{xx}(z)} + \frac{A_y^2}{\Delta_{yy}(z)}. \]  

(4-24)
In the intermediate range, the behavior of the degree of polarization of the beam propagating through the turbulent atmosphere is rather complicated. However, two special cases have already received attention in connection with free space propagation (Table. 4.1).

Table 4.1: The form of two electric cross-spectral density matrices of the source.

<table>
<thead>
<tr>
<th>Case I</th>
<th>$W(p'_1, p'<em>2, 0; \omega) = \begin{pmatrix} W</em>{xx}(p'_1, p'<em>2, 0; \omega) &amp; 0 \ 0 &amp; W</em>{yy}(p'_1, p'_2, 0; \omega) \end{pmatrix}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case II</td>
<td>$W(p'_1, p'<em>2, 0; \omega) = \begin{pmatrix} W</em>{xx}(p'_1, p'<em>2, 0; \omega) &amp; W</em>{xy}(p'_1, p'<em>2, 0; \omega) \ W</em>{yx}(p'_1, p'<em>2, 0; \omega) &amp; W</em>{xx}(p'_1, p'_2, 0; \omega) \end{pmatrix}$</td>
</tr>
</tbody>
</table>

We will now revisit these two cases.

Case I: $B_{xy} = B_{yx} = 0$.

With this choice of the beam parameters the electric cross-spectral density matrix at the source plane has only diagonal elements, i.e.

$$W_{ij}(p'_1, p'_2, 0; \omega) \equiv 0, \quad i \neq j. \tag{4-25}$$

From Eqs. (4-20) and (4-10) together with the condition (4-14) it follows that the degree of polarization of the beam at points on the axis is now given by the expression (see also [27])

$$\mathcal{P}(0, z; \omega) = \frac{A_x^2}{\Delta_{xx}^2} + \frac{A_y^2}{\Delta_{yy}^2}.$$
It seems worth mentioning that in this case (no correlations between mutually orthogonal components of the electric field in the source plane) and with \( A_x = A_y \), the source is unpolarized.

It then follows from Eq. (4-26), that if also \( \delta_{xx} = \delta_{yy} \), the beam remains unpolarized upon propagation. Further if \( \delta_{xx} \neq \delta_{yy} \) the degree of polarization does not take on a zero value.

Case II. \( B_{xy} \neq 0, B_{yx} \neq 0, \delta_{xx} = \delta_{yy}, \delta_{xy} \neq 0 \).

Next we consider another particular case of the electric cross-spectral density matrix (4-10). We assume that in the source plane the diagonal elements are equal to each other, i.e. that

\[
W_{xx}(\mathbf{p}_1, \mathbf{p}_2, 0; \omega) = W_{yy}(\mathbf{p}_1, \mathbf{p}_2, 0; \omega). \tag{4-27}
\]

We also assume that \( x \)- and \( y \) components of the electric field are correlated, i.e. that

\[
W_{ij}(\mathbf{p}_1, \mathbf{p}_2, 0; \omega) \neq 0, \quad i \neq j. \tag{4-28}
\]

From Eqs. (4-10), (4-20), (4-27) and (4-28) it follows that the degree of polarization at a point on the axis is, in this case, given by the expression

\[
P(0, z; \omega) = \frac{\sqrt{(A_x^2 - A_y^2)^2 + 4A_x^2 A_y^2 B_{xy}^2 \Delta_{xx}^4(z)}}{A_x^2 + A_y^2}. \tag{4-29}
\]

If Eqs. (4-27) and (4-28) are satisfied and if \( |B_{xy}| = 1 \), the source becomes fully polarized.

According to Appendix B of Ref. [32] the necessary condition \( \delta_{xx} = \delta_{yy} = \delta_{xy} \) must then also be satisfied for the beam to remain fully polarized through propagation. This leads to the conclusion that the beam generated by such a source remains fully polarized on propagation over any distance \( z \), both in free space and in atmospheric turbulence.
4.4 Far-zone behavior of the degree of polarization of a beam propagating in a turbulent atmosphere

Based on Tatarskii model (Eq. (3-21)) one readily finds that after sufficiently long propagation distance the elements of the cross-spectral density matrix of the Gaussian Schell-model beam are given by the expression [28]

\[
W_{ij}(p, z, \omega) \sim -\frac{A_i A_j B_{ij}}{T} z^{-3} - \frac{A_i A_j B_{ij} \alpha_{ij}}{T^2} z^{-4} + O\left(z^{-5}\right),
\]

(4-30)
as \(kz \to \infty\), \(O\) denoting the order of magnitude symbol and \(T\) depends on the atmospheric model [28]. As an example, for Tatarskii model \(T = 1.093 C_n^2 l_0^{-1/3} \sigma^{-2}\).

On substituting from Eq. (4-30) into the expression (4-20) for the degree of polarization one finds that, as \(kz \to \infty\),

\[
\mathcal{P}(p, z, \omega) \to \frac{[F(z)]^{1/2}}{G(z)},
\]

(4-31)

where

\[
F(z) = \left[ \frac{A_x^2 - A_y^2}{T} z^{-3} - \frac{A_x^2 \alpha_{xx} - A_y^2 \alpha_{yy}}{T^2} z^{-4} + O\left(z^{-5}\right) \right]^2 + 4 \left[ \frac{A_x^2 A_y^2 B_{xy}}{T} z^{-3} - \frac{A_x^2 A_y^2 B_{xy}}{T^2} z^{-4} + O\left(z^{-5}\right) \right]^2,
\]

(4-32)

\[
G(z) = \frac{A_x^2 + A_y^2}{T} z^{-3} - \frac{A_x^2 \alpha_{xx} + A_y^2 \alpha_{yy}}{T^2} z^{-4} + O\left(z^{-5}\right).
\]

(4-33)

It is clear from the formula (4-31), together with Eqs. (4-32) and (4-33), that the leading term (the term in the power of \(z^{-3}\)) does not depend on the coefficients for the effective beam
spread $\alpha_{ij}$, defined by Eq. (4-17). Moreover the “atmospheric coefficient” $T$ cancels out in the expression (4-31) in the limit as $kz \to \infty$. Hence the degree of polarization, given by the formula (4-31), becomes

$$P(p, z; \omega) \sim \frac{\sqrt{(A_x^2 - A_y^2)^2 + 4A_x^2 A_y^2 |B_{xy}|^2}}{A_x^2 + A_y^2} \quad \text{as } kz \to \infty. \quad (4-34)$$

On comparing equations (4-34) with the expression (4-19) we see that

$$P(p, z; \omega) \sim P(p, 0; \omega) \quad \text{as } kz \to \infty, \quad (4-35)$$

the expressions on both sides of this formula being independent of $p$. This result was obtained on the basis of the Tatarskii model of turbulence. If instead one considers the Kolmogorov model (Eq. (3-20)) one finds in place of Eq. (4-30) that the elements of the cross spectral density matrix are [28]

$$W_{ij}(p, z; \omega) = \frac{A_x A_y B_{xy}}{T z^{-16/5}} - \frac{A_x A_y B_{xy} \alpha_y}{T^2 z^{-22/5}} + O(z^{-28/5}), \quad (i = x, y, j = x, y). \quad (4-36)$$

By similar arguments as based on Eq. (4-30) one finds that the conclusion expressed by Eq. (4-35) again holds.

In free space the expression for the degree of polarization of a beam at points sufficiently far away from the source plane can be derived from the general formula (4-20) by setting for the beam spread $\Delta_{ij}$ the free-space value
where the $\alpha_{ij}$ are given by the formula (4-17). In this case $\mathcal{P}(p, z, \omega) \neq \mathcal{P}(0, z, \omega)$ and in general the far zone value of the degree of polarization depends on all initial parameters of the source.

The result expressed by Eq. (4-35) is the main conclusion of the preceding analysis. It shows that *after a sufficiently long distance of propagation through the atmosphere the degree of polarization of the beam returns to its initial value (its value in the source plane)*. Moreover, since the expression (4-34) depends only on the $A$- and $B$ coefficients, the degree of polarization of the beam after it traveled over a sufficiently long distance depends neither on the spectral degrees of correlation $\mu_{ij}$ of the Gaussian Schell-model source nor on the atmospheric turbulence. This conclusion is in agreement with a result derived in [27] for a more restricted class of beams.

### 4.5 Numerical examples

To illustrate the preceding analysis we calculated the variation in the degree of polarization of an electromagnetic Gaussian Schell-model beam along the beam axis for selected values of the source parameters. The results are shown in Figs. 4.2 – 4.6. The curves relating to propagation in free space are plotted for comparison.
Fig. 4.2: The change in the degree of polarization of an electromagnetic Gaussian Schell-model beam, calculated from Eq. (4-20). Curves A, B and C pertain to propagation in free space; D, E and F pertain to propagation in atmospheric turbulence. The source parameters were taken as: 
\[ \omega = 3 \times 10^{15} \text{ rad/m} \quad (\lambda = 0.628 \mu m), \quad A_x^2 = A_y^2 = 1, \quad B_{xy} = 0, \quad \sigma_x = \sigma_y = \sigma = 5 \text{cm}, \quad \delta_{xx} = 0.1 \text{mm}. \]
Curves A and D: \( \delta_{xy} = 0.2 \text{mm} \); B and E: \( \delta_{xy} = 0.5 \text{mm} \); C and F: \( \delta_{xy} = 1 \text{mm} \). The refractive index structure parameter was chosen as \( C_n^2 = 10^{-13} m^{-2/3} \).
Fig. 4.3: As for Fig. 2, but with $A_x^2 = 2$.

Fig. 4.4: As for Fig. 4.2, but with $A_x^2 = 5$. 
Figs. 4.2 – 4.4 show the changes in the degree of polarization for a Gaussian Schell-model beam for Case I (see Table 4.1). From these figures one can see the effect arising from the differences between values of r.m.s. widths $\delta_{xx}$ and $\delta_{yy}$. In particular, Fig. 4.2 shows the behavior of the degree of polarization of the beam generated by a completely unpolarized source ($A_x^2 = A_y^2 = 1$). It is to be noted that after sufficiently long propagation distance the beam becomes completely unpolarized again. Figs. 4.3 and 4.4 show the corresponding results for beams generated by partially polarized sources.

Figs. 4.5 and 4.6 show the changes in the degree of polarization of the beam, pertaining to Case II (see the Table) and illustrate the effect introduced by the difference between r.m.s. widths $\delta_{xx}$ and $\delta_{xy}$. 
Fig. 4.5: The change in the degree of polarization of an electromagnetic Gaussian Schell-model beam, calculated from Eq. (4-13). Curves A, B and C pertain to propagation in free space; D, E and F pertain to propagation in atmospheric turbulence. The source parameters were taken as: 
\[ \omega = 3 \times 10^{15} \text{rad/m} \ (\lambda = 0.628 \mu m), \quad A_x^2 = A_y^2 = \frac{1}{2}, \quad B_{xy} = \frac{1}{2}, \quad \sigma_x = \sigma_y = \sigma = 5 \text{cm}, \quad \delta_{xx} = 0.1 \text{mm}. \]

Curves A and D: \( \delta_{xy} = 0.2 \text{mm} \), B and E: \( \delta_{xy} = 0.5 \text{mm} \), C and F: \( \delta_{xy} = 1 \text{mm} \). The refractive index structure parameter was chosen as \( C_n^2 = 10^{-13} \text{m}^{-2/3} \).
It is of interest to note that in Case II, as a consequence of the presence of correlations between the $x$ and $y$ components of the electric field, the degree of polarization does not take on zero values along the propagation path, in contrast with the situation noted in Case I where it may vanish once or twice as the beam propagates.

In Figure 4.7 we illustrate the behavior of the degree of polarization of an electromagnetic Gaussian Schell-model beam both for propagation in turbulence, modeled in two models, and in free space with increasing distance from the source plane. The figure shows that
for propagation in free space the degree of polarization acquires a particular value after propagating a certain distance \( z = z_0 \) say) and it retains this value as the beam propagates further, i.e. for \( z > z_0 \). On the other hand in turbulent atmosphere, regardless of the turbulence model, the degree of polarization returns to its initial value (the value it has in the source plane) after it propagates over a sufficiently long distance.

![Diagram showing the change of degree of polarization](image)

**Fig. 4.7**: The change of the degree of polarization of a Gaussian Schell-model beam propagating in a turbulent atmosphere, calculated from Eq. (4-19). The parameters characterizing the source are \( \omega = 3 \times 10^{15} \text{ rad/sec} \) \( (\lambda = 0.628 \mu m) \), \( A_x^2 = A_y^2 = 0.5 \), \( B_{xy} = 0.2 \), \( \sigma_x = \sigma_y = \sigma = 5 \text{ cm} \), \( \delta_{xx} = \delta_{yy} = 0.1 \text{ mm} \). The parameters characterizing the atmosphere were chosen to be \( C_n^2 = 10^{-13} \text{ m}^{-2/3} \), \( l_0 = 5 \text{ mm} \).
4.6 Discussion

The two pronounced changes in the degree of polarization of an electromagnetic Gaussian Schell-model beam on the beam axis (illustrated in Figs. 4.2-4.6) may be explained in the following way. One can re-write Eq. (4-20) by using Eqs. (4-23) and (4-24) in the equivalent form

\[
P(0, z; \omega) = \frac{\sqrt{\left(\frac{A_x^2}{A_y^2} \frac{\Delta_{yy}^2(z)}{\Delta_{xx}^2(z)} - 1\right)^2 + 4|B_{xy}|^2 \frac{A_x^2}{A_y^2} \frac{\Delta_{yy}^4(z)}{\Delta_{xy}^4(z)}}}{\frac{A_x^2}{A_y^2} \frac{\Delta_{yy}^2(z)}{\Delta_{xx}^2(z)} + 1}, \tag{4-38}
\]

which in the source plane \( z = 0 \) reduces to

\[
P(0,0; \omega) = \frac{\sqrt{\left(\frac{A_x^2}{A_y^2} - 1\right)^2 + 4|B_{xy}|^2 \frac{A_x^2}{A_y^2}}}{\frac{A_x^2}{A_y^2} + 1}. \tag{4-39}
\]

It is evident from Eq. (4-39) that the degree of polarization of an electromagnetic Gaussian Schell-model beam in the source plane is completely defined by the ratio \( A_x^2 / A_y^2 \) and by \( B_{xy} \). However, as the beam propagates, the degree of polarization becomes also a function of the two ratios \( \frac{\Delta_{yy}^2(z)}{\Delta_{xx}^2(z)} \) and \( \frac{\Delta_{xy}^2(z)}{\Delta_{xy}^2(z)} \), as one can see from Eq. (4-38).

On propagation in free space the spread widths \( \Delta_{xx}(z), \Delta_{yy}(z) \) and \( \Delta_{xy}(z) \), which depend on the (non-equal) correlation coefficients \( \delta_{xx}, \delta_{yy} \) and \( \delta_{xy} \) respectively (see Eqs. (4-16) and (4-17)), start to increase at different rates. Hence the two ratios mentioned above, change with the distance \( z \) independently of each other and consequently affect the value of the degree of
polarization. However, in the far-zone all the beam spreads become quadratic in \( z \), i.e. they approach the values \( \alpha_{ij} z^2 \) \((i, j = x, y)\) and the relative spreads acquire the constant values \( \Delta_{yy}^2(z)/\Delta_{xx}^2(z) = \alpha_{xy}/\alpha_{xx} \) and \( \Delta_{yy}^2(z)/\Delta_{xy}^2(z) = \alpha_{yy}/\alpha_{xy} \). Hence the degree of polarization tends to its horizontal asymptote. This explains the only change of the degree of polarization in free-space propagation, which is caused by the correlation properties of the source. It takes place at relatively small distances from the source (see the dashed curves in Figs. 4.2 - 4.6).

When the beam propagates in the atmospheric turbulence only over a small distance from the source, the strength of the turbulence is negligible and it cannot overcome the change in polarization caused by the correlation properties of the source. However, as the beam propagates sufficiently far, the effect of the atmosphere becomes dominant, leading to the following asymptotic expression for beam spreads as \( k z \rightarrow \infty \) [28]:

\[
\Delta_{ij}^2 \approx 0.98 \left( \frac{C_n^2}{k} \right)^{6/5} k^{2/5} \sigma^{-2} z^{16/5}, \quad (i = x, y; \quad j = x, y).
\]

Using this formula it follows that \( \Delta_{yy}^2(z)/\Delta_{xx}^2(z) = \Delta_{yy}^2(z)/\Delta_{xy}^2(z) = 1 \). Hence the degree of polarization becomes independent of the distance \( z \) and remains a function of \( A_x^2 / A_y^2 \) and \( B_{xy} \) only, just as it is in the source plane. We may conclude that the turbulent atmosphere, becoming stronger with increasing distance, “washes out” all information about the correlation properties of the source. This explains why the degree of polarization acquires its original value (see the solid curves in Figs. 4.2 - 4.6), a fact that has already been addressed in Ref. [28].

We also note that there is a smooth continuous transition in the behavior of the degree of polarization between propagation in the atmospheric turbulence and in free space as the local strength of turbulence decreases. In fact, if the value of the refractive index structure parameter...
(the measure of turbulence strength) approaches zero, i.e. as $C_n^2 \to 0$, the curves representing the atmospheric propagation may be shown to tend to the corresponding free space curves. This fact is illustrated in Fig. 4.8, where the degree of polarization is plotted (with a particular set of source parameters) for several values of $C_n^2$.

Fig. 4.8 : Illustrating the continuous transition of the degree of polarization of a completely unpolarized beam propagating in the atmospheric turbulence and in free space for several values of the refractive index structure parameter $C_n^2$. The source parameters are chosen to be the same as those for the curves C and F in Fig. 4.2.
4.7 Conclusions

We have studied the changes in the degree of polarization of a stochastic electromagnetic Gaussian Schell-model beam propagating in free space and through atmospheric turbulence. Generally speaking the degree of polarization of a partially coherent electromagnetic beam propagating in a turbulent atmosphere is affected by two mechanisms. One is associated with the correlation properties of the source, which may be referred as “correlation-induced”; and the other is due to the atmosphere (“turbulence-induced”). While atmospheric turbulence was previously shown not to introduce appreciable polarization effects, the correlation properties of the source can lead to significant changes in the degree of polarization as the beam propagates. Since the polarization changes can be controlled to some extent by the choice of the source parameters, the theory we described may find applications to problems involving inverse scattering in random media, including the turbulent atmosphere, and the development of more efficient schemes for imaging by laser radars and for free-space optical communication systems.
CHAPTER 5

CHANGES IN THE STATE OF POLARIZATION OF
STOCHASTIC ELECTROMAGNETIC BEAMS
PROPAGATING THROUGH TURBULENT ATMOSPHERE

Recently changes in the state of polarization of a class of stochastic electromagnetic beams (so-called electromagnetic Gaussian Schell-model beams), propagating in free space have been investigated. In this chapter we extend the analysis to propagation of such beams in homogeneous, isotropic, non-absorbing atmospheric turbulence. We find that the effects of turbulence on the state of polarization are most significant when the atmospheric fluctuations are weak or moderate, while in strong regime of atmospheric fluctuations the state of polarization of the beam returns to its original state. Our results might find useful applications for sensing, imaging and communication through the atmosphere. The results of this chapter have been published in reference [60].

5.1 Introduction

In the last few years considerable progress has been made in the theory of polarization of electromagnetic fields. In particular it has been found both theoretically ([32], [54] and [61]) and experimentally [62] that the degree of polarization of a random electromagnetic beam may change on propagation, even in free space. The development of the unified theory of coherence and polarization ([9], [10] and [31]) makes it possible to study the changes not only in the degree
of polarization, but also in the complete state of polarization of such beams. In Ref. [16] the theory was applied to a broad class of stochastic electromagnetic beams, so-called electromagnetic Gaussian Schell-model beams ([28], [32])\(^3\).

In this chapter we apply the theory to investigate the changes in the state of polarization of electromagnetic Gaussian Schell-model beams, as they propagate in the turbulent atmosphere. The atmosphere is taken to be homogeneous and isotropic and it is also assumed that absorption by particles and aerosols is negligible. We only consider propagation paths over which the refractive index structure parameter is approximately constant (e.g. horizontal paths).

We will examine how all the polarization properties of the beam change as it propagates through different fluctuation regimes of atmospheric turbulence. We find, in particular, that although the state of polarization of the beam has a somewhat complicated behavior in the regimes of weak and moderate atmospheric fluctuations; it regains its original state (i.e. the polarization state in the source plane) in strong fluctuation regime. We illustrate the results by numerical examples.

5.2 Propagation of a random electromagnetic beam in the turbulent atmosphere

In Ref. [10] a propagation law governing the propagation of field correlations of random electromagnetic beams traveling in any linear deterministic or random medium was formulated

\(^3\) The analysis of Ref. [32] was carried out in the space-time domain, using the so-called “Beam Coherence-Polarization matrix”. Such a matrix is, however, restricted to description of quasi-monochromatic beams when only sufficiently small path differences are involved.
and shortly afterwards was applied to propagation through the turbulent atmosphere ([27], [28] and [54]). We will begin by first briefly reviewing the main formulas derived in those papers.

Suppose that a partially coherent beam propagates close to the z-axis from the plane \( z = 0 \) (which we will call the source plane) into the half-space \( z > 0 \), which contains the turbulent atmosphere. Let \( \mathbf{r} = (\rho, z) \) be the position vector at a point in the half-space \( z > 0 \), with \( \rho \) being a two-dimensional vector perpendicular to the direction of propagation of the beam (see Fig. 5.1).

![Diagram](image)

**Fig. 5.1**: Illustrating notation relating to the propagation of a beam through a turbulent atmosphere.

The second-order correlation properties of such a beam may be characterized by an electric cross-spectral density matrix [9]
\[
W(p_1, p_2, z; \omega) \equiv [W_{ij}(p_1, p_2, z; \omega)] = \left[ \langle E_i^*(p_1, z; \omega) E_j(p_2, z; \omega) \rangle \right], \quad (i = x, y; \quad j = x, y)
\]

(5-1)

where \( E_x(p, z; \omega) \) and \( E_y(p, z; \omega) \) are two mutually orthogonal components of the electric vector \( \mathbf{E}(r, \omega) \) perpendicular to the \( z \)-direction (axis of the beam) and the sharp brackets denote the average taken over an ensemble of realizations, in the sense of coherence theory in the space-frequency domain ([23], Sec. 4.7).

The elements of the cross-spectral density matrix at any two points \((p_1, z)\) and \((p_2, z)\) in a transverse plane \(z = \text{const} > 0\) may then be calculated from the elements of the matrix \( W(p_1^*, p_2, 0; \omega) \) in the source plane \(z = 0\) with the help of the extended Huygens-Fresnel principle ([15], see also [48]), viz.,

\[
W_{ij}(p_1, p_2, z; \omega) = \left( \frac{k}{2\pi z} \right)^2 \int \int d^2 \rho_1' \int \int d^2 \rho_2' W_{ij}(p_1^*, p_2^*, 0; \omega) \exp \left[ -i k \left( \frac{(p_1 - p_1')^2 - (p_2 - p_2')^2}{2z} \right) \right] \times \left\{ \exp \left[ \psi^* (p_1, p_1', z; \omega) + \psi(p_2, p_2', z; \omega) \right] \right\}_m,
\]

(5-2)

The expression in the angular brackets in Eq. (5-2) is the complex phase structure function of the fluctuations, caused by the atmospheric turbulence, the average, denoted by \( \langle \cdot \rangle_m \), being taken over the ensemble of the realizations of the atmosphere. This quantity may often be approximated by a quadratic function [15]:

\[
\left\{ \exp \left[ \psi^* (p_1, p_1', z; \omega) + \psi(p_2, p_2', z; \omega) \right] \right\}_m \approx \exp \left[ -\frac{(p_1' - p_2')^2}{\rho_0^2(z)} \right].
\]

(5-3)
Here

\[ \rho_0(z) = \left(0.55C_n^2 k^2 z\right)^{-3/5} \]  \hspace{1cm} (5-4)

is the spatial coherence radius of a spherical wave propagating in turbulence whose behavior is described by the Kolmogorov model, ([48], Sec. 6.4.4) and \( k = \omega/c \) is the wave number, \( c \) being the speed of light in vacuum and \( C_n^2 \) is the refractive index structure parameter ([48], Sec. 3.2.3), which characterizes the local strength of the atmospheric turbulence. For horizontal paths \( C_n^2 \) may be usually approximated by a constant.

Formulas (5-2), together with the chosen model for the electromagnetic source, can be used to determine all first-order (single point) correlation properties of the beam at any distance from the source plane.

5.3 Spectrum and polarization properties of a random electromagnetic beam

The single-point statistical properties of the stochastic electromagnetic beam include its spectrum and its polarization. Apart from a proportionality constant depending on the units used, the spectrum of the beam at the point \((\rho, z)\) is given by the expression

\[ S(\rho, z; \omega) = \text{Tr} \mathbf{W}(\rho, z; \omega), \]  \hspace{1cm} (5-5)

where \( \text{Tr} \) denotes the trace of the matrix.

The polarization properties of the beam at frequency \( \omega \) consist of the spectral degree of polarization and of the state of polarization of its polarized part. The spectral degree of polarization is defined by the formula (Ref. [9], Eq. (3))

\[ P(\rho, z; \omega) = \frac{1 - 4 \text{Det} \mathbf{W}(\rho, z; \omega)}{\left[\text{Tr} \mathbf{W}(\rho, z; \omega)\right]^2}, \quad 0 \leq P \leq 1, \]  \hspace{1cm} (5-6)
where $Det$ denotes the determinant. In general, $P$ changes with the distance $z$, the change depending on the type of source, which generates the beam and also on the correlation properties of the medium though which it travels as we discussed in details in chapter 4.

The state of polarization at any point is characterized by the parameters that specifying the polarization ellipse (i.e. its size, shape and orientation, see Fig. 5.2), which can be determined from the cross-spectral density matrix. It was shown in Ref. [16] (cf. [63]) that the angle $\theta$, known as the orientation angle, i.e. the angle that the major axis of the polarization ellipse makes with the x direction is given by the formula

$$
\theta(p, z; \omega) = \frac{1}{2} \arctan \left( \frac{2 \Re[W_{xy}(p, z; \omega)]}{W_{xx}(p, z; \omega) - W_{yy}(p, z; \omega)} \right), \quad -\pi / 2 \leq \theta \leq \pi / 2. \quad (5-7)
$$

Fig. 5.2 : Illustrating notation relating to the polarization ellipse of a random electromagnetic beam.

