Comparison Of Theoretical Models Of Power Spectral Density To The Experimental Value For Spectrum Of Irradiance Fluctuations

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COMPARISON OF THEORETICAL MODELS OF POWER SPECTRAL DENSITY TO THE EXPERIMENTAL VALUE FOR SPECTRUM OF IRRADIANCE FLUCTUATIONS

by

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B. S. University of Central Florida, 1986

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in the School of Electrical Engineering and Computer Science in the College of Engineering and Computer Science at the University of Central Florida Orlando, Florida

Fall Term
2008
ABSTRACT

A propagation experiment was designed, assembled, and conducted on an extended range to verify theoretical temporal models for weak to strong fluctuation theory. Laser light intensity was propagated over terrain at the Kennedy Space Center (Florida), and detected using optical receivers at a distance of 13.3 Km from the optical source. The intensity data from the experiment was used to generate an experimental Power Spectral Density (PSD) function.

The theoretical Mutual Coherence Function (MCF) and Wave Structure Function (WSF) as set forth by Andrews/Phillips [4], were evaluated to determine the effective relationship between the statistical moments of the random optical field and the laser light intensity. Two scales of interest were identified (refractive large-scale and diffractive small-scale) and plotted revealing the characteristic shape of each component.

In addition, statistical principles applied to the correlation/covariance function relationship and a graphical convolution process were used to generate a theoretical PSD function. Further, utilizing Taylor’s “frozen turbulence” hypothesis an analysis of the theoretical temporal covariance function was performed. Functional forms for refractive and diffractive log-irradiance components were developed and used to generate a second theoretical PSD function. Finally, the experimental and theoretical Power Spectral Density functions are plotted on the same graph and a comparison is performed.
The work performed during this study represents many long hours of experimentation, analysis, extensive mathematical rigor, and writing. Difficult and complex endeavors in life cannot be completed with anything less than a best effort. If something is to be attempted it should be approached with conviction, passion, and with the goal always in mind. Even in the rough times one must keep the faith, persevere, and draw on the traits instilled by wise and experienced elders that came before.

Mom and Dad - You can be proud of the fact that you have passed on your work ethic, your sense of dedication, and the goal oriented mind set it takes to bring a difficult task to completion. Thank you.

Alexander and Amanda - My wish for you is that the circle will be completed and I am able to pass on to you the perseverance, dedication, and commitment required in order to complete your undertakings, whatever they might be.

Leslie - Your encouragement, support, and love during the writing of this thesis means a great deal to me and I am very appreciative.

I am so blessed to have each of you in my life, I love you all, and I dedicate this work to you.

My feeling of frustration during this long, arduous journey has been surpassed by the sense of accomplishment at completion.

Craig/Dad
ACKNOWLEDGMENTS

I wish to acknowledge my professor and mentor Dr. Ron Phillips. He has been instrumental in my ability to “stay the course” through much adversity during this program of study. He is a wonderful teacher and patient advisor.

Dr. Phillips, thank you so much for all you have done for me. The path we covered was long indeed. I value the conversations we have had regarding this thesis and will always remember the times we have shared when the conversation strayed into other areas of interest or concern. I consider myself fortunate to have had the opportunity to study and learn under your tutelage. You have enriched my life.

Your Friend,

Craig
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CHAPTER 1

1.1 Overview

As the need for better, faster, and more reliable communication arises, the necessity to understand the physical effects of the atmosphere on a communications link becomes increasingly more important. To accommodate the rate demands of the future, or to further the ability to have portable communication links set up in remote areas or areas affected by disaster, we look to design optical systems to meet these demands. To achieve these goals one must understand the complexities of an optical wave traveling through the atmosphere.

Three primary processes affect an optical wave propagating through the atmosphere: absorption, scattering, and refractive index fluctuations (optical turbulence). Absorption and scattering of the beam are due to particles and aerosols found in the air and generally attenuate the power that ultimately arrives at a receiver. Optical turbulence causes irradiance fluctuations (i.e. intensity or a redistribution of the beam energy within a cross section of the beam), beam broadening (beyond the effects of diffraction), random variations of the position of the beam centroid called beam wander, and loss of spatial coherence of the optical wave.

Random fluctuations in the refractive index of the atmosphere are directly related to small temperature variations in the atmosphere caused by heating of the earth's surface. There are usually temperature gradients between the earth and the atmosphere residing over that earth with respect to time of day (or night) which causes light rays parallel to the surface to be
bent either up (earth hotter than atmosphere), or down, (earth cooler than atmosphere). These temperature variations lead to turbulent motion of the air due to winds and convection creating eddies or cells of varying scale sizes. Near the earth surface an eddy may range in size on the order of a millimeter (micro-scale) to a meter (macro-scale). The largest eddy size smaller than those at which turbulent energy is imparted defines an outer scale of turbulence, which near the ground is roughly comparable with the height above the ground. An effective inner scale of turbulence is associated with the smallest eddy size before energy is dissipated into heat. In the visible and near-IR regions of the frequency spectrum, the index of refraction fluctuations are caused almost entirely by temperature fluctuations. The outdoor propagation experiment upon which this paper is based uses an argon-ion laser operating in the visible light range, specifically, a wavelength of 514 nanometers (green light).

Optical turbulence in the atmosphere may be characterized by the Kolmogorov Power Spectral Density and three parameters, inner scale (lo), outer scale (Lo), and the structure parameter of refractive turbulence, $C_n^2$. The most critical element in the characterization of atmospheric turbulence along a propagation path is the refractive-index structure parameter. The propagation experiment was designed, assembled, and conducted on an extended range to verify temporal models. In the analysis that follows we will use measured data in an attempt to quantify and characterize the specific channel in which the experiment was conducted.
2.1 Experiment Description

Scintillation data was collected at two (2) different path lengths and on two (2) different days. The source of the optical beam was an argon-ion laser operating with a wavelength of 514 nm and located at the transmitting dome on the Innovative Science and Technology Experimentation Facility (ISTEF) laser range in both cases.

Figure 1 - ISTEF Facility
Case 1 of the experiment was conducted on 4/30/2003 with three (3) Photo Multiplier Tube (PMT) receivers mounted at the end of the ISTE laser range at a height of approximately 1 meter above the ground and a distance of 1 Km from the transmitter. Case 2 was conducted on 5/7/2003 with three (3) PMT receivers mounted on top of the Vehicle Assembly Building (VAB) located at Kennedy Space Center, Florida, at a height of approximately 160 meters above the ground and a distance of 13.3 Km from the transmitter source. The optical beam was transmitted for a period of approximately 5 minutes with background data recorded in each data file just before and just after the laser data was recorded. Three analog channels of a D/A computer data acquisition card, interfaced to a laptop computer, was used to record the incident laser power collected by the three (3) PMT receivers. The front end of each PMT receiver assembly was fitted with a 5.75 inch long / 0.3125 inch diameter tube and an optical filter with a 1 nm bandwidth and a transmission rating of approximately 50% (as measured by the manufacturer). The tube served as a sunshade and reduced the angle of view and the filter reduced the undesired incident light (background) entering the PMT’s. The sunshade and the optical filter drastically reduced the background light (i.e. sky radiance) level to prevent the PMT’s from operating in their region of saturation, and reduced the overall noise in the system. To assist in the alignment of the receiver to the transmitted beam, a riflescope was mounted on each PMT assembly. Control of the azimuth and elevation look angles was accomplished by installing each PMT atop a transit fixture. A custom made electronics box, which included a trans-impedance amplifier (TIA) and a Sallen-Key analog filter, was designed and interfaced by Stalder [15] to each PMT assembly enabling gain control and restricting each PMT to operate in the linear range.
\[ \theta_{VAB} \approx 117^\circ - 90^\circ \approx 27^\circ \]

\[ V_t \approx 6.2 \cos 27^\circ \approx 5.52 \text{ m/s} \]

<table>
<thead>
<tr>
<th>Range Tower ID</th>
<th>Tower Height (feet)</th>
<th>Wind Direction (Deg)</th>
<th>Wind Speed (Knots / m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0803</td>
<td>54</td>
<td>137</td>
<td>5.0 / 2.6</td>
</tr>
<tr>
<td>0313</td>
<td>492</td>
<td>117</td>
<td>12.0 / 6.2</td>
</tr>
</tbody>
</table>

\[ \theta_{ISTEF} \approx 137^\circ - 90^\circ \approx 47^\circ \]

\[ V_t \approx 2.6 \cos 47^\circ \approx 1.77 \text{ m/s} \]

Figure 2 – Experimental Layout
2.2 **Innovative Science and Technology Experimentation Facility**

The Innovative Science and Technology Experimentation Facility (ISTEF) is located on Merritt Island, Florida, on NASA’s Kennedy Space Center (KSC) property. ISTEF is an electro-optical observatory engaged in both developing and demonstrating innovative scientific approaches. It is owned by the Space and Naval Center, San Diego, California, and managed by the Computer Science Corporation, the site contractor. The Vertical Assembly Building (VAB) is located on the same side of the Banana River as the ISTEF site, giving ISTEF direct visual access to it while the laser transmission path passes over lush tropical vegetation and an industrial area of buildings. The KSC airspace is controlled and cleared for laser test activities, with established “keep-out” zones for both aircraft and watercraft. The physical location, its proximity to the two desired receiver locations, and the willingness of the facility personnel to support this experiment, made ISTEF an ideal choice to carry out this research.

2.3 **Vehicle Assembly Building**

The Vehicle Assembly Building (VAB) is located on NASA’s Kennedy Space Center, Merritt Island, Florida. The VAB is one of the largest buildings in the world. Its main purpose is to support Space Shuttle operations. The VAB covers 8 acres of ground and is surrounded by tropical vegetation, buildings, and water. It is 160 meters (525 ft) tall, 218 meters (716 ft) long, and 158 meters (518 ft) wide. The height, size, and location of the
VAB, coupled with the ability to optically view ISTEF, made it an ideal platform to mount
the PMT receivers for this laser scintillation experiment.

2.4 Experimental Power Spectral Density Function

The experiment was conducted over a period of several days and numerous “runs” were
performed with varying atmospheric conditions influencing the data during the individual
time periods. For this analysis a specific run was selected based on a number of
considerations including known wind speed during the data collection time, the absence of
saturation of the PMT’s by the light intensity, and the overall data quality. The “run”
selected was known as VAB Run 13 and the data was collected for a period of five (5)
minutes providing a set of data with several million samples. For baseline purposes, there
was thirty (30) seconds of calibration data collected before and after the run with the laser
turned off to provide information on the amount of background sky radiance collected. The
data values collected during the run had the mean value of the sky radiance subtracted from
the laser intensity values so as to provide a measure of only the laser light intensity, and
then the data set was trimmed to provide one (1) million samples for ease of computation.
The mean and variance of the signal was calculated and the mean value of the signal was
subtracted from each data point to create a zero mean data set. The scintillation index was
then calculated. The data set was parsed into sub-vectors and a Fast Fourier Transform was
performed on each sub-vector and the corresponding frequencies and magnitudes were
calculated. Reducing the data set to provide only half of the two-sided power spectrum was completed and the following figure was generated.

