1988

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ON K-γ-INSENSITIVE DOMINATION

by

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A dissertation submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in
the Department of Computer Science at
the University of Central Florida
Orlando, Florida

August 1988

Major Professor: Robert C. Brigham

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Abstract

A connected graph G is defined to be k-γ-insensitive if the domination number $\gamma(G)$ is unchanged when an arbitrary set of k edges is removed. The problem has been solved for $k = 1$. This Ph.D. dissertation focuses on finding extremal k-γ-insensitive graphs on p nodes, for $k \geq 2$. A graph is extremal if it has the minimum number of edges.

Two subproblems are considered. The first, which has been solved completely, specifies that the same set of nodes dominates each graph obtained from G by removing k edges. The second requires only that the graph G be connected. This is a much more difficult problem and represents the area of major contribution. Asymptotically correct values for the minimum number of edges e have been found for all $k \geq 2$ and all $\gamma \geq 2$ by establishing lower and upper bounds for e. The difference in these bounds is $O(\gamma^k)$ and is independent of p. The general results are improved when $k = 2$ and $\gamma \geq 3$, and exact solutions are given when $k = \gamma = 2$ and when $\gamma = 1$

with k arbitrary.

Considering applications for k-γ-insensitive graphs, we introduce the G-network, a new topological design which is a suitable architecture for point-to-point and interconnection networks. We show that the G-network has the following desirable characteristics: efficient routing, simple connections, small number of links and fault tolerance. Significant improvement in terms of routing performance and the number of edges is shown when the G-network is compared to the popular Barrel Shifter, Illiac and Hypercube networks.

To my mother, Francis Gilliam Haynes,

and to my grandfather, Warren G. Gilliam.

ACKNOWLEDGEMENTS

First and foremost, I would like to acknowledge the invaluable guidance and assistance of my advisor, Dr. Robert C. Brigham. He gave freely of his time to encourage, listen, correct and advise. This dissertation would not have been possible without him. In addition to being a great advisor, Dr. Brigham is a very special person whom I feel fortunate to know. I extend to Dr. Brigham my sincere thanks and admiration.

I would also like to thank the other members of my research committee, Dr. Ronald D. Dutton, Dr. Charles E. Hughes and Dr. George Papadourakis, for their time and assistance.

Special acknowledgements are due to Julie Carrington, Professor Frank Harary and Dr. Ratan Guha for their help in Chapters 7 and 8 of this dissertation.

I would like to thank Court Clara and the people of CCl for technical support and Susanne Payne for editorial assistance. I also appreciate the technical support provided by Ben Goldfarb, Tony Becker and Tony Travaglini.

I would like to thank my grandmother, Nell Gilliam, for always being there for me. Also, I want to thank my sister, Lisa Newman, and her family for their prayers and encouragement. In addition, I appreciate my father, U. G. Haynes, and his family. All my other family members and friends are to be commended for their moral support. In particular I would like to recognize: Dennis Gilliam and his family, Billy Gilliam and his family, Michelle Hawkins, Susan Rice, Janie Horn and Janna Boggs.

The companionship and assistance of my friends, Jennifer Burg, Donna Williams, Viva Wingate, Greg Schaper, Julie Carrington, Chris Walker, Jay Boland, Sajal, Semwal, Glenn Wendt, Bill Hughes and Greg Hanson, really made a difference in my life at UCF. I would particularly like to thank Ranga for his friendship and support throughout.

Finally, I would like to express my love and appreciation to my husband, Jack, for understanding and providing a supportive working environment.

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1. INTRODUCTION

This dissertation is concerned with a theoretical problem in graph theory. In recent years graph theory has emerged as a useful tool in the modeling of a number of computer science problems, including network design, routing, algorithm design, compiler design, Petri nets, complexity theory, deadlock detection, VLSI design, and processor interconnections in multiprocessor systems.

In his book, Graph Theory, Harary (1972), whom many recognize as the father of modern graph theory, links computer science to the growth in the interest in graph theory by establishing applications of graph theory in both communication science and computer technology. Harary's (1987) editorial in the Journal of Graph Theory stresses the importance of graph theory as a mathematical model in many scientific fields. However, he specifically spotlights computer science as follows:

> I have deliberately saved by far the currently most important area of application of graph theory for last. Of course I am referring to computer science. We are all aware that the computer/information revolution is only just beginning. It is not sufficiently emphasized that graph theory pervades all three major branches of computing: hardware, software, and theory. Among the topics within these three branches to which graph theoretic models are especially applicable are: computer languages, logic circuits and switching theory, computer networks and reliability, interconnection networks for parallel processors such as hypercubes, fault-tolerant and diagnostic graphs, VLSI (very large scale integration) design, and, of course, AI (artificial intelligence), in particular, semantic networks. (iv)

Tannenbaum (1981) devotes a chapter of his book, Computer Networks, to a direct application of graph theory in network design. A goal of network design is to connect desired sites with a minimum number of links in order to minimize costs. However, to achieve high reliability with unreliable components, the network must be redundant. Techniques from graph theory are paricularly useful in producing a minimum cost design that meets the specified requirements. Several books survey the applications of graph theory in different disciplines (Roberts 1978; Temperley 1981; Walther 1984; Wilson and Beineke 1979).

With an awareness of significant applications to computer science as well as to many other areas and an appreciation for the beautiful field of graph theory as motivation, we introduce the graph theoretical problem under investigation in this dissertation. All terms not specifically defined here can be found in Chartrand and Lesniak (1986).

All graphs considered are finite and undirected with no loops or multiple edges. For a graph G = (V,E) we denote the cardinality of V as p. An edge joining nodes u and v is uv and the degree of node v is d_v. The set of nodes which are adjacent to v is node v's open neighborhood denoted N(v). A graph G is connected if every pair of its nodes are joined by a path.

A subset of nodes D⊆V is a dominating set for a graph G if every node of G is either in D or is adjacent to some node of D. The domination number γ(G) is the minimum size of any dominating set. A connected graph is edge domination insensitive, or just γ-insensitive, if $\gamma(G) = \gamma(G-e)$ for any edge e, where G-e is the graph obtained from G by removing e. Dutton and Brigham (1988) consider the problem of finding extremal graphs having the γ-insensitive property. A γ-insensitive extremal graph on p nodes is one which has the minimum number of edges.

In this dissertation we extend the notion of γ-insensitivity by considering the removal of more than one edge. Thus we define a connected graph G to be k edge domination insensitive, or just k-γ-insensitive, if the domination number γ is unchanged when an arbitrary set of k edges is removed. The problem has been solved for $k = 1$ (Dutton and Brigham 1988). This dissertation focuses on finding extremal graphs, that is, graphs having the smallest number of edges required for any k-γ-insensitive graph on p nodes, for $k \geq 2$.

Several subproblems are possible. We consider two. The first insists that the same fixed set of γ nodes dominate G no matter which set of k edges is removed. In the second the only restriction is the initial connectedness of G.

According to Bollobás (1978) in his book Extremal Graph Theory, problems in the field of extremal graph theory tend to be difficult. The search for extremal k-γ-insensitive graphs proved to be no exception. As a result, the scope of this research is limited to finding exact values in the special cases when $\gamma = 1$ and $k \geq 2$ and when $\gamma = 2$ and $k = 2$. The minimum number of edges required in these cases was determined by establishing relevant structural properties of extremal graphs. Using restrictions provided by these structural properties and extensive counting arguments, we obtain the solution for $\gamma = 2$ and $k = 2$. The approach used for $\gamma = 2$ and $k = 2$ breaks down for $\gamma \geq 3$ and $k \geq 2$, and the exact result has proved elusive. Nevertheless, an asymptotically correct value, as p approaches infinity, for $E^k(p,\gamma)$ when $k \geq 2$ has been found. Upper and lower bounds for $E^k(p,\gamma)$ differing by $O(\gamma^k)$ establish the asymptotic result. Furthermore, the gap between the bounds is narrowed for $k = 2$ and $\gamma \geq 3$ achieving tighter bounds for all values of p and exact results for some small values of p.

One is always interested in possible applications of theoretical results. Network design represents one such area for the k-γ-insensitive property. A network corresponding to an extremal graph will be fault tolerant in terms of domination when any k links fail, and will have the smallest number of links among all such networks. That is, a network which can be represented by an extremal k-γ-insensitive graph has a minimized link cost and the property that some set of γ nodes can communicate directly in one hop with the other p-γ nodes even after k links fail.

We introduce a special 2-γ-insensitive graph called the G-network which is a suitable architecture for point-to-point and interconnection networks. The G-network has the following desirable characteristics: efficient and simple routing, small number of links and high degree of fault tolerance. As a point-to-point network with designated file servers, the G-network provides all nonisolated nodes with direct access to the file servers even if a node or up to two links fail. As an interconnection network, the G-network is more economical than each of the well-known Illiac Mesh, Barrel Shifter, and Hypercube networks in terms of the number of links (Hwang and Briggs 1984). Furthermore, the G-network shows a significant improvement over the others in the maximum number of routing steps required for any pair of nodes to communicate. Unlike most interconnection networks where the maximum number of routing steps required is dependent upon the number of processors (nodes), the maximum number of routing steps needed in the G-network is constant at four independent of the number of processors (nodes) and remains four when a single node or link fails. For massively parallel computation we construct a multilayered interconnection network by interconnecting copies of the G-network in parallel.

The features of these networks are an integral part of the graph theoretic design and remain relatively intact in spite of faults in the system. We note that the inherent design was a by-product of our search for extremal 2-γ-insensitive graphs.

The remainder of this dissertation will be organized as follows. In Chapter 2, we present a literature survey with two major sections. The first section considers the general area of domination theory in graphs. The second section examines the specific problem of edge insensitive domination and presents results from Dutton and Brigham's (1988) study of this concept. Chapter 3 includes the results for the first subproblem, the fixed dominating set case, and presents a modified version of this subproblem. We employ $E_f^k(p,\gamma)$ to represent the minimum number of edges required in this case.

The major contributions of this dissertation result from the second subproblem and are contained in Chapters 4, 5, and 6. $E^k(p,\gamma)$ represents the minimum number of edges required for this subproblem where the only restriction is connectedness of the original graph. Chapter 4 presents the asymptotically correct result for $E^k(p,\gamma)$ when $k \geq 2$ and the exact value when $k \geq 1$ when $\gamma = 1$. The exact results for $k = 2$ and $\gamma = 2$ are in Chapter 5. Chapter 6 tightens the bounds from Chapter 4 for $E^k(p,\gamma)$ when $k = 2$ and $\gamma \geq 3$.

Chapter 7 introduces the G-network and the multi-layered G-network as examples of applications of k-γ-insensitive graphs.

Related problems posed by Frank Harary (1988) are presented in Chapter 8. These problems consider when a graphical invariant changes or does not change due to the addition or removal of a node or an edge. Note that a 1-γ-insensitive graph is an extremal unchanging graph in terms of domination when an edge is removed. We present a survey of the known results along with our own research on the problem of changing and unchanging when the graphical invariant is the domination number $\gamma(G)$.

Chapter 9 concludes the dissertation and outlines suggested future work. The appendix contains a catalog of all 2-γ-insensitive graphs on $p \leq 10$ nodes.

2. A SURVEY OF DOMINATION LITERATURE

The work reported in this dissertation is a contribution to the general area of domination theory in graphs. The purpose of the first section of this literature survey is to highlight some of the recent developments in domination theory. However, our intent is not to give details of all aspects of domination; rather it is to provide a general background of domination that will set the stage for the second section's overview of edge domination insensitivity, which is the basis for our research.

2.1. Theory of Domination

The graphical invariant known as domination number, which is represented for graph G by the symbol $\gamma(G)$, has been studied extensively. According to Hedetniemi and Laskar (1987), more than 100 research articles on the subject of domination had been written in the four years prior to the publication of their paper. We shall present a brief survey of the literature on general domination theory in this section, which is subdivided into the following four subsections: applications, bounds on domination number, algorithms, and domination related concepts. Because of the volume of papers on domination theory and because our intent is to give just the flavor of the research in the general area of domination, we provide only a small sample of the current work for each of the subsections.

2.1.1. Applications

A graph having domination number γ corresponds to a network having p nodes where a minimum number, γ, of them can communicate directly (in one hop) with the remaining $p-\gamma$ nodes. Additional examples of applications for domination are given by Cockayne and Hedetniemi (1977). For example, they describe a situation in coding theory where the graphs have nodes which are n-dimensional vectors and two nodes are adjacent if they differ in exactly one component. Then the single error correcting codes correspond to the dominating sets of the graph which have certain additional properties. They also present the problem of keeping all nodes in a network under surveillance by a set of radar stations, the dominating set.

Similarly, Roberts (1978) suggests an interesting application in nuclear power plants. The plants (nodes) are positioned in various locations, and an arc is placed from location x to location y if it is possible for a watchman at x to observe a warning light at location y. What is the minimum number of guards necessary and where should they be located? The answer to this question corresponds to a minimum dominating set of a directed graph.

Hare, Hedetniemi, and Hare (1986) explore determining the domination of grid graphs. Grid graphs are frequently studied models of processor interconnections in multiprocessor VLSI systems.

A dominating set in such a graph represents a set of processors which can transmit to the remaining processors in one "hop."

The examples presented here are a mere sampling of the possible applications of domination theory. For more detail of applications in a wide variety of fields, including communication theory, computer science, psychology and political science see Cockayne (1976); Cockayne, Dawes and Hedetniemi (1980); Cockayne and Hedetniemi (1977); Brigham and Dutton (1988b); Hedetniemi and Laskar (1987); Hedetniemi, Laskar and Pfaff (1985); Roberts (1978); and Tannebaum (1981).

2.1.2. Bounds on the Domination Number

Relationships between graphical invariants represent an important area of research. The literature includes several bounds on the domination number in terms of other invariants, and many of these are cited in Allan, Laskar, and Hedetniemi (1984); Brigham and Dutton (1988a); Cockayne and Hedetniemi (1977); Hedetniemi and Laskar (1987); Laskar and Walikar (1981); and Marcu (1985). We note only a few of the more common bounds listed in these references. Let $G = (V,E)$ be a graph with domination number γ, minimum degree δ, and maximum degree Δ. Then

1. $p-e \leq \gamma \leq p-\Delta$
2. $e \leq (p-\gamma)(p-\gamma+2)/2$
3. $p/(\Delta+1) \leq \gamma$
4. If there are no isolated nodes, $\gamma \leq p/2$

5. If there are no isolated nodes, $\gamma \leq (p-\delta-2)/2$.

Fewer lower bounds than upper bounds exist for $\gamma(G)$ (Hedetniemi and Laskar 1987). A recent paper by Brigham and Dutton (1988a) establishes both lower and upper bounds for $\gamma(G)$.

Some reports consider finding bounds on domination number which are valid only for a specific family of graphs. For example, Cockayne, Hare, Hedetniemi, and Wimer (1985) determined both upper and lower bounds for the domination number of grid graphs. They concluded that the problem of determining the domination number of an nXn grid graph is closely related to the open problem of determining the minimum number of queens which are required to dominate an nXn chessboard (See Section 2.1.4).

2.1.3. Domination Algorithms

The problem of determining the domination number of an arbitrary graph is NP-complete (Garey and Johnson 1979). Hedetniemi and Laskar (1987) state that computing the domination number for several restricted classes of graphs, including planar, grid, perfect, chordal, split, bipartite, and line graphs, remains NP-complete. On the other hand, in the same paper they reference polynomial algorithms for finding the domination number of trees, forests and strongly chordal graphs.

Furthermore, Hetdetniemi, Laskar, and Pfaff (1986) present a linear time algorithm for finding a minimum dominating set in a cactus. Booth and Johnson (1982) show that the dominating set problem for chordal graphs and directed paths is NP-complete. However, they present a linear time algorithm that solves the problem for interval graphs. Moreover, Hedetniemi, Hedetniemi, and Laskar (1985) report that Gurevich, Stockmeyer, and Vishken have designed a polynomial time algorithm that is successful for graphs that differ from trees by only a fixed number of edges. Hare, Hedetniemi, and Hare (1986) solve the problem in polynomial time on kXn complete grids for any fixed k.

Other polynomial algorithms and NP-completeness results for domination and domination related concepts appear in Hare, Hedetniemi, and Hare (1986); Hedetniemi, Hedetniemi, and Laskar (1985); Hedetniemi and Laskar (1987); Hedetniemi, Laskar, and Pfaff (1986); and Pfaff, Laskar, and Hedetniemi (1984).

2.1.4. Domination Related Concepts

The literature provides many alternative approaches to the concept of domination. According to Hedetniemi and Laskar (1987) approximately 30 different notions of domination are currently known and "...many more can just as easily, and naturally, be defined" (18).

Conditions can be imposed on both the dominating set and the dominated set to yield new concepts. Then, as with the standard notion of domination, one is usually interested in the minimum sized dominating sets of the prescribed types. We mention only a few of the possibilities in this section.

The edge domination number of a graph G, $\gamma'(G)$, is the minimum number of edges in a set F such that every edge in E-F is adjacent to at least one edge in F. Like the node domination problem, the edge domination problem is NP-complete. For bounds and results involving $\gamma'(G)$ see Hedetniemi, Hedetniemi, and Laskar (1985); and Laskar and Peters (1985).

The independent domination number of a graph G, i(G), is the minimum cardinality of any dominating set in which all nodes of the dominating set are independent, that is, no edge joins any two of them. Independent domination is also NP-complete, and thus research has concentrated on establishing bounds for i(G) (Laskar and Walikar 1981) and in developing polynomial time algorithms to determine i(G) for certain families of graphs (Farber 1984; Pfaff, Laskar, and Hedetniemi 1984).

The concept of dominating cliques requires that the dominating set induce a complete subgraph (clique). This notion is in some sense the opposite of independent domination. However, an independent dominating set always exists for a graph whereas a clique dominating set may not. The literature contains characterizations of graphs possessing dominating cliques, bounds for the clique domination number, $\gamma_k(G)$, and algorithms for determining $\gamma_k(G)$ (Cozzens and Kelleher 1986). This type of domination has possible applications in network and communications theory where each node in a designated set of nodes has the ability to communicate directly with every other node in the set. Further, every node which is not in the set can communicate directly with at least one node in the set. In their paper Cozzens and Kelleher (1986) suggest that it might be important in setting up network links to provide this ability for a strong core group that needs to communicate directly with each other member of the core.

Another notion of domination which places a restriction on the dominating set is called connected domination. As the name suggests, the dominating set must induce a connected subgraph. Although this concept is in a sense less restrictive than clique domination, it could be applicable in network design where members of the core group must be connected by a path in the core. Of course this concept is defined only for connected graphs. Laskar and Peters (1983) establish several bounds for the connnected domination number, $\gamma_c(G)$.

A well known problem involving a dominating set, the queens problem, is to determine the smallest number of queens which can be placed on a chessboard so that every square is dominated by at least one queen. Among the many solutions to this problem, one solution requires that every queen be dominated by at least one other queen (Allan, Laskar, Hedetniemi 1984). This solution suggests yet another domination concept called total domination. A set of nodes D is a total dominating set if each node in V is adjacent to some node in D. Notice that this definition demands that each node in the set D must be adjacent to another node in D. We note that clique dominating sets and connected dominating sets having at least two nodes are total dominating sets. However, total dominating sets are not necessarily connected or clique dominating sets. Again we can speculate about the applications for this type of domination in a network which requires that each of the members of the core group has a direct link to at least one other member of the core group. Bounds involving total domination are given in Allan, Laskar, and Hedetniemi (1984); Cockayne, Dawes, and Hedetniemi (1980); and Krishnamoorthy and Murthy (1986).

Two additional interesting domination concepts, dominating cycles and paths, are closely related to the existence of Hamiltonian cycles and paths in graphs. A dominating cycle(path) is a cycle(path) of G for which every node of G is incident to at least one node of the cycle(path). A D-cycle(D-path) is defined by substituting the word edge for node in the above definition. Research considering a D-cycle(D-path) as a generalization of a Hamiltonian cycle(path) is presented in Clark, Colburn, and Erdös (1985); and Veldman (1983).

Brigham and Dutton (1988b) introduce the factor domination number. A graph $H = (V,E)$ has a t-factoring into factors G_1, G_2, ..., G_t if each graph $G_i = (V_i, E_i)$ has node set $V_i = V$ and the collection $\{E_1, E_2, \ldots, E_t\}$ forms a partition of E. A factor dominating set is a single subset of nodes which dominates each of the t factors of a graph H. The factor domination number, γ_f, is the size of a smallest such set. Observe that when $t = 1$ we can interpret γ_f as the ordinary domination number of H. Brigham and

Dutton (1988b) suggest possible applications of the factor domination number in communication networks. For example, the graph H could be considered as a communication network composed of t subnetworks (the factors G_i). Determining the factor domination number of a graph H and its factors representing such a structure yields the minimum number of "master" stations required so that a message can be transmitted from them to the remaining $p-\gamma_f$ stations in one hop, as long as at least one subnetwork is active. That is, these "master" stations can communicate directly to all desired sites as long as not all the subnetworks fail. Their paper establishes bounds for γ_f in terms of other graphical invariants.

Finally, we consider the concept of domination critical graphs. A node v is critical if $\gamma(G-v) < \gamma(G)$ and G is node domination critical if each node is critical. Brigham, Chinn, and Dutton (1988) envision applications in network theory:

> Such networks have the pleasant characteristics that (1) any processor can be in a minimum set of these "dominating" processors, and (2) the failure of any processor leaves a network which requires one fewer dominating processors. (2)

Closely related to this study are the works of Bauer, Harary, Nieminien, and Suffel (1983); and Sumner and Blitch (1983). Sumner and Blitch (1983) studied graphs where the domination number decreases when any edge is added. On a related problem, Dutton and Brigham (1988) define the edge domination insensitive property, which is the basis for our research. We postpone discussion of this property to the next section.

For additional domination concepts and related parameters, including irredundance number, domination pair number, and domatic number, see Chung, Graham, Cockayne, and Miller (1982); Favoron (1986); Hedetniemi, Hedetniemi, and Laskar (1985); Hedetniemi and Laskar (1987); Hedetniemi, Laskar, and Pfaff (1985); and Rall and Slater (1984).

2.2. Edge Domination Insensitivity

In the previous sections we have described briefly different concepts in domination. We now return to the specific domination related problem under consideration in this dissertation. It was Dutton and Brigham's (1988) study of the edge domination insensitive problem that laid the theoretical foundation for the present work. Thus we devote this entire section to reporting the results from their paper, which is the only existing paper in the literature on edge domination insensitivity (Dutton and Brigham 1988).

2.2.1. Problem Description and Applications

A connected graph G is edge domination insensitive if $\gamma(G) = \gamma(G\text{-}e)$ for any edge e of G. For brevity we shall say γ-insensitive. Dutton and Brigham (1988) consider the problem of finding extremal graphs having the γ-insensitive property. In this context graphs on p nodes are extremal if they have the smallest possible number of edges. In their quest to find extremal graphs, they define and solve three different subproblems. We shall present each of these subproblems in Section 2.2.2.

Since the same number of nodes dominate a γ-insensitive graph after removal of any edge, a network having the γ-insensitive property could be considered fault tolerant in terms of domination when a link goes down. In the words of Dutton and Brigham:

> It is interesting to speculate on applications for γ-insensitive graphs. One can, for example, contemplate minimum link communication networks having p stations where γ of them can transmit a message to the remaining $p-\gamma$ stations with no message traversing more than one communication link. For networks corresponding to γ-insensitive graphs this property is preserved whenever a single communication link fails. (2)

2.2.2. Results for Extremal γ-insensitive Graphs

The first subproblem considered by Dutton and Brigham (1988) insists that the same fixed set of γ nodes dominates G and G-e for all edges e of G. In this case they let $E_f(p,\gamma)$ represent the minimum number of edges needed in an extremal graph. The major result for the fixed case is given in the following theorem.

<u>Theorem A</u> (Dutton and Brigham 1988)

$$E_f(p,\gamma) = \begin{cases} 2p-2\gamma & \text{for } \gamma \geq 2 \text{ and } p \geq 3\gamma-2 \\ \text{undefined} & \text{otherwise.} \end{cases}$$

The second subproblem no longer requires that the same fixed set of nodes dominates, merely that some set of γ nodes will dominate G-e for any edge e of G. Letting $E(p,\gamma)$ represent the minimum number of edges required, we summarize their results for this problem in the following theorem.