It was also shown in Ref. [16] that the squares of the magnitudes of the major and the minor semi-axes of the polarization ellipse are given by the expressions
The degree of ellipticity of the polarization ellipse, defined by the formula
\[ \varepsilon = \frac{A_{\text{minor}}}{A_{\text{major}}}, \quad 0 \leq \varepsilon \leq 1, \]  
characterizes the shape of the polarization ellipse. It is unity for circular polarization and zero for linear polarization. For completeness we also give an expression for the area \( A \) of the polarization ellipse:
\[ A = \frac{\pi}{2} A_{\text{major}} A_{\text{minor}} = \frac{\pi}{2} \text{Im}[W_{xy}]. \]

### 5.4 Polarization properties of the beam generated by an electromagnetic Gaussian Schell-model source

We will now apply the results of the previous section to calculate the polarization properties of a particular class of stochastic beams, known as electromagnetic Gaussian Schell-model beams ([28], [32] and [54]). The elements of the cross-spectral density matrix of the field in the source plane, which generates such a beam, are given by the expressions
\[ W_{ij}(\mathbf{p}_1', \mathbf{p}_2', 0; \omega) = \sqrt{S_i(\mathbf{p}_1', 0; \omega)} \sqrt{S_j(\mathbf{p}_2', 0; \omega)} \eta_{ij}(\mathbf{p}_1' - \mathbf{p}_2', 0; \omega), \quad (i = x, y, j = x, y). \]  
\[ (5-12) \]

Here \( S_i^{(0)} \) represents the spectral density of the component \( E_i \) of the electric field and \( \mu_{ij}^{(0)} \) denotes the spectral degree of correlation between the components \( E_i \) and \( E_j \). Further, for such a source
\[ S_i \left( \mathbf{p}'_i, 0; \omega \right) = A_{i}^2 \exp \left( -\mathbf{p}'_i^2 / 2\sigma^2 \right), \quad (i = x, y), \quad (5-13) \]

\[ \eta_{ij} \left( \mathbf{p}'_2 - \mathbf{p}'_1, 0; \omega \right) = B_{ij} \exp \left[ \frac{\left( \mathbf{p}'_2 - \mathbf{p}'_1 \right)^2}{2\delta_{ij}^2} \right], \quad (i = x, y, \quad j = x, y). \quad (5-14) \]

In these expressions the parameters \( A_i, B_{ij}, \sigma \) and \( \delta_{ij} \) are all independent of position but may depend on the frequency. Since the matrix \( \mathbf{W}(\mathbf{p}'_1, \mathbf{p}'_2, 0; \omega) \) is non-negative definite (Ref. [23], Eq. (6.6-8)) and satisfies the relation

\[ W_{ij}(\mathbf{r}_1, \mathbf{r}_2; \omega) = W_{ji}^*(\mathbf{r}_2, \mathbf{r}_1; \omega), \quad (5-15) \]

the (generally complex) factors \( B_{ij} \) and the (real) variances \( \delta_{ij}^2 \) cannot be chosen arbitrarily. The relation (5-15) can be shown to imply that (Appendix A)

\[ B_{ij} \equiv 1 \quad \text{when} \quad i = j, \quad (5-16a) \]

\[ |B_{ij}| \leq 1 \quad \text{when} \quad i \neq j, \quad (5-16b) \]

\[ B_{ji} = B_{ij}^*, \quad (5-16c) \]

\[ \delta_{ji} = \delta_{ij}, \quad \text{when} \quad i \neq j. \quad (5-16d) \]

Moreover, non-negative definiteness of the cross-spectral density matrix implies the following double inequality ([64], see also [32]) \(^4\):

\[ \frac{1}{4\sigma_x^2} + \frac{1}{\delta_{xx}^2} \ll \frac{2\pi^2}{\lambda^2}, \quad \frac{1}{4\sigma_y^2} + \frac{1}{\delta_{yy}^2} \ll \frac{2\pi^2}{\lambda^2}. \]

---

\(^4\) In addition the source has to satisfy the condition that the field is beam-like. For free space the conditions are (Ref. [22])

\[ \frac{1}{4\sigma_x^2} + \frac{1}{\delta_{xx}^2} \ll \frac{2\pi^2}{\lambda^2}, \quad \frac{1}{4\sigma_y^2} + \frac{1}{\delta_{yy}^2} \ll \frac{2\pi^2}{\lambda^2}. \]
max\{\delta_{xx}, \delta_{yy}\} \leq \delta_{xy} \leq \min\left\{ \frac{\delta_{xx}}{\sqrt{|B_{xy}|}}, \frac{\delta_{xy}}{\sqrt{|B_{xy}|}} \right\}. \quad (5-17)

On substituting from the Eq. (5-13) and (5-14) into Eq. (5-12) one obtains the following expressions for the elements of the cross-spectral density matrix in the source plane:

\[
W_{ij}(\rho_1', \rho_2', 0; \omega) = A_i A_j B_{ij} \exp\left[-\left(\frac{\rho_1'^2 + \rho_2'^2}{4\sigma^2}\right)\right] \exp\left[-\frac{\left(\rho_1' - \rho_2'\right)^2}{2\delta_{ij}^2}\right],
\]

\[(i = x, y, j = x, y). \quad (5-18)\]

If \(\rho_1' = \rho_2' \equiv \rho'\) the formula (5-18) reduces to

\[
W_{ij}^{(0)}(\rho', 0; \omega) = A_i A_j B_{ij} \exp\left[-\frac{\rho'^2}{2\sigma^2}\right], \quad (i = x, y, j = x, y). \quad (5-19)\]

On substituting from Eq. (5-19) into Eqs. (5-7) – (5-8) and (5-10) one can readily determine the state of polarization of the source. In particular, one finds that the orientation angle, is given by the formula [16]

\[
\theta(\rho, 0; \omega) = \frac{1}{2} \arctan\left(\frac{2A_x A_y \text{Re}[B_{xy}]}{A_x^2 - A_y^2}\right) \quad (5-20)
\]

and the degree of ellipticity is given by the formula [16]

\[
\varepsilon(\rho, 0; \omega) = \sqrt{\left(A_x^2 - A_y^2\right)^2 + 4\left(A_x A_y |B_{xy}|\right)^2} - \sqrt{\left(A_x^2 - A_y^2\right)^2 + 4\left(A_x A_y \text{Re}[B_{xy}]\right)^2} \quad (5-21)
\]

Clearly, both \(\theta(\rho, 0; \omega)\) and \(\varepsilon(\rho, 0; \omega)\) are constants across such a source.
From the expressions of the elements of the cross-spectral density matrix at the source plane one can derive corresponding expressions for the cross-spectral density matrix valid at any distance \( z \) from the source. One only needs to substitute from Eq. (5-18) into the formula (5-2) and, after integrating over the source plane \( z = 0 \) with respect to \( \bm{\rho}'_1 \) and \( \bm{\rho}'_2 \) one obtains for the elements of the cross-spectral density matrix the expressions

\[
W_{ij}(\bm{\rho}_1, \bm{\rho}_2, z; \omega) = \frac{A_i A_j B_{ij}}{\Delta^2_{ij}(z)} \exp \left[ -\frac{\bm{\rho}_1^2 + \bm{\rho}_2^2}{4\sigma^2 \Delta^2_{ij}(z)} \right] \exp \left[ -\frac{(\bm{\rho}_1 - \bm{\rho}_2)^2}{2\delta^2 \Delta^2_{ij}(z)} \right] \exp \left[ -\frac{ik(\bm{\rho}_1^2 - \bm{\rho}_2^2)}{2\Phi_{ij}(z)} \right],
\]

\((i = x, y, j = x, y)\). \tag{5-22}

Here the quantities \( \Delta^2_{ij}(z) \), called sometimes “the beam expansion coefficients”, are given by the formula ([54], see also [32])

\[
\Delta^2_{ij}(z) = 1 + \alpha_{ij} z^2 + T z^{16/5}, \quad (i = x, y, j = x, y), \tag{5-23}
\]

where

\[
\alpha_{ij} = \frac{1}{(k\sigma)^2} \left( \frac{1}{4\sigma^2} + \frac{1}{\delta^2_{ij}} \right) \tag{5-24}
\]

and

\[
T = 0.98 \left( C_n^2 \right)^{6/5} k^{2/5} \sigma^{-2}. \tag{5-25}
\]

In these expressions \( k = \omega/c \) is the wave number of the beam and \( C_n^2 \) is, as before, the refractive index structure parameter. The quantities \( \Phi_{ij}(z) \) entering the last exponential factor in Eq. (5-22) are given by the formulas

\[
\Phi_{ij}(z) = z \left( 1 + \frac{1}{\Delta^2_{ij}(z)} \right). \tag{5-26}
\]

When \( \bm{\rho}_1 = \bm{\rho}_2 \equiv \bm{\rho} \) Eq. (5-22) takes the form
\[ W_{ij}(\mathbf{p}, \mathbf{p}; z; \omega) = \frac{A_i A_j B_{ij}}{\Delta_{ij}^2(z)} \exp \left[ -\frac{\rho^2}{2\sigma_i^2\sigma_j^2(z)} \right], \quad (i = x, y, j = x, y). \quad (5-27) \]

On substituting from Eqs. (5-23) – (5-25) and (5-27) into Eqs. (5-6) – (5-10) one can readily derive expressions for all the polarization characteristics of the beam at any distance \( z \) from the source plane. In particular, the orientation angle \( \theta \) and the degree of ellipticity \( \varepsilon \) along the beam axis (i.e. for \( \rho = 0 \)) are found to be given by the formulas

\[ \theta(0, z, \omega) = \frac{1}{2} \arctan \left( \frac{2A_x A_y \text{Re}[B_{xy}]\Delta_{xx}^2\Delta_{yy}^2}{(A_x^2\Delta_{xx}^2 - A_y^2\Delta_{yy}^2)} \right) \quad (5-28) \]

and

\[ \varepsilon(0, z, \omega) = \frac{\left( A_x^2\Delta_{yy}^2 - A_y^2\Delta_{xx}^2 \right)^2 + 4\left( A_x A_y B_{xy} \right)^2 \Delta_{xy}^2 + \left( A_x A_y B_{xy} \right)^4 \Delta_{xy}^2}{\sqrt{\left( A_x^2\Delta_{yy}^2 - A_y^2\Delta_{xx}^2 \right)^2 + 4\left( A_x A_y B_{xy} \right)^2 \Delta_{xy}^2 + \left( A_x A_y B_{xy} \right)^4 \Delta_{xy}^2}}, \]

where the argument \( z \) of the coefficients \( \Delta_{ij}^2 \) have been suppressed. An expression for the degree of polarization \( P(\mathbf{p}, z, \omega) \) has been derived in chapter 4.

### 5.5 Far-zone behavior of the state of polarization

We will now discuss the behavior of the state of polarization at large distances (\( kz \to \infty \)) from the source.

According to the analysis of Ref. [28] the asymptotic behavior of the elements of the cross-spectral density matrix as \( kz \to \infty \) is given by the expression
\[ W_{ij}(p,z,\omega) \sim \frac{A_i A_j B_{ij}}{T} z^{-16/5} - \frac{A_i A_j B_{ij} \alpha_{ij}}{T^2} z^{-22/5} + \ldots , (i = x, y, f = x, y) \]  

(5-30)

We note that if one retains only the first two terms in the expression the elements \( W_{ij}(p,z,\omega) \) do not depend on \( p \). On substituting the asymptotic expression (5-30) for the elements of the cross-spectral density matrix into Eq. (5-7) one obtains the following asymptotic formula for the orientation angle:

\[ \theta^{(\infty)}(p,z,\omega) = \frac{1}{2} \arctan \left( \frac{2 A_x A_y \text{Re}(B_{xy}) \left( z^{-16/5} - \alpha_{xy} z^{-22/5} + \ldots \right)}{(A_x^2 - A_y^2) z^{-16/5} - (A_x^2 \alpha_{xx} - A_y^2 \alpha_{yy}) z^{-22/5} + \ldots} \right), \quad \text{as } k z \to \infty. \]

(5-31)

It follows from this expression that for large values of \( z \) only the terms \( z^{-16/5} \) contribute. Ignoring higher-order terms in the asymptotic expansion one obtains for \( \theta^{(\infty)} \) the expression

\[ \theta^{(\infty)}(p,z,\omega) = \frac{1}{2} \arctan \left( \frac{2 A_x A_y \text{Re}(B_{xy})}{A_x^2 - A_y^2} \right) \]

\[ = \theta(p,0,\omega). \]

(5-32)

By a similar argument one can show that for sufficiently large values of \( z \), the degree of ellipticity (see Eq. (5-10)), \( \varepsilon^{(\infty)} \) say, also returns to its initial value, given by Eq. (5-21), i.e.

\[ \varepsilon^{(\infty)}(p,z,\omega) = \varepsilon(p,0,\omega). \]

(5-33)
Eqs. (5-32) and (5-33) imply that the state of polarization of the polarized portion of the beam is regained when the beam propagates sufficiently far from the source.

The results expressed by Eqs. (5-32) and (5-33) are valid when the beam propagates through the atmospheric turbulence, not in free space. The corresponding formulas for propagation in free space are derived in Appendix B.

5.6 Examples and discussion

Fig 5.3 shows several examples of the behavior of the polarization properties (the degree of polarization, the orientation angle and the degree of ellipticity) of a random electromagnetic beam, generated by an electromagnetic Gaussian Schell-model source. The changes of polarization properties are shown as a function of the Rytov variance $q$, defined by the formula

$$q = 1.23 C_n^2 k^{7/6} z^{1/6}$$  \hspace{1cm} (5-34)

rather than as a function of propagation distance $z$. The Rytov variance may be regarded to be a measure to the “global strength” of the atmospheric fluctuations.

\footnote{The far-zone behavior of the degree of polarization of the electromagnetic Gaussian Schell-model beams is discussed in Ref. [28] (see also [54]).}

\footnote{The Rytov variance $q$ (denoted sometimes by $\sigma_1^2$) is a measure of how the atmospheric fluctuations affect the intensity of a plane wave. For a stochastic electromagnetic beam such a parameter is not known; in that case the Rytov variance gives only a rough estimate of the global strength of atmospheric turbulence as the beam propagates.}
Fig. 5.3: Changes with the Rytov variance $q$, defined by Eq. (5-34), on propagation of (a) the degree of polarization; (b) the orientation angle; (c) the degree of ellipticity. The parameters characterizing the source and the strength of turbulence have been taken to have the values: $\omega = 3 \times 10^{15} \text{ rad/sec}$, $A_x = 1.3$, $A_y = 1$, $\delta = \pi / 6$, $\mid B_{xy} \mid = 0.2$, $\sigma = 2.5 \text{ cm}$, $\delta_{yy} = 7.5 \text{ mm}$, $\delta_{xy} = 10 \text{ mm}$, $\delta_{xx} = 5 \text{ mm}$ for curve (A), $\delta_{xx} = 6.5 \text{ mm}$ for (B), $\delta_{xx} = 9.5 \text{ mm}$ for (C) and $C_n^2 = 10^{-13} \text{ m}^{-2/3}$.

The turbulence is considered to be weak when $q << 1$, moderate when $q \sim 1$ and strong when $q >> 1^7$. Displaying the polarization properties of the beam in this manner one can see how they change not only quantitatively but also qualitatively as the beam passes through different fluctuating regimes of turbulence. In particular, one can see that the cumulative effect of source

\[^{7}\text{Such separation into the three ranges is not as artificial as might appear, because different phenomena (such as beam wander, scintillation) occur in these regimes. More importantly, different methods of analysis have to be applied in different regimes. For example, the method of small perturbations can only be applied for propagation in weak turbulence.}\]
correlations and turbulence becomes pronounced only in the moderate fluctuation regime, i.e. for $q \sim 1$. In strong turbulence, i.e. when $q >> 1$, this effect disappears, and the original polarization properties of the beam are regained, irrespective of their behavior in the moderate fluctuation regime, in agreement with the analytical results derived in Sec. 5.5.

Fig. 5.4 shows the evolution of the polarization ellipse generated by the source with parameters relating to the curves (A) in Fig. 5.3. Seven “snapshots” have been computed for different values of the parameter $q$. One can see the two types of changes: 1) in weak and moderate fluctuation regimes (the first four snapshots) the ellipse rotates and changes its shape; 2) in moderate-to-strong and in strong regimes (the last three snapshots) it gradually acquires the same shape as in the source plane. The actual values of the propagation distance $z$, are also indicated in the figure. One can see that both of these changes in the ellipsometric properties typically occur after propagation over distances of several kilometers. Since when propagating through the atmospheric turbulence the size of the polarization ellipse decreases very rapidly, one cannot illustrate its behavior along the whole propagation path by using the same scale for all the snapshots. For illustrative purposes the polarization ellipses in the successive figure are, therefore, shown on different scales.
Fig. 5.4: The evolution of the polarization ellipse on propagation. The parameters are the same as for the curve (A) in Fig. 5.3.

As already mentioned the results expressed by Eqs. (5-32) and (5-33) do not generally hold for propagation in free space. To illustrate this fact we compare in Fig. 5.5 the behavior of all the polarization properties in free space and in the turbulent atmosphere, for an initially
elliptically polarized, partially coherent beam for the same beam as in Fig. 5.3(a), curve (A), as a function of the propagation distance $z$. 

(a)
Fig. 5.5: Comparison between the changes of the polarization properties of a stochastic beam propagating in free space (curve (A)) and in the atmosphere (curve (B)). The parameters of the propagating beam are the same as in (A) in Fig. 5.3.
5.7 Conclusions

Through this chapter we have studied the changes in the state of the polarization of a stochastic electromagnetic Gaussian Schell-model beam propagating in atmospheric turbulence. We found that the effects of turbulence on the state of polarization are most significant when the atmospheric fluctuations are weak or moderate, while in strong regime of atmospheric fluctuations the state of polarization of the beam returns to its original state.

Because of the ability of some of the polarization properties to regain their initial values these properties could, perhaps, be used in modulation schemes for transmitting data through random media.
CHAPTER 6

DEPENDENCE OF THE DEGREE OF POLARIZATION ON THE DEGREE OF COHERENCE IN STOCHASTIC ELECTROMAGNETIC BEAMS

In this chapter we investigate the subtle relationship between coherence and polarization under more general circumstances based on the unified-theory of coherence and polarization. It is shown that two stochastic electromagnetic beams, which propagate from the source plane $z = 0$ into the half-space $z > 0$ may have different degrees of polarization throughout the half-space, even though they have the same sets of Stokes parameters in the source plane $z = 0$. This fact is due to the possible difference in the coherence properties of the field in that plane. The result is illustrated by an example. The results of this chapter have been published in reference [65].

6.1 Introduction

Some years ago D. James [57] and later others [16] and [61] showed that the degree of polarization of a stochastic electromagnetic field may change on propagation even in free space. More recently it was shown both theoretically [17], [18] and [66] and experimentally [19] that the degree of coherence of an electromagnetic beam at the pinholes in a Young’s interferometer affects the degree of polarization of the field in the interference pattern. These results are rather surprising, because until the publication of these relatively recent papers, none of the numerous investigations about polarization of light, carried out since the publication of Stokes’ classic
papers on polarization, published more than 150 years ago, contained even a hint that the degree of polarization of a light beam may change on propagation.

To obtain a better insight into this phenomenon, which might be called “coherence-induced changes of the degree of polarization” we analyze a somewhat more general problem than has been treated in the papers that we just cited. Specifically, we consider the following problem: A stochastic electromagnetic beam propagates from the plane $z = 0$ (which we will call the source plane), into the half space $z > 0$. We investigate the dependence of the degree of polarization at any plane $z > 0$ on the degree of coherence in the plane $z = 0$.

6.2 Generalized Stokes parameters

As is well known, the state of polarization of the field at any point $r$ in the half-space $z > 0$, at any frequency $\omega$, may be characterized by four Stokes parameters [44], which we will denote by $s_0(r, \omega)$, $s_1(r, \omega)$, $s_2(r, \omega)$ and $s_3(r, \omega)$. The degree of polarization is given by the well-known expression \[P(r, \omega) = \frac{\sqrt{s_1^2(r, \omega) + s_2^2(r, \omega) + s_3^2(r, \omega)}}{s_0(r, \omega)}.\] (6-1)

The Stokes parameters do not obey any propagation laws. However, as was recently shown, it is possible to determine how they change as the beam propagates by considering certain generalized Stokes parameters $S_{\alpha}(r_1, r_2, \omega)$, ($\alpha = 0,1,2,3$) introduced not long ago [67] and defined by the formulas

\[S_0(r_1, r_2, \omega) = \left\langle E_x^*(r_1, \omega)E_x(r_2, \omega) \right\rangle + \left\langle E_y^*(r_1, \omega)E_y(r_2, \omega) \right\rangle,\] (6-2a)
\[ S_1(r_1, r_2, \omega) = \left\langle E_x^*(r_1, \omega) E_x(r_2, \omega) \right\rangle - \left\langle E_y^*(r_1, \omega) E_y(r_2, \omega) \right\rangle, \]  
\[ S_2(r_1, r_2, \omega) = \left\langle E_x^*(r_1, \omega) E_y(r_2, \omega) \right\rangle + \left\langle E_y^*(r_1, \omega) E_x(r_2, \omega) \right\rangle, \]  
\[ S_3(r_1, r_2, \omega) = i \left[ \left\langle E_y^*(r_1, \omega) E_x(r_2, \omega) \right\rangle - \left\langle E_x^*(r_1, \omega) E_y(r_2, \omega) \right\rangle \right]. \]  

Here \( E_x(r, \omega) \) and \( E_y(r, \omega) \) are the Cartesian components of the complex electric field component of the electric vector at frequency \( \omega \) in two mutually orthogonal directions, perpendicular to the direction of the propagation of the beam and the angular brackets denote an ensemble average in the sense of coherence theory in the space-frequency domain [23].

The usual Stokes parameters are evidently special cases of these generalized Stokes parameters, viz.

\[ s_\alpha(r, \omega) = S_\alpha(r, r, \omega), \quad (\alpha = 0, 1, 2, 3). \]  

However, unlike the ordinary Stokes parameters, the generalized Stokes parameters obey precise propagation laws. For propagation in free space, one finds, that, in the paraxial approximation ([67], Eq. (7)),

\[ S_\alpha(r_1, r_2, \omega) = \int_D \int_D S_\alpha(p_1', p_2', \omega) K(p_1 - p_1', p_2 - p_2', z, \omega) d^2\rho_1' d^2\rho_2' \]  

Here \( p_1', p_2' \) are position vectors of two points in the source plane and \( K(p_1 - p_1', p_2 - p_2', z, \omega) \) is a paraxial propagator from the points \( p_1', p_2' \) in the source plane to two field points \( r_1 = (p_1, z) \) and \( r_2 = (p_2, z) \) at distance \( z \) from that plane. Explicitly

\[ K(p_1 - p_1', p_2 - p_2', z, \omega) = G^*(p_1 - p_1', z, \omega) G(p_2 - p_2', z, \omega) \]  

\( G \) being the free-space paraxial propagator of the Helmholtz operator, viz.,

\[ G(p', p, z) = -\frac{ik}{2\pi z} \exp[ik|p - p'|/2z] \]
\[ k = \omega / c, \quad c \text{ being the speed of light in vacuum}. \]

We recall that, according to Eq. (11) of Ref. [67], Eq. (13) the degree of coherence of the field at any two points \( r_1, r_2 \) in the half-space \( z > 0 \) is given by the expression

\[
\eta(r_1, r_2, \omega) = \frac{S_0(r_1, r_2, \omega)}{\sqrt{S_0(r_1, r_1, \omega)} \sqrt{S_0(r_2, r_2, \omega)}}. \tag{6-7}
\]

We see that the degree of coherence is entirely determined by a single generalized Stokes parameter, namely \( S_0(r_1, r_2, \omega) \).

### 6.3. Effects of the source coherence on the propagating beam polarization

Consider now two sources located in the plane \( z = 0 \) as shown in Fig. 6.1. We distinguish the parameters relating to them by superscripts 1 and 2. Suppose that the two sources have the same sets of Stokes parameters, i.e. that

\[
s^{(2)}_{\alpha}(p', \omega) = s^{(1)}_{\alpha}(p', \omega), \quad (\alpha = 0, 1, 2, 3), \tag{6-8}
\]

but that

\[
S^{(2)}_{\alpha}(p_1', p_2', \omega) \neq S^{(1)}_{\alpha}(p_1', p_2', \omega), \quad (\alpha = 0, 1, 2, 3) \tag{6-9}
\]
According to Eqs. (6-3), (6-4) and (6-9) we then have, at any point $r$ in the half-space $z > 0$,

$$s^{(2)}_{\alpha}(r, \omega) \neq s^{(1)}_{\alpha}(r, \omega), \quad (\alpha = 0, 1, 2, 3), \quad (6-10)$$

except possibly at some particular points.

Equation (6-8) implies that the two sources have the same sets of the four Stokes parameters, and hence, according to Eqs. (6-8) and (6-1), they have the same degree of polarization, i.e.

$$P^{(2)}(p', \omega) = P^{(1)}(p', \omega). \quad (6-11)$$

However, Eqs. (6-7) and (6-9) imply that the two sources will have different coherence properties, i.e. that

$$\eta^{(2)}(p'_1, p'_2, \omega) \neq \eta^{(1)}(p'_1, p'_2, \omega) \quad (6-12)$$
Let us now consider the field in the half-space \( z > 0 \). We see, at once, from Eq. (6-9) and the propagation law (6-4) that, in view of Eq. (6-10),

\[
s_{0}^{(2)}(\mathbf{r}, \omega) \neq s_{0}^{(1)}(\mathbf{r}, \omega),
\]

throughout that half-space. Consequently, we see from Eqs (6-1) and (6-10) that, in general,

\[
P^{(2)}(\mathbf{r}, \omega) \neq P^{(1)}(\mathbf{r}, \omega), \quad (z > 0)
\]

i.e. the degrees of polarization of the two beams will, in general, be different throughout the half-space into which they propagate.

