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PROSPECTIVE TEACHERS’ DEVELOPMENT OF WHOLE NUMBER CONCEPTS AND OPERATIONS DURING A CLASSROOM TEACHING EXPERIMENT

by

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A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Teaching and Learning Principles in the College of Education at the University of Central Florida Orlando, Florida

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Major Professor: Juli K. Dixon
ABSTRACT

A classroom teaching experiment was conducted to document prospective teachers’ development of whole number concepts and operations. The purpose of this mixed-methods study was to examine the collective understanding of prospective teachers in an elementary mathematics content course. Design research methodology, specifically a classroom teaching experiment was the methodology selected for this study since it allows learning to be documented in a classroom environment and is iterative in nature. A revised hypothetical learning trajectory and instructional tasks from a previous classroom teaching experiment were used in this study (Andreasen, 2006). Research about children’s development of whole number concepts and operations was used in developing instructional learning goals. In addition, research regarding prospective teachers’ development supported the instructional modification that all tasks would be presented and expected to be reasoned about in base-8. Both qualitative data and quantitative data were collected. Qualitative data included whole class dialogue that was videotaped and transcribed, as well as student work samples. Quantitative data included items from the Content Knowledge for Teaching Mathematics database that were administered prior to and subsequent to the instructional sequence in base-8 (Hill, Schilling, & Ball, 2005). It should be noted that the items selected from the database were in base-10. The emergent perspective served as the interpretive framework of the collected qualitative data. This perspective reflexivelycoordinates the social or group perspective simultaneously with psychological or individual perspective. As stated, this study sought to describe the communal mathematics understanding of prospective teachers in an elementary mathematics content course. Toulmin’s (1969) model of
argumentation and Rasmussen and Stephan’s three-phase methodology served to
document normative ways of group reasoning called classroom mathematical practices.
The following classroom mathematical practices were identified as taken-as-shared by
prospective teachers: (a) developing small number relationships using Double 10-Frames,
(b) developing two-digit thinking strategies using the open number line, (c) flexibly
representing equivalent quantities using pictures or Inventory Forms, and (d) developing
addition and subtraction strategies using pictures or an Inventory Form. Quantitative
results indicated that prospective teachers were able to apply mathematical
understandings grounded in base-8 to whole number concepts in base-10. In the end,
counting and calculating in base-8 provides a meaningful context for prospective teachers
to reconstruct their knowledge of whole number concepts and operations.
I would like to dedicate the following work to my extended family. I cannot express enough gratitude for their unyielding guidance, support and encouragement throughout my lifetime. Most of all, Tiffany, thank you for being my inspiration throughout this educational journey.
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### LIST OF ACRONYMS/ABBREVIATIONS

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<th>Acronym</th>
<th>Description</th>
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<tbody>
<tr>
<td>CKT-M</td>
<td>Content Knowledge for Teaching - Mathematics</td>
</tr>
<tr>
<td>CTE</td>
<td>Classroom Teaching Experiment</td>
</tr>
<tr>
<td>HLT</td>
<td>Hypothetical Learning Trajectory</td>
</tr>
<tr>
<td>MKT</td>
<td>Mathematical Knowledge for Teaching</td>
</tr>
<tr>
<td>NCTM</td>
<td>National Council of Teachers of Mathematics</td>
</tr>
<tr>
<td>PUFM</td>
<td>Profound Understanding of Fundamental Mathematics</td>
</tr>
<tr>
<td>TDE</td>
<td>Teacher Development Experiment</td>
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</table>
CHAPTER ONE: INTRODUCTION

In its 2000 publication *Principles and Standards for School Mathematics*, the National Council of Teachers of Mathematics (NCTM) identified gaining understanding of numbers and operations, developing number sense, and attaining fluency in arithmetic operations as the core components of mathematics education in elementary school. Emphasizing the long-lasting importance of studying number concepts and operations, Kilpatrick, Swafford, and Findell (2001) acknowledged,

> Number is a rich, many-sided domain whose simplest forms are comprehended by very young children and whose far reaches are still being explored by mathematicians. Proficiency with numbers and numerical operations is an important foundation for further education in mathematics and in fields that use mathematics. (p. 2)

Despite the importance of numerical concepts and operations, American teachers lack a deep understanding of elementary number concepts and operations, and, furthermore, this deficit is often conveyed to their students (Ma, 1999). Equally concerning is that prospective teachers are entering the profession without a firm understanding of mathematics, leaving them ill prepared to depart from the dull, rule-based mathematics instruction promoted during their own schooling (Ball, 1990). These factors lead to a workforce of teachers not adequately prepared, having at best a marginally sufficient understanding of mathematics (Kilpatrick et al., 2001). As a consequence, some teachers only instruct students *how* to perform arithmetic computations in the early elementary grades. However, only emphasizing students’ rote, procedural knowledge of these operations neglects a deeper conceptual understanding of
why the operation is mathematically valid (Ma, 1999). Neglecting to stress the need for a deeper conceptual understanding of mathematics leads to the perception that elementary mathematics requires little more than the teaching of the utility of numerical operations (Cobb, 1991). However, research has documented that there is more to teaching elementary mathematics than just straightforward procedural computations (Ball 1990; Ma, 1999).

In his influential research on teacher content knowledge, Shulman (1986) identified subject matter content knowledge and pedagogical content knowledge as types of understanding essential for teaching. Shulman explained that possessing subject-matter content knowledge includes an understanding of both the how and why of a subject whereas possessing pedagogical content knowledge leads to understanding of the knowledge unique to teaching a specific subject. Shulman asserted that in order to render subject matter content understandable to others, a teacher’s pedagogical content knowledge should include useful representations, illustrations, examples, and explanations that help facilitate learning. According to Shulman, a teacher possessing these types of knowledge can best promote students’ learning.

After Shulman published his research, other researchers began to examine the specific subject-matter content knowledge needed for teaching. Of particular interest to the current study is the subject-matter content knowledge required to teach elementary mathematics. Ball (1988, 1990) introduced the term substantive knowledge of mathematics to describe the necessary mathematical knowledge required for the teaching profession. Substantive knowledge of mathematics includes knowledge of underlying mathematical principles and meanings, attention to correct procedures and concepts, and understanding of the relationships and connections found within mathematics. In
addition, teachers require a depth of knowledge of mathematics that includes understanding what it means to know mathematics and perform mathematical operations, as well as possessing knowledge about mathematics as a field (Ball, 1990).

Ball's (1990) notions of substantive knowledge of mathematics and knowledge of mathematics became the foundations for what she and her colleagues later described as the mathematical knowledge for teaching (MKT) (Ball, Sleep, Bass, & Goffney, 2006). Ball et al.’s MKT concept further delineated the subject matter content knowledge defined by Shulman (1986) into three domains: common content knowledge, specialized content knowledge, and knowledge at the mathematical horizon. Common content knowledge is the mathematical knowledge most educated adults possess whereas specialized content knowledge is the mathematics knowledge teachers require to teach mathematics successfully (Hill, Schilling, & Ball, 2004). Knowledge at the mathematical horizon consists of the mathematics content that connects a student’s current experiences to future mathematical experiences (Ball, 1993). Together, these three domains emphasize an understanding of mathematics unique to the needs of teachers.

In the process of describing a teacher’s mathematical understanding that is “deep, broad, and thorough,” (Ma 1999, p. 120) Ma coined the concept of profound understanding of fundamental mathematics. Ma asserted that improving mathematics education for all children requires emphasizing future teachers’ profound understanding of fundamental mathematics during teacher preparation.

By describing the development of prospective teachers’ understanding of whole number concepts and operations, the current study focuses on mathematical understanding as described by Shulman (1987), Ball (1990), and Ma (1999). In this study, the emphasis of whole number tasks during a ten-day instructional unit is what
Shulman (1987) coined subject matter content knowledge. Research team members selected these tasks in order to deepen the participants’ profound understanding of fundamental mathematics, prior to the participants entering the teaching profession as Ma (1999) dictated. By emphasizing whole number concepts and operations in a meaningful manner, it was imagined that the tasks would cultivate the prospective teachers’ specialized content knowledge and would support the knowledge necessary to teach children in the future (Ball, 1990).

**Statement of Problem**

Although there has been much research into children’s development and understanding of whole number concepts and operations (Kamii, Livingston, & Lewis, 1996; Piaget, 1965; Steffe & Cobb, 1988), there has been limited research into prospective teachers’ development and understanding of these concepts. Prior research largely focused on concepts connected to division (Ball, 1990; Graeber, Tirosh, & Glover, 1989; Simon, 1993) although some research does exist related to place value and operations (Andreasen, 2006; McClain, 2003). In an attempt to increase understanding of prospective teachers’ development of whole number concepts and operations, the following research questions were investigated during a classroom teaching experiment:

1. What classroom mathematical practices become normative ways of thinking by prospective elementary school teachers during a whole number concepts and operations instructional unit?

2. Is there a statistically significant difference in group mean raw scores between prospective teachers’ pre- and post- whole number concepts and operations
situated in base-8 instructional unit administration of items selected from the

Content Knowledge for Teaching Mathematics (CKT-M) Measures database?

Qualitative and quantitative data were collected to document prospective
teachers’ mathematical development. The qualitative data included transcripts of
videotaped class sessions; field notes of research team members; audio-taped
conversations of research team meetings; and prospective teachers' work samples,
including class work assignments, homework assignments, and tests. The quantitative
data included items selected from the Content Knowledge for Teaching Mathematics
(CKT-M) Measures database developed by Hill et al. (2004).

Significance of Study

The purpose of this study was to document prospective teachers’ development of
whole number concepts and operations. Findings from Foundations for Success: The
Final Report of the National Mathematics Advisory Panel (2008) emphasized the
importance of studying prospective teachers’ development of whole numbers. Since
whole number concepts and operations are fundamental topics used by children to
understand other mathematical topics, the panel recommended that teacher education
programs should stress the importance of whole numbers. In addition, the panel
expressed the importance that prospective elementary teachers’ understand the
mathematical content they are responsible for teaching prior to them entering the
classroom.

The preparation of teachers is paramount to the success of their students (Ma,
1999). As the panel noted, teachers’ mathematical content knowledge is linked to
students’ achievement, and the impact of a series of effective or ineffective teachers dramatically impacts that achievement (National Mathematics Advisory Panel, 2008).

In order to study prospective teachers’ development of whole number concepts and operations a classroom teaching experiment was conducted in an elementary-education mathematics content course at a major university located in the southeastern United States. Cobb (2000) identified three major reasons to engage in classroom teaching experiments (CTE) includes the ability to observe: (a) instructional design and planning, (b) ongoing analysis of the classroom, and (c) retrospective analysis of data sources generated during the course of teaching. Classroom teaching experiments involve the creation and development of instructional sequences as well as the investigation of teaching and learning as it naturally occurs in the classroom setting. Researchers then use the documented results to help inform subsequent instructional plans and instructional decisions. Using this process, “theory is seen to emerge from practice and to feed back to guide it” (p. 308). A prior whole number instructional sequence described by Andreasen (2006) was influential in developing the tasks used during the current study. Furthermore, the items selected from the CKT-M add a quantitative dimension that documents learning by comparing group means raw scores at the beginning and the end of the instructional sequence.

Cobb (2000) summarized the underpinnings of classroom teaching experiments by articulating an emergent perspective. This interpretive framework allowed learning to be viewed simultaneously from the psychological perspective of the individual and the social perspective of the class. More specifically, the domains of the social perspective provide evidence of and describe the social dynamic in the classroom (Cobb & Yackel, 1996). The emergent perspective permits researchers to analyze data retrospectively in
order to identify learning that occurred and describe significant instances during the learning process (Yackel, 2001).

The current study used two methodologies for documenting prospective teachers’ development of whole number concepts and operations. The *constant comparative method* was used to identify and describe the regularities in classroom activity or *social norms* and the mathematically-based classroom activities or *sociomathematical norms* that occurred during a 10-day instructional unit (Glasser & Strauss, 1967). An *argumentation analysis* based on Toulmin’s argumentation model (1969) and a three-phase methodology articulated by Rasmussen and Stephan (2008) was conducted to identify normative classroom mathematical practices that shift in function in an argument or no longer need mathematical justification and as a result have become *taken-as-shared*. In addition, the prospective teachers were administered items from the CKT-M Measures database both prior to and following the instructional sequence. A dependent *t*-test was performed to analyze the group means (μ) of the two instances in time, which in this study were the pre- and post-instructional unit administration of items from the CKT-M Measures database (Hair, Anderson, Tatham, & Black, 1998). Performing this type of quantitative analysis allowed to for the research team to use an instrument in base-10 before and after instruction in base-8.

**Conclusion**

Instruction that creates a deep conceptual understanding of whole number concepts and operations is an essential part of the K-6 curriculum. Indeed, such an understanding serves as an important foundation for more advanced mathematical concepts (NCTM, 2000). The importance of children’s understanding of whole number
concepts and operations is well documented in the research literature. Despite the importance of children understanding whole number concepts and operations, the understanding of whole number concepts and operations that prospective teachers possess prior to teaching children has not been documented nearly as often. This study attempted to narrow the research gap between the limited research into prospective teachers’ knowledge of whole number concepts and operations and the greater quantity of research into elementary school children’s knowledge of the same concepts.

CHAPTER TWO discusses the literature addressing both children’s and prospective teachers’ development of whole number concepts and operations. Prior research, including results from previous classroom teaching experiments, is discussed as it relates to the current study and to refining, revising, and implementing the instructional sequence. CHAPTER THREE describes the methodology used in this study, which in addition to quantitative methods included measuring qualitative aspects of a classroom teaching experiment while grounding the findings in an interpretive framework based upon the emergent perspective (Cobb & Yackel, 1996). CHAPTER FOUR documents the qualitative and quantitative results of the study and CHAPTER FIVE discusses the implications of the study.
CHAPTER TWO: LITERATURE REVIEW

Historically, whole number concepts and operations have been taught for memorization rather than understanding. In his classic article on the meaning of arithmetic, Brownell (1947) argued,

To classify arithmetic as a tool subject, or as a skill subject, or as a drill subject is to court disaster. Such characterizations virtually set mechanical skills and isolated fact as the major learning outcomes, prescribe drill as the method of teaching, and encourage memorization through repetitive practice as chief or sole learning process. In such programs, arithmetical meanings of the kinds mentioned above have little or no place. Without these meanings to hold skills and ideas together in an intelligible, unified system, pupils in our school for too long a time have “mastered” skills which they do not understand, which they can use only in situations closely paralleling those of learning, and which they soon forget. (p. 11)

Almost 30 years later, Skemp (1976) distinguished between instrumental understanding and relational understanding. Skemp emphasized that when individuals possess instrumental understanding, they blindly follow mathematical “rules without reasons,” unlike individuals who possess relational understanding or the ability of knowing “what to do” and “why it is done” mathematically (p. 20).

Despite past attempts to call attention to Brownell and Skemp’s findings, many elementary school teachers continue to lack an understanding of whole number concepts and operations (Ma, 1999). In her influential comparison of Chinese and American elementary school teachers, Ma reported that teachers with procedural understanding of mathematical topics teach their students algorithmically, failing to make connections...
among mathematical topics. When these teachers attempt to make connections, they do so without substantial mathematical arguments. For example, although teachers may know how to subtract or multiply, they do not display mathematical understanding beyond the actions required to perform these operations. Such teachers directly contrast with those described as possessing *conceptual understanding*. Teachers possessing conceptual understanding can coordinate mathematical concepts, operations, and relationships to create a deep understanding of a mathematical topic. These teachers emphasize not only the actions required to perform mathematical operations but also their significance.

Because whole numbers and operations comprise much of elementary school mathematics and the importance of developing this knowledge base (NCTM, 2000), this chapter begins by addressing research into children’s development of whole number concepts and operations. The chapter then continues by addressing prospective teachers’ development of whole number concepts and operations. Finally, a *hypothetical learning trajectory* (HLT) or predicted pathway through which whole number concepts and operations may be acquired is introduced (Simon, 1995).

**Children’s Development of Whole Number Concepts**

Number development and counting are important mathematical foundations in children’s understanding of whole number concepts. Early explorations are often classified by intuitive, direct, and concrete experiences that eventually progress to more elaborate and mathematically sophisticated ways of using numbers, typically coinciding with symbolic notation manipulation at an abstract level (Kilpatrick et al., 2001).
Important landmarks of early number development include one-to-one correspondence, hierarchical inclusion, compensating, and part-whole relationships (Fosnot & Dolk, 2001). Based on her work with elementary students, Kamii (1985) defined the following important mathematical concepts: (a) **cardinality** as the concept that a number tells an individual the total amount because it is the last number counted (b) **one-to-one correspondence** as the concept of accounting for each object only once, and (c) **hierarchical inclusion** as the concept that numbers increase by exactly 1 each time while nesting within each other by this amount. Fuson, Grandua, and Sugiyama (2001) described children’s numerical connections between oral numbers, written numbers, and numerical quantities. These connections are made by strategies including disorganized counting, counting on fingers, recognizing patterns, relating words and numerals, and using manipulatives. When children attempt to understand quantity, finger usage often shows an initial correspondence between objects and quantity. Eventually, children continue to count with one-to-one correspondence and then transition into other counting strategies.

In their research, Steffe, Cobb, and von Glasserfeld (1988) indicated children’s early counting progresses through five distinct stages of activity, beginning with the counting of perceptual unit items and ending with the counting of abstract unit items. This progression starts with the creation of **perceptual unit items** as children use their perceptions to count every item as a unit followed by **figural unit items** as children begin counting items outside their immediate perceptual range. Third, children create **motor unit items** to coordinate their motor acts with either perceptual or figural unit items. Next, children create **verbal unit items** to coordinate the production of a unit item and the simultaneous speaking of a number word. When children are able to consider the
previously described units as objects that are counted, they are engaged in counting *abstract unit items*. These stages lead to the strategies used to perform addition and subtraction.

While working with kindergarten children, Baroody (1987) found that the children used both *concrete counting strategies* and *mental counting strategies*. Concrete counting strategies are used when countable objects are counted for each addend before all of the objects are counted for the total. Mental counting strategies are strategies that keep track of how far one should count from the cardinal number of the first addend. These strategies eventually evolve into addition and subtraction strategies, including those of counting backwards, skip counting, and taking steps of 10 (Fosnot & Dolk, 2001).

**Children’s Development of Whole Number Operations**

When faced with a computation, children choose among various methods to solve the problem, including using manipulatives to model the situation, inventing written procedures, drawing a picture, performing a mental calculation, choosing a known paper-and-pencil algorithm, or using technology (Carroll & Porter, 1998). Bass (2003) asserted that it is reasonable to use a single, clearly described generic solution method to solve mathematical problems that occur repeatedly. The use of written *algorithms*, or “procedures that can be executed in the same way to solve a variety of problems arising from different situations and involving different numbers”, is necessary for a myriad of reasons (Kilpatrick, et al. 2001, p.7). Because computations become more difficult to perform as numbers increase, it becomes necessary to keep track of the computation. Although calculators can often be used, paper-and-pencil algorithms are the most
reasonable method for certain numbers (Ashlock, 2006). If algorithms are built upon students' thinking and experiences, they can be used with meaning (Carroll & Porter, 1998). “Children can and do devise algorithms for carrying out multidigit arithmetic, using reasoning to justify their inventions and developing confidence in the process” (Kilpatrick et al., 2001, p. 7). Learning to use algorithms with meaning is an important part of developing mathematical proficiency.

Carroll and Porter (1997) noted that it is often beneficial to allow students to derive their own algorithms because different problems are better suited to “work” with certain numbers. In essence, students should possess procedural fluency, using a variety of procedures or strategies when working with numerical computations. Procedural fluency describes the use of mental or paper-and-pencil methods that are flexible, efficient, accurate, and generalizable based upon well-developed notions of number relationships and their properties (NCTM, 2000). Phillips (2003) explained, “Fluency will emerge as students use flexible strategies and demonstrate greater speed and accuracy” (p. 361).

Russell (2000) asserted that the foundation needed for fluency includes the understanding of operations and their relationship to each other; number relationships, including addition and multiplication “facts”; and a thorough understanding of the base-10 number system. Because this foundation demands more of children than simply following procedures, methods and procedures should be seen as tools to solve problems rather than the goals of mathematics instruction (Kilpatrick et al., 2001).

When children begin adding and subtracting, they typically follow a progression of development towards understanding. Initially, counting becomes abbreviated and faster before the properties of arithmetic are used to simplify computation (Kilpatrick et
Fosnot and Dolk (2001) identified important landmarks necessary for the development of number concepts and strategies for addition and subtraction. They include the notions of compensation, part-whole relationships, and the inverse relationship between addition and subtraction. In compensation, a problem such as $6 + 1 = 7$ becomes $5 + 2 = 7$ when 1 is removed from an addend and added to the other addend to conserve the sum. In part-whole relationships, children recognize that 2 and 5 are both included in 7. Finally, in recognizing the inverse relationship between addition and subtraction, children understand the mathematical connection between equations such as $5 + 2 = 7$ and $7 - 2 = 5$. Once students begin to explore multidigit addition and subtraction, unitizing becomes the central concept of place value. Unitizing occurs when 10 concurrently represents an entity—one group of 10—and 10 individual units (Cobb & Wheatley, 1988; NCTM, 2000). Addition and subtraction strategies stem from unitizing and the above landmarks, which together support children’s understanding of place value.

Carpenter, Fennema, Franke, Levi, and Empson (1999) summarized the strategies that children employ when attempting to solve whole number operation problems. When given context problems, children initially attempt to model the action or relationship that exists in a problem precisely; that is, children directly model the situations or relationships with physical objects. As their thinking matures, children eventually transition from direct modeling strategies using physical objects to counting strategies that mimic direct modeling strategies, but no longer require the use of physical objects. However, as students’ familiarity with strategies increases, they begin to represent solutions to problems that are not consistent with the structure of the problem. Furthermore, although children may rely on concrete counting strategies for a period,
computational practice and the absence of direct teaching leads them to invent advanced strategies, including mental strategies (Baroody, 1987).

Similarly, when given contextually based problems, children often develop flexible procedures that make sense to them by relying more on the numbers and the contexts provided and less on the established computational algorithms taught in most elementary classes. These flexible procedures are creative and varied yet reinforce the numerical concepts that underpin whole number computation (Carroll & Porter, 1998). Ma (1999) emphasized,

Being able to calculate in multiple ways means that one has transcended the formality of the algorithm and reached the essence of the numerical operations—the underlying mathematical ideas and principles. The reason that one problem can be solved in multiple ways is that mathematics does not consist of isolated rules, but connected ideas. Being able to and tending to solve a problem in more than one way, therefore, reveals the ability and the predilection to make connections between and among mathematical areas and topics. (p. 112)

After interviewing students in ten 4th-grade classes and three 2nd-grade classes, Carroll and Porter (1997) concluded that students’ natural tendencies often do not fit the standard algorithms taught in schools. When children are encouraged to develop their own meaningful computational strategies, they enhance their mathematical understanding of number sense and accuracy. Kamii, Lewis, and Livingston (1993) summarized the advantages of children developing their own procedures as (a) preventing children from forgoing their own thinking, (b) strengthening children’s understanding of place value, and (c) assisting children in developing better number sense.
By observing K – 3 school children in New York City, Madell (1985) identified various sophisticated subtraction procedures that children often create despite not being explicitly taught computational algorithms. Built upon their understanding of arithmetic operations, these procedures included (a) “counting on” from either the first or the greater addend, (b) addressing the larger portion of a number by computing from left to right, and (c) working with “easy” facts that involve numbers less than or equal to 10. One such subtraction procedure is shown in Figure 1. To solve 53 – 24, the child decomposed the minuend, 53, and subtrahend, 24, according to place values, choosing to address the larger portion of the number and operate from left to right. The child then subtracted 20 from 50 by focusing on the “tens” part of both the minuend and subtrahend to obtain the answer of 30. In the second step of the subtraction procedure, the child subtracted the “ones” portion of the minuend. Finally, the child added 3 to 26 to compensate when truncating or rounding down the tens portion of the minuend.

\[
\begin{align*}
53 - 24 &= 50 - 20 = 30 \\
30 - 4 &= 26 \\
26 + 3 &= 29
\end{align*}
\]

**Figure 1:** First subtraction procedure reported by Madell (1985).

Another sophisticated subtraction procedure reported by Madell (1985) is shown in Figure 2. Beginning in a similar manner to the procedure shown in Figure 1, the child subtracted from left to right by subtracting the “tens” part of both the minuend and subtrahend to obtain 30. The child then computed an “easy” fact of 4 – 3 to find the
excess amount of “ones” in the subtrahend. Finally, the child subtracted the computed excess, 1, from 30, resulting in the difference of 29.
After working with children in first and second grade, Kamii, Lewis, and Livingston (1993) identified a subtraction procedure, shown in Figure 3, similar to a procedure identified by Madell. Beginning with the larger portion of the minuend and subtrahend, the child subtracted the “tens,” resulting in an answer of 30. The child then proceeded to add the “ones” portion of the minuend to 30, temporarily resulting in 33. Finally, the student subtracted 4 from 33 to obtain the difference of 29.

**Figure 3: Subtraction procedure reported by Kamii et al. (1993).**

Kamii et al. also identified sophisticated addition procedures used when computing 18 + 17. By using left-to-right addition procedures, children using these strategies emphasized place value by decomposing addends during the computations. Each strategy began with the child adding 10 to 10 followed by the child addressing the “ones” place value in different ways. In the first strategy, shown in Figure 4, the child added 8 to 7 to obtain 15 then decomposed 15 into 10 + 5. Then the child added 10 to 20 to obtain 30 before adding 5, resulting in an answer of 35.
When adding 8 to 7 in the second strategy, shown in Figure 5, the child emphasized the part-whole relationship 8 and 2 have with 10 by decomposing the 7 into 2 and 5. Similar to the first strategy, 10 is added to 20 to obtain 30. Finally, the child added 5 to 30, resulting in an answer of 35.

