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Degree Of Aproximation Of Hölder Continuous Functions

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DEGREE OF APROXIMATION OF HÖLDER CONTINUOUS FUNCTIONS

by

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ABSTRACT

Pratima Sadangi in a Ph.D. thesis submitted to Utkal University proved results on degree of approximation of functions by operators associated with their Fourier series. In this dissertation, we consider degree of approximation of functions in $H_{\alpha,p}$ by different operators. In Chapter 1 we mention basic definitions needed for our work. In Chapter 2 we discuss different methods of summation. In Chapter 3 we define the $H_{\alpha,p}$ metric and present the degree of approximation problem relating to Fourier series and conjugate series of functions in the $H_{\alpha,p}$ metric using Karamata (K^{λ}) means. In Chapter 4 we present the degree of approximation of an integral associated with the conjugate series by the Euler, Borel and (e,c) means of a series analogous to the Hardy-Littlewood series in the $H_{\alpha,p}$ metric. In Chapter 5 we propose problems to be solved in the future.

This dissertation is dedicated to my parents, Allyn and Betty.

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CHAPTER 1: INTRODUCTION

A series is divergent if its sum diverges to infinity or oscillates finitely. Summability methods are used to assign a sum to series which oscillates finitely. Methods used to sum such series include Cesáro, Nörlund, Riesz, Abel, Euler, Borel, (e,c), and Karamata means. See Hardy [11] for all definitions and related results.

1.1 Some Basic Definitions and Theorems

Given $\sum a_n$ with partial sum s_n, let $t_n = p\, s_n$, where p is some transform of s_n. Let c be the collection of all convergent sequences. If $p: c \to c$ then p is said to be *conservative*.

If $s_n \to s$ implies that $p\, s_n \to s$ as $n \to \infty$ then p is said to be *regular.*

The *degree of convergence* of a summation method to a given function f is a measure of how fast t_n converges to f. This means that we need to find λ_n such that

$$\|t_n - f\| = O\left(\frac{1}{\lambda_n}\right), \tag{1.1}$$

where $\lambda_n \to \infty$ as $n \to \infty$.

Theorem A: Given any finitely oscillating sequence, there exists regular transformation which makes it summable [8].

Theorem B: Given any regular transformation, there exists a sequence which cannot be summed by that method [4].

Theorem C: Associated with each method is a limitation theorem which says what type of sequence can be summed by it [11].

A significant application of the summation methods is to Fourier series.

1.2 Fourier Series

Let $f \in L(0,2\pi)$ be periodic with period 2π. $C_{2\pi}$ is the collection of all continuous functions with period 2π.

The Fourier series of f is given by

$$f \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \tag{1.2}$$

where a_n and b_n are the Fourier coefficients.

The series conjugate to (1.2) is given by

$$f \sim \sum_{n=1}^{\infty}(a_n \sin nx - b_n \cos nx) \tag{1.3}$$

Zygmund [38] showed that if $f \in C_{2\pi} \cap Lip\ \alpha, 0 < \alpha \leq 1$ and s_n is the n^{th} partial sum of the Fourier series of f then

$$\|s_n(f;x) - f(x)\| = O\left(\frac{\log n}{n^a}\right). \tag{1.4}$$

1.3 Hardy-Littlewood Series

Let

$$\sum_{\mathrm{n}=0}^{\infty} A_n(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\,. \tag{1.5}$$

If

$$S_n^*(x) = \sum_{k=0}^{n-1} A_k(x) + \frac{1}{2} A_n(x) \tag{1.6}$$

then the Hardy-Littlewood series (HL-series), is defined as

$$\sum_{n=1}^{\infty} \frac{S_n^*(x) - f(x)}{n}. \tag{1.7}$$

Let

$$\sum_{\mathrm{n}=0}^{\infty} B_n(x) = \sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx). \tag{1.8}$$

If

$$\tilde{S}_n^*(x) = \sum_{k=1}^{n-1} B_k(x) = \frac{1}{2} B_n(x) \tag{1.9}$$

then the associated Hardy-Littlewood series is defined as

$$\sum_{n=1}^{\infty} \frac{\tilde{S}_n^*(x) - \tilde{f}(x)}{n}. \tag{1.10}$$

The convergence of the above series is addressed in a theorem due to Das et all. [6].

In this dissertation we shall determine the degree of convergence of certain means of the Fourier series, conjugate series, and the associated Hardy-Littlewood series of a function f to itself in $H_{\alpha,p}$. In Chapter 2 we present some background material. In Chapters 3 and 4 we show results related to our research. In Chapter 5 we make some concluding remarks and state a few problems to be solved in the future.

CHAPTER 2: BACKGROUND

Introduction

In this chapter we define the methods of summation that will be used throughout the dissertation. We also Give the definition of Hölder continuity. We state the degree of approximation of Hölder continuous functions by the K^{λ} means of their Fourier series and conjugate series due to Sadangi [29].

2.1 Methods of Summation

There are several methods of summing divergent series. We shall state several such methods:

Borel's Exponential mean: Let $\sum_{n=0}^{\infty} u_n(x)$ be an infinite series with sequence of partial sums $\{t_n(x)\}$. The Borel's exponential mean $B_p(t;x)$ of the sequence $\{t_n(x)\}$ is defined by

$$B_p(t;x) = e^{-p}\sum_{n=0}^{\infty} t_n(x)\frac{p^n}{n!}, (p > 0) \tag{2.1}$$

Euler mean: Given any sequence $\{t_n(x)\}$ its $(E,q), q > 0$, mean $E_n^q(t;x)$ is defined by

$$E_n^q(t;x) = (q+1)^{-n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k} t_k(x) \tag{2.2}$$

(e,c) mean: Let $\sum_{n=-\infty}^{\infty} c_n(x)$ be an infinite series with the partial sums $\{t_n(x)\}$. The (e,c), $(c > 0)$ mean $e_n^c(t;x)$ of $\{t_n(x)\}$ is defined by

$$e_n^c(t;x) = \sqrt{\frac{c}{\pi n}}\sum_{k=-\infty}^{\infty} t_{n+k}(x)\exp\left(-\frac{ck^2}{n}\right) \tag{2.3}$$

where it is understood that $t_{n+k}(x) = 0$, when $n + k < 0$.

K^λ mean: For $n = 1,2,3, \ldots$ and $0 \le m \le n$, we define the numbers $\begin{bmatrix} n \\ m \end{bmatrix}$ by

$$\prod_{v=0}^{n-1}(x+v) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} x^m \tag{2.4}$$

where

$$\prod_{v=0}^{n-1}(x+v) = x(x+1)\cdots(x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x)}$$

.

We shall use the convention that $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1$. The numbers $\begin{bmatrix} n \\ m \end{bmatrix}$ are known as the absolute values of the Stirling numbers of the first kind.

Let $\lambda > 0$. The K^λ mean $K_n^\lambda(t,x)$ of a sequence $\{t_n(x)\}$ is defined by

$$K_n^\lambda(t,x) = \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \lambda^k t_k(x) \tag{2.5}$$

If $K_n^\lambda(t,x) \to s$ as $n \to \infty$ we say that the sequence $\{t_n(x)\}$ is summable K^λ to s. The method K^λ is regular for $\lambda > 0$ and this case will be supposed throughout the present work.

2.2 Approximation by the K^λ Means of Fourier Series and Conjugate Series in the Hölder Metric

Let $C_{2\pi}$ be the space of all 2π periodic functions defined on $[0,2\pi]$ and let for $0 < \alpha \leq 1$ and for all x, y

$$H_\alpha = \{f \in C_{2\pi}: |f(x) - f(y)| \leq M|x - y|^\alpha\} \tag{2.6}$$

where M is a positive constant. The functions H_α are called Hölder continuous functions. The space H_α $(0 < \alpha \leq 1)$ is a Banach space [24] under the norm $||\cdot||_\alpha$:

$$\|f\|_\alpha = \|f\|_c + \sup_{x \neq y} \Delta^\alpha f(x,y)(f \in H_\alpha) \tag{2.7}$$

where $\|f\|_c$ denotes the sup norm of f with respect to x,

$$\Delta^\alpha f(x,y) = \frac{|f(x) - f(y)|}{|x - y|^\alpha}, x \neq y \tag{2.8}$$

and by convention

$$\Delta^0 f(x,y) = 0.$$

The metric induced by the norm $||\cdot||_\alpha$ on H_α is called the Hölder metric. It can be seen that

$$\|f\|_\beta \leq (2\pi)^{\alpha-\beta}\|f\|_\alpha \tag{2.9}$$

for $0 \leq \beta < \alpha \leq 1$. Thus $\{H_\alpha, ||\cdot||_\alpha\}$ is a Banach space which deceases as α increases, i.e.,

$$C_{2\pi} \supseteq H_\beta \supseteq H_\alpha \text{ for } 0 \leq \beta < \alpha \leq 1 \tag{2.10}$$

Alextis [1] studied the degree of approximation of the functions in H_α class by the Cesáro means of their Fourier series in the sup-norm. It was Prössodoorf [28] who initiated the work on the degree of approximation of functions in H_α by the Féjer mean of the Fourier series in the Hölder metric. This result has been generalized by Chandra [2], Mohapatra and Chandra [23], Singh [30], [31] using different methods. Chandra [2], [3] has studied the degree of approximation problem in the Hölder metric using Borel and Euler means. The degree of approximation of a function f in H_α has been studied by Das, Ghosh, and Ray [5] by using (e,c) means of Fourier Series in the Hölder metric.

The K^λ means were first introduced by Karamata [15]. Lototsky [20] reintroduced the special case $\lambda = 1$. Vuckovic [37] first studied the K^λ summability of the Fourier series. Sadangi [29] proved the following two theorems:

Theorem A Let $K_n^\lambda(f, x)$ be the K^λ mean of the Fourier series (1.2) of f at x. If $0 \leq \beta < \alpha \leq 1$ and $f \in H_\alpha$ then

$$\left\|K_n^\lambda(f,\cdot) - f\right\|_\beta = O(1)\frac{\log\log n}{(\log n)^{\alpha-\beta}} \tag{2.11}$$

Theorem B Let $h = \frac{\pi}{l(n)}$, where $l(n) = \frac{3}{2} + \lambda \sum_{k=1}^{n-1} \frac{1}{\lambda+k}$. Let $\widetilde{K}_n(f,x)$ be the K^λ mean of the series conjugate to the Fourier series of f at x. If $0 \le \beta < \alpha \le 1$ and $f \in H_\alpha$, then

$$\left\|\widetilde{K}_n^\lambda(f,\cdot) - \tilde{f}(\cdot,h)\right\|_\beta = O(1)\frac{\log\log n}{(\log n)^{\alpha-\beta}}, 0 \le \beta < \alpha \le 1 \tag{2.12}$$

2.3 The Measure of Convergence of the Euler, Borel, and (e,c) Means of a Series Associated with the Hardy-Littlewood Series in the Hölder Metric

Let

$$g(x) = \frac{2}{\pi}\int_{0^+}^{\pi} \psi_x(t)\frac{1}{2}\cot\left(\frac{1}{2}t\right)\log\left(\frac{1}{2}\csc\frac{1}{2}t\right)\, dt, \tag{2.13}$$

where

$$\psi_x(t) = \frac{1}{2}\{f(x+t) - f(x-t)\}. \tag{2.14}$$

Das, Ray, and Sadangi [7] obtained the rate of convergence of the associated Hardy-Littlewood series (1.10) to $g(x)$ in the Hölder metric:

Theorem. Let $\tilde{T}_n(x)$ be the *n*th partial sum of the Hardy-Littlewood series (1.7). Let

$0 \le \beta < \alpha \le 1$ and $f \in H_\alpha$. Then

$$\left\|\tilde{T}_{\boldsymbol{n}} - g\right\|_{\boldsymbol{\beta}} = O(1)\begin{cases}\dfrac{1}{n^{\alpha-\beta}}, \alpha-\beta \neq 1\\ \dfrac{\log n}{n}, \alpha-\beta = 1\end{cases} \tag{2.15}$$

Sadangi [29] obtained the degrees of approximation of $g(x)$ in the Hölder metric using the Euler, Borel, and (e, c) means of (1.7):

Theorem 1. Let $0 \leq \beta < \alpha \leq 1$ and let $f \in H_{\alpha}$. Then

$$||E_n^q(\tilde{T}) - g||_{\beta} = O(1)\begin{cases}\dfrac{1}{n^{\alpha-\beta}}, \alpha-\beta \neq 1\\ \dfrac{\log n}{n}, \alpha-\beta = 1\end{cases} \tag{2.16}$$

Theorem 2. Let $0 \leq \beta < \alpha \leq 1$ and let $f \in H_{\alpha}$. Then

$$||B_p(\tilde{T}) - g||_{\beta} = O(1)\begin{cases}\dfrac{1}{p^{\alpha-\beta}}, \alpha-\beta \neq 1\\ \dfrac{\log p}{p}, \alpha-\beta = 1\end{cases} \tag{2.17}$$

Theorem 3. Let $0 \leq \beta < \alpha \leq 1$ and let $f \in H_{\alpha}$. Then

$$\left\|e_n(\tilde{T}) - g\right\|_{\beta} = O(1)\begin{cases}\dfrac{1}{n^{\alpha-\beta}}, 0 < \alpha-\beta \leq \dfrac{1}{2}\\ \dfrac{1}{\sqrt{n}}, \quad \dfrac{1}{2} < \alpha-\beta \leq 1\end{cases} \tag{2.18}$$

CHAPTER 3: APPROXIMATION BY THE K^{λ} MEANS OF FOURIER SERIES AND CONJUGATE SERIES OF FUNCTIONS IN $H_{\alpha,p}$

3.1 Definitions and notations

Let $L_p[0,2\pi]$ be the space of all 2π-periodic integrable functions.

$$H_{\alpha,p} := \left\{ f \in L_p[0,2\pi] : \left(\int_0^{2\pi} |f(x+t) - f(x)|^p dx \right)^{\frac{1}{p}} \leq K|t|^{\alpha} \right\}, \tag{3.1}$$

where K is a positive constant. The space $H_{\alpha,p}$ $(p > 1, \alpha \leq 0 < 1)$ is a Banach space under the norm $|| \cdot ||_{\alpha,p}$:

$$\|f\|_{\alpha,p} := \|f\|_p + \sup_{t \neq 0} \frac{\|f(y+t) - f(y)\|_p}{|t|^{\alpha}}. \tag{3.2}$$

The metric induced by the norm $|| \cdot ||_{\alpha,p}$ on $H_{\alpha,p}$ is called Holder continuous with degree p. It can be seen that

$$\|f\|_{\beta,p} \leq (2\pi)^{\alpha-\beta} \|f\|_{\alpha,p}.$$

Since $f \in H_{\alpha,p}$ if and only if $\|f\|_{\alpha,p} < \infty$, we have,

$$L_p[0,2\pi] \supseteq H_{\beta,p} \supseteq H_{\alpha,p}, \qquad p > 1, \qquad 0 \le \beta < \alpha \le 1. \tag{3.3}$$

We write

$$\phi_x(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}$$

$$\psi_x(t) = \tfrac{1}{2}\{f(x+t) - f(x-t)\}$$

$$\tilde{f}(x,\varepsilon) = -\tfrac{2}{\pi}\int_\varepsilon^\pi \psi_x(t)\tfrac{1}{2}\cot\tfrac{1}{2}t\,dt$$

$$\tilde{f}(x) = \lim_{\varepsilon\to 0^+} \tilde{f}\,(x,\varepsilon),$$

whenever the limit exits.

