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ON A CALDERÓN PROBLEM IN FREQUENCY DIFFERENTIAL ELECTRICAL IMPEDANCE TOMOGRAPHY*

SUNGWHAN KIM[†] AND ALEXANDRU TAMASAN[‡]

Abstract. Recent research in electrical impedance tomography produced images of biological tissue from *frequency differential* boundary voltages and corresponding currents. Physically one is to recover the electrical conductivity σ and permittivity ϵ from the frequency differential boundary data. Let $\gamma = \sigma + i\omega\epsilon$ denote the complex admittivity, Λ_γ be the corresponding Dirichlet-to-Neumann map, and $\frac{d\Lambda_\gamma}{d\omega}|_{\omega=0}$ be its frequency differential at $\omega = 0$. If $\sigma \in C^{1,1}(\overline{\Omega})$ is constant near the boundary and $\epsilon \in C_0^{1,1}(\Omega)$, we show that $\frac{d\Lambda_\gamma}{d\omega}|_{\omega=0}$ uniquely determines $\nabla \cdot (\nabla \epsilon - \epsilon \nabla \ln \sigma) / \sigma$ inside Ω . In addition, if $\Lambda_\gamma|_{\omega=0}$ is also known, then ϵ and σ can be reconstructed inside. The method of proof uses the complex geometrical optics solutions.

Key words. Calderón problem, frequency differential electrical impedance tomography, complex geometrical optics

AMS subject classifications. 35R30, 35J25, 65N21

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1. Introduction. Electrical impedance tomography (EIT) aims to determine the electrical conductivity σ and permittivity distribution ϵ of a body from surface electrical measurements of voltages and corresponding currents. One major application is in medical imaging, where the change of the electrical properties of biological tissues with their physiological and pathological conditions is used to provide diagnostic information. Driven by its applications, considerable progress in both the engineering and mathematical facets of EIT has been achieved, and its development can be traced over the past two decades in the reviews [6, 3, 8, 2, 19].

We consider a conducting bounded body $\Omega \subset \mathbb{R}^n$, $n \geq 3$, with $C^{1,1}$ -boundary $\partial\Omega$. Its conductivity distribution $\sigma \in C^{1,1}(\overline{\Omega})$ is bounded away from zero, and its permittivity distribution $\epsilon \in C^{1,1}(\overline{\Omega})$. We assume that σ is constant near the boundary and ϵ is supported in Ω (these assumptions may be relaxed as explained later).

For a real valued function $f \in H^{1/2}(\partial\Omega)$ and an angular frequency ω the sinusoidal voltage $f(x) \cos(\omega t)$ is imposed at the boundary $\partial\Omega$. Then a time harmonic complex voltage potential u_ω distributes inside according to the problem

$$(1.1) \quad \nabla \cdot (\sigma + i\epsilon\omega) \nabla u_\omega = 0 \text{ in } \Omega \text{ and } u_\omega|_{\partial\Omega} = f.$$

The problem (1.1) has a unique complex (voltage potential) solution $u_\omega \in H^1(\Omega)$; see also Theorem 3.1 below. The exiting current

$$(1.2) \quad g_\omega := (\sigma + i\omega\epsilon) \mathbf{n} \cdot \nabla u_\omega$$

is measured ($\mathbf{n}$ is the unit outward normal) at the boundary, to define the Dirichlet-

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to-Neumann map

$$\Lambda_{\sigma+i\epsilon\omega} : f \mapsto g_\omega \in H^{-1/2}(\partial\Omega).$$

Originally formulated by Calderón [5] at the $\omega = 0$ frequency, the goal in EIT is to determine σ and ϵ from knowledge of $\Lambda_{\sigma+i\epsilon\omega}$. At zero frequency only σ is sought; in such a case the corresponding voltage potential v_0 is real valued and solves

$$(1.3) \quad \nabla \cdot (\sigma \nabla v_0) = 0 \text{ in } \Omega \text{ and } v_0|_{\partial\Omega} = f.$$

The Calderón problem (at $\omega = 0$) has been mostly settled in the affirmative and we refer to [19] for a state of the art solution. Of relevance to our work here, we mention the breakthrough result in [18], where $\Lambda_\sigma := \Lambda_\gamma|_{\omega=0}$ is shown to uniquely determine σ in three or higher dimensions, and the reconstruction method in [14] which allows for the $C^{1,1}$ -regularity assumed here. We note however that, while not explicitly stated, the results in [18, 14] extend to the complex admittivity $\gamma = \sigma + i\omega\epsilon$ to show that Λ_γ uniquely determines γ in three dimensions or higher. The analogous results in two dimensions at $\omega = 0$ were obtained in [15], with a nontrivial refinement in [1]. Also in two dimensions, but at an arbitrary fixed frequency ω (not necessarily zero) Λ_γ was shown recently to uniquely determine the complex admittivity γ in [4] (a previous result in [7] recovered γ for a sufficiently small imaginary part $\omega\epsilon$).