In the third strategy, shown in Figure 6, the child used a doubling strategy when adding 7 to 7 to obtain 14 by decomposing 8 into 7 and 1. The child then added the remaining 1 to 14, resulting in 15. As with the two previous strategies, the child added 10 to 20 to obtain 30 before adding 5, resulting in an answer of 35.
Huinker, Freckman, and Steinmeyer (2003) reported the following five strategies generated by 3rd-grade children to perform multidigit subtraction computation to solve the problem 674 – 328: (a) use a number that is easier to work with, (b) add up from the subtracted number, (c) subtract the number in parts, (d) subtract each place value, and (e) change the problem completely but maintain the difference between the numbers. John’s strategy, shown in Figure 7, emphasized subtracting a number, 330, that was “easier” to work with than the original subtrahend of 328. This resulted in a temporary difference of 344. In order to compensate for the “extra” 2 that he had subtracted when changing the subtrahend from 328 to 330, John added 2 to 344 to obtain 346.

\[
674 - 328 =
\]

\[
674 - 330 = 344
\]

\[
344 + 2 = 346
\]

**Figure 7: John’s subtraction strategy reported by Huinker et al. (2003).**

Jameese’s strategy of “adding up” to the minuend 674, shown in Figure 8, is similar to that of solving a missing addend subtraction problem. Jameese continually added missing amounts between 328 and 674. By adding the values 2, 300, 20, 4, and 20
sequentially to the subtrahend of 328, Jamese obtained the minuend of 674 before performing the calculation \(2 + 300 + 20 + 4 + 20\) to find the difference.

\[
674 - 328 = \\
328 + 2 = 330 \\
330 + 300 = 630 \\
630 + 20 = 650 \\
650 + 4 = 654 \\
654 + 20 = 674
\]

**Figure 8: Jamese’s subtraction strategy reported by Huinker et al. (2003).**

Keisha’s strategy, shown in Figure 9, was similar to a strategy reported by Madell, shown in Figure 1, because she emphasized place value by subtracting significant digits. Specifically, she decomposed 328 into 300 + 20 + 8 before subtracting 300 from 674 to obtain 374. She then subtracted 20 and 8 to obtain an answer of 346.

\[
674 - 328 = \\
674 - 300 = 374 \\
374 - 20 = 354 \\
354 - 8 = 346
\]

**Figure 9: Keisha’s subtraction strategy reported by Huinker et al. (2003).**

DeJuan’s strategy, shown in Figure 10, entailed finding an answer to a subtraction problem by subtracting in place values. He began by subtracting the “hundreds,” which led to initially subtracting 300 from 600, yielding 300. He then subtracted the “tens,” when subtracting 20 from 70, yielding 50. His next step was similar to a step described by Madell, shown in Figure 2; although when he subtracted the “ones,” he subtracted the greater value, 8, from the lesser value, 4, giving him –4. Finally, DeJuan added and
subtracted his place values by performing the operation $300 + 50 - 4$, which resulted in a difference of 346.

$$674 - 328 =$$

$$600 - 300 = 300$$

$$70 - 50 = 20$$

$$4 - 8 = -4$$

$$300 + 50 - 4 = 346$$

**Figure 10: DeJuan’s subtraction strategy reported by Huinker et al. (2003).**

The final strategy described by Huinker et al. was Robert’s subtraction strategy, shown in Figure 11. To solve $674 - 328$, Robert changed the problem completely but maintained the difference between the minuend and subtrahend by adding 2 to both of the numbers. Robert “made” the subtraction problem $676 - 330$, thus making the subtrahend an “easier” number to subtract.

$$674 - 328 =$$

$$676 - 330 = 346$$

**Figure 11: Robert’s subtraction strategy reported by Huinker et al. (2003).**

As shown by these examples, when children are allowed to invent addition and subtraction strategies, they strengthen their mathematical connections between place value, estimation, number sense, properties of operations, and conservation of number (Carroll & Porter, 1998; Huinker et al., 2003; Kamii et al., 1993; Madell, 1985). These connections in turn support the development of multiplication and division strategies.

When provided instruction that emphasizes thinking strategies, children are able to develop the proficiency necessary to multiply and divide (Kilpatrick et al., 2001).
Carpenter et al. (1999) summarized direct modeling strategies children apply when attempting to solve multiplication, measurement division, and partitive division problems with the multiplication fact family of 4, 6, and 24. The researchers stated, when using a direct modeling strategy to solve a multiplication scenario such as, “Bart has 4 boxes of pencils. There are 6 pencils in each box. How many pencils does Bart have all together?”, a child will make 4 groups with 6 counters in each group, and then count all the counters to obtain the answer (Carpenter et al, 1999, p. 39). The researchers also emphasized that children use direct modeling strategies when solving measurement division and partitive division problems. When solving a partitive division problem such as, “Bart has 6 boxes of pencils with the same number of pencils in each box. All together, he has 24 pencils. How many pencils are in each box?”, a child would divide the 24 counters into 6 groups with the same number of counters in each group. The child would then count the counters in one group to find the answer. Carpenter et al. (1999) also noted that children use a direct modeling strategy to solve a measurement division scenario like, “Bart has 24 pencils. They are packaged 6 pencils to a box. How many boxes of pencils does he have?”, by placing 24 counters into groups with 6 counters in each group, and then counting the number groups to find the answer. The above strategies support the development of counting strategies for multiplication and division.

Counting strategies for multiplication and measurement division often involve some form of skip counting. Counting is used because the number in each group is a known quantity composed of individual units or a composite unit, as defined by Steffe and Cobb (1988). Children typically add or subtract until they get to a certain number. The answer to a division problem is the number of times that the child counted, added, or subtracted the generating number, whereas the answer to a multiplication problem is the
total obtained after skip counting. It is more difficult to use strategies involving counting or adding to solve partitive division because the number in each group is unknown. Therefore, in order to use a counting strategy that corresponds to the action or relationship in the problem, children may use trial and error to determine which number to skip count or add; essentially, the child is seeking a number by which to count (Carpenter et al, 1999).

Fosnot and Dolk (2001b) identified counting by ones, skip counting, repeated addition, doubling, and using properties such as associative and distributive properties as ways students begin to develop strategies when multiplying. The researchers identified three of the strategies by analyzing elementary aged children’s approaches to the question, “How many cookies are there in three bags of cookies that hold six cookies each?” In the first strategy, Elijah used a “counting by ones” strategy to solve the problem. Specifically, Elijah counted out 1, 2, 3, 4, 5, and 6 to represent the first bag of cookies, followed by again counting out the same sequence of numbers to represent the second bag before finally counting out the same sequence to represent the third bag. Elijah then counted all three bags by ones to obtain an answer of 18. Elijah’s counting strategy resembles the concrete counting all strategy for addition described by Baroody (1987) in his work with kindergarten children.

In the second strategy, Richard relied on skip counting of 6, 12, and 18 to answer the question. Richard used the notion that one group of 6 is 6, two groups of 6 are 12, and three groups of 6 are 18. In essence, Richard used a composite unit of 6 as a unit that is itself composed of individual units (Steffe, 1994; Steffe et al., 1988). Whereas Richard’s strategy is multiplicative in nature, Nellie’s strategy, the third strategy, is additive because she performed the additive operation $6 + 6 + 6 = 18$. 


Elijah, Richard, and Nellie’s strategies differ from Cleo’s doubling strategy, shown in Figure 12. To answer the question, “How many cookies were in four bags, each containing eight cookies?” Fosnot and Dolk (2001b) described how Cleo represented 4 bags each having 8 cookies. She then added 8 to 8 twice, resulting in a sum of 16 for each addition. Cleo then added 16 + 16 to obtain 32.

![Diagram of Cleo's doubling strategy]

**Figure 12: Cleo’s doubling strategy reported by Fosnot and Dolk (2001b).**

Finally, Fosnot and Dolk (2001b) described Hannah’s use of the distributive property when answering the story problem, “How many turtles are in five boxes containing seven turtles in each?” After removing 2 turtles from each box, Hannah had 5 groups of 5, or, as she stated 5, 10, 15, 20, and 25. Hannah then multiplied 5 and 2 to obtain 10 before adding 25 to 10 to obtain 35, which can be represented as $5 \times (5 + 2)$.

As can be seen from the strategies presented, children’s understandings of whole number operations can be quite complex and varied. Therefore, prospective teachers must possess mathematical understandings equally complex and varied to accommodate students’ many different strategies. The current study attempted to describe the development of the understanding of whole number concepts and operations by prospective teachers using the base-8 number system. Because prospective teachers have developed familiarity of whole number concepts in base-10 (Hopkins & Cady, 2007), this study created a learning environment that simulates that which an elementary child may
experience when first experiencing base-10 whole number concepts and operations. The environment is based upon the plausibility that instruction in base-8 will promote enough disequilibrium in prospective teachers and allow the prospective teachers to reconceptualize their mathematical experiences when learning whole number concepts and operations (Gravemeijer, 2004).

Prospective Teachers’ Development of Whole Number Concepts and Operations

Only limited research has been conducted into prospective teachers’ development of whole number concepts and operations. Whereas researchers have documented prospective teachers’ understanding of rational number operations such as division (Ball, 1990; Graeber et al., 1989; Simon, 1993), few have emphasized a research focus on whole number concepts and operations (Andreasen, 2006; McClain, 2003).

As previously stated, a major emphasis of this study was to document the development of prospective teachers' mathematical understandings of whole number concepts and operations during an undergraduate elementary-mathematics content course. During a previous classroom teaching experiment (CTE) related to place value and operations, Andreasen (2006) discovered that three classroom mathematical practices emerged: unitizing, flexibility when representing numbers, and reasoning about operations. The instructional sequence identified by Andreasen was developed and implemented to help support prospective teachers' understandings of place value concepts and operations by situating tasks in both base-8 and base-10. However, as Hopkins and Cady (2007) emphasized, familiarity with base-10 can prevent adults from fully comprehending some whole number concepts. As a consequence, McClain (2003) found that prospective teachers developed tricks and shortcuts using base-10 to solve problems
in base-8, resulting in preservice teachers manipulating symbols rather than attempting to understand numerical quantities. To avoid this obstacle during the current study, all of the instructional tasks posed to the subjects were framed in base-8 and the subjects were expected to solve the problems solely using the base-8 number system. The rationale for using base-8 exclusively was two-fold, one, to create cognitive dissonance that forced prospective teachers to “relearn” whole number concepts and operations by using an unfamiliar numeration system. Second, since the prospective teachers explored whole number concepts and operations during base-8, the need to review the same mathematical topics in base-10 as in Andreasen (2006) became redundant.

McClain (2003) outlined learning goals for prospective teachers’ development of place value and multidigit addition and subtraction including prospective teachers developing: (a) an understanding of multiplicative relations within place value; and (b) ways of symbolizing addition and subtraction transactions in a Candy Factory scenario where candies were packaged in 1’s, 8’s, and 64’s, in base-10. The learning goals of the Teacher Development Experiment (TDE) she described were based upon research using children’s development of number concepts as well as previous attempts to use the Candy Factory scenario within the base-8 numeration system. In those prior attempts, many prospective teachers focused their time attempting to notate transactions correctly in base-8 rather than exploring the underlying mathematical concepts of place value. In doing so prospective teachers’ reasoning was not based upon the quantity represented by the notation, but was determined by ways to solve the problem using prior knowledge of base-10 computations. This misdirected focus resulted in the prospective teachers developing tricks and short cuts to facilitate their use of the notation. For example, preservice teachers solved computation problems by adding or subtracting in base-10 and
then adding 2 or subtracting 2 to the sum or difference, in order to obtain a solution that was in base-8. Whereas using number tricks produced a correct solution, the prospective teachers were merely manipulating symbols instead of acting on quantities resulting in class discussions focusing on the trick that converted the base-10 answer into correct base-8 answers, and not the conceptual understanding that supported their procedures. As a result, the preservice teachers did not develop invented algorithms or notational schemes. In addition, prospective teachers did not have a need to create drawings or inscriptions to support their notation, since their activity was procedural and not conceptual in nature.

For the above reasons, McClain (2003) chose to no longer frame tasks in base-8, instead she focused tasks on the composition of units that build from the value eight in base-10. McClain’s reasoning for modifying the context was that if the mathematics was trivial for the prospective teachers, the need to create ways to symbolize their transactions would not have emerged naturally. The goal was to build from the preservice teachers’ evolving notational schemes to support shifts in their understandings of place value and multidigit addition and subtraction so that they might develop conceptual understanding instead of mere proficiency with meaningless algorithms. McClain (2003) stated, “Preservice teachers would come to understand the mathematics in the context of their own problem solving efforts” (p.286). Upon entering the instructional sequence, McClain reported that prospective teachers had a very superficial understanding of place value and addition and subtraction including: (a) prospective teachers’ could identify a digit’s correct place value, but did not extend it into larger implications such as relations between place value and operations; and (b) prospective teachers’ understandings of
place value, addition and subtraction were grounded in rules for manipulating algorithms (McClain, 2003).

By using tasks in the Candy Shop, McClain (2003) emphasized quantifying collections by counting, creating different arrangements for the same quantity, and adding and subtracting quantities. McClain reported, pedagogical and subject matter content learning goals of the TDE became intertwined. One goal was to support prospective teachers’ subject matter content knowledge through the development of understanding of place value and multidigit addition and subtraction, while a concurrent goal was the development of prospective teachers’ pedagogical content knowledge. The pedagogical learning goal prompted prospective teachers to think about mathematics from a student’s perspective, and then reflect on how the activities could support children’s learning. In the end, the pedagogical learning goal created a dual perspective that was problematic for the prospective teachers. Since the prospective teachers were working to understand the mathematics themselves, they had not begun to understand how this could support children’s learning. However as the course unfolded, it became apparent to McClain that many of the prospective teachers were beginning to reconceptualize what it means to teach mathematics for understanding.

According to McClain, the goals the prospective teachers had for their classrooms were in the process of shifting from emphasizing correct procedures to developing children’s understandings. McClain believed prospective teachers’ emerging conceptual understandings related to place value and multidigit addition and subtraction initiated this shift. Since prospective teachers had to build imagery of a roll consisting of 8 pieces, their conception of a composite unit was fostered first through drawings and then symbolically. This allowed counting to yield a quantity based upon prospective teachers’
drawings not symbols devoid of understanding. McClain noted the importance that prospective teachers developed schemes to make a record of their own activity to solve tasks. The future teachers developed schemes that built on the understanding of the multiplicative structure that underlies place value. By creating different arrangements for the equivalent quantities, two distinct ways of solving the task emerged, by either continuing to pack or unpack the candies. Thus, prospective teachers performed the computations mentally or showing subtraction that yielded the correct quantity. Therefore, prospective teachers found ways to symbolize their activity of composing and decomposing various quantities. Next McClain stated the introduction of selling and buying transactions supported informal addition and subtraction algorithms. As such, many prospective teachers developed non-traditional, yet personally meaningful algorithms for addition and subtraction to symbolize their activity.

Likewise, Andreasen (2006) described a classroom teaching experiment conducted during an undergraduate mathematics content course investigating preservice teachers’ understanding of place value and operations. Like McClain (2003), Andreasen used research with children’s progression when learning place value and operations to identify the hypothetical learning trajectory and mathematical tasks. However, unlike McClain, a major portion of the instructional sequence was set in base-8 and prospective teachers were expected to think and reason entirely in base-8 instead of base-10. The instructional sequence included tasks for counting, unitizing, flexible representation of numbers, and development of operations (Andreasen, 2006). In Andreasen’s study, the instructional sequence concluded in base-10 providing opportunities for prospective teachers to discuss whole number operations and alternative strategies for operations.
Andreasen emphasized the necessity of this shift in instruction to further develop prospective teachers’ conceptual understanding of place value and operations as well as provide an opportunity to explore thinking strategies created by children. In the final phase of the HLT described by Andreasen, prospective teachers were transitioned from base-8 back into base-10 by the instructor in order to apply reasoning about place value and operations to understanding and developing strategies for whole number operations in base-10. Algorithms presented were based upon prospective teachers’ reasoning. Alternative algorithms presented by Andreasen included partial sums, column addition, partial differences, equal compensation, partial products, and partial quotients. Interestingly, these mathematical practices parallel children’s mathematical understandings of whole number concepts and operations (Carroll & Porter, 1998; Cobb & Wheatley, 1988; Fosnot & Dolk, 2001; Huinker et al., 2003; Kamii et al., 1993; Madell, 1985). Prospective teachers were required to explain and justify why the algorithms worked in addition to identify conceptual errors in student work and provide suggestions for correcting the mathematical errors. By attending to conceptual errors in base-10, the instructor focused on developing the prospective teachers’ specialized content knowledge for teaching while previous tasks in base-8 emphasized prospective teachers’ subject matter content knowledge.

Prospective teachers’ development of subject matter content knowledge was documented through social norms, sociomathematical norms, and classroom mathematical practices (Cobb & Yackel, 1996). Social norms of the class reported by Andreasen included: (a) the expectation of providing explanations and justifications of solutions and solution methods, (b) students making sense of other student’s solutions, and (c) students asking questions of classmates or the instructor. Sociomathematical
norms partially established during the class included: (a) criteria for different solutions and solution methods and (b) criteria for what constituted a good explanation. Both the social and sociomathematical norms supported three classroom mathematical practices that develop mathematical content knowledge of prospective teachers. The mathematical practices identified by Andreasen included: (a) unitizing, (b) flexibly representing numbers, and (c) reasoning with operations.

The current study builds upon the research presented by McClain (2003) and Andreasen (2006). Just as McClain and Andreasen, the current study used research involving children’s development of whole number concepts and operations to inform and revise a potential learning pathway focusing on prospective teachers’ development of whole number concepts and operations. In addition, all three studies implemented tasks in the Candy Shop scenario. Due to lack of mathematical success using base-8, McClain described mathematics learning goals that included addition and subtraction using a multiplicative structure occurring in base-10. In the study described by Andreasen instructional tasks included all basic operations in base-8 and base-10. The learning goals were influenced by both studies. However, in this study two major changes occurred: (a) all instructional tasks and conversations encountered by prospective teachers were in base-8, including ones that were encountered in base-10 in the other studies, (b) the usage of items from the, Content Knowledge for Teaching Mathematics Measures (CKT-M) which were administered before and after the instructional sequence in base-8. The changes allowed the research team to explore the possible impact an instructional sequence in base-8 may have on prospective teachers’ base-10 understanding. The current study builds upon and extends prior research regarding prospective teachers’
development of whole number concepts and operations by taking into account the learning goals described in the next section.

Hypothetical Learning Trajectory

Simon (1993) defined a hypothetical learning trajectory (HLT) as “the teacher’s prediction as to the path by which learning might proceed” (p. 35). Simon emphasized that although an expected tendency exists, the actual pathway is impossible to know in advance. Therefore, even though some students’ learning styles are similar in nature, idiosyncratic characteristics are present in each student’s learning. The learning trajectory is hypothetical because until students truly solve a problem, teachers cannot be certain what students will do or whether they will construct interpretations, ideas, and/or strategies. The HLT used in this study supported the refinement of Andreasen’s (2006) instructional sequence to better parallel the “big ideas” through which children progress when developing proficiency with respect to whole number concepts and whole number operations in elementary school. Due to the limited research regarding prospective teachers’ development of whole number concepts and operations, the current study presented research regarding children’s development of whole number concepts and operations to inform prospective teachers’ HLT. This approach is consistent with previous studies that used research conducted with elementary aged children to guide prospective teachers’ development of mathematics concepts (Andreasen, 2006; McClain, 2003). This research study attempted to document the collective learning of whole number concepts and operations that takes place in the elementary education classroom using the HLT suggested by Andreasen (2006), shown in Table 1.
Table 1: HLT for Place Value and Operations suggested by Andreasen (2006)

<table>
<thead>
<tr>
<th>HLT phase</th>
<th>Learning goal</th>
<th>Supporting tasks for instructional sequence</th>
<th>Supporting tools</th>
</tr>
</thead>
<tbody>
<tr>
<td>Phase One</td>
<td>Count and unitize objects efficiently</td>
<td>Counting strategies and representations, 10 frames</td>
<td>Snap cubes, 10 frames, and open number lines</td>
</tr>
<tr>
<td>Phase Two</td>
<td>Flexible representations of numbers</td>
<td>Candy factory scenario involving estimating, packing and unpacking candy, and inventory forms</td>
<td>Pictorial representations of boxes, rolls, and pieces and inventory forms</td>
</tr>
<tr>
<td>Phase Three</td>
<td>Operational fluency</td>
<td>Candy factory transactions, inventory forms, 10 frames, dot arrays, and context-based problems</td>
<td>Pictorial representations of boxes, rolls, and pieces; inventory forms; dot arrays; snap cubes; and open number lines</td>
</tr>
</tbody>
</table>

Classroom instruction began with foundational learning goals such as counting, unitizing, and flexibly composing and decomposing numbers. These learning goals in turn supported the development of methods, meanings, and properties as students develop computational fluency with the four basic number operations: addition, subtraction, multiplication, and division. A more thorough description of the instructional sequence will be described in the Methodology section of this study.

Conclusion

This chapter presented research regarding children’s development of whole number concepts and operations, including their progression from learning to use numbers (Fuson et al., 2001) to performing whole number operations flexibly (Carroll &
Porter, 1998; Huinker et al., 2003; Kamii et al., 1993; Madell, 1985). The chapter continued by presenting research and recommendations documented during teaching experiments regarding prospective teachers’ development of whole number concepts and operations (Andreasen, 2006; McClain, 2003) before concluding with an introduction to a HLT outlining a possible course of mathematics instruction.

The following chapter discusses the characteristics of design research before describing a classroom teaching experiment based upon this methodology. After describing the participants and the setting, the tasks in instructional sequence are described. This is followed by the qualitative and quantitative methods used for data collection during the study are explained, followed by the methods by which the collected qualitative and quantitative data were analyzed. The chapter concludes by discussing the trustworthiness and limitations of the study.
CHAPTER THREE: METHODOLOGY

To study the mathematical development of prospective teachers, a classroom teaching experiment was conducted in an elementary education mathematics content course at a major university in the southeastern United States. As previously stated, the following research questions were investigated during the classroom teaching experiment:

1. What classroom mathematical practices become normative ways of thinking by prospective elementary school teachers during a whole number concepts and operations instructional unit?

2. Is there a statistically significant difference in group mean raw scores between prospective teachers’ pre- and post- whole number concepts and operations situated in base-8 instructional unit administration of items selected from the Content Knowledge for Teaching Mathematics (CKT-M) Measures database?

This chapter begins by discussing the characteristics of design research, the methodology used in the study’s classroom teaching experiment, followed by a description of the participants and setting. The chapter continues by describing both the qualitative and quantitative data collection methods that were used, as well as the interpretive framework that guided the analysis of the qualitative data. This is followed by a discussion of the procedures used for analysis of the qualitative and quantitative data. Finally, trustworthiness and limitations of the study are discussed.

Design Research

outlined the characteristics of design research methodology. First, a major characteristic of design research is the use of methodologies that help develop theories about the process of learning. Second, design research is highly participatory in nature, allowing researchers to investigate possibilities for educational improvement in natural settings. Third, design research creates the necessary conditions for developing theories while placing them in a direct line of scrutiny with a process that is both prospective and reflective. It is prospective in the sense that research designs are implemented using a HLT or initial conjectures about how students acquire knowledge. Initially, conjectures about learning a topic and an instructional sequence of supporting that learning are formulated. During class sessions, the initial conjectures are reflected upon before being either substantiated or refuted so that new conjectures can be formed and tested. This reflection allows for both instantaneous changes to the instructional sequence as they occur in a class setting and for changes to future instructional tasks. Finally, design research is iterative, resulting in a cycle that is both continually inventive and revision focused. The theories developed are accountable to the design while they assist in the formulation of subsequent design experiments. Summarizing design experiments, Cobb (2000) concluded, “theory is seen to emerge from practice and to feed back to guide it” (p. 308).

One type of design research is a classroom teaching experiment. Undertaking a classroom teaching experiment allows for investigation of mathematics teaching and learning as it occurs in a classroom setting (Cobb, 2000). As presented at the end of Chapter 2, a classroom teaching experiment begins by constructing a hypothetical learning trajectory (HLT) supported by research. The creation and development of instructional sequences to support learning goals elaborated in the HLT are then
developed. In this study, a whole number concepts and operations instructional sequence was situated in base-8. As teaching and learning transpires in the classroom setting, researchers use the documented results to help inform subsequent instructional plans and decisions. This is another reason to conduct a classroom teaching experiment. It also allows the researchers an opportunity to retrospectively analyze significant instances of learning that occurred during the learning process (Yackel, 2001). For the presented reasons, a classroom teaching experiment was the methodology selected for the current study.

**Participants and Setting**

The participants in this research study were enrolled in an elementary education mathematics content course in a college of education at a major urban university in the southeastern United States. The students were primarily prospective teachers majoring in either elementary or exceptional education, although some members of the class did not declare a major prior to this study commencing. The class was comprised of 32 female students of which 18 were sophomores, 8 juniors, and 6 seniors.

The study was conducted in a four-credit elementary mathematics content course that is a prerequisite to a mathematics methods course that elementary education students are enrolled in during their first internship, however, it is the only mathematics education course in which exceptional education students are required to enroll. The class convened for 110-minute sessions twice per week during the 2007 spring semester. The whole number concepts and operations instructional sequence transpired during a 10-session unit. The students were taught using a problem-based curriculum that generally posed mathematical problems on which students worked individually or in small groups prior to
engaging in whole-class dialogue. To better facilitate small group interactions, tables in the classroom were arranged with four or five students per table. The research team of eight members was composed of the associate professor in mathematics education who was the course instructor, six doctoral students, and a visiting assistant professor mathematics education with a background in design research.