Let $S_n(f,x)$ and $\tilde{S}_n(f,x)$ respectively denote the nth partial sums of the series (1.2) and (1.3). It is known (Zygmund [38], p.50) that

$$S_n(f,x) - f(x) = \frac{2}{\pi}\int_0^\pi \phi_x(t)D_n(t)dt \tag{3.4}$$

$$\tilde{S}_n(f,x) = -\frac{2}{\pi}\int_0^\pi \psi_x(t)\widetilde{D}_n\,(t)dt, \tag{3.5}$$

where

$$D_n(t) = \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}, \qquad \widetilde{D}_n(t) = \frac{\cos\frac{1}{2}t - \cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t} \tag{3.6}$$

3.2 Main Results

In this chapter, we prove the following two theorems:

Theorem 1. Let $K_n^\lambda(f,x)$ be the K^λ mean of the Fourier series of f at x. If $p>1$, $0\le\beta<\alpha\le 1$, and $f\in H_{\alpha,p}$, then

$$\left\|K_n^\lambda(f,\cdot)-f\right\|_{\beta,p}=O\left(\frac{\log\log n}{(\log n)^{\alpha-\beta}}\right). \tag{3.7}$$

Theorem 2. Let $h=\frac{\pi}{l(n)}$, where $l(n)=\frac{3}{2}+\lambda\sum_{k=1}^{\lambda-1}\frac{1}{\lambda+k}$. Let $\widetilde{K}_n(f,x)$ be the K^λ mean of the series conjugate to the Fourier series of f at x. If $0\le\beta<\alpha\le 1$ and $f\in H_{\alpha,p}$, then

$$\left\|\widetilde{K}_n(f,x)-\tilde{f}(\cdot,h)\right\|_{\beta,p}=O\left(\frac{\log\log n}{(\log n)^{\alpha-\beta}}\right) \tag{3.8}$$

3.3 Additional Notations and Lemmas

We use the following additional notations:

$$G(t)=\phi_{y+u}(t)-\phi_y(t)$$

$$\tilde{G}(t)=\psi_{y+u}(t)-\psi_y(t)$$

$$L_n(y)=K_n^\lambda(f,y)-f(y)$$

$$\tilde{L}_n(y)=\widetilde{K}_n^\lambda(f,y)-\tilde{f}(y,h)$$

$$K_n(t)=\frac{\Gamma(\lambda)}{\Gamma(n+\lambda)}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\lambda^k\sin\left(k+\frac{1}{2}\right)t$$

$$\widetilde{K}_n(t) = \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \lambda^k \cos\left(k + \frac{1}{2}\right) t$$

$$\tilde{E}_n(t) = \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \lambda^k \widetilde{D}_k(t)$$

$$\rho_k(t) = (\lambda^2 + 2\lambda k \cos t + k^2)^{1/2}$$

$$\tan\theta_k = \frac{\lambda \sin t}{\lambda \cos t + k}$$

$$R(n,t) = \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \prod_{k=0}^{n-1} \rho_k(t)$$

$$l(n) = \frac{3}{2} + \lambda \sum_{k=0}^{n-1} \frac{1}{\lambda+k}$$

$$h = \frac{\pi}{l(n)}$$

We need the following lemmas for the proof of our theorems.

Lemma 1 Let $0 \leq \beta < \alpha \leq 1$. If $f \in H_{\alpha,p}$ then for $0 < t \leq \pi$ and $u \neq 0$

(i) $$G(t) = O(1) \begin{cases} t^{\alpha} \\ |u|^{\alpha} \\ |u|^{\beta} t^{\alpha-\beta} \end{cases}$$

(ii) $$\tilde{G}(t) = O(1) \begin{cases} t^{\alpha} \\ |u|^{\alpha} \\ |u|^{\beta} t^{\alpha-\beta} \end{cases}$$

Proof: The first and second estimates of (i) follow from the definition of $H_{\alpha,p}$ and $\phi_x(t)$.

Writing

$$|G(t)| = |G(t)|^{1-\beta/\alpha} |G(t)|^{\beta/\alpha}$$

and using the first two estimates of (i) we can derive the third one. Proof of (ii) is similar to that of (i).

Lemma 2 Let $0 \leq \beta < \alpha \leq 1$. If $f \in H_{\alpha,p}$ then for $x \neq y$

(i) $$G(t+h) - G(t) = O(1) \begin{cases} h^{\alpha} \\ |u|^{\alpha} \\ |u|^{\beta} h^{\alpha-\beta} \end{cases}$$

(ii) $$\tilde{G}(t+h) - \tilde{G}(t) = O(1) \begin{cases} h^{\alpha} \\ |u|^{\alpha} \\ |u|^{\beta} h^{\alpha-\beta} \end{cases}$$

Proof: Writing

$$G(t+h) - G(t) = \frac{1}{2}[f(y+u+t+h) - f(y+u+t)] +$$

$$[f(y+u-t-h) - f(y+u-t)] + [f(y+t+h) - f(y+t)] + [f(y-t-h) - f(y-t)]$$

and using the fact that $f \in H_{\alpha,p}$ we obtain the first estimate of (i). The remaining part of the proof is similar to that of Lemma 1 and hence it is omitted.

Lemma 3 Suppose that A and c are both positive constants. Let γ be any real number. Then as $\mu \to \infty$,

$$\int_{\pi/\mu}^{c} t^{\gamma} e^{-A\mu t^2}\, dt = O(1) \begin{cases} \mu^{-\gamma-1}, & \gamma < -1 \\ \log \mu, & \gamma = -1 \\ \frac{1}{\mu^{k}}, \ 2k-1 < \gamma \leq 2k, & k = 0,1,2,3, \ldots \\ \frac{1}{\mu^{\gamma-k}}, \ 2k \leq \gamma \leq 2k+1 \end{cases}$$

Lemma 4 Let $\theta_k, R(n,t), K_n(t)$ and $\widetilde{K}_n(t)$ be defined as in §3. Then

$$K_n(t) = R(n,t)\sin(\tfrac{3}{2}t + \textstyle\sum_{k=1}^{n-1}\theta_k)$$

$$\widetilde{K}_n(t) = R(n,t)\cos(\tfrac{3}{2}t + \textstyle\sum_{k=1}^{n-1}\theta_k)$$

Proof: By simple computation, we obtain

$$\widetilde{K}_n(t) + iK_n(t) = \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(\lambda e^{it})^k e^{\frac{1}{2}it}$$

$$= \frac{\Gamma(\lambda)e^{\frac{1}{2}it}}{\Gamma(n+\lambda)}\prod_{k=0}^{n-1}(\lambda e^{it} + k)$$

$$= \frac{\Gamma(\lambda)e^{\frac{1}{2}it}}{\Gamma(n+\lambda)}\prod_{k=0}^{n-1}[(\lambda\cos t + k) + i\lambda\sin t]$$

$$= \frac{\Gamma(\lambda)e^{\frac{1}{2}it}}{\Gamma(n+\lambda)}\prod_{k=0}^{n-1}\rho_k(t)e^{i\theta_k}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)}R(n,t)\exp\{i(\tfrac{3}{2}t + \textstyle\sum_{k=1}^{n-1}\theta_k)\}$$

from which the lemma follows.

Lemma 5 Let $0 < t < \pi$. Then for some positive constant A,

(i) $$R(n,t) = \begin{cases} O(1) \\ O(1)e^{-At^2\log n} \end{cases}$$

(ii) $$K_n(t) = \begin{cases} O(1) \\ O(1)e^{-At^2\log n} \end{cases}$$

(iii) $$\widetilde{K}_n(t) = \begin{cases} O(1) \\ O(1)e^{-At^2\log n} \end{cases}$$

(iv) $$\tilde{E}_n(t) = O(t^{-1})$$

Proof: $R(n,t)$ attains its maximum value for $t=0$ and it is easy to see that $R(n,0)=1$ and this ensures the first estimate of (i). Now

$$R(n,t)=\frac{\Gamma(\lambda)}{\Gamma(n+\lambda)}\prod_{k=0}^{n-1}(\lambda^2+2\lambda k\cos t+k^2)^{1/2}$$

$$=\frac{\Gamma(\lambda)}{\Gamma(n+\lambda)}\prod_{k=0}^{n-1}(\lambda+k)\,[1-\frac{4k\,sin^2\frac{1}{2}t}{(\lambda+k)^2}]^{1/2}$$

$$=\prod_{k=0}^{n-1}[1-\frac{4\lambda\,sin^2\frac{1}{2}t}{(\lambda+k)^2}]^{\frac{1}{2}}$$

$$=\exp\left[-\frac{1}{2}\sum_{k=1}^{n-1}\log\{1-\frac{4\lambda\,sin^2\frac{1}{2}t}{(\lambda+k)^2}\}^{-1}\right]. \qquad (3.9)$$

At this stage, we observe that

$$0<\frac{4\lambda k\,sin^2\frac{1}{2}t}{(\lambda+k)^2}<1$$

for $k=1,2,3,\ldots$ and $0<t<\pi$. As $\log(1-\theta)^{-1}\geq\theta$ for $0<\theta<1$ and $\sin\geq\frac{2x}{\pi}, 0\leq x\leq\frac{\pi}{2}$, we have

$$\sum_{k=1}^{n-1}\log\,\{1-\frac{4\lambda\,sin^2\frac{1}{2}t}{(\lambda+k)^2}\}^{-1}$$

$$\geq\sum_{k=1}^{n-1}\frac{4\lambda k\,sin^2\frac{1}{2}t}{(\lambda+k)^2}$$

$$\geq\frac{2\lambda t^2}{\pi^2}\sum_{k=1}^{n-1}\frac{k}{(\lambda+k)^2}$$

$$\geq At^2\log n\,, \qquad (3.10)$$

where A is some positive constant. Using (3.9) and (3.10) we obtain the second estimate of Lemma 5 (i), (ii), and (iii) follow from (i). As $\widetilde{D}_k(t) = O(t^{-1})$ (iv) follows at once.

Lemma 6 Let $0 < t \leq \pi/4$. Then

(i) $\sin(\frac{3}{2}t + \sum_{k=1}^{n-1}\theta_k) - \sin l(n)t = 0(t^3 \log n)$

(ii) $\cos(\frac{3}{2}t + \sum_{k=1}^{n-1}\theta_k) - \cos l(n)t = 0(t^3 \log n)$

Proof: We have

$$\left|\sin\left(\frac{3}{2}t + \sum_{k=1}^{n-1}\theta_k\right) - \sin l(n)t\right| \leq \left|(\frac{3}{2}t + \sum_{k=1}^{n-1}\theta_k) - l(n)t\right|. \qquad (3.11)$$

Next, we note that

$$0 < \frac{\lambda \sin t}{\lambda \cos t + k} < 1$$

whenever $0 < t \leq \pi/4$ and $k \geq 1$. Thus for $0 < t \leq \pi/4$,

$$\theta_k = \left[tan^{-1}\frac{\lambda \sin t}{\lambda \cos t + k} - \frac{\lambda \sin t}{\lambda \cos t + k}\right] + \left[\frac{\lambda \sin t}{\lambda \cos t + k} - \frac{\lambda\, t}{\lambda \cos t + k}\right] +$$

$$+\left[\frac{\lambda\, t}{\lambda \cos t + k} - \frac{\lambda\, t}{\lambda + k}\right] + \frac{\lambda\, t}{\lambda + k}$$

$$= O\left[\left(\frac{\lambda \sin t}{\lambda \cos t + k}\right)^3\right] + O\left[\frac{t^3}{\lambda \cos t + k}\right] + O\left[\frac{t^3}{(\lambda \cos t + k)(\lambda + k)}\right] + \frac{\lambda t}{\lambda + k}$$

$$= O\left(\frac{t^3}{k^3}\right) + O\left(\frac{t^3}{k^2}\right) + O\left(\frac{t^3}{k}\right) + \frac{\lambda t}{\lambda + k}$$

$$= \frac{\lambda t}{\lambda + k} + O\left(\frac{t^3}{k}\right), 1 \leq k \leq n - 1 \qquad (3.12)$$

Using (3.12), we have,

$$\frac{3}{2}t + \sum_{k=1}^{n-1} \theta_k = \frac{3}{2}t + \lambda t \sum_{k=1}^{n-1} \frac{1}{\lambda + k} + O(t^3) \sum_{k=1}^{n-1} \frac{1}{k}$$

$$= tl(n) + O(t^3 \log n) \tag{3.13}$$

Using (3.13) in (3.11), we obtain lemma 6(i). We omit the proof of (ii) as it is similar to that of (i).

Lemma 7

(i) $R'(n,t) = O(1) t \log n \, R(n,t), 0 < t \leq \pi/2$

(ii) $R(n, t+h) - R(n,t) = O(1) t e^{-At^2 \log n}, \; h \leq t \leq \pi/2$

Proof: We have

$$R(n,t) = \lambda \prod_{k=1}^{n-1} \rho_k(t)$$

and so by logarithmic differentiation,

$$R'(n,t) = R(n,t) \sum_{k=1}^{n-1} \frac{\rho'_k(t)}{\rho_k(t)}$$

$$= R(n,t) \sum_{k=1}^{n-1} \frac{(-\lambda k \sin t)}{(\rho_k(t))^2}$$

$$= O(1) t \, R(n,t) \sum_{k=1}^{n-1} \frac{1}{k} \quad (\because \rho_k(t) \geq k, 0 < t \leq \pi/2)$$

$$= O(1) t \log n \, R(n,t)$$

and this completes the proof of (i). By the Mean Value Theorem, for some ξ with $0 < \xi < 1$,

$$R(n, t+h) - R(n,t) = hR'(n, t + \xi h)$$

$$= O(1)h(t + \xi h)R(n, t + \xi h) \log n$$

$$= O(1)te^{-At^2 \log n}, \text{ for } h \le t \le {}^{\pi}/_{2}$$

The following lemma is due to Hardy, Littlewood, and Pólya [13].