Figure 3 illustrates the experimental Power Spectral Density function PSD, $S(f)$, as determined from light intensity data collected at the optical receivers during the experiment.
2.5 **Values Calculated from Experimental Data**

The values of the parameters given below were determined following the collection of data during the experiment previously described. The value of the path averaged refractive-index structure parameter was determined by Halbing [14], and was calculated using experimental data from one of the periods in which the PMT receiver experienced no saturation from the laser light intensity. All values given were used in the analysis of the temporal spectrum in this paper.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Path Length</td>
<td>$L = 13.3$ Km</td>
</tr>
<tr>
<td>Diameter of Receiver Aperture</td>
<td>$D = 0.001$ m</td>
</tr>
<tr>
<td>Refractive-index Structure Parameter</td>
<td>$C_n^2 = 8.5 \times 10^{-15}$ m$^{-2/3}$</td>
</tr>
<tr>
<td>Laser Wavelength</td>
<td>$\lambda = 514 \times 10^{-9}$ m</td>
</tr>
<tr>
<td>Average Wind Speed</td>
<td>$V_T \cong 3.6$ m/s</td>
</tr>
</tbody>
</table>

Figure 4 - Values Calculated/Used During the Experiment and Analysis
CHAPTER 3

3.1 Analytical Considerations

In this chapter a number of analytical considerations will be discussed. We want to determine the refractive and diffractive components of the turbulent field and also to determine to what extent the small and large eddies contribute to the turbulent effects. Both the small and large eddies are contributing factors to the resulting scintillation condition. While the small eddies are the significant contributors to the scattering effects (i.e. defocusing or diffraction) of atmospheric turbulence causing scintillation over short distances, the larger eddies also cause refraction and diffraction effects, however, over much larger distances. The presence of a large transverse coherence (small wave front wrinkle over large areas) condition results in scattering of the beam as it propagates through an atmospheric eddy, conversely for a small transverse coherence, it passes through an eddy with little affect (little to no scattering). As the beam passes through a path length encountering eddies of varying sizes, some small and some large, there will be path segments (distance) of scattering of the beam and segments of the beam traversing path segments without scattering taking place. This variation in scattering and non-scattering sets up a structure in which the large eddies form a modulation component due to the “intermittency” of the scattering effect. The calculation of the transverse wind speed is also a necessity. A trigonometric solution as applied to range wind data was used to provide the value of the transverse wind speed along the path of signal propagation.
3.2 Determination of the Transverse Wind Speed

To determine the temporal spectrum of irradiance fluctuations the transition frequency, $\omega_t$, must be known. The quantity $\omega_t$ physically represents the transition frequency at which the spectrum begins to decay under weak fluctuations and may be written as

$$\omega_t = \frac{V_T}{\sqrt{\frac{L}{k}}}$$  \hspace{1cm} (1)

where $V_T$ is the transverse wind speed; $L$ is the path length; $k$ is the wave number equal to $2\pi/\lambda$; $\lambda$ is the optical wavelength. The path length, $L$, is 13.3 Km and the optical wavelength, $\lambda$, is 514 nm. To complete the evaluation of the transition frequency the transverse wind speed, $V_T$, along the path of propagation, must be calculated. To determine the transverse wind speed, $V_T$, used herein, the wind speed data was obtained for measured speed at two separate instrumented towers, one at the transmitter end and one at the receiver end. The value of the transverse wind speed at the tower locations was calculated using a trigonometric solution. Figure 2 is a schematic diagram of the experimental layout showing relative geographical locations, the raw wind data values, the physical parameters of each tower, and the trigonometric calculation of the approximate transverse wind speed at the transmitter location (ISTEF) and at the optical receiver location (VAB). The approximate values of the transverse wind speed at the transmitter and receiver locations
were then used to calculate an “average” wind speed along the path of signal propagation. The approximated value of the average wind speed, $V_T$, is written as

$$V_T \cong \frac{5.52 \text{ m/s} + 1.77 \text{ m/s}}{2} \cong 3.6 \text{ m/s}$$

(2)

3.3 The Rytov Approximation

Neglecting polarization effects, the propagation of a monochromatic optical wave through a random medium is governed by the reduced wave equation (or Helmholtz equation)

$$\nabla U + k^2 n^2(R) U = 0$$

(3)

where $U(R)$ denotes the optical field at position $R = (x, y, z)$ and $n(R)$ is the random index of refraction. One of the popular, and most successful, solutions to Equation 3 is the well-documented [3,4] Rytov approximation. The Rytov approximation considers a free-space Guassian-beam wave at the receiver and first and second-order complex phase perturbations of the field, due to inhomogeneities, along the propagation path. The statistical moments of greatest interest including the mean field, the Mutual Coherence Function (MCF), and the general fourth-order moment (variance of the variance), may be determined by using the Rytov approximation. Under the Rytov approximation it is the first-order log amplitude and phase of the irradiance that is determined with the log-amplitude of an optical wave related to the random field by $\chi = 1/2\ln(I/A_0^2)$. Because of
this relationship it became customary to concentrate on the log-amplitude variance rather than directly to the scintillation index. The relationship between the log-amplitude variance and the scintillation index is given by

\[
\sigma_i^2 = \exp\left(4\sigma_x^2\right) - 1 = \exp\left(4\sigma_{\text{in}}^2\right), \quad \sigma_x^2 \ll 1
\]  

(4)

The conventional Rytov approximation is generally limited to weak fluctuation conditions or along relatively short propagation distances, however, a relatively simple model for irradiance fluctuations applicable in moderate to strong fluctuation regimes will be applied in this paper employing the modified Rytov approximation and assuming the received irradiance of the optical wave will be modeled as a modulation process in which small-scale (diffractive) fluctuations are modulated by statistically independent large-scale (refractive) fluctuations, and, the Rytov method for optical scintillation is valid, even over a longer propagation path and with a larger scintillation index, using a spatial frequency filter to properly account for the loss of spatial coherence of the optical wave in strong fluctuation conditions.

3.4 The Optical Scintillation Model (Weak Turbulence)

Throughout the study of optical scintillation it has been widely assumed that the model to best describe the irradiance (intensity) of an optical wave propagating through a random medium involves a modulation process in which the irradiance can be written as a simple
product. This usually involves the mean of one distribution being “smeared” by another
distribution. Assuming in the general model that the normalized (mean = 1) intensity can be
written as the product of two statistically independent random variables

\[ I = xy \]  

(5)

where \( < x > = < y > = 1 \) and therefore \( < I > = 1 \), where the angle brackets (i.e. < >) denote
an ensemble average (long-time average). It should be noted here that the mean value of
each component in Equation 5 might be different. This being true

\[ \langle I \rangle = \langle x \rangle \langle y \rangle \]  

(6)

and the second moment can be written

\[ \langle I^2 \rangle = \langle x^2 \rangle \langle y^2 \rangle \]  

(7)

From the study of statistics for random variables the variance can be expressed as

\[ \sigma^2 = \langle x^2 \rangle - 1 \quad \text{or} \quad \langle x^2 \rangle = 1 + \sigma^2 \]  

(8)
Now Equation 6 may be expressed as

\[ \langle I^2 \rangle = (1 + \sigma_x^2)(1 + \sigma_y^2) \]  

(9)

where \( \sigma_x^2 \) and \( \sigma_y^2 \) are the normalized variances of \( x \) and \( y \), respectively.

3.5 Scintillation Index Model

As an optical wave propagates through the atmosphere it will experience irradiance (intensity) fluctuations, or scintillation, caused by temperature variations in the atmosphere, resulting in index of refraction fluctuations (i.e. optical turbulence). The scintillation index is defined to be the normalized variance of irradiance (intensity) fluctuations as described by

\[ \sigma_i^2 = \frac{\langle I^2 \rangle - \langle I \rangle^2}{\langle I \rangle^2} = \frac{\langle I^2 \rangle}{\langle I \rangle^2} - 1 \]  

(10)

where the quantity \( I \), denotes the irradiance of the optical wave. By substituting Equation 9 into Equation 10, and noting that \( \langle I \rangle^2 = 1 \) (normalized) the implied scintillation index is

\[ \sigma_i^2 = \sigma_x^2 + \sigma_y^2 + \sigma_x^2\sigma_y^2 \]  

(11)
In the following analysis it is assumed that the x-component arises from the large-scale turbulent eddy effects and the y-component from statistically independent small-scale eddy effects. Note that Equation 11 is not a simple sum of refractive and diffractive effects except under weak fluctuations in which $\sigma_x^2 \ll 1$ and $\sigma_y^2 \ll 1$ so that $\sigma_j^2 \approx \sigma_x^2 + \sigma_y^2$. By using the relationship given in Equation 8 the large-scale and small-scale scintillation indexes are respectively given as Andrews/Phillips [4],

$$
\begin{align*}
\sigma_x^2 &= \exp(\sigma_{\ln x}^2) - 1 \\
\sigma_y^2 &= \exp(\sigma_{\ln y}^2) - 1
\end{align*}
$$

(12)

where $\sigma_{\ln x}^2$ and $\sigma_{\ln y}^2$ are called large-scale and small-scale log-irradiance variances, respectively. Consequently, the total scintillation index then takes on the form

$$
\sigma_j^2 = \exp(\sigma_{\ln I}^2) - 1 = \exp(\sigma_{\ln x}^2 + \sigma_{\ln y}^2) - 1
$$

(13)

The quantity $\sigma_{\ln I}^2 = \sigma_{\ln x}^2 + \sigma_{\ln y}^2$ is equivalent to the conventional variance of log-irradiance only under weak irradiance fluctuations ($\sigma_{\ln I}^2 \ll 1$) and in this case Equations 12 and 13 yield the results $\sigma_x^2 \approx \sigma_{\ln x}^2$, $\sigma_y^2 \approx \sigma_{\ln y}^2$, and $\sigma_j^2 \approx \sigma_{\ln I}^2 = \sigma_{\ln x}^2 + \sigma_{\ln y}^2$. When using the optical wave model of an infinite plane wave or spherical wave (point source) in the presence of weak irradiance fluctuations an approximation to the scintillation index, the Rytov approximation, can be used. The Rytov theory ignores diffraction effects and assumes smooth multiplicative perturbations in solving Maxwell’s equation for the vector amplitude.
of a propagating electromagnetic wave. The value of the scintillation index using the Rytov approximation may be calculated from the following equations:

\[
\sigma^2_i(L) = \sigma^2_i = 1.23C_n^2 k^{7/6} L^{11/6}, \quad \text{(unbounded plane wave)} \tag{14}
\]

\[
\sigma^2_i(L) = \beta_0^2 = 0.5C_n^2 k^{7/6} L^{11/6}, \quad \text{(spherical wave)} \tag{15}
\]

where \(\sigma^2_i\) and \(\beta_0^2\) are the Rytov variances, \(C_n^2 \text{m}^{-2/3}\) is the index of refraction structure parameter of the atmosphere, \(k = 2\pi/\lambda\) is the optical wave number, \(\lambda(\text{m})\) is wavelength, and \(L(\text{m})\) is the propagation path length between transmitter and receiver. The Rytov variance represents the normalized irradiance variance, or scintillation index, of an unbounded wave in weak fluctuations, but is otherwise considered a measure of optical turbulence strength when extended to strong fluctuation regimes by increasing either \(C_n^2\), or the path length \(L\), or both. In terms of this parameter, weak irradiance fluctuations are characterized by the condition \(\sigma^2_i < 1\), and moderate-to-strong irradiance fluctuations by \(\sigma^2_i \geq 1\).