### 6.4. Numerical example

We will illustrate the result of the preceding section by an example. Consider two electromagnetic Gaussian Schell-model sources with cross-spectral density matrices

\[
W^{(m)}(\mathbf{p}', \mathbf{p}'', \omega) = \begin{pmatrix}
W_{xx}^{(m)}(\mathbf{p}', \mathbf{p}'', \omega) & 0 \\
0 & W_{yy}^{(m)}(\mathbf{p}', \mathbf{p}'', \omega)
\end{pmatrix},
\]

where for the diagonal elements

\[
W_{ii}^{(m)}(\mathbf{p}', \mathbf{p}'', \omega) = A_i \exp \left[ -\frac{\mathbf{p}_i^2}{4\sigma_i^2} \right] \exp \left[ -\frac{(\mathbf{p}''_i - \mathbf{p}'_i)^2}{2\delta_i^{(m)}_i} \right], \quad (m = 1, 2, \quad i = x, y),
\]

\( A_i, \sigma \) and \( \delta_i^{(m)} \) being constants. Let us also assume that \( A_i = 1 \) and that

\[
\delta_{x}^{(1)} = \delta_{x}^{(2)} = \delta_{x} \neq \delta_{y}^{(1)} \neq \delta_{y}^{(2)}.
\]

The usual Stokes parameters of the two sources are

\[
\begin{cases}
    s_0^{(m)}(\mathbf{p}', \omega) = 2 \exp \left[ -\frac{\mathbf{p}'^2}{2\sigma_i^2} \right], \\
    s_{1,2,3}^{(m)}(\mathbf{p}', \omega) = 0,
\end{cases}
\]

\( 92 \)
i.e. they are the same for any point $\rho'$ of the source plane. Consequently, the degrees of polarization of the two sources are also the same, i.e. $P^{(1)}(\rho', \omega) = P^{(2)}(\rho', \omega) = 0$. The generalized Stokes parameters of the two sources are

\[
\begin{align*}
S_0^{(m)}(\rho'_1, \rho'_2, \omega) &= \exp \left[ -\frac{\rho'_1^2 + \rho'_2^2}{4\sigma^2} \right] \exp \left[ -\frac{(\rho'_2 - \rho'_1)^2}{2\delta_x^2} \right] - \exp \left[ -\frac{\rho'_1^2 + \rho'_2^2}{4\sigma^2} \right] \exp \left[ -\frac{(\rho'_2 - \rho'_1)^2}{2\delta_x^2} \right] \\
S_1^{(m)}(\rho'_1, \rho'_2, \omega) &= \exp \left[ -\frac{\rho'_1^2 + \rho'_2^2}{4\sigma^2} \right] \exp \left[ -\frac{(\rho'_2 - \rho'_1)^2}{2\delta_y^2} \right] - \exp \left[ -\frac{\rho'_1^2 + \rho'_2^2}{4\sigma^2} \right] \exp \left[ -\frac{(\rho'_2 - \rho'_1)^2}{2\delta_y^2} \right] \\
S_2^{(m)}(\rho'_1, \rho'_2, \omega) &= S_3^{(m)}(\rho'_1, \rho'_2, \omega) = 0
\end{align*}
\]

$(m = 1, 2)$.

From the definition of the spectral degree of coherence $\eta (r_1, r_2, \omega)$ one readily finds that for the two sources

\[
\eta^{(m)}(\rho'_1, \rho'_2, \omega) = \exp \left[ -\frac{(\rho'_2 - \rho'_1)^2}{2\delta_x^2} \right] + \exp \left[ -\frac{(\rho'_2 - \rho'_1)^2}{2\delta_y^2} \right], \quad (m = 1, 2).
\]

From Eqs. (6-19) and (6-20) and the condition (6-17) one can see that the generalized Stokes parameters and the spectral degrees of coherence of the two sources differ from each other.

Let us determine the usual Stokes parameters of the two sources at a point $(\rho_1 = 0, \rho_2 = 0, z > 0)$, i.e. an arbitrary point on the axis of the beam in the half-space $z > 0$. On substituting from Eq. (6-19) into Eq. (6-4) for the components of the usual Stokes parameters in the source plane, performing the integration and evaluating the resulting expression for the point $\rho_1 = \rho_2 = 0$ we find that
\begin{align*}
\quad s_0^{(m)}(0, z, \omega) &= \frac{1}{\Delta_x^2(z)} + \frac{1}{[\Delta_y^{(m)}(z)]^2} \\
\quad s_1^{(m)}(0, z, \omega) &= \frac{1}{\Delta_x^2(z)} - \frac{1}{[\Delta_y^{(m)}(z)]^2} \\
\quad s_2^{(m)}(0, z, \omega) &= 0 \\
\quad s_3^{(m)}(0, z, \omega) &= 0
\end{align*} 

(6-21)

with

\[ \Delta_x^2(z) = 1 + \frac{z^2}{k^2 \sigma^2} \left( \frac{1}{4 \sigma^2} + \frac{1}{\delta_x^2} \right) \] 

(6-22a)

and

\[ \Delta_y^{(m)2}(z) = 1 + \frac{z^2}{k^2 \sigma^2} \left( \frac{1}{4 \sigma^2} + \frac{1}{\delta_y^{(m)}^2} \right), \quad (m = 1, 2). \] 

(6-22b)

From conditions (6-17) and Eq. (6-22b) it follows that

\[ \Delta_y^{(1)}(z) \neq \Delta_y^{(2)}(z) \] 

(6-23)

and, consequently, we find from Eqs. (6-21) that

\[ s_0^{(2)}(0, z, \omega) - s_0^{(1)}(0, z, \omega) = \frac{1}{[\Delta_y^{(2)}(z)]^2} - \frac{1}{[\Delta_y^{(1)}(z)]^2} \neq 0, \] 

(6-24a)

\[ s_1^{(2)}(0, z, \omega) - s_1^{(1)}(0, z, \omega) = \frac{1}{[\Delta_y^{(2)}(z)]^2} + \frac{1}{[\Delta_y^{(1)}(z)]^2} \neq 0, \] 

(6-24b)

i.e. the Stokes parameters \( s_0 \) of the beams are different and the same is true about the Stokes parameters \( s_1 \), while the last two parameters are the same, viz. \( s_2^{(1)}(0, z, \omega) = s_2^{(2)}(0, z, \omega) = 0 \) and
Hence we have shown that the polarization properties of the beams generated by the two sources are different, at least along the axis. Equivalently, the degrees of polarization of the two beams along the axis are given by the formulas

$$P^{(m)}(0, z, \omega) = \left| \frac{1}{\Delta^2_x(z)} - \frac{1}{[\Delta^{(m)}_y(z)]^2} \right| \frac{1}{\Delta^2_x(z)} + \frac{1}{[\Delta^{(m)}_y(z)]^2}, \quad (m = 1, 2).$$

(6-25)

Fig. 6.2 shows how the degrees of polarization, calculated from Eq. (6-25), of the beams generated by two unpolarized sources vary along the axis.

Fig. 6.2 : The degree of polarization, calculated from Eq. (6-25), of the beams generated by two unpolarized sources along the axis as function of propagation distance $z$ from the source. The parameters of the two sources have been chosen as $\sigma = 1 cm$, $\delta_x = 0.1 mm$, $\delta_y^{(1)} = 0.1 mm$, $\delta_y^{(2)} = 1 mm$, $\lambda = 0.633 \mu m$. 

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6.5 Conclusions

In conclusion we may say that we have shown that two stochastic electromagnetic beams, which propagate from the plane $z = 0$ into the half-space $z > 0$ and that have the same Stokes parameters in that plane may, in general, have different degrees of polarization in that half-space. The difference is due to possible differences in the coherence properties of the source, information, which is not contained in the usual Stokes parameter description of the beam. Finally we mention that the coherence of the beam is not the only factor, which may be responsible for polarization changes. It may be shown that differences in the two-point correlations between the orthogonal components of the electric field in the source plane may also generate beams of different polarization properties, even if the usual Stokes parameters of the two sources are the same.
CHAPTER 7

OPTICAL HETERODYNE DETECTION OF STOCHASTIC ELECTROMAGNETIC BEAMS

In this chapter we present a general analysis for the optical heterodyne detection of stochastic electromagnetic beams. To describe the ensemble of quasi-monochromatic beams that are partially polarized and partially coherent, we use a recently developed matrix treatment [8]. We derive an expression for the signal-to-noise ratio (SNR) in terms of the beams coherence polarization matrices on the detector surface. Numerical examples are given for the SNR variation in the case of partially polarized, Gaussian Schell-model beams and the optimum detection is discussed in terms of the beam parameters of the local oscillator. The results of this chapter have been published in reference [68].

7.1 Introduction

As we mentioned previously most optical sources emit light by the process of spontaneous emission from a collection of excited atoms or molecules as in the case of the thermal light generation. Such radiation is consisting of a large number of independent contributions that fluctuate in the course of time. Even for the case of a highly stabilized laser beam, the light might become stochastic in nature upon propagating through a random medium such as turbulent atmosphere. Hence, radiation that reaches an optical receiver exhibits always random behavior. Moreover light interacts with matter in a fundamentally random (stochastic) manner, due to quantum effects; that is, light can be absorbed only in small discrete energy quanta.
Consequently statistical models are required to describe the interaction between light and matter in the detection process.

The interaction between light and matter can be characterized rigorously using the theory of quantum electrodynamics [69]. In this approach the electromagnetic fields are quantized and the postulates of quantum mechanics are used to describe the detection process. This approach is fundamental but is mathematically rather demanding and it does not give deep physical insight into the detection process. The other approach, which has been successfully used to describe the detection process, is the semi-classical approach [70]. In the semi-classical approach the electromagnetic field is treated classically but the interaction with the photo-detector is treated quantum mechanically. Whereas the semi-classical method is mathematically not as demanding as the quantum mechanical one, the two approaches are predicting the same statistical features of the detection process [71]. In this chapter we use the semi-classical approach to describe the interaction between light and the photo-detecting surface.

Optical detection is usually carried out by using photodetectors that convert the incident light into an electrical current. The photodetectors are divided into three main categories according to the physical effect that produce the detector response: photoconductive, photoemissive and photovoltaic [72]. The photodetectors can be used in optical receivers in two configurations, either through direct (incoherent) detection or coherent (heterodyne) detection [73]. Direct detection is a simple energy collection process for the optical energy incident on the optical surface. In the other technique, the coherent (heterodyne) detection, the received signal is combined with a locally generated optical signal, which is usually uncorrelated with the received signal. The interest in the optical heterodyne technique is motivated by its high spectral
resolution and its noise reduction capability, since it is limited by shot noise. On the other hand, an efficient operation requires strict spatial alignment.

A number of publications have addressed issues regarding the principles of the optical heterodyne technique and two examples are given below. Mandel and Wolf [74] examined the problem of optimally detecting a coherent light in the presence of unwanted background noise, using a semi-classical treatment. Building on an analogy between the detector and the transmitting antenna, Siegman [75] pointed out to the relationship between the detector area, the detector’s field of view and the signal wavelength, well-known as the antenna theorem for optical heterodyne detection. He also pointed out that, compared with the receiving antenna, the optical heterodyne detector is not sensitive to thermal radiation but it might be sensitive for detecting Doppler shifts in coherent scattered light.

In order to quantify the performance of a heterodyne detection system, one could determine the signal-to-noise ratio (SNR) or the heterodyne efficiency of the system and examples for such quantifications are given below. Lathi [76] derived an expression for the SNR of mixing inhomogeneous, partially coherent fields and, later Lathi and Nagel [77] gave criteria for choosing an optimum local oscillator field to detect certain optical field optimally. Fink [78] derived a general formula for the SNR of the heterodyne detection in terms of the received and the local oscillator signals and the detector size and shape. McGuire [79] gave an expression for the detected mean-square current of the beating component of mixing two partially coherent and quasi-monochromatic sources in terms of their mutual coherence functions. Tanaka and Saga [80] obtained the optimum conditions of the detecting system that maximize the heterodyne efficiency. Tanaka et al. [81] pointed out the relationship between the partially coherent beam
parameters and the detector dimensions, needed to maintain optimum efficiency. The aforementioned work dealt mainly with the scalar signals or beams. Recently, Gori [8] proposed a matrix formulation for dealing with quasi-monochromatic, partially polarized light when the beam is not completely coherent, which considers the most general form of representing vectorial optical quasi-monochromatic stochastic beams. As an example of such beams, Gori et al. [32] gave an expression for the polarization coherence matrix of the partially polarized Gaussian Schell-model (GSM) beams. They discussed the propagation of such beams in free space and they presented a criterion for the realizability conditions of such beams. Very recently, Piquero et al. [62] showed how one could realize this class of sources experimentally. In this chapter we derive an expression for the SNR when two beams of this class are mixed on a detector surface.

7.2 Theory and analysis

Let us assume two beams co-aligned in phase at a detector surface (z=0 plane for a Cartesian coordinate system) in such a way that there is no phase shift between them, so that they satisfy the maximum heterodyne efficiency configuration. One can write the instantaneous field components of both signals at the detector surface as

\[
U_s(p,t) = \left[ U_{sx}(p) \hat{x} + U_{sy}(p) \hat{y} \right] e^{j\omega_s t}, \tag{7-1}
\]

\[
U_s(p,t) = \left[ U_{sx}(p) \hat{x} + U_{sy}(p) \hat{y} \right] e^{j\omega_s t}, \tag{7-2}
\]

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where \( \mathbf{x} \) and \( \mathbf{y} \) are unit vectors. The process of mixing two optical signals produces an electrical current known as the photomixing current and the incremental value of this current at any point on the detector surface is given by [73]

\[
di(p, t) = \Re(p) \left[ U(p, t)U^*(p, t) \right]
\] (7-3)

where \( \di(p, t) \) is the incremental current, \( \Re(p) \) is the responsivity of the detector. \( U(p, t) \) is the total field at a point specified by the position vector \( p \) and the asterisk denotes the complex conjugate. It follows that at any point \( p \), the incremental current can be written in terms of the overlapping fields as

\[
di(p, t) = \Re(p) \left[ U_{ax}(p)U_{ax}^*(p) + U_{sx}(p)U_{sx}^*(p) + U_{ay}(p)U_{ay}^*(p) + U_{sy}(p)U_{sy}^*(p) \\
+ U_{ax}(p)U_{sx}^*(p)e^{j(\omega_x - \omega_0)t} + U_{sx}(p)U_{ax}^*(p)e^{-j(\omega_x - \omega_0)t} \\
+ U_{ay}(p)U_{sy}^*(p)e^{j(\omega_y - \omega_0)t} + U_{sy}(p)U_{ay}^*(p)e^{-j(\omega_y - \omega_0)t} \right]
\] (7-4)

Considering only the detected intermediate-frequency (IF) component of the current, which is oscillating in time and neglecting the dc part of the current, one obtains the formula

\[
di_{IF}(p, t) = \Re(p) \left[ U_{ax}(p)U_{ax}^*(p)e^{j(\omega_x - \omega_0)t} + U_{sx}(p)U_{ax}^*(p)e^{-j(\omega_x - \omega_0)t} \\
+ U_{oy}(p)U_{sy}^*(p)e^{j(\omega_y - \omega_0)t} + U_{sy}(p)U_{oy}^*(p)e^{-j(\omega_y - \omega_0)t} \right].
\] (7-5)

The total integrated spatially and instantaneously varying IF current is given by the expression

\[
i_{IF}(t) = \iiint \Re(p) \left[ U_{ax}(p)U_{ax}^*(p)e^{j(\omega_x - \omega_0)t} + U_{sx}(p)U_{ax}^*(p)e^{-j(\omega_x - \omega_0)t} \\
+ U_{oy}(p)U_{sy}^*(p)e^{j(\omega_y - \omega_0)t} + U_{sy}(p)U_{oy}^*(p)e^{-j(\omega_y - \omega_0)t} \right] d^2 p.
\] (7-6)
One could now define the time-averaged power corresponding to a single representation (realization) of the random field as $\bar{i_{IF}}^*i_{IF}$ where $i_{IF}$ is the instantaneous IF current and the bar denotes the time average over an interval short compared with the characteristic time of variation in $U_o(\rho,\tau)$ and $U_s(\rho,\tau)$. Hence, the total power is the ensemble average over all possible realization of the time average power, i.e.

$$P_{\text{total}} = \langle i_{IF}^*i_{IF} \rangle$$

$$\{\Re[(\rho_1)\Re(\rho_2)][\langle U_{ox}(\rho_1)U_{sx}^*(\rho_1)U_{ox}^*(\rho_2)U_{sx}(\rho_2) \rangle + \langle U_{ox}(\rho_1)U_{ox}^*(\rho_1)U_{oy}(\rho_2)U_{sy}(\rho_2) \rangle + \langle U_{sx}(\rho_1)U_{ax}^*(\rho_1)U_{ax}^*(\rho_2)U_{sx}(\rho_2) \rangle + \langle U_{sy}(\rho_1)U_{ox}^*(\rho_1)U_{sy}(\rho_2)U_{sy}(\rho_2) \rangle + \langle U_{oy}(\rho_1)U_{oy}^*(\rho_1)U_{ox}(\rho_2)U_{sy}(\rho_2) \rangle + \langle U_{sy}(\rho_1)U_{sy}^*(\rho_1)U_{sy}(\rho_2)U_{oy}(\rho_2) \rangle \rangle d^2\rho_1 d^2\rho_2.$$  

(7-7)

As can be seen, the total electrical power in Eq. (7-7) is proportional to the intensity-intensity correlations of the optical fields on the surface of the detector. It is important to note that the study of the averaged electrical power includes the fourth-order correlations of the optical fields. Next noting that in Eq. (7-7) we have each term and its complex conjugate, one could rewrite the equation as

$$P_{\text{total}} = \langle i_{IF}^*i_{IF} \rangle$$

$$\{\Re[(\rho_1)\Re(\rho_2)]\Re(\rho_1)\Re(\rho_2)[U_{sx}^*(\rho_1)U_{ox}(\rho_2)U_{sx}(\rho_1)U_{ox}(\rho_2)] \} + \Re(\rho_1)\Re(\rho_2)[U_{sx}(\rho_1)U_{ax}(\rho_2)U_{sx}(\rho_1)U_{ax}(\rho_2)] + \Re(\rho_1)\Re(\rho_2)[U_{sx}(\rho_1)U_{sy}(\rho_2)U_{sx}(\rho_1)U_{sy}(\rho_2)] + \Re(\rho_1)\Re(\rho_2)[U_{sy}(\rho_1)U_{oy}(\rho_2)U_{sy}(\rho_1)U_{oy}(\rho_2)] + \Re(\rho_1)\Re(\rho_2)[U_{sy}(\rho_1)U_{sy}(\rho_2)U_{sy}(\rho_1)U_{sy}(\rho_2)] \} d^2\rho_1 d^2\rho_2.$$  

(7-8)
Assuming further a Gaussian random process of zero mean for the random field components and applying the moment theorem for such distributions (c.f. [23], sect. 1.6.1), each ensemble average on the right-hand side of Eq. (7-8) factorizes and the averaged detected power becomes

\[
P_{\text{total}} = \left< i_{\text{IF}} i_{\text{IF}}^* \right> \]

\[
\{\Re(\rho_1)\Re(\rho_2) \times 2\Re\left[ \left< U_{xx}^*(\rho_1)U_{ox}(\rho_1) \right> \left< U_{xx}^*(\rho_2)U_{ox}(\rho_2) \right> \right] + \left< U_{xx}^*(\rho_1)U_{xx}(\rho_1) \right> \left< U_{ox}^*(\rho_2)U_{ox}(\rho_2) \right> + \left< U_{ox}^*(\rho_1)U_{ox}(\rho_1) \right> \left< U_{ox}^*(\rho_2)U_{xx}(\rho_2) \right> + \left< U_{ox}^*(\rho_1)U_{xx}(\rho_1) \right> \left< U_{ox}^*(\rho_2)U_{ox}(\rho_2) \right> \right.

\[
= \int \int \int \left[ \left< U_{xx}^*(\rho_1)U_{sy}(\rho_2) \right> \left< U_{ox}^*(\rho_2)U_{ox}(\rho_1) \right> \right] + \left[ \left< U_{sy}^*(\rho_1)U_{ox}(\rho_1) \right> \left< U_{ox}^*(\rho_2)U_{xx}(\rho_2) \right> \right] + \left[ \left< U_{sy}^*(\rho_1)U_{xx}(\rho_1) \right> \left< U_{ox}^*(\rho_2)U_{sy}(\rho_2) \right> \right] + \left[ \left< U_{ox}^*(\rho_1)U_{sy}(\rho_1) \right> \left< U_{ox}^*(\rho_2)U_{sy}(\rho_2) \right> \right])d^2 \rho_1 d^2 \rho_2.
\]

(7-9)

The expression for the total power can be further simplified if we consider that the received beam and the locally generated beam are mutually independent so that there is no correlation between the detected signal and the local oscillator signal; hence all the cross-correlation terms between the two overlapped signals vanish and one obtains

\[
P_{\text{total}} = \left< i_{\text{IF}} i_{\text{IF}}^* \right> \]

\[
\{\Re(\rho_1)\Re(\rho_2) \times 2\Re\left[ \left< U_{xx}^*(\rho_2)U_{ox}(\rho_1) \right> \left< U_{xx}^*(\rho_1)U_{ox}(\rho_1) \right> \right] + \left< U_{xx}^*(\rho_1)U_{xx}(\rho_1) \right> \left< U_{ox}^*(\rho_2)U_{ox}(\rho_2) \right> + \left< U_{ox}^*(\rho_1)U_{xx}(\rho_1) \right> \left< U_{ox}^*(\rho_2)U_{xx}(\rho_2) \right> \right.

\[
= \int \int \int + \left[ \left< U_{xx}^*(\rho_1)U_{sy}(\rho_2) \right> \left< U_{ox}^*(\rho_2)U_{ox}(\rho_1) \right> \right] + \left[ \left< U_{sy}^*(\rho_1)U_{ox}(\rho_1) \right> \left< U_{ox}^*(\rho_2)U_{xx}(\rho_2) \right> \right] + \left[ \left< U_{sx}^*(\rho_1)U_{sy}(\rho_2) \right> \left< U_{ox}^*(\rho_2)U_{sy}(\rho_2) \right> \right] + \left[ \left< U_{ox}^*(\rho_1)U_{sy}(\rho_1) \right> \left< U_{ox}^*(\rho_2)U_{sy}(\rho_2) \right> \right])d^2 \rho_1 d^2 \rho_2.
\]

(7-10)

One could rewrite the previous equation in terms of the beam coherence polarization (BCP) matrices of the two overlapped beams as
\[
\begin{align*}
P_{\text{total}} &= \left\langle \Im \left( iF \right) \right\rangle^* \\
&= \iint \{ \Re(\rho_1) \Re(\rho_2) \times (2 \text{Re}[\text{Tr}\{J_\rho(\rho_1, \rho_2) J_o(\rho_2, \rho_1)\}]) \} d^2\rho_1 d^2\rho_2, \\
&= \iint \{ \Re(\rho_1) \Re(\rho_2) \times (2 \text{Re}[\text{Tr}\{U_{ox}^*(\rho_2) U_{ox}(\rho_1) \} \} \} U_{ax}^*(\rho_1) U_{ax}(\rho_2) \} \\
&\quad + \{ U_{oy}^*(\rho_2) U_{oy}(\rho_1) \} \} U_{sy}^*(\rho_1) U_{sy}(\rho_2) \} \} d^2\rho_1 d^2\rho_2,
\end{align*}
\] (7-11)

where Tr denotes the trace. To check the validity of the previous expression we compare it with the power expression that has been derived previously for a scalar, partially coherent beam. Assuming that there are no cross-correlation terms in the expression of the BCP matrices, one immediately finds that

\[
P_{\text{total}} = \left\langle \Im \left( iF \right) \right\rangle^* \\
= \iint \{ \Re(\rho_1) \Re(\rho_2) \times (2 \text{Re}[\text{Tr}\{J_\rho(\rho_1, \rho_2) J_o(\rho_2, \rho_1)\}]) \} d^2\rho_1 d^2\rho_2,
\] (7-12)

which is equivalent to the results obtained for the scalar partially coherent signals case in [76].

In Eq. (7-11), \( P_{\text{total}} \) represents the value of the power carried in the detected beam. In order to find an expression for the SNR of such a system, the noise equivalent power (NEP) should be determined also and, assuming that the detector is operating at the shot noise limit, one could define the NEP as \( 2eBI_o \), where \( e \) is the electron change, \( B \) is the noise bandwidth of the IF filter and \( I_o \) is the average photo-current of the local oscillator defined as:

\[
I_o = \iint \Re(\rho) \text{Tr}[J_\rho(\rho, \rho)] d^2\rho.
\] (7-13)
Hence from Eq. (7-11) and Eq. (7-13), one can derive an expression for the SNR of the optical heterodyne system for partially polarized, partially coherent beams in terms of their BCP matrices as

\[
\text{SNR} = \frac{\iint \iint \{\Re(\rho_2)\Re(\rho_2) \times (2\Re[\text{Tr}\{J_x(\rho_1, \rho_2)J_o(\rho_2, \rho_1)\}])\}d^2 \rho_1 d^2 \rho_2}{2eB\iint \Re(\rho)\text{Tr}[J_o(\rho, \rho)] d^2 \rho}.
\]

(7-14)

Assuming that the responsivity does not vary with position then it could be defined as

\[\Re = e\eta_q / h\nu,\]

where \(\eta_q\) is the quantum efficiency of the photo-surface, \(h\) is Planck’s constant and \(\nu\) is the optical frequency. The corresponding expression for the signal-to-noise ratio for this responsivity could be normalized as follows:

\[
\text{SNR}^* = \frac{\text{SNR}}{\eta/(h\nu B)} = \frac{\iint \iint \Re[\text{Tr}\{J_x(\rho_1, \rho_2)J_o(\rho_2, \rho_1)\}]d^2 \rho_1 d^2 \rho_2}{\iint \text{Tr}[J_o(\rho, \rho)] d^2 \rho}.
\]

(7-15)

Eq. (7-15) constitutes the main result as it gives the normalized SNR of the system in terms of the BCP matrices of the two overlapped beams at the detector surface.

7.3 Results and discussion

As an illustration for the preceding analysis we will present a numerical example for the mixing process, assuming that both the detected signal and the local oscillator signal on the detector surface could be defined in the form of BCP matrix components of partially polarized GSM
beam as given in [32], where the elements of the matrix are defined at the detector surface (z=0) as

\[
J_{\alpha ij}(\rho_1, \rho_2) = I_{\alpha ij} \exp\left(-\frac{(\rho_1^2 + \rho_2^2)}{4\sigma_{I\alpha ij}^2} - \frac{(\rho_1 - \rho_2)^2}{2\sigma_{\mu ij}^2}\right),
\]

(7-16)

where \(\alpha = s\) (signal) or \(o\) (local oscillator) and \(i = (x, y), j = (x, y)\). \(\sigma_{I\alpha ij}\) is the intensity width and \(\sigma_{\mu ij}\) is the correlation width of the element \(ij\) of the BCP matrix of either the signal or the local oscillator.

The degree of polarization (DOP) of the partially polarized, partially coherent GSM beam at the detector surface is given by the formula [32]:

\[
DOP_\alpha(\rho, 0) = \left\{ \frac{(I_{\alpha xx} - I_{\alpha yy})^2}{I_{\alpha xx} + I_{\alpha yy}} \right\}^2 + \frac{4I_{\alpha xy}}{(I_{\alpha xx} + I_{\alpha yy})^2} \exp\left[ -\left( \frac{1}{\sigma_{\alpha xy}^2} - \frac{1}{\sigma_{\alpha xx}^2} \right) \rho^2 \right] \}^{1/2}
\]

(7-17)

Evidently, one could control the degree of polarization of a beam by changing some of its parameters and could also change the correlation widths and the intensity widths of different correlations \((xx, yy, xy\text{ and }yx)\).