Lemma 8 If $h(y, t)$ is a function of two variables defined for $0 \le t \le \pi,\ 0 \le y \le 2\pi,$ then

$$\left\| \int h(y, t)dt \right\|_p \le \int \|h(y, t)\|_p \, dt, \qquad p > 1$$

3.4 Proof of Theorem 1

Using (3.4) and notations of §3, we obtain

$$K_n^\lambda(f, x) = \frac{\Gamma(\lambda)}{\Gamma(n + \lambda)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \lambda^k S_k(f, x)$$

$$= \frac{\Gamma(\lambda)}{\Gamma(n + \lambda)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \lambda^k \left\{ \frac{2}{\pi} \int_0^\pi \phi_x(t) D_k(t) dt \right\}$$

$$+ f(x) \frac{\Gamma(\lambda)}{(n + \lambda)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \lambda^k$$

$$= \frac{2}{\pi} \int_0^{\pi} \frac{\phi_x(t)}{2 \sin \frac{1}{2} t} K_n(t)\, dt + f(x),$$

from which it follows that

$$L_n(x) = K_n^{\lambda}(f, x) - f(x) = \frac{2}{\pi} \int_0^{\pi} \frac{\phi_x(t)}{2 \sin \frac{1}{2} t} K_n(t) dt \tag{3.14}$$

Now, for fixed δ with $0 < \delta < {}^{\pi}/_{4}$, we write for $u \neq 0$

$$L_n(y+u) - L_n(y) = \frac{2}{\pi} \int_0^{\pi} \frac{G(t)}{2 \sin \frac{1}{2} t} K_n(t) dt$$

$$= \frac{2}{\pi} \left[\int_0^h + \int_h^{\delta} + \int_{\delta}^{\pi} \right] \frac{G(t)}{2 \sin \frac{1}{2} t} K_n dt$$

$$= \frac{2}{\pi} [P + Q + R] = I_1 + I_2 + I_3 \tag{3.15}$$

By Minkowski's inequality, we have,

$$\|L_n(y+u) - L_n(y)\|_p \leq \|I_1\|_p + \|I_2\|_p + \|I_3\|_p$$

And by Lemma 8, we may write,

$$\|I_1\|_p \leq \frac{2}{\pi} \int_0^h \|G(t)\|_p \left| \frac{K_n(t)}{2 \sin \frac{1}{2} t} \right| dt, \tag{3.16}$$

$$\|I_2\|_p \leq \frac{2}{\pi} \int_h^{\delta} \|G(t)\|_p \left| \frac{K_n(t)}{2 \sin \frac{1}{2} t} \right| dt\,, \tag{3.17}$$

$$\|I_3\|_p \leq \frac{2}{\pi}\int_{\delta}^{\pi}\|G(t)\|_p \left|\frac{K_n(t)}{2\sin\frac{1}{2}t}\right| dt. \tag{3.18}$$

By Lemma 1 (i) and Lemma 5 (ii), we obtain

$$\|I_1\|_p \quad = O\left(|u|^{\beta}\int_0^h t^{\alpha-\beta-1}dt\right) = O\left(\frac{|u|^{\beta}}{(\log n)^{\alpha-\beta}}\right). \tag{3.19}$$

and

$$\begin{aligned}\|I_3\|_p &= O\left(|u|^{\beta}\int_{\delta}^{\pi} t^{\alpha-\beta-1}e^{-At^2\log n}dt\right)\\ &= O\left(|u|^{\beta}e^{-A\delta^2}\log n\right)\\ &= O\left(\frac{|u|^{\beta}}{(\log n)^{\Delta}}\right), \Delta \text{ positive however large.}\end{aligned} \tag{3.20}$$

Using Lemma 4, we may write

$$\begin{aligned}\|I_2\|_p &\leq \int_h^{\delta}\|G(t)\|_p\left|\frac{1}{2\sin\frac{1}{2}t}-\frac{1}{t}\right||K_n(t)|dt + \int_h^{\delta}\frac{\|G(t)\|_p}{t}R(n,t)\sin l(n)\,dt\\ &+ \int_h^{\delta}\frac{\|G(t)\|_p}{t}R(n,t)\left\{\sin\left(\frac{3}{2}t+\sum_{k=1}^{n-1}\theta_k\right)-\sin l(n)t\right\}dt\\ &= I + J + K\end{aligned} \tag{3.21}$$

As $\left\{\frac{1}{2\sin\frac{1}{2}t}-\frac{1}{t}\right\} = O(t)$, using Lemma 1 (i) and Lemma 5 (ii), we obtain

$$I = O\left(|u|^{\beta}\int_h^{\delta} t^{\alpha-\beta+1}e^{-At^2\log n}dt\right)$$

$$= O\left(\frac{|u|^{\beta}}{\log n}\right) \tag{3.22}$$

by Lemma 3 (replacing μ by $\log n$). Now

$$J = \int_h^{\delta} \frac{\|G(t)\|_p}{|t|} |R(n,t) \sin l(n)t| dt \tag{3.23}$$

$$= \left[\int_h^{2h} + \int_{2h}^{\delta+h} - \int_{\delta}^{\delta+h}\right] \frac{\|G(t)\|_p}{|t|} |R(n,t) \sin l(n)t \,| dt$$

Replacing t by $t+h$ in the second integral of the above line, we obtain,

$$J = \int_h^{2h} \frac{\|G(t)\|_p}{|t|} |R(n,t) \sin l(n)t \mid dt$$

$$- \int_h^{\delta} \frac{\|G(t+h)\|_p}{|t+h|} |R(n,t+h) \sin l(n)t \mid dt$$

$$- \int_{\delta}^{\delta+h} \frac{\|G(t)\|_p}{|t|} |R(n,t) \sin l(n)t \mid dt \tag{3.24}$$

From (3.23) and (3.24), we get

$$2J = \int_h^{\delta} \left[\frac{\|G(t)\|_p}{|t|} |R(n,t)| - \frac{\|G(t+h)\|_p}{|t+h|} R(n,t+h)\right] |\sin l(n)t| dt$$

$$+ \int_h^{2h} \frac{\|G(t)\|_p}{|t|} |R(n,t) \sin l(n)t \,| dt$$

$$- \int_{\delta}^{\delta+h} \frac{\|G(t)\|_p}{|t|} |R(n,t) \sin l(n)t \,| dt$$

$$= L + M - N \tag{3.25}$$

By Lemma 1(i), Lemma 5 (i), we get

$$M = O\left(|u|^{\beta} \int_h^{2h} t^{\alpha-\beta-1} dt\right)$$

$$= O\left(\frac{|u|^{\beta}}{(\log n)^{\alpha-\beta}}\right) \tag{3.26}$$

and

$$N = O\left(|u|^{\beta} \int_{\delta}^{\delta+h} t^{\alpha-\beta-1} e^{-At^2 \log n} dt\right)$$

$$= O\left(|u|^{\beta} e^{-A\delta^2 \log n}\right)$$

$$= O\left(\frac{|u|^{\beta}}{(\log n)^{\Delta}}\right), \Delta \text{ positive however large} \tag{3.27}$$

We rewrite

$$L = \int_{h}^{\delta} \frac{\|G(t)\|_p - \|G(t+h)\|_p}{t} R(n,t) \sin l(n)t \, dt$$

$$+ \int_{h}^{\delta} \big(R(n,t) - R(n,t+h)\big) \frac{\|G(t+h)\|_p}{t} \sin l(n)t \, dt$$

$$+ \int_{h}^{\delta} \left(\frac{1}{t} - \frac{1}{t+h}\right) \|G(t+h)\|_p R(n,t+h) \sin l(n)t \, dt$$

$$= L_1 + L_2 + L_3 \tag{3.28}$$

By Lemma 1(i) and Lemma 2 (i), we obtain

$$L_1 = O\left(|u|^{\beta} h^{\alpha-\beta} \int_{h}^{\delta} \frac{dt}{t}\right)$$

$$= O\left(|u|^{\beta} \frac{\log\log n}{(\log n)^{\alpha-\beta}}\right) \tag{3.29}$$

By Lemma 1 (ii) and Lemma 7 (ii), we have

$$L_2 = O\left(|u|^{\beta} \int_{h}^{\delta} (t+h)^{\alpha-\beta} e^{-At^2 \log n} dt\right)$$

$$= O\left(|u|^{\beta} \int_{h}^{\delta} t^{\alpha-\beta} e^{-At^2 \log n} dt\right)$$

$$= O\left(\frac{|u|^{\beta}}{(\log n)^{\alpha-\beta}}\right) \tag{3.30}$$

using Lemma 3 (μ is replaced by $\log n$).

Using Lemma 1 (ii), Lemma 5 (i), and Lemma 3 (replacing μ by $\log n$), we have

$$L_3 = h\int_h^{\delta} \frac{\|G(t+h)\|_p}{t(t+h)} R(n, t+h)\sin l(n)t\, dt$$

$$= O\left(h|u|^{\beta}\int_h^{\delta} \frac{(t+h)^{\alpha-\beta-1}}{t} e^{-At^2\log n} dt\right)$$

$$= O\left(h|u|^{\beta}\int_h^{\delta} t^{\alpha-\beta-2} e^{-At^2\log n} dt\right)$$

$$= O|u|^{\beta}\begin{cases}\dfrac{1}{(\log n)^{\alpha-\beta}}, \alpha-\beta \neq 1\\ \dfrac{\log\log n}{\log n}, \alpha-\beta = 1\end{cases} \tag{3.31}$$

Collecting the results (3.25) through (3.31), we obtain,

$$J = O\left(|u|^{\beta}\frac{\log\log n}{(\log n)^{\alpha-\beta}}\right), 0 \leq \beta < \alpha \leq 1 \tag{3.32}$$

Now, using Lemma 1(i), Lemma 5 (i), and Lemma 6(i), we have

$$K = \int_h^{\delta} \frac{\|G(t)\|_p}{t} R(n,t)\left\{\sin\left(\frac{3}{2}t + \sum_{k=1}^{n-1}\theta_k\right) - \sin l(n)t\right\} dt$$

$$= O\left(|u|^{\beta}\log n \int_h^{\delta} t^{\alpha-\beta+2} e^{-At^2\log n}\, dt\right)$$

$$= O\left(\frac{|u|^{\beta}}{(\log n)^{\alpha-\beta}}\right), \tag{3.33}$$

by applying Lemma 3.

Collecting the results from (3.18) – (3.22), (3.32), and (3.33), we obtain

$$|L_n(y+u) - L_n(y)| = O\left(|u|^\beta \frac{\log\log n}{(\log n)^{\alpha-\beta}}\right), 0 \le \beta < \alpha \le 1 \tag{3.34}$$

which ensures that

$$\sup_{u\neq 0} \frac{|L_n(y+u) - L_n(y)|}{|u|^\beta} = O\left(\frac{\log\log n}{(\log n)^{\alpha-\beta}}\right), 0 \le \beta < \alpha \le 1 \tag{3.35}$$

Again $f \in H_{\alpha,p} \Longrightarrow \|\phi_x(t)\|_p = O(t^\alpha)$ and so proceeding as above, we obtain

$$\|L_n(\cdot)\|_p \ = O\left(\frac{\log\log n}{(\log n)^\alpha}\right), 0 < \alpha \le 1 \tag{3.36}$$

combining (3.35) and (3.36) we obtain (3.7) and this completes the proof of Theorem 1.

3.5 Proof of Theorem 2

Using (3.5) and notations of §3, we obtain

$$\widetilde{K}_n^\lambda(f,x) = \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \lambda^k \tilde{S}_k(f,x)$$

$$= -\frac{2}{\pi}\int_0^\pi \psi_x(t) \left(\frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \lambda^k \widetilde{D}_k(t)\right) dt$$

$$= -\frac{2}{\pi}\int_0^h \psi_x(t) \left(\frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \lambda^k \widetilde{D}_k(t)\right) dt$$

$$-\frac{2}{\pi}\int_h^\pi \psi_x(t) \left(\frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \lambda^k \right) \frac{1}{2}\cot\frac{1}{2}t \ dt$$

$$+\frac{2}{\pi}\int_h^\pi \frac{\psi_x(t)}{2\sin\frac{1}{2}t} \left(\frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \lambda^k \cos\left(k+\frac{1}{2}\right)t\right) dt$$

$$= -\frac{2}{\pi}\int_0^h \psi_x(t)\tilde{E}_n\,(t)dt + \tilde{f}(x,h) + \frac{2}{\pi}\int_h^\pi \frac{\psi_x(t)}{2\sin\frac{1}{2}t}\widetilde{K}_n(t)dt$$

which further ensures that

$$\tilde{L}_n(x) = \widetilde{K}_n^\lambda(f,x) - \tilde{f}(x,h)$$

$$= -\frac{2}{\pi}\int_0^h \psi_x(t)\tilde{E}_n(t)dt + \frac{2}{\pi}\int_h^\pi \frac{\psi_x(t)}{2\sin\frac{1}{2}t}\widetilde{K}_n(t)\,dt \tag{3.37}$$

Now for fixed δ with $0 < \delta < \pi/4$ and $x \neq y$, we write

$$\tilde{L}_n(y+u) - \tilde{L}_n(y) = -\frac{2}{\pi}\int_0^h \tilde{G}(t)\tilde{E}_n(t)dt + \frac{2}{\pi}\int_h^\delta \frac{\tilde{G}(t)}{2\sin\frac{1}{2}t}\widetilde{K}_n\,(t)dt$$

$$+\frac{2}{\pi}\int_\delta^\pi \frac{\tilde{G}(t)}{2\sin\frac{1}{2}t}\widetilde{K}_n(t)$$

$$= -\tilde{P} + \tilde{Q} + \tilde{R} \tag{3.38}$$

By Minkowski's inequality, we have

$$\|L_n(y+u) - L_n(y)\|_p \leq \|\tilde{P}\|_p + \|\tilde{Q}\|_p + \|\tilde{R}\|_p.$$

By Lemma 8,

$$\|\tilde{P}\|_p \leq \frac{2}{\pi}\int_0^h ||\tilde{G}\,(t)||_p\,|\tilde{E}_n(t)|dt \tag{3.39}$$

$$\|\tilde{Q}\|_p \leq \frac{2}{\pi}\int_h^\delta ||\tilde{G}\,(t)||_p \left|\frac{\widetilde{K}_n(t)}{2\sin\frac{1}{2}t}\right| dt \tag{3.40}$$

$$\|\tilde{R}\|_p \leq \frac{2}{\pi}\int_\delta^\pi ||\tilde{G}\,(t)||_p \left|\frac{\widetilde{K}_n(t)}{2\sin\frac{1}{2}t}\right| dt \tag{3.41}$$

By Lemma 1 (ii) and Lemma 5 (iv), we have

$$\|\tilde{P}\|_p = O\left(|u|^\beta \int_0^h t^{\alpha-\beta-1} dt\right)$$

$$= O\left(\frac{|u|^\beta}{(\log n)^{\alpha-\beta}}\right) \qquad (3.42)$$

Using Lemma 1 (ii) and Lemma 5 (iii), we have

$$\|\tilde{R}\|_p = O\left(|u|^\beta \int_\delta^\pi t^{\alpha-\beta-1} e^{-At^2 \log n} dt\right)$$

$$= O\left(|u|^\beta e^{-A\delta^2 \log n}\right)$$

$$= O\left(\frac{|u|^\beta}{(\log n)^\Delta}\right), \Delta \text{ positive however large} \qquad (3.43)$$

Now, adopting the lines of arguments similar to those used in estimating Q in the proof of Theorem 1, we can obtain

$$\|\tilde{Q}\|_p = O\left(|u|^\beta \frac{\log\log n}{(\log n)^{\alpha-\beta}}\right), 0 \le \beta < \alpha \le 1. \qquad (3.44)$$

Combining the results of (3.41) – (3.44), we obtain

$$\sup_{u \neq 0} \frac{\|\tilde{L}_n(y+u) - \tilde{L}_n(y)\|_p}{|u|^\beta} = O\left(\frac{\log\log n}{(\log n)^{\alpha-\beta}}\right), 0 \le \beta < \alpha \le 1. \qquad (3.45)$$

Similarly,

$$\|\tilde{L}_n(\cdot)\|_p = O\left(\frac{\log\log n}{(\log n)^\alpha}\right), 0 < \alpha \le 1. \qquad (3.46)$$

(3.8) follows from (3.45) and (3.46) and this completes the proof of Theorem 2.