Theorem B (Dutton and Brigham 1988)

$$E(p,\gamma) = \begin{cases} 3p-6 & \text{if } \gamma = 1 \text{ and } p \geq 3 \\ p-1 & \text{if } \gamma \geq 2 \text{ and } 2\gamma \leq p \leq 3\gamma-2 \\ p & \text{if } \gamma \geq 2 \text{ and } p = 3\gamma-1 \\ 2p-3\gamma & \text{if } \gamma \geq 2 \text{ and } p \geq 3\gamma \end{cases}$$

The final subproblem adds the restriction that G-e be connected for every edge e. Their results for this problem are summarized in the following theorem.

Theorem C (Dutton and Brigham 1988)

$$E_c(p,c) = \begin{cases} E(p,\gamma) = 3p-6 & \text{if } \gamma = 1 \text{ and } p \geq 3 \\ E(p,\gamma)+1 = p & \text{if } \gamma \geq 2 \text{ and } p = 3\gamma-2 \\ E(p,\gamma) = p & \text{if } \gamma \geq 2 \text{ and } p = 3\gamma-1 \text{ or } 3\gamma \\ E(p,\gamma)+1 = p+2 & \text{if } \gamma \geq 2 \text{ and } p = 3\gamma+1 \\ E(p,\gamma) = 2p-3\gamma & \text{if } \gamma \geq 2 \text{ and } p \geq 3\gamma+2 \end{cases}$$

The results for this subproblem are still unknown for $2\gamma \leq p \leq 3\gamma-3$. However, it is known that no γ-insensitive graphs exist when $p \leq 3\gamma-3$ if $\gamma = 2$, 3 or 4.

The reader is referred to Dutton and Brigham (1988) for illustrations of extremal γ-insensitive graphs in each of the three subproblems and for proofs of the theorems presented here.

In this dissertation we consider the more general problem of graphs having the γ-insensitive property when more than one edge, say k edges, are removed.

3. FIXED DOMINATING SET

The first of the two subproblems to be considered insists that the same fixed set of γ nodes dominate G and G-E' for all sets $E' \subseteq E$ where $|E'| = k$. Designate the fixed set of nodes by $V_1 = \{a_1, a_2, \ldots, a_\gamma\}$ and the remaining nodes by $V_2 = \{b_1, b_2, \ldots, b_{p-\gamma}\}$. We first observe that $\gamma \geq 2$ since a single node which dominates the entire graph cannot dominate the graph obtained by removing any k edges incident to it.

Lemma 3.1 defines characteristics of an extremal graph for this subproblem.

<u>Lemma 3.1</u>

Any extremal k-γ-insensitive graph with fixed dominating set V_1 and $E_f^k(p,\gamma)$ edges is a bipartite graph with partite sets V_1 and V_2. Furthermore, each node of V_2 has degree k+1.

<u>Proof</u>

V_1 is a fixed dominating set. Dominance is unaffected by edges between two nodes of V_1 or two nodes of V_2 and therefore no such edge is necessary. Any node $b_i \epsilon V_2$ must have degree at least k+1 so it can still be dominated by V_1 when k of its incident edges are removed. On the other hand, it is never necessary to have more than k+1 edges incident to b_i since b_i can be dominated via any unremoved edge. ∎

For $1 \le i \le \gamma$, define A_i to be those nodes of V_2 which are adjacent to $a_i \epsilon V_1$, i.e., $A_i = \{b_j | a_i b_j$ is an edge of G}.

<u>Theorem 3.1</u>

The fixed dominating set problem has no solution for $k \ge 2$.

<u>Proof</u>

Suppose G is an extremal bipartite graph with partite sets V_1 and V_2 such that V_1 is a fixed dominating set and $|V_1| = \gamma$. By Lemma 3.1 each node of V_2 has degree k+1. Thus $\gamma \ge k+1$ and it is possible to label the nodes of V such that $a_1, a_2, \ldots, a_{k+1}$ have a common neighbor, say b_1. A complete bipartite subgraph is induced by $\{a_1, a_2, \ldots, a_{k+1}\} \cup (A_1 \cap A_2 \cap \ldots \cap A_{k+1})$. Nodes a_1 and b_1 dominate this subgraph. Any node $b \epsilon V_2 - (A_1 \cap A_2 \cap \ldots \cap A_{k+1})$ must be dominated by some $a_i \epsilon V_1$, $i > k+1$, else it would be in the intersection. Thus $\{b_1\} \cup (V_1 - \{a_2, a_3, \ldots, a_{k+1}\})$ is a dominating set of size $2+(\gamma-k-1)$ $= \gamma-k+1$ which is less than γ, and hence creates a contradiction, when $k \ge 2$. Therefore, the problem as stated has no solutions unless $k = 1$. ∎

In view of Theorem 3.1, we consider a modified version of this subproblem where the fixed dominating set V_1 is not required to be a minimum dominating set if $k \ge 2$. A graph is said to be <u>k-edge</u>

domination fixed, or simply fixed, if V_1 will still dominate G when any k edges are removed.

Since V_1 can no longer be a fixed dominating set of size γ, we select a fixed dominating set of size $m_f > \gamma$. That is, we let $|V_1| = m_f$ and define $E_f^k(p,m_f)$ to be the minimum number of edges required in a k-edge domination fixed graph. The proof to the following lemma is identical to that of Lemma 3.1.

Lemma 3.2

Any extremal fixed graph is bipartite with partite sets V_1 and V_2. Furthermore each node of V_2 has degree k+1.

The following theorem is an immediate consequence of Lemma 3.2.

Theorem 3.2

$$E_f^k(p,m_f) = (k+1)(p-m_f).$$

Clearly $m_f \geq k+1$ since each node in V_2 has degree k+1. This condition is sufficient in order for the revised problem to have a solution.

Theorem 3.3

Extremal graphs exist for each value of m_f and k such that $m_f \geq k+1$ and $k \geq 2$.

Proof

Let $b_i \epsilon V_2$ be adjacent to any k+1 nodes of V_1. ■

Before presenting Theorem 3.4, we introduce relevant definitions and notation. A matching $M \subseteq E$ in G is a collection of edges no two of which share a common endpoint. Matching M saturates $X \subseteq V$ if every node of X is an endpoint of some edge in M. The open neighborhood of set $S \subseteq V$ is $N(S) = \{x \epsilon V | x$ is adjacent to at least one $s \epsilon S\}$.

We shall say that graph $H = (V,E)$ has t k-edge diminished subgraphs $G_1, G_2, ..., G_t$, called remainders, if each graph $G_i = (V_i, E_i)$ has node set $V_i = V$ and $E_i = E - \{e_1, e_2, ..., e_k\}$ for some set of k edges, $1 \le i \le t$. Subset D_r of V is a remainder dominating set if D_r is a dominating set for each G_i, $1 \le i \le t$, and the remainder domination number γ_r is the size of a smallest remainder dominating set. The following lemma will be useful in the proof to Theorem 3.4.

Lemma (Hall)

Let $G = (X,Y,E)$ be a bipartite graph where X and Y form the bipartite partition of V. Then there is a matching which saturates X if and only if $|N(S)| \ge |S|$ for all $S \subseteq X$.

The final theorem shows that a fixed dominating set is uniquely determined once m_f and k are specified.

Theorem 3.5

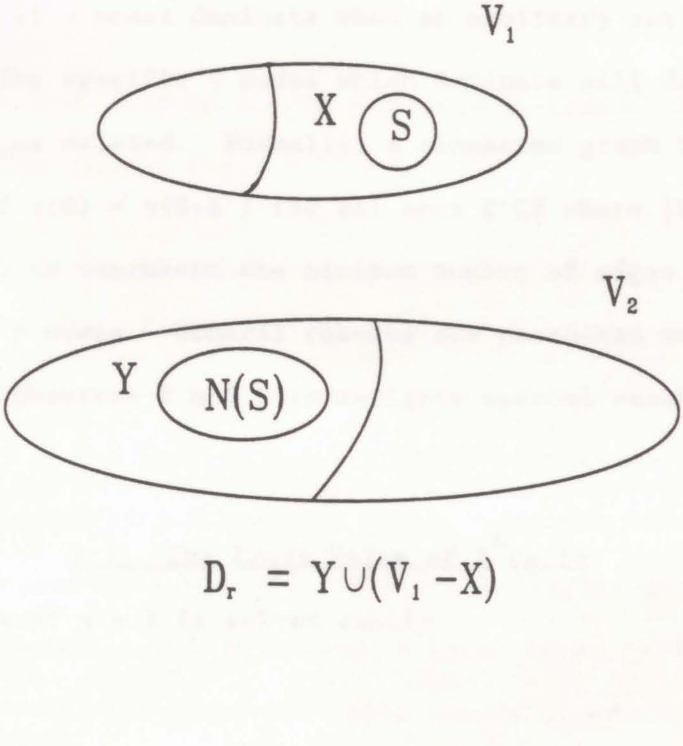
Let H be a connected k-edge domination fixed graph having $m_f = |V_1| \le |V_2|$ and the minimum number of edges. Then $\gamma_r = m_f$ and V_1 is the only minimum remainder dominating set.

Proof

By construction V_1 is a remainder dominating set for the bipartite graph H so $\gamma_r \le |V_1| = m_f$. Now let D_r be a minimum remainder dominating set different from V_1. See Figure 3.1. Every node $b_i \epsilon V_2 - D_r$ has all k+1 of its edges incident to $V_1 \cap D_r$ or there would be a remainder subgraph in which b_i is not dominated by D_r. It follows that all neighbors of $a_j \epsilon V_1 - D_r$ must be in $V_2 \cap D_r$. Note that $V_1 \cap D_r$ contains all nodes $a_j \epsilon V_1$ of degree at most k since there is a remainder subgraph which isolates a_j. Thus every node of $V_1 - D_r$ has degree at least k+1. Let $X = V_1 - D_r$, $Y = V_2 \cap D_r$, S be an arbitrary subset of X, E_s be the number of edges incident to S, and $E_{N(s)}$ be the number of edges incident to N(S). Then $(k+1)|N(S)| = E_{N(s)} \ge E_s \ge (k+1)|S|$ implying $|N(S)| \ge |S|$. By Hall's lemma there is a matching which saturates X. Thus $\gamma_r = |D_r| = |V_1 \cap D_r| + |V_2 \cap D_r| \ge |V_1 \cap D_r| + |V_1 - D_r| = |V_1| = m_f$. Hence $\gamma_r = m_f$ and it follows that $|V_1 - D_r| = |V_2 \cap D_r|$.

Notice that the number of edges considered so far is at least $(k+1)|V_1 - D_r| + (k+1)|V_2 - D_r| = (k+1)[|V_2 \cap D_r| + |V_2 - D_r|] = (k+1)|V_2|$ which is the minimum number possible. All of these edges are between X and Y and between $V_1 - X$ and $V_2 - Y$. Since the graph is extremal, these are the only edges. Because $V_2 \cap D_r \ne \emptyset$, the graph is disconnected. This establishes the contradiction which shows that V_1 is the only remainder dominating set. ■

This completes discussion of the first subproblem.



$$D_r = Y \cup (V_1 - X)$$

Figure 3.1

disconnected. This establishes the contradiction which shows that V_1 is the only remainder dominating set. ■

This completes discussion of the first subproblem.

4. SOME GENERAL RESULTS

In this chapter and the two following we consider the second subproblem described in Chapter 1 in which all that is required is that some set of γ nodes dominate when an arbitrary set of k edges is removed. The specific γ nodes which dominate will depend on the particular edges deleted. Formally, a connected graph G is k-γ-insensitive if $\gamma(G) = \gamma(G-E')$ for all sets $E' \subseteq E$ where $|E'| = k$. We employ $E^k(p,\gamma)$ to represent the minimum number of edges for such graphs having p nodes. General results are presented in this chapter while Chapters 5 and 6 investigate special cases in greater detail.

4.1. The Exact Value of $E^k(p,1)$

The case of $\gamma = 1$ is solved easily.

<u>Theorem 4.1</u>

Let $k \geq 1$. Then $E^k(p,1) = (2k+1)(p-k-1)$ if $p > 2k$ and is undefined if $p \leq 2k$.

<u>Proof</u>

Let G be a k-γ-insensitive graph having p nodes and $\gamma = 1$. Removal of any k edges involves at most 2k nodes of G. Suppose G has at most 2k nodes of degree p-1. Then k edges can be removed in such a way that none of these nodes can dominate the resulting graph, implying that the original graph is not k-γ-insensitive. Thus G must have at least 2k+1 nodes with degree p-1, so $E^k(p,1) \geq (1/2)[(2k+1)(p-1)+(p-2k-1)(2k+1)] = (2k+1)(p-k-1)$. Consider a graph with exactly 2k+1 nodes of degree p-1 and no other edges. This graph is k-γ-insensitive, so equality holds. ■

We note that Theorem 4.1 is a generalization of Dutton and Brigham's (1988) result for k = 1.

4.2. An Asymptotic Value for $E^k(p,\gamma)$

The remainder of the chapter is devoted to the derivation of an asymptotically correct value, as p approaches infinity, for $E^k(p,\gamma)$ when $k \geq 2$. Section 4.2.1 demonstrates an upper bound for $E^k(p,\gamma)$ by constructing a specific graph. Section 4.2.2 then develops some general properties of k-γ-insensitive graphs which are employed in Section 4.2.3 to compute a lower bound on $E^k(p,\gamma)$. The difference between the lower and upper bounds found will be independent of p, so the asymptotic result will be established.

4.2.1. The Upper Bound

To establish the upper bound we construct a family of k-γ-insensitive graphs under the assumption that $\gamma \geq k+1$ and $p \geq \gamma(k+1)$. Let $n = \lfloor (p-\gamma)/k \rfloor$ and $r = (p-\gamma) \bmod k$. Notice that $n \geq \gamma$. Graphs G = (V,E) having the desired properties are created as follows:

(1) $V = A \cup B_1 \cup B_2 \cup \ldots \cup B_n$ where $A = \{a_1, a_2, \ldots, a_\gamma\}$, $B_i =$

$\{b_{i1}, b_{i2}, \ldots, b_{ik}\}$ for $1 \le i \le n-1$ and $B_n = \{b_{n1}, b_{n2}, \ldots, b_{n,k+r}\}$.

(2) Each B_i, $1 \le i \le n$, induces a complete subgraph.

(3) Each b_{ij} is adjacent to exactly two nodes of A, one of which is a_1. The other is a_s for $s \ge 2$ subject to the restriction that at least k distinct a_s for $s \ge 2$ are adjacent to each B_i.

(4) Every a_s, $s \ge 2$, is adjacent to a b_{ij} for at least γ-k values of i.

It is straightforward to verify that G is connected, that b_{ij} and b_{it} are adjacent to distinct a_s for $s \ge 2$ and $i \le n-1$, that the set of nodes b_{nj} are adjacent to at least k distinct a_s for $s \ge 2$, and that G has $2(p-\gamma)+[\lfloor(p-\gamma)/k\rfloor-1][k(k-1)/2]+[(k+r)(k+r-1)]/2 \le (k+3)p/2-[(k+3)\gamma-2kr-r^2+r]/2$ edges. Figure 4.1 shows a graph having $p = 17$, $\gamma = 4$ and $k = 3$ which has been constructed according to the above specifications.

We must now show that G is k-γ-insensitive, and the next lemma is a first step.

Lemma 4.1

Any graph G constructed as above has domination number γ.

Proof

The set A dominates G so the domination number is at most γ. It remains to be shown that any dominating set D contains at least γ nodes. Certainly D contains at least γ-1 nodes since each a_s, $2 \le s$

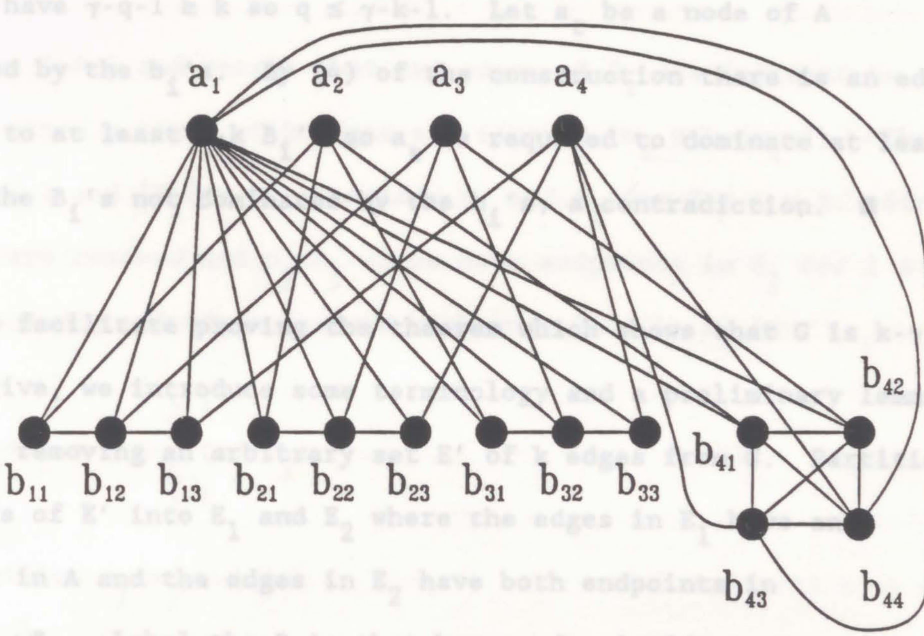


Figure 4.1

$\leq \gamma$, must be dominated either by itself or by an adjacent b_{ij}. But any b_{ij} can dominate only one such a_s. Thus if $a_1 \epsilon D$, $|D| \geq \gamma$. Suppose then that $a_1 \notin D$ and that $|D| = \gamma-1$. Let $b_1, b_2, \ldots, b_q$ designate the b_{ij}'s in D. Then these nodes of D can dominate at most q B_i's and q+1 a_i's, including a_1. All the remaining B_i's, and there must be at least one, have to be dominated by the b_i's. Thus we must have $\gamma-q-1 \geq k$ so $q \leq \gamma-k-1$. Let a_t be a node of A dominated by the b_i's. By (4) of the construction there is an edge from a_t to at least $\gamma-k$ B_i's so a_t is required to dominate at least one of the B_i's not dominated by the b_i's, a contradiction. ■

To facilitate proving the theorem which shows that G is k-γ-insensitive, we introduce some terminology and a preliminary lemma. Consider removing an arbitrary set E′ of k edges from G. Partition the edges of E′ into E_1 and E_2 where the edges in E_1 have an endpoint in A and the edges in E_2 have both endpoints in $B_1 \cup B_2 \cup \ldots \cup B_n$. Label the B_i's that have nodes incident to edges of E_1 by $S_1, S_2, \ldots, S_t$. Assume for now that $b_1 \epsilon S_1$ is incident to two edges of E_1. For $1 \leq i \leq t$ let n_i be the number of nodes in S_i that are incident to edges of E_1 and S_i' be the set of the n_i nodes of S_i which form the endpoints of these edges. Furthermore, let m_i be the number of edges from E_2 which are incident to nodes in S_i'. Define T_i to be set of nodes $v \epsilon S_i - S_i'$, such that v dominates the nodes of S_i'. Observe that T_i includes the nodes of S_i which are not incident to edges of E_1 or to the edges of E_2 which are incident to nodes of S_i'. Thus $|T_i| \geq k-(n_i+m_i)$. The following lemma states a useful fact about the size of T_i.

Lemma 4.2

Under the circumstances outlined above, $|T_i| \geq i$ for $1 \leq i \leq t$.

Proof

Let $i = 1$ and recall that two edges of E_1 are incident to $b_1 \epsilon S_1$. Since a total of k edges are removed, $m_1 \leq k-(n_1+1)$. Thus $m_1+n_1 \leq k-1$ and $|T_1| \geq 1$. Consider n_i and m_i for $2 \leq i \leq t$. Since k edges are removed and n_j+m_j edges have endpoints in S_j for $1 \leq j \leq i-1$, the number of edges removed with endpoints in S_i is

$$n_i+m_i \leq k-1-\sum_{j=1}^{i-1}(n_j+m_j). \qquad (1)$$

By definition $n_j+m_j \geq 1$ so

$$\sum_{j=1}^{i-1}(n_j+m_j) \geq i-1.$$

Therefore (1) implies $n_i+m_i \leq k-i$ so $|T_i| \geq k-(n_i+m_i) \geq i$. ■

We are now ready to present Theorem 4.2 which establishes the upper bound for $E^k(p,\gamma)$. The previously defined terminology will be employed in the proof.

Theorem 4.2

Let $\gamma \geq k+1$ and $p \geq \gamma(k+1)$. Then $E^k(p,\gamma) \leq (k+3)p/2-[(k+3)\gamma-2kr-r^2+r]/2$.

Proof

Let G be constructed as described above. By Lemma 4.1 G has domination number γ. Since G has the number of edges stated in the theorem, we need only show that G is k-γ-insensitive. As before, we remove an arbitrary set E′ of k edges, and now show the resulting graph still has domination number γ. If each b_{ij} has at least one edge to a node of A in G-E′, then A dominates. Thus we need to consider only the situation where at least one b_{ij}, say $b_1 \epsilon S_1$, has both edges between it and nodes in A removed. Notice that a_1 can dominate $(B_1 \cup B_2 \cup \ldots \cup B_n)-(S_1 \cup S_2 \cup \ldots \cup S_t)$. Furthermore, the nodes in S_i-S_i' have two edges to A so a_1 will also dominate them. By Lemma 4.2, $|T_i| \geq i$ for $1 \leq i \leq t$. Thus T_i has at least i nodes which are not incident to edges of E′. That is, T_1 has at least one node, say x_1, that is adjacent to a_1 and an a_i, say a_r. T_2 has at least two such nodes. Observe that at least one node of T_2, say x_2, dominates a_s where $s \neq 1$, r. Continuing in this manner we have t nodes x_1, x_2, ..., x_t such that each x_i is adjacent to a different a_j, $j \geq 2$. Let A′ be the set of a_j's, $j \geq 2$, dominated by $\{x_1, x_2, \ldots, x_t\}$. Then $A \cup \{x_1, x_2, \ldots, x_t\}-A'$ is a set of size γ which dominates G-E′, thereby showing G is k-γ-insensitive. ■

The edge bound given by Theorem 4.2 is maximized when r is at its largest value of k-1, which leads immediately to the following corollary.

<u>Corollary 4.1</u>

Let $\gamma \geq k+1$ and $p \geq \gamma(k+1)$. Then $E^k(p,\gamma) \leq (k+3)p/2-[(k+3)\gamma-3k^2+5k-2]/2$.

The corollary shows that for fixed k and γ the bound is asymptotically equal to $(k+3)p/2$. Section 4.2.3 will establish a lower bound asymptotically equal to the same expression, and this will complete the proof of the major result of this chapter.

4.2.2. Properties of k-γ-insensitive Graphs

The following theorems establish relevant structural properties which will be useful in Section 4.2.3 in developing the lower bound for $E^k(p,\gamma)$. Let N_i be the maximum possible number of nodes v of degree at most k which can have i common neighbors in a k-γ-insensitive graph, $1 \leq i \leq k$. The first theorem gives a bound for N_k.

<u>Theorem 4.3</u>

$N_k \leq 2$.

<u>Proof</u>

Suppose a_1, a_2, and a_3 are degree k nodes with common neighbors $b_1, b_2, \ldots, b_k$. See Figure 4.2. Remove the k edges

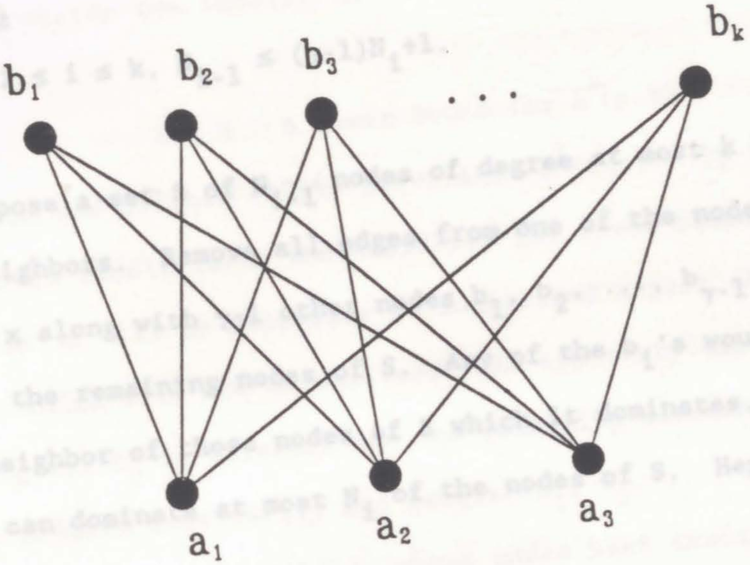


Figure 4.2

incident to a_1. Then a_1 must be in the dominating set D. None of a_i's neighbors can be in D or D-$\{a_1\}$ is a dominating set of size γ-1, a contradiction. Hence a_2 and a_3 are in D. But (D-$\{a_2, a_3\}$) $\cup\{b_1\}$ is a dominating set of size γ-1 for G, a contradiction. ■

<u>Theorem 4.4</u>

For $2 \le i \le k$, $N_{i-1} \le (\gamma-1)N_i+1$.