### 3.6 Gamma-Gamma Distribution Model (Strong Turbulence)

The development of the Probability Density Function (PDF) model of irradiance with parameters that directly relate to atmospheric conditions has already been documented in Andrews, Phillips [4], and will be used as required in this thesis. The following is a brief
description of that development. The PDF is consistent with scintillation theory and has, as a basis, the assumption that intensity is a product of two random variables, i.e. \( I = xy \), where \( x \) and \( y \) represent, respectively, large-scale and small-scale atmospheric effects. Both large-scale and small-scale irradiance fluctuations are governed, respectively, by the following gamma distributions:

\[
p_x(x) = \frac{\alpha (\alpha x)^{\alpha - 1}}{\Gamma(\alpha)} \exp(-\alpha x) \quad x > 0
\]

\[
p_y(y) = \frac{\beta (\beta y)^{\beta - 1}}{\Gamma(\beta)} \exp(-\beta y) \quad y > 0
\]

Using a change of variables where \( x \) is fixed and writing \( y = \frac{I}{x} \), the following conditional PDF is developed

\[
p_y\left(\frac{I}{x}\right) = \frac{\beta \left(\beta \frac{I}{x}\right)^{\beta - 1}}{x \Gamma(\beta)} \exp\left(-\beta \frac{I}{x}\right) \quad I > 0
\]

in which \( x \) is the (conditional) mean value of \( I \).
To obtain the unconditional irradiance distribution the average of the conditional PDF is formed over the gamma distribution for large-scale equation leading to Andrews/Phillips [4],

\[
p(1) = \int_0^\infty p_y \left( \frac{1}{x} \right) p_x(x)dx
\]

\[
= \frac{2(\alpha\beta)^{\frac{\alpha+\beta}{2}}}{\Gamma(\alpha)\Gamma(\beta)} I^{\frac{\alpha+\beta-1}{2}} K_{\frac{\alpha}{\beta}} \left( 2\sqrt{\alpha\beta I} \right) \quad I > 0
\]

called the gamma-gamma distribution. The gamma-gamma PDF is normalized in the sense that \(<I> = 1\). For the second moment we have \(<I^2> = \left( 1 + \frac{1}{\alpha} \right) \left( 1 + \frac{1}{\beta} \right)\), where the parameters of the gamma-gamma distribution with large-scale and small-scale scintillation are recognized to be

\[
\alpha = \frac{1}{\sigma_x^2}, \quad \beta = \frac{1}{\sigma_y^2}
\]

(20)

The total scintillation index is related to these parameters by

\[
\sigma_i^2 = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\alpha\beta}
\]

(21)
3.7 Covariance Function of Irradiance

To begin, the mean for the spatial variation of the complex random field is defined by

\[ < e(R) > = m(R) \]  \hspace{1cm} (22)

where once again the brackets \(< >\) denote an ensemble average. The associated covariance function is defined by the ensemble average

\[ B_e(R_1, R_2) = < e(R_1) - m(R_1) \overline{(e(R_2) - m(R_2))} > \]  \hspace{1cm} (23)

The random field is statistically homogeneous if its moments are invariant under a spatial translation i.e., the mean value \( m = < e(R) > \) is independent of the spatial position of \( R \) and the covariance functions depends only on \( R = R_2 - R_1 \). In this case \( B_e(R_1, R_2) = B_e(R_2 - R_1) \) or, equivalently,

\[ B_e(R) = < e(R_1) \overline{e(R_1 + R)} > - |m|^2 \]  \hspace{1cm} (24)

Thus, the notion of homogeneity is the spatial counterpart of stationarity in time. If the random field \( e(R) \) also has invariance properties under rotation, it is statistically isotropic. In this case, the covariance function depends only on the scalar distance \( \rho = |R_2 - R_1| \) (isotropic), i.e., \( B_e(R_1, R_2) = B_e(\rho) \) and the ensemble properties can be readily estimated by
spatial averaging. S.F. Clifford [10] determined the isotropic form of the covariance function for the Kolomogorov spectrum to be

$$B_{x,s}(\rho) = 2\pi \int_{0}^{\infty} d\kappa \kappa J_0(\kappa \rho) F_{x,s}(\kappa, 0)$$  \hspace{1cm} (25)$$

where $F_{x,s}(\kappa, 0)$ is $F_x(\kappa, 0)$ and $F_s(\kappa, 0)$, the amplitude and phase, respectively, of the Kolomogorov spectrum by

$$F_x(\kappa, 0) = \pi k^2 L \left[ 1 - \left( \frac{k}{\kappa^2 L} \right) \sin \left( \frac{\kappa^2 L}{k} \right) \right] \phi_n(\kappa)$$ \hspace{1cm} (26)$$

$$F_s(\kappa, 0) = \pi k^2 L \left[ 1 + \left( \frac{k}{\kappa^2 L} \right) \sin \left( \frac{\kappa^2 L}{k} \right) \right] \phi_n(\kappa)$$ \hspace{1cm} (27)$$

and $J_0(\kappa \rho)$ is a zero-order Bessel function of the first kind, defined by the relation

$$J_0(\kappa \rho) = \frac{1}{2\pi} \int_{0}^{2\pi} \exp[i\kappa \rho \cos \theta] \ d\theta$$ \hspace{1cm} (28)$$
Based on the assumed modulation process given in Equation 5 and shown in Andrews/Phillips [4] the irradiance covariance function can be expressed as a sum.

\[ B_x(\rho) = B_{x_x}(\rho) + B_{y_y}(\rho) + B_{x}(\rho)B_{y}(\rho) \]  
(29)

where \( \rho \) denotes separation distance between two points on a wave-front, \( B_x(\rho) \) is the covariance of the large-scale process, and \( B_y(\rho) \) is the covariance of the small-scale process. Analogous to the scintillation index, the covariance of irradiance Equation 13 can also be written Andrews/Phillips [4]

\[ B_x(\rho) = \exp[B_{ln_x}(\rho) + B_{ln_y}(\rho)] - 1 \]  
(30)

where \( B_{ln_x}(\rho) \) and \( B_{ln_y}(\rho) \) are the large-scale and small-scale log-irradiance covariances.

3.8 Physical Model for Amplitude Fluctuations

A closer investigation of the physical model of the amplitude fluctuations and how these amplitudes are affected by the turbulent cell or eddy will lead to a better understanding of the scintillation process. This investigation furthers an explanation of how the eddy size (i.e. large-scale, \( L_o \), and small-scale \( l_o \)) matters, and gives rise to the scintillation phenomena. The mixing of warm and cool air creates refractive turbulent cells or eddies.
The ground heats the air layer next to it and this heating causes the air to rise in a bubble form. The wind then shears the eddy as it rises causing the warm and cool air to be turbulently mixed. These swirling eddies act on a propagating optical wave by focusing and defocusing the beam’s energy as it passes through them. This scattering of the wave is dependent on the cell size. Small-scale contributions to scintillation are associated with turbulent cells smaller then either the first Fresnel zone, $\sqrt{L/k}$, or the coherence radius $\rho_o$, whichever is smallest. In contrast, turbulent cells larger then that of either the Fresnel zone, $L/k\rho_o$, the so-called “scattering disk”, generates large-scale fluctuations of the irradiance. Under the assumption of statistical isotropy for the atmosphere and the paraxial approximation, a single eddy can be modeled as if it were a “thin” Gaussian-shaped dielectric lens. The paraxial approximation is based on the notion that the propagation distance for an optical wave along the $z$-axis is much greater than the transverse spreading of the wave. The simplest physical model leading to optical scintillation is associated with a plane wave incident on a “sheet” of turbulent refractive cells for which the sheet thickness satisfies $\Delta z \ll L - z$. Figure 5 shows this concept pictorially.
By concentrating on a single eddy with radius, $R$, that is smaller than the coherence radius, and neglecting the effect of the random medium between the sheet and the receiver plane, beam wave analysis is used to express the on-axis amplitude in the receiver plane as

$$\frac{A}{A_0} = \frac{1}{\sqrt{\left(1 - \frac{z'}{f}\right)^2 + \left(\frac{2z'}{kR^2}\right)^2}} \quad z' = L - z$$  \hspace{1cm} (31)
The field beyond the aperture of a turbulent eddy is given by the expression, i.e. $z = 0$,

$$
\frac{A}{A_0} = \frac{1}{\sqrt{\left(1 - \frac{L}{f}\right)^2 + \left(\frac{2L}{kR^2}\right)^2}}
$$

(32)

where $A_0$ is the initial fluctuation amplitude, $f$ is the focal length of the eddy lens, $k$ is the optical wave number equal to $2\pi/\lambda$, $R$ is the radius of the eddy, and $L$ is the propagation path length beyond the eddy’s aperture. The effective focal length of the eddy lens is given by

$$
f = \frac{R/2}{\Delta n} = \frac{R}{2\Delta n} \quad \text{(meters)}
$$

(33)

where $R$ is the radius of the eddy and $\Delta n$ is the index of refraction of the eddy. The index of refraction $\Delta n$ is given as the root-mean-square (rms) index of refraction and is derived from the structure function relation, which states that under statistically homogeneous and isotropic turbulence, the index of refraction structure function exhibits the asymptotic behavior Andrews, Phillips [4], i.e.

$$
D_n(R) = \begin{cases} 
C_n^2 R^{2/3} & l_o \ll R \ll L_o \\
C_n^2 l_o^{-4/3} R^2 & R \ll l_o
\end{cases} \quad \text{[unitless]}
$$

(34)
where \( R \) is the scalar distance between two points in space, \( L_o \) is the large-scale turbulent eddy with size in the order of 1m, and \( l_o \) is the small-scale turbulent eddy with size in the order of 1mm. Therefore, the rms value of the index of refraction is

\[
\langle \Delta n \rangle_{r_m} = \sqrt{\langle \Delta n^2 \rangle} = C_n R^{1/3}
\] (35)

By substitution, and using the radius of the eddy expressed as a diameter (i.e. \( R = D/2 \)), the expression of the field beyond the aperture of an eddy becomes Andrews, Phillips [4],

\[
\frac{A}{A_0} = \frac{1}{\sqrt{\left[1 - \frac{2LC_n}{\left(\frac{D}{2}\right)^{2/3}}\right]^2 + \left(\frac{2L}{k\left(\frac{D}{2}\right)^2}\right)^2}}
\] (36)

Noting that the Near Field Distance (NFD) extends from the eddy’s aperture to a distance \( L \) given by

\[
L = \frac{kR^2}{2} = \frac{k\left(\frac{D}{2}\right)^2}{2} = \frac{kD^2}{8}
\] (37)

where \( D \) is the diameter of the eddy, and the distance to the focus of the eddy (FD) is given by
\[ L = \frac{R^{2/3}}{2C_n} = \left( \frac{D}{2} \right)^{2/3} \]

Figure 6 illustrates the relationship of the eddy size to the effect on a signal propagating through the respective eddies.

![Figure 6 - Amplitude Change of a Signal Passing Through Eddys of Varying Sizes](image)

Figure 6 shows four different eddies ranging in size from 1mm to 1m. Note as eddies grow larger in size the amplitude of the incident signal is less perturbed (i.e. more focused due to less scattering) than the signal passing through the smaller eddies. When the eddy size reaches approximately 1m in diameter the signal is not perturbed at all i.e., the amplitude of the transferred signal remains unchanged. This figure serves to illustrate that both small and
large eddies scatter (refract and diffract) the signal, however, the large eddies perturb the signal over much greater distances then do the small eddies.

3.9 Statistical Moments of the Optical Field

In an optical system operating through an atmospheric path it is often necessary to deal with stochastic or random optical fields. Such fields are created when a laser beam propagates through a medium such as the turbulent atmosphere that introduces effects that cause random variations to occur in the propagating field. These random variations lead to statistical fluctuations of the parameters of the optical field that can only be analyzed after associating proper statistics with the field itself.

In order to compare intensity data collected during the experiment to the theory developed by Andrews/Phillips [3,4] an effective relationship between the received intensity and the statistical moments of the random optical field is needed. The analyses that follows considers weak to strong turbulent regimes over short to long propagation distances.

The received wave will be characterized by the statistical moments of the random optical field \( U(r, L) \), where \( L \) is the propagating distance from the emitting aperture of the transmitter to the receiver and \( r \) is a vector in the receiver plane transverse to the propagation axis. The first moment \( <U(r, L)> \), where \( < > \) denotes an ensemble average,
describes the coherent portion of the field. The Mutual Coherence Function (MCF) of the wave is defined by the second moment

$$\Gamma_2(r_1, r_2, L) = \langle U(r_1, L)U^*(r_2, L) \rangle$$

where \( r_1 \) and \( r_2 \) are observation points in the receiver plane and \( U^*(r, L) \) denotes the complex conjugate field. For identical observation points, the MCF determines the mean irradiance from which turbulence induced beam spread is deduced. Also obtained from the MCF is the modulus of the complex degree of coherence that describes the loss of spatial coherence of an initially coherent wave. This loss of spatial coherence in a physical sense is caused by the large-scale (refractive effects) and small-scale (diffractive effects) of the turbulence in the atmosphere.