In the following, we study the variation of the normalized SNR with the detector radius for several cases for which we have assumed the same parameters for the detected signal while varying only one parameter of the local oscillator. In each case we computed numerically the corresponding normalized SNR to establish the condition of optimal detection (maximum SNR and minimum detector radius). In all the cases presented here the detected beam parameters have been assumed as follows: \(I_{sxx} = I_{syy} = 0.05, I_{sxy} = I_{syx} = 0.01, \sigma_{Is} = 0.5 \text{ mm}, \sigma_{\mu sxx} = \sigma_{\mu syy} = 0.2 \text{ mm}\) and \(\sigma_{\mu sxy} = \sigma_{\mu syx} = 0.25 \text{ mm}\).
In Fig. 7.1 we present the results of varying \( \sigma_{\mu_{ox}} = \sigma_{\mu_{oy}} \) while keeping the other parameters fixed. In this case we increased gradually the width of the correlation width in the xx and yy correlations for the local oscillator until the corresponding value of the detected signal is reached. As can be seen from Fig. 7.1, when the width of the correlation of the oscillator beam increases, approaching the same value of the detected signal, the SNR also increases. Note that there is an optimal detection radius that increases slightly.

Fig. 7.1: The normalized SNR versus the detector radius for different values of \( \sigma_{\mu_{ox}} = \sigma_{\mu_{oy}} \). The other parameters of the local oscillator beam are \( I_{ox} = I_{oy} = 0.5, I_{oxy} = I_{oxy} = 0.1, \sigma_{Io} = 1 \text{ mm} \) and \( \sigma_{\mu_{oxy}} = \sigma_{\mu_{oxy}} = 0.2 \text{ mm} \). The parameters of the detected beam are \( I_{sxx} = I_{syy} = 0.05, I_{sxy} = I_{syx} = 0.01, \sigma_{Is} = 0.5 \text{ mm}, \sigma_{\mu_{sxx}} = \sigma_{\mu_{syy}} = 0.2 \text{ mm} \) and \( \sigma_{\mu_{sxy}} = \sigma_{\mu_{syx}} = 0.25 \text{ mm} \).
In Fig. 7.2 we present the results of varying $\sigma_{\mu_{\text{oxy}}} = \sigma_{\mu_{\text{oyx}}}$ while keeping the other parameters fixed. In this case we increased the width of the correlation of the $xy$ and the $yx$ correlations for the local oscillator beam. As can be seen, when the width of the correlation of the oscillator beam increases, the detected SNR also increases. However, the optimal detection radius of the detector does not change significantly in this case. In general, the changes in the SNR are much smaller for this case than for the case demonstrated in Fig. 7.1. This happens because the amplitude of the intensity $I_{\text{oxy}}$, for $xy$ and $yx$ correlation is smaller than the amplitude of the intensity $I_{\text{oxo}}$, for the $xx$ and $yy$ correlations; hence the effect of those correlations ($xy$ and $yx$) on the improvement in the SNR is small.

![SNR vs Detector Radius](image)

**Fig. 7.2**: The normalized SNR versus the detector radius for different values of $\sigma_{\mu_{\text{oxy}}} = \sigma_{\mu_{\text{oyx}}}$. The other parameters of the local oscillator beam are $I_{\text{oxo}} = I_{\text{owy}} = 0.5$,
Next, we vary only $\sigma_{Io}$ while keeping the other parameters fixed. The results are shown in Fig. 7.3. In this case we increased the width of the intensity of the oscillator starting from a value smaller than the corresponding intensity width of the detected signal up to a value much larger than the corresponding value of the detected signal. In this case, as the beam width of the local oscillator decreases, the SNR increases but without reaching an optimum value for the detector’s radius. If the beam width of the local oscillator increases to approach the corresponding value of the detected beam or becomes larger, the SNR decreases and an optimum value is obtained for the detector radius, which does not depend on the beam width.

$\sigma_{I_{oxy}} = I_{O_{yx}} = 0.1, \sigma_{Io} = 1 \text{ mm and } \sigma_{I_{oxyy}} = \sigma_{I_{yox}} = 0.2 \text{ mm}$. The parameters of the detected beam are the same as in Fig. 7.1.

Fig. 7.3: The normalized SNR versus the detector radius for different values of $\sigma_{Io}$. The other parameters of the local oscillator beam are $I_{ox} = I_{oy} = 0.5$, $I_{oxy} = I_{oyx} = 0.1$, 

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\[ \sigma_{\mu_{ox}} = \sigma_{\mu_{oy}} = 0.2 \text{ mm and } \sigma_{\mu_{xy}} = \sigma_{\mu_{yx}} = 0.2 \text{ mm.} \] The parameters of the detected beam are the same as in Fig. 7.1.

Finally, in Fig. 7.4 we present the results of varying \( I_{oxy} = I_{oyx} \) while keeping the other parameters fixed. As the DOP is increased by increasing \( I_{oxy} = I_{oyx} \) for the oscillator beam, the detected SNR increases also and the optimal detector radius does not change significantly in this case. Note that for larger values of DOP the effect of the \( xy \) and \( yx \) correlations also increases; and hence the SNR then increases.

![Graph](image)

**Fig. 7.4**: The normalized SNR versus the detector radius for different values of \( I_{oxy} = I_{oyx} \). The other parameters of the local oscillator beam \( I_{ox} = I_{oy} = 0.5, \sigma_{Io} = 1 \text{ mm, } \sigma_{\mu_{ox}} = \sigma_{\mu_{oy}} = 0.2 \text{ mm and } \sigma_{\mu_{xy}} = \sigma_{\mu_{yx}} = 0.2 \text{ mm.} \) The parameters of the detected beam are the same as in Fig. 7.1.
7.4 Conclusions

An expression has been derived for the detected SNR from mixing two partially polarized, partially coherent beams. Examples have been given for the mixing process in the case of partially polarized GSM beams. The numerical examples demonstrate that the SNR of the heterodyne detection could be adjusted by controlling the local oscillator beam parameters. For the partially polarized, partially coherent beams, one has several degrees of freedom, which permit the detection to be optimally controlled. We have shown by some examples that the optimal values for the parameters of the local oscillator beam depend on specific characteristics of the detected beam.
CHAPTER 8
HETERODYNE EFFICIENCY OF A COHERENT DETECTION SYSTEM FOR PARTIALLY COHERENT BEAMS

In this chapter we consider the heterodyne efficiency as a measure of quality for a coherent detection system. The heterodyne efficiency reflects the matching between the received beam and the local oscillator beam on the detector surface and one could use this property for the alignment of the system. In the following we derive a general expression for the heterodyne efficiency of a detection system for beams at any state of coherence, assuming that the propagation direction for both signals (the received signal and the locally generated one) are slightly different. We derive an analytical expression for the heterodyne efficiency when mixing coherently, two partially coherent Gaussian Schell-model beams on a photo-detector surface. Numerical examples are given for the variation in the heterodyne efficiency with the detector radius and with the parameters of the overlapping beams. We show that the stability of partially coherent beams for the misalignment of the detection system is better than coherent beams.

8.1 Introduction

As we mentioned in the previous chapter the incoherent detection or the direct detection has poor noise rejection properties, especially for the detection of weak signals or when the detector has low sensitivity [82]. On the other hand heterodyne detection is a more powerful detection technique that has been used in different applications due to its noise reduction capabilities and also its high spectral resolution [20, 83, and 84]. But on the other hand a coherent detection
system requires strict alignment [21]. For example Siegman in his seminal paper [75] noted the trade-off between the angular field of view of the detector and its effective capture area, with their product limited by the square of the received wavelength.

Heterodyne (coherent) detection can be performed by mixing the received optical signal with a locally generated signal on a photo-detector surface and measuring the electrical output of the detector [73]. For efficient coherent mixing on the detector surface, the locally generated signal parameters should match the received signal parameters [85] and the heterodyne efficiency reflects this matching [86]. The heterodyne efficiency also has been considered a measure of quality of the heterodyne detection system to evaluate the performance of the coherent detection technique and compare it to the incoherent scheme [87]. In coherent detection systems the heterodyne efficiency has been considered as a metric to quantify the potential misalignment between the received signal and the locally generated one [88], which in turn reduces the signal-to-noise (SNR) ratio of the detection system.

The heterodyne efficiency of spatially fully coherent signals has been studied a long time ago. For example Fink [78] derived expressions for the SNR and the heterodyne efficiency in coherent detection system of deterministic fields. Effects of atmospheric conditions on the heterodyne efficiency have been studied for coherent laser radar (CLR) systems [89-90]. Several authors investigated the effect of misalignment or truncation of propagating beams on the heterodyne efficiency of CLR systems. For example Cohen [85] studied the effects of signal wave-front misalignment for various combinations of received signals and local oscillators. Tanaka and Ohta [91] studied the effect of the tilt and the offset of the received signal on the heterodyne efficiency of Gaussian beams and Tanaka and Saga derived the optimal conditions.
for maximum heterodyne efficiency in coherent systems in the presence of background radiation [80]. Chambers [86] gave a model for the heterodyne efficiency in the presence of wave-front distortion due to aberrations in CLR systems and later Delautre et al. [92] studied this experimentally, using spatial light modulators. Salzman and Katzir [93] described, using a circular symmetric receiver in a detection system, the heterodyne efficiency of the system as a product of several matrices that represent the effect of the parameters of the optical system such as defocusing and the Fresnel number of the optical system. The heterodyne efficiency of mixing partially coherent optical scalar signals has been studied by Tanaka et al. [81] for a well aligned detection system.

In this chapter we relax the alignment restrictions and assume that the propagation directions of the two beams are not parallel in order to investigate the sensitivity of the detection scheme to misalignment. We derive an expression for the heterodyne efficiency of mixing two quasi-monochromatic and spatially partially coherent beams on a detector surface in terms of the mutual coherence functions of the two beams in the presence of a slight shift between the propagation directions of the two signals. We derive an analytical expression for the heterodyne efficiency when the two overlapping beams are Gaussian Schell-model beams, which are the simplest analytical model of partially coherent beams. Finally, we present some numerical results to show the variation in the heterodyne efficiency with the parameters of the local oscillator beam and the angular shift between the propagation directions of the overlapped beams.

8.2 Heterodyne detection of partially coherent beams

Let us assume two quasi-monochromatic, partially coherent beams propagating normal to a detector surface (z = 0 plane for Cartesian coordinate system) but with a misalignment θ between
their directions of propagation as shown in Fig. 8.1. Let us further assume that the detector is perpendicular to the local oscillator beam’s propagation direction, one can then express the instantaneous field variables of both signals at the detector surface as

\[ U_o(p, t) = U_o(p)e^{j\omega_o t}, \quad (8-1) \]

\[ U_s(p, t) = U_s(p)e^{j\omega_s t}e^{jk\rho}, \quad (8-2) \]

where \( \omega_o \) and \( \omega_s \) are the central angular frequencies of the frequency modulated signals and \( k \) is the wave vector of the received signal.

![Diagram](image.png)

**Fig. 8.1**: Illustration for the notation of mixing two beams, with a shift \( \theta \) between their phase fronts, on a detector in the \( z = 0 \) plane.
The process of mixing two optical signals produces an electrical current known as the photomixing current and the incremental value of this current at any point \( \rho \) on the detector surface is given by [73]:

\[
di(\rho, t) = \Re(\rho)[U(\rho, t)U^*(\rho, t)],
\]

where \( di(\rho, t) \) is the incremental current, \( \Re(\rho) \) is the local responsivity of the detector, \( U(\rho, t) \) is the total field at point \( \rho \) and the asterisk denotes the complex conjugate. It follows that at any point \( \rho \), the incremental current can be expressed in terms of the overlapping fields as

\[
di(\rho, t) = \Re(\rho)[U_o(\rho)U_o^*(\rho) + U_s(\rho)U_s^*(\rho)
\]

\[
+ U_o(\rho)U_s^*(\rho)e^{j(\omega_o - \omega_s)t}e^{-jK\rho} + U_s(\rho)U_o^*(\rho)e^{-j(\omega_o - \omega_s)t}e^{+jK\rho}].
\]

\[\text{(8-4)}\]

Considering only the detected intermediate frequency (IF) component of the current which is oscillating in time and neglecting the direct current (DC) one finds that

\[
di(\rho, t) = \Re(\rho)[U_o(\rho)U_o^*(\rho)e^{j(\omega_o - \omega_s)t}e^{+jK\rho} + U_s(\rho)U_o^*(\rho)e^{-j(\omega_o - \omega_s)t}e^{-jK\rho}].
\]

\[\text{(8-5)}\]

The total integrated spatially and instantaneously varying IF current is given by the expression

\[
i_{IF}(t) = \int[\Re(\rho)[U_o(\rho)U_o^*(\rho)e^{j(\omega_o - \omega_s)t}e^{+jK\rho} + U_s(\rho)U_o^*(\rho)e^{-j(\omega_o - \omega_s)t}e^{-jK\rho}]}d^2\rho.
\]

\[\text{(8-6)}\]

One can define the time averaged power corresponding to one representation (realization) of the random field as \( \overline{i^*_{IF}} \), where the bar denotes the time average over an interval short compared with the characteristic time of variation of the fluctuating fields \( U_o(\rho) \) and \( U_s(\rho) \). Hence, the total power is the ensemble average over all the possible realizations of the time average power i.e.
\[ P_{\text{total}} = \left\{ i_{IF}(t) i_{IF}^*(t) \right\} \]
\[ = \int \int \int \Im(\rho_1) \Im(\rho_2) \left[ U_o(\rho_1) U_s^*(\rho_1) \exp(j \Delta \omega t + j K \rho_1 + C.C) \right] \left[ U_o(\rho_2) U_s^*(\rho_2) \exp(j \Delta \omega t + j K \rho_2 + C.C) \right] d^2 \rho_1 d^2 \rho_2, \]

(8-7)

where \( \Delta \omega = \omega_o - \omega_s \) and C.C denotes the complex conjugate. Eq. (8-7) reduces to the following form after performing the time averaging

\[ P_{\text{total}} = \int \int \int 2 \Re [\Im(\rho_1) \Im(\rho_2)] \left( U_o(\rho_1) U_s^*(\rho_1) U_o^*(\rho_2) U_s(\rho_2) \right) \exp(j (K \rho_1 - K \rho_2)) d^2 \rho_1 d^2 \rho_2. \]

(8-8)

Assuming further a Gaussian distribution with zero mean for the random field components and applying the moment theorem for such distributions, the ensemble average on the right hand side of Eq. (8-8) can be factorized and the averaged detected power becomes:

\[ P_{\text{total}} = \int \int [2 \Re \Im(\rho_1) \Im(\rho_2)] \left( U_o(\rho_1) U_s^*(\rho_1) U_o^*(\rho_2) U_s(\rho_2) \right) \exp(j (K \rho_1 - K \rho_2)) d^2 \rho_1 d^2 \rho_2. \]

(8-9)

The expression for the total power can be further simplified if one assumes that the received beam and the locally generated one are mutually independent so that there is no correlation between the detected signal and the local oscillator signal. All the cross-correlation terms between the two overlapped signals then vanish and one finds that:

\[ P_{\text{total}} = \int \int 2 \Re \Im(\rho_1) \Im(\rho_2) [\Gamma_o(\rho_1, \rho_2) \Gamma_s^*(\rho_1, \rho_2)] \exp(j (K \rho_1 - K \rho_2)) d^2 \rho_1 d^2 \rho_2, \]

(8-10)
where $\Gamma_s(\rho_1, \rho_2)$ and $\Gamma_o(\rho_1, \rho_2)$ are the mutual coherence functions for the overlapped beams on the detector surface. The above expression represents the useful power which appears at the output of the detection system. Assuming shot-noise limits for the detection system, which is a reasonable assumption for heterodyne detection, one obtains for the noise equivalent power (NEP) the expression [73]

$$NEP = 2eB[\Re(\rho)\Gamma_o(\rho, \rho) d^2 \rho, \quad (8-11)$$

where $B$ is the bandwidth of the intermediate frequency (IF) filter and $e$ is the electron charge.

One then obtains for the signal-to-noise ratio (SNR) of the system the expression

$$SNR = \frac{\|[\text{Re}(\rho_1)\Re(\rho_2)\Gamma_o(\rho_1, \rho_2)\Gamma_s^*(\rho_1, \rho_2)\exp(j(K_1, \rho_1 - K_2, \rho_2)) d^2 \rho_1 d^2 \rho_2}{2eB[\Re(\rho)\Gamma_o(\rho, \rho) d^2 \rho}$$

(8-12)

Further, assuming that the responsivity does not vary with the position on the detector, it can then be defined as $\Re = \frac{e\eta_q}{h\nu}$ at different positions, where $\eta_q$ is the quantum efficiency of the photo-surface, $h$ is the Planck's constant, and $\nu$ is the optical frequency [73]. Under this assumption one could define a normalized SNR by the expression [68]

$$SNR^* = \frac{\|[\text{Re}(\Gamma_o(\rho_1, \rho_2)\Gamma_s^*(\rho_1, \rho_2)\exp(j(K_1, \rho_1 - K_2, \rho_2)) d^2 \rho_1 d^2 \rho_2}{\|[\Gamma_o(\rho, \rho) d^2 \rho}$$

(8-13)

We note that the normalized SNR has the units (J.Hz) and does not depend on the detector parameters.
8.3 Heterodyne efficiency of partially coherent beams

The heterodyne efficiency can be defined as the ratio of the total coherent detected power as given in Eq. (8-10) to the maximum photo-mixing power available when the overlapping beams have the same spatial variation in their magnitudes and phases at the detector surface. The maximum power deduced from Eq. (8-10) is

\[ P_{\text{max}} = 2I_o I_s. \] (8-14)

Here \( I_o \) and \( I_s \) are the incoherent electrical currents corresponding to the local oscillator and the received signal respectively, they are given by the expressions

\[ I_o = \iint \Re(\rho) \Gamma_o(\rho, \rho) \, d^2 \rho, \] (8-15)

\[ I_s = \iint \Re(\rho) \Gamma_s(\rho, \rho) \, d^2 \rho. \] (8-16)

Hence the heterodyne efficiency for mixing two random beams may be defined as

\[
\eta_h = \frac{\iint \Re(\rho_1) \Re(\rho_2) [\Gamma_o(\rho_1, \rho_2) - \Gamma_s(\rho_1, \rho_2)] \exp(j(K \cdot \rho_1 - K \cdot \rho_2)) d^2 \rho_1 d^2 \rho_2}{\iint \Re(\rho) \Gamma_o(\rho, \rho) \, d^2 \rho \iint \Re(\rho) \Gamma_s(\rho, \rho) \, d^2 \rho}.
\] (8-17)

Using Eqs. (8-13) and (8-17) one can relate the heterodyne efficiency \( \eta_h \) to the normalized SNR of the detection system:

\[
\eta_h = \frac{\text{SNR}^*}{\iint \Re(\rho) \Gamma_s(\rho, \rho) \, d^2 \rho}.
\] (8-18)

Finally, under the assumption that the responsivity of the detector does not vary with the position, the corresponding expression for the heterodyne efficiency in this case is:
It is evident from Eq. (8-19) that the heterodyne efficiency represents an absolute measurement of the normalized coherent detection dependence on the beams parameters at the detector surface. It is to be noted that the heterodyne efficiency is independent of the detector parameters. The heterodyne efficiency could also be interpreted as the reduction of the optimum incoherent power on mixing two beams coherently on a detector surface [94]. Equation (8-19) represents the main results of this chapter. In the next section we will give an analytical solution for this equation when the two overlapping beams have a Gaussian Schell-model (GSM) form.

8.4 Heterodyne efficiency of partially coherent GSM beams

As an illustration for the preceding analysis, we will give an example for the mixing process, assuming that both the detected signal and the local oscillator signal on the detector surface are partially coherent GSM beams as given by [23], where the mutual coherence functions of both signals are defined at the detector surface (z=0) as:

\[
\Gamma_\alpha (\bm{\rho}_1, \bm{\rho}_2) = I_\alpha \exp \left[ -\frac{(\rho_1^2 + \rho_2^2)}{4\sigma^2_\alpha} - \frac{(\rho_1 - \rho_2)^2}{2\delta^2_\alpha} \right].
\] (8-20)

Here \( \alpha = s \) (signal) or \( o \) (local oscillator), \( \sigma \) is the intensity width of the signal and \( \delta \) is the coherence width of the signal. Now let us assume that the angle \( \theta \) is small so that \( \bm{k} \cdot \bm{\rho} \approx k \rho \cos(\phi) \) where \( d^2 \rho = \rho d\rho d\phi \) is the incremental area of the detector surface. Hence Eq. (8-19) may be written as follows
Further if we use the soft aperture approximation [82] one could replace the physical aperture (the circular one with radius R) with a Gaussian aperture (soft aperture) with an $e^{-1}$ decrease at $W^2 = (R^2/2)$ (W is the radius of the soft aperture). By using this approximation one finds that

$$\eta_h = \frac{\int_{\phi_2=0}^{2\pi} \int_{\phi_1=0}^{2\pi} \int_{\rho_2=0}^{R} \int_{\rho_1=0}^{R} \text{Re}[\Gamma_o(\mathbf{p}_1, \mathbf{p}_2)\Gamma_s^*(\mathbf{p}_1, \mathbf{p}_2)] \exp(jk\rho_1 \cos \phi_1 \theta - jk\rho_2 \cos \phi_2 \theta) \rho_1 \rho_2 d\rho_1 d\rho_2 d\phi_1 d\phi_2}{\int_{\phi_2=0}^{2\pi} \int_{\phi_1=0}^{2\pi} \int_{\rho_2=0}^{R} \int_{\rho_1=0}^{R} \Gamma_o(\mathbf{p}, \mathbf{p}) \rho d\rho d\phi d\theta}.$$  

(8-21)

For a mutual coherence function of Gaussian form the above integration can be performed analytically as shown in details in [Appendix C] and one can then express the heterodyne efficiency for this case as:

$$\eta_h = \frac{\int_{\phi_2=0}^{2\pi} \int_{\phi_1=0}^{2\pi} \int_{\rho_2=0}^{\infty} \int_{\rho_1=0}^{\infty} \text{Re}[\Gamma_o(\mathbf{p}_1, \mathbf{p}_2)\Gamma_s^*(\mathbf{p}_1, \mathbf{p}_2)] e^{-(\rho_1^2 + \rho_2^2)/W^2} \rho d\rho d\phi d\theta}{\int_{\phi_2=0}^{2\pi} \int_{\phi_1=0}^{2\pi} \int_{\rho_2=0}^{\infty} \int_{\rho_1=0}^{\infty} \Gamma_o(\mathbf{p}, \mathbf{p}) e^{-(\rho^2)/W^2} \rho d\rho d\phi d\theta}.$$  

(8-22)

$$\eta_h = \frac{(2\pi)^2 I_s I_o (e^{F^2/4\gamma_2} - e^{F^2 + \beta_1^2/4\gamma_1} + e^{-\beta_1^2/4\gamma_1}) e^{-F\beta_1/\gamma_1}}{(2\pi I_s/2)[1/2\sigma_s^2 + 2/R^2](2\pi I_o/2)[1/2\sigma_o^2 + 2/R^2]}.$$  

(8-23)

where $F=k\theta$, $k=2\pi/\lambda$ is the wave number, and the other constants are given by the expressions

$$\gamma_1 = \gamma_2 + \frac{\eta^2}{4\gamma_2},$$  

(8-24)
\begin{equation}
\beta_1 = \frac{\eta F}{2\gamma_2}, \tag{8-25}
\end{equation}

with

\begin{equation}
\gamma_2 = \frac{1}{4\sigma_s^2} + \frac{1}{2\delta_s^2} + \frac{1}{4\sigma_o^2} + \frac{1}{2\delta_o^2} + \frac{2}{R^2}, \tag{8-26}
\end{equation}

\begin{equation}
\eta = 2\left(\frac{1}{2\delta_s^2} + \frac{1}{2\delta_o^2}\right). \tag{8-27}
\end{equation}

From Eq. (8-23) one can see immediately the effect of the misalignment of two overlapped signals on the detector surface, as the heterodyne efficiency is degraded exponentially as the angle \(\theta\) is increased, as expected. Hence, the heterodyne efficiency is decreased as \(\theta\) increases and reaches zero as \(\theta\) approaches \((\lambda/D)\), where \(D\) is the diameter of the detector [72]. Therefore, one could see that the angle \(\theta\) also puts an upper limit on the detector radius, in such a way that the product of the diffraction-limited field of view (approximately equals \(\theta^2\)) and the area of the detector is equivalent to the square of the wavelength, according to the well-known antenna properties of the heterodyne receivers [75].

**8.5 Numerical results**

The heterodyne efficiency varies with the parameters of the received and the locally generated signal. We will now study the variation of the heterodyne efficiency with the detector radius for several cases for which we have assumed the same parameters for the detected signal, while varying only one parameter of the local oscillator. In each case we computed the corresponding heterodyne efficiency to establish the conditions for optimal detection.
In Fig. 8.2a we show the variation of the heterodyne efficiency with the detector radius for several values of the intensity widths $\sigma_o$ of coherent ($\delta_o = \infty$) locally generated signal and wavelength $\lambda = 0.6328 \, \mu m$, when $\theta=0$. The received signal, assumed to be coherent ($\delta_s = \infty$) with intensity width $\sigma_s =1 \, mm$ and wavelength $\lambda = 0.6328 \, \mu m$. The heterodyne efficiency equals to unity when the two overlapped signals are similar to each other. As one could see for small detector radius (i.e. <3 mm), increasing the intensity width $\sigma_o$ improves the mixing efficiency. In general $\sigma_o$ should be just appreciably larger than $\sigma_s$, for example ($\sigma_o = 10 \sigma_s$) as shown in [77].

The effect of the detector radius dominates for larger sizes (i.e. > 3mm) and the previous observations may be invalid. In Fig. 8.2b we show the variation of the heterodyne efficiency versus the detector radius for $\theta =0.0002 \, rad$ to show the effect of the angular shift between the two signals, which in general degrade the heterodyne efficiency. As shown in the Fig. 8.2b the heterodyne efficiency curves trend differs from that shown in Fig. 8.2a, as the efficiency in general increases as the the intensity width $\sigma_o$ decreases.
In Fig. 8.3a we show the variation of the heterodyne efficiency with the detector radius, for several values of the intensity widths $\sigma_o$ of coherent ($\delta_o = \infty$), locally generated signal and wavelength $\lambda = 0.6328 \, \mu\text{m}$, when $\theta=0$. The received signal assumed to be partially coherent ($\delta_s = 5 \, \text{mm}$) with intensity width $\sigma_s = 1 \, \text{mm}$ and with wavelength $\lambda = 0.6328 \, \mu\text{m}$. The heterodyne efficiency decreased slightly from unity when the two overlapped signals are similar to each other due to the effect of the correlation width of the received signal. As one can see, for small detector radius increasing the intensity width improves the mixing efficiency and the effect of the loss of coherence is not noticeable given the large value of $\delta_s$ in comparison with $\sigma_o$. Again $\sigma_o$ should be just appreciably larger than $\sigma_s$, for example $\sigma_o = 10\sigma_s$. The effect of the detector radius dominates for larger sizes (but smaller than the case shown in Fig. 8.2a) and the previous
observation could be invalid. In Fig. 8.3b we repeated the calculations for $\theta=0.0002$ rad to show the effect of the angular shift between the two signals.