<u>Proof</u>

Suppose a set S of N_{i-1} nodes of degree at most k have i-1 common neighbors. Remove all edges from one of the nodes of S, say x. Then x along with γ-1 other nodes $b_1, b_2, \ldots, b_{\gamma-1}$ must dominate the remaining nodes of S. Any of the b_i's would be an i-th common neighbor of those nodes of S which it dominates. Thus each of them can dominate at most N_i of the nodes of S. Hence $|S| = N_{i-1} \le (\gamma-1)N_i+1$. ■

Let $f(\gamma,k)$ be the number of nodes in V-D with degree at most k where G is a k-γ-insensitive graph with a minimum dominating set D. The following theorem gives an upper bound for $f(\gamma,k)$.

<u>Theorem 4.5</u>

$f(\gamma,k) \le 2(\gamma-1)^k+(\gamma-1)[((\gamma-1)^{k-1}+1)/(\gamma-2)]+1$.

<u>Proof</u>

From Theorem 4.4, $N_{i-1} \le (\gamma-1)N_i+1$ for $2 \le i \le k$, and $N_k \le 2$ by Theorem 4.3. It is straightforward to solve this recurrence for

N_1 to get $N_1 \leq 2(\gamma-1)^{k-1}+[((\gamma-1)^{k-1}+1)/(\gamma-2)]$.

Now suppose x is a node of degree at most k and remove all edges incident to it. Then x must be in the dominating set D along with γ-1 other nodes. Each of these γ-1 nodes can dominate at most N_1 nodes with degree at most k. Thus $f(\gamma,k) \leq (\gamma-1)N_1+1$. Simplifying yields the result. ■

4.2.3. A Lower Bound for $E^k(p,\gamma)$

Dutton and Brigham (1988) prove that $E^1(p,\gamma) = 2p-3\gamma$ for $p \geq 3\gamma \geq 6$. This result forms the basis for the lower bound for $E^k(p,\gamma)$, so it will be necessary to understand a part of their approach, which will now be outlined. An arbitrary minimum dominating set D_0 is selected and the remaining minimum dominating sets are ordered arbitrarily and labeled D_1, D_2, ..., D_m. The nodes of V-D_0 are partitioned into A_0 whose nodes have exactly one neighbor in D_0 and A′ whose nodes have at least two neighbors in D_0. It is clear that there are at least $p-\gamma+|A'|$ edges between D_0 and V-D_0. The count determined by Dutton and Brigham (1988) is obtained by finding one more edge associated with all nodes of A_0, except for those in a subset of size at most γ. This then adds at least $|A_0|-\gamma$ to the previous count to yield $p-\gamma+|A'|+|A_0|-\gamma = p-2\gamma+p-\gamma$ which gives the $2p-3\gamma$.

It is this last set of edges which provides the basis for the present work, and more detail about them must be described. Dutton and Brigham (1988) defined a partition of D_0 by $X_i = D_0 \cap D_2 \cap \ldots \cap D_{i-1}$-

D_i for $1 \le i \le m$ and $x_{m+1} = D_0 \cap D_1 \cap \ldots \cap D_m$. They then showed that for each node v of A_0 which is not in the special subset there is a unique associated incident edge e having both endpoints in $A_0 \cup A'$. Two possibilities exist. Either $v \in D_i$ for some i or the end node of e other than v is in D_i for some i. The following fact is immediate upon examining Dutton and Brigham's (1988) proof, although they do not state it: the D_i's in which at least one endpoint must be found are limited to those indices of i for which X_i is nonempty. This fact is one of the two key points to be employed below. The other arises from the structure of the subgraph G′ of G obtained by including only those edges counted by Dutton and Brigham (1988). The only nodes of G′ which are in $A_0 \cup A'$ and also have degree three or more must either be in a D_i where $X_i \neq \emptyset$ or must be the other endnode of an edge e associated with a node in such a D_i. It follows that the number of nodes of G′ which are in $A_0 \cup A'$ and also have degree three or more is at most twice the number which lie in D_i where $X_i \neq \emptyset$.

Our next theorem uses the above information to establish a lower bound. Again we employ $f(\gamma,k)$ to represent the number of nodes in V-D having degree at most k where D is a minimum dominating set.

<u>Theorem 4.6</u>

Let $k \ge 3$, $\gamma \ge 3$ and $p \ge \gamma^2+2\gamma+f(\gamma,k)$. Then $E^k(p,\gamma) \ge (k+3)p/2-[2(k+2)\gamma+(k-1)(\gamma^2+f(\gamma,k))]/2$.

Proof

Any k-γ-insensitive graph certainly must be 1-γ-insensitive, so the remarks preceding the theorem apply. The analysis used by Dutton and Brigham (1988) will be employed here, except that the dominating sets $D_1, D_2, \ldots, D_m$ are indexed so that $X_i \neq \emptyset$ for $1 \leq i \leq n$ and $X_i = \emptyset$ for $n+1 \leq i \leq m$. Thus the partition of D_0 is into the sets $X_1, X_2, \ldots, X_n, X_{m+1}$ where X_{m+1} may be empty. Now we find the maximum number of nodes in dominating sets $D_1, D_2, \ldots, D_n$ which can be in V-D_0, since these nodes determine the maximum possible number of nodes having degree three or more in the G′ subgraph discussed before. Since $X_i = D_0 \cap D_1 \cap \ldots \cap D_{i-1} - D_1$, there is at least one node of D_i in each of $X_{i+1}, X_{i+1}, \ldots, X_n$ so there are at most γ-(n-i) nodes of D_i in A_0. Then in $D_1 \cup D_2 \ldots \cup D_n - D_0$ there are at most

$$\sum_{i=1}^{n} (\gamma - n + i) = n\gamma - n^2 + n(n+1)/2.$$

Let $f(n) = n\gamma - n^2 + n(n+1)/2$. The derivative $f'(n) = \gamma - n + 1/2$ implying f(n) is increasing when $n < \gamma + 1/2$ and maximum when $n = \gamma + 1/2$. But $n \leq \gamma$ since the X_i's partition D_0, so f(n) is maximum when $n = \gamma$. Substituting we get $f(\gamma) = \gamma^2 - \gamma^2 + (\gamma^2 + \gamma)/2 = (\gamma^2 + \gamma)/2$. Thus there are at most $(\gamma^2+\gamma)/2$ nodes in V-D_0 which are in $D_1 \cup D_2 \cup \ldots \cup D_n$, and earlier remarks show that the number of degree three or higher nodes in G′ which are also in V-D_0 is at most $\gamma^2+\gamma$. Therefore at least $p-\gamma-(\gamma^2+\gamma)$ nodes in $V-D_0$ have degree at most two in G'. However, by Theorem 4.5 at most $f(\gamma,k)$ nodes of $V-D_0$ have degree less than or equal to k. Thus $p-\gamma-(\gamma^2+\gamma)-f(\gamma,k)$ nodes in $V-D_0$ must have in G a degree increased by at least k-1 over their degrees in G'. Therefore $E^k(p,\gamma) \geq 2p-3\gamma+(k-1)[p-\gamma-\gamma^2-\gamma-f(\gamma,k)]/2$. This expression is valid when $p \geq \gamma^2+2\gamma+f(\gamma,k)$ and reduces to the result. ∎

Since $f(\gamma,k)$ is independent of p, this lower bound is asymptotically equal to $(k+3)p/2$, the same asymptotic value as the upper bound of Theorem 4.2. From this we conclude the main result.

Theorem 4.7

$E^k(p,\gamma)$ is asymptotically equal to $(k+3)p/2$ as p approaches infinity.

5. THE EXACT VALUE OF $E^2(p,2)$

In the previous chapter an asymptotically correct result was established for $E^k(p,\gamma)$. In this chapter structural properties of extremal 2-2-insensitive graphs make it possible for us to find exact values for $E^k(p,\gamma)$ for the special case when $k = \gamma = 2$. First an upper bound for $E^2(p,\gamma)$ is presented in Section 5.1. Then Section 5.2 develops structural properties which are used in Section 5.3 to determine the desired lower bound and hence equality for $p \geq 11$. Finally, Section 5.4 reports results from a program which found all extremal 2-2-insensitive graphs on $p \leq 10$ nodes and summarizes the results for all values of p.

The proofs tend to be lengthy, often requiring a multitude of cases where the same ideas, specialized to the situation under consideration, are repeated. To avoid excessive duplication in the narrative, we adopt a shorthand notation. Such notation will be introduced as needed. Furthermore, a large number of illustrations are necessary to clarify the proofs, and in order to sustain continuity in the text all figures will be located at the end of this chapter.

5.1. An Upper Bound for $E^2(p,2)$

Our first theorem establishes an upper bound for $E^2(p,2)$ by constructing an appropriate 2-2-insensitive graph.

Theorem 5.1

$$E^2(p,2) \le \lfloor (5p-10)/2 \rfloor \quad \text{if } p > 4.$$

Proof

Construct a graph $G = (V,E)$ as follows:

$V = \{a_1, a_2, b_1, b_2, \ldots, b_{p-2}\}$,

$$E = \begin{cases} \{a_1b_i, a_2b_i \mid 1 \le i \le p-2\} \cup \{b_ib_{i+1} \mid i = 1,3,\ldots,p-3\} \\ \text{if } p \text{ even.} \\ \{a_1b_i \mid 1 \le i \le p-2\} \cup \{a_2b_i \mid 1 \le i \le p-4\} \cup \{a_2b_{p-2}\} \\ \cup \{b_ib_{i+1} \mid i = 1, 3, \ldots, p-6\} \cup \{b_{p-4}b_{p-3}, b_{p-3}b_{p-2}\} \\ \text{if } p \text{ odd.} \end{cases}$$

Figure 5.1 illustrates the constructions. The graph has $2(p-2)+(p-2)/2$ edges when p is even and $(p-2)+(p-3)+(p-1)/2$ edges when p is odd. In either event, G has $\lfloor (5p-10)/2 \rfloor$ edges. Clearly, $\{a_1, a_2\}$ dominates G and $\gamma = 2$.

We now prove G is 2-2-insensitive by removing arbitrary edges e_1 and e_2 from G and showing the domination number remains two. Let $D = \{a_1, a_2\}$. There are three possibilities.

Case 1 Both e_1 and e_2 are edges incident only to nodes in V-D. Then D dominates G-e_1-e_2.

Case 2 Edge e_1 is incident only to nodes in V-D and e_2 is incident to both D and V-D. Then D dominates G-e_1-e_2 unless p is odd and $e_2 = a_1b_{p-3}$. In this case $\{a_2, b_{p-4}\}$ dominates G-e_1-e_2.

Case 3 Both e_1 and e_2 have endpoints in D. Then D dominates

$G-e_1-e_2$ unless e_1 and e_2 have the same endpoint $b_i \epsilon V-D$, or p is odd and a_1b_{p-3} is one of the edges. In these situations $b_j \epsilon N(b_i)$ and one of a_1 and a_2 dominate $G-e_1-e_2$. ∎

5.2. Structural Properties of Extremal Graphs

We turn our attention to establishing a lower bound for $E^2(p,2)$. In this section we investigate useful structural properties of extremal graphs.

Lemma 5.1

If G is a 2-2-insensitive extremal graph on $p \geq 9$ nodes, then G has at most two nodes having degree one or two.

Proof

Let x, y, and z be three nodes of G having degree two or less, and $G-e_1-e_2$ be a graph where all the edges incident to x are removed. Then x is isolated and must be in the dominating set for $G-e_1-e_2$. Therefore some node v_1 must dominate all nodes of G-x, so v_1 has degree p-2. Furthermore, v_1 is not adjacent to x since $\gamma \neq 1$. See Figure 5.2(a). Similarly there are nodes v_2 and v_3 having degree p-2 and dominating with y and z, respectively. Note that v_1, v_2 and v_3 are distinct since $v_1 \notin N(x)$, $v_2 \notin N(y)$, and $v_3 \notin N(z)$. See Figure 5.2(b). Thus $E^2(p,2) \geq 3(p-2)-3 = 3p-9 > \lfloor(5p-10)/2\rfloor$ when $p \geq 9$, a contradiction. ∎

The proof to the next theorem is amenable to the employment of the following notation, which also will be used elsewhere in this paper. G' will always refer to a graph under consideration which is obtained from the original graph G by the removal of two edges.

(1) DS is to be read "degree sum" and refers to the sum of the degrees of all the nodes, i.e., $\Sigma\, d_v$.

(2) D(u,v) is to be read "Nodes u and v form a dominating set."

(3) (D{b,u}, p-2) is to be read "Node u is isolated in G' and thus must be in the dominating set of G'. Also there exists a node b such that $d_b = p-2$ in G' and {b,u} dominates G'. Further, $b \notin N(u)$ in G since $\gamma \neq 1$."

(4) $R(e_1, e_2)$ is to be read "Remove edges e_1 and e_2." The presence of e_2 is optional. Thus $R(e_1)$ is to be read "Remove edge e_1."

(5) $R(a_i x, a_i y : n)$ is to be read "Remove the edges $a_i x$ and $a_i y$. Then a_i or one of its neighbors must be in the dominating set. Since a_i's neighbor n dominates at least the nodes that a_i dominates, we may assume n is in the dominating set."

(6) $R(e_1, e_2 : \{v_1, v_2, \ldots, v_n\})$ is to be read "Remove edges e_1 and e_2. Then $v_1, v_2, \ldots, v_n$ become isolated nodes and hence must be in the dominating set."

(7) N(a,b : c) is to be read "Nodes a and b do not dominate G' because neither dominates c."

Often the proof will require the selection of a node x having a certain property. When the choice of a particular x is arbitrary, a specific node will be stated and marked with an "*" without further comment.

We are now ready to state the next theorem which is crucial to establishing a lower bound for $E^2(p,2)$.

Theorem 5.2

Let G be an extremal 2-2-insensitive graph on $p \geq 11$ nodes. Then $d_v \geq 3$ for all $v \in V$.

Proof

Since G is an extremal 2-2-insensitive graph, Theorem 5.1 implies $E^2(p,2) \leq \lfloor (5p-10)/2 \rfloor$. Any situation leading to more edges will yield a contradiction.

By Lemma 5.1, G has at most two nodes, say u and v, with degree one or two. There are five possible cases.

Case 1 $d_u = d_v = 1$.

Removing the two edges incident to u and v isolates u and v thus forcing them to be in the dominating set of G′, a contradiction when $p \geq 3$.

Case 2 $d_u = 1$ and $d_v = 2$.

2.a u is adjacent to v. Let x be v′s other neighbor. R(vu,vx : {u,v}), a contradiction when $p \geq 3$.

2.b u is not adjacent to v. Let x be u′s neighbor. R(ux) implying (D{y,u}, p-2). Let w be v′s other neighbor as shown in Figure 5.3(a). R(vw,vy) implying (D{z,v}, p-2). Since u has only one neighbor, $z = x$. Notice that this shows $w \neq x$. $R(ux,vy : \{u\})$ so u and either v or w must dominate G-ux-vy. It must be w which means $d_w = p-2$. See Figure 5.3(b). Then $DS \geq 3(p-2)+2+1+3(p-5) = 6p-18$ implying that $E^2(p,2) \geq 3p-9 > \lfloor (5p-10)/2 \rfloor$ when $p \geq 9$, a contradiction.

Case 3 $d_u = d_v = 2$.

3.a u is adjacent to v. Let x be u's other neighbor. $R(ux,uv)$ implying $(D\{y,u\}, p-2)$. See Figure 5.4(a). Similarly $R(vy,uv)$ implying $(D\{x,v\}, p-2)$. Note that $x \neq y$ since $x \in N(v)$. Label the other nodes $a_1, a_2, \ldots, a_{p-4}$ as shown in Figure 5.4(b). By Lemma 5.1 each a_i has degree at least three. Any additional edges must be between the a_i's. $R(a_1x, a_1y : *a_2)$. Now the only nodes which can dominate both u and v are u and v themselves. Thus, without loss of generality, assume $D(a_2,v)$. Hence a_2 has degree p-3 and $DS \geq 2+2+2(p-2)+(p-3)+3(p-5) = 6p-18$. Thus $E^2(p,2) \geq 3p-9 > \lfloor (5p-10)/2 \rfloor$, when $p \geq 9$.

3.b u is not adjacent to v. Let x and y be the neighbors of u. $R(ux,uy)$ implying $(D\{w,u\}, p-2)$, where $w \neq x$ and $w \neq y$. See Figure 5.5(a). Now remove the two edges incident to v and by a similar argument $(D\{z,v\}, p-2)$, $z \neq w$. Since z must be adjacent to u, we may without loss of generality assume $z = x$, as shown in Figure 5.5(b). Suppose v is not adjacent to y. $R(ux,vw)$ implying that we may take y and the other neighbor of v, say t, as a dominating set. Then $d_y+d_t \geq p$ since both y and t are adjacent to x and w. Hence $DS \geq 2(p-2)+p+2+2+3(p-6) = 6p-18$. Thus $E(p,2) \geq 3p-9$

$> \lfloor(5p-10)/2\rfloor$ if $p \geq 9$. Therefore we assume v is adjacent to y. Label the nodes other than u, v, w, x, and y as $a_1, a_2, \ldots, a_{p-5}$. See Figure 5.5(c). Any additional edges are between the a_i's. Now y is not adjacent to all a_i's since $\gamma \neq 1$. Thus $*a_1$ is not adjacent to y. $R(a_1x, a_1w)$ implying a_1 or $*a_2 \in N(a_1)$ must dominate with y since y is the only node which dominates both u and v. Thus DS $\geq$ $2+2+2(p-2)+p+3(p-6) = 6p-18$ and $E^2(p,2) \geq 3p-9 > \lfloor(5p-10)/2\rfloor$ when $p \geq 9$.

Case 4 $d_u = 1$ and all other nodes have at least degree three.

Let x be u's neighbor. R(ux) implying $(D\{y,u\}, p-2)$, $y \neq x$. See Figure 5.6(a). R(ux, xy) implying $(D\{z,u\}, p-2)$, $z \neq y$. See Figure 5.6(b). Since $\gamma \neq 1$, there exists a node w that is not adjacent to x. R(yw,zw). Then we may assume x dominates with w or a neighbor of w, say n. In the former case $d_x+d_w \geq p$ and in the latter $d_x+d_n \geq p$ since x, w, and n are adjacent to both y and z. Hence DS $\geq 1+2(p-2)+p+3(p-5) = 6p-18$. Thus $e \geq 3p-9 > \lfloor(5p-10)/2\rfloor$ if $p \geq 9$.

Case 5 $d_u = 2$ and all other nodes have degree at least three.

Let x and y be the neighbors of u. R(ux,uy) implying $(D\{w,u\}, p-2)$, $w \neq x$ and $w \neq y$. Label the other nodes $a_1, a_2, \ldots, a_{p-4}$ as shown in Figure 5.7. Note that each dominating set must include one of x, y and u in order to dominate u. R(wy,uy). We may assume x is the dominating set for G′, so one of the following cases must hold: (a) D(x,w), (b) D(x,y) and x is adjacent to y, (c) D(x,y) and x is not adjacent to y, (d) $D(x,a_i)$ and x is adjacent to y, and (e)

$D(x,a_i)$ and x is not adjacent to y.

We now examine these cases individually.

5.a $D(x,w)$. Note that xy must be an edge. See Figure 5.8. $R(wa_i,wa_j)$ where $i \neq j$ and a_i is not adjacent to x and a_j is not adjacent to y or a_i is not adjacent to both x and y. This must happen since $\gamma \neq 1$. Then some node $z \neq w$ must dominate with one of x,y, and u. Since x is adjacent to y and u, we may without loss of generality assume $D(x,z)$ where $z = y$ or $z = a_h$ for some h. Observe that in either case x and z dominate G-wy-uy. This implies that either Subcase 5.b or Subcase 5.d must hold. Thus this case will be seen to lead to a contradiction once it is shown that both 5.b and 5.d do.

5.b $D(x,y)$ and xy an edge. Then $d_x+d_y \geq p+2$ since $D\{x,y\}$, x is adjacent to y, and both x and y are adjacent to u and w. Thus $DS \geq 2+(p-2)+(p+2)+3(p-4) = 5p-10$. Hence $E^2(p,2) \geq (5p-10)/2 > \lfloor(5p-10)/2\rfloor$ when p is odd, and represents a contradiction in this event. Furthermore, if the lower bound for the DS increases, we have a contradiction when p is even. Thus assume no increase. Then each a_i has degree three and is adjacent to exactly one of x and y. As before, $\gamma \neq 1$ implies that $*a_1$ is not adjacent to y and $*a_2$ is not adjacent to x, so a_1 is adjacent to x and a_2 is adjacent to y. $R(a_1w,a_1x : a_j)$ implying $D(a_j,x)$ or $D(a_j,y)$. First assume $a_j = *a_3 \neq a_2$. See Figure 5.9(a). $N(a_3,x : a_2)$ so $D(a_3,y)$. Thus y must be adjacent to all a_i's except a_1 and possibly a_3, as shown in Figure 5.9(b). $R(a_2w,a_2y : *a_4)$ implying $D(a_4,x)$ or $D(a_4,y)$. But $N(a_4,y :$

a_1) and $N(a_4,x : a_5)$ so we must have $a_j = a_2$. Then a_2 must dominate $G-a_1w-a_1x$ with either x or y. Suppose $D(a_2,y)$. $R(a_3w,a_3y : a_4)$ implying $D(a_4,x)$ or $D(a_4,y)$. See Figure 5.9(c). Now $N(a_4,x : a_2)$ and $N(a_4,y : a_1)$. By a similar argument $\{a_2,x\}$ does not dominate. Thus we have the desired contradiction.

5.c $D(x,y)$ and x is not adjacent to y. Then DS ≥ $2+(p-2)+p+3(p-4) = 5p-12$. Thus the lower bound for the DS can increase by at most two without contradicting Theorem 5.1 and we are restricted to one of the following possibilities: (i) x and y are both adjacent to at most two of the same a_i's, (ii) at most two a_i's have degree four, (iii) one a_i has degree five, and (iv) x and y are adjacent to the same a_j and an a_i has degree 4 (i may equal j). Note that any of the above situations can involve at most six a_i's. In this context a node is "involved" if it is a node as described or is adjacent to such a node. Since $p \geq 11$ there exists at least one a_i, say a_1, which is not involved. Thus a_1 has degree three and a_1 is not adjacent to both x and y nor is it adjacent to a node with degree four or to a node with degree five. Thus there is a node $*a_2 \epsilon N(a_1)$ which has degree three and hence is adjacent to exactly one of x and y. Remove a_1w and the edge between a_1 and $\{x,y\}$. We may assume a_2 dominates with one of u, x, and y. $N(a_2,u : a_4)$. Suppose $D(a_2,x)$. Then a_2 is adjacent to y and x is adjacent to all a_i's except a_2 and possibly a_1. From possibility (i) above, there exist at least $p-8 \geq 3$ (since $p \geq 11$) a_i's, $i \geq 3$, which are adjacent to exactly one of x and y which means these a_i's are adjacent to x and not adjacent to y. Let $*a_3$ be such a node. See Figure 5.10. $R(wa_3, xa_3)$ implying either a_3 or $*a_4 \epsilon N(a_3)$ must dominate with one of u, x and y. $N(u, a_3 : a_2)$, $N(u, a_4 : a_2)$, $N(x, a_3 : a_2)$, $N(x, a_4 : a_2)$ and $N(y, a_3 : x)$. Thus $D(y, a_4)$ which means y must be adjacent to all nodes that are not in the neighborhood of a_4. Let m be the number of a_i's in the neighborhood of a_4. Then $1 \leq m \leq 3$ since a_4 has degree less than or equal to five. Thus y is adjacent to at least $p-4-2-(m+1) = p-7-m$ a_i's, $i \geq 5$. Hence x is adjacent to at least p-7-m of the same a_i's as y is and DS is increased by $m-1+(p-7-m) = p-8 \geq 3$, a contradiction. An analogous argument yields a contradiction for $D(y, a_2)$.