CHAPTER 4: THE MEASURE OF CONVERGENCE OF THE EULER, BOREL, AND (e,c) MEANS OF A SERIES ASSOCIATED WITH THE HARDY-LITTLEWOOD SERIES IN $H_{\alpha,p}$

4.1 Definitions and notations

Let $L_p[0,2\pi]$ be the space of all 2π-periodic integrable functions and for all t

$$H_{\alpha,p} := \left\{ f \in L_p[0,2\pi]: \left(\int_0^{2\pi} |f(x+t) - f(x)|^p dx \right)^{\frac{1}{p}} \leq K|t|^\alpha \right\}, \tag{4.1}$$

where K is a positive constant. The space $H_{\alpha,p}$ $(p > 1, \alpha \leq 0 < 1)$ is a Banach space under the norm $|| \cdot ||_{\alpha,p}$:

$$\|f\|_{\alpha,p} := \|f\|_p + \sup_{t \neq 0} \frac{\|f(y+t) - f(y)\|_p}{|t|^\alpha}. \tag{4.2}$$

The metric induced by the norm $|| \cdot ||_{\alpha,p}$ on $H_{\alpha,p}$ is called Hölder continuous with degree p. It can be seen that

$$\|f\|_{\beta,p} \leq (2\pi)^{\alpha-\beta} \|f\|_{\alpha,p}.$$

Since $f \in H_{\alpha,p}$ if and only if $\|f\|_{\alpha,p} < \infty$, we have,

$$L_p[0,2\pi] \supseteq H_{\beta,p} \supseteq H_{\alpha,p}, \qquad p > 1, \qquad 0 \leq \beta < \alpha \leq 1. \tag{4.3}$$

We write

$$\varphi_x(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}$$

$$\psi_x(t) = \frac{1}{2}\{f(x+t) - f(x-t)\}$$

$$\chi_x(t) = \int_t^\pi \varphi_x(u)\frac{1}{2}\cot\frac{1}{2}u\,du$$

$$\theta_x(t) = -\frac{2}{\pi}\int_0^t \psi_x(u)\frac{1}{2}\cot\frac{1}{2}u\,du$$

$$\tilde{f}(x) = -\frac{2}{\pi}\int_{0^+}^\pi \psi_x(t)\frac{1}{2}\cot\frac{1}{2}t\,dt \tag{4.4}$$

$$g(x) = \frac{2}{\pi}\int_{0^+}^\pi \psi_x(t)\frac{1}{2}\cot\left(\frac{1}{2}t\right)\log\left(\frac{1}{2}\csc\frac{1}{2}t\right)dt \tag{4.5}$$

$$\chi_x(0^+) = \int_{0^+}^\pi \varphi_x(u)\frac{1}{2}\cot\frac{1}{2}u\,du \tag{4.6}$$

$$h_x(t) = \frac{\pi}{2}\theta_x(t) - \frac{t}{2}\tilde{f}(x), \qquad 0 < t \leq \pi,$$

and defined elsewhere by periodicity with period 2π.

4.2 Main Results

It was Prössdorf [28] who initiated the work on the degree of approximations of the H_α class in the Hölder metric by Fejer means of the Fourier series. Chandra [2] obtained a generalization of Prössdorf's work on the Nörlund mean set-up. Later Mohapatra and Chandra [23] considered the problem by matrix means. Chandra [2], [3] also studied the degree of approximation of functions of the H_α class in the Hölder metric by their Fourier series using Borel's exponential means and Euler means. Das, Ojha, and Ray [5] have studied the degree of approximation of the integral

$$\chi_x(0^+) = \int_{0^+}^{\pi} \varphi_x(u) \frac{1}{2} \cot \frac{1}{2} u \, du$$

By the Euler, Borel, and (e,c) transforms of the HL-series in the Hölder metric.

Das, Ray, and Sadangi [6] obtained the following result on the rate of convergence of the series (1.10) to the integral $g(x)$ in the Hölder metric.

Let

$$\tilde{T}_n(x) = \sum_{k=1}^{n} \frac{\tilde{S}_n^*(x) - \tilde{f}(x)}{n}, n \geq 1$$

and zero otherwise. Let $E_n^q(\tilde{T}; x), B_p(\tilde{T}; x)$ and $e_n(\tilde{T}; x)$ be respectively the (E, q), Borel, and (e, c) means of $\{\tilde{T}_n(x)\}$.

We prove the following theorems.

Theorem 1. Let $0 \leq \beta < \alpha \leq 1$ and let $f \in H_{\alpha,p}$. Then

$$||E_n^q(\tilde{T}) - g||_{\beta,p} = O(1) \begin{cases} \frac{1}{n^{\alpha-\beta}}, \alpha - \beta \neq 1 \\ \frac{\log n}{n}, \alpha - \beta = 1 \end{cases} \tag{4.7}$$

Theorem 2. Let $0 \leq \beta < \alpha \leq 1$ and let $f \in H_{\alpha,p}$. Then

$$||B_p(\tilde{T}) - g||_{\beta,p} = O(1)\begin{cases} \dfrac{1}{p^{\alpha-\beta}}, \alpha - \beta \neq 1 \\ \dfrac{\log p}{p}, \alpha - \beta = 1 \end{cases} \tag{4.8}$$

Theorem 3. Let $0 \leq \beta < \alpha \leq 1$ and let $f \in H_{\alpha,p}$. Then

$$\left\| e_n(\tilde{T}) - g \right\|_{\beta,p} = O(1)\begin{cases} \dfrac{1}{n^{\alpha-\beta}}, 0 < \alpha - \beta \leq \dfrac{1}{2} \\ \dfrac{1}{\sqrt{n}}, \quad \dfrac{1}{2} < \alpha - \beta \leq 1 \end{cases} \tag{4.9}$$

In proving these theorems our main observation is that the kernels for Euler, Borel, and (e,c) means have some important characteristics in common even though they appear to be different. In what follows, we shall prove our theorems in a unified manner by taking full advantage of the common properties possessed by the kernels of Euler, Borel, and (e,c) means.

Recall the series (1.10). It is known from Zygmund [38] that

$$\tilde{S}_n^*(x) = -\frac{2}{\pi}\int_0^\pi \psi_x(t)\frac{1-\cos nt}{2\tan\frac{1}{2}t}\,dt$$

from which it follows that

$$\tilde{S}_n^*(x) - \tilde{f}(x) = \frac{2}{\pi}\int_0^\pi \psi_x(t)\frac{\cos nt}{2\tan\frac{1}{2}t}\,dt \tag{4.10}$$

For $n \geq 1$, we have, for the odd function $h_x(t)$,

$$c_n = \frac{2}{\pi}\int_0^h h_x(t)\sin nt\, dt$$

$$= \frac{2}{\pi}\left\{\int_0^\pi \frac{\pi}{2}\theta_x(t) - \frac{t}{2}\tilde{f}\,(x)\right\}\sin nt\, dt$$

$$= \frac{1}{\pi}\left[\{t\tilde{f}(x) - \pi\theta_x(t)\}\frac{\cos nt}{n}\right]_{t=0}^{\pi}$$

$$-\frac{1}{\pi}\int_0^\pi (\tilde{f}(x) + \psi_x(t)\cot\frac{1}{2}t)\frac{\cos nt}{n}\, dt$$

$$= -\frac{2}{n\pi}\int_0^\pi \psi_x(t)\frac{\cos nt}{2\tan\frac{1}{2}t}dt$$

$$= -\frac{\tilde{S}_n^*(x) - \tilde{f}(x)}{n} \qquad (4.11)$$

The series conjugate to $h(t) = \sum_{n=1}^{\infty} c_n \sin nt$ is $-\sum_{n=1}^{\infty} c_n \cos nt$ and hence, we have:

Proposition: The series (1.10) is the series conjugate to the Fourier series of the odd function $h_x(t)$ at $t = 0$.

In this case,

$$\tilde{T}_n(x) = -\sum_{k=1}^{n} c_k = -\frac{2}{\pi}\int_0^\pi h_x(t)\widetilde{D}_n(t)dt, \qquad (4.12)$$

where

$$\widetilde{D}_n(t) = \sum_{k=1}^{n}\sin kt\frac{\cos\frac{1}{2}t - \cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t} \qquad (4.13)$$

At this stage we may note that Das, Ojha, and Ray [5] have established the Fourier character of the HL-series (1.7).

4.3 Notations and Lemmas

Throughout the section, we use the following additional notations:

$$G(x,y) = g(x) - g(y)$$

$$\tilde{F}(x,y) = \tilde{f}(x) - \tilde{f}(y)$$

$$G(t) = \theta_x(t) - \theta_y(t)$$

We need the following lemmas for proof of our theorems.

Lemma 1. Let $0 \leq \beta < \alpha \leq 1$. If $f \in H_\alpha$, then for $0 < t \leq \pi$

$$||\psi_{y+u}(t) - \psi_y(t)||_p = O(t^\alpha) \tag{4.14}$$

$$= O(|u|^\alpha) \tag{4.15}$$

$$= O\left(|u|^\beta t^{\alpha-\beta}\right) \tag{4.16}$$

$$\theta_{y+u}(t) = O(t^\alpha) \tag{4.17}$$

$$\|G(t)\|_p = O\left(|u|^\beta t^{\alpha-\beta}\right) \tag{4.18}$$

Proof: The proof of (4.14) and (4.15) are omitted as they are immediate consequences of the definition of $\psi_y(t)$ and $H_{\alpha,p}$. Writing

$$\left|\psi_{y+u}(t) - \psi_y(t)\right| = |\psi_{y+u}(t) - \psi_y(t)|^{1-\beta/\alpha}|\psi_{y+u}(t) - \psi_y(t)|^{\beta/\alpha}$$

and using the estimates (4.14) and (4.15), we obtain (4.16).

As

$$\theta_x(t) = -\frac{2}{\pi}\int_0^t \psi_x(u)\frac{1}{2}\cot\frac{1}{2}u\,du\,,$$

estimate (4.17) follows from the fact that $\psi_x(u) = O(u^\alpha)$.

As

$$\|G(t)\|_p = \left\|-\frac{2}{\pi}\int_0^t \frac{\psi_{y+u}(\xi) - \psi_y(\xi)}{2\tan\frac{1}{2}(\xi)}d\xi\right\|_p,$$

estimate (4.18) follows by applying (4.16).

Lemma 2. Let $f \in H_{\alpha,p}$ and $0 \leq \beta < \alpha \leq 1$. Then

$$\|G(t+h) - G(t)\|_p = O\left(h|u|^\beta t^{\alpha-\beta-1}\right)$$

Proof. Applying the Mean value theorem and (4.16), we obtain for some θ with $0 < \theta < 1$

$$G(t+h) - G(t) = hG'(t+\theta h)$$

$$= h\left[\frac{2}{\pi}\left\{\psi_{y+u}(t+\theta h) - \psi_y(t+\theta h)\right\}\frac{1}{2}\cot\frac{1}{2}(t+\theta h)\right].$$

$$\|G(t+h)-G(t)\|_p = O\left(h|u|^\beta (t+\theta h)^{\alpha-\beta-1}\right)$$

$$= O\left(h|u|^\beta t^{\alpha-\beta-1}\right).$$

Lemma 3. Let $0 \le \beta < \alpha \le 1$ and let $f \in H_{\alpha,p}$. Then

(i) $\tilde{F}(x,y) = O\left(|u|^\beta\right)$

(ii) $G(x,y) = O\left(|u|^\beta\right)$

Proof. Since $\widetilde{F}(y+u,y) = \tilde{f}(y+u) - \tilde{f}(y) = \theta_{y+u}(\pi) - \theta_y(\pi) = G(\pi)$

Lemma 3(i) follows from (4.18). Using (4.16), we have

$$\|G(y+u,y)\|_p = \|g(y+u) - g(y)\|_p$$

$$\le \frac{2}{\pi}\int_0^\pi \left\|\psi_{y+u}(\xi) - \psi_y(\xi)\right\|_p \left|\frac{1}{2}\cot\frac{1}{2}\xi \log\frac{1}{2}\csc\frac{1}{2}\xi\ \right| d\xi$$

$$= O(1)|u|^\beta \int_0^\pi \xi^{\alpha-\beta-1}\log\frac{2\pi}{\xi}\,d\xi,$$

which ensures Lemma (ii) as the last integral is finite.

Lemma 4 (Das, Ojha, and Ray [5]). Suppose that A and δ are both positive constants. Let β be any real number. Then as $\lambda \to \infty$,

$$\int_{\pi/\lambda}^{\delta} t^\beta e^{-A\lambda t^2}\,dt = O\left(\lambda^{-\beta-1}\right), \beta < -1 \tag{4.19}$$

$$\int_{\pi/\lambda}^{\delta} t^\beta e^{-A\lambda t^2}\,dt = O(1)\log\lambda, \qquad \beta = -1 \tag{4.20}$$

$$\int_{\pi/\lambda}^{\delta} t^{\beta} e^{-A\lambda t^2} dt = O(1)\frac{1}{\lambda^k}, 2k-1 < \beta \le 2k, \;\; k = 0,1,2,\ldots \tag{4.21}$$

$$\int_{\pi/\lambda}^{\delta} t^{\beta} e^{-A\lambda t^2} dt = O(1)\frac{1}{\lambda^{\beta-k}}, 2k \le \beta \le 2k+1, k = 0,1,2,\ldots \tag{4.22}$$

Lemma 5. Let $c(\lambda, t)$ be defined for all $\lambda \ge 0$ and $0 \le t \le \pi$. Suppose that

(i) $c(\lambda, t) \ge 0$ for all $\lambda \ge 0$ and $0 \le t \le \pi$

(ii) $c(\lambda, t)$ is monotonic decreasing in t over $[0, \pi]$ for each positive constant A.

(iii) $c(\lambda, t) = O(e^{-A\lambda t^2})$ as $\lambda \to \infty, 0 < t \le \pi$ for some positive constant A.

(iv) $c(\lambda, t) - c(\lambda, t+h) = O\big(tc(\lambda, t)\big), \frac{\pi}{\lambda} < t \le \pi$, where $h = \pi/\lambda$.

Let $\theta_x(t)$ and $G(t)$ be respectively defined as in §1 and §4. If $f \in H_{\alpha,p}, 0 \le \beta < \alpha \le 1$ then for $0 < \delta \le \pi$

$$\text{a)} \;\; \int_{\frac{\pi}{\lambda}}^{\delta} \frac{G(t)}{t} c(\lambda, t) \sin \lambda t \, dt = O(1)|u|^{\beta} \begin{cases} \dfrac{1}{\lambda^{\alpha-\beta}} & \alpha - \beta \ne 1 \\ \dfrac{\log \lambda}{\lambda} & \alpha - \beta = 1 \end{cases} \tag{4.23}$$

$$\text{b)} \;\; \int_{\frac{\pi}{\lambda}}^{\delta} \frac{\theta_x(t)}{t} c(\lambda, t) \sin \lambda t \, dt = O(1)|u|^{\beta} \begin{cases} \dfrac{1}{\lambda^{\alpha}} & 0 < \alpha < 1 \\ \dfrac{\log \lambda}{\lambda} & \alpha = 1 \end{cases} \tag{4.24}$$

Proof of (a) Putting $h = \pi/\lambda$, we write

$$J = \int_h^\delta \frac{G(t)}{t} c(\lambda, t) \sin \lambda t \, dt \tag{4.25}$$

$$= \left(\int_{2h}^{\delta+h} + \int_h^{2h} - \int_\delta^{\delta+h} \right) \frac{G(h)}{t} c(\lambda, t) \sin \lambda t \, dt$$

$$- \int_h^\delta \frac{G(t+h)}{t+h} c(\lambda, t+h) \sin \lambda t \, dt + \int_h^{2h} \frac{G(t)}{t} c(\lambda, t) \sin \lambda t \, dt$$

$$- \int_\delta^{\delta+h} \frac{G(t)}{t} c(\lambda, t) \sin \lambda t \, dt \tag{4.26}$$

From (4.25) and (4.26) we obtain

$$2J = \int_h^\delta \left[\frac{G(t)}{t} c(\lambda, t) - \frac{G(t+h)}{t+h} c(\lambda, t+h) \right] sin \, \lambda t \, dt$$

$$+ \int_h^{2h} \frac{G(t)}{t} c(\lambda, t) \sin \lambda t \, dt - \int_\delta^{\delta+h} \frac{G(t)}{t} c(\lambda, t) \sin \lambda t \, dt$$

$$= P + Q - R \tag{4.27}$$

By Minkowski's Inequality,

$$2\|J\|_p \leq \|P\|_p + \|Q\|_p + \|R\|_p.$$

Since,

$$P = \int_h^\delta \left[\frac{G(t)}{t} c(\lambda, t) - \frac{G(t+h)}{t+h} c(\lambda, t+h) \right] \sin \lambda t \, dt$$

$$= \int_{\lambda}^{\delta} \frac{G(t) - G(t+h)}{t} c(\lambda, t) \sin \lambda t \, dt$$

$$+ \int_{h}^{\delta} \frac{G(t+h)}{t} [c(\lambda, t) - c(\lambda, t+h)] \sin \lambda t \, dt$$

$$+ \int_{\lambda}^{\delta} G(t+h) \left[\frac{1}{t} - \frac{1}{t+h}\right] c(\lambda, t+h) \sin \lambda t \, dt$$

$$= P_1 + P_2 + P_3 \tag{4.28}$$

By Minkowski's inequality,

$$\|P\|_p \leq \|P_1\|_p + \|P_2\|_p + \|P_3\|_p.$$

By Lemma 8 from Chapter 3 and the method of proof of Theorem 1 of Chapter 3, we have,

$$\|P_1\| = O(1)|u|^{\beta} h \int_{h}^{\delta} t^{\alpha-\beta-2} dt,$$

$$= O(1)|u|^{\beta} \begin{cases} h^{\alpha-\beta}, & \alpha - \beta \neq 1 \\ h \log h^{-1}, & \alpha - \beta = 1 \end{cases} \tag{4.29}$$

Where we have used Lemma 2 and (iii) of Lemma 5.