Fig. 8.3: The same as in Fig. 8.2 but the received beam was assumed to be partially coherent with a coherence width $\delta_s = 5$ mm.
In Fig. 8.4a we show the variation of the heterodyne efficiency with the detector radius for several values of the intensity widths $\sigma_o$ of coherent ($\delta_o = \infty$), locally generated signal and wavelength $\lambda = 0.6328 \ \mu m$, when $\theta = 0$. The received signal both assumed to be partially coherent ($\delta_s = 1 \ mm$) with intensity width $\sigma_s = 1 \ mm$ and wavelength $\lambda = 0.6328 \ \mu m$. Due to the large value of the coherence width, the two signals are completely different from each other and the observations noted in Fig. 8.2a and Fig. 8.3a are no longer valid. In this case, generally speaking, the heterodyne efficiency increases as the intensity width $\sigma_o$ of the locally generated signal decreases. In Fig. 8.4b the calculations were repeated for $\theta = 0.0002 \ rad$ to show the effect of the angular shift between the two signals.

In Fig. 8.5a we show the variation of the heterodyne efficiency with the detector radius for several values of the intensity widths $\sigma_o$ of coherent ($\delta_o = \infty$), locally generated signal and wavelength $\lambda = 0.6328 \ \mu m$, when $\theta = 0$. The received signal assumed to be partially coherent ($\delta_s = 0.5 \ mm$), with intensity width $\sigma_s = 1 \ mm$ and with wavelength $\lambda = 0.6328 \ \mu m$. Again due to the large value of the coherence width, the two signals are completely different from each other and the observations noted in connection with Fig. 8.2a and Fig. 8.3a are no longer valid. In this case, generally speaking, the heterodyne efficiency increases slightly as the intensity width $\sigma_o$ of the locally generated signal decreases. Fig. 8.5b presents results of calculations for $\theta = 0.0002 \ rad$ to showing the effect of the angular shift between the two signals.

In Fig. 8.6a we show the variation of the heterodyne efficiency ratio (HER), defined as the ratio of the heterodyne efficiency of the coherent detection system for $\theta = 0.0002 \ rad$ to the heterodyne efficiency when $\theta = 0$, with the detector radius.
Fig. 8.4: The same as in Fig. 8.2 but the received beam was assumed to be partially coherent with a coherence width $\delta_s = 1$ mm.
Fig. 8.5: The same as in Fig. 8.2 but the received beam was assumed to be partially coherent with a coherence width $\delta_s = 0.5$ mm.
Fig. 8.6: Heterodyne efficiency ratio (HER) as a function of the detector radius for several values of the coherence widths of the received signal ($\delta_s = \infty$ and $\delta_s = 0.5$ mm) that has an intensity width of $\sigma_s = 1$ mm. The locally generated beam was assumed to be coherent ($\delta_o = \infty$) with an intensity width of a) $\sigma_o = 0.5$ mm and b) $\sigma_o = 5$ mm.

The locally generated signal was assumed to be coherent ($\delta_o = \infty$), has intensity widths $\sigma_o = 0.5$ mm, and a wavelength $\lambda = 0.6328$ µm. The received signal was assumed to be at different states...
of coherence ($\delta_s = \infty$ and $\delta_s = 0.5$ mm), with intensity width $\sigma_s = 1$ mm and a wavelength $\lambda = 0.6328$ µm. Fig. 8.6b presents results of the HER calculations assuming that the locally generated signal is coherent ($\delta_o = \infty$), has intensity widths $\sigma_o = 5$ mm and a wavelength $\lambda = 0.6328$ µm, to show the effect of the intensity width of the locally generated signal. As shown in Fig. 8.6, the HER is larger of the less coherent beams than fully coherent beams especially at large detector sizes.

In summary the effect of the angle $\theta$ on the degradation of the heterodyne efficiency was severe for the coherent beams (Fig. 8.2b and Fig. 8.3b). The effect of the angle $\theta$ on the degradation of the heterodyne efficiency was less noticeable for the partially coherent cases (Fig. 8.4b and Fig. 8.5b). Hence, as also shown in Fig. 8.6, the less coherent beams are more stable to changes in the alignment of the coherent detection system than are the coherent beams. It is also worth mentioning that one could relax the coherence requirement of the local oscillator when the coherent detection system is used to detect partially coherent beams; we only need to have at least ($\delta_o = \delta_s$) but it is not necessary to have $\delta_o = \infty$ as for coherent beams detection.

**8.6 Conclusions**

An expression for the heterodyne efficiency of mixing two partially coherent beams with a small angular shift between their propagation directions has been derived as a measure of the quality of the coherent mixing of the beams. We derived an analytical expression for the heterodyne efficiency for the case of mixing two partially coherent Gaussian Schell-model (GSM) beams. The numerical examples demonstrate that the heterodyne efficiency of the coherent detection could be adjusted by controlling the local oscillator beam parameters. We have also shown by some examples that the optimal values for the parameters of the local oscillator beam depend on
specific characteristics of the detected beam. We demonstrated that partially coherent beams are more stable than fully coherent beams with respect to the misalignment of the detection system.
CHAPTER 9

HETERODYNE DETECTION OF PARTIALLY POLARIZED, PARTIALLY COHERENT BEAMS PROPAGATING IN FREE SPACE

In this chapter we evaluate the heterodyne efficiency of a heterodyne detection system for partially polarized, partially coherent, quasi-monochromatic beams. The Beam-Coherence-Polarization (BCP) matrix is used for the description of the statistical ensemble of this class of stochastic beams. We investigate the dependence of the efficiency of the detection process on the beams parameters as the beams propagate in free space. We discuss how the optimization of the detection system can be performed for received beams with different coherence and polarization properties by adjusting the corresponding properties of the local oscillator beam. The dependence of the mixing efficiency on the size of the receiving aperture is emphasized. We derive an analytical expression for the heterodyne efficiency in the case when both the received beam and the local oscillator beam belong to a broad class of so-called electromagnetic Gaussian Schell-model beams. Our analysis is illustrated by numerical examples.

9.1 Introduction

Optical heterodyne (coherent) detection is a powerful technique for the detection of weak signals or signals which are embedded into strong incoherent backgrounds [73, 95], because in such situations it can perform much better compared with a direct (incoherent) detection [82]. The heterodyne detection system is characterized by its capability of noise reduction and by its high
spectral resolution [83, 84]. Because of these advantages it has been extensively studied from different perspectives (see, for example, [74, 75]).

In order to estimate the performance of a heterodyne detection system one conventionally uses the signal-to-noise ratio (SNR) as a measure for the capability of the system to reject the noise inherent in it or the heterodyne efficiency as a metric to measure the mixing efficiency of the two overlapped beams on the detector surface. Fink [78] derived a general expression for the SNR and heterodyne efficiency of heterodyne detection of coherent beams, in terms of the intensity distributions of the mixed beams across the detector surface, taking into account the size and the shape of the detector as well. Lathi [76] derived an expression for the SNR for two linearly polarized, partially coherent beams, which are mixed on a detector surface; later Lathi and Nagal [77] also gave optimization criteria for the choice of the parameters of the local oscillator beam which maximizes the SNR when detecting such a beam. Recently, Salem and Dogariu [68] obtained an expression for the SNR when partially polarized, partially coherent beams are detected coherently. In [68], however, only a special case was treated in details, namely, when the mixed beams have uniform polarization across the detector surface. The heterodyne efficiency has been considered as a measure of quality of a heterodyne detection system to evaluate the performance of the coherent detection technique compared with the incoherent scheme [87]. It has been considered also to be a measure of the misalignment between the received beam and the locally generated one in coherent detection systems [87]. The heterodyne efficiency of a coherent detection system has been discussed in many publications in connection with the evaluation of the performance of coherent detection systems. For example Cohen [85] examined the effects of the phase-front misalignment between overlapped beams on
a detector surface, on the mixing efficiency. He studied the case of mixing fully coherent beams, for different distributions of the beams intensity. Tanaka and Ohta [91] have studied the effect of the tilt and the offset of the received signal on the heterodyne efficiency of Gaussian beams and Tanaka and Saga obtained optimal conditions for the maximum heterodyne efficiency in coherent detection systems in the presence of background radiation [80]. Recently Salem and Rolland studied the problem of mixing two partially coherent beams with small phase shift between their wave vectors coherently [96]. These authors emphasized the effect of this slight phase difference on the heterodyne detection of beams of any state of coherence.

It has been shown relatively recently that the coherence and the polarization properties of, quasi-monochromatic stochastic electromagnetic beams can appreciably change as the beam propagates, even in free space (cf. [97]). Moreover, the polarization of the beam can vary not only along the propagation distance but also in a direction perpendicular to it. Hence, even if the beam is generated with certain polarization properties, (usually being uniform across the transverse plane), after propagation, it will generally have somewhat different (and, moreover, usually not uniform) polarization properties across the surface of the detector.

In this chapter we generalize the definition of the heterodyne efficiency, given in Ref. [96] for scalar beams, to stochastic electromagnetic beams based upon the definition for the detected power of a heterodyne system of stochastic electromagnetic beams provided in [68]. We study the detection of beams propagating in free space from a remote source to a detector surface. Moreover, in order to carry out the most complete optimization of the detection system we will consider that the locally generated beam, akin the received beam, might also propagate a certain distance to the detector surface. The aim of this chapter is to examine how the heterodyne
efficiency of the system varies, not only with the size of the detector and the polarization properties of the mixed beams, but also with the propagation distances. Whilst the parameters of the received beam are not known, those of the local oscillator beam are fully controllable. We will show that the parameters of the local oscillator and the detector size can be chosen to maximize the mixing efficiency of the detection system.

The chapter is organized as follows: In section 9.2 we review free-space propagation of quasi-monochromatic, partially polarized beams of any state of coherence emphasizing the evolution of the polarization properties of the beam across its cross-section when the propagation distance increases. We will consider the propagation of a particular class of beams, the so-called electromagnetic Gaussian Schell-model (GSM) beams. In section 9.3 we introduce the heterodyne detection system in which both the received beam and the local oscillator beam are partially polarized, partially coherent, quasi-monochromatic beams, which reach the detector surface after free-space propagation from their generating sources. In section 9.4 we apply the definition of the heterodyne efficiency of a coherent system when the mixed beams are electromagnetic Gaussian Schell-model beams. We derive an analytical expression for the heterodyne efficiency when two electromagnetic GSM are combined coherently. Finally, in Section 9.5 we show by some numerical examples how different parameters of the beams affect the mixing efficiency in a coherent detection system. The optimization of the detection can be made by adjusting the size of the detector aperture and the properties of the local oscillator beam.

9.2 Free-space propagation of a stochastic electromagnetic beam

We begin by giving a brief review for the free-space propagation of the quasi-monochromatic partially polarized, partially coherent beams and we examine how the intensity and the
polarization through the cross-section of the beam in the transverse direction, change with increasing propagation distance. As we will see in Section 9.5, such changes have important implications on the detection system.

Suppose that the source is located in the plane $z = 0$ and generates a quasi-monochromatic, partially polarized, partially coherent beam propagating close to the positive $z$ direction (see Fig. 9.1).

![Fig. 9.1: Illustrating notation relating to the propagation of a beam in free space.](image)

The statistical ensemble of such a stochastic beam can be defined in terms of second-order correlation properties of the beam in the source plane using the so-called Beam Coherence-Polarization (BCP) matrix [25]

$$
\mathbf{J}(\mathbf{\rho}_1', \mathbf{\rho}_2') = \begin{pmatrix}
J_{xx}(\mathbf{\rho}_1', \mathbf{\rho}_2', 0) & J_{xy}(\mathbf{\rho}_1', \mathbf{\rho}_2', 0) \\
J_{yx}(\mathbf{\rho}_1', \mathbf{\rho}_2', 0) & J_{yy}(\mathbf{\rho}_1', \mathbf{\rho}_2', 0)
\end{pmatrix}, \quad (9-1)
$$
The elements of this matrix are the correlation functions between the mutually orthogonal components of the electric field, $E_x(p',0)$ and $E_y(p',0)$, at points with transverse position vectors $p'_1$ and $p'_2$ in the source plane at the same instant of time, i.e.,

$$J_y(p'_1, p'_2, 0) = \langle E_x^*(p'_1,0)E_y(p'_2,0) \rangle,$$

where the asterisk denotes the complex conjugate and the angular brackets denote the ensemble average. The elements of the BCP matrix of the beam $[J_y(p_1, p_2, z)]$, at distance $z$ from the source plane, can be calculated from the corresponding elements of the BCP matrix defined by Eqs. (9-1) and (9-2), i.e. $J_y(p'_1, p'_2, 0)$, using the propagation laws (cf. Ref. [30]) within the assumption of the paraxial approximation:

$$J_y(p_1, p_2, z) = \frac{1}{\lambda^2 z^2} \iint J_y(p'_1, p'_2, 0) \exp\left\{ -\frac{ik}{2z} \left[ (p_1 - p'_1)^2 - (p_2 - p'_2)^2 \right] \right\} d^2p'_1 d^2p'_2,$$

$$(i, j) = (x, y), \quad (9-3)$$

where $k = 2\pi/\lambda$ is the wave number of the beam, $\lambda$ is the central wavelength of the quasi-monochromatic wave and the integration extends over the source domain.

We will now assume that the source generates the so-called electromagnetic Gaussian Schell-model (GSM) beam. Following Ref. [30], the components of the BCP matrix characterizing such a beam in the plane of the source are given by the expressions

$$J_y(p'_1, p'_2, 0) = I_y \exp\left[ -\frac{(p'_1^2 + p'_2^2)}{4\sigma^2} \right] \exp\left[ -\frac{(p'_2 - p'_1)^2}{2\delta'^2} \right], \quad (i, j = x, y), \quad (9-4)$$
where $I_{ij}$ are the on-axis intensities, $\sigma$ is the r.m.s. width of the beam, $\delta_{ij}$ are the r.m.s. widths of the correlations. On substituting from Eq. (2-4) into Eq. (2-3) and performing the integration one obtains for the elements of the BCP matrix of the beam in the detector plane the expressions [25]

$$J_y(p_1, p_2, z) = \frac{I_y}{\Delta_y^2(z)} \exp\left(-\frac{p_1^2 + p_2^2}{4\sigma^2 \Delta_y^2(z)} \right) \exp\left(-\frac{(p_1 - p_2)^2}{2\delta_y^2 \Delta_y^2(z)} \right) \exp\left(-\frac{ik(p_1^2 - p_2^2)}{2R_y(z)} \right). \quad (9-5)$$

Here the beam expansion coefficients $\Delta_y^2(z)$ and the curvature coefficients $R_y(z)$ are given by the formulas

$$\Delta_y^2(z) = 1 + \left(\frac{z}{k\sigma}\right)^2 \left(\frac{1}{4\sigma^2} + \frac{1}{\delta_y^2}\right), \quad (9-6)$$

$$R_y(z) = z\left(1 + \frac{1}{\Delta_y^2(z)}\right). \quad (9-7)$$

Now, we will examine the variation of the intensity and the polarization properties of the beam with distance $z$ of propagation. The intensity of the beam at a point $P$, which is defined by the vector, $r = (\rho, z)$ is given by the expression

$$I(\rho, z) = Tr J(\rho, z), \quad (9-8)$$

where $Tr$ denotes the trace. The polarization properties of the beam consist of the degree of polarization and the state of polarization of the beam. The degree of polarization of the beam is defined by the formula [30]

$$P(\rho, z) = \sqrt{1 - \frac{4\text{Det} J(\rho, z)}{Tr J(\rho, z)^2}}, \quad 0 \leq P \leq 1, \quad (9-9)$$
where \( \text{Det} \) denotes the determinant. The state of polarization at any point within the beam cross-section is characterized by the parameters specifying the polarization ellipse (cf. [63]), which can be determined from the BCP matrix. The orientation angle \( \theta \) of the polarization ellipse, i.e. the angle which the major axis of the polarization ellipse makes with the x-direction, is given by the formula (cf. [16])

\[
\theta(p, z) = \frac{1}{2} \arctan \left( \frac{2 \text{Re}[J_{xy}(p, z)]}{J_{xx}(p, z) - J_{yy}(p, z)} \right), \quad -\pi/2 \leq \theta \leq \pi/2. \tag{9-10}
\]

The degree of ellipticity of the polarization ellipse can be defined by the formula [16]

\[
\varepsilon(p, z) = A_2(p, z)/A_1(p, z), \quad 0 \leq \varepsilon \leq 1, \tag{9-11}
\]

where \( A_1 \) and \( A_2 \) are the magnitudes of the major and of the minor semi-axes of the polarization ellipse respectively, given by the expressions

\[
A_{1,2}(p, z) = \left( \sqrt{(J_{xx} - J_{yy})^2 + 4|J_{xy}|^2} \pm \sqrt{(J_{xx} - J_{yy})^2 + 4\text{Re}[J_{xy}]^2} \right)^{1/2}/\sqrt{2} \tag{9-12}
\]

The degree of ellipticity \( \varepsilon(r, z) \) characterizes the shape of polarization ellipse; it is unity for circular polarization and zero for linear polarization.

In Fig. 9.2 we give an example for the variation of the intensity, the degree of polarization and the orientation angle of a stochastic electromagnetic beam versus the propagation distance \( z \) from the plane of the source and also the transverse distance \( \rho \) from the axial point of the beam as given by Eqs. (9-8) - Eq. (9-10). It was assumed that the propagating stochastic beam was generated by a linearly polarized Gaussian Schell-model source that has the parameters \( I_{xx} = 2.25, I_{yy} = 1, \sigma = 1 \text{ cm}, \delta_{xx} = 0.15 \text{ mm}, \delta_{yy} = 0.225 \text{ mm}, \delta_{xy} = 0.25 \text{ mm} \) and \( I_{xy} = 0.45 \).
From the contour plots one can readily see that, as the beam propagates sufficiently far from the source plane, its intensity distribution and also the distributions of its polarization properties change, and they are becoming non-uniform across the beam cross-section. This non-uniformity will impact the required properties of the local oscillator beam and the detector size. Also, since in a typical detection system the distance that the detected beam propagates from its source, say \( z_S \), is sufficiently large, i.e. \( z_S/\lambda > 1 \) (the detected beam is in its far zone), both its intensity and the polarization properties of the beam are affected by propagation (they are redistributed across the detector surface). Therefore, in general, the local oscillator beam at the detector surface should also exhibit far-zone characteristics, i.e. the distance between its source and the detector; say \( z_L \) should be sufficiently large, viz. \( z_L/\lambda > 1 \) in such a way that the parameters of the local oscillator beam should match the corresponding parameters of the received beam after propagating a distance \( z_s \) from the source plane. In other words, the detection system might be considered as having allowed (controllable) propagation path between the local oscillator and the detector plane. Such a system will now be introduced and analyzed.
Fig. 9.2: Contour plots for the variation of a) the intensity of the beam b) the degree of polarization of the beam c) the changes in the orientation angle, of a linearly polarized beam versus the propagation and the transverse distances of a beam propagating in free space. The parameters of the beam were taken as $\lambda = 0.6328 \mu m$, $I_{xx} = 2.25$, $I_{yy} = 1$, $\sigma = 1 \text{ cm}$, $\delta_{xx} = 0.15 \text{ mm}$, $\delta_{yy} = 0.225 \text{ mm}$, $\delta_{xy} = 0.25 \text{ mm}$ and $I_{xy} = 0.45$. 
9.3 Heterodyne detection of stochastic electromagnetic beams

![Diagram of heterodyne detection system]

Fig. 9.3: Schematic diagram for the heterodyne detection system. Different symbols are explained in the text.

A schematic diagram for the analyzed detection system is shown in Fig. 9.3. Two acousto-optics modulators AOM1 and AOM2 can be used to modulate the optical beams with angular frequencies $\omega_s$ and $\omega_L$ in the radio frequency (RF) range. C1 and C2 denote the generators of the random beams, which can be realized according to the methods proposed by Piquero et al. [61] or Shirai et al. [98]. Hence, the output of the generators C1 and C2 can be characterized by the BCP matrices of the correlation pairs with frequencies $\omega + \omega_s$ and $\omega + \omega_L$, where $\omega$ denotes the optical angular frequency. Suppose that the distances, which the two beams, the received and the local oscillator beams, travel to the detector from generators C1 and C2 are $z_s$ and $z_L$. 

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respectively. The expression for the signal-to-noise ratio (SNR) of the coherent mixing of stochastic electromagnetic beams, is given in [68] as

$$\text{SNR} = \frac{2 \text{Re} \int \int \Re(\rho_1)\Re(\rho_2)\text{Tr}[J^{(s)}(\rho_1,\rho_2,z_S)J^{(l)}(\rho_1,\rho_2,z_L)]d^2 \rho_1 d^2 \rho_2}{2eB \int \int \Re(\rho)\text{Tr}[J^{(l)}(\rho,\rho,z_L)]d^2 \rho}.$$  \hspace{1cm} (9-13)

where $J^{(s)}(\rho_1,\rho_2,z_S)$ and $J^{(l)}(\rho_1,\rho_2,z_L)$ are the BCP matrices of the detected beam and the local oscillator beam at the surface of the detector, Re denotes the real part and dagger denotes the Hermitian adjoint. In Eq. (9-13), $e$ is the electron charge, $B$ is the bandwidth of the intermediate frequency (IF) filter and $\Re(\rho)$ is the responsivity of the detector at the position $\rho$ on the detector surface. Eq. (9-13) can be modified to take into account any misalignment between the wavefronts of the two overlapped beams on the detector as given in [96] hence it can be expressed as

$$\text{SNR} = \frac{\int \int 2 \text{Re} \Re(\rho_1)\Re(\rho_2)\text{Tr}[J^{(s)}(\rho_1,\rho_2,z_S)J^{(l)}(\rho_1,\rho_2,z_L)]\exp(jK_1\rho_1 - jK_2\rho_2)d^2 \rho_1 d^2 \rho_2}{2eB \int \int \Re(\rho)\text{Tr}[J^{(l)}(\rho,\rho,z_L)]d^2 \rho}.$$  \hspace{1cm} (9-14)

where $K$ is the wave vector of the received signal, assuming that the local oscillator is propagating in a direction normal to the detector. Sometimes it is more appropriate to define a normalized SNR, a quantity, which does not depend on the detector parameters in general but depends on the parameters of the overlapped beams only. For stochastic electromagnetic beams mixing one can define the normalized SNR by the formula...
\[
\text{SNR}^* = \frac{\int D \text{Re} \text{Tr} \left[ J^{(s)}(p_1, p_2, z_S) J^{(l)}(p_1, p_2, z_L) \right] \exp(jK \cdot p_1 - jK \cdot p_2) d^2 \rho_1 d^2 \rho_2}{\int D \text{Tr} \left[ J^{(l)}(p, z_L) \right] d^2 \rho},
\]

(9-15)

where we assumed that the responsivity does not vary with the position on the detector, and it

can then be defined as \( \eta = \frac{e \eta_q}{hv} \) at different positions, where \( \eta_q \) is the quantum efficiency of the

photo-surface, \( h \) is the Planck's constant, and \( \nu \) is the optical frequency [73]. We note that the

normalized SNR has the units (J Hz) and does not depend on some detector parameters. To

evaluate the performance of the coherent detection system in this case and to show the effect of

the variation of different parameters on the detection process, it is more convenient to use the

heterodyne efficiency as a metric for the quality of the system in this case. Following the same

procedure as in [96] one could extend the definition of the heterodyne efficiency to the case of

stochastic electromagnetic beams to be written in the form

\[
\eta_h = \frac{\int D \text{Re} \mathfrak{R}(p_1) \mathfrak{R}(p_2) \text{Tr} \left[ J^{(s)}(p_1, p_2, z_S) J^{(l)}(p_1, p_2, z_L) \right] \exp(jK \cdot p_1 - jK \cdot p_2) d^2 \rho_1 d^2 \rho_2}{\int D \mathfrak{R}(p) \text{Tr} \left[ J^{(l)}(p, z_L) \right] d^2 \rho \cdot \int D \mathfrak{R}(p) \text{Tr} \left[ J^{(s)}(p, z_S) \right] d^2 \rho}.
\]

(9-16)

This equation reduces to the corresponding equation of the scalar beams mixing demonstrated in [96] when it has been applied to the scalar beams mixing. Equation (9-16) has maximum value when two co-aligned fully polarized electromagnetic beams are mixed together.
From Eq. (9-15) and (9-16) one can relate the heterodyne efficiency to the normalized SNR by the formula

\[
\eta_h = \frac{\text{SNR}^*}{\int_D \Re[\rho] \text{Tr}[J^{(s)}(\rho, \rho, z_s)] d^2 \rho}.
\]  

(9-17)

Assuming that the responsivity is constant across the detector, the corresponding expression for the heterodyne efficiency of the electromagnetic beams can be defined by the formula:

\[
\eta_h = \frac{\int_D \text{Re} \text{Tr}[J^{(s)} \rho_1 \rho_2, z_s] J^{(l)}(\rho_1, \rho_2, z_L) \exp(iK \rho_1 - iK \rho_2) d^2 \rho_1 d^2 \rho_2}{\int_D \text{Tr}[J^{(s)}(\rho, \rho, z_s)] d^2 \rho \cdot \int_D \text{Tr}[J^{(l)}(\rho, \rho, z_L)] d^2 \rho}.
\]  

(9-18)

As is evident from Eq. (9-18) the heterodyne efficiency can also be regarded as representing reduction of the optimum incoherent power upon mixing two beams coherently on a detector surface [94]. In the next section we give an analytical solution for this equation when the two overlapping beams belong to the broad class of Gaussian Schell-model (GSM).

### 9.4 Heterodyne detection of partially polarized, Gaussian Schell-model beams

When both the received beam and the local oscillator beam belong to a class of partially polarized, Gaussian Schell-model beams [30], an expression for the heterodyne efficiency can readily be derived. Suppose that the BCP matrices of the detected beam and the local oscillator beam at the detector surface have the elements [see Eqs. (9-5) – (9-7)]

\[
J^{(s)}_{ij}(\rho_1, \rho_2, z) = \frac{I^{(s)}_{ij}(z)}{\Delta^{(s)}_{ij}(z)} \exp\left(-\frac{(\rho_1^2 + \rho_2^2)}{4\sigma^{(s)}\Delta^{(s)}_{ij}(z)}\right) \exp\left(-\frac{(\rho_1 - \rho_2)^2}{2\sigma^{(s)}\Delta^{(s)}_{ij}(z)}\right) \exp\left(-\frac{i(k\rho_1^2 - k\rho_2^2)}{2R^{(s)}_{ij}(z)}\right),
\]

\[
\alpha = S, L, \quad (i, j = x, y),
\]

(9-19)
where superscripts \((S)\) and \((L)\) stand for the detected (signal) beam and the local oscillator beam.