<u>5.d</u> $D(x, *a_1)$ and xy an edge. See Figure 5.11(a). R(ux,wx) implying we may take y in the dominating set so as to dominate u. If D(x,y), for this case, then D(x,y) for G-uy-wy which was shown impossible by Subcase 5.b. Similarly D(w,y) can be eliminated as in Subcase 5.a. Therefore $D(y, a_j)$ for some j. First assume that $a_j \neq a_1$. Then $DS \geq 2+(p-2)+(p-1)+(p-1)+3(p-6) = 6p-20 > 5p-10$ if $p \geq 11$, a contradiction. Therefore assume $a_1 = a_j$. Then $D(x, a_1)$ and $D(y, a_1)$ mean both x and y must be adjacent to all a_i's that are not adjacent to a_1. Let m be the number of a_i's which are adjacent to a_1. Then $DS \geq 2+(p-2)+m+1+6+2(p-5-m)+3(p-5) = 6p-18-m$. A contradiction is avoided only when $6p-18-m \leq 5p-10$, i.e., when $m \geq p-8$. By definition $0 \leq m \leq p-5$ implies that $p-8 \leq m \leq p-5$. Thus a_1 is adjacent to at least $p-8 \geq 3$ a_i's, $i \neq 1$, and both x and y must be adjacent to at most $p-5-(p-8) = 3$ a_j's, $2 \leq j \leq p-4$. Let a_2, a_3

and a_4 be the nodes which, in addition to a_1, could possibly be adjacent to both x and y. Assume first that at least one of x and y is adjacent to each a_i, $i \neq 1$. Thus $DS \geq (6p-18-m)+m > 5p-10$ when $p \geq 9$. Therefore neither x nor y is adjacent to at least one a_i for $i \geq 5$ and $a_i \epsilon N(a_i)$. Since the degree of a_i is at least three and a_i is not adjacent to x or y, a_i must be adjacent to a_h for some $h \neq 1$. See Figure 5.11(b). $R(wa_i, a_1a_i)$. Then either a_i or $a_h \epsilon N(a_i)$ must dominate with one of x, y and u and we may as well assume x or y. Since both x and y dominate with a_1 and they play symmetric roles, assume $D(x,a_h)$ or $D(x,a_i)$. Note that x, y, a_h, a_i, and a_1 are all adjacent to w so a dominating set containing two of these nodes will have a degree sum greater than or equal to p-1. Recalling that $D(y,a_1)$ we have $DS \geq 2+(p-2)+2(p-1)+3(p-6) = 6p-20 > 5p-10$ when $p \geq 11$.

<u>5.e</u> $D(x,*a_1)$ and x is not adjacent to y. Then a_1y must be an edge to dominate y. See Figure 5.12(a). Suppose a_1 is adjacent to x. Then $DS \geq 2+(p-2)+(p+1)+3(p-4) = 5p-11$. If the lower bound for DS increases by more than one we have a contradiction. Thus at most one of y and the a_i's, $i \neq 1$, can have degree four or a_1 and x can both be adjacent to at most one of the same a_i's. These conditions involve at most three a_i's in addition to a_1. Since $p \geq 11$ at least three a_i's are not involved, so each of these has degree three and is adjacent to exactly one of a_1 and x. Because of the above restrictions it is straightforward to show that there are at least two of these a_i which are adjacent. Let a_2, a_3, and a_4 be

three of these nodes and a_2a_3 be an edge. Remove a_2w and the edge between a_2 and $\{x, a_1\}$. Then, without loss of generality, a_3 dominates with one of u, x, and y. $N(u,a_3 : a_4)$, $N(x,a_3 : y)$ and $N(y,a_3 : a_k$ for some k) since y is adjacent to at most one a_i besides a_1 and a_3 has degree 3.

Thus a_1 is not adjacent to x. See Figure 5.12(b). Then DS ≥ $2+(p-2)+(p-1)+3(p-4) = 5p-13$. Theorem 5.1 is contradicted if the lower bound for the sum increases by more than three. Therefore we can have only one of the following: (i) y can be adjacent to at most three a_i's, $i \neq 1$, (ii) x and a_1 can both be adjacent to at most 3 of the same a_i's, and (iii) at most three in any combination of (i) and (ii). Note that we do not exclude the possibility that some of the a_i's have degree greater than three, with a maximum possible degree of six. At least $p-8 \geq 3$ a_i's are not involved in any of the above conditions. Let a_2, a_3 and a_4 be nodes not adjacent to y and adjacent to exactly one of x and a_1. Remove a_2w and the edge incident to a_2 and $\{x,a_1\}$. Then either a_2 or $a_j \epsilon N(a_2)$, $j \neq 1$, must dominate with one of u, x and y. $N(u,a_2 : w)$. If $D(u,a_j)$, then a_j must be adjacent to p-5 a_i's and have degree at least $p-5+1 = p-4 \geq 7$. Hence the DS lower bound increases by at least four, a contradiction. $N(x,a_2 : y)$. $N(y,a_2 : a_k$ for some k) since a_2 and y together dominate at most five a_i's and there are at least seven. $N(y,a_j)$ since they dominate at most six of the p-6 a_i's and there are at least seven. Therefore $D(x,a_j)$ and a_jy and a_ja_1 must be edges in order to dominate y and a_1. See Figure 5.12(c). Hence DS

$\geq 2+(p-2)+(p-1)+4+4+3(p-6) = 5p-11$. To avoid a contradiction the DS bound can increase by at most one. Thus a_j can be adjacent to at most one of a_3 and a_4. Without loss of generality assume that a_3 is not adjacent to a_j. Then a_3 must be adjacent to x and hence is not adjacent to a_1. See Figure 5.12(d). $R(wa_3,xa_3)$ implying either a_3 or $a_k \epsilon N(a_3)$ must dominate with one of x, y, and u. $N(u,a_3 : a_1)$, $N(u,a_k : a_h$ for some h), $N(x,a_3 : y)$ and $N(y,a_3 : x)$. Hence either $D(x,a_k)$ or $D(y,a_k)$. Suppose $D(y,a_k)$. Then DS $\geq 2+(p-2)+(p-1)+4+(p-1)+3(p-7) = 6p-19 \geq 5p-10$ if $p \geq 9$. Thus $D(x,a_k)$ requiring edges $a_k y$ and $a_k a_1$ to dominate y and a_1. See Figure 5.12(e). Therefore DS $\geq 2+(p-2)+4+5+4+(p-1)+3(p-7) = 5p-9 > 5p-10$, a contradiction.

Each of the subcases resulted in a contradiction if $p \geq 11$, verifying that $d_v \geq 3$ for all $v \epsilon V$. ■

Next we present a series of lemmas leading to Theorem 5.3 which states that there is an extremal graph having two disjoint minimum dominating sets.

<u>Lemma 5.2</u>

Suppose a graph has at least two minimum dominating sets and any two such sets intersect. Then the dominating sets satisfy one of the following:

(1) there are exactly three dominating sets {x,y}, {x,z} and {y,z}, or

(2) all dominating sets contain a common node x, that is,

the dominating sets are $\{x,a_1\}$, $\{x,a_2\}$, ..., $\{x,a_t\}$.

Proof

Suppose no node appears in every dominating set. Let one dominating set be $\{x,y\}$ and a second be $\{x,z\}$. Since (2) does not hold, there must be a third dominating set which does not include x. In order to intersect the first two, it must be $\{y,z\}$. It is clear no further dominating sets are possible. ■

The existence of at least two minimum dominating sets as required by Lemma 5.2 is met automatically in 2-2-insensitive extremal graphs.

Lemma 5.3

Situation (1) of Lemma 5.2 cannot hold for extremal graphs if $p \geq 5$.

Proof

Assume (1) of Lemma 5.2 holds and let b_1, b_2, ..., b_{p-3} be the nodes of $G-\{x,y,z\}$. Each b_i must be adjacent to all of x, y, and z so it can be dominated when two edges between it and $\{x,y,z\}$ are removed. It follows that $\{x,b_1\}$ is a dominating set, which contradicts (1). ■

From this point on we assume we are discussing the situation described by (2) in Lemma 5.2. Let x, a_1, a_2, ..., a_t be the nodes which appear in dominating sets, with x being the common node. Of the remaining nodes let b_1, b_2, ..., b_m be adjacent to x and c_1, c_2,

..., $c_{p-t-1-m}$ be not adjacent to x. Note that no edges between nodes $b_1, b_2, \ldots, b_m, c_1, c_2, \ldots, c_{p-t-1-m}$ are necessary in extremal graphs. Furthermore, $t \geq 2$ in such graphs. Some elementary facts can now be stated.

<u>Lemma 5.4</u>

For extremal graphs we have:

(i) each a_i, $1 \leq i \leq t$, is adjacent to $c_1, c_2, \ldots, c_{p-t-1-m}$,

(ii) for any two nodes b_i, b_j there is an a_h adjacent to both.

<u>Proof</u>

(i) $D(x,a_i)$.

(ii) $G-xb_i-xb_j$ must be dominated by $\{x,a_h\}$ for some h. ■

<u>Lemma 5.5</u>

Let x be adjacent to r of the a_i, say $a_1, a_2, \ldots, a_r$. Then

(i) each a_i, $1 \leq i \leq r$, is adjacent to a_j, $r+1 \leq j \leq t$, and

(ii) each a_i, $r+1 \leq i \leq t$ is adjacent to every a_j, $j \neq i$.

<u>Proof</u>

(i) $D(x,a_i)$.

(ii) Node a_i is adjacent to $a_1, a_2, \ldots, a_r$ by (i). It is adjacent to a_j, $r+1 \leq j \leq t$, $i \neq j$, since $D(x,a_i)$. ■

<u>Lemma 5.6</u>

Situation (2) of Lemma 5.2 cannot hold if $t = 2$ and $p \geq 5$.

Proof

In this case there can be no c_i's and each b_i must be adjacent to all of x, a_1, and a_2. But then we arrive at a contradiction as in the proof of Lemma 5.3. ■

From now on we assume $t \geq 3$.

Lemma 5.7

Any extremal graph must have at least the following number of edges: $pt-(t-3)m-t^2/2-3t/2-r^2/2+3r/2$.

Proof

Each b_i has at least an edge to x and two edges to the a_j's [3m edges]; each a_i is adjacent to each c_j [t(p-t-1-m] edges); x is adjacent to r a_i's [r edges]; a_i, $1 \leq i \leq r$, is adjacent to a_j, $r+1 \leq j \leq t$, [r(t-r) edges]; and $a_{r+1}, a_{r+2}, \ldots, a_t$ form a complete subgraph [(t-r)(t-r-1)/2 edges]. Summing and simplifying yields the result. ■

Lemma 5.8

No extremal graph satisfying (2) of Lemma 5.2 exists when $t = 3$ and $p \geq 9$.

Proof

By Lemma 5.7 the number of edges required when $t = 3$ is $3p-9-r^2/2+3r/2$ where $0 \leq r \leq 3$. Now $-r^2/2+3r/2 \geq 0$ in this interval so the number of edges is at least $3p-9 > \lfloor (5p-10)/2 \rfloor$ if $p \geq 9$. ■

Thus we may assume $t \geq 4$. Since the expression of Lemma 5.7 is minimized for largest m, we have the following result.

Lemma 5.9

The value of m should be selected to be as large as possible.

In general Lemma 5.9 implies that m should be p-t-1. However, we shall see later that there is one case where it must be p-t-2.

Lemma 5.10

No extremal graph satisfying (2) of Lemma 5.2 exists when r = 0, 1 and $p \geq 9$.

Proof

If r = 0 the count of Lemma 5.7 reduces to $pt-(t-3)m-t^2/2-3t/2$. By Lemma 5.9, m should be taken as p-t-1 so the minimum number of edges is $pt-(t-3)(p-t-1)-t^2/2-3t/2 = 3p+t^2/2-7t/2-3$. This expression is smallest when t = 4 when it reduces to $3p-9 > \lfloor(5p-10)/2\rfloor$ if $p \geq 9$. If r = 1 the count of Lemma 5.7 reduces to $pt-(t-3)m-t^2/2-3t/2+1$. By an argument analogous to the r = 0 case the number of edges is at least $3p-8 > \lfloor(5p-10)/2\rfloor$ if $p \geq 7$. ■

Thus from now on we assume $r \geq 2$.

Lemma 5.11

The value of r should be selected to be as large as possible.

Proof

The expression of Lemma 5.7 is a decreasing function of r if r ≥ 2. ■

We cannot have $r = t$ and $m = p-t-1$ simultaneously since then x would have degree $p-1$ implying $\gamma = 1$. Therefore there are two combinations of interest: (i) $r = t-1$, $m = p-t-1$ and (ii) $r = t$, $m = p-t-2$. We shall deal with these separately. Let us first consider $r = t-1$ and $m = p-t-1$. In this case the count of Lemma 5.7 becomes $pt-(t-3)(p-t-1)-t^2/2-3t/2-(t-1)^2/2+3(t-1)/2 = 3p-t-5$.

Lemma 5.12

No extremal graph satisfying (2) of Lemma 5.2 and having fewer than $\lfloor (5p-10)/2 \rfloor$ edges exists when $r = t-1$ and $m = p-t-1$.

Proof

Suppose first that $t \leq p/2$. Then the minimum number of edges is at least $3p-t-5 \geq 3p-p/2-5 \geq \lfloor (5p-10)/2 \rfloor$. Now let $t > p/2$. Suppose all b_i's have degree at least four. Then, since all nodes have degree at least three by Theorem 5.2, we have $DS \geq 4(p-t-1)+[(p-t-1)+(t-1)]+3(t-1)+(t-1) = 5p-10$ so the lemma is true in this case. Suppose, therefore, that at least one b_i has degree three. Let it be adjacent to a_j and a_k. By Lemma 5.4(ii) every other b_i must be adjacent to at least one of a_j or a_k. If these other b_i's had degree three, at most $(p-t-1)+1$ a_i's would be adjacent to at least one b_i, leaving at least $(t-1)-(p-t) = 2t-p-1$ a_i's from $\{a_1,$

$a_2, \ldots, a_{t-1}\}-\{a_j, a_k\}$ not adjacent to a b_i. These all have edges to both x and a_t. To bring their degrees to at least three requires extra edges either between them or from b_i's having degree greater than three. In any event it will take at least $\lceil(2t-p-1)/2\rceil = \lfloor(2t-p)/2\rfloor$ more edges. These must be added to the base count of 3p-t-5 to show that the number of edges is at least $3p-t-5+\lfloor(2t-p)/2\rfloor = \lfloor(5p-10)/2\rfloor$. ■

Lemma 5.13

No extremal graph satisfying (2) of Lemma 5.2 and having fewer than $\lfloor(5p-10)/2\rfloor$ edges exists when $r = t-1$, $m = p-t-1$, and $p \geq 11$.

Proof

Follows from Lemmas 6, 8, 10 and 12. ■

Now we consider $r = t$ and $m = p-t-2$ for which the count of Lemma 5.7 becomes $pt-(t-3)(p-t-2)-t^2/2-3t/2-t^2/2+3t/2 = 3p-t-6$. Note that for x not to have degree p-1 we must have $p \geq t+2$.

Lemma 5.14

There is no extremal graph satisfying (2) of Lemma 5.2 when $r = t$, $m = p-t-2$ and $p \geq 5$.

Proof

In the basic structure outlined previously the subgraph induced by x, $a_1, a_2, \ldots, a_t$ is a star where x is the center node. Removal of any two edges of this star means that their two endpoints

a_i and a_j must either be adjacent to each other or both be adjacent to some third node a_h in order to dominate them both. Recall that no b_i is in a dominating set. Add on the minimum number of extra edges between the a_i's to ensure all are dominated no matter which two edges of the star are removed. Consider the subgraph induced by $a_1, a_2, \ldots, a_t$. Suppose this subgraph is disconnected, a_i and a_j are in distinct components, and xa_i and xa_j are removed. Then it is not possible for x and some a_h to dominate a_i and a_j. Hence the induced subgraph is connected and has at least t-1 edges in addition to those already counted, implying a minimum edge count of $(3p-t-6)+(t-1) = 3p-7 > \lfloor(5p-10)/2\rfloor$ if $p \geq 5$. ■

We are now ready for the disjoint dominating set theorem.

<u>Theorem 5.3</u>

When $p \geq 11$ there is at least one extremal graph having at least one pair of disjoint dominating sets.

<u>Proof</u>

From Theorem 5.1 we have an extremal graph with $\lfloor(5p-10)/2\rfloor$ edges, which has two disjoint dominating sets $\{a_1,b_i\}$ and $\{a_2,b_j\}$, $i \neq j$. It follows from the preceding lemmas that graphs without two disjoint dominating sets have at least $\lfloor(5p-10)/2\rfloor$ edges. ■

While Theorem 5.3 tells us there is an extremal graph having at least two disjoint minimum dominating sets when $p \geq 11$, Lemma

5.15 and Theorem 5.4 show there is an extremal graph having at most two.

Lemma 5.15

Let G be an extremal graph on $p \geq 11$ nodes. Then there do not exist four pairwise disjoint dominating sets.

Proof

Suppose there exist four pairwise disjoint dominating sets. Then $DS \geq 4(p-2)+4(p-8) = 8p-40$. Hence $e \geq 4p-20 > \lfloor(5p-10)/2\rfloor$ when $p \geq 11$. ■

We introduce additional shorthand notation to aid in the proof to Theorem 5.4.

ONE($v_1, v_2, ..., v_n$){:u} is to be read " At least one of {v_1, v_2, ..., v_n} must be in the dominating set for G' {so that u is dominated}." The ":u" is optional.

Theorem 5.4

(a) For p = 11 and p = 12 there is an extremal graph having at most two disjoint dominating sets, and

(b) for each $p \geq 13$ no extremal graph has three pairwise disjoint dominating sets.

Proof

By Lemma 5.15 there exist at most three pairwise disjoint dominating sets. Assume there are three and call them $D_1 = \{x_1,$

$x_2\}$, $D_2 = \{y_1, y_2\}$, and $D_3 = \{z_1, z_2\}$. Label the remaining nodes $a_1, a_2, \ldots, a_{p-6}$. Then $DS \geq 3(p-2)+3(p-6) = 6p-24$. Thus $e \geq 3p-12$. If it can be shown that an extremal graph requires at least one more edge, we will have $e \geq 3p-11 > \lfloor(5p-10)/2\rfloor$ when $p \geq 13$ and (b) will hold. Furthermore, $3p-11 = \lfloor(5p-10)/2\rfloor$ when $p = 11$ or $p = 12$. Since the constructions of Theorem 5.1 yield graphs having at most two disjoint dominating sets and these graphs have $\lfloor(5p-10)/2\rfloor$ edges, one can conclude the validity of (a). It follows that the theorem will be proved if we can demonstrate the necessity of at least one more edge.

Assume no additional edge exists. Since D_1, D_2 and D_3 are dominating sets, there must be six edges interconnecting them in such a way that each of the six nodes has degree two in the subgraph induced by $D_1 \cup D_2 \cup D_3$. It follows that this subgraph is either $2C_3$ or C_6 where the structure of the entire graph must be equivalent to (a) or (b) of Figure 5.13. Each node of $D_1 \cup D_2 \cup D_3$ dominates exactly three nodes from $D_1 \cup D_2 \cup D_3$: itself and its two neighbors. Since each a_i has degree three and by Lemma 5.15 any dominating set must include at least one node of $D_1 \cup D_2 \cup D_3$, each node in any dominating set must dominate at least three nodes in a row on a cycle of $D_1 \cup D_2 \cup D_3$.

<u>Case 1</u> The situation in Figure 5.13(a) exists.

$R(x_1y_1, y_1z_1 : *a_1)$. By the preceding argument a_1 is adjacent to x_1, y_1, and z_1. $R(x_1y_1, a_1z_1)$ implying that $ONE(x_1, y_1, a_1) : a_1$. But each of x_1, y_1, and a_1 can dominate at most two nodes of

$D_1 \cup D_2 \cup D_3$ in $G'-x_1y_1-a_1z_1$, a contradiction.

Case 2 The situation in Figure 5.13(b) exists.

$R(x_1y_1, y_1z_1 : *a_1)$. Then, since each dominating set must include one node of $D_1 \cup D_2 \cup D_3$, a_1 must dominate the nodes in one of the following sets: (1) $\{x_1, y_1, z_1\}$, (2) $\{x_2, y_1, z_1\}$ and (3) $\{x_1, y_1, z_2\}$. Consider (1). Then $D(a_1, y_2)$ and y_2 is adjacent to all a_i's except a_1 as shown in Figure 5.14. $R(x_1a_1, y_1z_1)$. $ONE(y_1, z_1, a_1):a_1$. This is a contradiction since each of these dominates at most two nodes in $D_1 \cup D_2 \cup D_3$. Sets (2) and (3) yield contradictions by analogous arguments.

Since all cases yield contradictions, at least one additional edge is required and the theorem is proven. ∎

It can be shown that there do not exist three pairwise disjoint dominating sets even for $p = 11, 12$. However, the proof is long and tedious. We omit it since Theorem 5.4 is sufficient for our needs, which are that the search for extremal graphs can be limited to ones having at most two disjoint dominating sets.

5.3. A Lower Bound for $E^2(p,2)$

In this section a lower bound for $E^2(p,2)$ is established which is equal to the upper bound of Theorem 5.1 when $p \geq 11$, thereby solving the problem for such p. A straightforward but not optimum lower bound is obtained first.

Lemma 5.16

$E^2(p,2) \geq (5p-16)/2$ if $p \geq 11$.

Proof

If $p \geq 11$, $d_v \geq 3$ for all nodes v by Theorem 5.2 and there is at least one extremal graph having one pair of disjoint dominating sets by Theorem 5.3. Each of the disjoint dominating sets has at least p-2 incident edges, so $DS \geq 2(p-2)+3(p-4) = 5p-16$. ∎

The bound of Lemma 5.16 is within three of the Theorem 5.1 result. We now show that at least three additional edges are required. Lemma 5.17 establishes a useful structural property.

Lemma 5.17

Let $D_1 = \{u,v\}$ and $D_2 = \{x,y\}$ be disjoint dominating sets of extremal graph G with connecting edges ux and vy, where G has fewer than $\lfloor(5p-10)/2\rfloor$ edges, and let $a_1, a_2, \ldots, a_{p-4}$ be the nodes of $G-(D_1 \cup D_2)$. If any two of uv, uy, xy and vx are also edges, then the degree of every a_i is three and each a_i, $1 \leq i \leq p-4$, is adjacent to exactly two nodes in $D_1 \cup D_2$.

Proof

Suppose that two of uv, uy, xy and vx are edges and that m a_i's have four edges to $D_1 \cup D_2$ and n have three edges. Then the number of edges is at least $\lceil 4+2(p-4)+2m+n+(p-4-m-n)/2 \rceil = \lceil 5p/2-6+3m/2+n/2 \rceil \geq \lfloor(5p-10)/2\rfloor$ if $m > 0$ or $n > 0$. Thus $m = n = 0$, and all a_i's have degree three and exactly two edges to $D_1 \cup D_2$. ∎

Before proceeding with the main theorem of this section, we introduce the following notation to facilitate the discussion. We continue to employ the shorthand notation introduced earlier.

(1) $R(a_i : n)$ is to be read "Remove the two edges between a_i and $D_1 \cup D_2$. Then a_i or one of its neighbors must be in the dominating set. Since a_i's neighbor n dominates at least the nodes that a_i does, we may assume n is in the dominating set."

(2) $I(b : v_1, v_2, \ldots, v_n)$ is to be read "If node b is in the dominating set, then the other node in the dominating set must dominate $v_1, v_2, \ldots, v_n$."

Furthermore, in any context where it makes more sense, uv may be read "u is adjacent to v" instead of "uv is an edge."

<u>Theorem 5.5</u>

$E^2(p,2) = \lfloor (5p-10)/2 \rfloor$ if $p \geq 11$.

<u>Proof</u>

Let G be an extremal graph with $DS \leq 5p-12$. By Theorem 5.3 G has at least two disjoint dominating sets $D_1 = \{u,v\}$ and $D_2 = \{x,y\}$ with the remaining nodes $a_1, a_2, \ldots, a_{p-4}$. By Lemma 5.16 G has at least $5p/2-8$ edges.

Consider the dominating set D for G-ux-vy. Since Theorem 5.4 shows that we need to consider only graphs with exactly two disjoint dominating sets and the nodes in D_1 and D_2 play symmetric roles, we may without loss of generality assume that $u \epsilon D$. Then there are five possibilities: (1) D = {u, v}, (2) D = {u, x}, (3) D = {u, y}, (4) D = {u, a_i} with a_i adjacent to u, and (5) D = {u, a_i} with a_i not adjacent to u.