By Lemma 1 and the definition of $c(\lambda, t)$, and using the method similar to that used to obtain (4.29), we get

$$\|P_2\|_p = O(1)|u|^{\beta} \int_{h}^{\delta} t^{\alpha-\beta-1} \, tc(\lambda, t) dt$$

$$= O(1)|u|^{\beta} \int_{h}^{\delta} t^{\alpha-\beta} c^{-A\lambda t^2} \, dt$$

$$= O(1)|u|^{\beta} h^{\alpha-\beta}, 0 \leq \beta < \alpha \leq 1, \tag{4.30}$$

using Lemma 4.

Adopting the technique similar to those used in splitting *J* into *P, Q,* and *R*, we can write

$$2P_3 = \int_h^\delta \left[G(t+h)\left(\tfrac{1}{t}-\tfrac{1}{t+h}\right)c(\lambda,t+h) - G(t+2h)\left(\tfrac{1}{t+h}-\tfrac{1}{t+2h}\right)c(\lambda,t+2h)\right]\sin\lambda t\,dt$$

$$+h\int_h^{2h}\frac{G(t+h)}{t(t+h)}c(\lambda,t+h)\sin\lambda t\,dt - \int_\delta^{\delta+h}\frac{G(t+h)}{t(t+h)}c(\lambda,t+h)\sin\lambda t\,dt$$

$$= L + M - N \tag{4.31}$$

By Minkowski's inequality,

$$2\|P_3\| \le \|L\|_p + \|M\|_p + \|N\|_p.$$

We have,

$$L = \int_h^\delta \left[G(t+h)\left(\frac{1}{t}-\frac{1}{t+h}\right)c(\lambda,t+h)\right.$$

$$\left. -G(t+2h)\left(\frac{1}{t+h}-\frac{1}{t+2h}\right)c(\lambda t+2h)\right]\sin\lambda t\,dt$$

$$= h\int_h^\delta \frac{G(t+h)-G(t+2h)}{t(t+h)}c(\lambda,t+h)\sin\lambda t\,dt$$

$$+h\int_h^\delta \frac{G(t+2h)}{t(t+h)}[c(\lambda,t+h)-c(t\,\lambda,t+2h)]\sin\lambda t\,dt$$

$$+h\int_h^\delta \frac{g(t+2h)}{t+h}\left(\frac{1}{t}-\frac{1}{t+2h}\right)c(\lambda,t+2h)\sin\lambda t\,dt$$

$$= L_1 + L_2 + L_3 \tag{4.32}$$

By Minkowski's inequality,

$$\|L\|_p \leq \|L_1\|_p + \|L_2\|_p + \|L_3\|_p.$$

By Lemma 2 and Lemma 8 of Chapter 3, we get

$$\|L_1\|_p \leq |h| \int_h^\delta \frac{\|G(t+h) - G(t+2h)\|_p}{|t(t+h)|} |c(\lambda, t+h) \sin \lambda t \,| dt$$

$$= O(1)|u|^\beta h^2 \int_h^\delta \frac{(t+h)^{\alpha-\beta-1}}{t(t+h)} e^{-At^2} dt$$

$$= O(1)|u|^\beta h^2 \int_h^\delta t^{\alpha-\beta-3} dt$$

$$= O(1)|u|^\beta h^{\alpha-\beta}, 0 \leq \beta < \alpha \leq 1 \tag{4.33}$$

Using Lemma 1 and properties of $c(\lambda, t)$, we get

$$\|L_2\|_p \leq |h| \int_h^\delta \frac{\|G(t+2h)\|_p}{|t(t+h)|} |[c(\lambda, t+h) - c(\lambda, t+2h)] \sin \lambda t \,| dt$$

$$= O(1)|u|^\beta h \int_h^\delta \frac{(t+2h)^{\alpha-\beta}}{t(t+h)} (t+h) c(\lambda, t+h) dt$$

$$= O(1)|u|^\beta h \int_h^\delta t^{\alpha-\beta-1} dt$$

$$= O(1)|u|^\beta h \,, 0 \leq \beta < \alpha \leq 1 \tag{4.34}$$

and

$$\|L_3\|_p \leq |h| \int_h^\delta \frac{\|G(t+2h)\|_p}{|t+h|} \left| \left(\frac{1}{t} - \frac{1}{t+2h} \right) c(\lambda, t+2h) \sin \lambda t \right| dt$$

$$= O(1)|u|^{\beta}h^{2}\int_{h}^{\delta}\frac{(t+2h)^{\alpha-\beta}}{t(t+h)(t+2h)}e^{-A\lambda t^{2}}dt$$

$$= O(1)|u|^{\beta}h^{2}\int_{h}^{\delta}t^{\alpha-\beta-3}dt$$

$$= O(1)|u|^{\beta}h^{\alpha-\beta}, 0 \le \beta < \alpha \le 1 \qquad (4.35)$$

Using Lemma 1 and the boundedness of $c(\lambda, t)$. We get

$$\|M\|_{p} \le \int_{h}^{2h}\frac{\|G(t+h)\|_{p}}{|t(t+h)|}|c(\lambda, t+h)\sin\lambda t\,|\,dt$$

$$= O(1)|u|^{\beta}h\int_{h}^{2h}t^{\alpha-\beta-2}dt$$

$$= O(1)|u|^{\beta}\begin{cases} h^{\alpha-\beta} & \alpha-\beta \ne 1 \\ h\log 2 & \alpha-\beta = 1\end{cases} \qquad (4.36)$$

and

$$\|N\|_{p} \le |h|\int_{\delta}^{\delta+h}\frac{G(t+h)}{|t(t+h)|}|c(\lambda, t+h)\sin\lambda t\,|dt$$

$$= O(1)|u|^{\beta}h\int_{\delta}^{\delta+h}t^{\alpha-\beta-2}dt$$

$$= O(1)|u|^{\beta}h, 0 \le \beta < \alpha \le 1 \qquad (4.37)$$

Collecting the results from (4.31) – (4.37), we obtain

$$\|P_{3}\|_{p} = O(1)|u|^{\beta}h^{\alpha-\beta}, 0 \le \beta < \alpha \le 1. \qquad (4.38)$$

Combining the results of (4.28), (4.29), (4.30), and (4.38), we have

$$\|P_3\|_p = O(1)|u|^\beta \begin{cases} h^{\alpha-\beta}, & \alpha-\beta \neq 1 \\ h\log h^{-1}, & \alpha-\beta = 1 \end{cases} \tag{4.39}$$

By Lemma 1, we have, for $0 \leq \beta < \alpha \leq 1$

)

$$\|R\|_p \leq \int_\delta^{\delta+h} \frac{\|G(t)\|_p}{|t|} |c(\lambda,t)||\sin\lambda t\,|dt \tag{4.40}$$

$$= O(1)|u|^\beta e^{-A\lambda\delta^2}$$

$$= O(1)|u|^\beta \lambda^{-\Delta} \tag{4.41}$$

for every positive Δ, however large. Collecting the above estimates for $P, Q,$ and R, we obtain

$$J = O(1)|u|^\beta \begin{cases} h^{\alpha-\beta}, & \alpha-\beta \neq 1 \\ h\log h^{-1}, & \alpha-\beta = 1 \end{cases}$$

and this completes the proof of part (a). We omit the proof of (b) because it is similar to that of part (a). The case where $\sin\lambda t$ is replaced with $\cos\lambda t$ can also be dealt with in a similar manner.

Lemma 6. If $f \in H_{\alpha,p}, 0 \leq \beta < \alpha \leq 1$, then as $\lambda \to \infty$

(a) $$\int_0^{\pi/\lambda} \frac{\|G(t)\|_p}{|t|} dt = \frac{O(1)|u|^\beta}{\lambda^{\alpha-\beta}}$$

(b) $$\int_0^{\pi/\lambda} \frac{||\theta_x(t)||_p}{|t|} dt = O(1)\lambda^{-\alpha}$$

Proof: The result follows from Lemma 1.

Lemma 7. Let $0 \leq \beta < \alpha \leq 1$ and let $f \in H_{\alpha,p}$. Then as $\lambda \to \infty$

(a) $$\int_\delta^{\pi} \frac{||G(t)||_p}{|t|} \left|e^{-A\lambda t^2}\right| dt = O(1)|u|^\beta \lambda^{-\Delta}$$

and

(b) $$\int_\delta^{\pi} \frac{||\theta_x(t)||_p}{|t|} e^{-A\lambda t^2} dt = O(1)\lambda^{-\Delta}$$

Proof: By Lemma 1

$$\int_\delta^{\pi} \frac{|G(t)|}{t} e^{-A\lambda t^2} dt = O(1)|u|^\beta \int_\delta^{\pi} t^{\alpha-\beta-1} e^{-A\lambda t^2} dt$$

$$= O(1)u^\beta e^{-A\lambda\delta^2}$$

$$= O(1)|u|^\beta \lambda^{-\Delta}, \Delta > 0$$

Part (b) can be dealt with in a similar fashion.

4.4 Proof of Theorem 1

We will use the following additional notations for the proof of Theorem 1.

$$l_n^E(x) = E_n^q(x) - g(x)$$

$$p_q^n(t) = (q+1)^{-n}(1+q^2+2q\cos t)^{\frac{n}{2}}$$

$$\Theta = \Theta(t) = \tan^{-1}\frac{\sin t}{q+\cos t}$$

$$P(n,t) = (q+1)^{-1}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\cos(k+\frac{1}{2})\,t, q>0$$

$$Q(n,t) = (q+1)^{-1}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\sin(k+\frac{1}{2})\,t, q>0$$

$$E(n) = (q+1)^{-n}\sum_{k=1}^{n}\binom{n}{k}q^{n-k}\sum_{\nu=k+1}^{\infty}\frac{(-1)^{\nu-1}}{\nu}$$

$$\lambda = \frac{n}{1+q}+\frac{1}{2}$$

We need the following lemmas.

Lemma 8. Let $0 < t \le \pi$. Then

$$p_q^n(n) \le e^{-Ant^2}, \tag{4.42}$$

where $A = 2q[\pi(1+q)]^{-2}$

Lemma 9. For $0 < t \le \pi$,

(i) $P(n,t) = p_q^n(t)\cos\left(n\Phi + \frac{1}{2}t\right)$ (4.43)

(ii) $P(n,t) = O(1)$ (4.44)

(iii) $E(n) = O(n^{-1})$ (4.45)

Proof: By simple computation, we have

$$P(n,t) + iQ(n,t) = (q+1)^{-1}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}e^{i\left(k+\frac{1}{2}\right)t}$$

$$= (q+1)^{-n}e^{i\frac{1}{2}t}\left(q+e^{it}\right)^{n}$$

$$= p_q^n(t)e^{i\frac{1}{2}t}\left[\cos\left(n\Phi + \frac{1}{2}t\right) + i\sin\left(n\Phi + \frac{1}{2}t\right)\right]$$

from which (ii) follows. As $\left|\cos\left(k+\frac{1}{2}\right)t\right| \le 1$ and $\sum_{k=0}^{n}\binom{n}{k}q^{n-k} = (1+q)^n$, estimate (iii) follows.

As

$$\sum_{\nu=k+1}^{\infty}\frac{(-1)^{\nu-1}}{\nu} = O(\frac{1}{k+1})$$

We have,

$$(q+1)^n E(n) = \sum_{k=1}^{n}\binom{n}{k}q^{n-k}\sum_{\nu=k+1}^{\infty}\frac{(-1)^{\nu-1}}{\nu}$$

$$= O(1)\sum_{k=1}^{n}\binom{n}{k}q^{n-k}\frac{1}{k+1} \tag{4.46}$$

Now

$$\sum_{k=1}^{n}\binom{n}{k}q^{n-k}\frac{1}{k+1} = \frac{1}{n+1}\sum_{k=2}^{n+1}\binom{n+1}{k}q^{n+1-k}$$

$$< \frac{1}{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k}q^{n+1-k}$$

$$= \frac{(q+1)^{n+1}}{n+1} \tag{4.47}$$

Using (4.47) in (4.46) we obtain (4.45).