### 3.10 Mutual Coherence Function and Wave Structure Function

Under weak fluctuation theory, using the Rytov approximation, a Mutual Coherence Function (MCF) and a complex degree of coherence, Equation 40 and Equation 41 respectively, was developed for a Guassian-beam wave by Andrews/Phillips [4], and written as

$$\Gamma_2(r_1, r_2, L) = \Gamma_2^0(r_1, r_2, L) \exp\left[\sigma_r^2(r_1, L) + \sigma_r^2(r_2, L) - T\right] \exp\left[-\frac{1}{2} \Delta(r_1, r_2, L)\right]$$

(40)
Physically, the quantity $\sigma_r^2(r, L)$ describes the atmospherically induced change in the mean irradiance profile in the transverse direction, the quantity $T$ (which is independent of $r$) describes the change in the on-axis mean irradiance at the receiver plane caused by turbulence, and $I_0(x)$ is a modified Bessel function. The real part of the complex degree of coherence, $\text{Re}[\Delta(r_1, r_2, L)] = D(r_1, r_2, L)$, is the Wave Structure Function (WSF) for a Gaussian-beam wave. In the limiting case of an unbounded plane wave the MCF, Equation 41, leads to the simpler expression Andrews/Phillips [4]

$$
\Gamma_z(p, r, L) = e^{-\frac{1}{2}D(p, L)}
$$

In analyzing the MCF and the WSF a set of physical criteria, Equation 43 and Equation 44 are employed.

$$
r = \frac{1}{2}(r_1 + r_2) \quad , \quad r = |r| \quad \text{Center of Gravity}
$$

$$
p = r_1 - r_2 \quad , \quad \rho = |p| \quad \text{Difference Vectors}
$$
In addition it will be assumed the received wave is an unbounded Plane Wave, i.e., the output plane (or receiver) beam parameters are, $\Lambda = 0$ (Fresnel ratio of the beam at the receiver - diffractive amplitude effects), $\Theta = 1$ (curvature of the beam at the receiver - refractive amplitude effects), and $\bar{\Theta} = 1 - \Theta$ (complimentary beam parameter relationship).

Also, an assumption is drawn that there are two spectral cutoff frequencies to be determined, the two cut-off frequencies are affected by large-scale (refractive) and small-scale (diffractive) turbulent eddies, and the two scales of interest are governed by the statistical theory of turbulence as set forth by Kolmogorov. For optical wave propagation, refractive-index fluctuations are caused almost exclusively by small fluctuations in temperature; therefore, pressure and humidity effects are neglected. It is generally accepted that the fundamental form of the spatial power spectrum of refractive index fluctuations is the same as that for temperature and, further, that the temperature fluctuations obey the same 2/3 power law associated with velocity fluctuations as shown by Kolmogorov using dimensional analysis. The Kolmogorov spectrum is theoretically valid only over the inertial sub-range and to justify its use in certain calculations over all wave numbers, it is ordinarily assumed that the outer scale is infinite and the inner scale negligibly small. The Kolmogorov spectrum is then described by

$$
\Phi_n(\kappa) = 0.033C_n^2 \kappa^{-11/3}, \quad 0 \leq \kappa \leq \infty \quad (45)
$$
Utilizing the previous assumptions, substituting the Kolmogorov spectrum into Equation 41, and noting that the double integral is no longer a function of $\xi$, the WSF is reduced to Andrews/Phillips [4],

$$\Delta(r_1, r_2, L) = (0.033)8C_n^3\pi^2k^2L\int_0^\infty\kappa^{-8/3}\left\{1 - J_0(\rho\kappa)\right\} d\kappa$$  \hspace{1cm} (46)

The result of this reduced WSF is in agreement with the analysis performed by S.F. Clifford [10], where it is shown that the total WSF is actually the sum of the log-amplitude and the phase structure functions. By using the method of the geometrical optics approximation, Andrews/Phillips [3], it can be shown that the log-amplitude structure function is approximately equal to zero and the phase structure function and the WSF are essentially the same. Considering the total WSF as a sum of the refractive and diffractive effects caused by the atmospheric eddies, and separating the reduced WSF integral into two parts, a refractive part (effects due to large-scale eddies), and a diffractive part (effects due to small-scale eddies), leads to Equation 47, or in terms of intensity ($I$), Equation 48, which can be evaluated in terms of the wave number ($k/L)^{1/2}$, i.e. the inverse of the Fresnel zone.

$$D(\rho, L)_{\text{Total}} = D(\rho, L)_{\text{refrac}} + D(\rho, L)_{\text{diff}}$$  \hspace{1cm} (47)

$$I_{\text{Total}} = I_{\text{refrac}} + I_{\text{diff}}$$  \hspace{1cm} (48)
The limits of integration for both the refractive and the diffractive integral representations will be the coherence radius of the plane wave, $\rho_{pl}$, in terms of the wave number $k$:

$$k = \frac{2\pi}{\lambda} \quad (49)$$

$$\rho_{pl} = \sqrt{\frac{k}{L}} = \sqrt{\frac{2\pi/\lambda}{L}} = \sqrt{\frac{2\pi}{L\lambda}} \quad (50)$$

A small wave number corresponds to a large eddy size that introduces a refractive effect on the beam wave, and conversely, a large wave number corresponds to a small eddy size which leads to diffraction of the beam wave.

The first integral will represent the effects due to the large eddy size (refractive contributor) and is associated with a small wave number. The basic form of the refractive integral is

$$I_{\text{refrac}} = C \int_0^{\infty} x^{-8/3} \left[ 1 - J_0(\rho x) \right] dx \quad (51)$$

where

$$C = (0.033)8C_n^2\pi^2k^2L \quad (52)$$
It should be noted that evaluation of this integral cannot be accomplished as a sum of integrals because a discontinuity exists when evaluating at the lower limit (i.e. division by zero in the first term). Therefore, by using a change of variables as

\[
y = \rho x \\
x = \frac{1}{\rho} y
\]

and writing the upper limit in terms of the wave number, the refractive integral becomes

\[
I_{\text{refrac}} = C \int_0^{\rho \sqrt{k/L}} \left( \frac{y}{\rho} \right)^{-8/3} \left[ 1 - J_0 \left( \frac{y}{\rho} \right) \right] \frac{1}{\rho} dy
\]

Noting that the integration does not depend on \( \rho \), it can be moved outside the integration, and the integral becomes

\[
I_{\text{refrac}} = C \rho^{5/3} \left[ \int_0^{\rho \sqrt{k/L}} y^{-8/3} \left[ 1 - J_0 (y) \right] dy \right]
\]
The solution to the integral in Equation 55 is given in terms of the generalized hypergeometric function [11] as

\[ I_{\text{refrac}} = \int_0^x x^n \{1 - J_0(x)\} \, dx = \frac{x^{n+3}}{4(n + 3)} \, _3F_3\left(1, \frac{n + 3}{2}; 2, 2, \frac{n + 5}{2}; -\frac{x^2}{4}\right) \]

(56)

Re : \( n \geq -3 \)

with the restriction that \( n \) must be greater than \(-3\) in the hyper-geometric function equation to provide a valid solution. The solution will be approximated using only the first two terms of the hyper-geometric function as the rest of the terms become relatively insignificant.

Therefore the solution for the refractive part of the WSF can be written as

\[ D(\rho, L)_{\text{refrac}} \simeq I_{\text{refrac}} \simeq 1.96 C_n^2 k^2 L \rho^2 \left(\sqrt{k/L}\right)^{1/3} \]

(57)
The second integral will represent the effects due to the small eddy size (diffractive contributor) and is associated with a large wave number. The basic form of the diffractive integral is

\[ I_{\text{diff}} = C \int_{x}^{\infty} x^{-8/3} \left[ 1 - J_0(\rho x) \right] dx \] (58)

where the value of \( C \) is given in Equation 52. Inspection of this integral reveals no discontinuity, as in the refractive case, and therefore can be solved as a difference of two integrals, i.e.

\[ I_{\text{diff}} = C \int_{x}^{\infty} x^{-8/3} dx - C \int_{x}^{\infty} x^{-8/3} J_0(\rho x) dx \] (59)

Again, using a change of variables as shown in Equation 53 and noting that the two integrals in Equation 59 are not a function of \( \rho \), we have

\[ I_{\text{diff}} = C p^{5/3} \left[ \int_{\rho \sqrt{k}/L}^{\infty} y^{-8/3} dy - \int_{\rho \sqrt{k}/L}^{\infty} y^{-8/3} J_0(y) dy \right] \] (60)
The first integral in Equation 60 is an integral of a function raised to a power and is easily solved. A tabulated solution exists to the second integral in Equation 60 and is given in terms of a hyper-geometric function as [11]

\[
I = \int_{x}^{\infty} x^n J_{\nu}(x) \, dx = 2^n \frac{\Gamma\left(\frac{\nu + n + 1}{2}\right)}{\Gamma\left(\frac{\nu - n + 1}{2}\right)} - \frac{x^{n+\nu+1}}{2^{\nu}(n+\nu+1)\Gamma(\nu+1)} \, _2F_1\left(\frac{n + \nu + 1}{2}, \frac{n + \nu + 3}{2}; -\frac{x^2}{4}\right) \quad (61)
\]

\[\text{Re} : n \leq 1/2\]

The result of solving both integrals represented in Equation 60 and using only the first two terms of the hyper-geometric function in the second integral yields

\[
D(\rho, L)_{\text{diff}} \approx I_{\text{diff}} \approx 2.61 C_n^2 k^2 L \rho^{5/3} - 1.96 C_n^2 k^2 L \rho^2 \left(\frac{\sqrt{k}}{L}\right)^{1/3} \quad (62)
\]

Using Equation 57 and Equation 62, which define the refractive and diffractive parts of the WSF respectively; the approximation to the total WSF can now be written as

\[
D(\rho, L)_{\text{Total}} \approx 1.96 C_n^2 k^2 L \rho^2 \left(\frac{\sqrt{k}}{L}\right)^{1/3} + 2.61 C_n^2 k^2 L \rho^{5/3} - 1.96 C_n^2 k^2 L \rho^2 \left(\frac{\sqrt{k}}{L}\right)^{1/3} \quad (63)
\]
Having determined the refractive and diffractive contributors to the wave structure function it is now desirable to represent them in terms of the Rytov variance. The Rytov variance physically represents the irradiance fluctuations associated with the unbounded plane wave under investigation. Weak fluctuations are associated with $\sigma_1^2 < 1$ and strong fluctuations with $\sigma_1^2 \gg 1$. Moderate fluctuation conditions are characterized by $\sigma_1^2 \sim 1$, and the condition $\sigma_1^2 \to \infty$ defines the saturation regime where normalized irradiance fluctuations reach a limiting value of unity. The value of the Rytov variance for a plane wave may be calculated as

$$\sigma_1^2 = 1.23C_n^2k^{7/6}L^{11/6}$$

(64)

where $C_n^2$ is the refractive-index structure parameter, $k = 2\pi/\lambda$ is the optical wave number, $\lambda$ is the wavelength, and $L$ is the propagation path length between transmitter and receiver.