The discussion of the five cases assumes certain points without stating them explicitly. Specifically, the theorem is true when the graph under discussion has DS ≥ 5p-11 since the number of edges is an integer implying that $(5p-11)/2 \geq \lfloor (5p-10)/2 \rfloor$. Furthermore, each case assumes an extremal graph with fewer than $\lfloor (5p-10)/2 \rfloor$ edges and then searches for a contradiction. In a situation where DS = 5p-12, demonstration of a need for another edge yields the desired contradiction.

<u>Case 1</u> D(u,v) for G-ux-vy.

Here we must have edges uy and vx in order for D to dominate x and y in G-ux-vy. Hence the edge count is increased to at least (5p/2)-6 which, since e is an integer, yields the desired result when p is odd. If p is even and any additional edges exist, the result also holds. Thus assume p is even and the edges described so far are the only ones. By Lemma 5.17 all a_i's are of degree three and have exactly two edges to $D_1 \cup D_2$. R(a_1 : *a_2). By Theorem 5.4, ONE(u,v,x,y). Without loss of generality, we may assume D(u,a_2). Consider the two possibilities: a_2 is adjacent to u and a_2 is not adjacent to u.

If a_2u, a_2 is not adjacent to v since a_2 has exactly two edges to $D_1 \cup D_2$. Therefore neither a_2 nor u dominates v unless there is an additional edge, in which case the result holds.

Suppose then that a_2 is not adjacent to u. Then a_2v since D(u,v) and a_2x or a_2y since D(x,y). See Figure 5.15(a). Also ua_i for $3 \le i \le p-4$. $R(a_3 : *a_4)$. ONE(u,v,x,y). $N(u,a_4 : a_2)$. Suppose $D(v,a_4)$. Then va_i for i = 1, 2 and $5 \le i \le p-4$. We have a contradiction since a_5 cannot be adjacent to both u and v. Suppose $D(x,a_4)$. If xa_4, another edge is necessary to dominate y and the result follows. Thus we may assume a_4 is not adjacent to x which means a_4y. Then xa_i for i = 1, 2 and $5 \le i \le p-4$. See Figure 5.15(b). $R(a_5 : *a_6)$. Again ONE(u,v,x,y). $N(u,a_6 : a_2)$, $N(v,a_6 : a_4)$, $N(x,a_6 : a_4)$ and $N(y,a_6 : a_2)$ so we have a contradiction. A symmetric argument shows that the case $D(y,a_4)$ also yields a contradiction.

<u>Case 2</u> D(u,x) for G-ux-vy.

At least two additional edges are required to dominate v and y in G-ux-vy. The possibilities are: (a) uv and uy, (b) uv and xy, (c) xv and uy, and (d) xv and xy. Observe that Subcases (a) and (d) are symmetric. Also, Subcase (c) and Case 1 are symmetric. To see this note that the proof to Case 1 used the fact that {u,v} dominates G-ux-vy, and this remains true in Subcase (c). Thus we need consider only Subcases (a) and (b).

For both (a) and (b) Lemma 5.17 can be employed to establish that the degree of $a_i = 3$ and each a_i has exactly two edges to $D_1 \cup D_2$. As before the result follows if another edge is necessary, so we assume we have only the edges described so far.

<u>2.a</u> Additional edges are uv and uy.

$R(a_1 : *a_2)$. Since $D(u,v)$, $D(x,y)$, and $D(u,x)$, a_2 is adjacent to (1) both u and x, or (2) both y and u, or (3) both v and x.

<u>2.a.1</u> There are edges a_2u and a_2x as shown in Figure 5.16(a). ONE(u,v,x,y). Consider the four possible dominating sets.

(i) $D(a_2,u)$. Then ua_i for $2 \le i \le p-4$. If a_1u, then $d_u = p-1$ and $\gamma = 1$, a contradiction. Therefore a_1v and a_1x since $D(u,v)$ and $D(u,x)$. See Figure 5.16(b). $R(a_3 : *a_4)$. ONE(u,v,x,y). $N(u,a_4 : a_1)$, $N(v,a_4 : a_2)$, $N(x,a_4 : v)$ and $N(y,a_4 : a_2)$ yielding the contradiction. Thus $D(a_2,u)$ is not possible.

(ii) $D(a_2,v)$. Then va_i for $3 \le i \le p-4$. See Figure 5.16(c). $R(a_3 : *a_4)$. ONE(u,v,x,y). $N(u,a_4 : a_5)$, $N(v,a_4 : a_2)$ and $N(y,a_4 : a_2)$. Thus $D(x,a_4)$ implying a_4y and xa_i for $i = 1$, 2 and $5 \le i \le p-4$. Now a_4v implies that a_4 is not adjacent to u and a_4y implies that a_4 is not adjacent to x. But $D(u,x)$, so we have a contradiction.

(iii) $D(x,a_2)$ is not possible since v is not dominated.

(iv) $D(y,a_2)$. Then ya_i for $3 \le i \le p-4$. $D(u,x)$ implies that u must be adjacent to all a_i's which are adjacent to y. Therefore ua_i for $2 \le i \le p-4$. Note that u is not adjacent to a_1 since $d_u \ne p-1$, so a_1v. Also $D(u,x)$ implies a_1x. See Figure 5.16(d). $R(a_3 : *a_4)$. ONE(u,v,x,y). $N(u,a_4 : a_1)$, $N(v,a_4 : a_2)$, $N(x,a_4 : a_5)$ and $N(y,a_4 : a_1)$. Thus $D(y,a_2)$ is not possible so Subcase 2.a.1 cannot occur.

<u>2.a.2</u> There are edges a_2u and a_2y.

Again ONE(u,v,x,y). Suppose $D(u,a_2)$. Then ua_i for $2 \le i \le$

p-4. Note that u is not adjacent to a_1 since $d_u \neq$ p-1. Then a_1v and a_1x since D(u,v) and D(u,x). See Figure 5.17. R(a_3 : *a_4). ONE(u,v,x,y). N(u,a_4 : a_1), N(v,a_4 : a_2), N(x,a_4 : a_2) and N(y,a_4 : a_1). Thus D(u,a_2) is not possible. Neither is D(v,a_2) since x is not dominated nor D(x,a_2) since v is not dominated nor D(y,a_2) since x is not dominated. Thus Subcase 2.a.2 is eliminated.

<u>2.a.3</u> There are edges a_2v and a_2x.

ONE(u,v,x,y). Consider the four possible dominating sets.

(i) D(u,a_2). Then ua_i for $3 \leq i \leq$ p-4. See Figure 5.18(a). R(a_3 : *a_4). ONE(u,v,x,y). N(u,a_4 : a_2), N(v,a_4 : a_5), N(x,a_4 : v) and N(y,a_4 : a_2) yielding the contradiction.

(ii) D(v,a_2). Then va_i for $2 \leq i \leq$ p-4. Since D(u,x), xa_i for $2 \leq i \leq$ p-4. Now $d_y \geq 3$ implies y must be adjacent to at least one a_i, so a_1y. Furthermore, D(u,x) implies a_1u. See Figure 5.18(b). R(a_3 : *a_4). ONE(u,v,x,y). N(u,a_4 : a_2), N(v,a_4 : a_1), N(x,a_4 : a_1) and N(y,a_4 : a_2) yielding the contradiction.

(iii) N(x,a_2 : y), a contradiction.

(iv) D(y,a_2). Then ya_i for $3 \leq i \leq$ p-4. D(u,x) implies that u must be adjacent to all a_i's that are adjacent to y. See Figure 5.18(c). R(a_3 : *a_4). ONE(u,v,x,y). N(u,a_4 : a_2), N(v,a_4 : x), N(x,a_4 : a_5) and N(y,a_4 : a_2) yielding the contradiction which eliminates Subcase 2.a.3 and hence completes the contradiction of Subcase 2.a.

<u>2.b</u> Additional edges are uv and xy.

R(a_1 : *a_2). ONE(u,v,x,y). The arguments for D(u,a_2) and

$D(x,a_2)$ will be analogous as will the arguments for $D(v,a_2)$ and $D(y,a_2)$. Hence we consider only $D(u,a_2)$ and $D(v,a_2)$. Suppose $D(u,a_2)$. Then ua_i for $3 \le i \le p-4$ and a_2y. $D(u,x)$ implies ua_2. See Figure 5.19(a). $R(a_3 : *a_4)$. ONE(u,v,x,y). Suppose $D(u,a_4)$. Then ua_i for $1 \le i \le p-4$. But Theorem 5.2 implies $d_v \ge 3$, so v must be adjacent to at least one a_i which is adjacent to u, a contradiction. $N(v,a_4 : a_2)$ and $N(x,a_4 : a_2)$. Suppose $D(y,a_4)$. Then ya_i for $i = 1$, 2 and $5 \le i \le p-4$. Since $D(u,x)$, ua_1 and ua_2 so ua_i for $1 \le i \le p-4$. See Figure 5.19(b). Again the fact that $d_v \ge 3$ leads to a contradiction which completes the elimination of $D(u,a_2)$. Suppose $D(v,a_2)$. Thus va_i for $3 \le i \le p-4$, and a_2x. Now $D(u,x)$ implies that x is adjacent to all a_i's which are adjacent to v. By Theorem 5.2 $d_y \ge 3$, so ya_1. Furthermore, $D(u,x)$ implies that ua_1. See Figure 5.19(c). $R(a_3 : *a_4)$. ONE(u,v,x,y). $N(u,a_4 : a_5)$, $N(v,a_4 : a_1)$, $N(x,a_4 : a_1)$ and $N(y,a_4 : a_2)$ yielding the contradiction which eliminates Subcase 2.b and thus all of Case 2.

<u>Case 3</u> D(u,y) for G-ux-vy.

Then uv and xy. Notice that in this case the edges between u, v, x and y are the same as the edges between u, v, x and y in Case 2.b. Furthermore, the D(u,x) in the Subcase 2.b is similar to the D(u,y) here. Thus this case can be treated in an analogous manner to Subcase 2.b.

<u>Case 4</u> $D(u,*a_1)$ for G-ux-vy where a_1u is an edge.

Then a_1x to dominate x in G-ux-vy. Neither v nor y is dominated by $\{u,a_1\}$ in G-ux-vy unless additional edges exist. The possibilities are (a) a_1v and a_1y, (b) uv and a_1y, (c) a_1v and uy and (d) uv and uy.

4.a Additional edges are a_1v and a_1y.

With the edges described so far, DS ≥ 5p-13. This implies that the degree of a_1 is at most five in any graph having fewer than $\lfloor(5p-10)/2\rfloor$ edges. Let a_1 be adjacent to a_2 if it is adjacent to any a_i. In any event ua_i for i = 1 and 3 ≤ i ≤ p-4. Since $d_v \geq 3$ v is adjacent to some node in addition to a_1 and y. It cannot be u or we would have Subcase 4.b and it cannot be x or DS ≥ 5p-11. Thus it must be an a_i.

4.a.1 $*a_3v$ and v is not adjacent to a_2. See Figure 5.20(a).

D(u,v) implies ua_2 and DS ≥ 5p-12. Node a_3 has three incident edges so it is not adjacent to any a_i. R(a_2 : $*a_4$). ONE(u,v,x,y). N(u,a_4 : v), N(v,a_4 : a_5) and N(x,a_4 : v). Thus D(y,a_4) implying a_4x and ya_i for i = 1, 3 and 5 ≤ i ≤ p-4. See Figure 5.20(b). R(a_5 : $*a_6$). ONE(u,v,x,y). N(u,a_6 : v), N(v,a_6 : a_4), N(x,a_6 : a_3) and N(y,a_6 : a_4) so we have a contradiction.

4.a.2 va_2. See Figure 5.20(c).

We may assume ua_2 is not an edge or the situation reverts to the previous case with the roles of a_2 and a_3 interchanged. D(u,a_1) implies a_1a_2. Hence DS ≥ 5p-12. R(a_3 : $*a_4$). N(u,a_4 : a_2), N(v,a_4 : a_5) and N(x,a_4 : v). Thus D(y,a_4) implying a_4x and ya_i, i = 1, 2 and 5 ≤ i ≤ p-4. See Figure 5.20(d). R(a_5 : $*a_6$). ONE(u,v,x,y). N(u,a_6 : a_2), N(v,a_6 : a_4), N(x,a_6 : a_2) and N(y,a_6 : a_4) yielding the contradiction.

<u>4.b</u> Additional edges are uv and a_1y.

$DS \geq 5p-13$ implying that the degree of a_1 is at most four. Let a_1 be adjacent to a_2 if it is adjacent to any a_i. In any event ua_i for $i = 1$ and $3 \leq i \leq p-4$. Since $d_v \geq 3$, v is adjacent to some other node. This additional neighbor of v must be an a_i or DS would be too large.

<u>4.b.1</u> va_2 and ua_2 is not an edge. See Figure 5.21(a).

$D(u,a_1)$ implies a_1a_2. Thus $DS \geq 5p-12$. $R(a_3 : *a_4)$. ONE(u,v,x,y). $N(u,a_4 : a_2)$, $N(v,a_4 : a_1)$ and $N(x,a_4 : v)$. Therefore $D(y,a_4)$ implying a_4x and ya_i, $i = 1$, 2 and $5 \leq i \leq p-4$. See Figure 5.21(b). $R(a_5 : *a_6)$. ONE(u,v,x,y). $N(u,a_6 : a_2)$, $N(v,a_6 : a_1)$, $N(x,a_6 : a_2)$ and $N(y,a_6 : a_4)$ so we have a contradiction.

<u>4.b.2</u> va_j and ua_j for some j.

$DS \geq 5p-12$ so neither a_1 nor a_j is adjacent to any a_i and there are no further edges to nodes of u, v, x and y. Hence ua_2. See Figure 5.21(c). Form G-uv-ux. No two of u, v, x and y dominate all of u, v, x and y so one of u, v, x, and y must dominate with some a_i. It cannot be u since no a_i dominates all of v, x and y. It cannot be v since the a_i would have to dominate at least p-5 a_i's which would make its degree too great. Suppose $D(x,a_i)$. Then $i = j$ and a_jy in order for all of u, v and y to be dominated. Furthermore, xa_i for $i \neq j$. See Figure 5.21(d). $R(a_2 : *a_3)$. ONE(u,v,x,y). $N(u,a_3 : y)$, $N(v,a_3 : a_4)$, $N(x,a_3 : v)$ and $N(y,a_3 : a_4)$ so we have a contradiction. Thus $D(y,a_i)$ where $i \in \{1, *2, j\}$, j

$\neq 2$, so ya_i for $4 \le i \le p-4$ and $i \neq j$. See Figure 5.21(e). If $D(y,a_1)$ then ya_j, ya_2 and ya_3. Since $d_x \ge 3$ x and y have a second common neighbor, a contradiction. Suppose $D(y,a_j)$ implying ya_2, ya_3, and a_jx. Consider G-ux-ua_j. The only pair of nodes from u, v, x and y which can dominate all of u, v, x and y is {v,x} and it cannot dominate G-ux-ua_j without increasing DS. ONE(v,x,a_j):a_j. Neither v nor x can be selected since $p \ge 11$ implies there is an a_i which is not dominated. N(a_j,u : y) and N(a_j,y : u) so we have a contradiction. Thus $D(y,a_2)$ implying a_2x and ya_j. See Figure 5.21(f). Consider G-ux-ua_2. As before no two of u, v, x and y dominate. ONE(x,*a_3):a_2. Note that since a_3 can dominate at least the nodes that a_2 can, perhaps by including the edge a_3x, we do not consider a_2. Selecting x causes several a_i's to not be dominated, so a_3 is chosen. ONE(u,v,y) N(u,a_3 : one of x and y) and N(v,a_3 : a_1). Thus $D(y,a_3)$ implying a_3x. Consider G-ux-a_1y. No two of u, v, x and y dominate. ONE(u,x,a_1):a_1. Again selecting x causes several a_i's to not be dominated. If u is selected we have $D(u,a_i)$ for some i and a_i must dominate x and y. But only a_1 can do this and a_1y is absent. N(v,a_1 : a_2) and N(y,a_1 : a_2) so we have a contradiction.

<u>4.c</u> Additional edges are a_1v and uy.

DS $\ge$ 5p-13 implying that the degree of a_1 is at most four. Let a_1 be adjacent to a_2 if it is adjacent to any a_i. In any event ua_i for i = 1 and $3 \le i \le p-4$.

<u>4.c.1</u> ua_2 is not an edge.

$D(u,v)$ implies va_2. $D(u,a_1)$ implies a_1a_2 and $DS \geq 5p-12$. See Figure 5.22(a). $R(a_3 : *a_4)$. $ONE(u,v,x,y)$. $N(u,a_4 : a_2)$, $N(v,a_4 : a_5)$, $N(x,a_4 : v)$ and $N(y,a_4 : a_1)$ yielding a contradiction.

4.c.2 ua_2.

Notice that node u is adjacent to all a_i. Furthermore, since $d_v \geq 3$, v must be adjacent to some a_j and $DS \geq 5p-12$. Thus a_1a_2 is not an edge and we may as well assume $j = 2$. See Figure 5.22(b). $R(a_3 : *a_4)$. $ONE(u,v,x,y)$. $N(u,a_4 : v)$, $N(v,a_4 : a_5)$, $N(x,a_4 : v)$ and $N(y,a_4 : a_1)$ so we have a contradiction.

4.d Additional edges are uv and uy.

$DS \geq 5p-12$. $D(a_1,u)$ implies a_1a_2 and ua_i for $i = 1$ and $3 \leq i \leq p-4$. See Figure 5.23. Notice $d_u = p-2$ so ua_2 is not an edge. Hence va_2. $R(a_3 : *a_4)$. $ONE(u,v,x,y)$. $N(u,a_4 : a_2)$, $N(v,a_4 : a_1)$, $N(x,a_4 : v)$ and $N(y,a_4 : a_1)$ so we have a contradiction which eliminates Subcase 4.d and completes the contradiction of Case 4.

Case 5 $D(u,a_1)$ and a_1 is not adjacent to u.

Then a_1x to dominate x in G-ux-vy. Since $D(u,v)$, a_1v. Either u or a_1 must dominate y in G-ux-vy so either a_1y or uy.

5.a a_1y. $DS \geq 5p-15$ so the sum can increase by at most three. Thus a_1 can have degree at most six or at most three a_i's can have degree exceeding three or, if $i \neq 1$, at most three a_i's can have three edges to $D_1 \cup D_2$. Therefore, since $p \geq 11$, there exist at least $p-4-3-1 \geq 3$ a_i's, $i \neq 1$, which have exactly two edges to $D_1 \cup D_2$, have degree three and are not adjacent to a_1. Let a_2, a_3 and a_4 be three such nodes. Then $D(u,a_1)$ implies ua_2, ua_3 and ua_4.

See Figure 5.24. $R(a_2 : a_j$ for some j, $3 \le j \le p-4)$. Consider the possibilities for a_j: (1) a_j has degree three with exactly two edges to $D_1 \cup D_2$, (2) a_j has degree four with two or three edges to $D_1 \cup D_2$, (3) a_j has degree five with two or three edges to $D_1 \cup D_2$, and (4) a_j has degree six with two edges to $D_1 \cup D_2$. In (3) a_j cannot have four edges to $D_1 \cup D_2$ and in (4) it cannot have three such edges since in either case DS would be too large.

<u>5.a.1</u> a_j has degree three with two edges to $D_1 \cup D_2$.

Since the degree of a_j is three, a_j is not adjacent to a_i, $i \ne 2$. Thus $D(u,a_1)$ implies ua_j and hence a_j is not adjacent to v. See Figure 5.25. Note that a_j could be a_3 or a_4. If it is either, assume without loss of generality that it is a_3. ONE(u,v,x,y).

(i) $N(u,a_j : a_1)$.

(ii) $N(v,a_j : a_4)$.

(iii) If $D(x,a_j)$, then additional edges are needed to dominate y and v. Recall that a_j is not adjacent to v so xv. If xy DS $\ge$ 5p-11, so $a_j y$. $D(x,a_j)$ implies xa_i, $i \ne 2$, j. See Figure 5.26. DS $\ge$ 5p-13, so a_1 can be adjacent to at most one a_i, $i \ne 2$, 3, 4, j. $R(a_4 : a_m)$. Note that $m \ne j$. ONE(u,v,x,y). $N(v,a_m : a_j)$, $N(x,a_m : a_j)$ and $N(y,a_m : a_r)$. There exists such an a_r since y can be adjacent to at most one a_i, $i \ne 1$, that is adjacent to x or a_m can be adjacent to at most one additional a_i for $i \ne 1$, 4. Thus $p \ge 11$ implies that at least $p-10 \ge 1$ nodes will not be dominated. Hence $D(u,a_m)$. Then $a_m a_1$ since a_1 is not adjacent to u and DS $\ge$ 5p-11. Therefore $\{x,a_j\}$ does not dominate.

(iv) The only remaining possibility is $D(y,a_j)$. Since a_j has degree three ya_i for $i \neq 2, j$. Then a_jx or yx to dominate x. First assume yx. See Figure 5.27(a). Considering $D(u,a_1)$ and $D(x,y)$, $DS \geq 5p-12$. Note that u and a_1 cannot be adjacent to the same a_i for $i \neq 1$, x and y cannot be adjacent to the same a_i, and each a_i for $i \neq 1$ has degree three. Now $d_v = 3$ implies that va_h for some $h \neq 1, 2, 3, 4, j$. Thus ua_i for $i \neq 1, h$. See Figure 5.27(b). $D(u,a_1)$ implies either ua_h or a_1a_h. In either event, using $D(u,v)$ and $D(x,y)$ $DS \geq 5p-12$, so a_1 is not adjacent to another a_i for $i \neq h$. $R(ux,a_1y)$. $ONE(v,x,a_1,a_h):a_1$. $I(v : u,x,a_i$ for $i \neq 1, h)$, a contradiction. $I(x : u,v,a_i$ for $i \neq 1, 2, j)$, a contradiction. $I(a_1 : u,y,a_i$ for $i \neq 1, h)$, a contradiction. $I(a_h$ where $a_1a_h :$ u,x,a_i for $i \neq 1, h)$ and $I(a_h$ where $ua_h : x,a_i$ for $i \neq h)$, both contradictions. Therefore x is not adjacent to y.

The only alternative is that a_jx must be the edge that dominates x. Using $D(u,a_1)$ and $D(x,y)$, $DS \geq 5p-14$. $R(ua_4,ya_4 :$ $a_m)$. See Figure 5.27(c). $ONE(u,v,x,y)$. $N(v,a_m : a_j)$ and $N(y,a_m :$ $a_j)$. Suppose $D(u,a_m)$. Then a_ma_1 to dominate a_1. $D(u,v)$ implies uv or a_mv. If uv $DS \geq 5p-11$, so a_mv. See Figure 5.27(d). $DS \geq 5p-13$. Thus there cannot be additional edges between u, v, x and y. Since $D(u,a_1)$ and $D(u,a_m)$, both a_1 and a_m must be adjacent to the a_i's that are not adjacent to u. Furthermore, DS increases by three for each such a_i so u must be adjacent to all a_i's except a_1 and a_m. Now a_1 can be adjacent to at most one other a_i, $i \geq 5$ and $i \neq j$, m, or at most one a_i, $i \geq 5$, can have three edges to $D_1 \cup D_2$, or a_m can be adjacent to at most one other a_i, $i \neq 1, 4$, in which case it cannot have a third edge to $D_1 \cup D_2$. R(ux,ya_1). ONE(v,x,a_1,a_m) : a_1. I(v : u,x,a_2,a_4,a_j), a contradiction. I(x : u,v,y,a_4,a_h for some h since both x and y can be adjacent to at most one more a_i), a contradiction. I(a_1 : u,y,a_i for $i \neq 1$, m), a contradiction. I(a_m : u,x,a_2,a_j) yielding the contradiction which eliminates D(u,a_m).

Thus D(x,a_m). Then a_2x since a_2 is not adjacent to a_m, and either a_mv or xv. Suppose xv. See Figure 5.27(e). Then DS $\geq$ 5p-13. Now x and a_m between them must be adjacent to p-9 $\geq$ 2 additional a_i's. Since y is adjacent to all a_i's, $i \neq 2$, j, either both x and y are adjacent to an a_i or a_m is adjacent to all the remaining a_i's. Either situation increases DS by one for each such a_i for a total increase of at least two, a contradiction. Hence a_mv. See Figure 5.27(f). D(u,a_1) implies ua_m or a_1a_m. In either event DS $\geq$ 5p-13. By the previous argument D(x,a_m) implies that the DS increases by p-9 $\geq$ 2, yielding the final contradiction which eliminates Subcase 5.a.1.

<u>5.a.2</u> Degree of a_j is four and a_j has two or three edges to $D_1 \cup D_2$.