Lemma 10. Let $\lambda = \frac{n}{1+q} + \frac{1}{2}, h = \frac{\pi}{\lambda}$, and $0 < t < \frac{\pi}{4}$. Then for $h \le t < \delta$

$$p_q^n(t+h) - p_q^n(t) = O(1)tp_q^n(t) \tag{4.48}$$

Proof: By the Mean Value Theorem, we have for some ξ with $0 < \xi < 1$

$$p_{\boldsymbol{q}}^{\boldsymbol{n}}(t+h) - p_q^n(t) = h\left[\frac{d}{dx}p_q^n(x)\right]$$

$$= \frac{-nhp_q^n(t+\xi h)}{1+q^2+2q\cos(1+\xi h)}\sin(t+\xi h)$$

$$= O(1)tp_q^n(t)$$

Lemma 11. Let $\lambda = \frac{n}{1+q} + \frac{1}{2}$ and $0 < \delta < \frac{\pi}{4}$. Then for $0 < t < \delta$

$$\cos\left(n\Phi + \frac{1}{2}t\right) - \cos\lambda t = O(nt^3) \tag{4.49}$$

Proof: We have

$$\left|\cos\left(n\Phi + \frac{1}{2}t\right) - \cos\lambda t\right| = \left|2\sin\frac{1}{2}\left(n\Phi + \frac{1}{2}t + \lambda t\right)\sin\frac{1}{2}\left(\lambda t - n\Phi - \frac{1}{2}t\right)\right|$$

$$\leq |\lambda t - n\Phi - \frac{1}{2}t|$$

$$= n|\Phi - \frac{t}{1+q}|$$

$$\leq \left[\left|\tan^{-1}\ \frac{\sin t}{q + cost} - \frac{\sin t}{q + cost}\right| + \left|\frac{\sin t}{q + cost} - \frac{t}{1+q}\right|\right]$$

$$= n\left[O\left(\left(\frac{\sin t}{q + \cos t}\right)^3\right) + O(t^3)\right]$$

$$= O(nt^3)$$

Proof of Theorem 1. Using (4.10), we have

$$\tilde{T}_n(x) = \sum_{k=1}^{n} \frac{\tilde{S}_k^*(x) - \tilde{f}(x)}{k}$$

$$= -\frac{2}{\pi}\int_0^\pi h_x(x)\widetilde{D}_x(t)dt$$

$$= -\frac{2}{\pi}\int_0^{\pi}\left[\frac{t}{2}\tilde{f}(x) - \frac{\pi}{2}\theta_x(t)\right]\widetilde{D}_n(t)dt$$

$$= -\frac{1}{\pi}\tilde{f}(x)\int_0^{\pi} t\,\widetilde{D}_n(t)dt - \int_0^{\pi}\theta_x(t)\,\widetilde{D}_n(t)dt$$

$$= -\tilde{f}(x)\sum_{k=1}^{n}\frac{\cos k\pi}{k} - \int_0^{\pi}\theta_x(t)\frac{\cos\frac{1}{2}t - \cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}\,dt$$

$$= \tilde{f}(x)\left[\log 2 - \sum_{\nu=n+1}^{\infty}\frac{(-1)^{\nu-1}}{\nu}\right] - \int_0^{\pi}\theta_x(t)\frac{1}{2}\cot\frac{1}{2}t\,dt + \int_0^{\pi}\theta_x(t)\frac{\cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}\,dt \quad (4.50)$$

Now,

$$\int_0^{\pi}\theta_x(t)\frac{1}{2}\cot\frac{1}{2}t\,dt = -\frac{2}{\pi}\int_0^{\pi}\frac{1}{2}\cot\frac{1}{2}t\,dt\left(\int_0^{t}\psi_x(u)\frac{1}{2}\cot\frac{1}{2}u\,du\right)$$

$$= -\frac{2}{\pi}\int_0^{\pi}\psi_x(u)\frac{1}{2}\cot\frac{1}{2}u\,du\int_u^{\pi}\frac{1}{2}\cot\frac{1}{2}t\,dt$$

$$= -\frac{2}{\pi}\int_0^{\pi}\psi_x(u)\frac{1}{2}\cot\frac{1}{2}u\left[\log\frac{1}{2}\csc\frac{1}{2}u + \log 2\,du\right]$$

$$= -g(x) + \tilde{f}(x)\,\log 2 \quad (4.51)$$

From (4.4) and (4.5), it follows that, for $n \geq 1$,

$$\tilde{T}_n(x) = g(x) + \int_0^{\pi}\theta_x(t)\frac{\cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}dt - \tilde{f}(x)\sum_{\nu=n+1}^{\infty}\frac{(-1)^{\nu-1}}{\nu} \quad (4.52)$$

Using (4.6) in (4.5), we obtain,

$$E_n^q(\tilde{T};x) = (q+1)^{-n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\tilde{T}_k\,(x)$$

$$= g(x)(q+1)^{-n}\sum_{k=1}^{n}\binom{n}{k} q^{n-k}$$

$$+\int_0^{\pi}\frac{\theta_x(t)}{2\sin\frac{1}{2}t}\left\{(q+1)^{-n}\sum_{k=1}^{n}\binom{n}{k}\; q^{n-k}\;\cos\left(k+\frac{1}{2}\right)t\right\}dt$$

$$-\tilde{f}(x)(q+1)^{-n}\sum_{k=1}^{n}\binom{n}{k} q^{n-k}\;\sum_{\nu=K=1}^{\infty}\frac{(-1)^{\nu-1}}{\nu}$$

$$= g(x)\left(1-\left(\frac{q}{1+q}\right)^n\right)+\int_0^{\pi}\frac{\theta_x(t)}{2\sin\frac{1}{2}t}\left\{P(n,t)-\left(\frac{q}{q+1}\right)^n\cos\frac{1}{2}t\right\}dt-\tilde{f}(x)E(n),\quad (4.53)$$

which ensures that

$$l_n^E(x) = E_n^q(\tilde{T};x) - g(x)$$

$$= -\left(\frac{q}{1+q}\right)^n g(x)+\int_0^{\pi}\frac{\theta_x(t)}{2\sin\frac{1}{2}t}P(n,t)dt$$

$$-\int_0^{\pi}\theta_x(t)\left(\frac{q}{1+q}\right)^n\frac{1}{2}\cot\frac{1}{2}t\;dt-\tilde{f}(x)E(n)\qquad (4.54)$$

Hence,

$$l_n^E(y+u)-l_n^E(y) = -\left(\frac{q}{1+q}\right)^n G(y+u,y)+\int_0^{\pi}\frac{G(t)}{2\sin\frac{1}{2}t}P(n,t)dt$$

$$-\left(\frac{q}{1+q}\right)^n \int_0^\pi \frac{G(t)}{2\tan\frac{1}{2}t} dt - E(n)\tilde{F}(x,y)$$

$$= -P(E) + Q(E) - R(E) + S(E). \tag{4.55}$$

By Minkowski's inequality,

$$\|l_n^E(y+u) - l_n^E(y)\| \le \|P(E)\|_p + \|Q(E)\|_p + \|R(E)\|_p + \|S(E)\|_p$$

By Lemma 3,

$$\|P(E)\|_p = \left(\frac{q}{1+q}\right)^n \||G(y+u,y)|\|_p = O\left(|u|^\beta\right)\left(\frac{q}{1+q}\right)^n \tag{4.56}$$

By Lemma 3 and Lemma 9 (iii)

$$\|G(y+u,y)\|_p = \left|-\tilde{F}(x,y)E(n)\right| = O\left(|u|^\beta n^{-1}\right) \tag{4.57}$$

Using Lemma 1, we get

$$\|G(y+u,y)\|_p \le \left(\frac{q}{1+q}\right)^n \int_0^\pi \frac{\|G(t)\|_p}{\left|2t\tan\frac{1}{2}\right|} dt$$

$$= O(1)|u|^\beta \left(\frac{q}{1+q}\right)^n \int_0^\pi t^{\alpha-\beta-1}\, dt$$

$$= O(1)|u|^\beta \left(\frac{q}{1+q}\right)^n \tag{4.58}$$

We put $\lambda = \frac{n}{1+q} + \frac{1}{2}$. Now for fixed δ with $0 < \delta < \pi/4$ we split the integral $Q(E)$ as follows:

$$\|Q(E)\|_p \le \left[\int_0^{\pi/\lambda} + \int_{\pi/\lambda}^{\delta} + \int_\delta^\pi\right] \frac{\|Q(E)\|_p}{\left|2\sin\frac{1}{2}t\right|}\ |P(n,t)|\, dt$$

$$\le \|I(E)\|_p + \|J(E)\|_p + \|K(E)\|_p, \tag{4.59}$$

by Minkowski's inequality.

By Lemma 9 (ii) and Lemma 6,

$$\|I(E)\|_p \le \int_0^{\frac{\pi}{\lambda}} \frac{\|G(t)\|_p}{\left|2\sin\frac{1}{2}t\right|} |P(n,t)| dt$$

$$= O(1)|u|^\beta \int_0^{\pi/\lambda} \frac{|G(t)|}{t} dt$$

$$= \frac{O(1)|u|^\beta}{n^{\alpha-\beta}} \tag{4.60}$$

By Lemma 8, Lemma 9, and Lemma 7, we obtain

$$\|K(E)\|_p \le \int_\delta^\pi \frac{\|G(t)\|_p}{2\left|\sin\frac{1}{2}t\right|} \left|p_q^n(\mathrm{t}) \cos\left(n\Phi + \frac{1}{2}t\right)\right| dt$$

$$= O(1)|u|^\beta \int_\delta^\pi \frac{|G(t)|}{|t|} e^{-Ant^2} dt$$

$$= \frac{O(1)|u|^\beta}{n^\Delta}, \Delta > 0, \qquad \text{however large} \tag{4.61}$$

We write

$$\|J(E)\|_p \le \int_{\frac{\pi}{\lambda}}^\delta \frac{\|G(t)\|_p}{\left|2\sin\frac{1}{2}t\right|} |P(n,t)| dt$$

$$= \int_{\frac{\pi}{\lambda}}^\delta \|G(t)\|_p \left(\left|\frac{1}{2\sin\frac{1}{2}t} - \frac{1}{t}\right|\right) |\, p_q^n(t)| \left|\cos\left(n\Phi + \frac{1}{2}t\right) dt\right|$$

$$+ \int_{\frac{\pi}{\lambda}}^\delta \frac{G(t)}{t} p_q^n(t) \cos\lambda t \, dt$$

$$+ \int_{\frac{\pi}{\lambda}}^\delta \frac{G(t)}{t} p_q^n(t) \left\{\cos\left(n\Phi + \frac{1}{2}t\right) - \cos\lambda t\right\} dt$$

$$= J_1(E) + J_2(E) + J_3(E) \tag{4.62}$$

Using Lemma 1, Lemma 8, Lemma 4 and the fact that $\frac{1}{2\sin\frac{1}{2}t} - \frac{1}{t} = O(t)$, we obtain

$$\left\|J_1(E)\right\|_p \leq \int_{\frac{\pi}{\lambda}}^{\delta} \left\|G(t)\right\|_p \left\{\left|\frac{1}{2\sin\frac{1}{2}t} - \frac{1}{t}\right|\right\} \left|p_q^n(t)\cos(n\Phi + \frac{1}{2}t)\right| dt$$

$$= O(1)|u|^\beta \int_{\pi/\lambda}^{\delta} t^{\alpha-\beta+1} e^{-Ant^2} dt$$

$$= O(1)|u|^\beta \int_{\pi/n}^{\delta} t^{\alpha-\beta+1} e^{-Ant^2}$$

$$= \frac{O(1)|u|^\beta}{n^{-1}} \tag{4.63}$$

Using Lemma 1, Lemma 8, Lemma 11, and Lemma 4, we have

$$\left\|J_3(E)\right\|_p \leq \int_{\pi/\lambda}^{\delta} \frac{|G(t)|}{|t|} \left|p_q^n(t)\right| \left|\cos\left(n\Phi + \frac{1}{2}t\right) - \cos\lambda t\right| dt$$

$$= O(1)|u|^\beta n \int_{\pi/\lambda}^{\delta} t^{\alpha-\beta+2} e^{-Ant^2} dt$$

$$= O(1)|u|^\beta n \int_{\pi/n}^{\delta} t^{\alpha-\beta+2} e^{-Ant^2} dt$$

$$= \frac{O(1)|u|^\beta}{n^{\alpha-\beta}} \tag{4.64}$$

Collecting the estimates for $P(E),\ S(E),\ R(E),\ I(E),\ K(E),\ J_1(E)$ and $J_3(E)$ from (4.56), (4.57), (4.58), (4.60), (4.61), (4.63), and (4.64), we obtain

$$\|l_n^E(y+u) - l_n^E(y)\|_p \leq \frac{O(1)|u|^\beta}{n^{\alpha-\beta}} + J_2(x) \tag{4.65}$$

For $\lambda = \frac{n}{1+q} + \frac{1}{2}$

$$p_q^n(t) = p_q^{\left(\lambda - \frac{1}{2}\right)(1+q)}(t) = c(\lambda, t).$$

Therefore, we may write

$$\|l_n^E(y+u) - l_n^E(y)\|_p \leq \int_{\frac{\pi}{\lambda}}^{\delta} \frac{\|l_n^E(y+u) - l_n^E(y)\|_p}{|t|} \left|p_q^n(t) \cos \lambda t \right| dt$$

$$= \int_{\frac{\pi}{\lambda}}^{\delta} \frac{\|G(t)\|_p}{|t|} |c(\lambda, t) \cos \lambda t \,| dt. \tag{4.66}$$

Note that $c(\lambda, t)$ satisfies (i), (ii), (iii), and (iv) of Lemma 5. Therefore,

$$\|J_2(E)\|_p = O(1)|u|^\beta \begin{cases} \frac{1}{\lambda^{\alpha-\beta}}, \alpha - \beta \neq 1 \\ \frac{\log \lambda}{\lambda}, \alpha - \beta = 1 \end{cases}$$

$$= O(1)|u|^\beta \begin{cases} \frac{1}{n^{\alpha-\beta}}, \alpha - \beta \neq 1 \\ \frac{\log \lambda}{n}, \alpha - \beta = 1 \end{cases} \tag{4.67}$$

Which in conjunction with (4.65) gives us

$$\sup_{u \neq 0} \left|\Delta^\beta l_n^E(y+u, y)\right| = \sup_{u \neq 0} \frac{\|l_n^E(y+u) - l_n^E(y)\|_p}{|u|^\beta}$$

$$= O(1) \begin{cases} \frac{1}{n^{\alpha-\beta}}, & \alpha - \beta \neq 1 \\ \frac{\log n}{n}, & \alpha - \beta = 1 \end{cases} \tag{4.68}$$

Again $f \in H_{\alpha,p} \Rightarrow \|\theta_x(t)\|_p = O(t^\alpha)$ and so using Lemma 5(b), 6(b), 7(b), and proceeding as above, we obtain

$$\|l_n^E(\cdot)\|_p = O(1)\begin{cases} \dfrac{1}{n^\alpha}, 0 < \alpha < 1 \\ \dfrac{\log n}{n}, \quad \alpha = 1 \end{cases} \tag{4.69}$$

Combining (4.68) and (4.69), we get (4.10) and this completes the proof of Theorem 1.

4.5 Proof of Theorem 2

We will use the following additional notations and Lemmas for the proof of Theorem 2.

$$l_p^\beta(x) = B_p(\tilde{T}; x) - g(x)$$

$$H_p(t) = e^{-p}\sum_{n=0}^{\infty}\frac{p^n}{n!}\cos(n+\frac{1}{2})t$$

$$G_p(t) = e^{-p}\sum_{n=0}^{\infty}\frac{p^n}{n!}\sin\left(n+\frac{1}{2}\right)t$$

$$\lambda = p + \frac{1}{2}$$

$$B(p) = e^{-p}\sum_{n=1}^{\infty}\frac{p^n}{n!}\sum_{k=n+!}^{\infty}\frac{(-1)^{k-1}}{k}$$

Lemma 12. Let $0 < \delta < \pi/4$ and let $A = 2/\pi^2$. Then

(i) $e^{-p(1-\cos t)} = O(e^{-Apt^2})$ (4.70)

(ii) $e^{-p(1-\cos t)} - e^{-p(1-\cos(t+\pi/p))} = O(1)te^{-Apt^2}$ (4.71)

Lemma 13 For $0 < t \leq \pi$

(i) $H_p(t) = e^{-p(1-\cos t)} \cos(p \sin t + \frac{1}{2}t)$ (4.72)

(ii) $H_p(t) = O(1)$ (4.73)

(iii) $B(p) = O(\frac{1}{p})$ (4.74)

Proof: By simple computation we have

$$H_p(t) + iG_p(t) = e^{-p} \sum_{n=0}^{\infty} \frac{p^n}{n!} e^{i\left(n+\frac{1}{2}\right)t}$$

$$= e^{-p} e^{i\frac{1}{2}t} \sum_{n=0}^{\infty} \frac{\left(pe^{it}\right)^n}{n!}$$

$$= e^{-p} e^{i\frac{1}{2}t} e^{p\, e^{it}}$$

$$= e^{-p(1-\cos t)} e^{i(p \sin t + \frac{1}{2}t)},$$

which ensures (i). (ii) follows from (i).