Recent research in [9, 11, 12, 16, 17] produce physiologically relevant images by using the frequency dependent behavior of the complex potential u_ω . These new methods are known as frequency differential electrical impedance tomography (fdEIT). Physically one imposes boundary voltages at two distinct frequencies and measures a difference between corresponding boundary exit currents. Despite the apparent usefulness in medical diagnostics, the quantities behind the images in fdEIT are not so well understood. In this paper we take a first step towards explaining what can be quantitatively obtained by fdEIT. We formulate the problem in terms of the frequency differential operator at the boundary: What can be obtained from knowledge of $\frac{d\Lambda_\gamma}{d\omega}|_{\omega=0}$?

To simplify notation, let $\mathcal{D}$ be defined for real valued functions $f \in H^{1/2}(\partial\Omega)$ by

$$(1.4) \quad \mathcal{D}(f) := \left. \frac{d}{d\omega} \right|_{\omega=0} \Lambda_{\sigma+i\omega\epsilon}(f),$$

and then extended by complex linearity:

$$\mathcal{D}(f + ig) := \mathcal{D}(f) + i\mathcal{D}(g).$$

Our main result, Theorem 2.2, shows that $\mathcal{D} : H^{1/2}(\Omega) \rightarrow H^{-1/2}(\Omega)$ is a well-defined bounded operator which uniquely determines the function $\nabla \cdot (\nabla\epsilon - \epsilon\nabla \ln \sigma)/\sigma$ inside, in three or higher dimensions. The method of proof uses the complex geometrical optics solutions in [18]. We note that only the action of $\mathcal{D}$ on real valued functions is needed.

If, in addition, the Dirichlet-to-Neumann map $\Lambda_\sigma := \Lambda_\gamma|_{\omega=0}$ is also known, we are able to recover separately the conductivity σ and the permittivity ϵ inside Ω . Moreover, in this case we do not need to assume σ constant near the boundary. Specifically, if $\sigma \in C^{1,1}(\overline{\Omega})$ then its boundary values $\sigma|_{\partial\Omega}$ and its normal derivative $\frac{\partial\sigma}{\partial\mathbf{n}}|_{\partial\Omega}$ can be recovered from Λ_σ as shown in [14] for this regularity (and earlier in [13] and [18] for C^∞ -conductivity). Then σ can be extended with preserved $C^{1,1}$ -regularity

to the whole space while making it constant near the boundary $\partial\tilde{\Omega}$ of a neighborhood $\tilde{\Omega}$ of Ω . As shown in [15] the Dirichlet-to-Neumann map can be transferred from $\partial\Omega$ to $\partial\tilde{\Omega}$ (the fact that we deal with a complex valued coefficient does not change the proof from the real case). Therefore the assumption that σ is constant near the boundary $\partial\Omega$ does not restrict generality. Similarly, the assumption that ϵ has compact support in Ω can be replaced by the knowledge of the boundary values $\epsilon|_{\partial\Omega}$ and its normal derivative $\frac{\partial\epsilon}{\partial\mathbf{n}}|_{\partial\Omega}$; see also (4.4).

With $v_\omega := \Re(u_\omega)$ and $h_\omega := \Im(u_\omega)$ denoting the real, respectively, imaginary part of the voltage potential obtained for a real valued boundary data f , the problem (1.1) can be rewritten as a Dirichlet problem for the coupled system:

$$(1.5) \quad \nabla \cdot \left[\begin{pmatrix} \sigma & -\omega\epsilon \\ \omega\epsilon & \sigma \end{pmatrix} \begin{pmatrix} \nabla v_\omega \\ \nabla h_\omega \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega, \quad \begin{pmatrix} v_\omega \\ h_\omega \end{pmatrix} \Big|_{\partial\Omega} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

Not only do v_ω and h_ω have a nonlinear dependence on the conductivity σ , permittivity ϵ , and angular frequency ω , but also their intrinsic mutual relation makes this dependence difficult to investigate directly from (1.5).