**Data Collection**

Whole-class conversations were videotaped during all of the sessions by directing three video cameras on different areas of the classroom. Two video cameras focused on the classroom interactions that took place between the instructor and the students while the third video camera focused on the solutions displayed at the front of the classroom. These solutions were either written on the whiteboard or projected via the document camera. The videotaped class conversations then were transferred to DVD and transcribed. In addition, student artifacts such as class work, homework, and tests were collected and photocopied. Together transcribed and student artifacts were used to document qualitative findings. Field notes composed by the research team were used to help describe the learning that took place as observed by each member. These field notes helped the research team make instructional decisions by allowing them to reflect on daily classroom interactions and plan future learning experiences. Finally, data were collected from the administration of selected items from the Content Knowledge for Teaching Mathematics (CKT-M) Measures database (Hill et al., 2004) both before and after participation in the instructional unit. This data provides an opportunity to explore the possible impact that an instructional sequence in base-8 may have on prospective
teachers’ base-10 understanding. Together these data helped document the prospective elementary teachers’ development of whole number concepts and operations.

Instructional Sequence

As stated in the Literature Review, the current study builds upon and extends prior research regarding prospective teachers’ development of whole number concepts and operations by taking into account the learning goals described in the HLT. However, as Hopkins and Cady (2007) stated, familiarity with base-10 may limit prospective teachers from learning whole number concepts and operations with more depth. What becomes imperative is “instruction that helps students develop their current ways of reasoning into more sophisticated ways of mathematical reasoning” (Gravemeijer, 2004, p. 106). Studies using Realistic Mathematics Education (RME) help students develop sophisticated mathematical ways of reasoning through three heuristics: guided reinvention, didactical phenomenology, and emergent modeling. Guided reinvention is the key principle in RME. By placing students in an environment where a particular concept of mathematics was developed, the students are able to “reinvent” the concept themselves (Gravemeijer, 2004). As stated, using base-10 with prospective teachers would limit their possibility of “reinventing” the same important mathematical “big ideas” that children experience. As such, all instruction documented during the study occurs in base-8. This allowed the prospective teachers to “relearn” whole number concepts and operations in base-8 similarly to children developing whole number concepts and operations in base-10.

The second heuristic of RME, didactical phenomenology, focuses on how mathematical experiences are perceived and how they may create opportunities for
learning (Gravemeijer, 2004). Throughout the ten-day instructional unit, context problems helped students grow mathematically by creating opportunities for the prospective teachers to develop their understanding of whole number concepts and operations. Initial tasks that supported important mathematical concepts such as counting and unitizing eventually supported tasks that allowed students to compose and decompose numbers. In the end, tasks completed in previous phases supported students’ procedural fluency.

In the final heuristic of RME, emergent modeling helps “students model their own informal mathematical activity. The aim is that the model with which the students model their own informal activity gradually develops into a model for more formal mathematical reasoning” (Gravemeijer, 2004, p. 117). During the study, the teacher introduces pedagogical content tools (PCT), or a device, graph, equation or statement that a teacher used to connect students’ thinking while moving the mathematical agenda forward. Two different types of PCT are transformational records, and generative alternatives. A transformational record is notation, diagrams, or graphical representation of a student’s thinking that later is used by the students to answer new problems (Rasmussen & Marrongelle, 2006). Another type of pedagogical content tool is generative alternatives or symbolic expressions as well as graphical representations used by a teacher to foster social norms that generate student justifications (Rasmussen & Marrongelle, 2006).

The overarching goal of the hypothetical learning trajectory was to impact prospective teachers’ subject matter content knowledge regarding whole number concepts and operations. In order to accomplish this, the HLT consisted of three phases. During Phase 1 of the instructional unit, tasks emphasized counting and unitizing. Phase
focused on students flexibly representing numbers by composing and decomposing numbers. The third and final phase of the instructional unit focused on procedural fluency or flexible, accurate, efficient algorithms used to perform addition, subtraction, multiplication, or division.

**Instructional Tasks**

The instructional sequence utilized the base-8 numeration system. The numerals 0, 1, 2, 3, 4, 5, 6, and 7 were used to create numbers in the base-8 numeration system in which all whole number instructional tasks were situated. In an attempt to avoid confusion between the numeration systems, a unique number naming convention was used to distinguish base-8 numbers from base-10 numbers. According to the base-8 vernacular developed by the research team, numbers were grouped in ones (1’s), oneee-zeros (10’s), and one-hundreees (100’s). For example, according to the numerical sequence in base-8, the number following seven (7) would be oneee-zero (10) whereas the number following sevenee-seven (77) is oneee-hundree (100). Some tasks were situated within the context of a Candy Shop in which oneee-zero (10) pieces of candy equaled one (1) roll, and oneee-zero (10) rolls of candy equaled one (1) box of candy. Because all of the instructional tasks were situated in base-8, all numbers described in the study are elements of the base-8 number system, whereas numbers in base-10 will be identified as such.

Initial tasks during Phase 1 of the instructional sequence familiarized students with the base-8 numeration system called “Eight World.” Students verbally skip counted forwards and backwards by various numbers. For example, the instructor would have students begin on the number 3 and skip count by 2’s. This helped students create the
base-8 number sequence mentally. Counting also allowed students to confront benchmark numbers in the sequence by attempting to figure out what number came after numbers like 7, 37, and 77. The instructor then displayed the pedagogical content tool, 10-Frames, an example of which is shown in Figure 13, using the overhead projector. The instructor turned the projector on for three to five seconds so the students could see the frame before turning the projector off. The instructor then asked the students, “How many dots are in the frames?”

![Double 10-Frames representing 10.](image)

Figure 13: Double 10-Frames representing 10.

The next tasks emphasizing counting were context problems. Students solved addition and subtraction problems, such as:

**Marc has 12 marbles. He purchased 31 more at the store. How many marbles does he have in total?**

The teacher documented the students’ thinking by using an open number line, like the one shown Figure 14. The instructor introduced this transformational record as a means to document student thinking as well as provide the students a tool that they could later use in solving other problems.
The next tasks situated in “Mrs. Wright’s Candy Shop” were important in developing the multiplicative structure of base-8. In the opening Candy Shop Task, students were asked to estimate the number of “candies” represented by Unifix cubes in a clear bag; students’ estimates ranged from 35 to 112 in base-8. In order to count the 125 candies the bag contained, several students suggested grouping pieces into rolls similarly to the Candy Shop. In Mrs. Wright’s Candy Shop, candy is packaged using boxes, rolls and pieces as shown in Figure 15.

Figure 15: Box, roll, and piece

Oneee-zero (10) pieces of candy are packaged into one (1) roll, and oneee-zero (10) rolls of candy are packaged into one (1) box of candy. This multiplicative structure helped students begin to unitize (Cobb & Wheatley, 1988). In turn counting and unitizing supported subsequent tasks presented during Phase 2 of the instructional sequence. During Phase 2, instructional tasks emphasized flexibility of representing numbers as students composed and decomposed equivalent amounts of candy by drawing boxes, rolls and pieces. However, as amounts of candy increased, a more efficient way to record...
equivalent amounts was necessary; at this point of the unit, the instructor introduced another transformational record called an Inventory Form, shown in Figure 16.

<table>
<thead>
<tr>
<th>Boxes</th>
<th>Rolls</th>
<th>Pieces</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Figure 16: Inventory Form**

Learning goals and pedagogical content tools from both Phase 1 and Phase 2 of the HLT were important in supporting learning goals of the final phase of the HLT.

During Phase 3 of the HLT, the Candy Shop scenario provided opportunities for students to begin to develop meaningful methods for addition, subtraction, multiplication, and division. Initially, the instructor projected various configurations of Boxes, Rolls, and Pieces and the total amount of candies then asked students to describe the missing amount of candy. This instructional task allowed students to cultivate mental computational strategies for addition and subtraction. Subsequent tasks, both within context and context free, familiarized students with addition and subtraction algorithms. The instructor then used 10-Frames to introduce and support the development of multiplication and division. The following task continued to develop multiplication concepts by situating students in a Candy Shop in which a broken machine is producing candy. Another task that supported development of multiplication was an “Egg Carton Scenario.” In this scenario, various dimensions of a prototype carton supported various computational strategies including using pictures and algorithms. The final tasks presented during Phase 3 of the instructional sequence included story problems from which the instructor expected students to develop accurate, flexible and efficient algorithms for multiplication and division. Along with instructional tasks during Phases 1
and 2, the instructional tasks in Phase 3 supported prospective teachers’ development of whole number concepts and operations.

Interpretive Framework

The study employed qualitative research methods to document a semester-long classroom teaching experiment conducted in an undergraduate mathematics content course addressing prospective elementary school teachers’ mathematical content knowledge. Learning was viewed from the emergent perspective that coordinates learning from both the individual or psychological perspective and the group or social perspective (Cobb, 2000; Cobb & Yackel, 1996). Because the psychological perspective cannot be truly isolated from a social learning situation, the emergent perspective depends upon students reorganizing their own mathematical understanding as they contribute to evolving group mathematical practices (Cobb & Yackel, 1996; Yackel & Cobb, 1996). Therefore, both individual sense making and group processes should be equally important in describing the classroom dynamic (Cobb, 2000; Yackel, 2001).

Using the emergent perspective introduced by Cobb and Yackel (1996), the interpretive framework differentiated learning in the classroom environment, see Table 2. Both the social and psychological perspectives were broken down into three domains. The social perspective consists of social norms, sociomathematical norms, and classroom mathematical practices whereas the psychological perspective consists of the individual’s beliefs about his or her role, others’ roles, and the general nature of mathematical activity; mathematical beliefs and values; and mathematical conceptions and activity. As the social perspective was the primary focus of data collection and analysis for this study, this chapter continues by defining the three domains that comprise this perspective.
Social norms, which characterize regularities in the collective classroom activity, are jointly established by the teacher and students; as members of the classroom community, each individual’s meanings are formed from interpretations of social interactions (Yackel, 2001). These social norms apply to any subject regardless of content area. Through student-teacher interactions, social norms may include explanations and justifications of one’s answers using content-specific argumentation and student challenges to the thinking of other students (Yackel & Cobb, 1996). More specifically, social norms are formed when students (a) explain and justify solutions, (b) make sense of explanations given by other students, (c) indicate agreement and disagreement with other students’ solutions, and (d) raise questions when a conflict in interpretations or solutions becomes apparent (Cobb & Yackel, 1996).

The second domain of the interpretive framework is that of sociomathematical norms or normative classroom understandings that are mathematically based (Yackel & Cobb, 1996). Sociomathematical norms encompass that which constitutes “a different mathematical solution, a sophisticated mathematical solution, an efficient mathematical solution, and an acceptable mathematical explanation” (Cobb & Yackel, 1996, p. 178). Yackel and Cobb (1996) demonstrated that elementary school children could assume the

Table 2: Interpretive Framework

<table>
<thead>
<tr>
<th>Social perspective</th>
<th>Psychological perspective</th>
</tr>
</thead>
<tbody>
<tr>
<td>Social norms</td>
<td>Beliefs about one's role, others’ roles, and general nature of mathematical activity in school</td>
</tr>
<tr>
<td>Sociomathematical norms</td>
<td>Mathematical beliefs and values</td>
</tr>
<tr>
<td>Classroom practices</td>
<td>Mathematical conceptions and activity</td>
</tr>
</tbody>
</table>
obligation of making mathematical sense of their peers’ explanations. The children often challenged each other to move beyond procedural responses to strategy-based solutions as well as to use language that emphasized mathematical significance. Notably, these challenges to norms served as negotiation points to describe communal learning.

The third domain of the interpretive framework is that of classroom mathematical practices. Often, individuals express mathematical explanations and justifications when using pedagogical content tools. As a result, researchers can observe taken-as-shared mathematical ideas. One way classroom mathematical practices become taken-as-shared are when backings and/or warrants are no longer included during a mathematical justification whereas in the past the backing and/or warrants were included. Another way ideas become taken-as-shared is if any part of an argument shifts in function by serving a different role in an argument (Cobb, & Yackel, 1996; Rasmussen & Stephan, 2008; Yackel, 2001; Toulmin, 1969). This realization of shared classroom development defines classroom practices. Although these practices do not guarantee the learning of every student in the classroom setting, they do provide an overview of collective mathematical learning in a class (Cobb, 1998). Together, social norms, sociomathematical norms, and classroom practices framed the social perspective that was used for qualitative analysis during this study. In addition, quantitative data was collected using items from the CKT-M Measures database outlined in the following section.

Items from the CKT-M Measures Database

Mathematical knowledge for teaching is the professional mathematics knowledge that separates teaching from other occupations (Ball, Hill, & Bass, 2005). It is this mathematical knowledge that is the focus of the CKT-M Measures database. The
researchers developed the CKT-M Measures database by creating and analyzing a database of items that they believed represented the mathematical content knowledge necessary to teach elementary mathematics and the role content knowledge impacts children’s learning (Hill, Schilling, & Ball, 2004; Phelps, Hill, Ball, & Bass, 2006). Multiple instruments focused on elementary number concepts and operations (K – 6) were developed from items in the database, each having reliabilities ranging from .72 -.81 (Phelps, Hill, Ball, & Bass, 2005). Items from the Elementary Number Concepts and Operations instruments can be differentiated into two types of knowledge: common knowledge of content and specialized knowledge of content. Common content items include mathematical knowledge such as computing, making accurate mathematical statements and the ability to solve problems correctly (Ball, Hill, & Bass, 2005). An example of a common knowledge of content item is shown in Figure 17.

Ms. Domínguez was working with a new textbook and she noticed that it gave more attention to the number 0 than her old book. She came across a page that asked students to determine if a few statements about 0 were true or false. Intrigued, she showed them to her sister who is also a teacher, and asked her what she thought.

Which statement(s) should the sisters select as being true? (Mark YES, NO, or I’M NOT SURE for each item below.)

<table>
<thead>
<tr>
<th></th>
<th>Yes</th>
<th>No</th>
<th>I’m not sure</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) 0 is an even number.</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>b) 0 is not really a number. It is a placeholder in writing big numbers.</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>c) The number 8 can be written as 008.</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

**Figure 17: Common knowledge of content item identified by Ball, Hill, and Bass (2005)**
The researchers also described items that emphasized specialized knowledge of content to include representing mathematical ideas and operations, providing alternative representations, proposing explanations, and evaluating inventive student solutions; an example of a specialized knowledge of content item is shown in Figure 18 (Ball, Hill, & Bass, 2005; Phelps, Hill, Ball, & Bass, 2006). Specialized knowledge of content items are consistent with classroom social norms in which student’s explained and justified solutions and made sense of another’s unique solution method, as well as the sociomathematical norm what makes a different solution.

<table>
<thead>
<tr>
<th>Student A</th>
<th>Student B</th>
<th>Student C</th>
</tr>
</thead>
<tbody>
<tr>
<td>35 × 25</td>
<td>35 × 25</td>
<td>35 × 25</td>
</tr>
<tr>
<td>125</td>
<td>175</td>
<td>25</td>
</tr>
<tr>
<td>+ 75</td>
<td>+ 700</td>
<td>150</td>
</tr>
<tr>
<td>875</td>
<td>875</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td></td>
<td>+ 600</td>
</tr>
<tr>
<td></td>
<td></td>
<td>875</td>
</tr>
</tbody>
</table>

Which of these students is using a method that could be used to multiply any two whole numbers?

<table>
<thead>
<tr>
<th>Method A</th>
<th>Method B</th>
<th>Method C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

**Figure 18: Specialized knowledge of content item identified by Ball, Hill, and Bass (2005)**

Items from the CKT-M Measures database were selected for this study, since the items allowed the research team to explore the possible impact an instructional sequence in base-8 may have on prospective teachers’ base-10 understanding. For this study, 25 items were selected from the CKT-M Measures database for administration to the
students both before and after their participation in an instructional unit. Prior to the selection of the 25 items, a member of the research team reviewed all elementary number concepts and operations items, only selecting items that stressed whole number concepts and operations. It should be noted that items from the database included content knowledge items about rational number concepts and operations; since the development of whole number concepts and operations were the focus of this rational number items were not considered. The member of the research team then compiled items based on their emphasis on whole number concepts and operations in accordance with the three learning goals outlined in the HLT, selecting both common content knowledge and specialized content knowledge items. The compiled items were then reviewed by the team member and the classroom instructor to ensure the items selected supported learning goals and supporting tasks of the instructional sequence. After reviewing all the items, 25 items were selected and administered to the students in the class; nine of the items were common content knowledge items and sixteen of the items were specialized content knowledge items.

Data Analysis

Both qualitative and quantitative methods were used to analyze the data collected during the instructional sequence. Two methodologies were used to analyze the qualitative data collected during the classroom teaching experiment. The constant comparative method described by Glasser and Strauss (1967) was first used to identify social and sociomathematical norms in the classroom environment. The second methodology, described by Rasmussen and Stephan (2008) and based upon Toulmin’s argumentation model (1969), was used to analyze whole-class discourse and establish
classroom mathematical practices that had become taken-as-shared during instruction on whole number concepts and operations. Finally, a dependent \( t \)-test was performed to analyze quantitative data.

Documenting Social and Sociomathematical Norms

In their groundbreaking research, Glasser and Strauss (1967) introduced the constant comparative method to analyze qualitatively collected data. The first step is comparing incidents and organizing data into categories. This step is followed by integrating categories and the properties of these categories. Next, a story integrating the categories is written before the researcher writes the theory. Another approach consistent with the constant comparative method was outlined by Cobb and Whitenack (1996) when they described a methodology to conduct longitudinal analyses of classroom transcripts. The methodology begins with researchers analyzing the video recordings and transcripts of each class session episode by episode. During the second phase, conjectures are ventured to develop chronological developmental categories over time. Finally, the researchers synthesize independent chronologies to create a unified chronology that serves as the basis for written case studies. Both of these methodologies were used to identify social and sociomathematical norms in the present study.

The current study followed a specific procedure to identify both social and sociomathematical norms in the undergraduate elementary education classroom. First, all of the 10 class sessions were videotaped to record the dialogue that occurred between the instructor and students. Second, each videotaped class session was transferred to DVD format and transcribed. Third, two members of the research team independently read the transcripts of each class session to identify, code, and verify each instance of a social or
sociomathematical norm. Fourth, each instance of a norm was compared to previous and subsequent incidents of the same norm to ascertain norm patterns. Finally, the researchers produced a synthesized overview of the social and sociomathematical norms. This methodology is consistent with the four-stage constant comparative method described by Glasser and Strauss (1967) and the methodology used for conducting longitudinal analysis described by Cobb and Whitenack (1996).

Documenting Classroom Mathematical Practices

Rasmussen and Stephan (2008) developed a methodology to document taken-as-shared communal learning. Based upon Toulmin’s (1969) model of argumentation, this methodology is used to document normative ways of group reasoning during a classroom experience. First, a student states a claim, usually in response to a problem or mathematical statement. The student usually supports the claim by providing data, most often in the form of a method or mathematical relationship. The student then uses warrants, or elaborations to validate the data to the claim. Finally, the student provides backing, a response to why an argument should be accepted. In order to provide an example of each element of an argument, Rasmussen and Stephan (2008) described responses of 4th grade students asked to find the area of a 4 by 7 rectangle. In their example, Jason made the claim 28. Jason then provided data to support his claim by stating that he multiplied the length times the width. The researchers then presented Jason’s warrant, in which he declared that multiplying length times width as the way to find area. Jason’s warrant was stated to connect his claim and data. Finally, Jason provided the backing, 4 rows of 7 squares, as a justification to “why” the warrant is valid. However, problem-based classroom interactions are usually not this straightforward;
there are often simultaneous claims, data, or warrants to a given problem. As a result, researchers must sift through each of these claims and their corresponding data, warrants, and backings to document justified and rejected claims (Rasmussen & Stephan, 2008). In order to describe these complex interactions, Rasmussen and Stephan returned to the 4th grade example where students were asked to find the area of a 4 by 7 rectangle. The researchers declared that the children in the class used the claim and data provided by Jason. However, multiple warrants were provided by the children in the class including: (a) counting all the squares one by one, (b) adding 4 seven times, (c) adding 7 four times, and (d) multiplying 4 by 7. In the example, children providing multiple warrants while making the same claim and providing the same data encapsulates the complexity of documenting of classroom mathematical practices.

As a result, Rasmussen and Stephan (2008) developed a three-phase approach that served as the method for documenting whole-class discussion in this study. In the first phase, every class discussion was videotaped and transcribed. The transcripts were used to create an argumentation log to indicate each time a claim was made by a student. The second phase of analysis used data from the argumentation log to document claims over time. By examining data across all class sessions, the researchers could identify which mathematical ideas had become taken-as-shared. Taken-as-shared mathematical practices occur most frequently when backings and/or warrants no longer occur during a mathematical justification as they had before or if any part of an argument shifts in function, such as when what was once a claim now serves as data, a warrant, or backing. In order to document such progressions, the researchers developed a mathematical ideas chart for each class session. This chart allowed the researchers to chart the mathematical ideas that were discussed and must be further investigated in the future, mathematical
ideas that had become taken-as-shared, and additional comments that had either theoretical or practical connections. Each lesson was compared to the previous days’ lesson so that the researchers could chart the progression of ideas that had become taken-as-shared. Finally, the researchers compiled all of the taken-as-shared mathematical ideas and incorporated them into general mathematical activity used by members of the class. This taken-as-shared mathematical activity serves as the basis for determining classroom mathematical practices (Cobb, 1998; Cobb & Yackel, 1996; Rasmussen & Stephan, 2008; Yackel & Cobb, 1996).

Analyzing Items from the CKT-M Measures Database

A dependent t-test was conducted to analyze the results from the pre- and post-instructional unit administrations of items selected from the CKT-M Measures database. This method of statistical analysis was selected in order to analyze the group means of the participants at two instances in time, in this case before and after participation in an instructional unit (Hair, Anderson, Tatham, & Black, 1998).

The null hypothesis of this study was that there is no statistically significant difference between the results from the pre- and post-instructional unit administrations of the CKT-M Measures items.

Trustworthiness

Trustworthiness and credibility are of the utmost importance when conducting qualitative research studies (Lincoln & Guba, 1986; Rossman & Rallis, 2003). Several strategies were employed to ensure the trustworthiness and credibility of the current study. First, the research team maintained a prolonged engagement in the setting that provided the team with lengthy and intensive contact with the students in the elementary education
mathematics classroom (Lincoln & Guba, 1986; Rossman & Rallis, 2003). Members of the research team were present at every class session during the 10-day whole number concepts and operations unit. Prolonged engagement in the classroom setting allowed the researchers to pursue persistent observation and take in-depth field notes. The second strategy used to ensure trustworthiness and rigor was engaging in a community of practice (Rossman & Rallis, 2003). This community consisted of the course instructor of record, an education faculty member, and the doctoral students. All of the members of this community discussed the learning goals of the class sessions, research supporting instructional tasks, and possible courses of action to support learning goals. Finally, all data were analyzed by multiple research team members to minimize researcher bias (Lincoln & Guba, 1986; Rossman & Rallis, 2003). Two researchers independently examined the transcripts and compared instances of claims, data, warrants, and backings. In the process of verifying or refuting each instance of a claim, the researchers were able to produce the argumentation log.

In order to attend to transferability, a thick descriptive narrative about the mathematical development follows in Chapter 4 so that judgments may be made by other researchers who may want to apply all or part of the findings elsewhere. All of the strategies described were employed to establish trustworthiness and credibility, and thus good research practice (Rossman and Rallis, 2003).

**Limitations**

This study faced several possible limitations. First, because the students were administered the same items from the CKT-M Measures database, test-retest reliability may have been limited; when students take identical tests, growth of learning may not be
completely attributable to learning, but attributable to the students’ familiarity with the items. Another limitation was the modest number of students participating in the study. A third limitation was that the study emphasized documentation of the social perspective of the classroom rather than documentation of the social and psychological perspective in tandem.

**Conclusion**

This chapter began by presenting characteristics of design research before describing a classroom teaching experiment based upon this methodology. After describing the participants and the setting, the qualitative and quantitative methods used for data collection during the study were explained, followed by the methods by which the qualitative and quantitative data were analyzed. Using the emergent perspective, the classroom learning environment was actively and reflectively analyzed to describe learning from the social perspective (Cobb, 2000). Finally, the chapter concluded by discussing the trustworthiness and limitations of the study.

The following chapter describes the findings of the classroom teaching experiment used to document prospective teachers’ development of whole number concepts and operations. This documentation will include social norms, and sociomathematical norms that support classroom mathematical practices. Also described are quantitative results of a dependent *t*-test of CKT-M items administered before and after the instructional unit.
CHAPTER FOUR: FINDINGS

The purpose of this study was to document prospective teachers’ shared development of whole number concepts and operations in an elementary education mathematics classroom. Prospective teachers’ development of the above concepts were described using social norms, sociomathematical norms, and taken-as-shared classroom mathematical practices (Cobb & Yackel, 1996; Yackel & Cobb, 1996). In addition, a dependent t-test was conducted to analyze the results from the pre- and post-instructional unit administrations of items selected from the Content Knowledge for Teaching Mathematics (CKT-M) Measures database linking quantitative results to qualitatively documented mathematical development.