Now,

$$B(p) = e^{-p} \sum_{n=1}^{\infty} \frac{p^n}{n!} \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{k}$$

$$= O(1) e^{-p} \sum_{n=1}^{\infty} \frac{p^n}{n!\,(n+1)}$$

$$= O(1) \frac{e^{-p}}{p} \sum_{n=1}^{\infty} \frac{p^{n+1}}{(n+1)!}$$

$$= O(1) \frac{e^{-p}}{p} e^{p}$$

$$= O\left(\frac{1}{p}\right)$$

Lemma 14. Let $\lambda = p + \frac{1}{2}$ and $0 < \delta < \pi/4$. Then for $0 < t < \delta$.

$$\cos(p \sin t + \frac{1}{2} t) - \cos \lambda t = O(pt^3) \tag{4.75}$$

Proof: Expressing the difference $\cos(p \sin t + \frac{1}{2} t) - \cos \lambda t$ as a product and making use of the fact that $\sin t - t = O(t^3)$ the estimate (4.75) can be established.

Proof of Theorem 2. From (4.52), we get for $n \geq 1$

$$\tilde{T}_n(x) - g(x) + \int_0^\pi \theta_x(t) \frac{\cos(n + \frac{1}{2} t)}{2 \sin \frac{1}{2} t} dt - \tilde{f}(x) \sum_{\nu=n+1}^{\infty} \frac{(-1)^{\nu-1}}{\nu} \tag{4.76}$$

Hence the Borel's exponential mean $B_p(\tilde{T}_n; x)$ of $\{\tilde{T}_n(x)\}$ is given by

$$B_p(\tilde{T}_n; x) = e^{-p} \sum_{n=1}^{\infty} \frac{p^n}{n!} \left[g(x) + \int_0^\pi \theta_x(t) \frac{\cos\left(n + \frac{1}{2}\right) t}{2 \sin \frac{1}{2} t} dt - \tilde{f}(x) \sum_{\nu=n+1}^{\infty} \frac{(-1)^{\nu-1}}{\nu} \right]$$

$$= (1 - e^{-p}) g(x) + \int_0^\pi \frac{\theta_x(t)}{2 \sin\left(\frac{1}{2} t\right)} (H_p(t) - e^{-p} \cos(\frac{1}{2} t)) dt - \tilde{f}\,(x) B(p),$$

which ensure that

$$l_p^\beta(x) = B_p(\tilde{T}, x) - g(x)$$

$$= e^{-p} g(x) + \int_0^\pi \frac{\theta_x(t)}{2 \sin \frac{1}{2} t} H_p(t) dt - e^{-p} \int_0^\pi \frac{\theta_x(t)}{2 \tan \frac{1}{2} t} dt - \tilde{f}(x) B(p) \tag{4.77}$$

Therefore,

$$\left\| l_n^\beta(y + u) - l_n^\beta(y) \right\|_p = \left\| e^{-p} G(x, y) + \int_0^\pi \frac{G(t)}{2 \sin \frac{1}{2} t} H_p(t) dt \right.$$

$$\left. - e^{-p} \int_0^\pi \frac{G(t)}{2 \tan \frac{1}{2} t} dt - \tilde{F}(x, y) B(p) \right\|_p$$

$$:= \|P(B)+Q(B)-R(B)-S(B)\|_p \tag{4.78}$$

By Minkowski's inequality,

$$\left\| l_n^{\beta}(y+u)-l_n^{\beta}(y) \right\|_p \le \|P(B)\|_p+\|Q(B)\|_p+\|R(B)\|_p+\|S(B)\|_p$$

Using the estimates for $G(y+u,y), G(t), \tilde{F}(x,y)$ and $B(p)$ and adopting the technique employed for deriving the estimates for $P(E), R(E),$ and $S(E)$ in §5, it can be shown that

$$\|P(B)\|_p = O(1)|u|^{\beta}\, e^{-p}, \tag{4.79}$$

$$\|R(B)\|_p = O(1)|u|^{\beta} e^{-p}, \tag{4.80}$$

and

$$\|S(B)\|_p = O(1)|u|^{\beta} p^{-1} \tag{4.81}$$

We put $\lambda = p+\frac{1}{2}$. Now for fixed δ with $0<\delta<\pi/4$, we write

$$\|Q(B)\|_p \le \left\| \left[\int_0^{\frac{\pi}{\lambda}} + \int_{\frac{\pi}{\lambda}}^{\delta} + \int_{\delta}^{\pi} \right] \frac{G(t)}{2\sin\frac{1}{2}t}\, H_p(t)dt \right\|_p$$

$$\le \|I(B)\|_p+\|J(B)\|_p+\|K(B)\|_p\ , \tag{4.82}$$

By Lemma13 (ii), Lemma 6, and Lemma 8 of Chapter 3,

$$\|I(B)\|_p \le O(1)\int_0^{\frac{\pi}{\lambda}} \frac{\|G(t)\|_p}{|t|}dt = \frac{O(1)|u|^{\beta}}{p^{\alpha-\beta}} \tag{4.83}$$

By Lemma 12, Lemma 13(i), Lemma 7, and Lemma 8 of Chapter 3,

$$\|K(B)\|_p \le \int_{\delta}^{\pi} \frac{\|G(t)\|_p}{\left|2\sin\frac{1}{2}t\right|} \left|e^{-p(1-\cos t)}\right| \left|\cos(p\sin t+\frac{1}{2}t)\right| dt$$

$$= O(1)\int_{\delta}^{\pi} \frac{\|G(t)\|_p}{|t|} e^{-Apt^2}\, dt$$

$$= O(1)\frac{|u|^{\beta}}{p^{\Delta}} \quad ,\Delta \text{ positive however large} \tag{4.84}$$

We write

$$\| J(B)\|_p = \left\| \int_{\frac{\pi}{\lambda}}^{\delta} \frac{G(t)}{2\sin\frac{1}{2}t} e^{-p(1-\cos t)} \cos(p\sin t + \frac{1}{2}t)dt \right\|_p$$

$$= \left\| \int_{\pi/\lambda}^{\delta} G(t)\left(\frac{1}{2\sin\frac{1}{2}t} - \frac{1}{t}\right) e^{-p(1-\cos t)} \cos(p\sin t + \frac{1}{2}t)dt \right.$$

$$\left. + \int_{\frac{\pi}{\lambda}}^{\delta} \frac{G(t)}{t} e^{-p(1-\cos\, t)} \cos\lambda t\, dt + \int_{\frac{\pi}{\lambda}}^{\delta} \frac{G(t)}{t} e^{-p(1-\cos t)} \{\cos(p\sin t + \frac{1}{2}) - \cos\lambda t\}\, dt \right\|_p$$

$$\le \|J_1(B)\|_p + \|J_2(B)\|_p + \|J_3(B)\|_p\, , \tag{4.85}$$

by Minkowski's inequality.

Using Lemma 1, Lemma 13 (i), Lemma 14, and Lemma 4 and proceeding as in the proof of $J_1(E)$ and $J_3(E)$, it can be shown that

$$\|J_1(B)\|_p = O(1)|u|^{\beta} p^{-1} \tag{4.86}$$

and

$$\|J_3(B)\|_p = \frac{O(1)|u|^{\beta}}{p^{\alpha-\beta}} \tag{4.87}$$

Collecting the estimates for $P(B),\ S(B),\ R(B),\ I(B),\ K(B),\ J_1(B)$ and $J_3(B)$ from (4.79), (4.80), (4.81), (4.83), (4.84), (4.86), and (4.87), we obtain

$$\left\| l_n^{\beta}(y+u) - l_n^{\beta}(y) \right\|_p \le \|J_2(B)\|_p + O(1)\frac{|u|^{\beta}}{p^{\alpha-\beta}} \tag{4.88}$$

For $\lambda = p + \frac{1}{2}$ (i.e., $p = \lambda - \frac{1}{2}$), the expression $e^{-p(1-\cos t)}$ reduces to $e^{-\left(\lambda-\frac{1}{2}\right)(1-\cos t)} = c(\lambda, t)$

In view of Lemma 12 the function $c(\lambda, t)$ satisfies all the requirements of Lemma 5 and hence

$$\|J_2(B)\|_p = \left\| \int_{\frac{\pi}{\lambda}}^{\delta} \frac{G(t)}{t} e^{-p(1-\cos t)} \cos \lambda t \, dt \right\|_p$$

$$= \left\| \int_{\frac{\pi}{\lambda}}^{\delta} \frac{G(t)}{t} c(\lambda, t) \cos \lambda t \, dt \right\|_p$$

$$= O(1)|u|^{\beta} \begin{cases} \dfrac{1}{p^{\alpha-\beta}}, & \alpha - \beta \neq 1 \\ \dfrac{\log p}{p}, & \alpha - \beta = 1 \end{cases}, \tag{4.89}$$

by the method used previously.

From (4.88) and (4.89), we obtain,

$$\left\| l_n^{\beta}(y+u) - l_n^{\beta}(y) \right\|_p \le O(1)|u|^{\beta} \begin{cases} \dfrac{1}{p^{\alpha-\beta}}, & \alpha - \beta \neq 1 \\ \dfrac{\log p}{p}, & \alpha - \beta = 1 \end{cases}$$

which ensures that

$$\sup_{u \neq 0} \left| \Delta^{\beta} l_p^{\beta}(y+u, y) \right| = \sup_{u \neq 0} \frac{|l_p^{\beta}(y+u) - l_p^{\beta}(y)|}{|u|^{\beta}}$$

$$= O(1) \begin{cases} \dfrac{1}{p^{\alpha-\beta}}, & \alpha - \beta \neq 1 \\ \dfrac{\log p}{p}, & \alpha - \beta = 1 \end{cases} \tag{4.90}$$

When $f \in H_{\alpha,p}$ it can be shown that

$$\left\| l_n^{\beta}(\cdot) \right\|_p = \sup_{-\pi \le x \le \pi} \left| l_p^{\beta}(x) \right| = O(1) \begin{cases} \dfrac{1}{p^{\alpha}}, & 0 < \alpha < 1 \\ \dfrac{\log p}{p}, & \alpha = 1 \end{cases} \tag{4.91}$$

From (4.90) and (4.91), we obtain (4.8) and this completes the proof of Theorem 2.

4.6 Proof of Theorem 3

We will use the following notations and Lemmas for Theorem 3.

$$\theta(n) = \sqrt{\frac{c}{\pi n}} \sum_{k=-(n-1)}^{\infty} e^{-\frac{ck^2}{n}}$$

$$K_n(t) = \sqrt{\frac{c}{\pi n}} \left\{ 1 + 2 \sum_{k=1}^{n-1} e^{-\frac{ck^2}{n}} \cos kt \right\}$$

$$L_n(t) = \sqrt{\frac{c}{\pi n}} \sum_{k=n}^{\infty} e^{-\frac{ck^2}{n}} \cos\left(n + k + \frac{1}{2}\right) t$$

$$l_n^e(x) = e_n(\tilde{T}; x) - g(x)$$

$$e(n) = \sqrt{\frac{c}{\pi n}} \sum_{k=-(n-1)}^{\infty} e^{-\frac{ck^2}{n}} \sum_{\nu=n+k+1}^{\infty} \frac{(-1)^{\nu-1}}{\nu}$$

$$\lambda = n + \frac{1}{2}, \qquad A = \frac{1}{4c}, \qquad h = \pi/\lambda$$

We need the following Lemmas.

Lemma 15. Let $c > d > 0$. Then

$$K_n(t) = \exp(-nAt^2) + \psi(n) \tag{4.92}$$

where

$$\psi(n) = O(e^{-dn})$$

Lemma 16. For $c > 0$

$$\text{(i)} \quad L_n(t) = O(1)\frac{e^{-cn}}{\sqrt{n}} \tag{4.93}$$

$$\text{(ii)} \quad e(n) = O\left(n^{-\frac{1}{2}}\right) \tag{4.94}$$

Proof:

(i) We have

$$L_n(t) = \sqrt{\frac{c}{\pi n}} \sum_{k=n}^{\infty} e^{-\frac{ck^2}{n}} \cos\left(n + k + \frac{1}{2}\right) t$$

$$= O(n^{-\frac{1}{2}}) \sum_{k=n}^{\infty} e^{-\frac{ck^2}{n}}$$

$$= O\left(n^{-\frac{1}{2}}\right) \int_n^{\infty} \frac{n}{2cx} \frac{d}{dx} \left\{- \exp\left(-\frac{cx^2}{n}\right)\right\} dx$$

$$= O\left(n^{-\frac{1}{2}}\right) e^{-cn} .$$

(ii) Clearly,

$$\sum_{\nu=n+k+1}^{\infty} \frac{(-1)^{\nu-1}}{\nu} = O(1)\frac{1}{n+k+1} \text{, whenever } n+k+1 > 0$$

and so

$$\sqrt{\frac{\pi n}{c}}\, e(n) = \sum_{k=-(n-1)}^{\infty} e^{-\frac{ck^2}{n}} \sum_{\nu=n+k+1}^{\infty} \frac{(-1)^{\nu-1}}{\nu}$$

$$= O(1) \sum_{k=-(n-1)}^{\infty} \frac{e^{-\frac{ck^2}{n}}}{n+k+1}$$

$$= O(1)\left[\frac{1}{n+1} + \sum_{k=1}^{n-1} \frac{e^{-\frac{ck^2}{n}}}{n-k+1} + \sum_{k=1}^{\infty} \frac{e^{-\frac{ck^2}{n}}}{n+k+1}\right]$$

$$= O(1)\left[\frac{1}{n+1} + S_1 + S_2\right] \tag{4.95}$$

As $e^{-\frac{ck^2}{n}} \leq n/ck^2$, we have

$$S_2 \leq \frac{n}{c}\sum_{k=1}^{\infty} \frac{1}{(n+k+1)k^2} = O(1) \tag{4.96}$$

Lastly,

$$S_1 = \sum_{k=1}^{M} \frac{e^{-\frac{ck^2}{n}}}{n-k+1} + \sum_{k=M+1}^{n-1} \frac{e^{-\frac{ck^2}{n}}}{n-k+1}, \qquad M = \left[\frac{n}{2}\right]$$

$$= O(1)\frac{1}{n-M+1}\sum_{k=1}^{M} e^{-\frac{ck^2}{n}} + O(1)e^{-\frac{c(M+1)^2}{n}} \sum_{k=1}^{n} \frac{1}{n-k+1}$$

$$= O(1)\frac{M}{n-M+1} + O(1)e^{-\frac{cn}{4}} \log n$$

$$= O(1) \tag{4.97}$$

From (4.95), (4.96), and (4.97) the second part of the Lemma follows.