Recall that each of a_2, a_3, and a_4 has degree three and is adjacent to u. See Figure 5.28. Considering the degrees of D(u,a_1) and D(x,y), DS $\geq$ 5p-13. ONE(u,v,x,y).

(i) N(v,a_j : at least one of a_3 and a_4).

(ii) Suppose D(u,a_j). Then a_ja_1 since u is not adjacent to a_1. The degree of a_j is four so a_j is not adjacent to any other a_i.

Hence ua_i for $i \neq 1$, j. Furthermore, either uv or a_jv to dominate v and either uy or a_jy to dominate y. If either uv or uy, $DS \geq 5p-11$. Thus a_jv and a_jy. See Figure 5.29(a). $R(a_3 : a_h, h \neq 1, 2, j)$. $ONE(u,v,x,y)$. $N(u,a_h : a_j)$, $N(v,a_h : a_2)$ and $N(x,a_h : a_j)$. Thus $D(y,a_h)$ and y must be adjacent to all a_i's that are not adjacent to a_h, and either a_hx or yx. If yx $DS \geq 5p-11$, so a_hx. Then $DS \geq 5p-13$. Thus a_h can be adjacent to at most one other a_i, say a_s, if any, where clearly $s \neq 2$, j and also $s \neq 1$ since this increases DS by two. Then ya_i for $i \neq 3$, h, s. See Figure 5.29(b). Now a_1 is adjacent to at most one additional a_i, and this can occur only if a_h is not adjacent to an a_s. Thus designate by a_s the extra node, if any, which is adjacent to either a_h or a_1. Furthermore, if there is no such a_s, at most one a_i, $i \neq 1$, can have three edges to $D_1 \cup D_2$. $R(ux,a_1y)$. $ONE(v,x,a_1,a_j) : a_1$. $I(v : u,x,a_2)$, a contradiction. $I(x : u,v,y,a_2)$, a contradiction. $I(a_1 : u,y,a_i$ for $i \neq 1$, j, $s)$, a contradiction. $I(a_j : u,x,a_i$ for $i \neq 1$, 2, $j)$ yielding the contradiction for $D(u,a_j)$.

(iii) Next assume $D(x,a_j)$. Then v and y must be dominated by $\{x, a_j\}$. If xy and xv, then $DS \geq 5p-10$. Suppose xy and a_jv. Since $D(u,a_1)$ either ua_j or a_1a_j. In either event, $DS \geq 5p-11$. Suppose a_jy and xv. Then $DS \geq 5p-12$. Thus a_j can have only two edges to $D_1 \cup D_2$ without increasing DS, so a_j is adjacent to a_s, each a_i has degree three for $i \neq j$, and xa_i for $i \neq 2$, j and s. Note s could be 3 or 4. If it is either, let it be 3. $D(u,a_1)$ implies ua_i, $i \neq 1$. See Figure 5.30(a). $R(a_4 : a_h)$. $ONE(u,v,x,y)$. $N(u,a_h : a_1)$,

$N(v,a_h : a_2)$, $N(x,a_h : y)$ and $N(y,a_h : a_5$ where a_5 is not a_h) yielding the contradiction.

Therefore a_jy and a_jv. $D(u,a_1)$ implies ua_j or a_1a_j. Thus, since the degree of a_j is four, $D(x,a_j)$ implies xa_i for $i \neq 2, j$. See Figure 5.30(b). First assume ua_j. See Figure 5.30(c). DS $\geq$ 5p-13. $R(a_3 : a_r$ where $r \neq 2, j)$. ONE(u,v,x,y). $N(v,a_r : a_2)$, $N(x,a_r : a_j)$ and $N(y,a_r : a_t)$. There exists such an a_t since y can be adjacent to at most one additional a_i which is also adjacent to x or a_r can be adjacent to at most one more a_i. Since $p \geq 11$, there is at least one a_i not dominated. Hence $D(u,a_r)$. Then a_1a_r to dominate a_1 and DS $\geq$ 5p-11. Thus a_1a_j and a_j is not adjacent to u. See Figure 5.30(d). Then DS $\geq$ 5p-13. $R(a_3 : a_r)$. ONE(u,v,x,y). $N(u,a_r : a_j)$, $N(v,a_r : a_2)$, $N(x,a_r : a_j)$ and $N(y,a_r : a_h$ for some h ≥ 4) yielding the contradiction which eliminates $D(x,a_j)$ as a possibility.

(iv) The only possibility still to consider is $D(y,a_j)$. Then u and x must be dominated by $\{y, a_j\}$. If yu and yx, DS $\geq$ 5p-10. If either yx and a_ju or yu and a_jx, then using $D(u,a_1)$ and $D(x,y)$, DS $\geq$ 5p-11. Thus a_ju and a_jx. Since a_j can be adjacent to at most one of a_3 and a_4, we may assume a_j is not adjacent to a_3. Then $D(y,a_j)$ implies ya_3. See Figure 5.31(a). $R(a_3u, a_3y : a_t$, $t \neq 1, j)$. ONE(u,v,x,y). $N(v,a_t : a_2)$.

Suppose $D(u,a_t)$. Then a_ta_1 to dominate a_1. $D(x,y)$ implies a_tx or a_ty. If a_tx, then $D(y,a_j)$ implies a_ja_t or ya_t. If ya_t DS $\geq$ 5p-10. If a_ja_t DS $\geq$ 5p-11. Therefore a_t is not adjacent to x and hence $a_t y$. D(u,v) implies $a_t u$ or $a_t v$. If $a_t u$, using $D(u,a_1)$ and D(x,y), DS ≥ 5p-11, so $a_t v$. DS ≥ 5p-12. Thus a_1 is not adjacent to another a_i, $i \neq t$, so ua_i for $i \neq 1, t$. If a_j has three edges to $D_1 \cup D_2$, DS increases. Therefore $a_j a_r$ for some $r \neq 1, t$. $D(y,a_j)$ implies ya_i for $i \neq 2, j, r$. See Figure 5.31(b). $R(ux,a_1 y)$. ONE(u,v,x,y). $I(u : v,x,y,a_1)$, a contradiction. $I(v : u,x,a_i, i \neq 1, t)$, a contradiction. $I(x : u,v,y,a_i, i \neq 1, 2, j, r)$, a contradiction. $I(y : u,x,a_1,a_j)$, a contradiction. Thus $\{u, a_t\}$ does not dominate $G-a_3u-a_3y$.

Suppose $D(x,a_t)$. Then xy or $a_t y$ to dominate y and xv or $a_t v$ to dominate v. If xy and xv, DS ≥ 5p-10. If xy and $a_t v$, then $a_t u$ or $a_t a_1$. In either event, DS ≥ 5p-10. Next assume $a_t y$ and xv. See Figure 5.31(c). Since D(u,v), either ua_t or va_t. However, va_t (and not ua_t) implies $a_1 a_t$ and DS ≥ 5p-10. Thus ua_t. See Figure 5.31(d). DS ≥ 5p-12. Hence no a_i, $i \neq 1$, is adjacent to both x and y, and each a_i has degree three for $i \neq j$. $D(x,a_t)$ implies xa_i for $i \neq 3, t$ and $D(y,a_j)$ implies ya_i, $i \neq 2, r, j$ where $a_r \in N(a_j)$, showing x and y have a common neighbor a_i for $i \geq 2$, a contradiction. Therefore $a_t y$ and $a_t v$. See Figure 5.31(e). $D(u,a_1)$ implies ua_t or $a_1 a_t$. First assume $a_1 a_t$ so DS ≥ 5p-12. $D(u,a_1)$ implies ua_i for $i \neq 1, t$, and each a_i has degree three for $i \neq 1, j, t$. Now a_j cannot have another edge to $D_1 \cup D_2$, so $a_j a_r$ for some $r \neq 1, 2, 3, t$. $D(y,a_j)$ implies ya_i for $i \neq 2, j, r$. No a_i, $i \neq 1$, can be adjacent to both x and y. However, for some $h \neq 1, 2, 3, j, r, t$, there is an a_h adjacent to neither a_t nor a_j. Hence $D(y,a_j)$ and

$D(x,a_t)$ imply ya_h and xa_h, a contradiction. Thus ua_t and DS ≥ 5p-12. Then no a_i, $i \neq 1$, is adjacent to both x and y. Furthermore a_1 is not adjacent to any a_i, so ua_i for $i \neq 1$ and $a_j a_r$ for some $r \neq 1$, 2, 3, t. $D(x,a_t)$ implies xa_i for $i \neq 3$, t, and $D(y,a_j)$ implies ya_i for $i \neq 2$, j, r. Hence there is an a_h, $h \geq 2$, adjacent to both x and y, yielding the contradiction which eliminates $D(x,a_t)$.

Therefore $\{y, a_t\}$ must dominate $G\text{-}a_3u\text{-}a_3y$, implying $a_t a_j$ or ya_j. First let $a_t a_j$. Then DS ≥ 5p-12 using $D(u,a_1)$ and $D(x,y)$. Thus a_t is not adjacent to another a_i and each a_i has degree three for $i \neq j$, t. $D(u,a_1)$ implies ua_i for $i \geq 2$. $D(y,a_t)$ implies ya_i for $i \neq j$, t and $a_t x$. Now $d_v = 3$ implies that v must be adjacent to an a_i, $i \neq 1$, and v is not adjacent to a_2, a_3, a_4, a_j and a_t, so $*a_h v$. See Figure 5.31(f). $R(ux,a_1y)$. $ONE(v,x,a_1):a_1$. $I(v : u,x,a_i, i \neq 1, h)$, a contradiction. $I(x : u,v,y,a_i$ for $i \neq 1$, j, t), a contradiction. $I(a_1 : u,y,a_i$ for $i \neq 1)$, a contradiction. Thus $a_t a_j$ is not an edge, so ya_j. Then ya_t since $D(a_j,y)$. Again $D(y,a_t)$ implies $a_t x$. See Figure 5.31(g). Hence DS ≥ 5p-11. Therefore $\{y, a_t\}$ does not dominate, yielding the contradiction which eliminates Subcase 5.a.2.

<u>5.a.3</u> a_j has degree five with two or three edges to $D_1 \cup D_2$.

Using $D(u,a_1)$ and $D(x,y)$, DS ≥ 5p-12. Thus all a_i's, $i \neq 1$, j, have degree three and no a_i, $i \neq 1$, is adjacent to both x and y or both u and a_1. Consider the possible dominating sets.

(i) $D(u,a_j)$ implies $a_j a_1$. Furthermore $a_j v$ and $a_j y$ since either uv or uy increases DS. If $a_j u$ or $a_j x$ the DS increases.

Hence a_ja_s for some s. Since a_s has degree three and is adjacent to one of u and v and one of x and y, a_s is not adjacent to a_1. Thus $D(u,a_1)$ implies a_su, and $D(u,a_j)$ implies ua_i for $i \neq 1$, j. See Figure 5.32. Therefore a_1 is not adjacent to any additional a_i since both ua_i and a_1a_i increase DS. Thus each a_i for $i \neq 1, 2, j$, s has degree three, two edges to $D_1 \cup D_2$ and a degree three neighbor in $V\text{-}D_1 \cup D_2$. $R(ux,a_1y)$. $ONE(v,x,a_1,a_j):a_1$. $I(v : u,x,a_i$ for $i \neq 1$, j), a contradiction. $I(x : u,v,y,a_j)$, a contradiction. $I(a_1 :$ u,y,a_i for $i \neq 1$, j), a contradiction. $I(a_j : u,x,a_i$ for $i \neq 1, 2$, j, s), a contradiction. Therefore $\{u, a_j\}$ does not dominate.

(ii) $D(v,a_j)$. Notice v has degree three and a_j has degree five so only ten nodes are dominated, a contradiction for $p \geq 11$.

(iii) $D(x,a_j)$. Then a_jy and a_jv since either xy or xv would increase the DS. Note a_j is not adjacent to x or DS increases. See Figure 5.33(a). $D(u,a_1)$ implies either ua_j or a_1a_j to dominate a_j. In either case $DS \geq 5p\text{-}12$, so no a_i, $i \neq 1$, is adjacent to both u and v. Then a_ja_r for some $r \neq 1$. Notice that r could be 3 or 4. If it is either, let it be 4. In the original DS computed for Subcase 5.3 d_v is counted as three, so $D(u,v)$ implies ua_i for $i \neq 1$, j. $D(x,a_j)$ implies xa_i for $i \neq 2$, j, r. See Figure 5.33(b). Recall that a_1 and u are not adjacent to the same a_i. $R(a_4 : a_m$ where $m \neq 1$, j, r). $ONE(u,v,x,y)$. $N(u,a_m : a_1)$, $N(v,a_m : a_2)$, $N(x,a_m : a_j)$ and $N(y,a_m : a_s$ for some $s \neq 1, 2, 4, j, m, r)$ yielding the contradiction which eliminates $D(x,a_j)$ as a possibility.

(iv) $D(y,a_j)$. Then a_ju and a_jx since either yu or yx increases DS. See Figure 5.34(a). Since a_j has degree five, it must be adjacent to two other nodes. Note that y cannot be one of these nodes since this would increase the initial DS count which used D(x,y). Thus either a_j is adjacent to two a_i's in addition to a_2 or a_j is adjacent to v and one a_i in addition to a_2. First assume a_j is not adjacent to v and $a_j a_r$ and $a_j a_t$. Since $a_j a_1$ increases the DS, a_j is not adjacent to a_1. Hence $a_r \neq a_1$ and $a_t \neq a_1$. Each a_i has degree three for $i \neq 1, j$, implying $a_r u$ and $a_t u$. See Figure 5.34(b). DS $\geq$ 5p-13, so a_1 can be adjacent to at most one a_i, say a_m, if any, where $m \neq r, t$ or DS would increase. First suppose $a_1 a_m$ and hence a_m is not adjacent to u. Thus $a_m v$. D(y,a_j) implies ya_i for $i \neq 2, j, r, t$. See Figure 5.34(c). DS $\geq$ 5p-12. Recall that the original DS counted three for d_v, so v is not adjacent to another a_i and ua_i for $i \neq 1, m$. R(ux,ya_1). ONE(v,x,a_1,a_m) : a_1. I(v : u,x,a_i for $i \neq 1, m$), a contradiction. I(x : u,v,y,a_i for $i \neq 1, 2, j, r, t$), a contradiction. I(a_1 : u,y,a_i for $i \neq 1, m$), a contradiction. I(a_m : u,x,a_i for $i \neq 1, m$), a contradiction. Therefore a_1 has degree three and is not adjacent to any a_i's. D(u,a_1) implies ua_i for $i \neq 1$. Again $d_v = 3$ implies v is adjacent to an a_h in addition to a_1. Since a_h has degree three and is adjacent to u and v, a_h is not adjacent to an a_i. Hence D(y,a_j) implies ya_i for $i \neq 2, j, r, t$. See Figure 5.34(d). R(ux,a_1y). ONE(v,x,a_1) : a_1. I(v : u,x,a_i for $i \neq h$), a contradiction. I(x : u,v,y,a_i for $i \neq 1, 2, j, r, t$), a contradiction. I(a_1 : u,y,a_i for $i \neq 1$), a contradiction. Thus a_j

is not adjacent to two nodes a_r and a_t.

Therefore a_jv and a_ja_r. DS ≥ 5p-12. Thus a_1 has degree three and is not adjacent to any a_i's. Then $D(u,a_1)$ implies ua_i for $i \neq 1$. Further $D(y,a_j)$ implies ya_i for $i \neq 2, j, r$. See Figure 5.34(e). Without loss of generality let $a_r = a_3$ if it is either a_3 or a_4. $R(a_4 : a_m, m \neq 1,(r))$. $ONE(u,v,x,y)$. $N(u,a_m : a_1)$, $N(v,a_m : a_2)$, $N(x,a_m : a_s$ for some $s \neq 1, 2, j, r)$ and $N(y,a_m : a_j)$ yielding the contradiction which eliminates Subcase 5.a.3.

<u>5.a.4</u> a_j has degree six with two edges to $D_1 \cup D_2$.

$D(u,a_1)$ and $D(x,y)$ imply DS ≥ 5p-11 yielding the contradiction which eliminates Subcase 5.a.4 and hence completes elimination of Subcase 5.a.

<u>5.b</u> uy. Recall a_1x and a_1v. Let m a_i's have four edges to $D_1 \cup D_2$ and n have three edges. Then the number of edges is at least $3+2(p-4)+2m+n+(p-4-m-n)/2 = (5p-14+3m+n)/2 > \lfloor (5p-10)/2 \rfloor$ if $m > 0$ or $n > 2$. Therefore, if the edge count is smaller than $\lfloor (5p-10)/2 \rfloor$, at most two a_i's can have three edges to $D_1 \cup D_2$ and no a_i has four edges to $D_1 \cup D_2$, $1 \leq i \leq p-4$. DS ≥ 5p-14. Thus DS can increase by at most two. It follows that a_1 is adjacent to at most three a_i's since degree three was counted for a_1 in DS. Therefore u must be adjacent to at least $p-8 \geq 3$ a_i's. The DS also implies that at most two a_i's have degree four or at most one a_i has degree five. Notice that any situation which adds two to DS involves at most six a_i's.

First consider the special case where a_1 is not involved in increasing the degree sum and $*a_3$ and $*a_6$ have degree four. Then

ua_i, $i \neq 1$, h where a_h is a neighbor of a_1. Note that p must be at least twelve for this case to occur. See Figure 5.35. Then using either D(u,v) and D(x,y) or D(u,a_1) and D(x,y), DS $\geq$ 5p-12. Thus each a_i for $i \neq 3$, 6 has degree three and all a_i have exactly two edges to $D_1 \cup D_2$. Furthermore, $a_h v$ since $d_v = 3$ and u and v can have no common a_i neighbor. R(ux,uy). ONE(u,v,x,y). No two of u, v, x and y dominate. I(u : v,x,y), a contradiction. I(v : a_i for $i \neq 1$, h), a contradiction. I(x : u,v,y), a contradiction. I(y : x,u,a_1) yielding the contradiction for this situation.

In all situations except the previous one, since p $\geq$ 11, there exists *a_2 which is not involved in increasing DS, and is not a_1 or adjacent to a_1. Then a_2 must have degree three and be adjacent to *a_3 with degree three, where a_3 is not adjacent to a_1. If six a_i's were involved in increasing the DS by two or four a_i's were involved and a_1 is not involved, p is forced to be at least twelve. D(u,a_1) implies ua_2 and ua_3. See Figure 5.36(a). R(a_2 : a_3).
ONE(u,v,x,y).

N(u,a_3 : a_1).

Suppose D(v,a_3). Then va_i for $i \neq 2$, 3. Each a_i, $i \neq 1$, must be adjacent to u or a_1. Let m be the number of a_i's which are adjacent to a_1, $1 \leq m \leq 3$. Then u must be adjacent to p-5-m a_i's and hence both u and v are adjacent to p-5-m-2 of the same a_i's. Then DS $\geq$ 6p-22 $\geq$ 5p-11 if p $\geq$ 11.

Assume D(x,a_3) implies xa_i for $i \neq 2$, 3 and vx since a_3 is not adjacent to v. DS $\geq$ 5p-12 implies each a_i has degree three and no a_i can be adjacent to both u and v or both x and y. Hence a_1 must be adjacent to exactly one a_i, say $*a_4$, and $D(u,a_1)$ implies ua_i for $i \neq 1, 4$. Furthermore, a_3y since x is not adjacent to y. See Figure 5.36(b). $R(a_5 : *a_6)$. ONE(u,v,x,y). $N(u,a_6 : a_1)$, $N(v,a_6 : a_2)$, $N(x,a_6 : a_3)$ and $N(y,a_6 : a_4)$ yielding the contradiction.

Thus $D(y,a_3)$. Then ya_1 to dominate a_1. Considering $D(u,a_1)$ and $D(x,y)$, $DS \geq 5p-12$. Hence x is not adjacent to y, so a_3x and ya_i for $i \neq 2, 3$. See Figure 5.36(c). Now $d_v = 3$ so v must be adjacent to a_j, $j > 3$. $D(u,a_1)$ implies ua_j or a_1a_j. First let ua_j. Then $DS \geq 5p-12$. Thus each a_i has degree three, no other a_i can be adjacent to both u and v or both x and y, and a_j is not adjacent to an a_i. Since a_1 is not adjacent to an a_i, $D(u,a_1)$ implies ua_i for $i \neq 1$. See Figure 5.36(d). $R(a_5 : *a_6)$. ONE(u,v,x,y). $N(u,a_6 : a_1)$, $N(v,a_6 : a_2)$, $N(x,a_6 : a_j)$ and $N(y,a_6 : a_3)$, a contradiction. Finally, assume a_ja_1 and a_j is not adjacent to u. $DS \geq 5p-12$. Therefore no a_i, $i \neq 1$, has three edges to $D_1 \cup D_2$ and each a_i, $i \neq 1$, has degree three. $D(u,a_1)$ implies ua_i for $i \neq 1, j$. See Figure 5.36(e). $R(a_5 : *a_6)$. ONE(u,v,x,y). $N(u,a_6 : a_1)$, $N(v,a_6 : a_3)$, $N(x,a_6 : a_j)$ and $N(y,a_6 : a_3)$ yielding the contradiction which eliminates Subcase 5.b and hence completes elimination of case 5.

All cases where an extremal graph might have fewer than $\lfloor (5p-10)/2 \rfloor$ edges have been examined and each leads to a contradiction. Thus $E^2(p,2) \geq \lfloor (5p-10)/2 \rfloor$, if $p \geq 11$, and equality follows from Theorem 5.1. ■

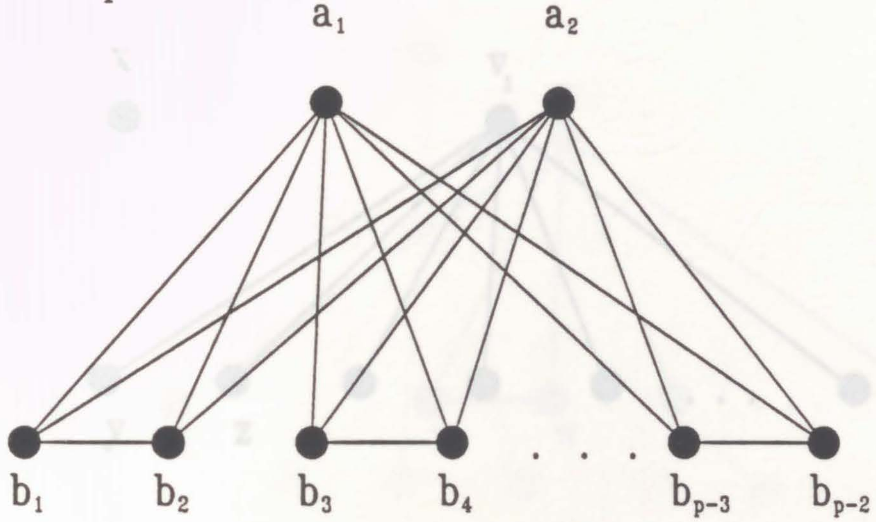
5.4. The Value of $E^2(p,2)$ for All p

Theorem 5.5 gives $E^2(p,2)$ for $p \geq 11$. Since G is connected $\gamma \leq p/2$ (Marcu 1985), and hence the problem is undefined for $p < 4$. For $4 \leq p \leq 10$ we utilized a magnetic tape supplied by R. C. Read from the University of Waterloo that catalogues all graphs on 10 nodes or less. A program was written to search the tape to find extremal 2-2-insensitive graphs. Results from the search are shown in Figure 5.37 and indicate that the value of $E^2(p,2)$ given by Theorem 5.5 actually holds for $p \geq 9$. For complete results from the search, see the appendix which lists the adjacency matrices of all 2-2-insensitive graphs having $p \leq 10$ nodes. All values of $E^2(p,2)$ are now summarized in the following theorem.