Key to this work, in Theorem 3.1 we identify the regime of frequencies

$$(1.6) \quad |\omega| < \left\| \frac{\epsilon}{\sigma} \right\|_{L^\infty(\Omega)}^{-1}$$

in which the family of operators $\omega \mapsto \Lambda_{\sigma+i\epsilon\omega}$ is analytic in the strong operator topology (from $H^{1/2}(\Omega)$ to $H^{-1/2}(\Omega)$). This analytic dependence allows for a recurrence type of decoupling. Also the frequency differential boundary operator can be made more explicit:

$$(1.7) \quad \mathcal{D}(f) = i\epsilon \frac{\partial v_0}{\partial \mathbf{n}} + i\sigma \frac{d}{d\omega} \Big|_{\omega=0} \frac{\partial h_\omega}{\partial \mathbf{n}} = i \frac{\epsilon}{\sigma} \Big|_{\partial\Omega} \Lambda_\sigma(f) + i\sigma \frac{d}{d\omega} \Big|_{\omega=0} \frac{\partial h_\omega}{\partial \mathbf{n}}.$$

2. Statement of results. The main result is formulated in terms of the complex geometrical optics solutions of Sylvester and Uhlmann in [18] whose existence is recalled below both for convenience and to set notation. The coefficients σ , and ϵ assumed constant near the boundary, are extended by (the corresponding) constant on the complement of Ω .

For $\delta \in \mathbb{R}$, the weighted norm $\|f\|_{L_\delta^2}^2 := \int_{\mathbb{R}^n} |f(x)|^2 (1 + |x|^2)^\delta dx$ is used. For $\mathbf{k}, \eta, \mathbf{l} \in \mathbb{R}^n$ with $\mathbf{k} \cdot \eta = \mathbf{k} \cdot \mathbf{l} = \mathbf{k} \cdot \eta = 0$, and $|\eta|^2 = \frac{|\mathbf{k}|^2}{4} + |\mathbf{l}|^2$, consider the vectors

$$(2.1) \quad \begin{aligned} \xi_1(\eta, \mathbf{k}, \mathbf{l}) &:= \eta - i \left(\frac{\mathbf{k}}{2} + \mathbf{l} \right), \\ \xi_2(\eta, \mathbf{k}, \mathbf{l}) &:= -\eta - i \left(\frac{\mathbf{k}}{2} - \mathbf{l} \right). \end{aligned}$$

Note that $\xi_1 \cdot \xi_1 = \xi_2 \cdot \xi_2 = 0$, $|\xi_1|^2 = |\xi_2|^2 = 2(|\frac{\mathbf{k}}{2}|^2 + |\mathbf{l}|^2)$, and $\xi_1 + \xi_2 = -i\mathbf{k}$. We restate their result in the variant below.

THEOREM 2.1 (Theorem 2.3 in [18]). *Let $n \geq 3$ and $\sigma \in C^{1,1}(\overline{\Omega})$ be constant near the boundary. For $-1 < \delta < 0$ there are two constants $R, C > 0$ dependent only on δ , $\|\Delta\sqrt{\sigma}/\sqrt{\sigma}\|_{L^\infty(\Omega)}$, and Ω such that, for $\xi_j \in \mathbb{C}^n$, $j = 1, 2$ as in (2.1) with*

$$|\mathbf{l}| > R,$$

there exist $w(\cdot, \xi_j) \in H_{loc}^1(\mathbb{R}^n)$ solutions of $\nabla \cdot \sigma \nabla w(\cdot, \xi_j) = 0$ in $\mathbb{R}^n$, of the form

$$(2.2) \quad w(x, \xi_j) = e^{x \cdot \xi_j} \sigma^{-1/2} (1 + \psi(x, \xi_j))$$

with

$$(2.3) \quad \|\psi(\cdot, \xi_j)\|_{L_\delta^2} \leq \frac{C}{|\xi_j|}.$$

The main result which will be proven in section 4 is the following.

THEOREM 2.2. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be an open domain with $C^{1,1}$ -boundary, $\sigma \in C^{1,1}(\overline{\Omega})$ be constant near the boundary, and $\epsilon \in C_0^{1,1}(\Omega)$. Recall the frequency differential map $\mathcal{D}$ defined in (1.4). For each $\mathbf{k} \in \mathbb{R}^n$, let $f_j := w(\cdot, \xi_j)|_{\partial\Omega}$, $j = 1, 2$ be the traces of the complex geometrical optics solutions in (2.2). Then*

$$(2.4) \quad \mathcal{F} \left[\frac{\nabla \cdot (\nabla \epsilon - \epsilon \nabla \ln \sigma)}{\sigma} \right] (\mathbf{k}) = \lim_{|\mathbf{l}| \rightarrow \infty} -2i \int_{\partial\Omega} \mathcal{D}(f_1) f_2 \, ds,$$

where $\mathcal{F}$ denotes the Fourier transform.

We stress here that only the action of $\mathcal{D}$ on real valued functions is needed in Theorem 2.2 above.

If the Dirichlet-to-Neumann map Λ_σ (at frequency $\omega = 0$) is also known, then σ can be recovered inside as shown in [14]. As a corollary to Theorem 2.2 one is also able to reconstruct ϵ inside.