As described in Chapter 3, following every class session during the semester, the research team convened to discuss learning that transpired during the class session as well as to discuss the learning goals for the upcoming session. This gave the research team the opportunity to reflect on student learning, and, if necessary, to adjust instructional tasks to better meet the learning goals outlined in the hypothetical learning trajectory, described in Chapter 2. Also discussed during research team meetings were the establishment of social norms and sociomathematical norms, both of which supported the development of prospective teachers’ taken-as-shared mathematical understanding of whole number concepts and operations.

Throughout the ten-day instructional sequence, context problems were introduced to emphasize mathematical learning goals while also establishing and maintaining social and sociomathematical norms of the classroom. Negotiation of social and sociomathematical norms between the instructor and the prospective teachers were a
primary focus of class sessions during the first week of the semester and continued until the end of the semester. Through continual negotiation of social and sociomathematical norms, the instructor emphasized mathematical learning goals of the learning trajectory related to whole number concepts and operations.

Social Norms

Social norms are one of the three domains that constitute the social perspective of a classroom setting. Social norms include students: (a) explaining and justifying solutions; (b) attempting to make sense of explanations given by other students; (c) indicating agreement and disagreement with other students’ solutions; and (d) questioning when a conflict in interpretations or solutions becomes apparent (Cobb & Yackel, 1996). It is important to note that social norms are negotiated between the instructor and the class during whole class dialogue. After analysis the following social norms were identified: students (a) explained and justified their solutions, and (b) made sense of another student’s solutions. The following chapter contains conversations that detail the negotiation of these norms between the instructor and students in the class. While other dialogue exists in supporting the norms, selected dialogue documents the interaction that took place when establishing each of the norms.

Students Explaining and Justifying Solutions

The first social norm negotiated during the classroom teaching experiment was students’ explaining and justifying their solutions. Explaining and justifying refers to the obligation of members of the class to describe the solution process used to solve a problem. During the negotiation, the classroom instructor initially placed this norm in the
foreground of conversation; however, over time students in the class sustained this norm (Andreasen, Roy, Safi, Tobias, & Dixon, 2008; Dixon, Andreasen, & Stephan, in press).

During the initial stage of analysis, two researchers independently coded all incidents of students explaining and justifying their solution processes to minimize researcher bias. Throughout the ten-day whole number concepts and operations unit, members of the research team identified more than one hundred instances of the social norm. After which, each coded instance was compared to previous instances of the social norm to document the negotiation of the norm between the instructor and students. The following five whole class episodes detail students assuming the responsibility of explaining and justifying solutions.

The first step in the negotiation of the social norm, explaining and justifying, becoming taken-as-shared was for the instructor to bring attention to the norm by placing it in foreground of conversation. During the first class session of the instructional sequence the instructor introduced Double 10-Frames to assist students in making sense of counting in base-8. The instructor “flashed” the transparency with the Double 10-frame shown in Figure 19 using the overhead projector; after which the instructor posed the question “how many dots?” were in the frames. The “flashing” technique consisted of the instructor turning on the projector for 3 seconds or less and then turning off the projector so that the students had to rely on visualizing the image from memory. As the subsequent conversation unfolded, the instructor elicited methods used by students to determine the total number of dots.
Figure 19: Double 10-Frames

Instructor: ...So take a look up here, we’re going to do another one. [Instructor flashes Double 10-Frames containing 3 and 5]. I’ll ask you, after everyone has had a chance to think about it then I’ll say how many? Cause I’m seeing some of you doing this. [Counting in the air] You can’t see it but you are counting in the air? Keep your hands up tall, how many of you that counted in the air, counted by ones? Okay, put your hands down. How many of you did not count by ones? What did you count by?

Student: I counted them 3, 4, 1

Instructor: You guys see where she got 3, 4, 1? If I just turn it back on you’d see it. 3, 4, 1. Is that what you did?

Student: Yeah

Instructor: So you did 3, 4, 1. 3 plus 4 is

Student: 10, no, no it’s 7,

Instructor: Is that how you did it? You said 3, 4, 1, but what did you do with 3, 4, 1?

Student: Add them up, 3 plus 4 plus 1 is 10.

Instructor: So did you know it? How do you know 3 plus 4 plus 1 is 10?

Student: I am getting faster at doing it.

Instructor: What is she getting faster at doing?

Jackie: Like, 7 to 10?

Laughing

Instructor: That’s one thing, but she knew 3 plus 4 was 7 probably because of base-10, so I guess you’re right, it would be know 7 plus 1 is 10. She’s learning 7 plus 1 is 10. Good. Yes, how did you do it?

Student: I moved the last dot over there and then knew the whole thing across is 10. No over there, in the missing box and all across is 10. [student motions from left to right]
In the above conversation, the instructor emphasized the strategies students used to arrive at 10 dots in the frames. The instructor indicated the importance of students recognizing different solution strategies, but also asserted the expectation that students were required to explain their thinking, evident in the above quote, “If I had said to you, do this, you would have been okay, whatever, but you hear it from another student, and, it’s powerful. It is a great strategy in teaching, so we are going to do that a lot, ask you to share your thinking.”

During the next episode, the instructor and students began to negotiate what it meant to explain and justify a solution. During the second day of the instructional unit, tasks included context problems used to help students derive individual computational strategies when solving them. In the following episode, members of the class were responding to the following problem.

**There were 62 children in the band. 36 were boys and the rest were girls. How many girls were in the band?**

Instructor: …How can I start? Cordelia.

Cordelia: I came up with 20…

Instructor: I am not asking for the answer. **How can I start?**

Cordelia: Well, like take 62 and 36 and make 10’s; six 10’s minus three 10’s is 30 and then you still have the 2 and the 6.

Initially, Cordelia responded by stating her answer, this is a typical response to problems posed in mathematics classrooms. However, rather than allowing Cordelia to solely provide the answer to the context problem, the instructor emphasized the importance of the process Cordelia used to find the answer. By asking Cordelia, “**How can I start?**”, the
instructor shifted the emphasis when responding to a problem. As a result, Cordelia explained how she arrived at her solution with the process as well as her answer. Until this point during the instructional unit, the instructor often prompted students to identify the pathway they used to solve a problem, an example of which is shown in the previous dialogue. That is until the members of the class began to take responsibility in maintaining this social norm.

Later during the second day of the instructional unit, a pivotal incidence occurred illustrating the shift in responsibility. In the following episode, a student described how she solved the following problem without prompting from the instructor.

There are 51 seagulls on the beach, 22 flew away. How many are still on the beach?

Edith: Should I go up there?
Instructor: Do you want to go up there or do you want me to write it for you?
Edith: It might be easier…
Instructor: Certainly for me.
Edith: Well, I went to the numbers and I saw…
Instructor: Okay, pay attention to Edith please.

Edith: I saw that it was 51 seagulls minus 22 so I had to figure out that. And I realized that if I took, if I added one to 51 I could easily from that number subtract 30 to get 22, so then all I did is took 30 minus the 1 that I added originally and got 27.

![Figure 20: Edith’s Solution to 51 - 22.](image)

The importance of the above dialogue must be stated, since it is the first time during the instructional sequence that a student initiated how she arrived at an answer without being
prompted by the instructor. In her explanation, Edith used the open number line, shown in Figure 20, to record her thinking.

By the fifth session, the obligation that students explain and justify their solution process had become taken-as-shared by members of the class. In Figure 21, students were required to use mental math to find the missing amount of candy when the following problems was displayed on the document camera:

![Diagram of candies](image)

There are 51 total candies.

**Figure 21: How many candies are missing from 51?**

Instructor: …What did you do Beth?

Beth: I just subtracted 20 from 51, and minused 3 from that, to get 26.

Instructor: That’s a great example of what she did, she explained it. Can you say that again louder?

Beth: I saw that there were 2 rolls, so I know that equaled 20 so I minused 20 from 51 and got 31, and then I subtracted 3 to get 26.

Instructor: Okay. That time you included justification for part.

Beth: Oh.

Instructor: The first time you just said what you did, and I was going to use that as a good example to say great explanation, now add a justification, but then you did anyway.

Similarly, to the previously presented episode where Edith automatically described how she solved the problem, Beth automatically explained how she thought about finding the missing candies. In addition, when prompted by the instructor, Beth explained “how” she solved the problem, as well as justified “why” her solution was mathematically valid.

This episode documents the shift in obligation that members of the classroom community
viewed explaining and justifying to include how they arrived at their solutions along with a mathematical justification of those solutions.

By the seventh day of the instructional unit, members of the class articulated their obligation when explaining and justifying a solution.

Suzy: I was just going to say, didn’t …we just take the rules of multiplying? It doesn’t matter which order they are in you know. You are teaching the rules of multiplication.

Instructor: What do you guys think about that?

Claire: **You are going to have to explain and justify it.**

Suzy: Yeah, but that’s okay though; it’s fine

Instructor: Sure?

Suzy: Tell you how to justify it? Sure, get me a math book. I don’t know the exact definition, but I do know from prior knowledge that

Instructor: You know from prior memorization

Suzy: Yeah, from memorization that it doesn’t matter which order.

Instructor: So if you know that, **what should she be able to do** – if she knows that?

Suzy: **Explain and justify.**

In the presented dialogue, Claire emphasized that in order for Suzy to use the commutative property, an explanation and justification would be required. In addition to Claire’s emphasis on sustaining the norm, Suzy, herself, stated if she knew something from prior knowledge, she should be able to provide an explanation and justification. This is important in itself because members of the class had determined that in order to provide an answer, a student was obligated to provide the process as well as an answer when explaining and justifying their solution.

**Making Sense of an Explanation Given by Another Student**

The second social norm negotiated during the classroom teaching experiment was students’ making sense of explanations given by others. As with explaining and
justifying, the establishment of this norm was initially placed in the foreground by the classroom instructor, however over time this norm was sustained by students in the class (Andreasen, Roy, Safi, Tobias, & Dixon, 2008; Dixon, Andreasen, & Stephan, in press).

The initial step when analyzing this norm was for two researchers to independently code all incidents of students making sense of another student’s solution. Throughout the ten-day whole number concepts and operations unit, members of the research team identified more than one hundred instances of the social norm. During the second stage of documenting this norm, each coded instance was compared to previous instances of the social norm to document the negotiation of the norm between the instructor and students as well as to establish the norm. The following classroom conversations illustrate the establishment of the social norm, students making sense of another student’s explanation.

During the first class session of the instructional unit, the instructor cultivated an environment where students were obligated to make sense of explanations made by their classmates. The first instance of the instructor fostering this obligation occurred during the following conversation about the number of dots in the Double 10-Frames shown in Figure 22.

![Figure 22: Double 10-Frame.](image)

Instructor: How do you know?
Winnie: Cause, if you do what, we were at 5, 6, 7, 0, 1, 2.

Instructor: **You guys follow what she just did**, 5, 6, 7, 0?

Students: Yeah, yes, yeah.

Olympia: **I followed**.

By explicitly asking students if they “follow what” Winnie is doing the instructor initiated the students’ responsibility of making sense of solutions. This initial classroom assertion, along with the instructor’s response to Kassie’s statement “Yeah, I’m confused” in the following passage provided opportunity for the class and instructor to continue negotiating this norm.

Instructor: Anyone?

Kassie: **Yeah, I am confused**.

Instructor: Okay, so thank you for saying that, hold on. I’m confused is a very important statement in here, because now it means we need to help you. We meaning you guys need to help.

The significance of the instructor reframing Kassie’s lack of understanding is important for two reasons. First, the instructor was creating a safe classroom environment where a student’s lack of understanding was viewed as an opportunity for learning. Second, by making the class responsible to help any student that does not understand the instructor created the expectation that students must make sense of explanations in order to assist classmates that do not understand.

During the second day of the instructional unit, the members of the class and the instructor began to negotiate this norm while discussing the following context problem:

**Marc had 12 roses. He bought 37 more. How many did he buy altogether?**

Suzy: I just see it as 51 just knowing 7 and 2 goes over…

Instructor: 7 and 2 goes over?

Suzy: Goes up to the next whole number, you know.
Instructor: Aren’t they all whole numbers. 1, 2, 3, 4, 5, 6, those are whole numbers; so what do you mean goes over?

Suzy: I’m not making sense?

Instructor: No.

Suzy: Okay, well then I don’t know how to explain it.

Instructor: Can someone help Suzy explain that?

Olympia: I guess she looked at the 7 and the 2 and she probably counted 7 then to 10, then 11, and then you leave that as a 5 and I guess that she probably wanted, did you?

Suzy: I did…and then you have the 10.

Instructor: Got it.

Olympia: Yeah, then you just brought it over

Instructor: 10, just hanging out?

Laughing

Instructor: What?

Jessica: I understand what she is saying.

Instructor: What is she saying?

Jessica: It’s not that clear, but I know what she is saying.

Instructor: Tell me.

Jessica: She just explained it

Instructor: Here is the thing; it doesn’t count as knowing what she is saying unless you can explain it. So then ask a question to help you explain it or explain it.

The above conversation was significant in the instructor negotiating this norm with the students in the class. In the above dialogue, Jessica’s inability to explain Suzy’s solution provided the instructor an opportunity to explicate the norm. By the end of the conversation, the instructor clearly stated that in order to understand a peer’s solution, another member of the class must possess the ability to explain the process used by that peer. This explanation provided the foundation from which students could maintain the norm.
By the third day of the instructional unit, an important shift in obligation of maintaining the norm occurred during Nancy’s explanation of her solution, shown in Figure 23, to a problem requiring students to draw an equivalent representation of 246 candies using boxes, rolls, and pieces.

Figure 23: Nancy’s representation of candies of 246 candies

Nancy: So if this answer was a problem on our homework, and I was explaining my example, I would say that how I got to like I unraveled two of the rolls, like really be specific? …Okay, I think that’s right. I always second-guess myself when I get up here. Okay, so what I did, I first…

Instructor: Okay can you tell us how many you have drawn there?

Nancy: Yeah, I have, 1 box, okay, I have 1 box, 14 rolls, and 6 pieces.

Instructor: Thanks.

Nancy: And what I did is I took apart 1 of the boxes, that’s right, I took apart 1 of the boxes, so first I had, that’s how many rolls that there were to start with. And I took apart, one of my boxes to have to show 10 rolls and those are my pieces that I didn’t mess with. I left those alone. My question to you is how would I explain it so that way people could understand exactly what I did cause I feel like I am jibber-jabbing right now?

Instructor: What do you guys think jibber-jabber or makes sense?

Students: Makes sense.

Instructor: So I could ask anyone of you what she did and you could tell us? Jackie.

Jackie: Well I was going to say she said to write it down and convert it, she said she wrote it down and converted it.

Instructor: Okay, that’s a good point I am going to address that. Don’t leave yet. So I know just what you did, that we as a class agreed that we know just what you did. That’s a great explanation then.
The presented dialogue is important because Nancy asked the class to determine if her explanation made sense or if it was “jibber-jabber.” Nancy’s concern with understandability of her explanation and answer is significant because the students had assumed the responsibility of sustaining this norm. In addition, since Jackie was able to explain Nancy’s solution, members of the class understood that it was their responsibility to make sense of a peer’s solution.

This shift in responsibility of maintaining the norm was taken-as-shared in the following conversation. The question being explored was about a student’s subtraction process, shown in Figure 24, used to solve the following story problem:

**There were 312 candies in a Candy Shop. 165 candies melted. How many candies were left?**

![Figure 24: 312 – 165.](image)

Edith: In a way, I think of it like the number line, like you add on to it and you subtract that distance. That is how I think of it, like the distance. **Does that make better sense?** Cause you add on to the distance between the two numbers so then you have to subtract it again.

Claire: The distance.

Edith: Not the actual numbers, but just the distance between the numbers, do you want me to draw it?

Instructor: Please do.

Edith. Okay.

Edith: Okay so you start off by adding 10, like you’re moving it, like you’re shifting the distance from here all the way out to here, so then you add a 10 down here.

Caroline: **I thought you said subtract though that’s why, you were saying subtract.**

Edith: Yeah, because …

Student: You are not subtracting you’re adding. You’re making it larger.

70
Edith: but, this is how I am thinking of it. I am trying to justify why it makes sense to me. I’m not saying to subtract the number, you add the number, but you’re subtracting like from the distance. Okay so this, okay, because this is the original difference (student gesturing with hands) then you are adding on to the distance, so you have to subtract from that distance by adding on. Does that make sense?

Nancy: Um-hum.

Instructor: Nancy, you’re saying yes; what did she say to help you?

Nancy: Her hands helped me, it’s the shift you’re changing the numbers, like you said, but the distance is staying the same, you’re just adding on and adding on, but it is still the same distance from right here as it was from right here, where we shifted 10 and we shifted 10.

Edith: Is anyone still confused?

Caroline: I get that, I already got that but I just don’t get where you’re saying you’re adding and subtracting.

Edith: Oh yeah because you …

Caroline: Now you’re subtracting again.

Edith: Well, don’t think numbers, think the distance, if that will help because you’re adding then you have to subtract from that difference to get to the original distance between the two numbers, because you added 10, so it’s a greater distance, so you have to subtract ten from that difference by increasing the number value.

Caroline: All right.

Edith: Does everybody understand?

Instructor: Claudia.

Claudia: I was just going to ask is that what Jackie was thinking?

Jackie: Yes.

The above dialogue exemplifies many of the characteristics regarding this social norm. First, Edith asked if “that made better sense” when describing what she thought was occurring. In addition, it should be noted that the instructor was more of a facilitator of the dialogue that took place, whereas students sustained the conversation based on their understanding of the subtraction procedure. Finally, and perhaps most importantly, Claudia returned to Jackie who started the conversation and asked her if that is what she was thinking about when solving the problem.
After analysis the following social norms were identified: students (a) explained and justified their solutions, and (b) made sense of another student’s solutions. The presented conversations detailed the negotiation that transpired between the instructor and students in the class. While other dialogue exists in supporting the above norms, selected dialogue documents the interaction that took place that initiated and subsequently supported the norms being taken-as-shared. The other two social norms, students indicating agreement or disagreement, and raising questions when a conflict arises, did not surface during class dialogue (Cobb & Yackel, 1996). One possible reason why they did not was the tendency for students to present multiple correct solution pathways and if incorrect answers surfaced, the answer was incorrect due to a computational error whereas the process used was sound. Throughout the instructional sequence, the identified social norms supported the sociomathematical norms described in the next section.

Sociomathematical Norms

The second domain of the social perspective is sociomathematical norms (Cobb & Yackel, 1996). Sociomathematical norms are normative based mathematical classroom understandings specific to students’ mathematical understandings; they include what constitutes an acceptable, different, efficient, and sophisticated mathematical solution (Yackel & Cobb, 1996). During whole class dialogue, the instructor and members of the class negotiate sociomathematical norms together, and consequently the sociomathematical norms are unique to that classroom dynamic. In the current study, students negotiated what was an acceptable solution and were able to identify what constitutes a different solution to a given problem. The following conversations are
classroom conversations that detail the negotiation that took place between the instructor and members of the class when establishing these sociomathematical norms.

Acceptable Solution

The first sociomathematical norm negotiated during the classroom teaching experiment was what constitutes an acceptable solution. This norm refers to the level of specificity members of the class used to describe their solution process. During the negotiation, the norm was initially placed in the foreground of conversation by the classroom instructor, however over time this norm was sustained by students in the class (Andreasen, Roy, Safi, Tobias, & Dixon, 2008; Dixon, Andreasen, & Stephan, in press).

During the initial stage of documenting this norm researchers coded all incidents of students describing an acceptable solution. Throughout the ten-day whole number concepts and operations unit, members of the research team independently identified instances of the sociomathematical norm. After which, each coded instance was compared to previous instances of the norm to document the criterion of the norm. The following five classroom episodes illustrate the negotiation of an acceptable solution.

The first step in establishing the criteria of an acceptable solution was for the instructor to bring attention to the norm by placing it in the foreground of conversation. By day two of the instructional sequence, students began assuming the responsibility of explaining their answers. By doing so, they also were negotiating what it meant to state an acceptable solution; the following dialogue allowed the instructor to explicate this sociomathematical norm.

Instructor: Other questions? If you saw a picture of this and it said a student did this can you explain what the student did? Yeah, Jackie.
Jackie: Well I think part of the problem when you’re trying to explain what somebody else did is even if you can look at the picture, explain how you would use that picture to get the right answer that is not necessarily the way they did it, it is not necessarily their thought process.

Instructor: Very good point, which is why, one of the many reasons I have you explain and justify what you did so I can actually know your thought process. … I need to know your thought process, like you just said and why your thought process is okay and that would include your explanation and your justification. What was your thought process and why is what you did okay. Why was your thought process okay?

During class conversations that occurred up to this point in the unit, a student’s explanation of “how” she solved a problem was the criterion for an acceptable answer. However, the instructor introduced the importance of justifying “why” the solution is mathematically valid. By doing so, the instructor exhibited a conceptual orientation as defined by Thompson, Philipp, Thompson, and Boyd (1994). The instructor’s goal was deepen students’ mathematical reasoning through classroom discourse, and not emphasize discourse as recap of rote procedures evidenced when a teacher possess a calculational orientation (Thompson et al., 1994).

The negotiation of the preciseness of a solution became a point of emphasis evidenced by the instructor’s concluding request in the previous dialogue as well as the following conversation that occurred during the third class session emphasizing whole number concepts. In the following task presented in Figure 25, students represented 246 candies by drawing two different illustrations using boxes, rolls, and pieces.

![Image of 246 candies](https://via.placeholder.com/150)

**Figure 25: 246 candies**

Instructor: … When you write about your strategy, I need to know which one it is, I need to know about that level of specificity… So your job is to reread your statement; could an outside person, could someone else really know what I did? Yes.

Nancy: So if this answer was a problem on our homework, and I was explaining my example, I would say that, how I got to like, I unraveled two of the rolls, like really be specific?
In the presented dialogue, the instructor points out when describing an acceptable solution process it should be detailed enough that someone could really understand what was done. Therefore, in this particular problem, Nancy noted that an acceptable answer would have to describe the process of unraveling the rolls to reach that level of specificity.

By the sixth class session, the responsibility for presenting an acceptable solution shifted from the instructor to the students. In the following dialogue, without prompting from the instructor Mary automatically explained and justified her solution when determining the solution to $166 + 277$ shown in Figure 26.

\[
\begin{array}{c}
1 \quad 6 \quad 7 \\
+ \quad 2 \quad 7 \quad 7 \\
\hline
3 \quad 0 \quad 0 \\
1 \quad 5 \quad 0 \\
1 \quad 6 \\
\hline
4 \quad 6 \quad 6
\end{array}
\]

**Figure 26:** $167 + 277 = 466$.

Instructor: Okay, so who wants to tell us how they got [their answer]; what I would write if I was the student here? Someone who hasn’t shared today. Mary.

Mary: You take the 100’s or the 10’s …

Instructor: The what?

Mary: The 100’s column …

Instructor: The 100’s, okay.

Mary: Yeah.

Instructor: And.

Mary: You have the 2 plus 1 which is 3, and because, since there is boxes or whatever would be 300, and then you take the 10 column and you add the 7 and 6 which is, which would come out to
be, oh, 15, but because you’re looking at it like boxes, rolls, and pieces, or 15 rolls would equal, one box, 5 rolls and 0 pieces, and then you go to the …

Instructor: Are we okay so far? Stop us at any point. Okay.

Mary: then you go to the 1’s column and add the 7 and the 7, which would come up to be, 16, so you take the 1 and put it in the rolls column, because, 10 pieces equals 1 roll, and 6 would stay in the pieces column, and from there you would add, that would be 6 pieces, 6 rolls, and 4 boxes.

Instructor: So, hum. Do you guys agree with how Mary did this? I like the way you provided your justification as you went. That was …

Caroline: She’s good.

It should be noted, when presenting her solution Mary simultaneously fulfilled both criteria of an acceptable solution by explaining “how” she arrived at her answer as well as justifying “why” her solution is valid.

During the seventh session, the shift in responsibility from instructor to members of the class was sustained. In the following episode Claire emphasized that in order for Suzy to use the commutative property, an explanation and justification would be required.

Suzy: I was just going to say, didn’t …we just take the rules of multiplying? It doesn’t matter which order they are in you know. You are teaching the rules of multiplication.

Instructor: What do you guys think about that?

Claire: You are going to have to explain and justify it.

Suzy: Yeah, but that’s okay though; it’s fine

Instructor: Sure?

Suzy: Tell you how to justify it? Sure, get me a math book. I don’t know the exact definition, but I do know from prior knowledge that

Instructor: You know from prior memorization

Suzy: Yeah, from memorization that it doesn’t matter which order..

Instructor: So if you know that, what should she be able to do – if she knows that?

Suzy: Explain and justify.
In addition to Claire’s emphasis on sustaining the norm, Suzy, herself, stated if she knew something from prior knowledge, she should be able to provide an explanation “how” and justification “why.” This is important in itself because members of the class had taken the responsibility of maintaining the norm. This is also evident in the conversation about the following context problem:

**Katrina brings 52 marbles to school to give to her friends. She plans to give each of 10 friends the same number of marbles. How many marbles will each friend get? Will Katrina have any marbles left? If so, how many?**

Edith: Well she was, drew out a grouping of dots, and like the total number and then separated them by, or divided it into 10 groups.

Instructor: Is that what you did?

Laura: Um-huh.

Instructor: So, how did you know where to start the next row?

Laura: I just counted, started counting them out. Like …

Instructor: Point how you drew it so we can see what you did.

Laura: I, first I drew out all of the dots.

Instructor: Show me; what was the first dot that you drew? The second one,

Laura: I drew groups of 10, all the way down; I made 5 groups of 10 and then the 2 extra.

Instructor: Did you see how she, how many of you thought she drew it all the way across? Yeah look at, almost everyone thought you went all the way across, and so did I. So you drew, one group of 10 like a 10-Frame.

Laura: Right.

Instructor: Another 10-Frame, another 10_Frame, and then tell us about those lines.