Theorem 5.6

$$E^2(p,2) = \begin{cases} \text{undefined} & \text{if} \quad p \leq 3 \\ 4 & \text{if} \quad p = 4 \\ 7 & \text{if} \quad p = 5 \\ 9 & \text{if} \quad p = 6 \\ 11 & \text{if} \quad p = 7 \\ 13 & \text{if} \quad p = 8 \\ \lfloor (5p-10)/2 \rfloor & \text{if} \quad p \geq 9 \end{cases}$$

p even:



p odd:

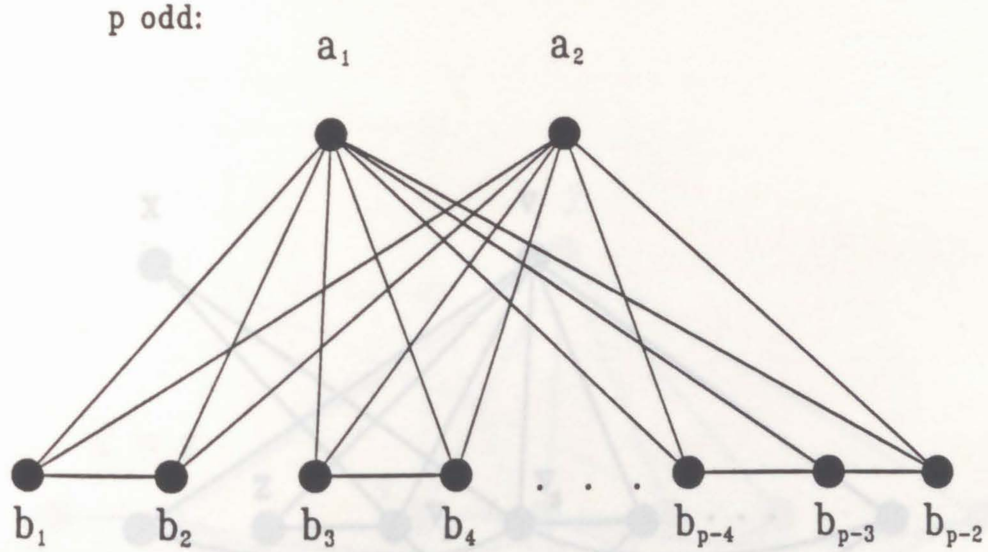


Figure 5.1

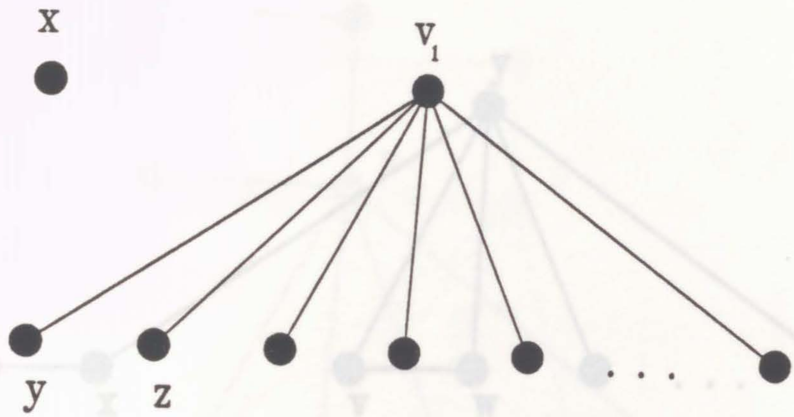


Figure 5.2(a)

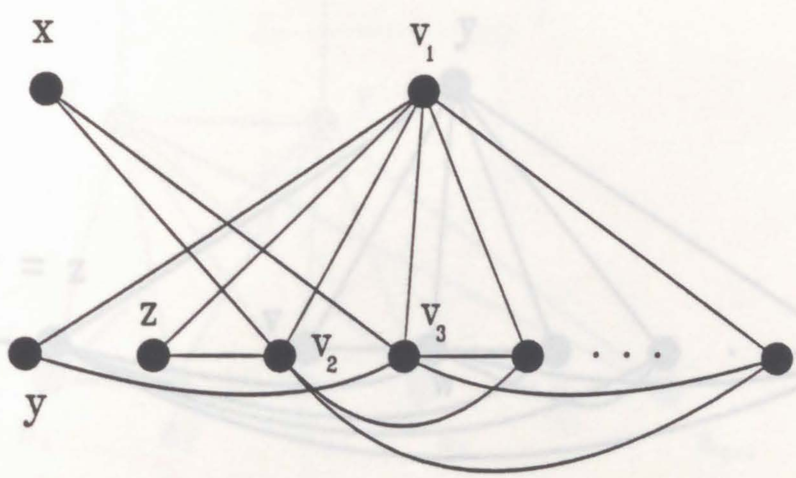


Figure 5.2(b)

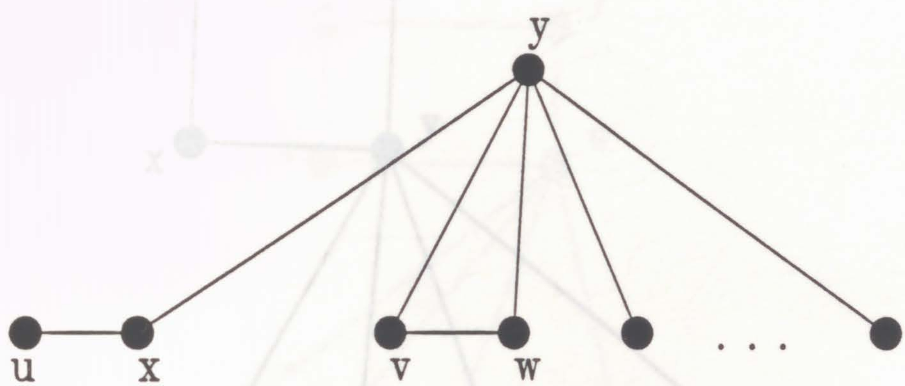


Figure 5.3(a)

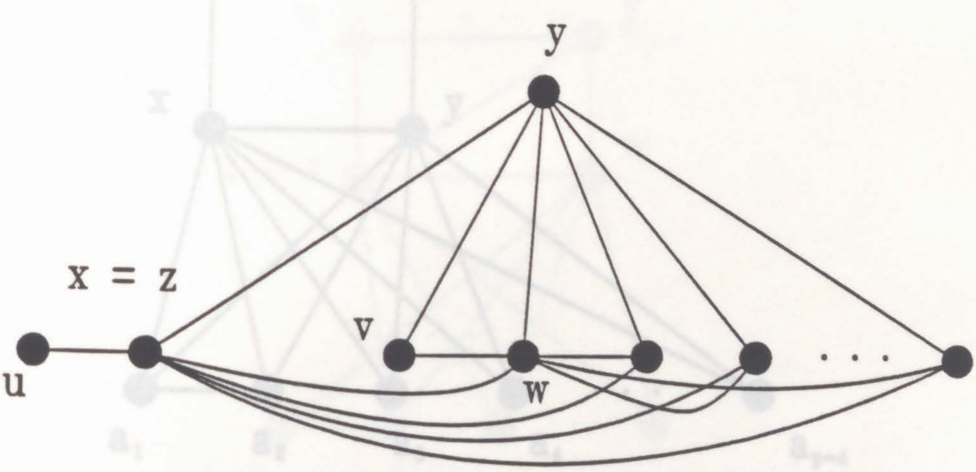


Figure 5.3(b)

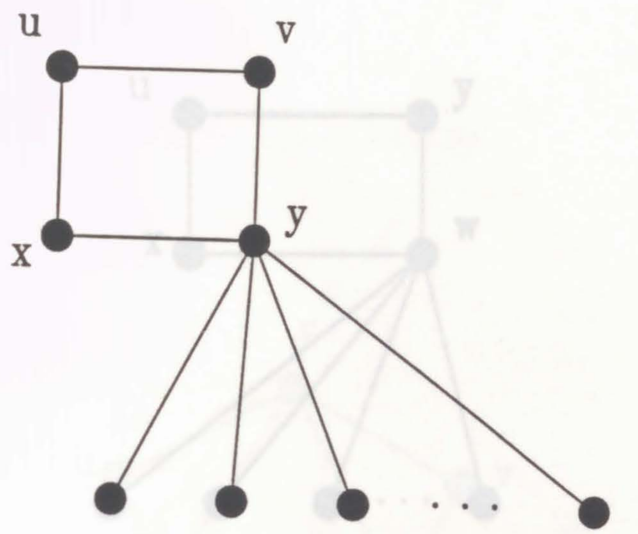


Figure 5.4(a)

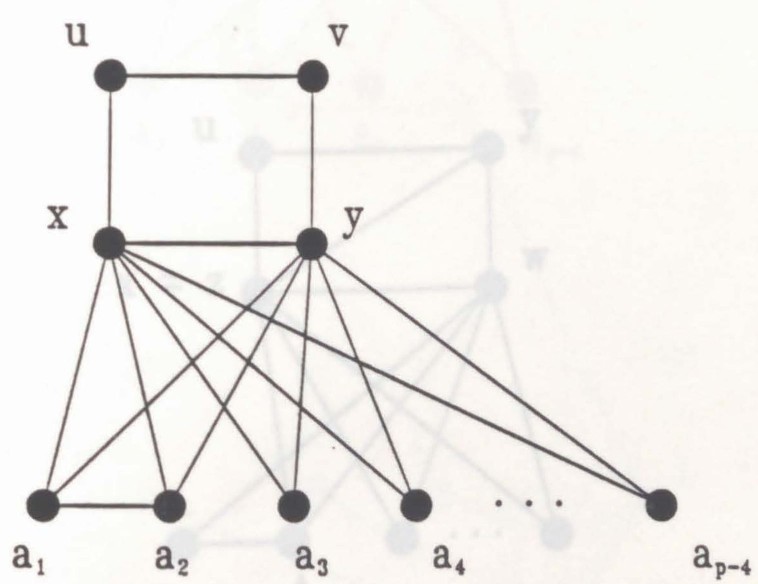


Figure 5.4(b)

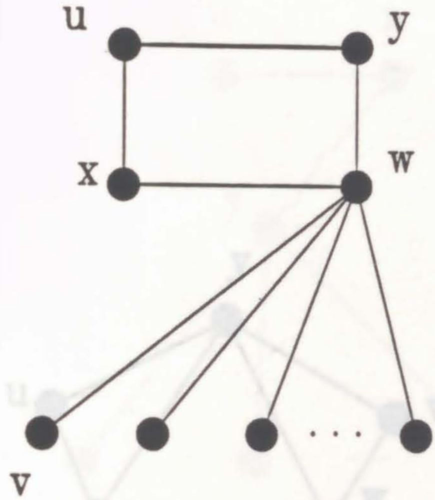


Figure 5.5(a)

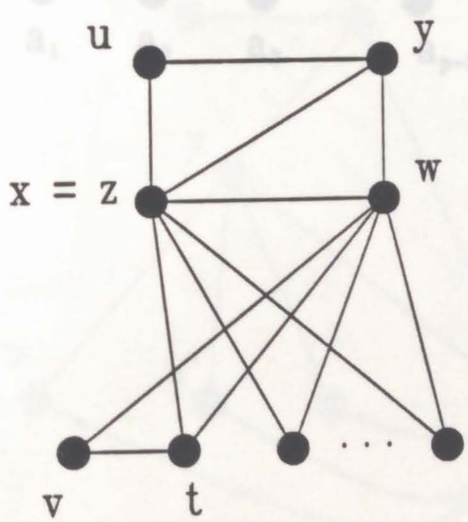


Figure 5.5(b)

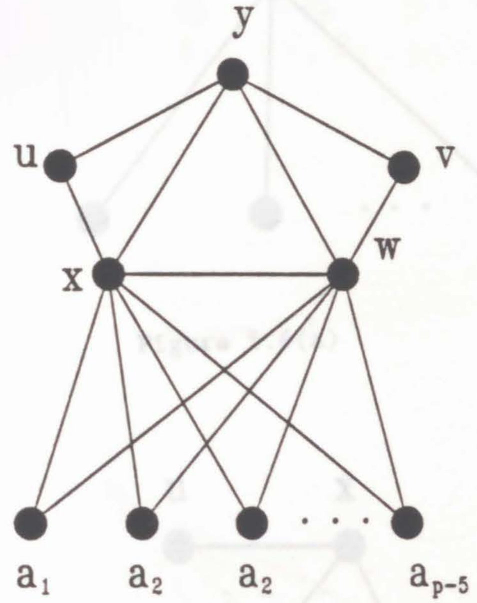


Figure 5.5(c)

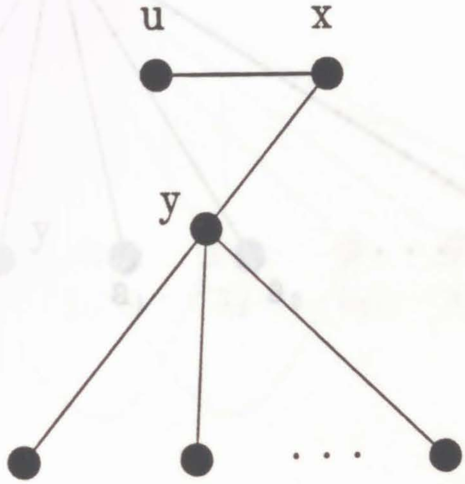


Figure 5.6(a)

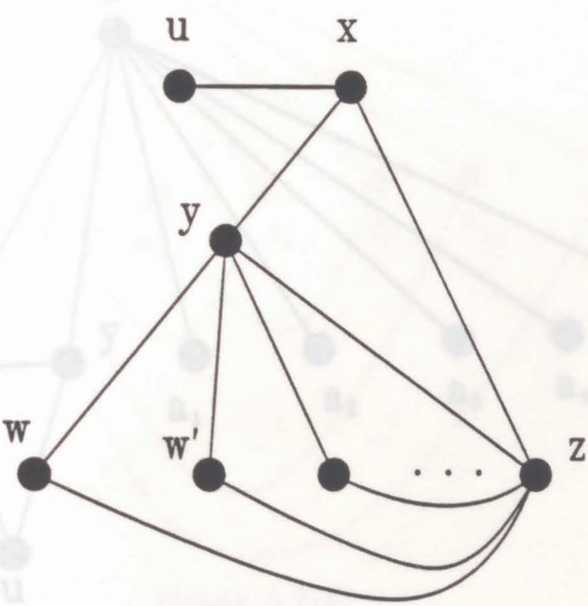


Figure 5.6(b)

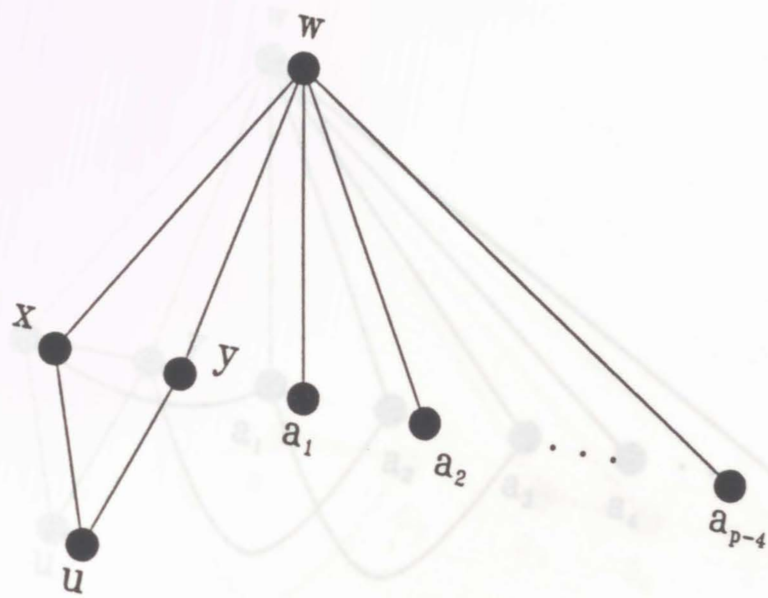


Figure 5.7

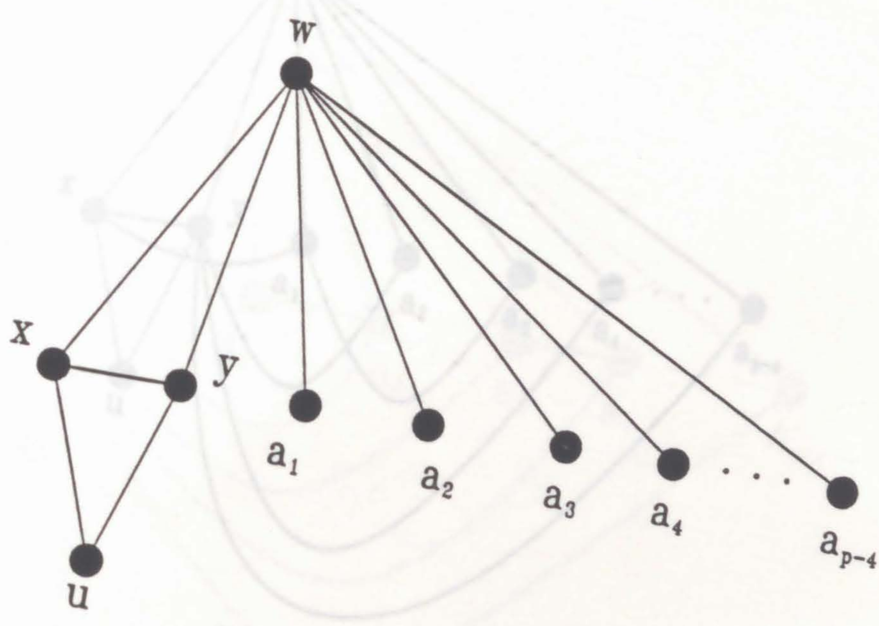


Figure 5.8

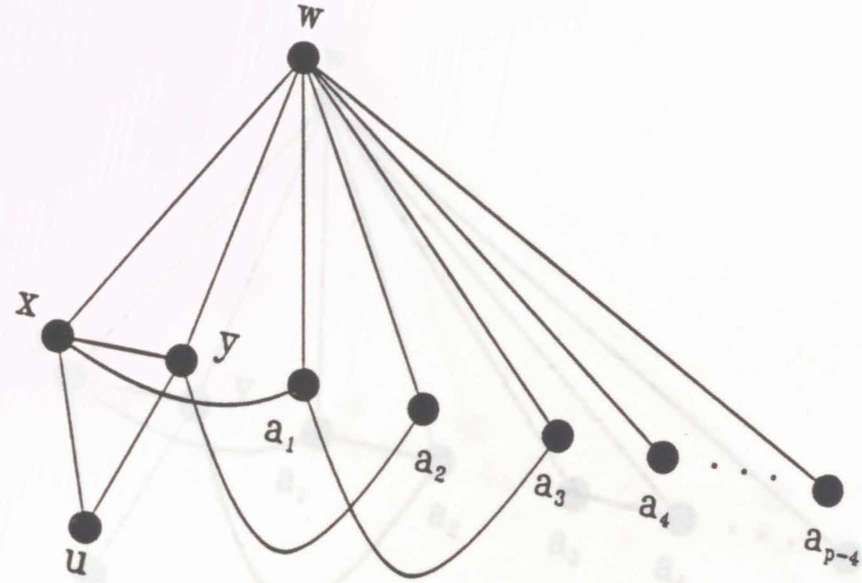


Figure 5.9(a)

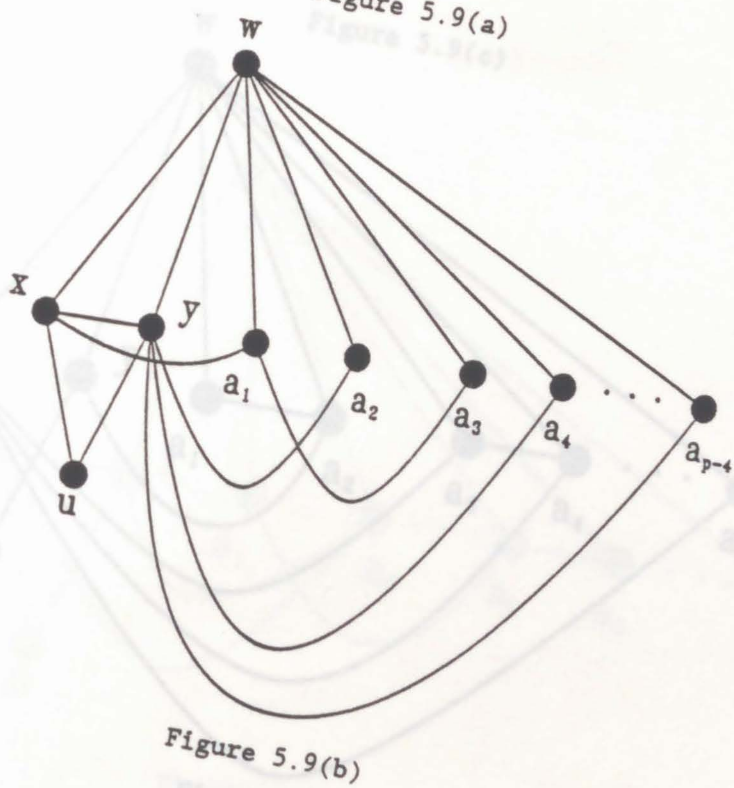


Figure 5.9(b)

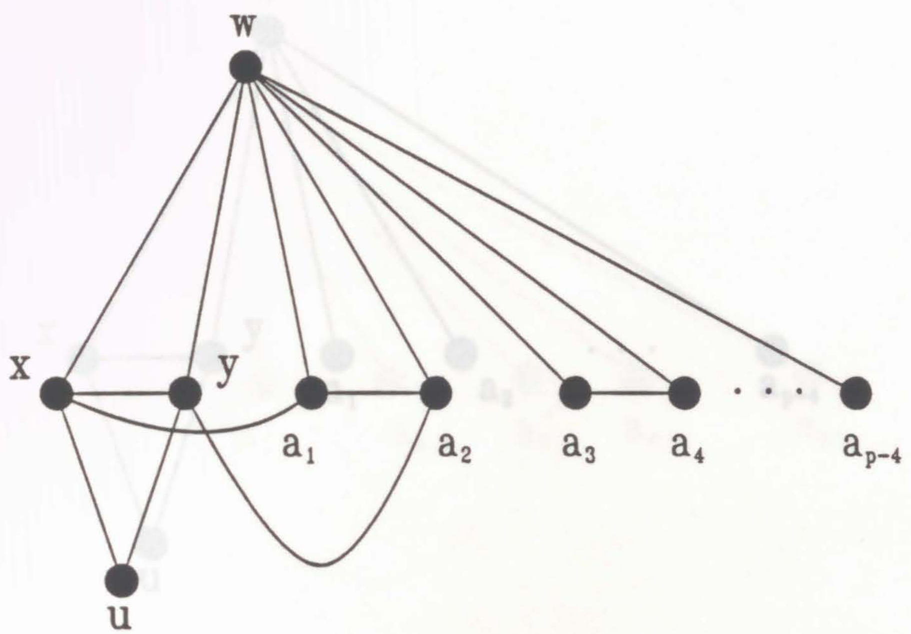


Figure 5.9(c)

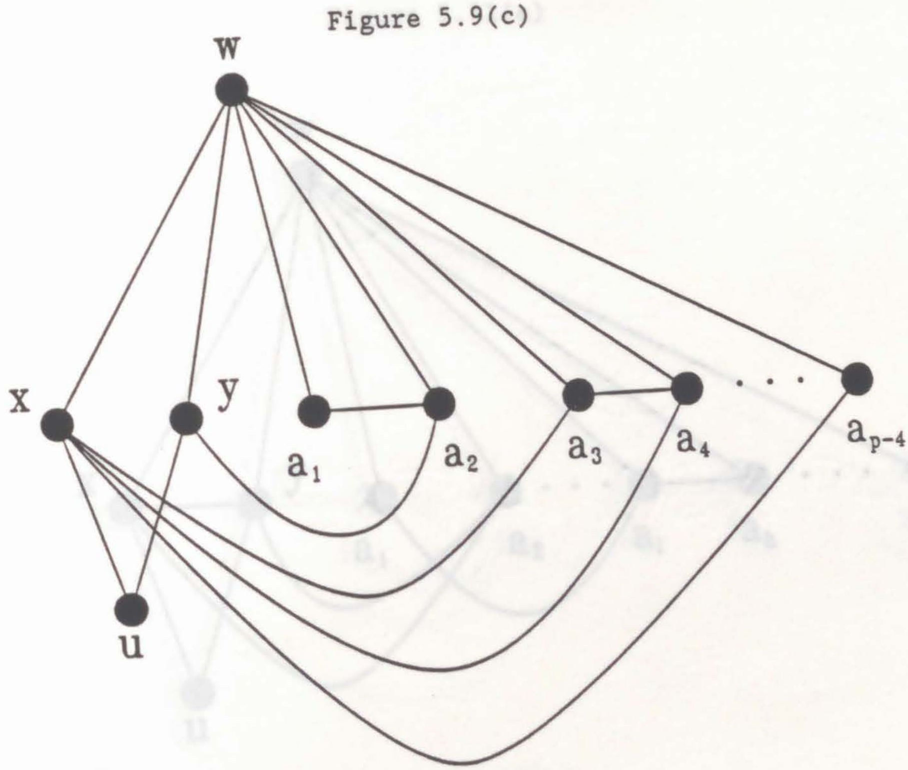


Figure 5.10

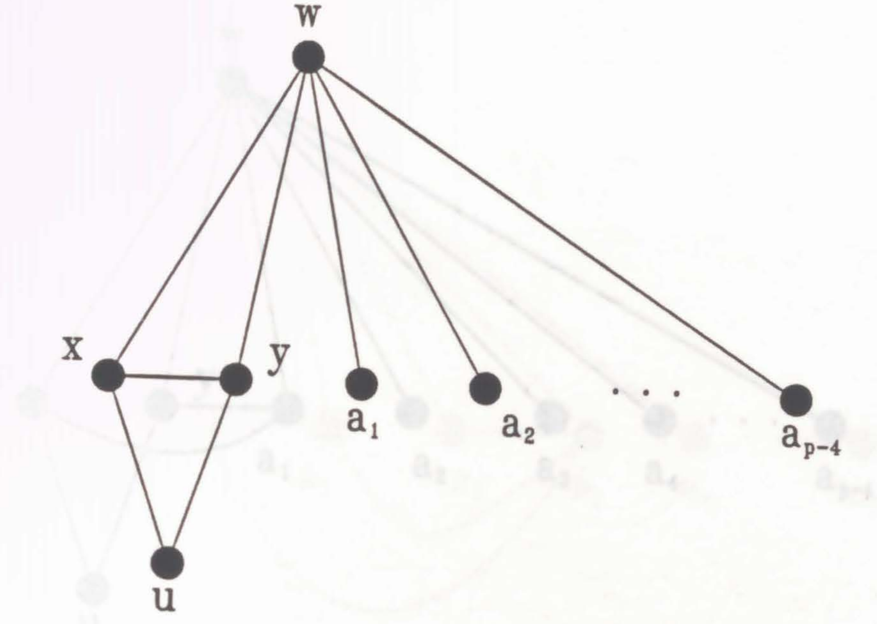


Figure 5.11(a)