COROLLARY 2.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be an open domain with smooth $C^{1,1}$ -boundary. Assume $\sigma \in C^{1,1}(\overline{\Omega})$ constant near the boundary and $\epsilon \in C_0^{1,1}(\Omega)$. Then σ and ϵ inside Ω can be reconstructed from knowledge of Λ_σ and $\mathcal{D}$.*

3. Analytic dependence in frequency. In this section we prove the analytic dependence in the frequency of $\omega \mapsto \Lambda_{\sigma+i\omega\epsilon}$ in the strong operator topology from $H^{1/2}(\Omega) \rightarrow H^{-1/2}(\Omega)$. The assumptions on the coefficients are slightly relaxed. By $C^{0,1}(\Omega)$ we denote the space of Lipschitz continuous maps.

THEOREM 3.1. *Let $\sigma \in L^\infty(\Omega)$ bounded away from zero and $\epsilon \in C^{0,1}(\Omega)$. Assume that ω lies in the frequency range (1.6). Then the Dirichlet problem (1.1) has a unique solution $u_\omega \in H^1(\Omega)$ with the following series representation:*

$$(3.1) \quad \begin{aligned} v_\omega(x) &:= \Re(u_\omega(x)) = \sum_{k=0}^{\infty} v_{2k}(x) \omega^{2k} \quad \text{and} \\ h_\omega(x) &:= \Im(u_\omega(x)) = \sum_{k=1}^{\infty} h_{2k-1}(x) \omega^{2k-1}, \end{aligned}$$

where the summation is convergent in the $H^1(\Omega)$, v_0 is the solution to (1.3), and for $k = 1, 2, \dots$ the following recurrence holds:

$$(3.2) \quad \begin{cases} \nabla \cdot (\sigma \nabla h_{2k-1}) = -\nabla \cdot (\epsilon \nabla v_{2(k-1)}) & \text{in } \Omega, \\ \nabla \cdot (\sigma \nabla v_{2k}) = \nabla \cdot (\epsilon \nabla h_{2k-1}) & \text{in } \Omega, \\ h_{2k-1}|_{\partial\Omega} = v_{2k}|_{\partial\Omega} = 0. \end{cases}$$

Assuming the recurrences in (3.2) hold we establish first some basic estimates.

LEMMA 3.1. Let v_{2k} and h_{2k-1} , $k = 1, 2, \dots$ be defined in (3.2). Then, for $k = 1, 2, \dots$, we have

$$(3.3) \quad \left[\int_{\Omega} \sigma |\nabla v_{2k}|^2 dx \right]^{\frac{1}{2}} \leq \left\| \frac{\epsilon}{\sigma} \right\|_{L^{\infty}(\Omega)}^{2k} \left[\int_{\Omega} \sigma |\nabla v_0|^2 dx \right]^{\frac{1}{2}}$$

and

$$(3.4) \quad \left[\int_{\Omega} \sigma |\nabla h_{2k-1}|^2 dx \right]^{\frac{1}{2}} \leq \left\| \frac{\epsilon}{\sigma} \right\|_{L^{\infty}(\Omega)}^{2k-1} \left[\int_{\Omega} \sigma |\nabla v_0|^2 dx \right]^{\frac{1}{2}}.$$

Proof. Let us fix a positive natural number k . From (3.2) the divergence theorem implies that

$$\int_{\Omega} \sigma |\nabla h_{2k-1}|^2 dx = - \int_{\Omega} \epsilon \nabla v_{2k-2} \cdot \nabla h_{2k-1} dx.$$

Cauchy's inequality applied to the right-hand side above yields

$$\int_{\Omega} \sigma |\nabla h_{2k-1}|^2 dx \leq \left\| \frac{\epsilon}{\sigma} \right\|_{L^{\infty}(\Omega)} \left[\int_{\Omega} \sigma |\nabla v_{2k-2}|^2 dx \right]^{\frac{1}{2}} \left[\int_{\Omega} \sigma |\nabla h_{2k-1}|^2 dx \right]^{\frac{1}{2}},$$

and thus

$$(3.5) \quad \left[\int_{\Omega} \sigma |\nabla h_{2k-1}|^2 dx \right]^{\frac{1}{2}} \leq \left\| \frac{\epsilon}{\sigma} \right\|_{L^{\infty}(\Omega)} \left[\int_{\Omega} \sigma |\nabla v_{2k-2}|^2 dx \right]^{\frac{1}{2}}.$$