Laura: Those lines are just how I divided it up to get the 10 friends the equal amount of marbles.

Instructor: How did you know where to draw the lines?

Laura: I guessed.

Instructor: So you guessed that there would be 5 in each?

Laura: Yes.

Instructor: Okay, questions for Laura? Now it’s, yeah Alex.

Alex: **So what’s the justification for it? We know the how, but what is the justification for it?**
As was typical in conversations, the instructor prompted the class to ask questions after a student provided an answer. In this case, after Laura provided an answer to the above question, Alex’s insistence that Laura also provide a justification for her answer supports the criteria that in addition to an explanation “how,” a solution is only complete if a justification “why” is provided. As a result, students identified an acceptable solution to a problem to include the method how it was solved as well as the mathematical reason why it is valid.

Different Solution

Another sociomathematical norm negotiated between the instructor and students in the class during the classroom teaching experiment was what constitutes a different mathematical solution. The social norms: students explaining and justifying their solutions and students making sense of explanations given by other students supported the development of this sociomathematical norm. As a result, many solution processes were presented to given problems. The establishment of this sociomathematical norm initially was placed in the foreground of conversation by the classroom instructor, however over time this norm was sustained by students in the class (Andreasen, Roy, Safi, Tobias, & Dixon, 2008; Dixon, Andreasen, & Stephan, in press).

As with the previously presented social and sociomathematical norms, the initial stage of documenting this norm was to code incidents of students explaining different mathematical solutions. During the ten-day whole number concepts and operations unit, members of the research team independently identified more than seventy-five instances
of the sociomathematical norm. During the second stage, each coded instance was compared to previous instances of the norm to document the establishment of the norm between the instructor and students. The following classroom conversations illustrate the establishment of what constitutes a different solution.

During the first session of the whole number concepts and operations unit, the instructor introduced a series of Double 10-Frames, one of which is shown in Figure 27. The instructor projected a series of these frames one-by-one, for up to 3 seconds each, and prompted students to use mental math to describe the number of dots projected onto the screen. The frames were used to develop students’ counting and early computational strategies in base-8. After students shared the number of dots they saw, the instructor asked to describe how they determined the answer.

![Figure 27: Double 10 Frames](image)

Instructor: **How did you know?**

Claudia: I just remembered the pattern like, and I just counted it, like in my head how I pictured it.

Instructor: What do you mean?

Claudia: Like, I looked at it and remembered what flashed on the screen and I just counted the picture in my mind.

Instructor: Did you count 1, 2, 3, 4, 5, 6, 7, 10?

Claudia: Yes.
Instructor: **How many of you counted each one?** Raise your hand. Please raise your hand so I can see. Okay, so some of you did it, thank you, so that’s one strategy. **How else did it was 10?** Yes.

Caroline: In my mind I saw 4 and 4 and said eight, so 10, ha.

Instructor: So you got 4 plus 4 and did that thing you are not allowed to do and went back. **How many of you did that not allowed to do thing?** Okay, don’t any more.

Laughing

Instructor: How many of you said 4 plus 4 is 10? Straight, you did. So I see 4, I see 4, I know 4 and 4 is 10. That’s where you want to get to, this wouldn’t be a problem if you had grown up here (referring to base-8), but you grew up in that other place and so it’s a problem. But, how did you see 4, this I can see how you saw 4, how can you see 4 here when I did this quickly?

Katherine: I saw that there is one missing at the bottom kind of,

Instructor: Here?

Katherine: No

Instructor: Right here?

Katherine: Right,

Instructor: Okay, so you move this one here? **How many of you did that?** Ok, how many of you moved this one in your mind here? Okay, so about six. How many of you said 1 plus 3 is 4? Raise your hand. Isn’t it interesting, how we are just saying how many and there 10 and there are that many ways of getting there. **I wonder if we just left it showing, if you would have done it differently than you have.**

Initially this norm was placed in the foreground by the instructor by verbally restating the mathematical strategies that Claudia, Caroline, and Katherine used. This was a by-product of the instructor and students negotiating social norms such as explaining and justifying a solution, and having students make sense of a peer’s solution; the teacher created a learning environment in which students were expected to share the ways they were thinking about the problem mathematically.

In the above dialogue, the instructor supported this sociomathematical norm in multiple ways. First, instead of allowing for the result to be sufficient to the question “How many dots?”, the instructor focused on the methods students used to solve the problem. By doing so the instructor elicited different mathematical ways students could
arrive at the same solution. In addition, the instructor emphasized the expectation that if a student solved the problem differently that it was important to share the method with the class. The question, “How many of you did that?” created an environment that if a student’s solution method was not the same, there was at least one other method that should be presented. This allowed the instructor to identify a possible dilemma before it escalated. By expecting students to present multiple mathematical solution processes and not just answers, the instructor emphasized the expectation that students reason mathematically in base-8 instead of adding in base-10 and converting back to base-8 like Caroline suggested in the presented dialogue. This expectation was crucial to students’ development of whole number concepts and operations since Caroline’s method was similar to prospective teachers’ “tricks” in which they manipulated symbols instead of acting on quantities described in McClain’s (2003) study.

Once the expectation that students must reason exclusively in base-8 was made clear, the students and instructor continued to explore what it meant to be a different solution. In the following episode, the instructor flashed the Double-10 frame shown in Figure 28.

Figure 28: Double 10 Frames

Instructor: Okay here we go, ready? This is what I am looking at. (Teacher gestures at counting in the air)
Laughing, Students discussing

Instructor: Alright Cordelia, How did you get it?

Cordelia: Well you have a whole one full so that is 10 and then you have a half so that is plus 4, 14, and plus 1, 15

Instructor: How many of you did it just like that?

Student: Full, 4, 1

Instructor: Full, 4, 1 is another way to describe it. How many of you counted by ones? Took a while, eventually those counting by ones strategies are not efficient enough to keep up and then you start working on developing other strategies. How many of you used counting by ones strategies for some of it but not all of it. Beth how could you use counting by ones for just some of it?

Beth: I would, I just naturally count up by ones.

Instructor: There are a lot of your students will naturally keep counting by ones. Jane.

Jane: I think you just start at 10 and add 5.

Instructor: or 10, 11, 12, 13, 14, 15… How many of you did 10 plus 5? How many of you did 10 counted on by ones to get to 15? Anyone do it a way that we haven’t talked about? How did you do it?

Kassie: I knew the whole thing was 20 and I saw 3 empty boxes, so I subtracted the three.

Instructor: How did you know what 20 – 3 was?

Kassie: I counted.

Instructor: You counted backwards. How many of you did that; started at 20 and counted backwards by ones? Anyone do it slightly different? Completely different? A lot of different strategies with this one.

In the above conversation, the instructor asked an important question in cultivating this sociomathematical norm. By asking students, “How many of you did it just like that?” the instructor fostered the obligation that students were responsible to present a different solution process. This allowed students to identify and describe solution processes including the counting strategies counting up by ones, and counting on from the larger described by Baroody (1987) in his study with kindergarten children. Also presented were an addition and a subtraction strategy explained by Cordelia and Kassie.
respectively. Eventually, these counting strategies were incorporated into computational procedures used to solve story problems.

In the following scenario, students solved the following story problem mentally. After the problem was presented, the teacher recorded various students’ thinking shown in Figure 29, and Figure 30.

**Marc had 12 roses. He bought 37 more. How many did he buy altogether?**

Instructor: …So how else could we do it? Yes.

Claire: 37 plus 2.

Instructor: Okay, so if I do 37 plus 2.

Claire: It’s 41 …

Instructor: How did you know?

Claire: Cause, I just

Instructor: So 37, 40, 41. We’re okay with that? Okay.

Claire: Then you add 10, and it would be 51.

Instructor: How did you know?

Claire: I added, 10.

Laughing

![Diagram of Claire’s method to solve 12 + 37](image)

**Figure 29: Claire’s method to solve 12 + 37**

Instructor: What kind of past experiences in this class have helped us get from 41 to 51? What?

Claudia: Counting by 10’s.

Instructor: Counting by oneee-zero, remember we did that skip counting on Monday and here we are kind of skip counting. Okay, anything else? Questions for Claire? Raise your hands if you did it just like that. Raise your hands tall if you did it just like that. Thank you. Okay, put your hands down. **Okay, that means several of you did it differently.** Who wants to share? Yes.
Cordelia: Well, we already had like oneee-two over there so I counted from the 37, 40, 41, 42…

Instructor: Oh, so you counted by ones on this

Cordelia: …45, 46, 47 …

Instructor: Okay, wait, I have got to catch up to you, you said this was 37 and then what?

Cordelia: 40, 41, 42, 43, 44, 45, 46, 47, 50, 51.

Instructor: And were you done there?

Cordelia: Yes.

Instructor: How did you know you were done?

Cordelia: Cause we had already said the top was equal to 12.

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![Figure 30: Cordelia’s method to solve 12 + 37](image)

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Instructor: Okay, thank you. What I am asking her to do is first explain what she did, which she did, I used this and this was 37 and then justify why what she did was okay; we had, because we had already established above here already represented oneee-two, so that is how I knew that I could do what I explained. You are going to be required to do an explanation and that justification so I am trying to point that out to you now. Questions about that? How are those different? I went 37, 40, 41, 42; we established this as 12, so I knew when I was done. **How are those different?** Yeah.

Caroline: Counting by ones and counting by groups.

Instructor: Okay, so that is the difference of these two. Excellent, **because these are different** and they’re different because this one we counted by groups, this one you counted by ones. How many of you agree that makes them different solution strategies? Raise your hand if you think they are not different. **I am interested in sharing those differences**, so that’s good.

The above conversation is important for two reasons. Until this point in the instructional unit, students were only required to identify different solutions. However, in the above dialogue the instructor deliberately began to shift the responsibility in identifying differences to the students by asking, “How are those different?” In addition, the
instructor supported this shift by beginning to emphasize the conceptual underpinnings of a solution (Thompson, Philipp, Thompson, & Boyd, 1994). This initial shift allowed Caroline to identify different counting strategies used by other classmates to solve the problem.

The instructor and students began to negotiate what constitutes a different solution during the following conversation occurring during day three of the instructional unit. The students and instructor were discussing the following question:

**The local power company was buying cookies for their employees. They ordered some cookies on Monday and 243 cookies on Tuesday. They were billed for 422. How many did they order on Monday?**

Instructor: Who wants to share their solution of how to solve the problem?

Kassie: I’ll be brave, I don’t know if I got the right answer.

Instructor: Well I appreciate it and it’s okay if you didn’t get the right answer.

Kassie: I’ll show you how I attempted it.

Instructor: What do we do if she didn’t get the right answer?

Student: Help her.

Instructor: Thank you. Let her know if you don’t understand what she is doing. There you go.

Kassie: Alright, basically what I did is I started out with 243, well whatever that is…

Instructor: What is that?

Kassie: 243. And I just started by going with 10’s, and then I got to this point and I was like if I go up another 10 I’ll mess up the number line, so I just jumped with, 5 and I got to, no not that yet, this would be, how would you…

Student: 300.

Kassie: There you go, then I did to 400, so that would be plus a 100, and then I continued jumping by 10, I didn’t want to jump by anything too big, or 10, and then plus 2. And then I added it all up, these would be 10’s and I got 157. And that’s how I did it, yeah?

Edith: Isn’t there one too many 10’s?

Kassie: I might of. Yeah, I did, one too many, sorry.
Figure 31: Kassie’s method to solve 243 + ___ = 422

Instructor: Questions? Don’t go yet I do have a question. Before I ask that question, how many of you solved it just like that? So how many of you solved it differently? … Who is going to be the second to share a different solution strategy for this problem? Thank you…

Edith: Okay, well I started out with 422 and I added 1 more just to make it end with the same number and then in the end I’ll subtract it again, but I went back 20, you have 43…

Instructor: What did you get there? Sorry.

Edith: 403.

Instructor: Okay.

Edith: Then I’ve got 100, that’s 303. I subtracted 30, 253, and then I just subtracted 10 more and got 243, and that was 160 that I went back, so then I subtract one and get 157, the one that I added. Any questions?

Instructor: Can someone tell us why she started at 423?

Mary: She wanted the last number to be the same as the ending.

Instructor: Is that it?

Edith: Uh-huh.

160-1=157

Figure 32: Edith’s method to solve 243 + ____ = 422

Instructor: Great, thank you very much. are there questions though? Who can describe how these solutions are different? These solution processes. How are they different? Jane.

Jane: One added until they got to the answer and one subtracted.

Instructor: So what did the first one start with?
Jane: The first one used 243 how much you had on Monday…

Instructor: Okay.

Jane: and they kept adding until they got 157 on top to get to 423. The other one subtracted till it got to 243.

Instructor: And where was the answer in each case?

Jane: On top. [of the open number line]

Instructor: Did anyone have the answer so that it wasn’t on top? … Who wants to share?

Jane: I will.

Instructor: Okay. Go ahead, and we’ll let this one go after we do this one more.

Jane: So like instead of adding or subtracting on top, I took away what we ordered on Monday and this is what I got, so this is the final answer.

Instructor: Questions for Jane? How many of you did it just like that?

Olympia: I just grouped different.

Instructor: You what?

Olympia: I just grouped.

Instructor: So you grouped instead of 10’s at a time, you grouped

Olympia: 20’s.

Instructor: Okay. Great, great. So you could say those solutions are similar, you just grouped differently, but between this solution and this solution, pretty different. [pointing to Edith’s and Jane’s solutions] So at some time I may say what I’d like you to do is solve a problem in different ways, and rather than changing it from 200 and then 20, these are different, very different ones.

In order to solve the above problem, the prospective teachers presented three different subtraction strategies. The strategies included adding up to the sum, shown in Figure 31,
finding the difference between the start and endpoint, shown in Figure 32, and subtracting the starting point from the ending point, shown in Figure 33. Also noted in the conversation is Olympia’s realization that she used the same strategy as Jane, but used different numbers. This point in the negotiation supports the notion that the students in the class were beginning to comprehend that using an operation not consistent with the presented problem, addition in a subtraction context, constituted a different solution, whereas using the same operation with different numbers would not constitute a different solution.

At the end of the third session of the whole number unit, the instructor continued to encourage students to make sense of what constitutes a different solution by explicating the following.

Instructor: So like if you use the number line I’m looking at what we can hardly see on the board from the beginning if you started by subtracting by 10 and then 30 that to me isn’t really different than I started with subtracting 40. You guys understand that is the same strategy with different numbers. As students continued to discuss what constitutes a different solution, students in the class began to embrace the norm as evidenced in the following statement that occurred on the sixth day of the instructional unit.

Katherine: I like that there is three different ways to do it. We learned one way …, maybe if they had three different ways …

Katherine points out that there are multiple ways to solve a problem and that if people had different ways to solve a problem that it might provide additional learning opportunities.

As the eighth session progressed evidence that students began to take responsibility and sustain this sociomathematical norm is evident in the following dialogue. In this dialogue, Katherine is describing “how many” eggs a prototype egg carton with the dimensions 6 by 12 could hold.

Katherine: Well I did it and just counted them all every one. **Then I was trying to figure out another way to do it** and I did 5 by 6 is 36, so like in the last problem, and …

Instructor: Okay.

Katherine: …so 6 by 12 is double of 5 by 6.

Instructor: How do you know that?

Katherine: Because put it, you could put it as twelve, if you could just, well underneath 5 by 6, if you put it as 6 by 12, I mean 12 by 6, you see that 5 and 12, well 12 is double of 5, and so you just double your answer from the first one.

Instructor: But this one is 6 times 12 not 12 times 6. Look 6 times 12.

Katherine: It’s the same. It’s the same.

Instructor: How do you know?

Katherine: Because it depends on which way you want to count it.

Instructor: What do you mean?

Katherine: Like, I could, I, like she did before she counted the 5’s, and then you said some people may have counted the 6’s, you could count the 6’s or count the 12’s and you get the same answer.

Initially, Katherine used the strategy, counting up by ones, which many students successfully used when solving problems using the Double-10 Frames near the beginning of the unit. However, since classroom dialogue emphasized what represents a different solution, she began “trying to figure out another way to do it.” By sustaining this norm, she used a prior problem to reason with the numbers presented in the problem and used a doubling strategy to solve the problem.

To describe what constitutes a different solution required that students explain and justify their solutions as well as make sense of other students’ solutions. The classroom instructor initiated the establishment of this sociomathematical norm, however through negotiation between the instructor and students, members of the class sustained this norm. As a result, students identified two criteria for being a different solution. The
first criterion was counting by ones and counting by groups. The second criterion was using an operation not consistent with the context of the problem.

**Efficient Solution**

Although what constitutes an efficient solution was not taken-as-shared, there were noteworthy conversations that transpired regarding the sociomathematical norm. An efficient solution is one that produces a solution with minimal steps. During the following day five dialogue, the instructor brought this norm to the attention of students in the class.

Instructor: So the question is, is it okay that we are still using the number line? *Something that you want to start thinking about in the way you solve problems is the way that I’m solving them efficient?* If you feel that the way you’re solving them is efficient, just be prepared to tell how it’s efficient. If it is not efficient start moving towards what you think might be more efficient ways of solving the problems. Yeah, that was not an answer. At least I’m consistent.

At this point in the instructional sequence, many students were using the open number line to document their thought processes when solving a story problem in base-8. However, as students became more comfortable reasoning in base-8, the necessity of using the open number line to solve story problems no longer was apparent. This allowed the instructor the opportunity to prompt students to solve context problems in a more efficient way.

However, conversations between the instructor and members of the class did not yield a taken-as-shared perception of what constitutes an efficient solution as shown in the following dialogue.

Instructor: …How many of you did it just like this? Raise your hand. Wow they’re spread out pretty evenly how you did these problems. Who did it, anyone do it differently? *Which one do you think is most efficient?*

Students whispering

Instructor: What do you think Beth?

Beth: I think that one.

Instructor: This one? Why? What makes it efficient to you?
Beth: Well, I mean, I did it the same, but *I just did the numbers*, I didn’t really think about the boxes, rolls, and stuff cause like ...

Instructor: So what did you think about instead of boxes, rolls, and pieces?

Beth: Just numbers.

Instructor: Just numbers. How did you label these?

Beth: I just put 236 plus 146, and added?

Instructor: And so how did you explain this?

Beth: I guess I can explain it saying a box, this, I actually don’t think about it like that.

Instructor: So you think about it, but because you are required to justify, how could you justify if you don’t think boxes, rolls, and pieces, how could you justify this? You can’t. How can she? Help her. Yeah Jackie.

Jackie: Well you have to start with not thinking of it as ones, tens, hundreds [referring to base-10], but 1’s, 10’s, and 100’s [referring to base-8].

Instructor: Does that help you Beth. Help her.

Jackie: Well I’m thinking if you think of it as regular numbers [base-10], ones, tens, and one hundreds, then you count up to ten and go to eleven and that’s not what we’re doing in base-8 so if you think of it as 1’s, 10’s, …

Instructor: So they still look like this? 1’s, 10’s, and 100’s, we just say them differently.

Jackie: Right.

Instructor: Beth so how, do you want to help?

Claire: Well I was just going to say you can’t say the 1 as a 1 if you wrote out 10 that might help to know that it is a 10.

Instructor: What?

Claire: Because, it’s in 10’s place …

Instructor: This one you’re talking about?

Claire: Yeah, so you …

Instructor: So you want to do this?

Claire: Yeah, well 10 plus 30, cause that is really what it is …

Instructor: So I get 40?

Claire: Yeah, and then you would add the 40, right?

Instructor: So then it would be, and then I would add?

Claire: I would just, I would just make the 1 equal to 10.
Instructor: The 1 is equal to 10 right, so maybe your justification could say that, I’m afraid if you write a 10 here, what’s going to happen?

Claire: Then you’ll add 10 to 3 or 13 …

Instructor: If a 10 is in this place, what’s its value?

Claire: Oh, 100.

Instructor: Right. 100, but I see what you’re talking about, and when we expand out this we can say instead of this, you remember this came up a while ago, we can say 200 plus 30 plus 6. We can say those things. Okay. So Beth how would you justify this?

Beth: Could you just say there was like 236 pieces instead of like 2 boxes, 3 rolls, and 6 pieces?

Instructor: Okay, you could. Then what? So I add the 6 and the 6, and I get 14 pieces. Claire talk louder.

Caroline: You condensed them, you grouped them. Instead of having 14 pieces of candy over here, you just went ahead and wrapped up 10 of those pieces to a roll, so you’re adding a roll to the roll’s column.

Instructor: And if you don’t want to use boxes, rolls, and pieces terminology, what can we call this column?

Caroline: 10.

Instructor: 10 column. So when we do that we grouped up 10 of those pieces and turned it into a 100, or into a 10, I’m sorry, and added those. You need to be able to justify that, if you are uncomfortable practice writing it out when you’re solving these problems individually and share with each other or the class. We need to continue to justify those. So you thought, how many of you thought this one was the most efficient. Winnie why do you think this is the most efficient?

Winnie: I don’t know, if you just look at it, it looks in my opinion like easier to understand, it looks neater all together, like the first one you did I understand how she got that, and why she did that, but I can see where if you were trying to show kids that, how it can get confusing, and I mean the number line is good but eventually you are going to have get away from that and start putting it like that in my opinion.


Edith: Well I think it just depends on the person, because like for me I don’t think that last way is the most efficient because I get confused right now adding them up like that. Like I have to do like Claire did, and do the pictures, like cause you didn’t draw all the pictures for that one, so …

Instructor: I didn’t draw the pictures.

Edith: …it is not exactly like we both thought out.

Instructor: Right.

Edith: So for me, I think that way is more efficient, just because I can actually see it, and make sense of it …

Instructor: So this, starting with the pictures?
Edith: Yes, but I think …

Instructor: She is saying that I cheated, and she is right.

Laughing

Edith: **I think it varies from person to person.**

Instructor: Okay, okay. True. Yeah, Claudia.

Claudia: Like I started doing it that way …

Instructor: The way at the end.

Claudia: yeah the inventory way or the way at the end, and then like I realized that I got the wrong answer so then I switched the picture way and I got it a lot faster and got it right so I think it goes back to what Edith said.

Instructor: Which was?

Claudia: **That it just depends on the person.**


Caroline: I just kind of think the Inventory Sheet or that way is better because it sets you up for harder problems that you might do later. Addition, subtraction, multiplication, division.

In the above conversation, criteria for what constitutes an efficient solution included: seeing it better, and just the numbers, however, by the end of the conversation multiple students expressed that criteria for what constitutes an efficient solution could not be reached since an efficient solution depends on the person solving the problem.

During the ten-day instructional unit, criteria for sociomathematical norms became apparent. Through negotiation between the instructor and the members of the class, an acceptable solution included both “how” an answer was reached as well as “why” the answer is mathematically valid. Criteria for what constitutes a different solution included counting by various quantities, as well as using different operations to arrive at an answer. Whereas, dialogue about efficient solutions occurred, criteria for what constitutes an efficient solution had not become taken-as-shared.
Classroom Mathematical Practices

Classroom mathematical practices are the final of the three domains that constitute the social perspective of a classroom setting (Cobb & Yackel, 1996). Classroom mathematical practices are mathematical ideas that no longer require mathematical justification or have shifted in function in an argument and have become what is termed taken-as-shared (Cobb, & Yackel, 1996; Rasmussen & Stephan, 2008; Yackel, 2001). As Cobb (1998) stated these practices do not document the learning of every student in the classroom environment, but provide an overview of the collective mathematical learning in a class. Since this study documents the collective learning of prospective teachers during a classroom teaching experiment, classroom mathematical practices are presented through instructional tasks, and ways of symbolizing.

A three-phase methodology developed by Rasmussen and Stephan (2008) was used to document classroom mathematical practices. The methodology is based upon Toulmin’s (1969) model of argumentation, and is used to document normative ways of group reasoning during a classroom experience. To use the methodology, whole class dialogue is coded according to its function using Toulmin’s (1969) model of argumentation. According to Toulmin’s model, a statement’s purpose serves at least one of four functions in an argument: claim, data, warrant, or backing. After being posed with a question, a student makes a claim, responding to a problem or mathematical prompt. Then the student supports the claim by providing data, most often in the form of a method or mathematical relationship. The third function in an argument called a warrant, it is provided to validate the data to the claim; this can explicit in conversation or be implied. Finally, the student provides backing, which is a response to why an argument
should be accepted. The following classroom dialogue from the current study illustrates statements serving the functions in an argument.

Instructor: …What did you do Beth?
Beth: I just subtracted 20 from 51, and minused 3 from that, to get 26. (data and claim)
Instructor: That’s a great example of what she did, she explained it. Can you say that again louder?
Beth: I saw that there were 2 rolls, so I know that equaled 20 so I minused 20 from 51 and got 31, and then I subtracted 3 to get 26. (data, warrant, claim)
Instructor: Okay. That time you included justification for part.
Beth: Oh.
Instructor: The first time you just said what you did, and I was going to use that as a good example to say great explanation, now add a justification, but then you did anyway.