The WSF in terms of the Rytov variance and the Fresnel zone is

$$D(\rho, L)_{\text{Total}} \cong 1.6\sigma_1^2\left(\frac{\rho}{\sqrt{L/k}}\right)^2 + 2.1\sigma_1^2\left(\frac{\rho}{\sqrt{L/k}}\right)^{5/3} - 1.6\sigma_1^2\left(\frac{\rho}{\sqrt{L/k}}\right)^2$$

(65)
By substituting Equation 64 into Equation 63 the MCF for a plane wave can be expressed as a product of exponentials and written in terms of the Rytov variance as

\[
\Gamma_2(\rho, r, L) = \exp\left[ -\frac{1}{2} D(\rho, L)_{\text{Total}} \right] \\
= \exp\left[ -\frac{1}{2} \left( 1.6\sigma_1^2 \left( \frac{\rho}{\sqrt{L/k}} \right)^2 \right) \right] \\
\times \exp\left[ -\frac{1}{2} \left( 2.1\sigma_i^2 \left( \frac{\rho}{\sqrt{L/k}} \right)^{5/3} - 1.6\sigma_i^2 \left( \frac{\rho}{\sqrt{L/k}} \right)^2 \right) \right]
\]  

(66)

Plots of the refractive and diffractive Mutual Coherence Functions using the Wave Structure Functions as provided by Equation 57 and Equation 62 are graphically illustrated in Figure 7. The value of the refractive index structure parameter \( C_n^2 \), used in Equation 57 and Equation 62 to plot the theoretical approximation for the Mutual Coherence Functions (refractive and diffractive) in strong fluctuations and illustrated in Figure 7, was calculated from analysis performed by Halbing [14] during this propagation experiment.
Theoretical Approximation For the Mutual Coherence Function in Strong Fluctuations

Figure 7 - Refractive (Large-scale) and Diffractive (Small-scale) MCF

- Wave Length $\lambda = 0.514 \times 10^{-6}$
- Path Length = 13.3 Km
- Wave Number $k = 2\pi\lambda^{-1}$
- Receiver Aperture Diameter $D = 0.001$ m
- Transverse Wind Velocity $V_T = 3.6$ m/s
- Refractive Index Structure Parameter $C_n^2 = 8.5 \times 10^{-15}$
- Rytov Variance $\sigma^2 = 70.5$
3.11 Power Spectral Density Function Using Graphical Convolution

The focus of this section is to use the temporal covariance function as developed by Andrews/Phillips [4], i.e.,

\[ B_j(\tau, D) = \exp[ B_{ln,r}(\tau, D) + B_{ln,y}(\tau, D)] - 1 \]  \hspace{1cm} (67)

and through the analysis, linearly model the mutual coherence function using the characteristic refractive and diffractive components of the mutual coherence functions as shown in Figure 7. The desired result from the analysis is to yield a theoretical power spectral density function \( S(f) \) using a graphical method that accounts for the turbulence introduced by both the refractive and diffractive contributors that affect a propagating optical signal. The Power Spectral Density function (PSD) is a measure of the strength of variations (energy) as a function of frequency. A plot of the PSD function shows at which frequencies variations are strong and at which frequencies variations are weak. The principles of statistics, specifically the relationship between the correlation function \( R(\tau) \) (i.e. a measure which describes the strength and direction of a linear relationship between two random variables) and the covariance function \( B(\tau) \) (i.e. the measure of how much two random variables change together), will provide an analytical path to develop the theoretical PSD.
In general,

$$R(\tau) = B(\tau) + m^2$$

As shown in the previous equation $R(\tau)$ is equal to $B(\tau)$ summed with the square of the expected value $m$ (mean of a random variable). Previously, in order to create a zero mean data set ($m = 0$), the experimentally gathered data had the mean value subtracted from each data point (Ref paragraph 2.4). Using the zero mean data set an experimental PSD was determined. To be consistent here, a zero mean will be assumed for this analysis when determining the theoretical PSD. With mean equal to zero the correlation and covariance functions are equal to each other, or

$$R(\tau) = B(\tau) \quad \text{with } m = 0$$

Since the correlation function and the covariance function are equal to each other the Fourier transforms of the respective functions are also equal to each other,

$$\mathcal{F}[R(\tau)] = \mathcal{F}[B(\tau)]$$
Utilizing the Einstein-Wiener Khinchin theorem [1], which states that the power spectral density of a wide-sense stationary random process is given by the Fourier transform of the autocorrelation function, the PSD can be written

\[ S(f) = \mathcal{F}[R(\tau)] = \mathcal{F}[B(\tau)] \]

The unit of the PSD function is energy per unit frequency and energy within a specific frequency range can be obtained by integrating PSD within that frequency range. Further, by integrating the PSD function when \( R(\tau = 0) \) the variance can be represented by the area under the curve as

\[ R(\tau = 0) = \int S(f) \, df = \sigma^2 \quad \text{i.e., the area under the curve} \]

or likewise, the covariance function \( B(\tau = 0) \) can be written

\[ B(\tau = 0) = \int S(f) \, df = \sigma^2 \quad \text{i.e., the area under the curve.} \quad (68) \]

A Fourier transform duality principle that defines two functions \( g_1(t) \) and \( g_2(t) \) in the time domain as equal to the convolution of the Fourier transform of the same two functions will also be utilized herein.
If

\[ \mathcal{F}[g_1(t)] = G_1(f) \quad \text{and} \quad \mathcal{F}[g_2(t)] = G_2(f) \]

then

\[ \mathcal{F}[g_1(t)g_2(t)] = G_1(f) \cdot G_2(f) \]

Convolution is an important concept in Fourier theory. Because of a mathematical property of the Fourier transform, referred to as the convolution theorem, it is convenient to carry out calculations involving convolutions. Mathematically, a convolution is defined as the integral over all space of one function at \( f \) times another function at \( f-f \). The integration is taken over the variable \( f \), typically from minus infinity to infinity over all the dimensions. The convolution integral is written

\[ G_1(f) \cdot G_2(f) = \int G_1(f) G_2(f-f) \, df \quad \text{limits } -\infty \text{ to } \infty \]

The mathematical operation in the convolution integral has simple graphical interpretations. First, plot \( G_1(f) \) and then "flip and shift" \( G_2(f-f) \) on the \( f \) axis, where \( f \) is fixed. Second, multiply the two signals and compute the signed area of the resulting function of \( f \) to obtain \( G_1(f) \cdot G_2(f) \). This process can be repeated for every value of \( f \) of interest.
Now, using the described statistical principles, the convolution process, and the duality principle, the determination of the theoretical PSD function can be facilitated using linear modeling and a graphical convolution technique. The mathematical steps to accomplish this task are as follows and the graphical process can be seen in Figure 8.

The PSD is equal to the Fourier transform of the covariance function and by substituting the temporal covariance function developed by Andrews/Phillips [4],

\[
S(f) = \mathcal{F} [B(\tau)] = \mathcal{F} [\exp(B_{\ln x}(\tau, D) + B_{\ln y}(\tau, D))] - 1
\]

or

\[
S(f) = \mathcal{F} [\exp(B_{\ln x}(\tau, D))] \mathcal{F} [\exp(B_{\ln y}(\tau, D))] - \mathcal{F} [1]
\]

Utilizing the Fourier transform duality the power spectral density function can be written,

\[
S(f) = S[\exp(B_{\ln x}(f))] \ast S[\exp(B_{\ln y}(f))] - \delta(f)
\]  \hspace{1cm} (69)
The graphical representation of the mathematical steps is illustrated in Figure 8. Figure 8a1 (left side) and 8a2 (left side) show a piecewise linear representation (solid lines) of the respective temporal covariance functions $B_y(\tau)$ and $B_x(\tau)$ (dotted lines). The area under the temporal covariance function in both cases is equal to the area under their respective linear representations. The characteristic shapes of the respective diffractive and refractive components (dotted lines) shown in Figure 8a1 and 8a2 were developed in paragraph 3.10 and can be seen in Figure 7. By taking the Fourier transform of the respective covariance functions $B_y(\tau)$ and $B_x(\tau)$ the PSD components $S_y(f)$ and $S_x(f)$, are obtained. Mathematically

$$S_y(f) = \mathcal{F}[B_y(\tau)] \quad \text{and} \quad S_x(f) = \mathcal{F}[B_x(\tau)]$$

where

$$B_y(\tau) = \exp[B_{lny}(\tau, D)] \quad \text{and} \quad B_x(\tau) = \exp[B_{lnx}(\tau, D)]$$

Note that the value of the cut-off frequency for $S_y(f)$ is $f_2 = 1/\tau_2$, and the cut-off frequencies for $S_x(f)$ are $f_1 = 1/\tau_1$ and $f_3 = 1/\tau_3$. Also, due to the power conservation characteristic of the Fourier transform, the amplitudes of the respective functions, $S_y(f)$ and $S_x(f)$, is chosen such that the area under the linear representations of $B_y(\tau)$ and $B_x(\tau)$ is maintained.

Figure 8b illustrates the process of the graphical convolution. $S_y(f)$ is plotted on the graph. $S_y(f)$ is then "flipped and shifted" on the $f$ axis while keeping $f$ fixed. $S_y(f)$ and $S_x(f)$ are then multiplied together at specific values of $f$ that are of interest.
The result of the convolution is depicted in Figure 8c yielding the total theoretical PSD function \( S(f) \). The multiplication of \( S_y(f) \) and \( S_x(f) \) results in the calculation of the area under the product (amplitude value), which is then assigned to the plot on which the convolution process is being performed. During the convolution process the power spectrums, \( S_y(f) \) and \( S_x(f) \), need to be considered as two-sided spectrums, therefore the amplitudes of the individual spectrums are divided by two and the relative relationship between the frequencies remain in tact. The resultant amplitude values and two breakpoint frequencies are annotated on the plot.
Figure 8 - Graphical Convolution Leading to Theoretical PSD Function
3.12 The Wave Structure Function in Weak to Strong Fluctuations

As shown previously, the Rytov perturbation method can be used to derive analytic expressions governing the statistical quantities of a propagating optical wave through the earth’s atmosphere. It is generally accepted that the validity of these expressions is restricted to the weak fluctuation regime, for sufficiently small separation distances. It has been shown using experimental results of the wave structure function, as a function of fluctuation strength, for a fixed value of the separation distance, that the Rytov method does not accurately model the behavior of the WSF in moderate to strong fluctuation regimes. In a paper published by Cynthia Y. Young etal. [9] an effective atmospheric spectral model was mathematically developed by multiplying the Kolmogorov spectral model (with inner and outer scale effects ignored) by an appropriate spatial filter function which eliminates the effects of the intermediate scale sizes. The waveform discussed here is an unbounded plane wave. In the presence of weak fluctuations, Young etal. [9] substituted the Tatarski and Hill spectral models, Andrews/Phillips [4], into the WSF as given in Equation 41. An analysis was then performed by substituting an “effective” Tatarski spectrum Andrews/Phillips [4], into the WSF for the weak to strong fluctuations.
Young et al. [9], used a WSF of the form

\[
D(\rho, L) = D_x(\rho, L) + D_y(\rho, L) \quad \rho \rangle \rangle l_o
\]  

(70)

where subscript, \( x \), refers to refractive effects and subscript, \( y \), refers to diffractive effects.

The refractive part of the WSF is given as

\[
D_x(\rho, L) = 1.47 \sigma_i^2 \left( \frac{\rho^2 k \eta_{x(pl)}^{1/6}}{L} \right) \left( 1 + 0.058 \frac{\rho^2 k \eta_{x(pl)}}{L} \right)^{-1/6}
\]

(71)

and the diffractive part of the WSF is given as

\[
D_y(\rho, L) = 2.37 \sigma_i^2 \left( \frac{\rho^2 k}{L} \right)^{5/6} - 1.47 \sigma_i^2 \left( \frac{\rho^2 k \eta_{y(pl)}^{1/6}}{L} \right) \left( 1 + 0.058 \frac{\rho^2 k \eta_{y(pl)}}{L} \right)^{-1/6}
\]

(72)

The Rytov variance again is given by

\[
\sigma_i^2(L) = \sigma_i^2 = 1.23 C_n^2 k^{7/6} L^{11/6}, \quad \text{(unbounded plane wave)}
\]

(73)
The non-dimensional parameters, $\eta_x(p_l)$ and $\eta_y(p_l)$, which incorporate the plane wave large-scale and small-scale frequency cut-offs, $\kappa_x(p_l)$ and $\kappa_y(p_l)$, are

$$\eta_x(p_l) = \frac{L\kappa_x^2}{k} = \frac{2.61}{1 + 1.11\left(\sigma_1^2\right)^{6/5}}$$  \hspace{1cm} (74)$$

and

$$\eta_y(p_l) = \frac{L\kappa_y^2}{k} = 3\left[1 + 0.69\left(\sigma_1^2\right)^{6/5}\right]$$  \hspace{1cm} (75)$$

Utilizing Equation 70 through Equation 75, as determined by Young et al. [9], in conjunction with the parameters of this experiment (i.e. $L = 13.3$ Km, wave number $k = \frac{2\pi}{\lambda}$, and refractive index structure parameter $C_n^2 = 8.5 \times 10^{-15}$ m$^{-2/3}$), a plot of the “Young’s” WSF was generated and can be seen in Figure 9. It is noted that the diffractive contributor of the WSF is in general agreement with analysis previously shown in Figure 7, however, the refractive component “crosses over” the diffractive component and displays a “non-physical” characteristic that is in disagreement with the expected behavior.
Figure 9 - Refractive and Diffractive Contributors to the WSF in Moderate Fluctuations
3.13 Taylor’s “Frozen Turbulence” Hypothesis

The theory to this point has described optical turbulence in terms of the spatial statistics. In the performance of the experiment described in this paper it was the temporal statistics that were actually measured due to the limitation of measuring the light intensity at only one spatial point. For this reason, an investigation of the relationship between the temporal and spatial statistics becomes a central issue. As a basis for this relationship it is necessary to rely on Taylor’s “frozen turbulence” hypothesis. G. I. Taylor (1921) says that temporal variations of meteorological quantities at a point are produced by advection of these quantities by the mean wind speed flow and not by the changes in the quantities themselves. This can be illustrated by the way dye spreads in a turbulent flow or the idea of clouds moving at a particular speed with little change in shape over small time periods.