Similarly we obtain

$$(3.6) \quad \left[\int_{\Omega} \sigma |\nabla v_{2k}|^2 dx \right]^{\frac{1}{2}} \leq \left\| \frac{\epsilon}{\sigma} \right\|_{L^{\infty}(\Omega)} \left[\int_{\Omega} \sigma |\nabla h_{2k-1}|^2 dx \right]^{\frac{1}{2}}.$$

By induction, the estimates (3.3) and (3.4) follow. $\square$

Proof of Theorem 3.1. The coupled system (1.5) is equivalent to the two elliptic equations

$$(3.7) \quad \nabla \cdot (\sigma \nabla v_{\omega}) = \nabla \cdot (\omega \epsilon \nabla h_{\omega}) \quad \text{in } \Omega,$$

and

$$(3.8) \quad \nabla \cdot (\sigma \nabla h_{\omega}) = -\nabla \cdot (\omega \epsilon \nabla v_{\omega}) \quad \text{in } \Omega.$$

We seek solutions in the ansatz

$$(3.9) \quad v(x, \omega) := \sum_{k=0}^{\infty} v_k(x) \omega^k \quad \text{and} \quad h(x, \omega) := \sum_{k=0}^{\infty} h_k(x) \omega^k.$$

Let us assume first that the series representations in (3.9) are convergent in $H^1(\Omega)$. If (3.7) and (3.8) are satisfied, then

$$0 = \nabla \cdot (\sigma \nabla v_0) + \sum_{k=0}^{\infty} \nabla \cdot (\sigma \nabla v_{k+1} - \epsilon \nabla h_k) \omega^{k+1}$$

and

$$0 = \nabla \cdot (\sigma \nabla h_0) + \sum_{k=0}^{\infty} \nabla \cdot (\sigma \nabla h_{k+1} + \epsilon \nabla v_k) \omega^{k+1},$$

where the divergence is taken in the weak sense. In particular we obtain

$$(3.10) \quad \nabla \cdot (\sigma \nabla v_0) = 0, \quad \nabla \cdot (\sigma \nabla h_0) = 0 \quad \text{in } \Omega,$$

and, for $k = 0, 1, 2, \dots$,

$$(3.11) \quad \nabla \cdot (\sigma \nabla v_{k+1}) = \nabla \cdot (\epsilon \nabla h_k), \quad \nabla \cdot (\sigma \nabla h_{k+1}) = -\nabla \cdot (\epsilon \nabla v_k) \quad \text{in } \Omega.$$

By our assumption, both series are convergent in $H^1(\Omega)$, and their sums have well-defined traces in $H^{1/2}(\partial\Omega)$, which are the corresponding sums of the traces of the terms. Now $v(x, \omega) = f(x)$ and $h(x, \omega) = 0$ for $x \in \partial\Omega$ yield $v_0|_{\partial\Omega} = f$ and $h_0|_{\partial\Omega} = 0$, and, for $k = 1, 2, \dots$,

$$h_{2k-1}|_{\partial\Omega} = v_{2k}|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega.$$

Note that v_0 is the solution of (1.3) and $h_0 \equiv 0$.

Conversely, for $f \in H^{1/2}(\partial\Omega)$, let v_0 be the solution of (1.3) and define two sequences of functions $\{v_k\}_0^\infty$ and $\{h_k\}_1^\infty$ via the recurrence (3.2). From Lemma 3.1 it follows that for any $k = 1, 2, \dots$,

$$\begin{aligned} \|\nabla v_{2k}\|_{L^2(\Omega)} \omega^{2k} &\leq \frac{1}{\sqrt{m}} \left[\int_{\Omega} \sigma |\nabla v_{2k}|^2 dx \right]^{\frac{1}{2}} \omega^{2k} \\ &\leq \frac{1}{\sqrt{m}} \left\| \frac{\omega \epsilon}{\sigma} \right\|_{L^\infty(\Omega)}^{2k} \left[\int_{\Omega} \sigma |\nabla v_0|^2 dx \right]^{\frac{1}{2}} \\ &\leq \sqrt{\frac{M}{m}} \|\nabla v_0\|_{L^2(\Omega)} \left\| \frac{\omega \epsilon}{\sigma} \right\|_{L^\infty(\Omega)}^{2k}, \end{aligned}$$

and

$$\begin{aligned} \|\nabla h_{2k-1}\|_{L^2(\Omega)} \omega^{2k-1} &\leq \frac{1}{\sqrt{m}} \left[\int_{\Omega} \sigma |\nabla h_{2k-1}|^2 dx \right]^{\frac{1}{2}} \omega^{2k-1} \\ &\leq \frac{1}{\sqrt{m}} \left\| \frac{\omega \epsilon}{\sigma} \right\|_{L^\infty(\Omega)}^{2k-1} \left[\int_{\Omega} \sigma |\nabla v_0|^2 dx \right]^{\frac{1}{2}} \\ &\leq \sqrt{\frac{M}{m}} \|\nabla v_0\|_{L^2(\Omega)} \left\| \frac{\omega \epsilon}{\sigma} \right\|_{L^\infty(\Omega)}^{2k-1}. \end{aligned}$$

If ω satisfies (1.6), the $H^1(\Omega)$ -convergence of the series well defines $v(x, \omega)$ and $h(x, \omega)$ in (3.9) to be the unique solutions of (1.5). $\square$

The result below further clarifies the frequency differential operator $\mathcal{D}$ in (1.4).