Formula (9-18) can be re-written in the form

\[
\eta_h = \frac{\sum_{i,j=x,y} P_{c_{ij}}}{\left[ \sum_{i=x,y} \sum_{i,x,y} P_{d_{ii}} \right]^{1/2}},
\]

(9-20)

where the coherent power of the heterodyne mixing \(P_{c_{ij}}\) and the incoherent power \(P_{d_{ii}}^{(\alpha)}\) are given by the expressions

\[
P_{c_{ij}} = \int D \text{Re}J_{ij}^{(s)}(\rho_1, \rho_2, z_S)J_{ij}^{(l)}(\rho_1, \rho_2, z_L)\exp(jK\cdot\rho_1 - jK\cdot\rho_2)d^2\rho_1d^2\rho_2,
\]

\((i, j) = (x, y)\)  \hspace{2cm} (9-21)

and

\[
P_{d_{ii}}^{(\alpha)} = \int D J_{ii}^{(\alpha)}(\rho, z_{LO})d^2\rho, \hspace{1cm} (i = x, y), \alpha = (L, S).
\]

(9-22)

In Eqs. (9-21), and (9-22) the symbol \(D\) indicates that the integration extends over the area of the detector with a hard-aperture of diameter \(D\). In order to simplify the integration it is reasonable to approximate the diameter of the hard aperture \(D\) by an aperture or radius \(W\), sometimes called Gaussian or “soft” aperture, using the relation

\[
W^2 = D^2/8.
\]

(9-23)

Next, by multiplying the integrand by the exponential cut-off factor \(\exp[-(\rho_1^2 + \rho_2^2)/W^2]\) and extending the integration to infinity (c.f. [82]), one can rewrite Eqs. (9-15) and (9-16) in the forms:
\[ P_{ij} = 2 \text{Re} \int_{\mathbb{R}} \sum_{j=1}^{N} J_{ji}^{(s)}(p_1, p_2, z_S) J_{ij}^{(l)}(p_1, p_2, z_L) \exp \left[ -\frac{\rho_1^2 + \rho_2^2}{W^2} \right] \exp(jK \cdot p_1 - jK \cdot p_2) d^2 \rho_1 d^2 \rho_2, \]

\[ (i, j = x, y) \] (9-24)

and

\[ Pd_{ij}^{(\alpha)} = \int_{\mathbb{R}} J_{ij}^{(\alpha)}(p, p, z_\alpha) \exp \left[ -\frac{\rho^2}{W^2} \right] d^2 \rho, \quad (i = x, y) \text{ and } \alpha = (L, S). \] (9-25)

Using the approximation given by Eq. (9-23) and assuming that the angle \( \theta \) between the wave fronts of the two overlapped beams, is very small, hence \( k \cdot p \approx k \rho \cos(\phi) \), analytic expression for the heterodyne efficiency can be derived as indicated in Appendix D. It follows from Appendix D that the analytical expressions for the coherent power of the heterodyne mixing \( P_{C_{ij}} \) and the incoherent power \( Pd_{ij}^{(\alpha)} \) are;

\[ P_{C_{ij}} = (2\pi)^2 \frac{J_{ij}^{(s)}}{(\Delta(S)_{ij})^2} \frac{1}{(\Delta(L)_{ij})^2} \left( e^{\frac{F^2}{4\gamma_2}} / 2\gamma_2 \right) \left( e^{-\frac{F^2 + \beta_1^2}{4\gamma_1}} / 2\gamma_1 \right) e^{-\frac{FB_1}{\gamma_1}}, \]

\[ (i, j) = (x, y) \] (9-26)

where

\[ \gamma_2 = a_{ij} - ib_{ij}, \] (9-27)

\[ \eta = 2c_{ij}, \] (9-28)

\[ \gamma_1 = (a_{ij} + ib_{ij}) + \frac{\eta^2}{4\gamma_2}, \] (9-29)

\[ \beta_1 = \frac{\eta F}{2\gamma_2}, \] (9-30)
\[ a_{ij} = \frac{1}{\Delta_{ij}^{(s)^2}} \left( \frac{1}{4\sigma_{ij}^2} + \frac{1}{2\delta_{ij}^{(s)^2}} \right) + \frac{1}{\Delta_{ij}^{(c)^2}} \left( \frac{1}{4\sigma_{ij}^2} + \frac{1}{2\delta_{ij}^{(c)^2}} \right) + \frac{1}{W^2}, \tag{9-31} \]

\[ b_{ij} = k \left( \frac{1}{R_{ij}^{(l)}} - \frac{1}{R_{ij}^{(s)}} \right), \tag{9-32} \]

\[ c_{ij} = \frac{1}{\delta_{ij}^{(s)^2} \Delta_{ij}^{(c)^2}} + \frac{1}{\delta_{ij}^{(l)^2} \Delta_{ij}^{(c)^2}}, \tag{9-33} \]

\[ F = k\theta. \tag{9-34} \]

and

\[ P_{d_{ii}^{\alpha}} = \frac{2\pi (I_{ij}^{\alpha} / \Delta_{ii}^{(\alpha)^2})}{2\left( \frac{1}{2\sigma_{ii}^{(\alpha)^2}} + \frac{1}{W^2} \right)}, \quad (i, j) = (x, y), \quad \alpha = (L, S). \tag{9-35} \]

We discussed the effect of the angular phase shift \( \theta \) on the detection efficiency of beams of any state of coherence in chapter 8. Here we study the effects due to propagation of electromagnetic beams; hence we consider that the two beams are co-aligned on the detector surface.

### 9.5 Results and discussion

In the following we compute the variation of the heterodyne efficiency of the coherent mixing of two stochastic beams as a function of the aperture size of the detector using Eq. (9-26). The effects of the parameters of the overlapping beams on the mixing efficiency are discussed. We begin our investigations by checking the effect of the wave-front curvature of the propagating beams on the heterodyne efficiency. For this purpose we consider the mixing of two scalar and
coherent beams on the detector surface. The received beam was assumed to be scalar and coherent one with \( \lambda = 0.6328 \, \mu m \) and has 5 cm intensity width (\( \sigma_s \)), we assumed also that the beam propagated in free space for several distances (0, 1m, 10m). The local oscillator was chosen to be scalar and coherent with \( \lambda = 0.6328 \, \mu m \), 5 cm intensity width and propagating in free space also. We assumed first that the local oscillator has planar wave-front and we checked the effect of the variation in the propagation distance of the received beam on the heterodyne efficiency as shown in Fig. 9.4a. One could see that when \( Z_s = 0 \) (i.e. the received beam did not propagate) the two beams were in match and the heterodyne efficiency was unity and for small propagation distances the mismatch between the two overlapped beam was large and hence the heterodyne efficiency was less than the case of the longer propagation distance. In Fig. 9.4b we show the effect of the propagation of the local oscillator beam itself on the heterodyne efficiency, where we assumed \( Z_L = 1m \). One can see that the mixing efficiency has been decreased due to the wave-front curvature of the local oscillator beam.

![Graph showing heterodyne efficiency](image_url)
Fig. 9.4 : Variation of the heterodyne efficiency ($\eta_h$) with the detector radius, the received beam was assumed to be scalar and coherent with $\lambda = 0.6328 \mu m$ and has 5 cm intensity width ($\sigma$) and propagating for a distance 0, 1 m, 10 m in free space. The local oscillator was chosen to be scalar and coherent with $\lambda = 0.6328 \mu m$, 5 cm intensity width and propagating for a distance a) 0 m b) 1 m in free space.

As is well-known it is possible, by using a suitable lens system and adjusting the lengths before and after the lens, to compensate the wave-front curvature by the added phase from the lens [44]. We showed the importance of equalizing the phase of the propagating beams and next we will drop the effect of this wave-front curvature, assuming that the wave-front curvature has been corrected by using suitable optics in the detection system.

Next, we consider the detection of unpolarized, partially coherent beams that propagate in free space. The results for this case are shown in Fig. 9.5. The received beam was assumed to be unpolarized with $\lambda = 0.6328 \mu m$, 5 cm intensity width ($\sigma_s$), $I_{Sxx} = I_{Syy} = 0.1$, $\delta_{sx}=0.1$ mm, $\delta_{sy}=0.5$ mm, we assumed also that the beam propagated in free space for several distances (0, 100m,
The local oscillator parameters were chosen as $\lambda = 0.6328 \, \mu\text{m}$, $I_{Lxx} = 1$, $I_{Lyy} = 1$, $\sigma_L = 5 \, \text{cm}$ and we considered that the correlation widths are taking different values. In Fig. 9.5a we assumed that the local oscillator was fully coherent ($\delta_{Lxx} = \delta_{Lyy} = \infty$) and has correlations in two orthogonal directions. As one can see, the heterodyne efficiency in this case does not exceed the value 0.5 and it increases as the propagation distance of the received beam increases, hence according to van-Cittert Zernike theorem the coherence of the beam increases upon propagation in free space [44]. Then the matching between the two overlapping beams will increase. In Fig. 9.5b we checked the effect of decreasing the correlation widths of the local oscillator beam, where we assumed that $\delta_{Lxx} = \delta_{Lyy} = 5 \, \text{mm}$. As is clear the difference between Fig. 5b and Fig. 5a is very minor. These results show that we could relax the requirement of the coherence of the local oscillator as long as its correlation width is at least 10 times larger than the corresponding correlation width of the received beam.

![Graph](a)
Fig. 9.5: Variation of the heterodyne efficiency ($\eta_h$) with the detector radius. The received beam was assumed to be unpolarized with $\lambda = 0.6328\ \mu m$, 5 cm intensity width ($\sigma_s$), $I_{sx} = I_{sy} = 0.1$, $\delta_{sx} = 0.1\ \text{mm}$, $\delta_{sy} = 0.5\ \text{mm}$ and propagating for a distance 0, 100 m, 1000 m in free space. The local oscillator parameters were chosen as $\lambda = 0.6328\ \mu m$, $I_{Lxx} = 1$, $I_{Lyy} = 1$, $\sigma_L = 5\ \text{cm}$ and a) $\delta_{Lxx} = \delta_{Lyy} = \infty$  b) $\delta_{Lxx} = \delta_{Lyy} = 5\ \text{mm}$.

Finally, we checked the case of the detection of partially polarized, partially coherent beams as given in Fig. 9.6. The parameters of the received beam were assumed to be $\lambda = 0.6328\ \mu m$, $\sigma_s = 5\ \text{cm}$, $I_{sx} = I_{sy} = 0.5$, $I_{sxy} = 0.125$, $\delta_{sx} = \delta_{sy} = 0.1\ \text{mm}$, $\delta_{sxy} = 0.5\ \text{mm}$, we assumed also that the beam propagated in free space for several distances (0, 100m, 1000m). The local oscillator parameters were chosen as $\lambda = 0.6328\ \mu m$, $I_{Lxx} = I_{Lyy} = I_{Lxy} = 5$, $\sigma_L = 5\ \text{cm}$ and we considered that correlation widths are equal as the case for the fully polarized GSM beam but they are taking different values in each figure. In Fig. 9.6a we assumed that the local oscillator is coherent as we chose $\delta_{Lxx} = \delta_{Lyy} = \delta_{Lxy} = \infty$. One can see that as the beam propagate in free-space the heterodyne efficiency becomes more homogeneous over larger aperture size but its
magnitude reduces due to free-space diffraction. The aperture averaging does not improve the performance of the detection system as shown. In Fig. 9.6b we checked the effect of decreasing the correlation widths of the local oscillator beam, where we assumed that $\delta_{Lxx} = \delta_{Lyy} = \delta_{Lxy} = 5$ mm. The minor differences between Fig. 9.6a and Fig. 9.6b show that one could relax the requirement of the coherence of the local oscillator by using a suitable partially coherent beam. In Fig. 9.6c we assumed that $\delta_{Lxx} = \delta_{Lyy} = \delta_{Lxy} = 0.5$ mm. As is clear in the figure the heterodyne efficiency decreased significantly with respect to the results seen previously in Fig. 9.6a. One can see that, hence we decreased the correlation widths of the local oscillator to unacceptable degree, as in this example, the heterodyne efficiency deteriorated drastically.

![Graph](a)
Fig. 9.6: Variation of the heterodyne efficiency ($\eta_h$) with the detector radius. The parameters of the received beam were assumed to be $\lambda = 0.6328 \, \mu m$, $\sigma_s = 5 \, cm$, $I_{sx} = I_{sy} = 0.5$, $I_{sxy} = 0.125$, $\delta_{sxx} = \delta_{syy} = 0.1 \, mm$, $\delta_{sxy} = 0.5 \, mm$ and propagating distance $z_s = 0, 100 \, m, 1000 \, m$ in free space. The local oscillator parameters were chosen as fully polarized beam has $\lambda = 0.6328 \, \mu m$, $I_{Lxx} = I_{Lyy} = I_{Lxy} = 5$, $\sigma_L = 5 \, cm$ and a) $\delta_{Lxx} = \delta_{Lyy} = \delta_{Lxy} = \infty$ b) $\delta_{Lxx} = \delta_{Lyy} = \delta_{Lxy} = 5 \, mm$ c) $\delta_{Lxx} = \delta_{Lyy} = \delta_{Lxy} = 0.5 \, mm$. 
9.6 Conclusions

An expression for the heterodyne efficiency of mixing two stochastic electromagnetic beams with small angular shift between their propagation directions has been derived as measure for the quality of the coherent mixing of such beams. The effect of the change of the beams parameters in free-space has been considered. We derived analytical expression for the heterodyne efficiency for the case of mixing two stochastic electromagnetic Gaussian Schell-model (GSM) beams. The numerical examples demonstrate that the heterodyne efficiency of the coherent detection could be adjusted by controlling the corresponding parameters of the local oscillator beam parameters. We have also shown by some examples that the optimal values for the parameters of the local oscillator beam depend on specific characteristics of the detected beam.
CHAPTER 10
DETECTION OF PARTIALLY COHERENT BEAMS
PROPAGTING IN TURBULENT ATMOSPHERE

In this chapter we evaluate the signal-to-noise ratio (SNR) of detection systems for quasi-monochromatic, spatially partially coherent sources in the transmitter, which generate beams propagating in the turbulent atmosphere. Use of partially coherent beams in the transmitter has been suggested sometime ago for suppressing the severe effects of intensity fluctuations caused by the atmospheric turbulence. We compare the performance of an incoherent (direct) detection system with a coherent (heterodyne) detection on using either of them at the receiver of a communication system. The dependence of the SNR on the size of the receiving aperture is also emphasized for each case, as indication for the aperture averaging process. We derive an analytical expression for the SNR for the case when the detected beams belong to a broad class of so-called Gaussian Schell-model beams. Our calculations are illustrated by numerical examples.

10.1 Introduction

Free-space laser propagation is a relatively inexpensive and easy realizable substitute for fiber optics links, whenever the latter is neither feasible nor practical, especially as possible solution for the so-called last mile problem, to increase the bandwidth supplied to the end users of an optical communications network [11]. The free-space laser link introduces secure and broadband link but, on the other hand, it suffers from the effects of atmospheric turbulence through the
communication channel. The influence of the atmosphere may be very severe, because it causes beam spread, beam wandering and intensity fluctuations (scintillation) at the receiver. Several schemes have been suggested to mitigate the effects of atmospheric turbulence on the communication link, such as aperture averaging [12], use of adaptive optics [13] and use of partially coherent rather than coherent light on the transmitter [14]. It was found that the use of partially coherent light improves the performance of the free-space link in many cases and mitigates the effect of turbulence in a reasonable way [15]. The latter approach is promising compared to the other methods. It has been found that there are some advantages of using partially coherent beams rather than fully coherent beams in some applications [33, 59]. Some investigations recently carried in [59] and [99] showed that the use of a partially coherent source at the transmitter decreases the intensity fluctuations, and consequently the bit error rate (BER) was also decreased, when direct detection technique was used. It is well-known that optical heterodyne (coherent) detection is a powerful technique used for detection of weak signals or signals embedded into strong incoherent backgrounds [73, 95] because in such situations it can perform much better compared with direct (incoherent) detection [82]. Moreover, the heterodyne detection system is characterized by its capability to reduce noise and by its high spectral resolution [83] and [84]. In order to estimate the performance of a detection system one conventionally uses the signal-to-noise ratio (SNR) as a metric to measure the capability of the system to reject the noise inherent in it [76] or the heterodyne efficiency as another metric to measure the mixing efficiency of two overlapped beams on a detector surface [80].

In this chapter we use the normalized SNR to characterize the performance of both coherent detection and direct detection when they are used for detection of beams generated by
sources at any state of coherence, propagating in turbulent atmosphere. The chapter is organized as follows: In Section 10.2 we review propagation of quasi-monochromatic partially coherent beams in atmospheric turbulence and we emphasize the evolution of the properties of the beam on propagation. We only consider propagation of a particular class of beams, the well-known Gaussian Schell-model (GSM) beams. In Section 10.3 we analyze the direct detection system and we evaluate the shot-noise limited normalized SNR for GSM beams detection. In section 10.4 we introduce the heterodyne detection system and derive an analytical expression for the normalized SNR of GSM beams detection. Finally, in Section 10.5 we compare the performances of the coherent versus incoherent detection systems by several numerical examples. The effect of the aperture averaging on both schemes is also discussed.

10.2 Propagation of partially coherent beams in turbulent atmosphere

We begin with a brief review of propagation of partially coherent, quasi-monochromatic beams in turbulent atmosphere and we examine how the intensity and the degree of coherence of the beam along the axis of the beam is changing with increasing axial propagation distance and distance in the transverse plane. Such changes directly affect the output of the detection system.

Let us begin by considering a monochromatic linearly polarized electric field defined as follows

\[ E(\rho, z, t) = U(\rho, z)e^{-j\omega t} \]  \hspace{1cm} (10-1)

and propagating in the turbulent atmosphere. Here \( E(\rho, z) = U(\rho, z)e^{jkz} \), \( U(\rho, z) \) represents the complex amplitude of the electric field component at a point \( (\rho, z) \) [see Fig. 10.1], \( k = \frac{\omega}{c} \) is the
wave number, and $c$ is the speed of light in vacuum. The electric field $E$ satisfies the inhomogeneous scalar Helmholtz equation [44];

$$\nabla^2 U(p,z) + k^2 n^2(p) U(p,z) + 2\nabla(U(p,z),\nabla \ln(n(p))) = 0$$  \hspace{1cm} (10-2)$$

The last term on the left hand side of Eq. (10-2) represents depolarization of the propagating beam for the vector case. At optical wavelengths in turbulent atmosphere where the size of the smallest eddy is much larger than the wavelength, the scattering is mostly in the forward direction in a small solid angle close to the beam axis. We may then neglect this term [36] and Eq. (10-2) becomes;

$$\nabla^2 U(p,z) + k^2 n^2(p) U(p,z) = 0$$  \hspace{1cm} (10-3)$$

Fig. 10.1 : Illustrating notation relating to the propagation of a beam in turbulent atmosphere.

Eq. (10-3) is a stochastic equation because the refractive index is a random variable. Suppose that the field is beam-like and propagates in turbulent atmosphere from the plane $z = 0$ into the half-space $z > 0$ close to z-axis. Let $r = (p,z)$ be the position vector at a point in the half-space
$z > 0$, $\mathbf{p}$ is denoting a two-dimensional transverse vector perpendicular to the direction of propagation of the beam. The beam at any point in the half-space $z > 0$ can be expressed by the following expression, based on, the so-called extended Huygens-Fresnel principle ([48], Sec. 12.2):

$$E(\mathbf{p}, z) = -\frac{ik \exp(ikz)}{2\pi} \left[ \int \frac{\exp\left(\frac{i(k\mathbf{p} - \mathbf{p}')}{2z}\right)}{2\pi} \exp\left(\psi(\mathbf{p}, \mathbf{p}', z)d^2\mathbf{p}'\right) \right],$$  

(10-4)

where $E(\mathbf{p}', 0)$ is the electric field at the point $(\mathbf{p}', 0)$ and $\psi$ is a random complex phase that represents the effect of the turbulent atmosphere on a monochromatic spherical wave. For partially coherent beams we represent the beam by a correlation function $\Gamma(\mathbf{p}_1, \mathbf{p}_2, z)$ between the electric fields at two points in any transverse plane $z$. The correlation function can be defined as follows

$$\Gamma(\mathbf{p}_1, \mathbf{p}_2, z) = \langle E^*(\mathbf{p}_1, z)E(\mathbf{p}_2, z) \rangle.$$

(10-5)

The correlation function between the electric fields at two points $(\mathbf{p}_1, z)$ and $(\mathbf{p}_2, z)$ in a transverse plane $z = \text{const} > 0$ may be obtained on substituting Eq. (10-4) into Eq. (10-5) and one then finds that

$$\Gamma(\mathbf{p}_1, \mathbf{p}_2, z) = \left(\frac{k}{2\pi}\right)^2 \int d^2\mathbf{p}_1 \int d^2\mathbf{p}_2 \Gamma(\mathbf{p}_1', \mathbf{p}_2', 0) \exp\left[-\frac{i(k(\mathbf{p}_1 - \mathbf{p}_1')^2 - (\mathbf{p}_2 - \mathbf{p}_2')^2)}{2z}\right]$$

$$\times \left\langle \exp\left[\psi^*(\mathbf{p}_1', \mathbf{p}_1', z) + \psi(\mathbf{p}_2, \mathbf{p}_2', z)\right] \right\rangle_m,$$

(10-6)

where $\langle \ldots \rangle_m$ denotes averaging over the statistical ensemble of the random medium.
Consider a planar, secondary, Gaussian Schell-model source [23, Sect. 5.4.2], located in the plane \( z = 0 \) and propagating into the half-space \( z > 0 \) containing a random, statistically homogeneous, isotropic medium. Such a source is characterized by the following correlation function:

\[
\Gamma(p_1', p_2', 0) = I \exp \left( -\frac{(p_1'^2 + p_2'^2)}{4\sigma_2^2} \right) \exp \left( -\frac{|p_2' - p_1'|^2}{2\sigma_\mu^2} \right),
\]

(10-7)

where \( I, \sigma_\mu, \) and \( \sigma_\mu \) are positive constants. In order that such a source generates a beam, the condition

\[
\frac{1}{\sigma_\mu^2} + \frac{2}{\sigma_\mu^2} << k^2
\]

(10-8)

must be satisfied, ([22, Sect. 5.6.4]). The ensemble average over the realizations of the atmospheric turbulence can be approximated by the following expression [46]

\[
\left\langle \exp \left[ \psi^*(-p_1', p_2', z) + \psi(p_2', p_1', z) \right] \right\rangle_m =
\exp \left( -\frac{\pi^2 k^2 z}{3} \int_0^\infty K^3 \Phi_n(K) dK \left\{ (p_2 - p_1)^2 + (p_2 - p_1')(p_2' - p_1') + (p_2' - p_1')^2 \right\} \right),
\]

(10-9)

where \( K \) is the turbulence wavenumber and \( \Phi_n(K) \) is the power spectrum function of the refractive index fluctuations. For Gaussian-Schell model source and by using the approximation given in Eq. (10-9) an analytical solution for Eq. (10-6) may shown to be represented in the form [100]
\[
\Gamma(\rho_1, \rho_2, z) = \frac{I}{\Delta^2(z)} \exp\left(-\frac{(\rho_1^2 + \rho_2^2)}{4\sigma_\mu^2 \Delta^2(z)}\right) \exp\left[-|\rho_2 - \rho_1|^2 \left(\frac{1}{2\sigma_\mu^2 \Delta^2(z)} + M(1 + \sigma_\mu^2) - \frac{M^2 z^2}{2k^2 \sigma_\mu^2 \Delta^2(z)}\right)\right] \exp\left(\frac{ik(\rho_2^2 - \rho_1^2)}{2R(z)}\right),
\]

(10-10)

where

\[
\Delta^2(z) = 1 + \left(\frac{z}{k\sigma_\mu^2}\right)^2 + \frac{2Mz^2}{k^2 \sigma_\mu^2},
\]

(10-11)

\[
M = \frac{\pi^2 k^2 z}{3} \int_0^{\infty} K^3 \Phi_n(K) dK,
\]

(10-12)

\[
R(z) = \frac{k^2 \sigma_\mu^2 \Delta^2(z) z}{k^2 \sigma_\mu^2 \Delta^2(z) + Mz^2 - k^2 \sigma_\mu^2},
\]

(10-13)

\[
\frac{1}{\delta^2} = \frac{1}{4\sigma_\mu^2} + \frac{2}{\sigma_\mu^2}.
\]

(10-14)

One can see from Eq. (10-10) that the amplitude of the correlation function decreases as the beam propagates in the medium and the intensity width becomes broader. The expansion factor or the broadening term given by Eq. (10-6) depends on diffraction effects and also on the atmospheric turbulence. The degree of coherence of the beam changes according to two mechanisms that compete with each other, namely free-space propagation which leads to the increase in the spectral degree of coherence according to van-Cittert Zernike and the effect of turbulence on the other hand that leads to a decrease in the degree of coherence caused by the randomness of the atmospheric medium. Finally, the wave-front curvature changes from planar at the source to spherical, according to the Huygens-Fresnel principle, and in the far zone the wave-front becomes planar again.
10.3 Incoherent detection of partially coherent beams propagating in turbulent atmosphere

In this section we study the direct detection of partially coherent beams propagating in turbulent atmosphere. We calculate the SNR of the shot-noise limited direct detection system shown in Fig. 10.2. The incremental output current due to the optical signal at any point of the detector is given by [73]

\[ di(\rho, t) = R(\rho) \left[ E^*(\rho, t)E(\rho, t) \right] \]

(10-15)

where \( di(\rho, t) \) is the incremental current and \( R(\rho) \) is the responsivity of the detector. \( E(\rho, t) \) is the electric field component at a point specified by position vector \( \rho \) and the asterisk denotes the complex conjugate.