There are essentially two time scales of concern in each of the two examples cited. For example, one time scale for the motion of the atmosphere across an observation path and the second time scale associated with dynamics of the turbulence (i.e., eddies). The first time scale, that due to advection, can be estimated by \( \frac{L_0}{V_T} \), where \( L_0 \) is the outer scale of turbulence and \( V_T \) is the mean wind velocity transverse to the observation path, and is of the order of 1s [3]. The second time scale is associated with eddy turnover time, about 10s [3], and because it is much slower than the first, can ordinarily be neglected in comparison to the mean wind flow. Therefore, under the Taylor “frozen turbulence” hypothesis,
turbulent eddies are treated as frozen in space and moved across the observation path by the mean wind velocity component $V_T$.

3.14 Temporal Spectrum of Irradiance Fluctuations

Based on Taylor frozen turbulence hypothesis, spatial statistics can be converted to temporal statistics by knowledge of the average wind speed transverse to the direction of propagation. In the case of a plane wave, this is accomplished by setting $\rho = (V_T)\tau$ in the covariance function where $V_T$ is the average transverse wind velocity described previously. By assuming an incident plane wave and letting $r_1 = -r_2$ the definition of the temporal covariance function Andrews/Phillips [4] is

$$B_1(\tau, D) = 8\pi^2k^2L \int_0^\infty \kappa \Phi_\kappa(\kappa)J_0(\kappa V_T \tau) \exp \left( -\frac{D^2 \kappa^2}{16} \right) \left( 1 - \cos \frac{L \kappa^2 \zeta}{k} \right) d\kappa d\zeta \quad (76)$$

where $D$ represents the aperture of the receiver.
Based on the Wiener-Khinchin theorem [1], (i.e. power spectral density of a wide-sense stationary random process is given by the Fourier transform of the autocorrelation function), the temporal spectrum of irradiance fluctuations $S_I(\omega)$ is defined by the Fourier transform of the temporal covariance function

$$S_I(\omega, D) = 2 \int_{-\infty}^{\infty} B_I(\tau, D)e^{-i\omega\tau}d\tau = 4 \int_{0}^{\infty} B_I(\tau, D)\cos(\omega\tau)d\tau$$

(77)

where $B_I(\tau, D)$ is the temporal covariance function described in Equation 76. Since $B_I(\tau, D)$ is an “even” function we can use the cosine portion of $e^{-i\omega\tau}$. The extra factor of two (2) in the transform integral is a consequence of considering only positive frequencies. Using the Fourier transform pair the inverse transform yields

$$B_I(\tau, D) = \frac{1}{2\pi} \int_{0}^{\infty} S_I(\omega, D)\cos(\omega\tau)d\tau$$

(78)

Re-writing the temporal covariance function under weak-to-strong fluctuations Andrews/Phillips [4]

$$B_I(\tau, D) = \exp[B_{lnx}(\tau, D) + B_{liny}(\tau, D)] - 1$$

(79)

where $B_{lnx}(\tau, D)$ and $B_{liny}(\tau, D)$ represent the large-scale and small-scale log-irradiance covariance’s, respectively.
For the zero inner scale case, the large-scale (refractive) log-irradiance temporal covariance for an unbounded plane wave is defined by [4]

\[
B_{\text{in}x}(\tau, D) \approx \frac{0.49\sigma_i^2}{\left(1 + 0.65d^2 + 1.11\sigma_i^{12/5}\right)^{7/6}} _1F_1\left(7/6; 1; -\frac{\omega_i^2 \tau^2 \eta_x}{4 + d^2 \eta_x}\right)
\]

(80)

where \( \omega_i = V_T / \sqrt{L / k}, \eta_x = \frac{2.61}{1 + 1.11\sigma_i^{12/5}}, d = \sqrt{\frac{kD^2}{4L}}, \sigma_i^2 \) is the Rytov variance defined in equation (64), and \(_1F_1(a; c; x)\) is a confluent hypergeometric function of the first kind.

For zero inner scale the small-scale (diffractive) log-irradiance temporal covariance for an unbounded plane wave is given by [4]

\[
B_{\text{in}y}(\tau, D) \approx \frac{0.51\sigma_i^2}{\left(1 + 0.90d^2 + 0.62d^2 \sigma_i^{12/5}\right)^{5/6}} \left(\omega_i^2 \tau^2 \eta_y\right)^{5/12} K_{5/6}(\omega_i \tau \sqrt{\eta_y})
\]

(81)

where \( \omega_i = V_T / \sqrt{L / k}, \eta_y = 3\left(1 + 0.69\sigma_i^{12/5}\right), d = \sqrt{\frac{kD^2}{4L}}, \sigma_i^2 \) is the Rytov variance defined in Equation (64), \( K_\nu(.) \) is a modified Bessel function of the second kind of order \( \nu \) [8].
Figure 10 and Figure 11 illustrate the effects of different values of the transverse wind speed on the equations for the large-scale (refractive) and small-scale (diffractive) temporal covariance for an unbounded plane wave with zero inner scale. Each plot is shown with all parameters of the equation remaining constant while the value of the average transverse wind velocity, $V_T$, is varied (i.e. 3.0 m/s, 3.6 m/s, and 4.0 m/s). As noted earlier the value of $V_T$ during the experiment was determined to be 3.6 m/s and is the reference point in the plots.

![Refractive Temporal Covariance (B_{lnx})](image)

Figure 10 - Large-scale Log Irradiance Temporal Covariance ($B_{lnx}$) as a Function of the Transverse Wind Velocity $V_T$
Figure 11 - Small-scale Log Irradiance Temporal Covariance ($B_{\text{lny}}$) as a Function of the Transverse Wind Velocity $V_T$

Refractive Index Structure Parameter $C_n^2 = 8.5 \times 10^{-15}$
Rytov Variance $\sigma_r^2 = 70.5$
In order to facilitate the theoretical calculation of the temporal spectrum of irradiance fluctuations using Equations 79, 80, and 81, approximations to the confluent hyper-geometric function in Equation 80 and the Bessel function in Equation 81 were developed. The exponential series representation of the confluent hyper-geometric function of the first kind is given by

\[
_{1}F_{1}(a; a; x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(a)_n} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x
\] (82)

and

\[
_{1}F_{1}(a; c; -x) = 1 - \frac{a \ x}{c \ 1!} + \frac{a(a+1) x^2}{c(c+1) 2!} - ...
\]

Also the exponential series is given by

\[
e^{-ax} = 1 - a \frac{x}{1!} + a^2 \frac{x^2}{2!} - ... + a^n \frac{x^n}{n!} + ...
\]

If parameter c in the second equation in Equation 82 is equal to 1, the first two terms of the confluent hyper-geometric function and the exponential series are identical and the third term differs only in the coefficient.
Consequently, the confluent hyper-geometric function can be approximated by the exponential function and the large-scale log irradiance covariance may be written

\[ a = 7/6, \quad c = 1, \quad x = \frac{\omega_2^2 \eta_x}{4 + d^2 \eta_r} \]

\[ B_{\text{ln},x}(\tau, D) \cong C_x \exp \left[ -\frac{7}{6} \frac{\omega_2^2 \eta_x}{4 + d^2 \eta_r} \tau^2 \right] \]

where

\[ C_x = \frac{0.49 \sigma_i^2}{\left( 1 + 0.65 d^2 + 1.11 \sigma_i^{12/5} \right)^{7/6}} \]

Figure 12 illustrates the theoretical large-scale (refractive) log-irradiance temporal covariance for an unbounded plane wave as given in Equation 80 without the approximation to the confluent hyper-geometric function, and as given in Equation 83 using the approximation to the confluent hyper-geometric function.
Figure 12 - Theoretical Refractive Covariance (B_{lnx}) Compared to Representation of Theoretical Refractive Covariance Using Hyper-geometric Function Approximation

Rylov Variance $\sigma_1^2 = 70.5$

Transverse Wind Velocity $V_T = 3.6 \text{ m/s}$
To approximate the modified Bessel Function of the second kind and its coefficient in Equation 81, it is necessary to consider both the small value approximation and the large value approximation, i.e.

\[
\text{small value; } y \to 0^+ \quad K_\nu(y) \approx \frac{\Gamma(\nu)}{2} \left(\frac{2}{\nu}\right)^\nu \quad \nu > 0
\]

(84)

\[
\text{large value; } y \to \infty \quad K_\nu(y) \approx \frac{\pi}{\sqrt{2y}} \exp[-y] \quad \text{for any } \nu.
\]

By writing the Bessel function and its coefficient from Equation 81 as

\[
\left(\omega^2 \tau^2 \eta_y\right)^{5/12} K_{5/6}\left(\omega \tau \sqrt{\eta_y}\right) = C_K \ K_{5/6}(y) \tag{85}
\]

where

\[
C_K = \left(\omega \sqrt{\eta_y} \tau\right)^{5/6}
\]

\[
y = \omega \tau \sqrt{\eta_y}
\]
Using the definitions in Equation 84, the small value and large value approximations of the Bessel function are written

\[
C_k K_{5/6}(y) \approx \Gamma \left( \frac{5}{6} \right) 2^{-1/6} \quad \text{for small } \tau
\]

(86)

\[
C_k K_{5/6}(y) \approx \sqrt{\frac{\pi}{2}} \left( \omega_i \tau \left( \sqrt{\eta_y} \tau \right)^{1/3} \exp \left[ -\omega_i \sqrt{\eta_y} \tau \right] \right) \quad \text{for large } \tau
\]

By multiplying the small value Bessel approximation by the large value Bessel approximation the value of the small-scale log irradiance covariance may be written as

\[
B_{m,y}(\tau, D) \equiv C_y \left[ \Gamma \left( \frac{5}{6} \right) 2^{-1/6} \left[ \sqrt{\frac{\pi}{2}} \left( \omega_i \sqrt{\eta_y} \tau \right)^{1/3} \exp \left[ -\omega_i \sqrt{\eta_y} \tau \right] \right] \right]
\]

(87)

where

\[
C_y = \frac{0.51 \sigma_i^2 \left( 1 + 0.69 \sigma_i^{12/5} \right)^{-5/6}}{1 + 0.90 d^2 + 0.62 d^2 \sigma_i^{12/5}}
\]

63
Equation 87 is a good engineering approximation to the small-scale log irradiance covariance, however, by using numerical iteration a better approximation was achieved and is written as

\[
B_{\ln, y}(\tau, D) \approx C_y \left[ \Gamma \left( \frac{5}{6} \right) 2^{-1/6} \right] \left[ \frac{15}{16} \exp \left[ -\frac{8}{11} \omega_y \sqrt{\eta_y \tau} \right] \right]
\]

(88)

\[
\approx C_{y1} \exp \left[ -\frac{8}{11} \omega_y \sqrt{\eta_y \tau} \right]
\]

where

\[
C_{y1} = C_y \left[ \Gamma \left( \frac{5}{6} \right) 2^{-1/6} \right] \frac{15}{16}
\]