COROLLARY 3.2. *Let $\epsilon \in C^{0,1}(\overline{\Omega})$, and $\sigma \in L^\infty(\Omega)$ with $\text{essinf}_{\Omega} \sigma > 0$. For $f, g \in H^{1/2}(\partial\Omega)$ real valued we have*

$$(3.12) \quad \mathcal{D}(f + ig) = i\epsilon \frac{\partial v_0}{\partial \mathbf{n}} + i\sigma \frac{\partial h_1}{\partial \mathbf{n}},$$

where v_0 solves $\nabla \cdot (\sigma \nabla v_0) = 0$ in Ω , and $v_0|_{\partial\Omega} = f + ig$, and h_1 solves

$$\nabla \cdot (\sigma \nabla h_1) = -\nabla \cdot (\epsilon \nabla v_0) \text{ in } \Omega \text{ and } h_1|_{\partial\Omega} = 0.$$

Proof. If $g = 0$ the corollary follows directly from the definition (1.4) and the fact that $\frac{d}{d\omega} \frac{\partial h_\omega}{\partial \mathbf{n}}|_{\omega=0} = \frac{\partial h_1}{\partial \mathbf{n}}$, with h_1 defined in the recurrence (3.2). If g is arbitrary, the result follows from the complex linearity of the two terms in the right-hand side of (3.12). $\square$

From Theorem 3.1 it follows that v_0 is the zeroth order term of the series expansion of the real part v_ω and h_1 is the first order term of the series expansion of the imaginary part h_ω . Moreover, when $\omega \ll 1$ is small, and $\omega_1, \omega_2 = O(\omega)$, we have

$$(3.13) \quad \frac{v_{\omega_1} - v_{\omega_2}}{\omega_1 - \omega_2} = v_2(\omega_1 + \omega_2) + O(\omega^3) \text{ and } \frac{h_{\omega_1} - h_{\omega_2}}{\omega_1 - \omega_2} = h_1 + O(\omega^2),$$

where h_1 and v_2 satisfy the Poisson's problems

$$(3.14) \quad \begin{cases} \nabla \cdot (\sigma \nabla h_1) = -\nabla \cdot (\epsilon \nabla v_0) & \text{in } \Omega, \\ \nabla \cdot (\sigma \nabla v_2) = \nabla \cdot (\epsilon \nabla h_1) & \text{in } \Omega, \\ h_1|_{\partial\Omega} = v_2|_{\partial\Omega} = 0. \end{cases}$$

4. Proof of Theorem 2.2 and its corollary. For $j = 1, 2$ let $w_j(x) := w(x; \xi_j)$ be the complex geometrical optics (CGO) solutions corresponding to a fix vector $\mathbf{k} \in \mathbb{R}^n$ as provided by Theorem 2.1. For w_1 in the CGO above, let h_1 be the solution of the Poisson equation

$$\nabla \cdot (\sigma \nabla h_1) = -\nabla \cdot (\epsilon \nabla w_1) \text{ in } \Omega \text{ and } h_1|_{\partial\Omega} = 0.$$

First we carry out the calculation without the assumption that $\epsilon = 0$ near the boundary to emphasize the fact that knowledge of ϵ and its normal derivative at the boundary suffices.

Following the explicit formula for $\mathcal{D}$ in Corollary 3.2 we obtain

$$(4.1) \quad \begin{aligned} \int_{\partial\Omega} \mathcal{D}(w_1)w_2 ds &= i \int_{\partial\Omega} \left(\epsilon \frac{\partial w_1}{\partial \mathbf{n}} + \sigma \frac{\partial h_1}{\partial \mathbf{n}} \right) w_2 ds \\ &= i \int_{\Omega} \sigma \nabla h_1 \cdot \nabla w_2 + \epsilon \nabla w_1 \cdot \nabla w_2 \, dx. \end{aligned}$$

Since $h_1 = 0$ on $\partial\Omega$ and w_2 solves the conductivity equation $\nabla \cdot \sigma \nabla w_2 = 0$, the first integral on the right-hand side of (4.1) is zero so that

$$(4.2) \quad \int_{\partial\Omega} \mathcal{D}(w_1)w_2 \, ds = i \int_{\Omega} \epsilon \nabla w_1 \cdot \nabla w_2 \, dx.$$