Lemma 17. For the functions $e_n(t)$, $K_n(t)$ and $L_n(t)$, we have,

$$e_n(t) = K_n(t)\cos\left(n+\frac{1}{2}\right)t + L_n(t). \tag{4.98}$$

Proof:

We have,

$$e_n(t) = \sqrt{\frac{c}{\pi n}} \sum_{k=-(n-1)}^{\infty} e^{-\frac{ck^2}{n}} \cos\left(n+k+\frac{1}{2}\right)t$$

$$= \sqrt{\frac{c}{\pi n}} \sum_{k=-(n-1)}^{n-1} e^{-\frac{ck^2}{n}} \cos\left(n+k+\frac{1}{2}\right)t + \sqrt{\frac{c}{\pi n}} \sum_{k=n}^{\infty} e^{-\frac{ck^2}{n}} \cos\left(n+k+\frac{1}{2}\right)t$$

$$= \sqrt{\frac{c}{\pi n}} \left[\sum_{k=1}^{n-1} e^{-\frac{ck^2}{n}} \left\{\cos\left(n+k+\frac{1}{2}\right)t + \cos\left(n-k+\frac{1}{2}\right)t\right\} + \cos\left(n+\frac{1}{2}\right)t\right] + L_n(t),$$

which ensures that

$$\sqrt{\frac{\pi n}{c}}\left(e_n(t) - L_n(t)\right) = \sum_{k=1}^{n-1} e^{-\frac{ck^2}{n}} \left\{\cos\left(n+k+\frac{1}{2}\right)t + \cos\left(n-k+\frac{1}{2}\right)t\right\} + \cos\left(n+\frac{1}{2}\right)t$$

$$= 2\left[\sum_{k=1}^{n-1} e^{-\frac{ck^2}{n}} \cos kt + 1\right] \cos(n+\frac{1}{2})\, t$$

$$= \sqrt{\frac{\pi n}{c}}\, K_n(t) \cos\left(n+\frac{1}{2}\right)t,$$

from which (4.98) follows.

Proof of Theorem 3:

Collecting the expression for $\tilde{T}_n(x)$ from (57), we have,

$$e_n(\tilde{T};x) = \sqrt{\frac{c}{\pi n}} \sum_{k=-\infty}^{\infty} e^{-\frac{ck^2}{n}} \tilde{T}_{n+k}(x)$$

$$= \sqrt{\frac{c}{\pi n}} \sum_{k=-n+1}^{\infty} e^{-\frac{ck^2}{n}} \left\{g(x) + \int_0^{\pi} \theta_x(t) \frac{\cos\left(n+k+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}\, dt - \tilde{f}\,(x) \sum_{\nu=n+k+1}^{\infty} \frac{(-1)^{\nu-1}}{\nu}\right\}$$

$$= \theta(n)g(x) + \int_0^{\pi} \frac{\theta_x(t)}{2\sin\frac{1}{2}t} e_n(t)dt - \tilde{f}(x)e(n). \tag{4.99}$$

Thus,

$$l_n^e(x) = e_n\big(\tilde{T}; x\big) - g(x)$$

$$= (\theta(n) - 1)g(x) + \int_0^{\pi} \frac{\theta_x(t)}{2\sin\frac{1}{2}t}\, e_n(t)dt - \tilde{f}(x)e(n)$$

which further ensures that

$$\| \, l_n^e(y+u) - l_n^e(y) \|_p$$

$$= \left\| (\theta(n) - 1)G(x,y) + \int_0^{\pi} \frac{G(t)}{2\sin\frac{1}{2}t} e_n(t)dt - \tilde{F}(x,y)e(n) \right\|_p \tag{4.100}$$

Using Lemma 17 and Lemma 15, we can rewrite (4.100) as follows:

$$\|l_n^e(y+u) - l_n^e(y)\|_p \le \|(\theta(n) - 1)G(y+u,y)\|_p + \int_0^{\pi} \frac{\|G(t)\|_p}{\left|2\sin\frac{1}{2}t\right|} |K_n(t)| \left|\cos\left(n + \frac{1}{2}\right)t\right| dt$$

$$+ \int_0^{\pi} \frac{\|G(t)\|_p}{\left|2\sin\frac{1}{2}t\right|} |L_n(t)| dt + \left|\tilde{F}(y+u,y)e(n)\right|$$

$$\le \|(\theta(n) - 1)G(y+u,y)\|_p + \int_0^{\pi} \frac{\|G(t)\|_p}{\left|2\sin\frac{1}{2}t\right|} e^{-Ant^2} \left|\cos\left(n + \frac{1}{2}\right)t\right| dt$$

$$+\left\|\psi(n)\int_0^{\pi} G(t)\frac{\cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}dt\right\|_p + \left\|\tilde{F}(x,y)e(n)\right\|_p + \int_0^{\pi}\frac{\|G(t)\|_p}{\left|2\sin\frac{1}{2}t\right|}|L_n(t)|dt$$

$$= \|P(e)\|_p + \|Q(e)\|_p + \|R(e)\|_p + \|S(e)\|_p + \|T(e)\|_p \qquad (4.101)$$

As $\theta(n) - 1 = O(n^{-\frac{1}{2}})$, using Lemma 3 (ii), we have

$$\|P(e)\|_p = \|(\theta(n)-1)G(y+u,y)\|_p = O(1)|u|^{\beta}n^{-\frac{1}{2}} \qquad (4.102)$$

By Lemma 1 and Lemma 15,

$$\|R(e)\|_p = \left\|\psi(n)\int_0^{\pi}\frac{G(t)}{2\sin\frac{1}{2}t}\cos\left(n+\frac{1}{2}\right)t\,dt\right\|_p$$

$$= O(1)|u|^{\beta}e^{-dn}\int_0^{\pi} t^{\alpha-\beta-1}dt$$

$$= O(1)|u|^{\beta}e^{-dn}. \qquad (4.103)$$

By Lemma 1 and Lemma 16 (i),

$$\|T(e)\|_p \le \int_0^{\pi}\frac{\|T(e)\|_p}{|2\sin\frac{1}{2}t|}|L_n(t)|dt$$

$$= O(1)|u|^{\beta}\frac{e^{-cn}}{\sqrt{n}}\int_0^{\pi} t^{\alpha-\beta-1}dt$$

$$= O(1)|u|^{\beta}\frac{e^{-cn}}{\sqrt{n}}. \qquad (4.104)$$

Using Lemma 3 (i) and Lemma 16 (ii), we obtain

$$\|S(e)\|_p = \left\|\tilde{F}(x,y)e(n)\right\|_p = O(1)|u|^{\beta}n^{-\frac{1}{2}} \qquad (4.105)$$

Collecting the results from (4.101) – (4.105), we get

$$\| l_n^e(y+u) - l_n^e(y)\|_p \le \|Q(e)\|_p + O(1)|u|^\beta n^{-\frac{1}{2}} + O(1)|u|^\beta n^{\beta-\alpha} \qquad (4.106)$$

We put $\lambda = n + \frac{1}{2}$. Now for fixed δ with $0 < \delta < \pi/4$ we split the integral as follows:

$$\|Q(e)\|_p \le \left[\int_0^{\pi/\lambda} + \int_{\pi/\lambda}^{\delta} + \int_{\delta}^{\pi}\right] \frac{\|G(t)\|_p}{\left|2\sin\frac{1}{2}t\right|} \left|e^{-Ant^2}\right| \left|\cos\left(n+\frac{1}{2}\right)t\right| dt$$

$$= \|I(e)\|_p + \|J(e)\|_p + \|K(e)\|_p \qquad (4.107)$$

Following the same lines of argument used in obtaining estimates for $I(B)$ and $K(B)$ in §4, it can be shown that for $0 \le \beta < \alpha \le 1$,

$$\|I(e)\|_p = O(1)\frac{|u|^\beta}{n^{\alpha-\beta}} \qquad (4.108)$$

$$\|K(e)\|_p = O(1)\frac{|u|^\beta}{n^\Delta}, \Delta > 0 \qquad (4.109)$$

Next, we write

$$\|J(e)\|_p \le \int_{\frac{\pi}{\lambda}}^{\delta} \|G(t)\|_p \left(\left|\frac{1}{2\sin\frac{1}{2}t} - \frac{1}{t}\right|\right) \left|e^{-Ant^2}\right| \left|\cos\left(n+\frac{1}{2}\right)t\right| dt$$

$$+ \int_{\pi/\lambda}^{\delta} \frac{\|G(t)\|_p}{|t|} e^{-Ant^2} \left|\cos\left(n+\frac{1}{2}\right)t\right| dt$$

$$= \|J_1(e)\|_p + \|J_2(e)\|_p \qquad (4.110)$$

Using Lemma 1, Lemma 4, and proceeding as in the proof of $J_1(E)$, it can be shown that

$$\| J_1(e)\|_p = O(1)|u|^\beta n^{-1}. \qquad (4.111)$$

From (4.106) – (4.111), it follows that

$$\| l_n^e(y+u) - l_n^e(y)\|_p \leq \|J_2(e)\|_p + O(1)|u|^\beta n^{-\frac{1}{2}} + O(1)|u|^\beta n^{\beta-\alpha} \tag{4.112}$$

For $\lambda = n + \frac{1}{2},\ e^{-Ant^2} = e^{-A\left(\lambda-\frac{1}{2}\right)t^2} = c(\lambda, t)$

Clearly, $c(\lambda, t)$ satisfies the conditions of Lemma 5, and hence,

$$\|J_2(e)\|_p \leq \int_{\pi/\lambda}^{\delta} \frac{\|G(t)\|_p}{|t|} e^{-A\left(\lambda-\frac{1}{2}\right)t^2} |\cos \lambda t\,| dt$$

$$= \int_{\pi/\lambda}^{\delta} \frac{\|G(t)\|_p}{|t|} |c(\lambda, t) \cos \lambda t\,| dt$$

$$= O(1)|u|^\beta \begin{cases} \dfrac{1}{n^{\alpha-\beta}}, \alpha - \beta \neq 1 \\ \dfrac{\log n}{n}, \alpha - \beta = 1 \end{cases} \tag{4.113}$$

From (4.112) and (4.113), it follows that

$$\left\|\Delta^\beta l_n^e(y+u, y)\right\|_p = \left\|\frac{l_n^e(y+u) - l_n^e(y)}{u^\beta}\right\|_p$$

$$= O(1) \begin{cases} \dfrac{1}{n^{\alpha-\beta}}, 0 < \alpha - \beta \leq \dfrac{1}{2} \\ \dfrac{1}{\sqrt{n}}, \dfrac{1}{2} \leq \alpha - \beta \leq 1 \end{cases} \tag{4.114}$$

Again, $f \in H_{\alpha,p} \Rightarrow \|\theta_x(t)\|_p = O(t^\alpha)$, and so proceeding as above, we obtain

$$\|l_n^e(\cdot)\|_p = O(1) \begin{cases} \dfrac{1}{n^\alpha}, 0 < \alpha \leq \dfrac{1}{2} \\ \dfrac{1}{\sqrt{n}}, \dfrac{1}{2} \leq \alpha \leq 1 \end{cases} \tag{4.115}$$

Now (4.9) follows from (4.114) and (4.115) and this completes the proof of Theorem 3.

CHAPTER 5: CONCLUSION AND FUTURE PROBLEMS

5.1 Conclusion

In this dissertation we have extended the result of Pratima Sadangi [29] to obtain estimates in $H_{\alpha,p}$ norm, $p > 1$. These estimates were obtained by applying a result from *Inequalities,* Hardy, Littlewood, and Polya [13] and a modification of the methods used by Sadangi [29]. However, there still remains other results which can be obtained. These future problems are mentioned below:

5.2 Problem 1

In 2007 R. A. Lasuriya [17] used a modified version of the distance function in the space H_α to obtain degree of convergence of functions in H_α.

Suppose that $\mathbb{R} = (-\infty, \infty)$ and $C(\mathbb{R})$ is the space of uniformly continuous and bounded functions $f(\cdot)$ on the whole real axis with norm

$$\|l_n^e(\cdot)\|_{C(\mathbb{R})} = \sup_{-\infty<x<\infty} |f(x)|.$$

Denote by $H_{\omega^*}(\mathbb{R})$ the set of all functions $f(\cdot) \in C(\mathbb{R})$ satisfying the condition

$$\sup_{\substack{-\infty<x,y<\infty \\ x\neq y}} h^{\omega^*}(f; x, y) < \infty,$$

where

$$h^{\omega^*}(f; x, y) = \frac{|f(x) - f(y)|}{\omega^*(|x-y|)}, \qquad h^0(f; x, y) = 0$$

and, for $t \geq 0, \omega^*(t)$ is a nondecreasing function. We can show that $H_{\omega^*}(\mathbb{R})$ is a Banach space with respect to the generalized Hölder norm

$$\|l_n^e(\cdot)\|_{\omega^*(\mathbb{R})} = \|l_n^e(\cdot)\|_{C(\mathbb{R})} + \sup_{\substack{-\infty<x,y<\infty \\ x\neq y}} h^{\omega^*}(f;x,y). \tag{5.1}$$

Suppose that $H_\omega(\mathbb{R})$ is the set of functions $f(\cdot) \in C(\mathbb{R})$ satisfying the condition

$$\sup_{\substack{-\infty<x,y<\infty \\ x\neq y}} h^{\omega}(f;x,y) = \sup_{\substack{-\infty<x,y<\infty \\ x\neq y}} \frac{|f(x)-f(y)|}{\omega(|x-y|)} < \infty$$

and contained in the space $H_{\omega^*}(\mathbb{R})$, where. For $t \geq 0$, $\omega(t)$ is a nondecreasing function. In particular, setting

$$\omega^*(t) = t^\beta, \;\; \omega(t) = t^a, \qquad 0 \leq \beta < \alpha \leq 1,$$

for $H_{\omega^*}(\mathbb{R})$ we obtain the space

$$H_\beta(\mathbb{R}) = \{f \in C(\mathbb{R}): |f(x)-f(y)| \leq l(f)|x-y|^\beta, \qquad \forall x,y \in \mathbb{R}\}$$

with Hölder norm

$$\|f\|_{\beta(\mathbb{R})} = \|f\|_{C(\mathbb{R})} + \sup_{\substack{-\infty<x,y<\infty \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\beta}$$

and for the set $H_\omega(\mathbb{R})$ we have

$$H_\alpha(\mathbb{R}) = \{f \in C(\mathbb{R}): |f(x)-f(y)| \leq l(f)|x-y|^\alpha, \qquad \forall x,y \in \mathbb{R}\}$$

$$H_\alpha(\mathbb{R}) \subset H_\beta(\mathbb{R}).$$

Theorem [17]. Suppose that $0 \le \beta < \eta \le 1$. Then, for any $f \in H_\omega(\mathbb{R})$, the following relation holds:

$$\|F_\lambda(f;x) - f(x)\|_{\omega^*(\mathbb{R})} = O(1) \sup_{\substack{-\infty<x,y<\infty \\ x\neq y}} \frac{\big(\omega(|x-y|)\big)^{\frac{\beta}{\eta}}}{\omega^*(|x-y|)} \left[\frac{1}{\lambda}\int_1^\lambda \left(\omega\left(\frac{1}{\sigma}\right)\right)^{1-\frac{\beta}{\eta}} d\sigma\right], \lambda > 1 \quad (5.2)$$

where $O(1)$ is a quantity uniformly bounded in the parameter λ and depending, in general, on $f(\cdot) \in H_\omega(\mathbb{R})$.

This leads to the following problem:

Problem 1. Consider functions in L_p and define $H_{\omega,,p}(\mathbb{R})$ as the notion of distance in that space. We then obtain an estimate for $F_\lambda(f,x) - f(x)$ in that metric.

5.3 Problem 2

In 1996 in a Ph.D. thesis submitted to Utkal University Ojha [27] proved results for degree of convergence of functions associated with their Fourier series. These results are analogous to the results of Sadangi [29] considered in the present dissertation. Our next objective is to solve the following:

Problem 2. Prove analogous theorems for Fourier series of those proved in Chapter 4 of the present dissertation.

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