![Fig. 10.2: Schematic diagram for an incoherent detection system of partially coherent beams propagating in the turbulent atmosphere.](image)

The total integrated spatially and temporally varying current is given by the expression

\[ i(t) = \int \int R(\rho) \left[ E^*(\rho, t)E(\rho, t) \right] d^2 \rho. \]

(10-16)
The time-averaged electrical power corresponding to a single representation (realization) of the random field is \( \bar{ii} \), the bar denoting the time average over an interval short compared with the characteristic time of the variations of \( E(\rho, t) \). Hence, the total power is the ensemble average over all possible realization of the time averaged power, i.e.

\[
P_{\text{total}} = \left\langle i(t)i^*(t) \right\rangle = \int \int \int \Re(\rho_1)\Re(\rho_2)\left\langle E(\rho_1)E^*(\rho_1)E(\rho_2)E^*(\rho_2) \right\rangle d^2\rho_1 d^2\rho_2.
\]

(10-17)

Assuming further that the field components can be represented by a Gaussian random process of zero mean and applying the moment theorem for such processes (c.f. [23], sect. 1.6.1), the ensemble average on the right-hand side of Eq. (10-17) factorizes as follows:

\[
\left\langle E(\rho_1)E^*(\rho_1)E(\rho_2)E^*(\rho_2) \right\rangle = \left\langle E(\rho_1)E^*(\rho_1) \right\rangle \left\langle E(\rho_2)E^*(\rho_2) \right\rangle
\]

\[
+ \left\langle E^*(\rho_1)E(\rho_2) \right\rangle \left\langle E^*(\rho_2)E(\rho_1) \right\rangle.
\]

(10-18)

The preceding equation can be expressed as

\[
\left\langle E(\rho_1)E^*(\rho_1)E(\rho_2)E^*(\rho_2) \right\rangle = I(\rho_1)I(\rho_2) + |\Gamma(\rho_1, \rho_2)|^2,
\]

(10-19)

and the averaged detected power becomes

\[
P_{\text{total}} = \left\langle i(t)i^*(t) \right\rangle = \int \int \Re(\rho_1)\Re(\rho_2)\left[ I(\rho_1)I(\rho_2) + |\Gamma(\rho_1, \rho_2)|^2 \right] d^2\rho_1 d^2\rho_2.
\]

(10-20)

\( P_{\text{total}} \) is the power carried by the detected beam and in order to find an expression for the SNR of such a system, the noise equivalent power (NEP) should also be determined. Assuming that the detector is operating in shot-noise limits, one can define the NEP as \( 2eBI_0 \), where \( e \) is the
electron change, $B$ is the noise bandwidth of the filter at the output of the detector and $I_o$ is the average photo-current of the local oscillator defined as

$$I_o = \iiint \Re[p] \left| \Gamma(p, p) \right|^2 d^2 \rho, \quad (10-21)$$

then one can define the SNR of the optical detection system by

$$SNR = \frac{\iiint \Re[p_1] \Re[p_2] (I(p_1) I(p_2) + \left| \Gamma(p_1, p_2) \right|^2) d^2 \rho_1 d^2 \rho_2}{2eB \iiint \Re[p] \left| \Gamma(p, p) \right|^2 d^2 \rho}. \quad (10-22)$$

If we assume that the responsivity does not vary with the position, it can be defined as $\Re = e \eta_q / h \nu$ at every point, where $\eta_q$ is the quantum efficiency of the photo-surface, $h$ is Planck’s constant, and $\nu$ is the optical frequency. The corresponding expression for the signal-to-noise ratio (SNR) for this responsivity may be normalized as follows

$$SNR^* = \frac{\frac{SNR}{\eta / h \nu B}}{D} = \frac{\iiint (I(p_1) I(p_2) + \left| \Gamma(p_1, p_2) \right|^2) d^2 \rho_1 d^2 \rho_2}{2 \iiint D \left| \Gamma(p, p) \right|^2 d^2 \rho}. \quad (10-23)$$

We note that the normalized SNR has the units (J Hz) and does not depend on the detector parameters. In Eq. (10-23) the integrations extend over the area of the detector with a hard-aperture diameter $D$. It is more convenient to approximate the diameter of a hard aperture $D$ by an infinite aperture that has an $e^{-1}$ decrease at radius $W$, sometimes called Gaussian or “soft” aperture, defined in terms of $D$ by the expression

$$W^2 = D^2 / 8. \quad (10-24)$$
The exponential cut-off factor \( \exp\left[-\left(\rho_1^2 + \rho_2^2\right)/W^2\right] \) will be present in the integrand. Hence the integration limits can, to a good approximation, be extended to infinity (c.f. [82]), which is more suitable for the Gaussian fields calculations. One can then rewrite Eq. (10-23) in the form

\[
SNR^* = \frac{\text{SNR}}{\eta/h_B} = \frac{\int \left[ (I(p_1)I(p_2) + |\Gamma(p_1,p_2)|^2)^2 \exp\left[-\left(\rho_1^2 + \rho_2^2\right)/W^2\right]d^2\rho_1 d^2\rho_2}{2\int \left[ |\Gamma(p,p)|^2 + \left(\rho^2\right)/W^2\right]d^2\rho},
\]

(10-25)

where the integrations extend over the limit of the soft aperture (an infinite one). An analytical expression for the normalized SNR for the case of GSM beams can be found, as shown in Appendix E, and the SNR is given by the expression

\[
SNR^* = \frac{E_1 \frac{\pi^2}{c(c - d^2/a^2)} + E_2 \frac{\pi^2}{2c}}{2E_3 \frac{\pi}{a}},
\]

(10-26)

where

\[
E_1 = E_2 = \left(\frac{I}{\Delta^2(z)}\right)^2,
\]

(10-27)

\[
a = \frac{1}{2\sigma_i^2\Delta^2(z)} + \frac{1}{W^2},
\]

(10-28)

\[
b = 2\left[\frac{1}{2\sigma_i^2\Delta^2(z)} + M(1 + \sigma_i^2) - \frac{M^2 z^2}{2k^2 \sigma_i^2 \Delta^2(z)}\right],
\]

(10-29)

\[
c = a + b,
\]

(10-30)

\[
d = 2b,
\]

(10-31)
\[ E_s = \frac{I}{\Delta^2(z)} \]  

(10-32)

10.4 Coherent detection of partially coherent beams propagating in turbulent atmosphere

In this section we will study the coherent detection of partially coherent beams propagating in the turbulent atmosphere. We calculate the SNR of the shot-noise limited coherent detection system shown in Fig. 10.3, for beams propagating in the turbulent atmosphere.

Fig. 10.3 : Schematic diagram for a coherent detection system of partially coherent beams propagating in the turbulent atmosphere.

An expression has been derived already for the normalized SNR for such a detection system previously in [96], assuming the overlapping of two quasi-monochromatic, spatially partially
coherent beams propagate in a direction normal to a detector surface but with misalignment $\theta$ between their directions of propagation as shown in Fig. 10.4.

The normalized SNR is readily found to be

$$ SNR^* = \frac{\iiint \text{Re}[\Gamma_0(\rho_1, \rho_2), \Gamma_1^*(\rho_1, \rho_2)] \exp(K \cdot \rho_1 - K \cdot \rho_2) d^2 \rho_1 d^2 \rho_2}{\iint \Gamma_o(\rho, \rho) d\rho^2}. $$

Fig. 10.4 : Illustration for the notation of mixing two beams, with a shift $\theta$ between their phase fronts, on a detector in the $z = 0$ plane.

where $k$ is the wave vector of the received signal, $\Gamma_0(\rho_1, \rho_2)$ and $\Gamma_1(\rho_1, \rho_2)$ are the mutual coherence functions of the overlapped beams on the detector surface. When the two overlapped beams are GSM beams the following expression for the normalized SNR is obtained as has been shown previously in chapter 8.
\[ \text{SNR}^* = \frac{(2\pi)^2 I_s I_0 e^{2} (e^{\frac{F^2}{4\gamma_2}} - e^{\frac{F^2 + \beta^2}{4\gamma_1}}) e^{\frac{F\beta}{\gamma_1}}}{(2\pi I_0 / 2(\frac{1}{2\sigma_o^2} + \frac{1}{W^2}))}, \]  

(10-34)

where \( F = k_0, k = 2\pi/\lambda \) is the wave number, and

\[ \gamma_2 = \frac{1}{4\sigma_s^2 \Delta^2(z)} + \frac{1}{2\delta_s^2 \Delta^2(z)} + \frac{1}{4\sigma_o^2} + \frac{1}{2\delta_o^2} + \frac{1}{W^2}, \]  

(10-35)

\[ \eta = 2(\frac{1}{2\delta_s^2 \Delta^2(z)} + \frac{1}{2\delta_o^2}), \]  

(10-36)

\[ \gamma_1 = \gamma_2 + \frac{\eta^2}{4\gamma_2} \]  

(10-37)

\[ \beta = \frac{\eta F}{2\gamma_2} \]  

(10-38)

10.5 Results and discussions

We began our investigations by calculating the effect of the propagation distance of the optical beam in the turbulent atmosphere on the SNR of both coherent and incoherent systems. First, the source was assumed to be coherent. Fig. 10.5a shows the case of no propagation; one can see that the normalized SNR of the coherent detection system is much better than the incoherent one (The normalized SNR of the incoherent detection system has been multiplied by factor \( 10^6 \)). Fig. 10.5b shows the case when the beam propagates a distance 1 km in the turbulent atmosphere.
Fig. 10.5: The variation of the normalized SNR (SNR*) versus the detector radius for both detection techniques. The normalized SNR of the incoherent detection system has been multiplied by factor $10^6$. The source was assumed to be coherent ($\delta_s=\infty$) has wavelength $\lambda = 0.628 \ \mu\text{m}$ and intensity width $\sigma_s = 5 \ \text{mm}$. The local oscillator source was assumed to be coherent ($\delta_o=\infty$) with wavelength $\lambda = 0.6328 \ \mu\text{m}$ and intensity width $\sigma_o = 5 \ \text{mm}$. The parameters of the atmospheric turbulence were assumed as follows; the inner size parameter $l_0 = 5 \ \text{mm}$ and $C_n^2 = 10^{-17} \ \text{m}^{-2/3}$. The propagation distance of the source were assumed as a) $z_s = 0$ and b) $z_s = 1 \ \text{km}$. 
The coherent detection performs better than the incoherent one, especially for small detector radii. As the beam propagates further in the turbulent atmosphere, it looses its coherence as a result of the randomness of the turbulent atmosphere; hence the mismatch between the local oscillator and the received beam increases, and consequently, the normalized SNR of the coherent detection system decreases. The aperture averaging improves the SNR schemes, especially the incoherent one.

Next, we check the effect of the turbulence strength on the normalized SNR, as shown in Fig. 10.6. Fig. 10.6a shows the case of weak turbulence ($C_n^2 = 10^{-15} \text{ m}^{-2/3}$) and Fig. 10.6b shows the case of the strong turbulence ($C_n^2 = 10^{-14} \text{ m}^{-2/3}$). The normalized SNR of the incoherent detection system has been multiplied by a factor $10^2$. As expected, the normalized SNR of the coherent detection system decreases as the turbulence strength increases. The performance of the incoherent detection improves compared with the coherent one as the turbulence strength increases, especially for large detector sizes. The coherent detection system performs better than the incoherent one at small detector sizes.

![Graph showing the effect of turbulence strength on SNR](image)
Fig. 10.6: The variation of the normalized SNR ($SNR^*$) versus the detector radius for both detection techniques. The normalized SNR of the incoherent detection system has been multiplied by factor $10^2$. The source was assumed to be coherent ($\delta_s=\infty$) has wavelength $\lambda = 0.628 \, \mu m$ and intensity width $\sigma_s = 5 \, mm$. The propagation distance of the source in the turbulent atmosphere was taken as 3 km. The local oscillator source was assumed to be coherent ($\delta_o=\infty$) with wavelength $\lambda = 0.6328 \, \mu m$ and intensity width $\sigma_o = 5 \, mm$. The parameters of the atmospheric turbulence were taken as follows; the inner size parameter $l_o = 5 \, mm$ and a) $C_n^2 = 10^{-15} \, m^{-2/3}$ b) $C_n^2 = 10^{-14} \, m^{-2/3}$.

Finally, we check the effect of the diffuser placed in front of the laser source at the transmitter of the communication system. Fig. 10.7a shows the case when the source was assumed to be partially coherent, with coherence width of $\delta_s = 5 \, mm$. Fig. 10.7b shows the case when the source was assumed to be less coherent than in the previous case. In Fig. 10.7b, the coherence width of the source was taken as $\delta_s = 3 \, mm$. The normalized SNR of the incoherent detection system has been multiplied by factor $10^3$. One can see that the ratio of the normalized SNR of the coherent detection system to the incoherent detection system decreases as the coherence width of the source decreases, as a result of the increase of the mismatch between the
received beam and the locally generated beam. The coherent detection behaves better than the incoherent one for small detector sizes even for the case when the source is less coherent. The aperture averaging improves the incoherent detection SNR more than the coherent one.

(a)
Fig. 10.7 : The variation of the normalized SNR (SNR*) versus the detector radius for both detection techniques. The normalized SNR of the incoherent detection system has been multiplied by factor $10^3$. The propagating source was assumed to be partially coherent one has wavelength $\lambda = 0.6328 \, \mu m$ and intensity width $\sigma_s = 5 \, mm$. The propagation distance of the source in the turbulent atmosphere was taken as 1km. The local oscillator source was assumed to be coherent ($\delta_o = \infty$) with wavelength $\lambda = 0.628 \, \mu m$ and intensity width $\sigma_o = 5 \, mm$. The parameters of the atmospheric turbulence were taken as follows; the inner size parameter $l_o = 5 \, mm$ and $C_n^2 = 10^{-17} \, m^{-2/3}$. The coherence width of the propagating source was assumed as a) $\delta_s = 5 \, mm$ and b) $\delta_s = 3 \, mm$.

10.6 Conclusions

An expression for the normalized SNR of a coherent detection system and an incoherent detection one has been derived in order to compare their performance. The source was assumed to be partially coherent, propagating in turbulent atmosphere. We derived analytical expressions for the normalized SNRs for the case of Gaussian Schell-model (GSM) beams, subject to the
restriction that the detected electric fields can be described as Gaussian stochastic processes. We compared the performance of a coherent detection system versus an incoherent one for many situations when detecting beams propagating in different media. The effect of the aperture averaging over the normalized SNR has been emphasized for both detection schemes.
CHAPTER 11

SUMMARY OF CONTRIBUTIONS AND CONCLUSIONS

The work presented in this dissertation considers the propagation of optical beams at any state of coherence in linear media, deterministic or random, and also it considers the detection of the received beams at the receiver of an optical communications system. The need to analyze such systems arises from the fact that most of the physically realizable optical sources are radiating in a random manner given the random nature of the radiation of the large number of atoms, which constitute the source. Also, the interaction of the coherent light with a lot of natural and synthetic materials produces optical fields, which are fluctuating randomly. As a consequence the optical fields that one encounters, in general, in most of the applications are fluctuating or random ones.

Through the propagation in the atmosphere, the electromagnetic beams can be exposed to atmospheric turbulence, which causes fluctuations in the refractive index through the passage of the beam. Such a fluctuating medium might severely degrades any signal propagating through it, especially it causes intensity fades of the signal. Within the framework of the scalar-coherence theory, demonstrated in Sec. 2.1, one can describe the changes of the properties of any stochastic field such as the spectral density and the spectral degree of coherence on propagation in any linear medium, deterministic or random. In chapter 3 we described the propagation of the scalar partially coherent beams in the atmospheric turbulence in order to characterize the changes of their parameters on propagation in such a fluctuating medium. As noted by many researchers previously, there are some advantages of using partially coherent beams rather than fully coherent ones in some applications. For example the usage of stochastic beams at the transmitter
instead of deterministic ones has been suggested sometime ago to suppress the severe effects of intensity fluctuations due to atmospheric turbulence. We showed that\(^8\) partially coherent beams should be seriously considered for use in any long path propagation scheme through turbulent atmosphere such as one frequently encounters in remote sensing, in the use of communication systems, and in guiding, hence they are less affected by changes in the atmospheric conditions. We considered also the detection of beams at any state of coherence to quantify the effect of the beam coherence on the detection process. We generalized\(^9\) the expression of the heterodyne efficiency of mixing two partially coherent beams for the case of having a small angular shift between the propagation directions of beams at any state of coherence. We demonstrated that the heterodyne efficiency of the coherent detection could be adjusted by controlling the local oscillator beam parameters. We also showed that the stability of the partially coherent beams for the misalignment of the detection system is better than coherent beams by comparing the heterodyne efficiency of the coherent detection system for each case. The used detection scheme at the receiver is very important to quantify the received signal efficiently; hence it determines the signal-to-noise ratio (SNR), which affects directly the bit error rate (BER) of the received


signal. In our work we compared\textsuperscript{10} the performance of an incoherent (direct) detection system versus a coherent (heterodyne) detection one upon the use of either of them at the receiver of communication system of beams propagating in turbulent atmosphere and namely we evaluated the SNR for each case. The effect of the aperture averaging on the SNR has been emphasized for both detection schemes.

The aforementioned scalar-coherence theory ignored the vector nature of stochastic fields, which should be taken into account for some applications such as the ones which depend on the change of the polarization of the field. Recently generalization for the scalar-coherence theory including the vector aspects of the stochastic beams has been formulated and it is well-known as the unified theory of coherence and polarization of stochastic beams. The use of the unified theory of coherence and polarization makes it possible to study both the coherence properties and the polarization properties of stochastic electromagnetic beams on propagation in any linear media. The central quantity in this theory is a $2 \times 2$ matrix that describes the statistical ensemble of any stochastic electromagnetic beam in the space-frequency domain or its Fourier transform to describe the beam in space-time domain. However, not every matrix is representing a source, which will generate a beam-like field. In this dissertation we derived\textsuperscript{11} the conditions that the cross-spectral density matrix of a so-called planar, secondary, electromagnetic Gaussian

\textsuperscript{10} M. Salem and J. Rolland, ”Detection of partially coherent beams propagating in turbulent atmosphere” , Submitted to Journal of modern optics.

Schell-model source has to satisfy in order to generate a beam. These conditions are likely to be of importance in connection with the design of diffusers, which generate highly directional beams for various applications, such as guiding, aiming, and communications.

The effects of atmospheric turbulence on partially coherent beams have been studied till very recently only within the framework of the scalar theory and, consequently, such treatments cannot provide any information about the polarization properties of the beam. In this dissertation, using the unified theory of coherence and polarization we showed\textsuperscript{12} the effects of turbulent atmosphere on the degree of polarization of a partially coherent electromagnetic beam, which propagates through it and we compared with the propagation in vacuum. We also showed\textsuperscript{13} analytically that the degree of polarization of a beam generated by a partially polarized Gaussian-Schell model source, which propagates through atmospheric turbulence tends to the same value as it has at the source plane in the far-zone. In this dissertation we studied also the changes of the state of polarization of a partially coherent electromagnetic beam as an extension of the previously developed analysis for the propagation of such beams in free space. We found\textsuperscript{14} that


the effects of turbulence on the state of polarization are most significant when the atmospheric fluctuations are weak or moderate, while in strong regime of atmospheric fluctuations the state of polarization of the beam returns to its original state, whilst for free-space propagation the state of polarization does not tend to its original state.

In this dissertation we investigated the subtle relationship between the coherence and the polarization under more general circumstances based on the unified-theory of coherence and polarization. We showed\textsuperscript{15} that the coherence properties of the beam are not included in the usual Stokes parameters and one needs to use the so-called generalized Stokes parameters to include such properties in Stokes parameters. More precisely we showed that two stochastic electromagnetic beams, which propagate from the source plane $z = 0$ into the half-space $z > 0$ may have different degrees of polarization throughout the half-space, even though they have the same sets of Stokes parameters in the source plane $z = 0$, due to the possible difference in the coherence properties of the fields in that plane.

The detection of the optical signals is important hence it affects the fidelity of the communications system. In this dissertation we presented a general analysis for optical heterodyne detection of stochastic electromagnetic beams. We derived\textsuperscript{16} an expression for the

\textsuperscript{15} M. Salem, O. Korotkova and E. Wolf, “Can two planar sources with the same sets of Stokes parameters generate beams with different degrees of polarization?”, *Opt. Letts.* **31** (20), 3025-3027 (2006).

signal-to-noise ratio (SNR) when two stochastic electromagnetic beams are mixed coherently on a detector surface in terms of the space-time domain representation of the beams, the beam coherence polarization matrices. We have shown also that the optimal values for the parameters of the local oscillator beam depend on specific characteristics of the detected beam. Finally, we evaluated\textsuperscript{17} the heterodyne efficiency of a heterodyne detection system for stochastic beams propagating in vacuum and we discussed the dependence of the heterodyne efficiency of the detection process on the changes in the beam parameters as the beam propagates more distance. Moreover the dependence of the mixing efficiency on the size of the receiving aperture, as measure for the aperture averaging process, has been studied.

\textsuperscript{17} M. Salem and J. Rolland, ”Heterodyne detection of partially polarized, partially coherent beams propagating in free space” , Submitted to \textit{Opt. Commun}.
APPENDIX A

PROOF OF THE INEQUALITY $|\eta_{ij}(r_1, r_2, \omega)| \leq 1$.

We have the obvious inequality

$$\left\langle \left| a_i E_i(r_1, \omega) + a_j E_j(r_2, \omega) \right|^2 \right\rangle \geq 0, \quad (A-1)$$

where $a_i$ and $a_j$ are arbitrary constants. This inequality implies that

$$a_i^* a_i \left\langle E_i^*(r_1, \omega) E_i(r_1, \omega) \right\rangle + a_j^* a_j \left\langle E_j^*(r_2, \omega) E_j(r_2, \omega) \right\rangle + a_i^* a_j \left\langle E_i^*(r_1, \omega) E_j(r_2, \omega) + a_i a_j^* \left\langle E_j^*(r_2, \omega) E_i(r_1, \omega) \right\rangle \geq 0. \quad (A-2)$$

We may express (A2) in the form

$$a_i^* a_i S_i(r_1, \omega) + a_j^* a_j S_j(r_2, \omega) + a_i^* a_j W_{ij}(r_1, r_2, \omega) + a_i a_j^* W_{ji}(r_1, r_2, \omega) \geq 0. \quad (A-3)$$

It follows from the definition of the off-diagonal elements of the cross spectral density matrix that $W_{ij}(r_1, r_2, \omega) = W_{ji}(r_2, r_1, \omega)$. Using this fact and a well-known property of a non-negative definite quadratic form [101] it follows that the determinant

$$\begin{vmatrix} S_i(r_1, \omega) & W_{ij}(r_1, r_2, \omega) \\ W_{ij}^*(r_1, r_2, \omega) & S_j(r_2, \omega) \end{vmatrix} \geq 0, \quad (A-4)$$

implying that

$$|\eta_{ij}(r_1, r_2, \omega)| \equiv \frac{|W_{ij}(r_1, r_2, \omega)|}{\sqrt{S_i(r_1, \omega) S_j(r_2, \omega)}} \leq 1 \quad (A-5)$$

for all values of the arguments and the suffixes.
APPENDIX B

FORMULAS FOR THE ORIENTATION ANGLE AND FOR THE DEGREE OF ELLIPTICITY OF A BEAM PROPAGATING IN FREE SPACE.

In this Appendix we derive the far-zone expressions ($kz \to \infty$) for the orientation angle and for the degree of ellipticity of an electromagnetic Gaussian Schell-model beam propagating in free space. It follows from Eqs. (5-23), with $T = 0$, and from Eq. (5-27) that the elements of the cross-spectral density matrix become, in the asymptotic limit as $kz \to \infty$,

$$W_{ij}^{(\infty)}(p, z, \omega) = \frac{A_i A_j B_{ij}}{\alpha_{ij}^2} z^{-2} - \frac{A_i A_j B_{ij}}{\alpha_{ij}^2} \left(1 + \frac{\rho^2}{2\sigma^2}\right) z^{-4} + \ldots,$$

$$\left(i = x, y, j = x, y\right). \quad (B-1)$$

On substituting from Eq. (B-1) into Eq. (5-7) for the elements of the cross-spectral density matrix one obtains for the orientation angle of the polarization ellipse of the beam the expression

$$\theta^{(\infty)}(p, z, \omega) = \frac{1}{2} \arctan \left( \frac{2A_x A_y \text{Re}(B_{xy})}{\left(\frac{A_x^2}{\alpha_{xx}} - \frac{A_y^2}{\alpha_{yy}}\right) z^{-2} - \left(\frac{A_x^2}{\alpha_{xx}} - \frac{A_y^2}{\alpha_{yy}}\right) \left(1 + \frac{\rho^2}{2\sigma^2}\right) z^{-4} + \ldots} \right), \quad (B-2)$$

where the parameters the $\alpha_{ij}$ are defined by Eq. (5-24). If one retains only the leading terms, expression (B-2) reduces to
\[ \theta^{(\infty)}(p, z, \omega) = \frac{1}{2} \arctan \left( \frac{2A_x A_y \text{ Re}[B_{xy}] \alpha_{xx} \alpha_{yy}}{A_x^2 \alpha_{yy} - A_y^2 \alpha_{xx} \alpha_{xy}} \right), \quad (B-3) \]

which shows the far-zone behavior of the orientation angle. Similarly, using the asymptotic expansion (B-1) one can show that the far-zone behavior of the degree of ellipticity, defined by Eq. (5-10), is given by the expression:

\[ e^{(\infty)}(p, z; \omega) = \sqrt{\frac{(A_x^2 \alpha_{yy} \alpha_{xy} - A_y^2 \alpha_{xx} \alpha_{xy})^2 + 4(A_x A_y \alpha_{xx} \alpha_{yy} B_{xy})^2 - (A_x^2 \alpha_{yy} \alpha_{xy} - A_y^2 \alpha_{xx} \alpha_{xy})^2 + 4(A_x A_y \alpha_{xx} \alpha_{yy} \text{ Re}[B_{xy}])^2}{(A_x^2 \alpha_{yy} \alpha_{xy} - A_y^2 \alpha_{xx} \alpha_{xy})^2 + 4(A_x A_y \alpha_{xx} \alpha_{yy} B_{xy})^2 + (A_x^2 \alpha_{yy} \alpha_{xy} - A_y^2 \alpha_{xx} \alpha_{xy})^2 + 4(A_x A_y \alpha_{xx} \alpha_{yy} \text{ Re}[B_{xy}])^2}}. \quad (B-4) \]

One can see from formulas (B-3) and (B-4) that the orientation angle and the degree of ellipticity of the polarization ellipse depend on the correlation properties of the source via the parameters \( \alpha_{ij} \) in the beam expansion coefficient (5-23). The expression for the degree of polarization of the beam propagating in free space was derived in Ref. [28], Eq. (4.8).

Formula (B-1) implies that if at least two leading terms of the series are retained then all polarization properties of the beam in the far zone depend only on the transverse distance \( p \) from the axis.
APPENDIX C

DERIVATION OF THE HETERODYNE EFFICIENCY OF GSM BEAMS MIXING.