Now use $2\nabla w_1 \cdot \nabla w_2 = [\Delta(w_1 w_2) - (w_1 \Delta w_2 + w_2 \Delta w_1)]$ and the fact that the w_j 's also solve

$$\Delta w_j + \nabla \ln \sigma \cdot \nabla w_j = 0 \quad \text{in } \Omega,$$

to obtain

$$\begin{aligned} \int_{\Omega} \epsilon \nabla w_1 \cdot \nabla w_2 \, dx &= \frac{1}{2} \int_{\partial\Omega} \left(\epsilon \frac{\partial(w_1 w_2)}{\partial \mathbf{n}} - \frac{\partial \epsilon}{\partial \mathbf{n}}(w_1 w_2) \right) ds \\ &\quad + \frac{1}{2} \int_{\Omega} [(\Delta \epsilon)(w_1 w_2) + \epsilon \nabla \ln \sigma \cdot \nabla(w_1 w_2)] \, dx. \end{aligned}$$

Using the Green's formula in the last integral and the assumption that σ is constant near the boundary we obtain

$$(4.3) \quad \int_{\Omega} \epsilon \nabla w_1 \cdot \nabla w_2 \, dx = \frac{1}{2} \int_{\partial\Omega} \left(\epsilon \frac{\partial(w_1 w_2)}{\partial \mathbf{n}} - \frac{\partial \epsilon}{\partial \mathbf{n}} (w_1 w_2) \right) ds \\ + \frac{1}{2} \int_{\Omega} \nabla \cdot (\nabla \epsilon - \epsilon \nabla \ln \sigma) (w_1 w_2) \, dx.$$

From (4.2) and (4.3) we have that

$$(4.4) \quad \int_{\Omega} \nabla \cdot (\nabla \epsilon - \epsilon \nabla \ln \sigma) w_1 w_2 \, dx = -2i \int_{\partial\Omega} \mathcal{D}(w_1) w_2 \, ds \\ - \int_{\partial\Omega} \left(\epsilon \frac{\partial(w_1 w_2)}{\partial \mathbf{n}} - \frac{\partial \epsilon}{\partial \mathbf{n}} (w_1 w_2) \right) ds.$$

If we now use the assumption of ϵ being support in Ω , (4.4) further simplifies to

$$(4.5) \quad \int_{\Omega} \nabla \cdot (\nabla \epsilon - \epsilon \nabla \ln \sigma) w_1 w_2 \, dx = -2i \int_{\partial\Omega} \mathcal{D}(w_1) w_2 \, ds.$$

By the choice of ξ_j 's in the complex geometrical optics w_j 's, we have

$$\int_{\Omega} \frac{\nabla \cdot (\nabla \epsilon - \epsilon \nabla \ln \sigma)}{\sigma} e^{-ix \cdot \mathbf{k}} (1 + \psi(x, \xi_1))(1 + \psi(x, \xi_2)) \, dx = -2i \int_{\partial\Omega} \mathcal{D}(w_1) w_2 \, ds.$$

Since the integrand in the left-hand side above is supported in Ω , the integral can be taken over the entire space $\mathbb{R}^n$. The decay estimates (2.3) then yield

$$\mathcal{F} \left[\frac{\nabla \cdot (\nabla \epsilon - \epsilon \nabla \ln \sigma)}{\sigma} \right] (\mathbf{k}) = \lim_{|\mathbf{k}| \rightarrow \infty} -2i \int_{\partial\Omega} \mathcal{D}(w_1) w_2 \, ds,$$

where $\mathcal{F}$ is the Fourier transform in $\mathbb{R}^n$. This completes our proof of Theorem 2.2.

The proof of Corollary 2.1 relies on the results in [14] which show that the traces of the geometrical optics solutions used in Theorem (2.2) can be recovered from a singular integral equation at the boundary: Since $\sigma \in C^{1,1}(\overline{\Omega})$ is recovered inside from Λ_{σ} , we may assume without loss of generality that $\sigma = 1$ near the boundary. Then Lemmas 2.7 and 2.12(b) in [14] yield that the traces $f_j := w(\cdot, \xi_j)|_{\partial\Omega}$ at the boundary are the unique solutions to the equation

$$(4.6) \quad f_j = e^{ix \cdot \xi_j} - \left(S_{\xi_j} \Lambda_{\gamma} - B_{\xi_j} - \frac{1}{2} I \right) f_j, \quad j = 1, 2,$$

where the boundary operators S_{ξ} and B_{ξ} are the single and double layer potentials

$$S_{\xi} f(x) = \int_{\partial\Omega} G_{\xi}(x, y) f(y) ds \text{ and } B_{\xi} f(x) = p.v. \int_{\partial\Omega} \frac{\partial G_{\xi}}{\partial \mathbf{n}}(x, y) f(y) ds$$

associated with the Fadeev–Green kernel

$$G_{\xi}(x) = \frac{e^{ix \cdot \xi}}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{ix \cdot \eta}}{\eta^2 + 2\xi \cdot \eta} d\eta.$$