Figure 13 illustrates the theoretical small-scale (diffractive) log-irradiance temporal covariance for an unbounded plane wave as given in Equation 81 without the approximation to the modified Bessel Function of the second kind, and as given in Equation 87 using the approximation for the modified Bessel Function of the second kind, and then as given in Equation 88 using a numerical fit method to improve the accuracy of the model.
Figure 13 - Theoretical Diffractive Covariance ($B_{yy}$) Compared to Representation of Theoretical Diffractive Covariance Using Bessel Function and Numerical “Best Fit”
3.15 Functional Forms of the Theoretical Temporal Covariance Functions

To continue with the analysis of the temporal spectrum of irradiance the theoretical large-scale and small-scale log irradiance covariances may be viewed in their functional form as

\[ B_{\text{ln}x}(\tau, D) \equiv C_{x} \exp \left[ -\frac{\tau^2}{R_x} \right] \quad \text{Refractive} \]

\[ B_{\text{ln}y}(\tau, D) \equiv C_{y1} \exp \left[ -\frac{\tau}{D_y} \right] \quad \text{Diffractive} \quad (89) \]

where

\[ C_x = \frac{0.49\sigma^2_i}{\left(1 + 0.65d^2 + 1.11\sigma^2_i \right)^{7/6}} \quad , \quad R_x = \frac{6}{7} \left( \frac{4 + d^2\eta_x}{\omega^2\eta_x} \right) \]

\[ C_{y1} = C_y \left[ \int \frac{5}{6} 2^{-1/6} \right] \frac{15}{16} \quad , \quad D_y = \left( \frac{11}{8\omega_i \sqrt{\eta_y}} \right) \]

By substituting these functional forms into the temporal spectrum of irradiance fluctuations in Equation 77, \( S_i(\omega) \) may be written as

\[ S_i(\omega, D) = 4 \int_{0}^{\infty} \left\{ \exp \left[ C_x \exp \left[ -\frac{\tau^2}{R_x} \right] + C_{y1} \exp \left[ -\frac{\tau}{D_y} \right] \right] - 1 \right\} \cos(\omega \tau) d\tau \quad (90) \]
In this form the temporal spectrum integral is difficult to evaluate. To facilitate the computation the integral will be transformed to a sum of integrals. Using a series representation and retaining only the lower order terms the exponential in Equation 90 can be re-written

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}
\]

and

\[
e^x \left[ c_x \exp \left( -\frac{\tau^2}{R_x} \right) + c_{y1} \exp \left( -\frac{\tau}{D_y} \right) \right] \approx 1 + c_x \exp \left( -\frac{\tau^2}{R_x} \right) + c_{y1} \exp \left( -\frac{\tau}{D_y} \right) + c_x c_{y1} \exp \left( -\frac{\tau^2}{R_x} \right) \exp \left( -\frac{\tau}{D_y} \right)
\]

By substituting Equation 91 into Equation 90 the temporal spectrum integral can be written

\[
S_I(\omega, D) = \frac{4}{\omega} \int_0^{\infty} \left[ 1 + c_x \exp \left( -\frac{\tau^2}{R_x} \right) + c_{y1} \exp \left( -\frac{\tau}{D_y} \right) + c_x c_{y1} \exp \left( -\frac{\tau^2}{R_x} \right) \exp \left( -\frac{\tau}{D_y} \right) - 1 \right] \cos(\omega t) dt (92)
\]
Simplification of Equation 92 leaves the solution to the temporal spectrum integral as a sum of three integrals as

\[
S_I(\omega, D) = 4 \int_0^\infty C_x \exp \left[ -\frac{\tau^2}{R_x} \right] \cos(\omega \tau) d\tau + 4 \int_0^\infty C_y \exp \left[ -\frac{\tau}{D_y} \right] \cos(\omega \tau) d\tau + 4 \int_0^\infty C_x C_y \exp \left[ -\frac{\tau^2}{R_x} \right] \exp \left[ -\frac{\tau}{D_y} \right] \cos(\omega \tau) d\tau
\] (93)

The solutions to all three definite integrals are tabulated and may be found in a table of integrals [17]. The first two integrals are straightforward and the solutions are

\[
4 \int_0^\infty C_x \exp \left[ -\frac{\tau^2}{R_x} \right] \cos(\omega \tau) d\tau = 2C_x \sqrt{\pi R_x} \exp \left[ -\frac{\omega^2}{4(1/R_x)} \right]
\] (94)

\[
4 \int_0^\infty C_y \exp \left[ -\frac{\tau}{D_y} \right] \cos(\omega \tau) d\tau = 4C_y D_y \left[ \frac{1}{1 + \frac{\omega^2}{(1/D_y)^2}} \right]
\]
The solution to the third integral is more rigorous and will be discussed. The third integral is of the form given and its tabulated solution [13] is

\[
\int_0^\infty \exp\left[-\beta x^2 - \nu x\right] \cos(b x) \, d x =
\]

\[
\frac{1}{4\sqrt{\beta}} \left\{ e^{\frac{(\nu-i b)^2}{4\beta}} \left[ 1-\Phi\left(\frac{\nu-i b}{2\sqrt{\beta}}\right)\right] + e^{\frac{(\nu+i b)^2}{4\beta}} \left[ 1-\Phi\left(\frac{\nu+i b}{2\sqrt{\beta}}\right)\right] \right\} \quad \text{Re } \beta > 0
\]

(95)

where the asymptotic representation is

\[
\Phi(x) = 1 - \frac{1}{\pi} e^{-x^2} \sum_{k=0}^{n-1} (-1)^k \frac{\Gamma(k+1/2)}{x^{2k+1}}
\]

The series expansion of the asymptotic representation is

\[
\Phi(x) = 1 - \frac{1}{\pi} e^{-x^2} \left[ \frac{\Gamma(1/2)}{x} - \frac{\Gamma(3/2)}{x^3} + \frac{\Gamma(5/2)}{x^5} - \ldots \right]
\]

(96)

retaining only the first term in the series and with \( \Gamma(1/2) = \sqrt{\pi} \)

\[
\Phi(x) = 1 - \frac{1}{\pi} e^{-x^2} \left[ \frac{\sqrt{\pi}}{x} \right]
\]
By the properties of complimentary functions [17] Equation 97 and Equation 98 can be written

\[
1 - \Phi \left( \frac{v - ib}{2\sqrt{\beta}} \right) = 1 - \left\{ \frac{1}{\pi} \left[ \frac{\sqrt{\pi}}{v - ib} \right] \right\} \left[ \frac{\sqrt{\pi}}{v - ib} \right]^{\frac{(v - ib)^2}{4\beta}}
\]

(97)

And

\[
1 - \Phi \left( \frac{v + ib}{2\sqrt{\beta}} \right) = 1 - \left\{ \frac{1}{\pi} \left[ \frac{\sqrt{\pi}}{v + ib} \right] \right\} \left[ \frac{\sqrt{\pi}}{v + ib} \right]^{\frac{(v + ib)^2}{4\beta}}
\]

(98)
By substituting Equations 97 and 98 into Equation 95 the solution to the third integral becomes

$$\int_0^\infty \exp\left[-\beta x^2 - \nu x\right] \cos(bx) dx =$$

$$\frac{1}{4\sqrt{\beta}} \left\{ \left(\frac{u-ib}{4\beta}\right)^2 \frac{2\sqrt{\beta} \sqrt{\pi}}{\pi(u-ib)} e^{\frac{(u-ib)^2}{4\beta}} + e^{\frac{(u+ib)^2}{4\beta}} \right\}$$

The first two exponentials in Equation 99, when multiplied together, yields a value of 1. Similarly the second two exponentials in Equation 99, when multiplied together also yields a value 1. When performed, these two multiplications effectively remove all the exponentials from the right side of Equation 99. Factoring out the like terms from inside the brackets yields a coefficient and a complex conjugate pair inside the brackets.
Further, by using complex conjugate simplification (i.e. the numerator and denominator of the first fraction inside the brackets is multiplied by the denominator of the second fraction in the brackets and vice versa) and a cancellation of like terms with opposite signs, the generic solution to the third integral reduces to

\[
\int_0^\infty \exp\left[-\beta x^2 - \nu x\right] \cos(bx) dx = \frac{1}{4} \sqrt{\pi} \left\{ \frac{2\sqrt{\beta \pi}}{\nu - ib} \right\} + \left\{ \frac{2\sqrt{\beta \pi}}{\nu + ib} \right\} \\
= \frac{1}{4} \sqrt{\beta \pi} \left[ \frac{1}{\nu - ib} + \frac{1}{\nu + ib} \right] \\
= \frac{\nu}{\nu^2 + b^2} 
\] (100)

The specific solution for the third integral in Equation 93 can now be written

\[
4 \int_0^\infty \frac{d}{dx} \left[ \frac{-x^2}{R_x} \right] \exp\left[-\frac{\tau^2}{R_x}\right] \exp\left[-\frac{\tau}{D_y}\right] \cos(\omega \tau) d\tau = 4C_x C_y \int_0^\infty \frac{1}{1 + \frac{\omega^2}{(1/D_y)^2}} \\
= \frac{1}{R_x}, \quad \nu = \frac{1}{D_y}, \quad b = \omega 
\] (101)

where

\[ \beta = \frac{1}{R_x}, \quad \nu = \frac{1}{D_y}, \quad b = \omega \]

were used in the generic solution.
Now, the solutions for each of the three integrals in Equation 93, as given in Equation 94 and Equation 101, can be summed together to yield the approximation to the theoretical temporal spectrum of the irradiance fluctuations of the signal, i.e.,

\[
S_f(\omega, D) \approx 2C \sqrt{\frac{\pi R}{x}} \exp \left[ -\frac{\omega^2}{4(1/R_x)} \right]
\]

\[
+ 4C_y D_y \left[ \frac{1}{1 + \frac{\omega^2}{(1/D_y)^2}} \right]
\]

\[
+ 4C_x C_y D_y \left[ \frac{1}{1 + \frac{\omega^2}{(1/D_y)^2}} \right]
\]

(102)

Combining term 2 and term 3 of Equation 102 using factoring yields the approximation to the theoretical temporal spectrum of irradiance fluctuations

\[
S_f(\omega, D) \approx 2C \sqrt{\frac{\pi R}{x}} \exp \left[ -\frac{\omega^2}{4(1/R_x)} \right] + 4C_y D_y \left[ 1 + C_x \right] \left[ \frac{1}{1 + \frac{\omega^2}{(1/D_y)^2}} \right]
\]

(103)
Figure 14 illustrates the theoretical model of the temporal spectrum of irradiance fluctuations (i.e. Power Spectral Density Function), $S(f)$, as given in Equation 77 using the analysis, the functional forms of the theoretical large-scale and small-scale log irradiance covariances, and the mathematical approximations developed herein.

Theoretical Power Spectral Density Function

![Theoretical Power Spectral Density Function](image)

- Amplitude Values Theoretical (Math w/ Approximations)
- Transverse Wind Velocity $V_T = 3.6$ m/s
- Rytov Variance $\sigma_r^2 = 70.5$

Figure 14 - Theoretical Power Spectral Density Function from Functional Forms of Log Irradiance Covariance (Mathematical Approximations Used)
CHAPTER 4

4.1 Summary of Assumptions and Conditions Used in the Analysis

The following is a summary of the assumptions and conditions for the theory and experiment that were adhered to in order to facilitate the analysis leading to the development of the theoretical and experimental Power Spectral Density functions.

1. The received irradiance of the optical wave was modeled as a modulation process in which small-scale (diffractive) fluctuations are multiplicatively modulated by statistically independent large-scale (refractive) fluctuations.

2. The Rylov method for optical scintillation is valid, even over a longer propagation path and with a larger scintillation index, with the introduction of a spatial frequency filter to properly account for the loss of spatial coherence of the optical wave in strong fluctuation conditions.

3. The wave under consideration for the development of the theoretical math model for the temporal spectrum of irradiance fluctuations was limited to a plane wave with zero inner scale.
4. The estimated refractive index structure parameter value used in the calculations was \( C_n^2 = 8.5 \times 10^{-15} \text{ m}^{-2/3} \) and was determined using the data gathered during the experiment, Halbing [14].