In this Appendix we derive the analytical solution of the heterodyne efficiency as given in Eq. (8-22) for the case of GSM beams. Let us begin by performing the integration

$$ I = \int_{\phi_2=0}^{2\pi} \int_{\phi_1=0}^{2\pi} \int_0^{\infty} \int_0^{\infty} \text{Re}\left[ \Gamma_o (\rho_2 \cdot \rho_2) \Gamma_\pi^* (\rho_1 \cdot \rho_1) \right] e^{-\left( \rho_1^2 + \rho_2^2 \right)/W^2} \left[ \exp\left( j \left( k \rho_1 \cos \phi_1 - k \rho_2 \cos \phi_2 \right) \right) \right] \rho_1 \rho_2 d\rho_1 d\rho_2 d\phi_1 d\phi_2. $$

(C-1)

For mutual coherence functions which are Gaussian the integration could be written as follows:

$$ I = \int_{\phi_2=0}^{2\pi} \int_{\phi_1=0}^{2\pi} \int_0^{\infty} \int_0^{\infty} \text{Re}\left[ I_1 \Gamma_o e^{-A(r_1^2 + r_2^2)} e^{2B \rho_1 \rho_2 \cos \phi_1 \phi_2} \right] e^{-\left( r_1^2 + r_2^2 \right)/W^2} \left[ \exp\left( j \left( k r_1 \cos \phi_1 - k r_2 \cos \phi_2 \right) \right) \right] r_1 r_2 dr_1 dr_2 d\phi_1 d\phi_2,$$

(C-2)

where

$$ A = \frac{1}{4\sigma_x^2} + \frac{1}{2\delta_x^2} + \frac{1}{4\sigma_o^2} + \frac{1}{2\delta_o^2} + \frac{2}{R^2}, $$

(C-3)

$$ B = \frac{1}{2\delta_x^2} + \frac{1}{2\delta_o^2}. $$

(C-4)

Let us set also $F = k\theta$, $F' = jk\theta$ and $C = 2B\rho_1 \rho_2$. The integration can be written as

$$ I = \int_{\rho_1=0}^{\infty} \int_{\rho_2=0}^{\infty} \int_{\phi_2=0}^{2\pi} \int_{\phi_1=0}^{2\pi} \text{Re}\left[ I_1 \Gamma_o e^{-A(\rho_1^2 + \rho_2^2)} e^{C \cos(\phi_1 - \phi_2)} \right] \left[ \exp\left( F'(\rho_1 \cos \phi_1 - \rho_2 \cos \phi_2) \right) \right] \rho_1 \rho_2 d\rho_1 d\phi_1 d\rho_2 d\phi_2.$$

(C-5)
Let us begin by performing the integration over $\phi$. One can rearrange $I$ as follows

$$I = \int_{\rho_1 = 0}^{\infty} \int_{\rho_2 = 0}^{\infty} I_o e^{-A(p_1^2 + p_2^2)} e^{-F'\rho_2 \cos \phi_2} \rho_1 \rho_2 d\phi_2 d\rho_2 d\rho_1$$

(C-6)

We used the identities [48] and [102]

$$\int_{\phi_1 = 0}^{2\pi} \exp([\cos \phi_1 (C \cos \phi_2 + F' \rho_1) - \sin \phi_1 (C \sin \phi_2)]) d\phi_1$$

where $I_0(N^2 + M^2 - 2MN \cos(\phi_2 - \psi)) = \sum_{-\infty}^{\infty} (-1)^m I_m(N) I_m(M) \cos m(\phi_2 - \psi)$,

(C-8)

where $I_m$ is the modified Bessel function of order $m$ and $\psi = \pi$. The integration becomes

$$I_1 = 2\pi \sum_{-\infty}^{\infty} (-1)^m \int_{\rho_1 = 0}^{\infty} \int_{\rho_2 = 0}^{\infty} \Re[I_o e^{-A(p_1^2 + p_2^2)}] I_m(C) I_m(F' \rho_1) \rho_1 \rho_2 d\rho_2 d\rho_1$$

(C-9)

By using the identity [102]

$$\int_{\phi = 0}^{2\pi} e^{(\alpha \cos \phi + \beta \sin \phi)} \cos m(\phi - \psi) d\phi = 2\pi I_m(\sqrt{\alpha^2 + \beta^2}) \cos m(\phi - \xi),$$

(C-10)

where $\xi = \tan^{-1}(\frac{\beta}{\alpha})$. The integration becomes
\[ I = (2\pi)^2 \sum_{-\infty}^{\infty} (-1)^m \cos(m \psi) \int_{\rho_1=0}^{\infty} \operatorname{Re}[I_s I_0 e^{-A(\rho_1^2 + \rho_2^2)}] \]

\[ I_m(C) I_m(F'\rho_1) I_m(F'\rho_2) \rho_1 \rho_2 d\rho_2 d\rho_1 \]

(C-11)

On substituting with the value of the modified Bessel function of order \( m \) in terms of Bessel function [48] as follows;

\[ I_m(F'\rho_1) = j^{-m} J_m(-F\rho_1), \]

(C-12)

\[ I_m(F'\rho_2) = j^{-m} J_m(-F\rho_2). \]

(C-13)

The integration can be written as

\[ I = (2\pi)^2 I_s I_0 \sum_{-\infty}^{\infty} (-1)^m \cos(m \psi) \int_{\rho_1=0}^{\infty} \operatorname{Re}[e^{-A(\rho_1^2)} J_m(-F\rho_1) \rho_1 d\rho_1 \int_{\rho_2=0}^{\infty} r_2 e^{-A(\rho_2^2)} J_m(F\rho_2) I_m(2B\rho_1 \rho_2) d\rho_2. \]

(C-14)

The integration over \( r_2 \) can be performed by using the identity [103]

\[ \int_{x=0}^{\infty} x e^{-\alpha x^2} I_1(\gamma x) dx = \frac{1}{2\alpha} e^{-\frac{(\beta^2 - \gamma^2)}{4\alpha}} J_1(\frac{\beta\gamma}{2\alpha}). \]

(C-15)

The integration becomes

\[ I = (2\pi)^2 I_s I_0 (e^{-\frac{F^2}{4\gamma^2}}) \sum_{m=-\infty}^{\infty} (-1)^m \cos(m \psi) \int_{\rho_1=0}^{\infty} e^{-(A + \frac{\eta^2}{4\gamma^2}) \rho_1^2} J_m(-F\rho_1) J_m(-\frac{\eta F}{2\gamma^2}) \rho_1 d\rho_1. \]

(C-16)
where

\[ \gamma_2 = A, \quad (C-17) \]

\[ \eta = 2B. \quad (C-18) \]

The integration over \( r_1 \) can be performed using the identity [103]

\[ \int_{x=0}^{\infty} x e^{-\mu^2 x^2} J_1(\alpha x) J_1(\beta x) dx = \frac{1}{2\mu^2} e^{-4\mu^2 I_1(B\gamma)} \quad (C-19) \]

The integration becomes

\[ I = (2\pi)^2 I_s I_o \left( e^{\frac{F^2}{4\gamma_2}} / 2\gamma_2 \right) \left( e^{\frac{F^2 + \beta_i^2}{4\gamma_1}} / 2\gamma_1 \right) \sum_{m=-\infty}^{\infty} (-1)^m \cos(m\psi) I_m \left( \frac{F\beta_i}{\gamma_1} \right), \quad (C-20) \]

where

\[ \gamma_1 = A + \frac{\eta^2}{4\gamma_2}, \quad (C-21) \]

\[ \beta_i = \frac{\eta F}{2\gamma_2}. \quad (C-22) \]

Finally using the identity [102]

\[ \sum_{m=-\infty}^{\infty} (-1)^m I_m \left( \frac{F\beta_i}{\gamma_1} \right) \cos(m\psi) = e^{\frac{F\beta_i}{\gamma_1} \cos(\psi)}. \quad (C-23) \]

The integration \( I \) can be expressed in closed form.
The integrations in the denominator of Eq. (8-22) take the form

$$I = (2\pi)^2 I_s I_o (e^{-\frac{F^2}{4\gamma_2}} / 2\gamma_2)(e^{-\frac{F^2+\beta_1^2}{4\gamma_1}} / 2\gamma_1)e^{-\frac{F\beta_1}{\gamma_1}}.$$  \hfill (C-24)

For a Gaussian mutual coherence function one could write the integration as follows;

$$I_d = \int_{\phi=0}^{2\pi} \int_{\rho=0}^{\infty} \Gamma(\rho, \rho) e^{-\frac{\rho^2}{2\sigma^2}} \frac{\rho^2}{W^2} \rho d\rho d\phi.$$  \hfill (C-25)

Integrating over $\phi$ we obtain

$$I_d = 2\pi \int_{\rho=0}^{\infty} e^{-\frac{\rho^2}{2\sigma^2}} \frac{\rho^2}{W^2} \rho d\rho d\phi.$$  \hfill (C-26)

The integration over $r$ could be performed using the identity [103]

$$\int_{x=0}^{\infty} x e^{-\mu^2 x^2} dx = \frac{1}{2\mu^2}.$$  \hfill (C-27)

Hence we may express $I_d$ in closed form as

$$I_d = \frac{2\pi}{2[\frac{1}{2\sigma^2} + \frac{1}{W^2}]}.$$  \hfill (C-28)

By using Eq. (C-24) and Eq. (C-29), an expression for the heterodyne efficiency of GSM beams mixing can be readily found as

$$\eta_h = \frac{(2\pi)^2}{(2\pi I_s / [2[\frac{1}{2\sigma^2} + \frac{2}{R^2}])}(2\pi I_o / [2[\frac{1}{2\sigma^2} + \frac{2}{R^2}])}. \hfill (C-30)$$
APPENDIX D

DERIVATION OF THE HETERODYNE EFFICIENCY OF ELECTROMAGNETIC GSM BEAMS MIXING.

In this Appendix we give details of the analytical solution for the heterodyne efficiency of electromagnetic GSM beams. We begin by performing the integration

\[
P_{cij} = \frac{2I_{ij}^{(s)} I_{ij}^{(l)}}{\Delta_{ij}^{(s)} \Delta_{ij}^{(l)} \Delta_{ij}^{(\omega)} \Delta_{ij}^{(\omega)}} \Re \int \int_{W} \exp \left( -\frac{\rho_{1}^{2} + \rho_{2}^{2}}{4\sigma^{(s)} \Delta_{ij}^{(s)}} \exp \left( -\frac{(\rho_{1} - \rho_{2})^{2}}{2\delta_{ij}^{(s)} \Delta_{ij}^{(s)}} \exp \left( \frac{ik(\rho_{1}^{2} - \rho_{2}^{2})}{2R_{ij}^{(s)}} \right) \right) \right) \times \exp \left( -\frac{\rho_{1}^{2} + \rho_{2}^{2}}{4\sigma^{(l)} \Delta_{ij}^{(l)}} \exp \left( -\frac{(\rho_{1} - \rho_{2})^{2}}{2\delta_{ij}^{(l)} \Delta_{ij}^{(l)}} \exp \left( \frac{-ik(\rho_{1}^{2} - \rho_{2}^{2})}{2R_{ij}^{(l)}} \right) \right) \right) \exp \left( \frac{-\rho_{1}^{2} + \rho_{2}^{2}}{W^{2}} \right) \right)
\]

\[
\exp \int_{W} (K_{\rho_{1}} - jK_{\rho_{2}}) d\rho_{1} d^{2}\rho_{2},
\]

\[i, j = x, y. \quad (D-1)\]

This formula can be compactly written as

\[
P_{cij} = \frac{2I_{ij}^{(s)} I_{ij}^{(l)}}{\Delta_{ij}^{(s)} \Delta_{ij}^{(l)} \Delta_{ij}^{(\omega)} \Delta_{ij}^{(\omega)}} \Re \int_{W} \exp \left[ (a_{ij} + ib_{ij}) \rho_{1}^{2} \right] \rho_{1} d\rho_{1} d\rho_{2}
\]

\[
\exp \int_{W} (a_{ij} - ib_{ij}) \rho_{2}^{2} \exp \left[ c_{ij} \rho_{1} \rho_{2} \cos(\phi_{1} - \phi_{2}) \right] \exp (jk\rho_{1} \cos \phi_{1} - jk\rho_{2} \cos \phi_{2}) d\phi_{1} d\phi_{2} d\rho_{1} d\rho_{2},
\]

\[i, j = x, y. \quad (D-2)\]

where the coefficients \(a_{ij}, b_{ij}\) and \(c_{ij}\) are

\[
ad_{ij} = \frac{1}{\Delta_{ij}^{(s)}} \left( \frac{1}{4\sigma^{(s)} \Delta_{ij}^{(s)}} \right) + \frac{1}{\Delta_{ij}^{(l)}} \left( \frac{1}{2\delta_{ij}^{(l)} \Delta_{ij}^{(l)}} \right) + \frac{1}{W^{2}}, \quad (D-3)
\]
\[ b_{ij} = \frac{k}{2} \left( \frac{1}{R_{ij}^{(s)}} - \frac{1}{R_{ij}^{(s)}} \right) \]  
\[ (D-4) \]

\[ c_{ij} = \frac{1}{\delta_{ij}^{(s)} \Delta_{ij}^{(s)} 2} + \frac{1}{\delta_{ij}^{(s)} \Delta_{ij}^{(s)} 2} . \]  
\[ (D-5) \]

Let us denote also \( F = k\theta \), \( F = ik\theta \) and \( C_{ij} = c_{ij}\rho_1\rho_2 \). The integration may be written as

\[ P_{cij} = \int_{\rho_1 = 0}^{\infty} \int_{\rho_2 = 0}^{\infty} \int_{\phi_1 = 0}^{2\pi} \int_{\phi_2 = 0}^{2\pi} \text{Re} \left[ I_{ij}^{(s)} I_{ij}^{(t)} e^{-(a_{ij} + ib_{ij})\rho_1^2} e^{-(a_{ij} - ib_{ij})\rho_2^2} C_{ij} \cos(\phi_1 - \phi_2) \right] \]

\[ \left[ e^{F'(\rho_1 \cos \phi_1 - \rho_2 \cos \phi_2)} \right] \rho_1 \rho_2 d\phi_1 d\phi_2 d\rho_2 d\rho_1 . \]

\[ (D-6) \]

To evaluate \( P_{cij} \), we perform the integration over \( \phi_1 \) and by rearranging the terms of \( P_{cij} \) one finds

\[ I = \int_{\rho_1 = 0}^{\infty} \int_{\rho_2 = 0}^{\infty} \int_{\phi_1 = 0}^{2\pi} \text{Re} \left[ I_{ij}^{(s)} I_{ij}^{(t)} e^{-(a_{ij} + ib_{ij})\rho_1^2} e^{-(a_{ij} - ib_{ij})\rho_2^2} e^{-F'\rho_2 \cos \phi} \right] \rho_1 \rho_2 d\phi_2 d\rho_2 d\rho_1 \]

\[ \left[ e^{2\pi \frac{\cos \phi_1 (C_y \cos \phi_2 + F'\rho_1) - \sin \phi_1 (C_y \sin \phi_2)}{d\phi_1} \right] d\phi_1 . \]

\[ (D-7) \]

Using the identities ([48] and [102]).
\[
I_0 \left( \sqrt{N^2 + M^2 - 2MN \cos(\phi_2 - \psi)} \right) = \sum_{-\infty}^{\infty} (-1)^m I_m (N) I_m (M) \cos (\phi_2 - \psi),
\]

where \( I_m \) is the modified Bessel function of order \( m \) and \( \psi = \pi \). The integration gives

\[
P_{cij} = 2\pi \sum_{-\infty}^{\infty} (-1)^m \int_{\rho_1=0}^{\infty} \int_{\rho_2=0}^{\infty} \text{Re} \left[ I_{ij}^{(s)} I_{ij}^{(t)} e^{-(a_j + ib_j) \rho_1^2} e^{-(a_j - ib_j) \rho_2^2} I_m (C_{ij}) I_m (F' \rho_1) \rho_1 \rho_2 d \rho_1 d \rho_2 \right]
\]

By using the identity [102]

\[
\int_{\phi=0}^{2\pi} e^{[F' \rho_2 \cos \phi_2]} \cos (\phi_2 - \psi) d \phi_2 = 0
\]

where \( \xi = \tan^{-1} \left( \frac{\beta}{\alpha} \right) \), the integral becomes

\[
P_{cij} = 2\pi (2\pi)^2 \sum_{-\infty}^{\infty} (-1)^m \cos (m \psi) \int_{\rho_1=0}^{\infty} \int_{\rho_2=0}^{\infty} \text{Re} \left[ I_{ij}^{(s)} I_{ij}^{(t)} e^{-(a_j + ib_j) \rho_1^2} e^{-(a_j - ib_j) \rho_2^2} I_m (C_{ij}) I_m (F' \rho_1) I_m (F' \rho_2) \rho_1 \rho_2 d \rho_1 d \rho_2 \right]
\]

By substituting the value of the modified Bessel function of order \( m \) in terms of Bessel function [48] one has

\[
I_m (F' \rho_1) = j^{-m} J_m (-F \rho_1),
\]
The integral can be written as

\[ P_{cij} = (2\pi)^2 I_i^{(s)} I_j^{(l)} \sum_{\rho_2} \sum_{\rho_1} (-1)^m \cos(m \psi) \int_{-\infty}^{\infty} \frac{e^{-((a_{ij}+ib_{ij})\rho_1^2)}}{\rho_1} \int_{-\infty}^{\infty} \frac{e^{-((a_{ij}-ib_{ij})\rho_2^2)}}{\rho_2} J_m(-F\rho_1)J_m(-F\rho_2) J_m(-2c_{ij}\rho_1\rho_2) d\rho_1 d\rho_2. \] 

The integration over \( \rho_2 \) can be performed by using the identity [103]

\[ \int_{-\infty}^{\infty} xe^{-\alpha x^2} I_1(\gamma x) dx = \frac{1}{2\alpha} e^{\frac{(\beta^2-\gamma^2)}{4\alpha}} J_1\left(\frac{\beta\gamma}{2\alpha}\right). \]

The integral becomes

\[ P_{cij} = (2\pi)^2 I_i^{(s)} I_j^{(l)} \left( e^{-\frac{F^2}{4\gamma_2}} \right) \sum_{m=-\infty}^{\infty} \sum_{\rho_1} (-1)^m \cos(m \psi) \int_{-\infty}^{\infty} \frac{e^{-((a_{ij}+ib_{ij})\rho_1^2)}}{\rho_1} \int_{-\infty}^{\infty} \frac{e^{-((a_{ij}-ib_{ij})\rho_2^2)}}{\rho_2} J_m(-F\rho_1)J_m(-\frac{\eta F}{2\gamma_2}) J_m(-2c_{ij}\rho_1\rho_2) d\rho_1 d\rho_2, \]

where

\[ \gamma_2 = a_{ij} - ib_{ij}, \]  

\[ \eta = 2c_{ij}. \]

The integration over \( r_1 \) can be performed, using the identity [103]
and one finds that

\[ P_{cij} = (2\pi)^2 I_{ij}^{(s)} I_{ij}^{(c)} \left( e^{-F^2/4\gamma_2} / 2\gamma_2 \right) \left( e^{-F^2 + \beta_1^2/4\gamma_1} / 2\gamma_1 \right) \sum_{m=-\infty}^{\infty} (-1)^m \cos(m\psi) I_m \left( F\beta_1 / \gamma_1 \right), \]  

(D-21)

where

\[ \gamma_1 = (a_{ij} + ib_{ij}) + \frac{\eta^2}{4\gamma_2}, \]  

(D-22)

\[ \beta_1 = \frac{\eta F}{2\gamma_2}. \]  

(D-23)

Finally using the identity [102]

\[ \sum_{m=-\infty}^{\infty} (-1)^m I_m \left( F\beta_1 / \gamma_1 \right) \cos(m\psi) = e^{\frac{F\beta_1}{\gamma_1} \cos(\psi)}. \]  

(D-24)

The integral \( P_{cij} \) can be written in closed form:

\[ P_{cij} = (2\pi)^2 I_{ij}^{(s)} I_{ij}^{(c)} \left( e^{-F^2/4\gamma_2} / 2\gamma_2 \right) \left( e^{-F^2 + \beta_1^2/4\gamma_1} / 2\gamma_1 \right) e^{-(F\beta_1 / \gamma_1)}. \]  

(D-25)

The integrals in the denominator of Eq. (9-20) take the form

\[ P_d = \int_{\phi=0}^{2\pi} \int_{\rho=0}^{\rho_0} \Gamma(p, q) e^{-\rho^2 / W^2} \rho d\rho d\phi. \]  

(D-26)

For Gaussian mutual coherence function case, one expresses the integral as follows;
\[ P_d = \int_{\phi=0}^{2\pi} \int_{\rho=0}^\infty e^{-\frac{\rho^2}{2\sigma^2} - \frac{\rho^2}{W^2}} \rho d\rho d\phi. \quad (D-27) \]

Integrating over \( \phi \) one finds that

\[ P_d = 2\pi \left[ e^{-\frac{\rho^2}{2\sigma^2} - \frac{\rho^2}{W^2}} \right]_{\rho=0}^\infty \rho d\rho d\phi. \quad (D-28) \]

The integration over \( r \) can be performed using the identity [103]

\[ \int_0^\infty x e^{-\mu^2 x^2} dx = \frac{1}{2\mu^2}. \quad (D-29) \]

Hence the integral can be expressed in closed form as;

\[ P_d = \frac{2\pi}{2 \left[ \frac{1}{2\sigma^2} + \frac{1}{W^2} \right]} \]. \quad (D-30)
APPENDIX E

DERIVATION OF THE SNR OF INCOHERENT DETECTION SYSTEM FOR GSM BEAMS.

In this Appendix we calculate the expression of the normalized SNR of an incoherent detection system given in Eq. (10-25) for the case of GSM beams. Let us begin by performing the integration

\[
I_1 = \iiint \left| \Gamma(\rho_1, \rho_2) \right|^2 d^2 \rho_1 d^2 \rho_2.
\]  

(E-1)

For the case of a GSM beam the integrand can be expressed as follows

\[
\left| \Gamma(\rho_1, \rho_2, z) \right|^2 = \left( \frac{I}{\Delta^2(z)} \right)^2 \exp \left( -\frac{(\rho_1^2 + \rho_2^2)}{2\sigma_i^2 \Delta^2(z)} \right) \exp \left( -2\left[ \frac{1}{2\sigma_{\rho}^2 \Delta^2(z)} + M (1 + \sigma_i^2) - \frac{M^2 z^2}{2k^2 \sigma_i^2 \Delta^2(z)} \right] (\rho_2 - \rho_1)^2 \right).\]

(E-2)

Using the aforementioned soft aperture approximation, see Eq. (10-24), the integral in Eq. (E-1) can be expressed as

\[
I_1 = \int_\infty^{-\infty} \int_\infty^{-\infty} \left| \Gamma(\rho_1, \rho_2) \right|^2 \exp \left( -\frac{(\rho_1^2 + \rho_2^2)}{W^2} \right) d^2 \rho_1 d^2 \rho_2.
\]  

(E-3)

On substituting from Eq. (E-2), Eq. (E-3) becomes

\[
I_1 = \int_\infty^{-\infty} \int_\infty^{-\infty} E_1 \exp(-a(\rho_1^2 + \rho_2^2)) \exp(-b((\rho_1^2 + \rho_2^2 - \rho_1 \cdot \rho_2)) d^2 \rho_1 d^2 \rho_2.
\]  

(E-4)
where

\[ E_1 = \left( \frac{I}{\Delta^2(z)} \right)^2, \]  

(E-5)

\[ a = \frac{1}{2\sigma_i^2\Delta^2(z)} + \frac{1}{W^2}, \]  

(E-6)

\[ b = 2\left[ \frac{1}{2\sigma_i^2\Delta^2(z)} + M(1 + \sigma_i^2) - \frac{M^2z^2}{2k^2\sigma_i^2\Delta^2(z)} \right]. \]  

(E-7)

The integral (E-4) can be expressed in the form

\[ I_1 = \int \int E_1 \exp(-c(p_1^2 + p_2^2)) \exp(-d(p_1 \cdot p_2)) d^2\rho_1 d^2\rho_2, \]  

(E-8)

where

\[ c = a + b, \]  

(E-9)

\[ d = 2b. \]  

(E-10)

We may re-write the integral \( I_1 \), as

\[ I_1 = \int \int E_1 \exp(-c p_1^2) d^2\rho_1 \int \int \exp(-c p_2^2) \exp(-d(p_1 \cdot p_2)) d^2\rho_2. \]  

(E-11)

The integration can be performed by using the following identity [103]

\[ \int \exp(-\beta^2r^2) \exp(-j\mathbf{q}_1 \cdot \mathbf{r}) d\mathbf{r} = \frac{\sqrt{\pi}}{\beta^2} \exp(-\frac{q_1^2}{4\beta^2}), \]  

(E-12)

where

\[ \beta^2 = c, \]  

(E-13)

\[ \mathbf{q}_1 = jd\rho_1. \]  

(E-14)

The integral in Eq. (E-11) becomes
\[ I_1 = E_i \frac{\pi}{c} \int_{-\infty}^{\infty} \exp\left(-\left(c - \frac{d^2}{2c}\right)\rho_1^2\right) d^2 \rho_1. \]  

(E-15)

Using an incremental area expressed in the cylindrical coordinates, the last equation can be expressed in the form

\[ I_1 = E_i \frac{\pi}{c} \int_{\rho_1=0}^{\infty} \int_{\phi=0}^{2\pi} \exp\left(-\left(c - \frac{d^2}{2c}\right)\rho_1^2\right) \rho_1 d\rho_1 d\phi. \]  

(E-16)

Performing the integration over \( \phi \), Eq. (E-16) becomes

\[ I_1 = E_i \frac{2\pi^2}{c} \int_{\rho_1=0}^{\infty} \exp\left(-\left(c - \frac{d^2}{2c}\right)\rho_1^2\right) \rho_1 d\rho_1. \]  

(E-17)

The integration over \( \rho_1 \) can be performed using the identity \([103]\)

\[ \int_{x=0}^{\infty} xe^{-\alpha^2x^2} dx = \frac{1}{2\alpha^2}. \]  

(E-18)

The integral in (E-17) has the value

\[ I_1 = E_i \frac{\pi^2}{c(c - \frac{d^2}{2c})}. \]  

(E-19)

Let us evaluate the other integral in the normalized SNR numerator (Eq. (10-25))

\[ I_2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma(p_1, p_1) \Gamma(p_2, p_2) \exp\left(-\left(\frac{p_1^2 + p_2^2}{W^2}\right)\right) d^2 \rho_1 d^2 \rho_2. \]  

(E-20)

For GSM beam, Eq. (E-20) can be expressed in the form
\[ I_2 = E_2 \int_{-\infty}^{\infty} \exp(-a\rho^2) d^2 \rho_1 \int_{-\infty}^{\infty} \exp(-a\rho^2) d^2 \rho_2, \quad (E-21) \]

where

\[ E_2 = \left( \frac{I}{\Delta^2(z)} \right)^2, \quad (E-22) \]

and \( a \) is defined before in Eq. (E-6)

The integral in Eq. (E-21) can be calculated by using incremental areas expressed in cylindrical coordinates and using the identity given in Eq. (E-18). Following the outlined procedure to find the integral (E-17), one finds that

\[ I_2 = E_2 \frac{\pi \rho^2}{a^2}. \quad (E-23) \]

The integration in the denominator of Eq. (10-25) takes the form

\[ I_3 = \int_{-\infty}^{\infty} \Gamma(\rho, \rho) \exp\left(-\frac{2\rho^2}{W^2}\right) d^2 \rho. \quad (E-24) \]

For GSM beam, Eq. (E-24) can be written as follows

\[ I_3 = E_3 \int_{-\infty}^{\infty} \exp(-a\rho^2) d^2 \rho, \quad (E-25) \]

where

\[ E_3 = \frac{I}{\Delta^2(z)}, \quad (E-26) \]

and \( a \) is defined before in Eq. (E-6)
The integral in Eq. (E-25) can be calculated by using incremental area expressed in the cylindrical coordinates and using the identity given in Eq. (E-18). Following the outlined procedure to find the integral (E-17), one finds that

\[ I_3 = E_3 \frac{\pi}{a}. \]  

(E-27)
APPENDIX F

PUBLICATIONS BY THE AUTHOR


LIST OF REFERENCES