Once the traces of w_1 , and w_2 are determined, the right-hand side of (2.4) is determined. By Fourier inversion, we then determine the essentially bounded function

$$(4.7) \quad Q[\sigma, \epsilon] := \frac{\nabla \cdot (\nabla \epsilon - \epsilon \nabla \ln \sigma)}{\sigma}.$$

With σ and Q known, the permittivity ϵ is the unique solution of the Dirichlet problem

$$(4.8) \quad \Delta \epsilon - \nabla \epsilon \cdot \nabla \ln(\sigma) - \epsilon \Delta \ln(\sigma) = Q\sigma \text{ in } \Omega, \quad \epsilon|_{\partial\Omega} = 0.$$

This finishes the proof of Corollary 2.1.

5. Concluding remarks. We formulated a Calderón-type problem using frequency differential $\mathcal{D} := \frac{d\Lambda_\gamma}{d\omega}|_{\omega=0}$ of the Dirichlet-to-Neumann map at $\omega = 0$. Provided that σ is (an unknown) constant near the boundary and ϵ is supported in Ω , we showed that the frequency differential uniquely determines Q in (4.7) relating the conductivity σ to the permittivity ϵ . However, if the Dirichlet-to-Neumann map at $\omega = 0$ is also available, then σ and ϵ can be recovered inside. We note here that ϵ need not be supported in Ω , since the quantity Q in (4.7) can still be recovered if $\epsilon|_{\partial\Omega}$ and its normal derivative $\frac{\partial \epsilon}{\partial \mathbf{n}}$ are known at the boundary, according to (4.4).

Our results yield the following effect of the admittivity $\sigma + i\omega\epsilon$ on the complex voltage potential u_ω . The real part $\Re(u_\omega)$ is influenced mainly by the conductivity σ , whereas the imaginary part $\Im(u_\omega)$ is influenced by the combination $Q[\sigma, \epsilon]$ in (4.7).

There are infinitely many pairs σ, ϵ which yield the same quantity Q in (4.7). More precisely, let $Q \in L^\infty(\Omega)$ be in the range of the combination in (4.7). For an arbitrary $f \in C^{1,1}(\bar{\Omega})$, let σ_f be any solution of the transport equation

$$\nabla \ln \sigma_f \cdot \nabla f = Q - \Delta f.$$

Then the pair (σ_f, ϵ) with $\epsilon = f\sigma_f$ yields the same Q , independent of f .

If the actual value of σ at the boundary is not known, since $Q[\lambda\sigma, \lambda\epsilon] = Q[\sigma, \epsilon]$ for any $\lambda > 0$, the recovered quantity is not sensitive to the contrast in the pair of coefficients. However, the boundary data $\mathcal{D}$ can distinguish the difference in scale between the conductivity σ and the permittivity ϵ since $Q[\sigma, \lambda\epsilon] = \lambda Q[\sigma, \epsilon]$.

In practice the angular frequency ω is not arbitrarily small. However, due the scaling $\gamma = \sigma + i(t\omega)\frac{\epsilon}{t}$, we apply the results above to $t\omega$, with t small. This scaling is meaningful at angular frequencies of up to a few kHz, where the scaling factor is the permittivity of the vacuum $\epsilon_0 = 8.8 \times 10^{-12} F/m$, since then $\omega\epsilon_0$ is still numerically small.

From a numerical perspective, (3.13) shows a difference in scale (of order 1) between the real and imaginary parts of the complex voltage potential at small frequency (or $\epsilon_0\omega$ as explained above). They imply that $\mathcal{D}$ is approximated at $O(\omega^2)$ by the difference quotient at two small frequencies without a need to distinguish the real from the imaginary part of the voltage potential.

Following from (1.7), in fdEIT it is the quotient $\frac{\epsilon}{\sigma^2}|_{\partial\Omega}$ at the boundary which scales the boundary information about the admittivity inside. In particular, when $\epsilon|_{\partial\Omega} = 0$ it is only the imaginary part of the voltage potential which carries the information about the coefficients from inside to the frequency differential data at the boundary. In such a case we can still expect to recover the quantity Q in (4.7).

While the formulated problem is still severely ill-posed, these theoretical results are expected to help in understanding the quantitative feature of fdEIT.

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