5. Using wind data recorded via two weather instruments located on the Kennedy Space Center, a trigonometric solution was used to determine the Average Transverse Wind speed across the slant path of the experimental “channel”. That value was calculated to be \( \omega_t = 3.6 \text{ m/s} \).

6. An incident wave is affected differently depending on the size of the turbulent eddy through which it propagates as shown in Figure 6. The illustration shows eddies ranging in size from 1mm to 1m. As eddies grow larger in size the amplitude of the incident signal is less perturbed (i.e. more focused due to less scattering) than the signal passing through the smaller eddies. When the eddy size reaches approximately 1m in diameter the signal is not perturbed at all i.e., the amplitude of the transferred signal remains unchanged. Both small and large eddies scatter (refract and diffract) the signal, however, the large eddies perturb the signal over much greater distances then do the small eddies.
4.2 Analytical Approaches to Model Power Spectral Density

An effort was made using three different approaches to develop a Power Spectral Density (PSD) mathematical model to represent the light intensity characteristics of an incident optical signal at the optical receivers. The three independent approaches are described in the following paragraphs.

In addition to the three analytical approaches described herein, a paper published by Young etal. [9] was considered. The paper developed an effective atmospheric spectral model by multiplying the Kolmogorov spectral model (inner and outer scale turbulent effects ignored) by an appropriate spatial filter function to eliminate the turbulent effects of the intermediate scale sizes. The paper investigated several waveforms, however, only the unbounded plane wave will be discussed here to facilitate a comparative analysis. Young etal. [9], substituted the Tatarski and Hill spectral models given by Andrews/Phillips [4] into equation (41) which describes the WSF in weak fluctuations. An analysis was then performed by substituting an effective Tatarski spectrum given by Andrews/Phillips [4] into the WSF for the weak to strong fluctuations.

Work performed during this research included a review of the analysis performed by “Young”. In addition, two mathematical cases examined by “Young” were re-examined to validate the previous work. Case I used the identical analysis and the math equations were
reformed. Case II eliminated the use of an approximation for the hyper-geometric function that had been used in Case I. Case II investigated whether or not the hyper-geometric function approximation was introducing error in the calculations. Both math models for Case I and Case II were validated, however, when the result was plotted (ref Figure 9, moderate fluctuations), the analysis gave results that seemed “non-physical”. The curve representing the diffractive (small scale) contribution to the WSF in moderate scintillation “crossed over” the curve representing the refractive (large scale) contribution to the WSF. This was an unexpected result. Additionally, the expected “exponential” characteristic shape of the diffractive curve as shown in Figure 7 was different then the curve for the diffractive component according to “Young” as shown in Figure 9.

4.3 Approach 1 - Mutual Coherence Function and Wave Structure Function

Approach 1 investigated the Mutual Coherence Function (MCF) and the Wave Structure Function (WSF). By using the method of the geometrical optics approximation Andrews/Phillips [3] the log-amplitude structure function is approximately equal to zero and the phase structure function and the WSF are essentially the same. The WSF integral was separated into two parts; one representing the effects due to large-scale eddies (refractive) and one representing the effects due to small-scale eddies (diffractive). The limit of integration, used for both the refractive and diffractive integral representations, was
the coherence radius of the plane wave as defined in terms of the wave number. The integrals representing the refractive and diffractive contributors to the power spectral density of a propagating light beam were individually evaluated. The details of the evaluations are delineated in paragraph 3.10. In solving the integrals an approximation to a hyper-geometric function was used which resulted in plotting the approximation for the theoretical Mutual Coherence Function shown in Figure 7. The solution to each integral was then summed together and written in terms of the Rytov variance as seen in Equation 66.

While approach 1 did not yield a total theoretical PSD function model that could be plotted, the graphical representations and the characteristic shapes of the refractive and diffractive components of the MCF were valuable and were utilized in approach 2 and approach 3.
Approach 2 - Graphical Convolution Method

Approach 2 used the temporal covariance function developed by Andrews/Phillips [4], principles of statistics (i.e. correlation function, covariance function) a duality principle of the Fourier transform, and a piecewise linear representation of the diffractive and refractive Mean Coherence Functions (MCF). Using these principles a graphical convolution method was performed to produce a theoretical Power Spectral Density (PSD) Function.

The graphical representation of the mathematical steps is illustrated in Figure 8. Figure 8a1 (left side) and 8a2 (left side) show a piecewise linear representation (solid lines) of the respective temporal covariance functions $B_y(\tau)$ and $B_x(\tau)$ (dotted lines). The area under the temporal covariance functions in both cases is equal to the area under their respective linear representations. The characteristic shapes of the respective diffractive and refractive components (dotted lines) shown in Figure 8a1 and 8a2 were developed in paragraph 3.10 and are shown in Figure 7. By taking the Fourier transform of the respective covariance functions $B_y(\tau)$ and $B_x(\tau)$ the PSD components $S_y(f)$ and $S_x(f)$, were obtained.

The value of the cut-off frequency for $S_y(f)$ is $f_2 = 1/\tau_2$, and the cut-off frequencies for $S_x(f)$ are $f_1 = 1/\tau_1$ and $f_3 = 1/\tau_3$. Due to the power conservation characteristic of the Fourier transform, the amplitudes of the respective functions, $S_y(f)$ and $S_x(f)$, was chosen such that the area under the linear representations of $B_y(\tau)$ and $B_x(\tau)$ was maintained.
Figure 8b illustrates the process of the graphical convolution. \( S_x(f) \) was plotted on the graph. \( S_y(f) \) was then "flipped and shifted" on the f axis while keeping \( f \) fixed. \( S_y(f) \) and \( S_x(f) \) were multiplied together at specific values of \( f \) that were of interest.

The result of the convolution is depicted in Figure 8c yielding the total theoretical PSD function \( S(f) \). The multiplication of \( S_y(f) \) and \( S_x(f) \) resulted in the calculation of the area under the product (amplitude value), which was then assigned to the plot on which the convolution process was performed. During the convolution process the power spectrums \( S_y(f) \) and \( S_x(f) \) were considered as two-sided spectrums, therefore, the amplitudes of the individual spectrums were divided by two and the relative relationship between the frequencies remained in tact. The resultant amplitude values and two breakpoint frequencies of the PSD \( S(f) \) were annotated on Figure 8c.

Approach 2 produced a theoretical PSD function that is shown in Figure 8.
4.5 **Approach 3 – Theoretical Temporal Covariance Functional Forms**

Approach 3 investigated the relationship between the temporal and spatial statistics. As a basis for that relationship the G. I. Taylor “frozen turbulence” hypothesis Andrews/Phillips [4] was used to rewrite the temporal covariance function in terms of the transverse wind speed. Taylor states that temporal variations of meteorological quantities at a point are produced by advection of these quantities by the average wind speed flow and not by the changes in the quantities themselves. An average transverse wind speed (3.6 m/s) was calculated from range data collected during the experiment and used to convert spatial statistics to temporal statistics.

The Wiener-Khinchin theorem [1] (i.e. the power spectral density of a wide-sense stationary random process is given by the Fourier transform of the autocorrelation function) provided the mathematical link between the Power Spectral Density and the temporal covariance function. The relationship between the Power Spectral Density and the temporal covariance function relationship is shown in Equation 77, and again after an application of a Fourier transform pair in Equation 78.

Equation 79 (temporal covariance function), Equation 80 (large-scale (refractive) log-irradiance temporal covariance function), and Equation 81 (small-scale (diffractive) log-
irradiance temporal covariance function) furnished by Andrews/Phillips [4], were evaluated as follows.

The log-irradiance covariance expressions were evaluated utilizing the calculated transverse wind speed, integral calculus and advanced algebra. In addition, mathematical approximations for a confluent hyper-geometric function (refractive case), a modified Bessel function of the second kind (diffractive case), and a numerical “best fit” method (diffractive case) were employed to develop functional forms of the theoretical temporal covariance function. Figure 12 and Figure 13 illustrate the graphical plot of the theoretical refractive and diffractive covariance functions, respectively. Also, over-layed on each figure are the graphical representations of the refractive and diffractive covariance functions as developed using the mathematical approximations.

Approach 3 produced a theoretical PSD function that is shown in Figure 14.
4.6 **Comparison of Power Spectral Density Functions**

All three approaches to determine a theoretical model of the Power Spectral Density (PSD) are described in sections 4.3 through 4.5 for reference. The following is a comparative analysis of approach 2 (convolution method) and approach 3 (functional forms method) that were productive yielding a theoretical model for the PSD. Approach 1 did not yield a theoretical PSD model and is not represented here. The results of approach 2 and approach 3 were presented in Figure 8 and Figure 14, respectively. To better illustrate the similarities and the differences in the independent analytical approaches, the results of each approach are plotted together in Figure 15. In addition to the results of the two independently obtained theoretical PSD functions, the resultant PSD function obtained from the intensity data collected during the experiment is also plotted in Figure 15. The overlaying of the three different PSD functions facilitates the ability to perform a comparative analysis.
Figure 15 - Comparison of Power Spectral Density Function Results

Figure 15 illustrates three independently obtained Power Spectral Density functions plotted using the same points of reference on the x and y logarithmic axes. The three graphs on the plot include the PSD function obtained using the measured amplitude of the light intensity data collected in the laser experiment (thin line), the theoretical PSD result generated using mathematical approximations and functional forms of the theoretical model as developed by Andrews/Phillips [4] (dashed line), and the graphical convolution result using linear approximations to the refractive and diffractive components of the Mutual Coherence Function (MCF) (heavier line).
In all three cases it can be noted that the results display a similar characteristic with respect to the general shape of the curves. The peak amplitude in the lower frequency region for all three curves is a value in the range of 0.040 - 0.050. Also, each of the three curves displays the existence of two (2) break-point frequencies. The first breakpoint occurs in the range from 8 Hz to 10 Hz and the second breakpoint occurs in the frequency range of approximately 200 Hz to 700 Hz. While these breakpoint frequencies do not occur at exactly the same place for each of the curves represented, they do illustrate a reasonable approximation in each case and overall the values remain within family. The roll-off rate of the experimental PSD compares well with the PSD obtained from the graphical convolution method. The PSD function obtained from the functional forms method has a roll-off rate that is much steeper than the other two PSD functions at the first breakpoint frequency resulting in a much deeper cut in the theoretical amplitude values in the frequency band from 10-30 Hz.

Also, the roll-off for the functional form theoretical PSD at the second break-point frequency is again steeper than the roll-off for the other two PSD functions. This leads to the existence of frequencies with less amplitude in the frequency band from 40 to 400 Hz.

A number of assumptions and mathematical approximations were used when analyzing the theoretical models. The small value and large value approximation for the modified Bessel function of the second kind was used in the development of the theoretical diffractive covariance. Further, to refine the representation of the theoretical diffractive covariance, a
numerical iteration “best fit” method was utilized. In addition, the first order terms of an exponential series were used to represent the confluent hyper-geometric function to facilitate the calculation of the theoretical refractive covariance. Use of these aforementioned course approximations, for the more rigorous mathematical terms in this analysis, could likely account for differences in the outcome of the theoretical power spectral density as compared to the power spectral density developed from the experimental data. This mathematical approach, while not rigorous in detail, still resulted in a good first order engineering approximation for the theoretical power spectral density function.
APPENDIX: EQUIPMENT USED
Equipment Used in the Experiment

Toshiba Satellite 1.6 Ghz Intel Pentium 4 Processor Laptop computer

National Instruments Data Acquisition Card (DAQ Card 1200)

Custom made TIA’s (ref. thesis by John Stalder)

Coherent Innova 90 Argon Ion Laser (wavelength 514nm, green)

Hamamatsu Type R1387 Photo Multiplier Tube (optical receiver)

Andover Optical Filter (1nm bandwidth at 514nm, 50% efficient, eliminate background)

Keithly 247 high voltage power supply

National Instruments Labview version 6.0 (data collection and analysis)

Mathcad 11 mathematical tool (data analysis)
REFERENCES