In the presented dialogue, prospective teachers were asked to subtract 51 – 23. Beth made the claim 26 as an answer to the question. She initially provided data to support her claim by stating that she “subtracted 20 from 51, and minused 3 from that” to arrive at 26. After the instructor asked Beth to repeat her explanation, Beth provided a warrant linking the claim and data, when stating, “I saw that there were 2 rolls, so I know that equaled 20.” However as Rasmussen and Stephan (2008) pointed out, classroom interactions are complex and usually not straightforward; since there are often simultaneous claims, data, or warrants to a given problem. Consequently, researchers must sift through each of these claims and their corresponding data, warrants, and backings to document accepted and rejected claims (Rasmussen & Stephan, 2008). In order to organize the complexities of classroom mathematical practices, Rasmussen and Stephan (2008) developed a three-phase approach that served as the methodology to document whole-class discussion in this study. In the first phase, every whole class discussion during the whole number concepts and operations instructional sequence was videotaped and transcribed. The transcripts were used to create an argumentation log to
indicate each claim, data, warrant, and backing made by students. The second phase of analysis used the argumentation log to document claims over time. By examining claims across all class sessions, the researchers could identify which mathematical ideas had become taken-as-shared. Taken-as-shared mathematical ideas occur two ways. One way classroom mathematical practices become taken-as-shared are when backings and/or warrants are no longer included during a mathematical justification whereas in the past the backing and/or warrants were included. Another way ideas become taken-as-shared is if any part of an argument shifts in function, such as when what was once a claim now serves as data, a warrant, or backing. In order to document the progressions, the researchers developed a mathematical ideas chart for each class session. This chart allowed the researchers to chart: (a) the mathematical ideas and concepts that were discussed and must be further investigated in the future, (b) the mathematical ideas that had become taken-as-shared, and (c) additional comments that had either research or practical connections. Each lesson was compared to the previous days’ lesson so that the researchers could document the progression of ideas that had become taken-as-shared. Finally, the researchers compiled all of the taken-as-shared mathematical ideas and incorporated them into general mathematical activity used by members of the class. This taken-as-shared mathematical activity serves as the basis for determining classroom mathematical practices (Cobb, 1998; Cobb & Yackel, 1996; Rasmussen & Stephan, 2008; Yackel & Cobb, 1996). After analysis of data from the current study, the following classroom mathematical practices of prospective teachers emerged: (a) developing small number relationships using Double 10-Frames, (b) developing two-digit thinking strategies using the open number line, (c) flexibly representing equivalent quantities using pictures or Inventory Forms, and (d) developing addition and subtraction strategies
using pictures or an Inventory Form. Along with prospective teachers’ ways of symbolizing, the following sections will include an overview of the instructional sequence and the classroom mathematical practices that emerged.

Phase One of the Instructional Sequence

Initial tasks during Phase One of the instructional sequence were created to support the learning goals: counting and unitizing in base-8 (Andreasen, 2006). During the initial instructional tasks, the instructor prompted students to count forwards and backwards by various numbers in order to familiarized students with the base-8 numeration system called “Eight World.” The tasks also helped students transition from base-10 and support the instructor’s expectation that they reason solely in base-8. Other tasks involving counting also allowed students to confront benchmark numbers in base-8 when encountering numbers that followed 7 and 77.

Tasks presented during the first day of the instructional unit introduced a series of Double 10-Frames, an example of which is shown in Figure 21. One by one, the instructor would turn the overhead projector on for up to three seconds so the students could see the frames before turning the projector off. The instructor then asked the students, “How many dots are in the frames?”

![Double 10-Frames representing 10](image)

Figure 21: Double 10-Frames representing 10
Other tasks during Phase One of the instructional sequence emphasized prospective teachers developing counting strategies in base-8 when solving context problems, such as:

**Marc has 12 marbles. He purchased 31 more at the store. How many marbles does he have in total?**

The instructor documented the prospective teachers’ thinking by using an open number line, like the one shown Figure 22. It is referred to as an “open” number line since the numbers are not placed on the line in predetermined locations but are added to the line to represent the mathematical moves in the given solution to the particular problem. The instructor introduced this transformational record to document prospective teachers’ thinking as well as to provide them with a tool that they could later use in solving other context problems.

![Open Number Line Example](image)

**Figure 22: Instructor’s documentation of a student’s thinking on an open number line**

The classroom mathematical practices that evolved during this phase were: (a) developing small number relationships using Double 10-Frames, and (b) developing two-digit thinking strategies using the open number line.

**Developing Small Number Relationships Using Double 10-Frames**

A vital foundation to developing whole number concepts and operations is counting (Baroody, 1987; Steffe, Cobb, & Von Glassefeld, 1988). As prospective teachers engaged in tasks created to support counting, developing small number
relationships using Double 10-Frames emerged. Two distinct ways of prospective teachers’ thinking surfaced during this practice: counting by ones, and counting from a group of 10. The episodes that follow document the practice as it became taken-as-shared.

During the first class session of the instructional sequence, the instructor introduced a series of Double 10-Frames. A Double 10-frame is a pedagogical content tool that allowed students to make sense of quantities in base-8. The frames also assisted students in recognizing part-whole relationships, and helped students to begin developing counting strategies in base-8. After the instructor displayed a transparency with the Double 10-frame, the instructor asked the question “how many?” dots were in the frames. The subsequent conversation unfolded, the instructor elicited methods used by students’ to determine the total number of dots in the Double 10-frame shown in Figure 23.

![Figure 23: Double 10 Frames](image)

Instructor: So I’m going to say how many? And you’re going to say. We’re okay with that? Okay, so let me show you what I am really going to do. What I am really going to do is show them to you on this [Instructor turns on overhead projector for three seconds and then turns it off] and then I am going to say how many? And you’re going to say how many. … Are we ready? Here’s the thing, if you are not looking, it’s not going to fun cause you have not seen it. How many?

Students: 10. (claim)

Instructor: Anyone disagree? … Uh, how did you know?

Claudia: I just remembered the pattern like, and I just counted it, like in my head how I pictured it. (data)
Instructor: What do you mean?

Claudia: Like, I looked at it and remember what flashed on the screen and I just counted the picture in my mind. (data)

Instructor: Did you count 1, 2, 3, 4, 5, 6, 7, 10? (warrant)

Claudia: Yes.

In the above dialogue, Claudia stated that she used counting (data) to determine that there were 10 (claim) dots in the frame. At that instance, the instructor took the opportunity to clarify the counting process Claudia used by asking if she counted by ones from 1 to 10, and as a result provided a warrant linking the claim, 10, and data, I counted. However not long after Claudia’s solution, *counting by ones* became a way for students to describe the reasoning they used to solve problems as evidenced in Jackie’s solution to the Double 10-Frames shown in Figure 24.

![Double-10 Frames containing 11](image)

**Figure 24: Double-10 Frames containing 11**

Instructor: Did we talk about how we saw this one? Who hasn’t shared? Let’s have a new sharer share. Jackie.

Jackie: Well, I just, I mean I didn’t see like a picture or anything, I just counted but with the knowledge that you don’t go to eight, you go to 10, I counted 1, 2, 3, 4, 5, 6, 7, 10, 11. (data)

Instructor: And, I am going to call that you “counted by ones,” how many of you “counted by ones”? Put your hands up.

In the presented dialogue, Jackie described counting by ones, “1, 2, 3, 4, 5, 6, 7, 10, 11” to state how many dots were in the frames. In this episode, “counted by ones” shifted in
function to serve as data, as opposed to the previous episode where it served as a warrant.

As stated, one way for a classroom mathematical practice to be considered taken-as-shared is if a claim, data, warrant, or backing shift in function in an argument, thus, counting by ones is a normative way prospective teachers reasoned using the Double 10-Frames.

Another way students reasoned using Double 10-Frames was to count on after making a group of 10 by realizing that one half of the Double 10-Frame or two rows in the Double-Frames equals 10, examples of which are shown Figure 25.

![Figure 25: Two ways students describe 10 in Double-10 Frames](image1)

In the following episode, Cordelia described the total number of dots in the Double-10 Frame shown in Figure 26.

![Figure 26: Double 10-Frame containing 15](image2)

Instructor: All right Cordelia, How did you get it?
In her solution, Cordelia did not furnish a warrant when stating she had “a whole one full.” In addition, the other members of the class did not challenge Cordelia in order to provide a warrant connecting her claim and data, thus supporting this evolving practice grouping by 10 when using Double 10-Frames as becoming taken-as-shared.

Prospective teachers’ grouping by 10 also was evident as students described the total number of dots in the Double 10-Frames shown in Figure 27; however, instead of identifying a group of 10 as a single full Frame, Jessica identified 10 as two full rows of a Double 10-Frame, as exemplified in the following dialogue:

![Figure 27: Double 10-Frame containing 11](image)

Jessica: I took the second column, I took the bottom 2 and moved them up to the top, just cause we know the first two rows equal [10] … (data)

Instructor: Like this?

Jessica: Yeah, and we already know those equal 10. (data)

In Jessica’s strategy, she was able to group 10 by shifting the bottom 2 dots in the Frame located on the right to the top two rows of the Frame on the left. Similar to Cordelia’s solution presented prior to this episode, Jessica did not present a warrant with her solution strategy nor did members of the class require her to do so. Thus, Jessica
presented another strategy supporting the evolving practice of *grouping by 10* when using Double 10-Frames as becoming taken-as-shared.

The presented classroom mathematical practice developing small number relationships using Double 10-Frames evolved as students *counted by ones* and *grouped by 10* when engaged in tasks that supported their understanding of the base-8 number system. This practice supported the prospective teachers’ development of two-digit thinking strategies when using an open number line.

Developing Two-Digit Thinking Strategies Using the Open Number line

During the second day of the instructional sequence instructional tasks in context were presented for students to solve, for example:

**Marc has 12 marbles. He purchased 31 more at the store. How many marbles does he have in total?**

As prospective teachers described their reasoning, the instructor documented their responses on an open number line, like the one shown Figure 28. Over time, the students began to use the open number line to support their thinking when solving context problems.

![Simplified Open Number Line](image)

**Figure 28: Instructor’s documentation of a student’s thinking on an open number line**

The classroom mathematical practice developing counting strategies using the open number line emerged while prospective teacher encountered context problems. Three ways of thinking: (a) counting by 1’s, (b) counting by groups of 10’s and groups of 1’s,
One of way that the prospective teachers participated in the mathematical practice of developing counting strategies using the open number line was by counting by 1’s. In the following episode, Cordelia presents the method in which she reasoned to solve:

**Marc had 12 roses. He bought 37 more. How many did he buy altogether?**

Cordelia: Well, we already had 12 over there so I counted from the 37, 40, 41, 42… (data)

Instructor: Oh, so you counted by ones on this

Cordelia: …45, 46, 47 … (data)

Instructor: Okay, wait, I have got to catch up to you, you said this was 37 and then what?

Cordelia: 40, 41, 42, 43, 44, 45, 46, 47, 50, 51. (data and claim)

Instructor: And were you done there?

Cordelia: Yes.

Instructor: How did you know you were done?

Cordelia: Cause we had already said the top was equal to 12.

As Cordelia described her thinking, the instructor recorded her thinking on the open number line shown in Figure 29.

![Figure 29: Cordelia’s solution to 12 + 37](image)

The presented episode details classroom dialogue in which a member of the class described counting by 1’s using an open number line. To solve the context problem, Cordelia begins with the greater of the two addends and counts by ones until she arrives at her claim, 51. Since Cordelia did not supply a warrant during her solution process, nor
did any member of the class challenge Cordelia’s response, *counting by 1’s* became
taken-as shared as students developed counting strategies using the open number line. It
is plausible that the roots of this strategy can be traced directly to students counting by
1’s using Double-10 Frames.

A second way that the prospective teachers participated in the classroom
mathematical practice of developing counting strategies using the open number line was
by *counting by 10’s and 1’s* as the classroom mathematical practice continued to develop.
During the following two episodes, the strategy counting by 10’s and 1’s was
documented. In response to the following context problem, Claire described the open
number line she drew, shown in Figure 30.

**Marc had 12 roses. He bought 37 more. How many did he buy altogether?**

Claire: 37 plus 2. (data)

Instructor: Okay, so if I do 37 plus 2.

Claire: It’s 41 … (data)

Instructor: How did you know?

Claire: Cause, I just

Instructor: So 37, 40, 41. We’re okay with that? Okay.

Claire: Then you add 10, and it would be 51. (data and claim)

Instructor: How did you know?

Claire: I added, 10. (data)

Laughing

![Figure 30: Claire’s method to solve 12 + 37](image-url)
Instructor: What kind of past experiences in this class have helped us get from 41 to 51?

Claudia: Counting by 10’s. (warrant)

At the end of the episode, Claudia presented a warrant, “counting by 10’s,” to the instructor’s challenge about the validity of “adding 10” to arrive at the claim 51. However, during the second day of the instructional sequence, Betty used the counting strategy counting by 10’ and 1’s to solve the following story problem.

**There were 62 children in the band. 36 were boys and the rest were girls. How many girls were in the band?**

Betty: ...I went from 36 and went to 42 counting up singles and then 42, 52, 62 and knew that was 20 and then added the 4. (data and claim)

In her solution, Betty initially counted by “singles” so that the missing amount would end with the same digit that the total number of band members had. After adding four as she “went from 36 to 42,” Betty added by 10’s, to arrive at the total number of members in the band. As stated earlier, if a presented argument no longer requires a justification to support its mathematical validity, it has become a taken-as-shared practice. Similarly, to counting by 1’s strategies, the roots of counting by 10’s and 1’s using an open number line can be linked to students counting by 10’s and 1’s using Double-10 Frames.

As students became more comfortable solving base-8 context problems, their evolving solution strategies became more inventive. The final way that the prospective teachers participated in the mathematical practice of developing counting strategies using the open number line was by finding a number to make a computation easier. This strategy surfaced during this practice. In the following episode, Edith described how she solved the following problem by drawing the open number line, shown in Figure 31.

**There are 51 seagulls on the beach, 22 flew away. How many are still on the beach?**

Edith: Should I go up there?
Instructor: Do you want to go up there or do you want me to write it for you?

Edith: It might be easier…

Instructor: Certainly for me.

Edith: Well, I went to the numbers and I saw…

Instructor: Okay, pay attention to Edith please.

Edith: I saw that it was 51 seagulls minus 22 so I had to figure out that. And I realized that if I took, if I added one to 51 I could easily from that number subtract 30 to get 22, so then all I did is took 30 minus the 1 that I added originally and got 27. (data and claim)

Figure 31: Edith’s Solution to 51 - 22.

Edith described adding 1 to the total number of seagulls so that she could “easily” subtract the 22 seagulls that flew away. An indication that this practice had evolved was that Edith’s solution did not contain a warrant nor did members of the class request mathematical validation. Thus, Edith presented another strategy, finding a number that makes a computation easier, which supported the evolving classroom mathematical practice of developing two-digit thinking strategies using the open number line as taken-as-shared.

The classroom mathematical practices that evolved during the first phase of the instructional sequence supported linear based reasoning in base-8, however, unitizing, which involves collections based reasoning, was not realized until in Phase Two of the instructional sequence.
Phase Two of the Instructional Sequence

The learning goal for Phase Two of the instructional sequence was flexibly representing numbers (Andreasen, 2006). On day three of the instructional sequence, the instructor framed instructional tasks situated in “Mrs. Wright’s Candy Shop.” In Mrs. Wright’s Candy Shop, candy can be collected into package types called boxes, rolls and pieces as shown in Figure 32.

![Figure 32: Boxes, roll, and pieces](image)

Pieces of candy are packaged into rolls, and rolls of candy are packaged into boxes of candy; 10 pieces equals 1 roll, 10 roll equals 1 box.

During Phase Two, instructional tasks emphasized flexibly representing numbers by having prospective teachers draw equivalent amounts using boxes, rolls and pieces. Students accomplished these tasks by realizing that 1 roll was simultaneously 10 pieces, vital in understanding unitizing (Cobb & Wheatley, 1988). However, as amounts of candy increased, a more efficient way to record equivalent amounts was necessary; at this point during the unit, the instructor introduced an Inventory Form, shown in Figure 33.

<table>
<thead>
<tr>
<th>Boxes</th>
<th>Rolls</th>
<th>Pieces</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Figure 33: Inventory Form**
The classroom mathematical practice that evolved during this phase was flexibly representing equivalent quantities using pictures of boxes, rolls and pieces, or Inventory Form.

**Flexibly Representing Equivalent Quantities Using Pictures or Inventory Forms**

During Phase Two of the instructional sequence, the instructor introduced instructional tasks emphasizing prospective teachers’ ability to flexibly represent a number. The instructor did so by having students investigate ways to find equivalent amounts of candy. In order to support this mathematical development, the instructor introduced and prospective teachers utilized two different pedagogical content tools, boxes, rolls, pieces, and Inventory Forms. In the dialogue that follows, prospective teachers’ development of normative ways of thinking using pictures of boxes, rolls and pieces when engaged in the Candy Shop tasks is presented. This development is followed by similar normative ways reasoning used by prospective teachers when using Inventory Forms. It must be stated again that during Candy Shop tasks, pieces of candy are packaged into rolls, and rolls of candy are packaged into boxes of candy; 10 pieces equals 1 roll, 10 roll equals 1 box.

When an individual simultaneously views ten, as one group of ten or as ten ones that individual is able to unitize (Cobb & Wheatley, 1988). This concept is at the core of tasks created to support the learning goal, flexibly representing numbers. Two ways of thinking that supported this concept of unitizing emerged during this classroom mathematical practice: composing a quantity, and decomposing a quantity using pictures of boxes, rolls, and pieces. The episodes that follow document the practice as it became taken-as-shared.
The following Candy Shop task required students to draw an equivalent representation of 246 candies using pictures of boxes, rolls, and pieces, shown in Figure 34.

![Figure 34: 246 candies represented by 2 Boxes, 4 Rolls, and 6 Pieces](image)

In Cordelia’s solution, she stated that in order to solve the problem, she first needed to determine the total amount of candy represented in the picture.

Cordelia: Well, I figured that before I try and repackage everything that I should know how many I have all together [Figure 31]. So I counted the boxes, 200, and then, 40, I came up with 246. This is my rolls from a box. Okay, that is 246, cause I got… (data)

Instructor: Talk a little louder.

Cordelia: This is 246 cause I’ve got 100 here, 100 in each of these rows and 40 and 6 single pieces. And then I drew it like this. So then I got 100, and then 10 rolls of 10, and 46 individual pieces. (data and claim)

Once Cordelia determined that there were 246 total candies, she was able to compose an equivalent quantity by drawing 1 box, 10 rolls and 46 pieces, shown in Figure 35.

![Figure 35: Cordelia’s representation of 246](image)
Since Cordelia’s solution did not require a warrant from the members of the class, it supported the evolving practice composing an equivalent quantity using pictures of boxes, roll, and pieces as being taken-as-shared.

As the class conversation continued, Nancy described finding an equivalent quantity by decomposing the original quantity. A picture of Nancy’s solution is shown in Figure 36.

![Figure 36: Nancy’s representation of candies of 246 candies](image)

Instructor: Okay can you tell us how many you have drawn there?

Nancy: Yeah, I have 1 box, 14 rolls, and 6 pieces. (claim)

Instructor: Thanks.

Nancy: And what I did is I took apart 1 of the boxes, that’s right, I took apart 1 of the boxes, so first I had, that’s how many rolls that there were to start with. And I took apart, one of my boxes to have to show 10 rolls and those are my pieces that I didn’t mess with. I left those alone …

To explain how she arrive at 1 box, 14 rolls, and 6 pieces Nancy described how she “took apart” one of the boxes “to show 10 rolls.” Nancy realized that one box was also 10 rolls, meaning that the quantity would not change even if she drew a different picture representing 246. In the dialogue, Nancy did not provide a warrant nor did members of the class request mathematical justification. As a result, decomposing a quantity to find
an equivalent quantity using boxes, rolls, and pieces became a taken-as-shared classroom mathematical practice.

Prospective teachers illustrated a similar development of flexibly representing quantities using an Inventory Form. As candy totals increased in quantity, a more efficient way to record equivalent amounts was necessary. As a result, the instructor introduced Inventory Forms. Decomposing a quantity using Inventory Forms surfaced during this classroom mathematical practice. The episodes that follow document the practice as it became taken-as-shared.

At the beginning of the fourth day of instruction, the instructor displayed the Inventory Form, shown in Figure 37, as a way for prospective teachers to record candy totals as well as explore different equivalent amounts of candy.

<table>
<thead>
<tr>
<th>Boxes</th>
<th>Rolls</th>
<th>Pieces</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

**Figure 37: Inventory Form with 1 Box, 3 Rolls, and 4 Pieces**

Instructor: Is there a particular one that you would like to go over or are you fine with not going over them? Questions? All right then. It’s my sense then that you are comfortable representing these candies in different ways. How many of you are getting a little tired of drawing boxes, rolls, and lots of pieces? Yeah you know Mrs. Wright did also, and she felt that it wasn’t such a great way to keep track of what she was doing in the Candy Shop. So she had this efficiency study done and she was told that the time you are spending in counting could be much reduced if you found a different way to record the amount of candy, record the inventory amount in your shop. So she said all right fine, I’ll do something else, and this is what she came up with. I’ll keep track of my boxes, rolls, and pieces by just counting them,..., she counted these boxes, these pieces of candy, and counted, and counted, and counted, and organized them, and figured it out she wrote this, what did she find? After she organized them, what did she find? What does that represent?

Caroline and other students: 134. (claim)

Instructor: 134?

Students: Um-hum.

Instructor: How else could she have found 134? In pictures, what else would it have looked like?

Cordelia: Well, it could’ve been 1 box. (claim)

Instructor: Could’ve been 1 box, okay.

Laughing
Cordelia: 3 rolls and 4 pieces. (claim)

Instructor: Yeah, okay maybe I didn’t need to ask that question.

Laughing

To continue to engage students and their development of flexibly representing quantities using an Inventory Form, the instructor rewrote the Inventory Form containing 1 box, 3 rolls, and 4 pieces to an Inventory Form that had 1 box and 34 pieces, shown in Figure 38.

<table>
<thead>
<tr>
<th>Boxes</th>
<th>Rolls</th>
<th>Pieces</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>34</td>
</tr>
</tbody>
</table>

**Figure 38: Inventory Form with 1 Box and 34 Pieces**

Instructor: What if she recorded that way? What do you suppose she would have found?

Suzy: You could unwrap the rolls and 30 pieces. (data)

Instructor: Yeah, bring that up.

Suzy: You unwrap rolls and add to the pieces. (data)

Instructor: How many pieces?

Suzy: 30. (data)

Instructor: 30 to the 4. Okay, I am not going to do that, [draw pieces on the board] but yeah, cause we decided it’s not so efficient to do that, to sit here and make all these dots, yeah that’s what she would have found; more or less candy than that way of recording it?

After Cordelia stated, “1 box, 3 rolls, and 4 pieces” as a possible representation, Suzy suggested to “unwrap rolls and add to the pieces.” It should be noted that students in the class realized that unwrapping the rolls as suggested would produce conservation in quantity and not change the total, described as the conversation continued.

Students: The same.

Instructor: The same, not more or less, the same amount of candy, different way of representing it using our inventory form, and we’re going call this an Inventory Form for now, because Mrs. Wright used it to keep track of her inventory. … This is just one representation I could have done for this one. Can someone, don’t start filling out you inventory form yet. What are some other representations I could have filled out for this much candy? What are some other representations I could have done? Raise your hands and tell me. Mary.
Mary: You could fill 134 in pieces. (claim)

Instructor: Could I have?

Students: Yeah.

Instructor: Okay, could you come draw it, just kidding.

Laughing

Instructor: Okay, what else could I have done? Raise your hand to tell me what else I could have done. … Barbara could you tell us, who didn’t have her hand raised, could you tell us, give us an example, another way of writing this?

Barbara: You could put, take 10, 13 rolls, and then 4 pieces. (claim)

Instructor: What questions should I ask Barbara and should I have asked the people that spoke before you? What questions should I ask you, Barbara?

Barbara: I don’t know.

Jackie: Why you did that?

Instructor: Okay.

Barbara: Because I broke down a box into rolls, and 10 rolls equals 1 box. (data and warrant)

After Mary and Barbara’s answers, Jackie conveyed that an acceptable solution required justification, to which Barbara responded, “I broke down a box into rolls, and 10 rolls equals one box.” Later in the conversation, students made sense of their classmates’ solutions.

Instructor: Okay, how about another way? Someone else. We’ve represented this, what else? Go ahead.

Suzy: 1 box, 2 rolls, 14 pieces. (claim)

Instructor: How did she get that? Jessica how did she get that?

Jessica: She broke down 1 of the rolls and made [them] into pieces. (data)

Instructor: When you break down a roll, how do you know?

Jessica: You have to add 10 pieces to the pieces you already have. (warrant)

Instructor: Any others? It would be a neat question to ask how many different ways could you represent that.
Understanding “why” is at the core of students’ understanding of decomposing a quantity using an Inventory Form, making Jackie’s request for a warrant “Why did you do that?” important in the decomposition of a quantity into an equivalent amount.

Later during the class session, students continued to reason with Inventory Forms. In the following episode, students were prompted to find two equivalent representations of the Inventory Form representing 457 candies, shown in Figure 39.

<table>
<thead>
<tr>
<th>Boxes</th>
<th>Rolls</th>
<th>Pieces</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>5</td>
<td>7</td>
</tr>
</tbody>
</table>

**Figure 39: Inventory Form representing 457 candies**

Cordelia: I did, I wrote the boxes, and then I … (data)

Instructor: So how many were there?

Cordelia: 4, and then I took apart 1 roll so that would be 4 … (claim and data)

Instructor: 4?

Cordelia: rolls, and then I got more pieces, and I got 17 pieces.

When describing one of her representations, Cordelia described taking a roll apart resulting in “more pieces” in her claim, 4 boxes, 4 rolls, and 17 pieces. Cordelia did not furnish a warrant when describing breaking down a roll and adding it to the pieces as had occurred earlier in the conversation, thus making decomposing quantities using an Inventory Form a taken-as-shared classroom mathematical practice.

As presented in the above episodes, the instructor introduced instructional tasks emphasizing prospective teachers’ ability to flexibly represent numbers. The instructor did so by having students investigate ways to find equivalent amounts of candy utilizing the pedagogical content tools, boxes, rolls, pieces, and Inventory Forms. The presented dialogue documented prospective teachers’ development of normative ways of decomposing numbers into equivalent quantities when reasoning with pictures of boxes, rolls, pieces or Inventory Forms.
Phase Three of the Instructional Sequence

The learning goal of Phase Three of the instructional sequence was to support prospective teachers’ fluency with whole number operations. During Phase Three of the HLT, the Candy Shop scenario provided opportunities for students to begin to develop meaningful methods for addition, subtraction, multiplication, and division. Initially, the instructor projected various pictorial configurations of boxes, rolls, and pieces and then asked students to describe the missing amount of candy. This initial instructional task during Phase Three allowed students to cultivate mental computational strategies for addition and subtraction. Subsequent tasks, including context and non-context problems, familiarized students with addition and subtraction algorithms. The instructor then used 10-Frames to introduce and support the development of multiplication and division. The following task supported the development of multiplication concepts by situating students in a Candy Shop in which a broken machine was producing candy. Another task that supported development of multiplication was an “Egg Carton Scenario.” In this scenario, various dimensions of a prototype carton supported various computational strategies including using pictures and algorithms. The final tasks presented during Phase Three of the instructional sequence included context problems from which the instructor expected students to develop accurate, flexible and efficient algorithms for multiplication and division. Previously taken-as-shared mathematical practices from Phases One and Two of the instructional sequence supported classroom mathematical practices that emerged during Phase Three of the instructional sequence. The classroom mathematical practice that became taken-as-shared during Phase Three was prospective teachers developing addition and subtraction strategies using pictures or Inventory Forms.
Developing Addition and Subtraction Strategies Using Pictures or Inventory Forms

As students engaged in tasks created to support the development of computational strategies, the classroom mathematical practice that emerged was developing addition and subtraction strategies using pictures or Inventory Forms. These prospective teachers’ normative ways of reasoning occurred when the prospective teachers reasoned mathematically using pictures of boxes, rolls, and pieces, and Inventory Forms. The documented ways of reasoning were supported by the previously documented classroom mathematical practices: (a) developing two-digit thinking strategies using the open number line, and (b) flexibly representing equivalent quantities using pictures or Inventory Forms. Three ways of thinking: decompose a quantity to subtract, compose a quantity to subtract, and adding up to find the missing amount surfaced as prospective teachers reasoned using pictures of boxes, rolls, and pieces, and Inventory Forms. The episodes that follow document the practice as it became taken-as-shared.

In the previously presented classroom mathematical practice, the members of the class utilized the warrant that 10 pieces equals 1 roll, and 10 roll equals 1 box to compose or decompose equivalent quantities using boxes, rolls, and pieces. This is important because the emphasis was on not only the package type, but also the quantity that the package contained. In addition, this justification helped identify the classroom mathematical practice, students were able to decompose and compose quantities to perform subtraction and addition using pictures of boxes, rolls, and pieces. Three strategies emerged as students solved the following context problem:

There were 62 lemon candies in the candy shop. After a customer bought some there were only 25 lemon candies in the shop. How many lemon candies did they buy?
In the first solution presented, Claire used Candy Shop terminology as she described using rolls and pieces to solve the context problem. The instructor recorded her thinking and drew the picture represented in Figure 40.

![Diagram](image)

**Figure 40: Instructor’s written record of Claire’s solution**

Claire: Alright. I drew 6 rolls. (data)

Instructor: Okay.

Claire: And 2 pieces. And then to find out how many there were I took away 25, so I had to convert 1 of the rolls to pieces. (data)

Instructor: So you didn’t draw 25 here, you just took it away from here?

Claire: Yeah. (data response to question)

Instructor: Okay. So what did you do I’m sorry.

Claire: I made that 1 roll into pieces. (data)

Instructor: Okay. I know, I know. Okay.

Claire: And then I took away 2 rolls. (data)

Instructor: So you took the rolls away first.

Claire: Yeah, and then I crossed out 5 pieces. And that left me with 35. (data and claim)

In her solution, Claire was unable to take away 25 candies from 62, until she converted 1 roll into pieces. Claire then subtracted the 2 rolls and 5 pieces from the decomposed total 5 roll and 12 pieces to arrive at her claim, 35.
Edith also described using rolls and pieces to solve the context problem; however, instead of decomposing a roll into pieces, Edith composed 62 as 5 rolls and 12 pieces. The picture Edith used is represented in Figure 41.

![Figure 41: Instructor’s written record of Edith’s solution](image)

Edith: I started out, I knew that if I drew, 6 rolls I would have to break them down, like break one apart to get the 5 pieces for 25, so I just drew 5 rolls and 12 pieces, and then I took the 5 pieces and then took the 2 rolls. (data)

Instructor: So what was different about how Edith solved the problem? Jane.

Jane: She did pieces first.

Instructor: She did, she took away her pieces first. What else is different? Jackie.

Jackie: She didn’t draw the 6 rolls and then convert 1 into pieces. She just kind of did that in her mind, she drew 5 rolls to start with.

Instructor: She configured the candy to start with, she configured it differently.

As stated during the previously presented classroom mathematical practices, one way for a classroom mathematical practice to become taken-as-shared is that an explanation no longer requires a justification by the classroom community (Rasmussen & Stephan, 2008). The classroom mathematical practices, decomposing a quantity to subtract using pictures of boxes rolls and pieces, and composing a quantity to subtract using pictures of boxes, rolls, and pieces emerged as members of the class did not require Claire nor Edith to provide warrants to their solution processes.

As the conversation continued, a third way of using pictures of boxes, rolls, and pieces to perform computations surfaced. In this strategy, Caroline added up to the total
candies, thus finding the missing amount of candy. The episode that follows documents the practice as it became taken-as-shared.

Caroline: Just instead of subtracting from 62, I added to 25 until I got to … (data)

Instructor: So did you use pictures?

Caroline: and I counted. Yeah. (data)

Instructor: So you made 25.

Caroline: Okay, I just found. I had 25 in my head. I went … (data)

Instructor: Okay, I have 25 …

Caroline: Yeah, and then 26, 27, 30, 31, 32; And then I just counted up three 10’s. (data)

Instructor: So did you start, 32, 42?

Caroline: 52, 62. (data and claim)

In her solution, Caroline initially counted by pieces so that the missing amount would end with the same digit that the total number of candies had. After adding the pieces as she “counted up three 10’s,” or three rolls she arrived at the total number of candies the customer bought. As with the previously presented strategies if a presented argument no longer requires a justification to support its mathematical validity, it has become a taken-as-shared practice. The roots of this practice can be traced back to similar strategies prospective teachers used when counting by 10’s and 1’s using the open number line and counting by 10’s and 1’s when using the Double-10 Frames. For example, Emily used the notion of finding an easier number to compute with when solving a context problem using an open number line. This type of reasoning was also documented as students solved problems using an Inventory Form. One normative way of prospective teachers’ thinking, documented during this practice, was regrouping using an Inventory Form. The episodes that follow documents the practice as prospective teachers reasoned with Inventory Forms.
On the sixth day of the instructional unit, students worked individually on the following context problem:

**This many candies were in the store room.**

<table>
<thead>
<tr>
<th>Boxes</th>
<th>Rolls</th>
<th>Pieces</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>6</td>
</tr>
</tbody>
</table>

The factory makes 146 more candies. How many candies are in the store room now?

In order to answer the question, Claire described using boxes, rolls, and pieces by “turning them into numbers,” shown in Figure 42.

<table>
<thead>
<tr>
<th>B</th>
<th>R</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 + 1 = 3</td>
<td>3 + 4 = 7</td>
<td>6 + 6 = 14</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

**Figure 42: Instructor’s written record of Claire’s solution**

Instructor: So when you originally drew out your pieces, you drew out 6 pieces and 6 more pieces?

Claire: Yeah. (data)

Instructor: Okay, and then they would have been actually, how many pieces altogether?

Claire: 14. (data)

Instructor: Okay. And how many rolls altogether?

Multiple Students: 7. (data)

Instructor: And how many boxes?

Multiple Students: 3. (data)

Instructor: And then you said you converted them; To keep track of them? Okay. … Is that the order you did it?

Claire: Well I guess, okay, no, okay, and then I turned into numbers and then I worked with the numbers, so I had the 14 pieces, the 7 rolls and the 3 boxes, and then I did what we did with the Inventory Forms, and I put the pieces to rolls, and rolls to boxes. (warrant)

In her solution, Claire provided a warrant, “I put the pieces to rolls, and rolls to boxes,” connecting her method of solving the problem, the data, to her claim. Claire’s process
was followed by Caroline’s solution process using Inventory Forms emphasizing place value can be seen in Caroline’s solution shown in Figure 43.

<table>
<thead>
<tr>
<th>Boxes</th>
<th>Rolls</th>
<th>Pieces</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>+1</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>4</td>
</tr>
</tbody>
</table>

**Figure 43: Instructor’s written record of Caroline’s solution**

Caroline: Instead of starting with the boxes, well, instead of writing out the candies, I did all Inventory Sheets. (data)

Instructor: So you made an Inventory Sheet?

Caroline: Yeah.

Instructor: And you.

Caroline: Wrote 2 for the boxes, 3 for the rolls, 6 for the pieces. 1 for the boxes, 4 for the rolls, 6 for the pieces, and I added starting from the pieces. (data)

Instructor: What did you write down here?

Caroline: I wrote 4 pieces, and I added a roll. (data)

Instructor: So you never wrote the 1 here?

Caroline: No I just added 1 onto the rolls. (data)

Instructor: Okay.

Caroline: And then I got 10 rolls, so I just wrote 0, and put a 1 for the boxes. (data)

Instructor: Wait, wait, wait, you almost started to justify.

Caroline: I didn’t know if you were ready for …

Instructor: Am I ready for justification?

Caroline: No, I don’t think so; I think you said try to separate them, so …

Instructor: Okay, so you were getting ready to do it after?

Caroline: Yeah.

Instructor: Oh, okay, okay.

Caroline: So yeah. And it’s okay to carry those over because 10 pieces equals 1 roll, and so instead of writing the 14, I wrote 1 roll and 4 pieces. The same thing with the rolls to boxes, I converted them. (warrant)
Instructor: How many of you did it, what? I thought somebody said something. How many of you did it just like this? Raise your hand. Wow! They’re spread out pretty evenly how you did these problems. Who did it, anyone, do it differently? Which one do you think is most efficient?

In each of the above solution methods, the students described ways they thought of solving the addition problem using place value. Each student used a solution method describing her understanding of addition. Claire described how she added in columns prior to regrouping her boxes, rolls, or pieces, and Caroline described the traditional addition algorithm method using mathematically appropriate language. Important in establishing that the presented computational strategies were still not taken-as-shared was that a justification was still being provided when adding two numbers together.

As students were solving the non-context problem in the following episode, shown in Figure 44, a warrant when adding two numbers using an Inventory Form no longer was provided, thus establishing it as a taken-as-shared practice.

<table>
<thead>
<tr>
<th>Boxes</th>
<th>Rolls</th>
<th>Pieces</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>-2</td>
<td>7</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>3</td>
</tr>
</tbody>
</table>

Figure 44: 417 – 253 using Inventory Forms

Nancy: For me I was trying to challenge myself a little more because the pictures are easy for so I’m trying (inaudible) more difficult approach.

Instructor: Okay, so you are trying a more difficult approach, not using pictures makes it more difficult.

Nancy: For me, yeah.

Instructor: How many of you agree with that, not making pictures is a way of making it more difficult? So no pictures equals more difficulty.

Olympia: I have a question.

Instructor: Yeah, I think I said to you and you said we upgraded. What did you mean by that?

Olympia: Well to me subtraction is the hardest like that, so I always have to check my subtraction with addition.

Instructor: Okay.
Olympia: Because just for addition it is a little easier to count on your fingers and make sure it’s all right, and subtraction going backwards you could kind of get lost,

Instructor: Got it. So how did you solve this one?

Olympia: Okay. Oh, since we started out with 7; it was a little easier I guess. I know 3 from 7 is 4, and then … (data)

Instructor: Put the 4 here?

Olympia: Yes. I know I can’t take away 5 from 1 so I have to break up a box. (data)

Instructor: Okay.

Olympia: And when you break down the box, you take away 1 box, cause you’re breaking down 1 box, so now you are left with 3 in the box column, and it becomes 10 rolls in the rolls column, no it doesn’t, you, you bring ten rolls over so now it becomes 11 rolls … (data)

Instructor: Okay.

Olympia: And now, you take the 5 rolls from the 11 rolls, that becomes 4, and 2 boxes from 3 boxes is 1. And now I checked it with addition, and I did 2 boxes, 5 rolls, 3 pieces plus 1 box, 4 rolls, 4 pieces. I know 3 plus 4 is 7, and 5 plus 4 is 11, and I know 2 boxes plus 1 box is 3, and … I added the 1 in the, the first one, with the 3, and I got 4 so it becomes 4 boxes, 1 roll, 7 pieces. (data and claim)

In the presented dialogue, Olympia determined the accuracy of her answer to a subtraction problem with addition. In her “check,” Olympia described the addition steps she used to arrive at the minuend, 417. As stated if a presented argument no longer requires a justification to support its mathematical validity, it has become a taken-as-shared practice. This is the case in the regrouping strategy using Inventory Forms. As stated when prospective teachers reasoned with pictures of boxes, rolls, and pieces, the roots of prospective teachers’ reasoning with Inventory forms when adding or subtracting can be traced back to counting strategies such as counting on by groups when utilizing an open number line or the Double-10 Frames to solve a context problem.

Conclusion

Although classroom mathematical practices do not document the learning of every student in the classroom environment, they do provide an overview of the
collective mathematical learning in a classroom environment (Cobb, 1998). As prospective teachers engaged in instructional tasks while reasoning with pedagogical content tools, taken-as-shared classroom mathematical practiced evolved. After coding each statement during whole class dialogue and analyzing the functions of those statements, the following normative ways of mathematical thinking evolved: (a) developing small number relationships using Double 10-Frames, (b) developing two-digit thinking strategies using the open number line, (c) flexibly representing equivalent quantities using pictures or Inventory Forms, and (d) developing addition and subtraction strategies using pictures or an Inventory Form. Each of the taken-as-shared classroom mathematical practices reinforced or were supported by counting strategies, using easier numbers for computations, composing and decomposing equivalent quantities that emerged during the three phases of the instructional sequence. By the end of the instructional sequence, the practices described in this section impacted prospective teachers’ understanding of whole number concepts and operations in base-10.

**Results from Content Knowledge for Teaching Mathematics (CKT-M) Measures**

At the beginning of the instructional sequence, prospective teachers were administered 25 items from the Content Knowledge for Teaching Mathematics (CKT-M) Measures. Students’ results were graded either correct or incorrect on the items obtaining a number of correct items that were entered into the SPSS statistical software package. At the end of the semester the same items were administered to the class and entered into the SPSS statistical software package. These measures were taken in response to the research question:
Is there a statistically significant difference in group mean raw scores between prospective teachers’ pre- and post- whole number concepts and operations situated in base-8 instructional unit administration of items selected from the Content Knowledge for Teaching Mathematics (CKT-M) Measures database?

A paired sample or dependent-t test was used to analyze matched pairs of data. This statistical measure was selected because the data are normally distributed and the same group is tested over time.

$H_0$: There is no difference in mean scores in pre- and post- administrations of items from the Content Knowledge for Teaching Mathematics Measures.

Table 3: Paired Sample Statistics

<table>
<thead>
<tr>
<th>Pair</th>
<th>Pretest</th>
<th>Posttest</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>13.09</td>
<td>15.47</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mean</th>
<th>N</th>
<th>Std. Deviation</th>
<th>Std. Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>13.09</td>
<td>32</td>
<td>2.480</td>
<td>.438</td>
</tr>
<tr>
<td>15.47</td>
<td>32</td>
<td>3.090</td>
<td>.546</td>
</tr>
</tbody>
</table>

Table 4: Paired Samples Test

<table>
<thead>
<tr>
<th>Pair 1</th>
<th>Pretest - Posttest</th>
<th>Mean</th>
<th>Std. Deviation</th>
<th>Std. Error Mean</th>
<th>95% Confidence Interval of the Difference</th>
<th>t</th>
<th>df</th>
<th>Sig. (2-tailed)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>-2.375</td>
<td>2.673</td>
<td>.473</td>
<td>-3.399 - 1.399</td>
<td>-5.026</td>
<td>31</td>
<td>.000</td>
</tr>
</tbody>
</table>

There was a statistically significant difference ($t = -5.026$, df = 31, $p < 0.01$) in mean score between the two administrations, shown in Table 4. The administration of the pre- test were reported with $\mu_1 = 13.09$ and $\sigma_1 = 2.48$, whereas the administration of the post- test were reported with $\mu_2 = 15.47$ and $\sigma_2 = 3.09$, shown in Table 3. The 95%
confidence interval for the true mean difference is $-3.339 < \mu < -1.411$. As a result of
the whole number instructional sequence in base-8, a statistically significant increase in
items from the CKT-M ($\mu_1 = 13.09$ to $\mu_2 = 15.47$) was observed, $t(31) = -5.026$, $p < 0.01$.

Conclusion

According to Cobb (1998), classroom mathematical practices provide an
overview of collective mathematical learning in a class. The classroom mathematical
practices documented in this study include prospective teachers: (a) developing small
number relationships using Double 10-Frames, (b) developing two-digit thinking
strategies using the open number line, (c) flexibly representing equivalent quantities
using pictures or Inventory Forms, and (d) developing addition and subtraction strategies
using pictures or an Inventory Form. Quantitative findings revealed that there was a
statistically significant mean difference in pre-, post- instructional unit administrations of
the Content Knowledge for Teaching Mathematics items (CKT-M).

In the following chapter, concluding thoughts connecting results from the current
classroom teaching experiment to presented research in the literature review is presented.
In addition, implications of the current study to future research possibilities will be
described.
CHAPTER FIVE: CONCLUSION

The purpose of this study was to document prospective teachers’ shared development of whole number concepts and operations in an elementary education mathematics classroom. In order to investigate this shared development a classroom teaching experiment was conducted, which allowed members of the research team to study mathematics teaching and learning in a classroom setting (Cobb, 2000).

The development of prospective teachers’ understanding of whole number concepts and operations was emphasized in this study for two reasons. First, since it is a core component of elementary school mathematics, the importance of whole number concepts and operations on future mathematics learning has been stressed (National Mathematics Advisory Panel, 2008; NCTM, 2000; Kilpatrick, Swafford, & Findell, 2001). Second, despite its significance, prospective elementary teachers are entering the profession without a depth of knowledge regarding whole number concepts and operations necessary to support the unique understandings children have of these fundamental mathematical concepts (Ball, 1990; Kilpatrick, Swafford, & Findell, 2001; Ma, 1999; National Mathematics Panel, 2008; Shulman, 1986).

As Hopkins and Cady (2007) stated, familiarity with base-10 number concepts may limit adults’ understanding of the number concepts that they are learning. As a result, McClain (2003) and Andreasen (2006) described classroom teaching experiments (CTEs) in which instructional tasks were situated in a “Candy Shop” scenario in base-8. The scenario allowed prospective teachers to reconceptualize their mathematical understandings by promoting cognitive dissonance while engaging in the tasks in base-8 (Gravemeijer, 2004).
Just as in the research conducted by McClain (2003) and Andreasen (2006), the current study used research describing children’s development of whole number concepts and operations to inform the hypothetical learning trajectory (HLT), described in CHAPTER TWO. However, as Simon (1995) stated the HLT is a projected learning pathway of the students in a class, and is different from, the actualized learning trajectory. The actualized learning trajectory, shown in Table 5, realized during prospective teachers’ development of whole number concepts and operations is different in several ways from the hypothetical learning trajectory presented in the second chapter of this study. First, unitizing and counting were identified as learning goals of Phase One of the HLT. Prospective teachers engaged in tasks and reasoned with tools to support those goals. As the students reasoned with tools during Phase One, counting strategies became evident in the taken-as-shared practices that occurred during Phase One: (a) developing small number relationships using Double 10-Frames, and (b) developing two-digit thinking strategies using the open number line. Whereas it may seem that prospective teachers were unitizing during Phase One of the instructional sequence when they counted by groups of 10, that was not the case. The prospective teachers did not view 10 simultaneously as both one group of 10 and 10 individual units. It was not until the prospective teachers reasoned with boxes, rolls, pieces and Inventory Forms during Phase Two of the instructional sequence, did they unitize and understand that numbers can simultaneously represent 1 roll or 10 pieces, or as 10 rolls or 1 box (Cobb & Wheatley, 1996). It was also during the second phase of the actualized learning trajectory that the research team to support unitizing as a learning goal added additional tasks, bolded in Table 5. As a result, it is apparent that unitizing should be a learning goal during Phase Two of the instructional sequence since it was the keystone of taken-as-
shared mathematical practice during Phase Two: flexibly representing quantities using pictures or Inventory Forms.

**Table 5: Actualized Learning Trajectory**

<table>
<thead>
<tr>
<th>HLT phase</th>
<th>Learning goal</th>
<th>Supporting tasks for instructional sequence</th>
<th>Supporting tools</th>
</tr>
</thead>
<tbody>
<tr>
<td>Phase One</td>
<td>Counting</td>
<td>Counting; Skip Counting; Open number line Problems; Counting Problem Set #1; Counting Problem Set #2</td>
<td>Double 10-Frames, and Open number line</td>
</tr>
<tr>
<td>Phase Two</td>
<td>Unitizing, Flexible representations of numbers</td>
<td>Estimating with Snap Cubes; Candy Shop 1; <strong>Candy Shop Exercise 1 (Exit Question)</strong>; Candy Shop Exercise #2; Torn Forms</td>
<td>Snap Cubes; Boxes, rolls and pieces; Inventory forms</td>
</tr>
<tr>
<td>Phase Three</td>
<td>Invented Computational Strategies</td>
<td>Candy Shop 2; Candy Shop Inventory; Candy Shop Addition and Subtraction; Inventory Forms for addition and subtraction (in context); Inventory Forms for addition and subtraction (out of context); 10 Dot Frames; Broken Machine; Multiplication Scenario, Multiplication Word Problems; Division Word Problems</td>
<td>Boxes, rolls, and pieces; Inventory forms; dot arrays; and Open number line</td>
</tr>
</tbody>
</table>

Since design experiments are iterative, a cycle that is both continually inventive and revision focused results. The theories developed are accountable to the design while they assist in the formulation of subsequent design experiments (Cobb, 2000; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). The emphasis on revision was noted during Phase Three of the actualized learning trajectory, when prospective teachers developed computational strategies for addition and subtraction. McClain (2003) reported that prospective teachers used tricks and shortcuts, by working in base-10 and converting
answers back to base-8, to solve problems. When performing tasks the prospective
teachers’ emphasis was manipulation of digits rather than understanding numerical
quantities. Andreasen (2006) also described prospective teachers’ understanding related
to whole number concepts and operations; tasks presented were initially set in base-8 and
followed by tasks in base-10 allowing for prospective teachers to apply reasoning about
place value and operations to better understand strategies for whole number operations in
base-10. As a by-product of tasks being set in both bases, many tasks were repetitious. A
major change to the instructional sequence included the decision to require students to
think in and engage in all instructional tasks solely using base-8. The instructor was
scrupulous in requiring that students describe their mathematical reasoning using base-8.
The final addition to the instructional sequence was the pre- and post- instructional unit
administration of items selected from the Content Knowledge for Teaching Mathematics
(CKT-M) Measures database. This provided a quantitative analysis not reported by the
other studies, allowing the research team to explore the possible impact an instructional
sequence in base-8 may have on prospective teachers’ base-10 understanding.

In order to describe the prospective teachers’ development, both qualitative and
quantitative analyses were conducted. Qualitative research methods included the
constant-comparative method articulated by Glasser and Strauss (1967), Toulmin’s
method of argumentation, and Rasmussen and Stephan’s (2008) three phase-
methodology. The constant-comparative method was used to identify various social
norms, sociomathematical norms during the CTE. Toulmin’s method of argumentation
(1969), Rasmussen and Stephan’s three-phase methodology (2008) were used to
document taken-as-shared classroom mathematical practices. Quantitative research
included a statistical analysis of students’ scores on items selected from Content
Knowledge for Teaching Mathematics (CKT-M) database described by Hill, Schilling, and Ball (2004) prior and subsequent to a ten-day instructional unit emphasizing whole number concepts and operations, to ascertain the impact of an instructional sequence in base-8 on prospective teachers’ base-10 understanding.

Qualitative findings included the social norms students explained and justified their solutions, and made sense of other individual’s solution methods. The sociomathematical norms identified during the current study was what constitutes an acceptable as well as what is a different solution. Finally, classroom mathematical practices that became taken-as-shared were prospective teachers: (a) developing small number relationships using Double 10-Frames, (b) developing two-digit thinking strategies using the open number line, (c) flexibly representing equivalent quantities using pictures or Inventory Forms, and (d) developing addition and subtraction strategies using pictures or an Inventory Form. In addition, the quantitative findings revealed that there was a statistically significant difference in pre-, post- instructional unit administrations of the CKT-M. Together these results indicate that counting and calculating in base-8 provided a meaningful context for prospective teachers to reconstruct their knowledge of whole number concepts and operations in base-10, supported by a statistically significant increase in mean scores before and after the instructional sequence.

The findings are important for several reasons. First, the expectation that all students reason in base-8 allowed prospective teachers to experience mathematics in a way similar to ways in which children experience learning whole number concepts and operations in base-10. In fact, it has been documented in this study that many prospective teachers reasoned in base-8 in similar ways to children reasoning in base-10. For
example, the prospective teachers developed counting strategies when figuring out how many dots were contained in Double-10 Frames. Counting by ones and counting by groups of 10 were among the strategies employed by prospective teachers to arrive at the total number of dots in the pedagogical content tool, Double-10 Frames. These counting strategies are similar to those described in research with children (Baroody, 1987; Carpenter et al., 1999). The similar reasoning also is apparent when the prospective teachers developed computational strategies for addition and subtraction problems.

Prospective teachers decomposed an addend or decomposed the minuend to find the sum or difference respectively. Strategies developed by prospective teachers in which place value was emphasized were similar to those developed by children in research reported by Madell (1985), Kamii, Lewis, and Livingston (1993), and Huinker, Freckman, and Steinmeyer (2003). For example, the strategy employed by the prospective teacher, Kassie, in which she added up in parts to find the solution to the missing addend problem is similar to the adding up strategy that Jamese used to solve $674 - 328$ (Huinker et al., 2003). Another example of a prospective teacher reasoning in base-8 that is similar to a child’s reasoning in base-10 was Claire’s record of adding by place value using Inventory Forms. In her solution, Claire added each place value, followed by recomposing the partial sums to find the result. This strategy is similar to a child’s addition strategy when adding $18 + 17$ in which the child initially added to find the partial sums 20 and 15, and then added the partial sums to find the result of 35 (Kamii, Lewis, & Livingston, 1993).

Finally, a third prospective teacher’s strategy that was similar to a child’s base-10 strategy was Edith’s number line strategy in which she added a number to the minuend to make the computation “easier.” Edith’s strategy was similar to John’s subtraction strategy
in which he added a number to the minuend to solve a simpler computation followed by adjusting the result to compensate the number that he added (Huinker et al., 2003).

It also should be noted that by learning mathematics in a classroom environment that emphasizes social and sociomathematical norms described by Cobb and Yackel (1996), the prospective teachers in this study experienced learning mathematics differently than they might have in the past. Sustaining social and sociomathematical norms allowed the prospective teachers to develop profound understanding of fundamental mathematics (Ma, 1999). In addition, the norms permitted the instructor and class to discuss mathematics methods while simultaneously learning whole number content matter. Finally, it could seem that instruction in base-8 and a quantitative analysis in base-10 are incongruent; however, statistical significance in mean averages illustrates that instruction in base-8 impacts subject matter content knowledge in base-10. While these results are promising, there still is a necessity to conduct further research with prospective teachers’ development of whole number concepts and operations.

Implications

This study attempted to add to the limited amount of research involving prospective teachers’ development of whole number concepts and operations. Important in developing tasks was research regarding the development of children’s understanding of whole number concepts and operations. As stated, design experiments are iterative, resulting in a cycle that is both continually inventive and revision focused (Cobb et al. 2003). A design experiment assists in the formulation of future design experiments. Findings from the current iteration of a whole number and operation CTE support the following suggestions for revisions in subsequent CTEs.
The analysis of the current study provides an opportunity for several areas of subsequent research. First, provide more instructional time to see if ideas during tasks emphasizing multiplication and division become taken-as-shared classroom mathematical practices (Cobb, 1998; Cobb & Yackel, 1996; Rasmussen & Stephan, 2008; Yackel & Cobb, 1996). Also, since normative ways of reasoning occurred using various tools, it may be plausible to emphasize one of the tools. For example, similar ways of reasoning occurred when prospective teachers drew pictures of boxes, rolls, and pieces or used an Inventory Form to flexibly represent equivalent quantities. In future iterations, the instructor may emphasize the more abstract representation that occurs when reasoning with an Inventory Form simultaneously to students’ representations using boxes, rolls, and pieces. Second, analysis during the current study was limited to the social perspective of the emergent perspective described by Cobb and Yackel (1996). In order to document a more complete description of the classroom environment, the individual perspective of members of the class should be analyzed. A third area that this research can be used to support future research is to document the impact of the instructional sequence longitudinally. For example, continue research with prospective teachers who completed the whole number concepts and operations instructional sequence during the content course in their course of study. Some avenues that may be explored include documenting the interactions between the prospective teachers and elementary-grade students when teaching whole number concepts and operations. This will allow researchers to study prospective teachers’ content knowledge and pedagogical content knowledge simultaneously. Eventually, it would also be beneficial to connect prospective teachers’ whole number content knowledge with the knowledge of the future students they teach.
Finally, developing a chain of signification so that theoretical implications may be studied. As shown, various research opportunities stem from the current study.

**Conclusion**

The hypothetical learning trajectory used by Andreasen (2006) was refined to develop prospective teachers’ understanding of whole number concepts and operations. In the end, the instructional tasks supported the following taken-as-shared classroom mathematical practices in which prospective teachers: (a) developing small number relationships using Double 10-Frames, (b) developing two-digit thinking strategies using the open number line, (c) flexibly representing equivalent quantities using pictures or Inventory Forms, and (d) developing addition and subtraction strategies using pictures or an Inventory Form. Findings show that there was a statistically significant difference in mean scores on items from CKT-M prior to and subsequent to the instructional sequence. These presented results have implications regarding prospective teachers’ subject matter content knowledge.
Base-8 100’s Chart
Counting Problem Set #1

1. There were 54 people in a movie theatre. 6 more entered the theatre. How many people watched the movie?

2. Marc has 12 marbles. He purchased 31 more at the store. How many marbles does he have in total?

3. There were 62 children in the band. 36 were boys and the rest were girls. How many girls were in the band?

4. There are 51 seagulls on the beach, 22 flew away. How many are still on the beach?

5. Greg has 45 playing cards. Melissa has 35 playing cards. How many cards do they have together?

6. There are 111 students at Jackson Elementary, 64 are on a field trip. How many students are still at school?
1. Before lunch, you sold 37 cookies. After lunch you sold 45 cookies. How many cookies did you sell in the day?

2. Johnny bought 23 cookies. Steve bought some cookies, too. Johnny and Steve bought 52 cookies in all. How many cookies did Steve buy?

3. The local power company was buying cookies for their employees. They ordered some cookies on Monday and 243 cookies on Tuesday. They were billed for 422 cookies. How many did they order on Monday?

4. Mrs. Johnson brought 53 cookies for her class. She gave 25 cookies to Mr. Jones. How many cookies did Mrs. Johnson have left?

5. Jessica decided to share all her cookies with her friends. She gave 27 cookies to one friend and 43 cookies to another friend. How many cookies did she have to start with?

6. Susie had 154 crayons. She gave some crayons to her friend. She had 127 crayons left. How many did she give to her friend?

7. The Candy Shop made 237 cookies last night. The Cookie Company made 372 cookies last night. How many more cookies did the Cookie Company make than the Candy Shop?
Candy Shop 1

You own a candy shop in Base-8 World. Candy comes packaged in boxes, rolls, and individual pieces.

There are 10 candy pieces in a roll and 10 rolls in a box.

Use this information to complete the following:

1. Show two different ways to represent the following:
2. Show two different ways to represent the following:

3. Show two different ways to represent the following:
4. Show two different ways to represent 426 candies.

5. Show two different ways to represent 277 candies.

6. Show two different ways to represent 652 candies.
Torn Forms

Which forms represent the same quantities of candies? Write the quantities for each using single digits in each column.

<table>
<thead>
<tr>
<th>Form A</th>
<th>Form B</th>
</tr>
</thead>
<tbody>
<tr>
<td>B  R  P</td>
<td>B  R  P</td>
</tr>
<tr>
<td>1  36  7</td>
<td>0  57  37</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Form C</th>
<th>Form D</th>
</tr>
</thead>
<tbody>
<tr>
<td>B  R  P</td>
<td>B  R  P</td>
</tr>
<tr>
<td>3  27  37</td>
<td>0  0  347</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Form E</th>
<th>Form F</th>
<th>Form G</th>
</tr>
</thead>
<tbody>
<tr>
<td>B  R  P</td>
<td>B  R  P</td>
<td>B  R  P</td>
</tr>
<tr>
<td>2  6  107</td>
<td>0  27  57</td>
<td>2  33  77</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Form H</th>
<th>Form I</th>
<th>Form J</th>
</tr>
</thead>
<tbody>
<tr>
<td>B  R  P</td>
<td>B  R  P</td>
<td>B  R  P</td>
</tr>
<tr>
<td>2  10  47</td>
<td>2  17  77</td>
<td>2  32  107</td>
</tr>
</tbody>
</table>
Candy Shop Inventory

1. This many lemon candies are in the candy shop.

Mrs. Wright makes 23 more lemon candies. How many lemon candies are in the candy shop now?

2. This many chocolate candies are in the candy shop.

How would you unpack some of these candies so that you can sell 35 chocolate candies? How many chocolate candies will be left in the candy shop?

3. This many orange candies were in the candy shop.
How would you unpack some of these candies so that you can sell 42 orange candies?
How many orange candies will be left in the candy shop?
Candy Shop Addition and Subtraction

1. There were 46 tangerine candies in the candy shop. Ms. Wright made 24 more tangerine candies. How many tangerine candies are in the shop now?

2. There were 62 lemon candies in the candy shop. After a customer bought some there were only 25 lemon candies left in the shop. How many lemon candies did they buy?

3. There were 34 grape candies in the candy shop. After Ms. Wright made some more there were 63 grape candies in the shop. How many more grape candies did she make?

4. There were 53 orange candies in the candy shop. A customer buys 25 candies. How many orange candies are in the shop now?
Inventory Forms for Addition and Subtraction (In Context)

1. This many candies were in the store room.

<table>
<thead>
<tr>
<th>Boxes</th>
<th>Rolls</th>
<th>Pieces</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>6</td>
</tr>
</tbody>
</table>

The factory makes 146 more candies. How many candies are in the store room now?

2. This many candies are in the store room.

<table>
<thead>
<tr>
<th>Boxes</th>
<th>Rolls</th>
<th>Pieces</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0</td>
<td>4</td>
</tr>
</tbody>
</table>

A customer orders 136 candies. How many candies will be left in the store room?

3. This many candies are in the store room.

<table>
<thead>
<tr>
<th>Boxes</th>
<th>Rolls</th>
<th>Pieces</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

A customer orders 145 candies. How many candies will be left in the store room?

4. This many candies are in the store room.

<table>
<thead>
<tr>
<th>Boxes</th>
<th>Rolls</th>
<th>Pieces</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>5</td>
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</tbody>
</table>

The factory makes 256 more candies. How many candies are in the store room now?
Inventory Forms for Addition and Subtraction (Out of Context)

<table>
<thead>
<tr>
<th>Boxes</th>
<th>Rolls</th>
<th>Pieces</th>
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<tbody>
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<th>Rolls</th>
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<td>6</td>
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<td>-2</td>
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<tr>
<td>-2</td>
<td>5</td>
<td>3</td>
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<tr>
<th>Boxes</th>
<th>Rolls</th>
<th>Pieces</th>
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</thead>
<tbody>
<tr>
<td>2</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>+2</td>
<td>5</td>
<td>3</td>
</tr>
</tbody>
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<td>2</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>+2</td>
<td>4</td>
<td>6</td>
</tr>
</tbody>
</table>
Broken Machine

Ms. McLaughlin operates a machine that puts 10 sticks of gum in a pack.

1. How many sticks of gum are in 5 packs?

The machine breaks down and now only puts 7 sticks in a pack.

2. How many sticks of gum are in 5 packs?

3. How many sticks of gum are in 7 packs?

After Ms. McLaughlin tries to fix the machine, she makes a mistake and now the machine only puts 6 sticks in a pack.

4. How many sticks of gum will be in 5 packs?

5. How many sticks of gum will be in 6 packs?

Mr. Strawberry has a machine for putting candies into bags. The machine is made to put 20 candies in each bag. The machine is not working correctly and is putting only 17 candies in each bag.

6. How many candies does the machine use for 5 bags?

7. How many candies does the machine use for 6 bags?
Mr. Strawberry tries to fix the machine but it gets worse. Now it puts only 16 candies in each bag.

8. Now how many candies does the machine use for 6 bags?

9. How many candies does the machine use for 3 bags?

Later that day, Mr. Strawberry tries to fix the machine yet again. Now it puts 22 candies in each bag.

10. How many candies does the machine use for 10 bags?

11. How many candies does the machine use for 6 bags?
Multiplication Scenario

A marketing team has created three new prototypes for an egg carton. Explain and justify how many eggs would fit in each carton?

a. 5 by 6 egg carton

b. 6 by 12 egg carton

c. 3 by 16 egg carton
Multiplication Word Problems

1. Mrs. Wright lines up 12 pieces of candy into rows for gift packages. Each package has 16 rows of the 12 pieces. How many candies are in a gift package?

2. Mr. Smith’s eighth grade class was performing *Romeo and Juliet*. The school theater had 23 rows with 15 seats in each row. How many people will be able to see the play?

3. A farmer plants a rectangular garden that measures 27 feet along a side and 13 feet along the adjacent side. How many square feet did the farmer plant?
**Division Word Problems**

Solve each of the problem situations below. Draw pictures if that will help you understand the situation and solve the problem. Remember we are still in 8 world.

1. Katrina brings 52 marbles to school to give to her friends. She plans to give each of 10 friends the same number of marbles. How many marbles will each friend get? Will Katrina have any marbles left? If so, how many?

2. Jason has 43 pencils to share with some of his class. There are 5 students in his class that he would like to give his pencils to. How many pencils does each friend get?

3. Sarah has 125 candies. She wants to give each of her friends 12 candies. How many friends can she share with? Does she have any candies left for herself? If so, how many?

4. Micah has some friends he wants to share his stickers with. He has 236 stickers. How many friends can he share them with if he wants each friend to get 14 stickers? How many stickers, if any, does Micah have left?
Create Your Own Base-8 Problems

For this activity, you will be creating four word problems in base-8. Below, write one word problem for addition, one for subtraction, one for multiplication, and one for division. Once you have created these problems, solve them on a separate sheet of paper. After you have found the solutions, trade with someone else in the class and solve theirs.

1.

2.

3.

4.
APPENDIX B: SAMPLE ARGUMENTATION
**Question:** Marc had 12 roses. He bought 37 more. How many did he buy altogether?

<table>
<thead>
<tr>
<th>Claim</th>
<th>Data</th>
<th>Challenge</th>
<th>Warrant</th>
<th>Backing</th>
</tr>
</thead>
<tbody>
<tr>
<td>51</td>
<td>Wouldn’t it be like 51? Because if you think about 3 plus 1 is 4 and 7 plus 2 is 1.</td>
<td>But that is not a 3.</td>
<td>Oh, I am doing it like the whole and changing.</td>
<td></td>
</tr>
<tr>
<td>51</td>
<td>37 plus 2. It’s 41. Then you add 10, and it would be 51.</td>
<td>What kind of past experiences in this class have helped us get from 41 to 51?</td>
<td>Counting by 10.</td>
<td></td>
</tr>
<tr>
<td>51</td>
<td>Well, we already had like 12 over there; so I counted from the 37, 40, 41, 42, 43, 44, 45, 46, 47, 50, 51.</td>
<td>How did you know you were done?</td>
<td>Cause we had already said the top was equal to 12.</td>
<td></td>
</tr>
<tr>
<td>51</td>
<td>I did 10 plus 10 and put them together... 2 plus 7, which was 11, right? Yeah.</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
December 18, 2006

Judi Dixon, Ph.D.
University of Central Florida
Teaching and Learning Principles
ED 129
Orlando, FL 32816-1250

Dear Dr. Dixon:

With reference to your protocol 406-4020 entitled, "Prospective Teachers' Development of Understanding of Mathematical Concepts during a Classroom Teaching Experiment," I am enclosing your record's approved, expedited document of the IRB Form you had submitted to our office. This study was approved on 12/1/06. The expiration date for this study will be 12/31/2007. Should there be a need to extend this study, a Continuing Review Form must be submitted to the IRB Office for review by the Chairman or full IRB at least one month prior to the expiration date. This is the responsibility of the investigator.

Please be advised that this approval is given for one year. Should there be any amendments or administrative changes to the already approved protocol, they must also be submitted to the Board through use of the Amendments Modification Request Form. Changes should not be initiated until written IRB approval is received. Adverse events should be reported to the IRB as they occur.

Should you have any questions, please do not hesitate to call me at 407-823-2901.

Please accept our best wishes for the success of your endeavor.

Cordially,

Joanne Sturtevant
IRB Office

12201 Research Parkway • Suite 501 • Orlando, FL 32826-3246 • 407-823-3778 • Fax 407-823-3299
THE UNIVERSITY OF CENTRAL FLORIDA  
INSTITUTIONAL REVIEW BOARD (IRB)  

IRB Committee Approval Form  

PRINCIPAL INVESTIGATOR(S):  Jul Dixon, Ph.D.  

PROJECT TITLE: Prospective Teachers' Development of Understanding of Mathematical Concepts during a Classroom Teaching Experiment  

[ ] New project submission  [ ] Initial submission was approved by expedited review  

[ ] Continuing review of another project  [ ] Initial submission was approved by expedited review  

[ ] Study expires  [ ] Suspension of enrollment in study  

Chair:  
Expedited Approval  

Dated:  2/10/06  

[ ] Exempt  

Dated:  

[ ] Exempt  

[ ] Initial submission was approved by expedited review  

Expiration Date:  12/31/07  

[ ] Waiver of documentation of consent approved  

[ ] Waiver of HIPAA Authorization approved  

IRB Reviewers:  

Signed:  Dr. Craig Van Slyke, Chair  

Signed:  Dr. Sophia Dziegielewski, Vice-Chair  

NOTES FROM IRE CHAIR (IF APPLICABLE):
UCF IRB Protocol Submission Form

Please type this form using the Microsoft Word template. Send as mail. Allow a minimum of 2-3 weeks for the approval process. A letter of approval will be mailed to you once approved. Information on the form must match information on the grant application, dissertation or thesis, consent forms or letters, and flyers for recruitment. There are no deadlines for submission of minimal risk requests as they are reviewed at least weekly. It is decided by the IRB that the study involves greater than minimal risk or involving research, the complete IRB packet must be submitted by the 1st business day of the month in consideration at the monthly IRB meeting. At the time of investigator is Staying, Master’s Candidate or Doctoral Candidate.

1. Title of Protocol: Prospective Teachers’ Development of Understanding of Mathematical Concepts during a Classroom Teaching Experience

2. Principal Investigator(s): [List the faculty supervisor as both the Principal Investigator and the Society supervisor (if student) if both students are doing the research. List students as co-investigator(s)]

   Signature
   Name: Mr./Ms./Mrs./Dr. (chose one)
   Employee ID or Student PID #: Degree:
   Title:
   Department:
   College:
   E-Mail:
   Telephone:
   Facsimile:

3. Supervisor(s):

   Signature
   Name: Mr./Ms./Mrs./Dr. (chose one)
   Employee ID or Student PID #: Degree:
   Title:
   Department:
   College:
   E-Mail:
   Telephone:
   Facsimile:
   Home Telephone:

   Signature
   Name: Mr./Ms./Mrs./Dr. (chose one)
   Employee ID or Student PID #: Degree:
   Title:
   Department:
   College:
   E-Mail:
   Telephone:
   Facsimile:
   Home Telephone:
4. Collaborating institution(s) and researcher(s) (identify the institution and its FWA number, if known. List the names of collaborating researchers and briefly describe their roles in the study. Provide contact information. If the collaborating institution receives federal funds and does not have a federalwide assurance, a completed UCF Individual Investigator Agreement is required prior to approval.)

N/A

5. Dates of proposed project (cannot be retroactive) From: 01/08/2007 To: 01/08/2008

6. Source of funding for the project (grants, title, agency, account/proposal # or “Unfunded”):

Unfunded

7. Scientific purpose of the investigation (dissertation or thesis is not the scientific purpose):

The purpose of this study is to investigate prospective elementary teachers’ reasoning about and understanding of number concepts and operations and geometry and measurement using instructional sequences related to these mathematical ideas.

8. Describe the research methodology in non-technical language (the UCF IRB needs to know what will be done with or to the research participants – include audio/video taping – explain the who, what, when, where, why and how of the procedures you wish to implement)

Participants will be undergraduate students enrolled in MAE 3801: Instructional Mathematics for Elementary School. Consent forms will be collected from each prospective participant prior to data collection.

At the beginning of the Spring 2007 semester, an elementary mathematics content knowledge test (see attached) will be administered to students enrolled in one section of MAE 3801. The test will be administered at the beginning and end of the semester in a pre-post format. This test will not be used as a grade for the class. To ensure anonymity on the test, students will be randomly assigned an ID number. The list of numbers given to students will be kept in a locked cabinet and destroyed upon completion of the post-test such that no identifying record will exist after the study is over to link a student with a particular ID number. A sample of students will also be interviewed several times throughout the semester to document their growth in elementary mathematics content knowledge (see attached sample interview questions). Interviews will be semi-structured since pre-selected questions will be asked of all students, however, individual interviews will vary as a result of interviewee responses. Interviews will be audio-taped and videotaped. The instructor and students in MAE 3801 will be audio-taped and videotaped during each class session to capture instructor/student interactions as well as student/student interactions. A research team including students enrolled in doctoral programs in the College of Education and other faculty in the College of Education will observe each class and take field notes to further document these interactions. Artifacts including students’ coursework will be collected to further inform the research study. Student course work will be collected, photocopied, and originals returned to the student with the exception of the final exams. The research team will meet between class sessions. These meetings will be audio-taped.
9. Describe the potential benefits and anticipated risks and the steps that will be taken to minimize risks and protect participants (risks include physical, psychological, social or economic harm - if there are no direct benefits and/or no risks, state that).

There are no anticipated benefits or risks to participation.

10. Describe how participants will be recruited, how many you hope to recruit, the age of participants, and proposed compensation (if any). When recruiting college students, you should state here that "Participants will be 18 years of age or older" if you want to avoid the need for a parental consent form.

Initially, students enrolled in a 9:30 a.m. section of MAE 2801 will be asked to participate in the research study. If all students agree, then research will be conducted during this section. However, if not all students agree, then students in an 11:30 a.m. section of MAE 2801 will be asked to participate in the study. Those students that choose to participate in the study will remain in the section. Those students that choose to not participate in the study will be rescheduled in a concurrent offering of the same course. It is anticipated that all participants will be 18 years of age or older. If they are not, parental consent and child assent will be obtained prior to conducting research.

11. Describe the informed consent process (include a copy of the informed consent document - if a waiver of documentation of consent is requested to make the study completely anonymous, include a consent form or informational letter with no signature lines or reference to signing).

Informed consent will be obtained from the students prior to data collection via a signed letter. If any students are under the age of 18, their parent/guardian will be contacted to obtain consent and the student will give assent via signed letter. See attached for consent and assent letters.

12. Describe any protected health information (PHI) you plan to obtain from a HIPAA-covered medical facility or UCF designated HIPAA component (include the completed UCF HIPAA Authorization Form or the UCF HIPAA Waiver of Authorization Form, giving the details of the planned use or disclosure of the PHI. See the UCF IRB Web page for HIPAA details and forms).

NA

I approve this protocol for submission to the UCF IRB. [Signature]

Department Chair/Director Date

Cooperating Department (if more than one Dept. involved) [ ]

Department Chair/Director Date

Note: If required signatures are missing, the form will be returned to the PI unprocessed.
APPENDIX D: STUDENT INFORMED CONSENT LETTER
January 8, 2007 (Student Consent)

Dear Student:

I am conducting a study, the purpose of which is to investigate the ways in which prospective elementary school teachers understand number and operation and geometry and measurement concepts. I am asking you to participate in this study because you have been identified as a student in one of the elementary mathematics content courses at UCF. Researchers will observe and videorecord class sessions of the Instructional Mathematics for Elementary School (IMAE 3201). Selected groups may also be audiotaped during class discussions.

Selected students will be asked to participate in several interviews lasting no longer than 15 minutes each. You will not have to answer any questions if you do not wish to answer. The interviews will be conducted at your convenience on campus after we have received a copy of the signed consent form from you. With your permission, we would like to audio and video tape these interviews.

Only the research team will have access to the audio and video tapes, which may be professionally transcribed, removing any identifiers during transcription. The tapes will then be kept in a locked file cabinet. After we have received a copy of this signed consent form, you will be asked to complete a questionnaire about your mathematics content knowledge at the beginning and the end of the course. Your name will not appear on the questionnaire, but a unique code will be used for identification purposes. Only the researchers will have access to the identification codes which will be destroyed after the end of course questionnaires. Copies of your course assignments may be used in presentation and publications related to this study. Your name will be kept confidential and will not be revealed in the final manuscript(s) or any related presentations.

There are no anticipated risks, compensation or other direct benefits to you as a participant in this study. You are free to withdraw your consent to participate and may discontinue your participation in the study at any time without consequence.

If you have any questions about this research project, please contact Dr. Jill T. Dillen at (407) 823-4140 or jdillen@mail.ucf.edu. Questions or concerns about research participants’ rights may be directed to the IRB Coordinator, Institutional Review Board (IRB), University of Central Florida (UCF), 12321 Research Parkway, Suite 301, Orlando, Florida 32826-5246. Telephone number is (407) 823-9301 and the fax number is (407) 823-9299. The hours of operation are 8:30 am until 5:00 pm, Monday through Friday, except on University of Central Florida official holidays.

Please sign and return one copy of this letter. A second copy is provided for your records. By signing this letter, you give us permission to videotape you and report your responses anonymously in the final manuscript(s). You also give us permission to use videotape segments as a part of related publications and presentations.

Sincerely,

Jill Dillen, Ph.D.

[Signature]

I have read the procedure described above for this research study.

I agree to participate in the research.

I do not agree to participate in this research.

I confirm that I am 18 years or older.

Participant ___________________________ Date ____________

[Stamp: University of Central Florida Institutional Review Board]
LIST OF REFERENCES


Proceedings of the Conference of the International Group for the Psychology of
Mathematics Education, Utrecht, Netherlands.