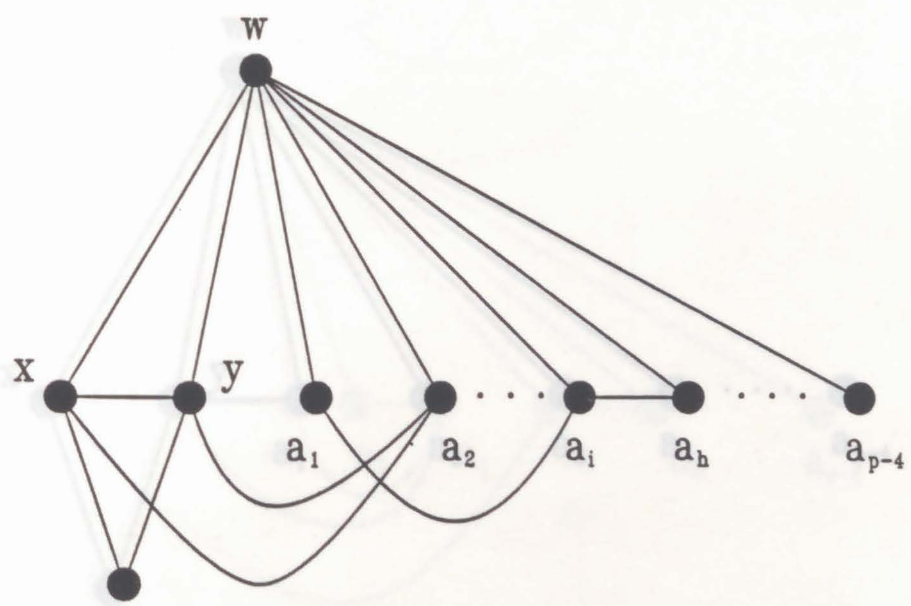


Figure 5.11(b)

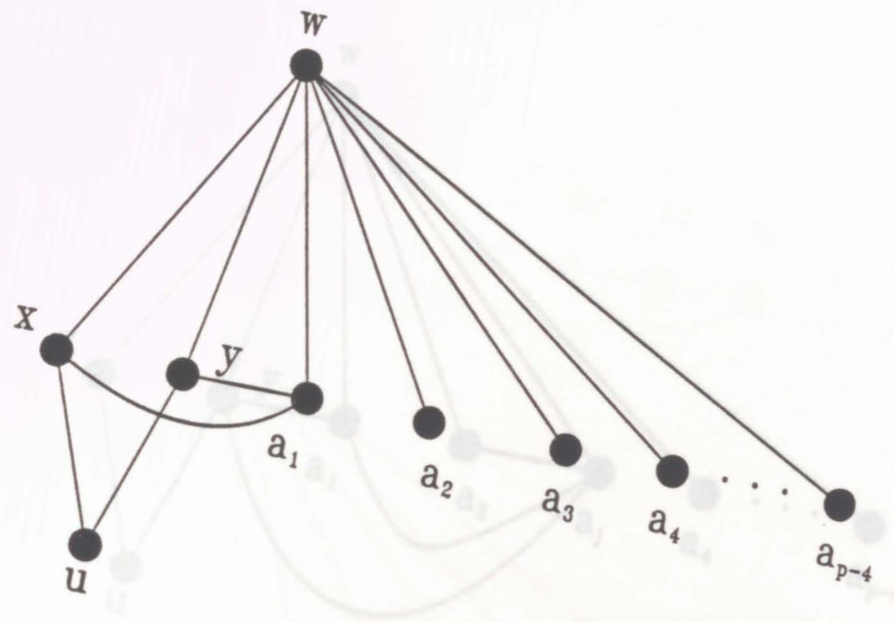


Figure 5.12(a)

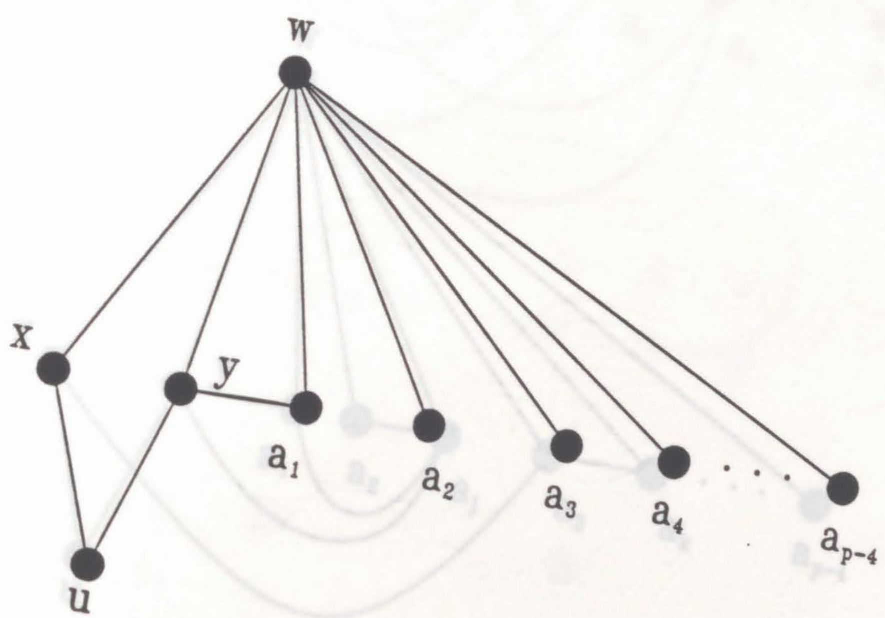


Figure 5.12(b)

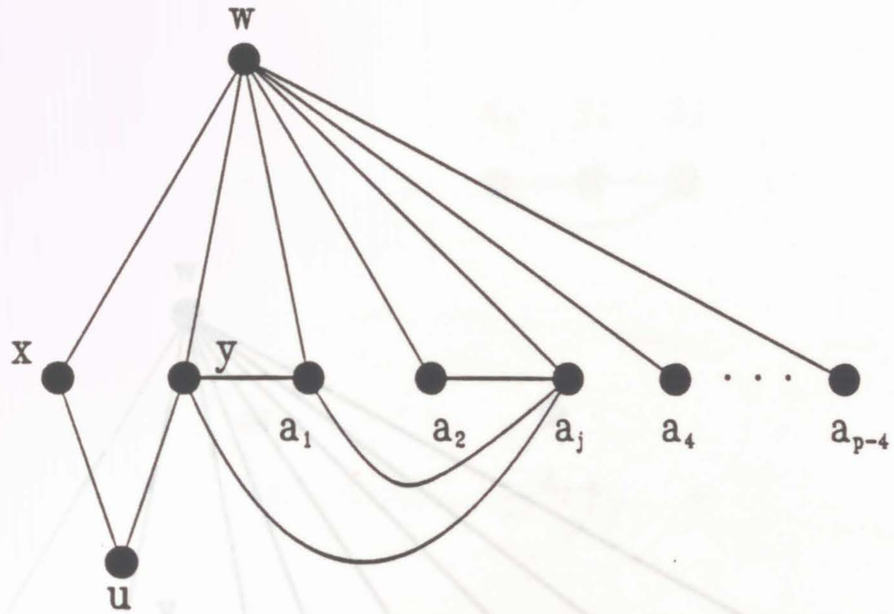


Figure 5.12(c)

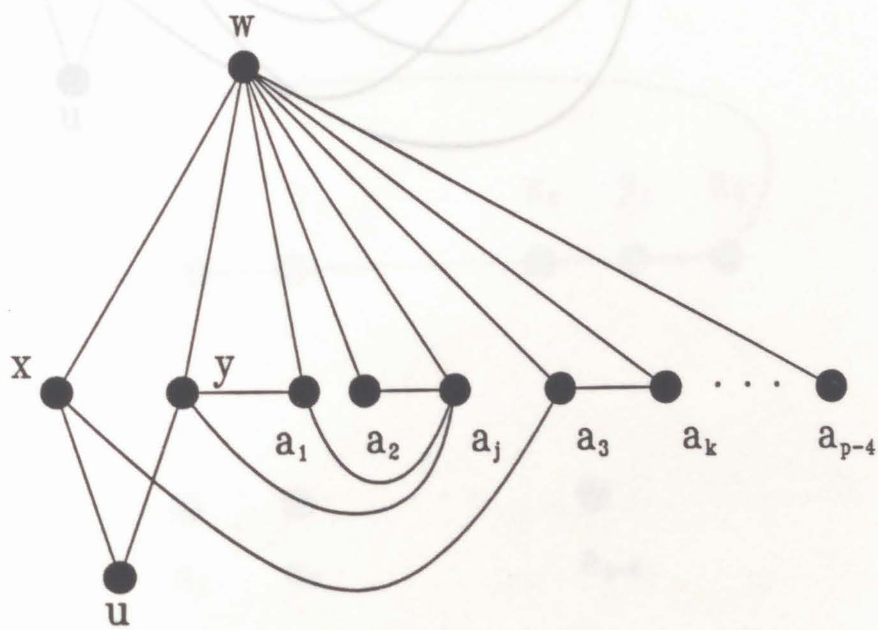


Figure 5.12(d)

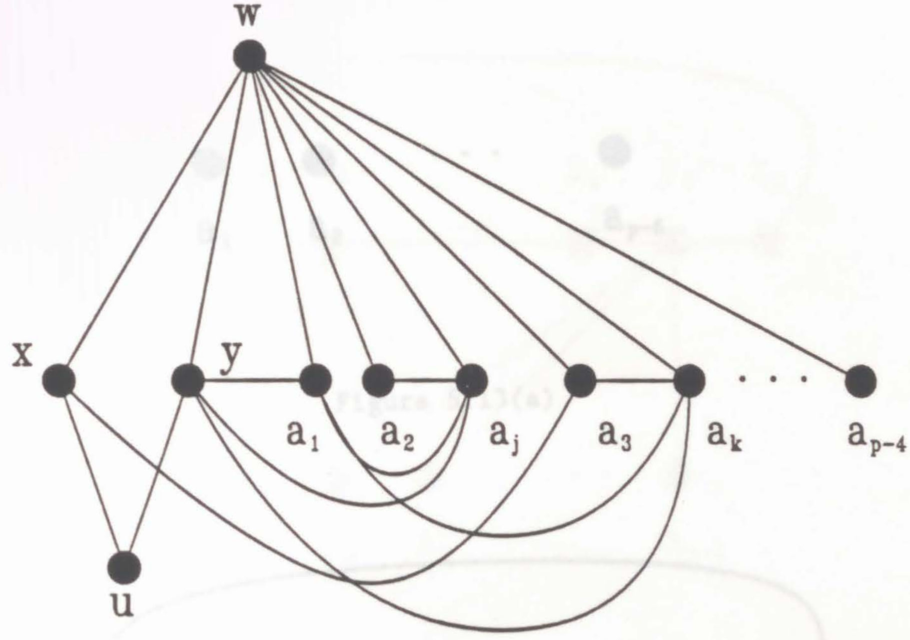


Figure 5.12(e)

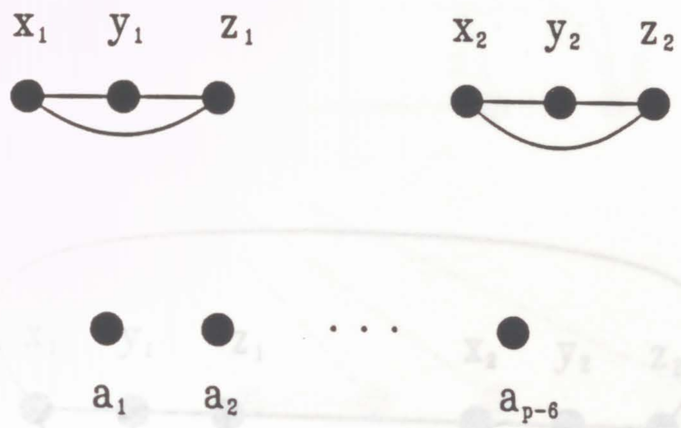


Figure 5.13(a)

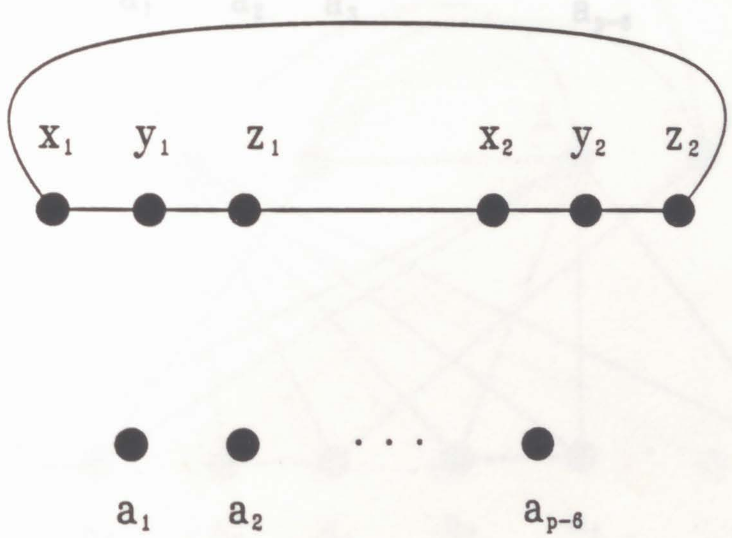


Figure 5.13(b)

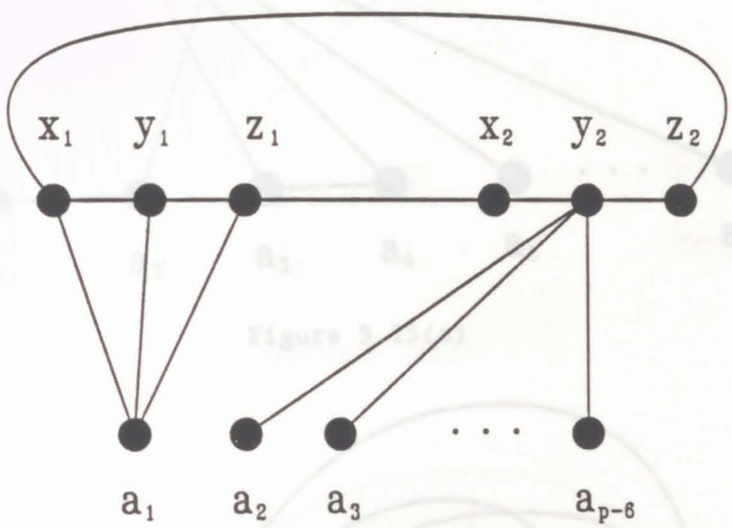


Figure 5.14

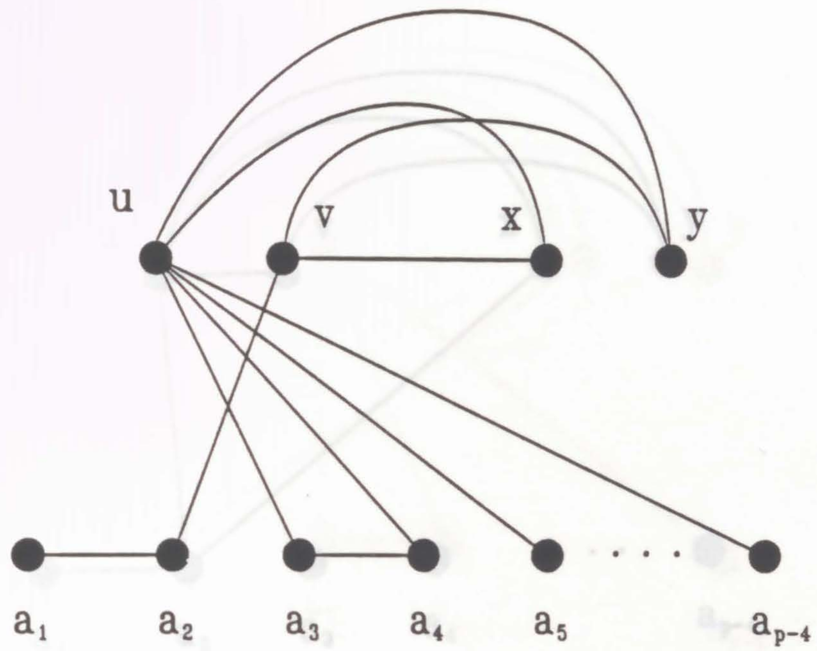


Figure 5.15(a)

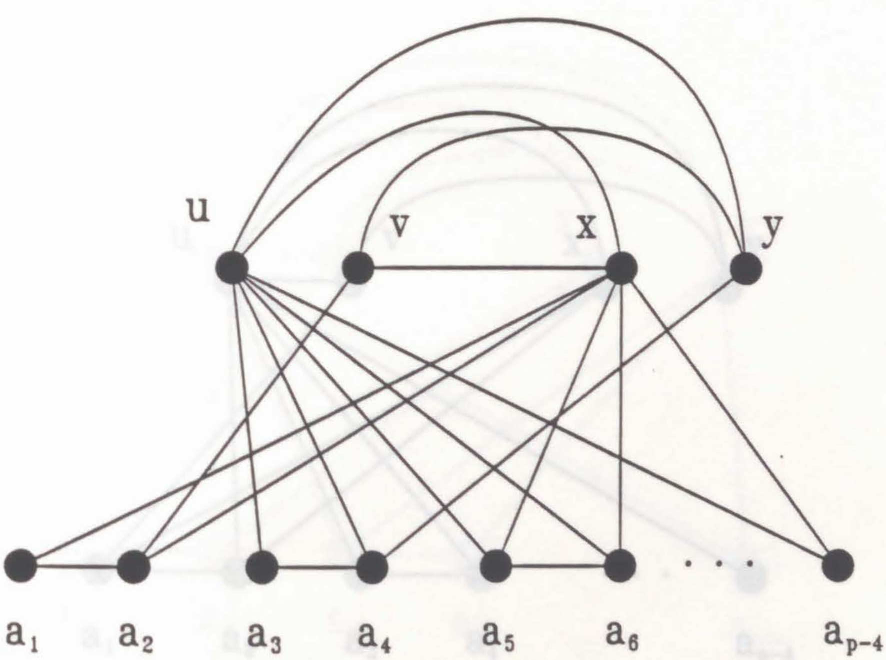
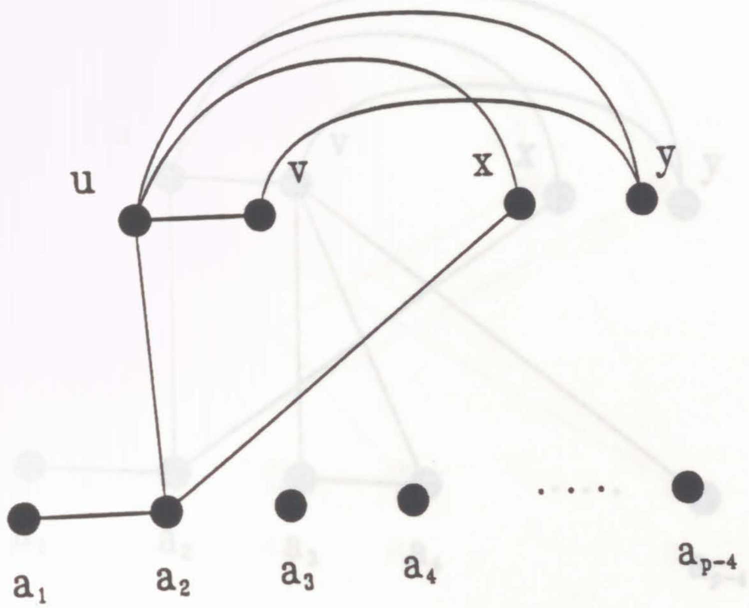


Figure 5.15(b)



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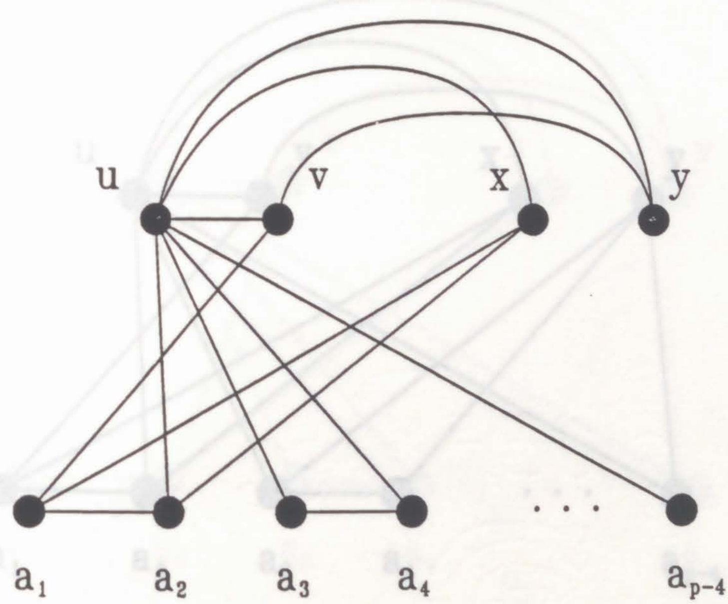


Figure 5.16(b)

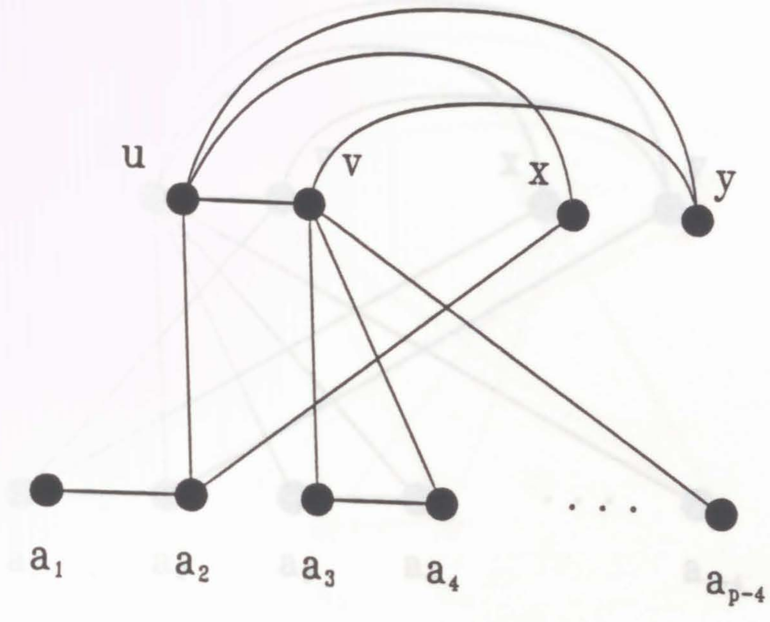


Figure 5.16(c)

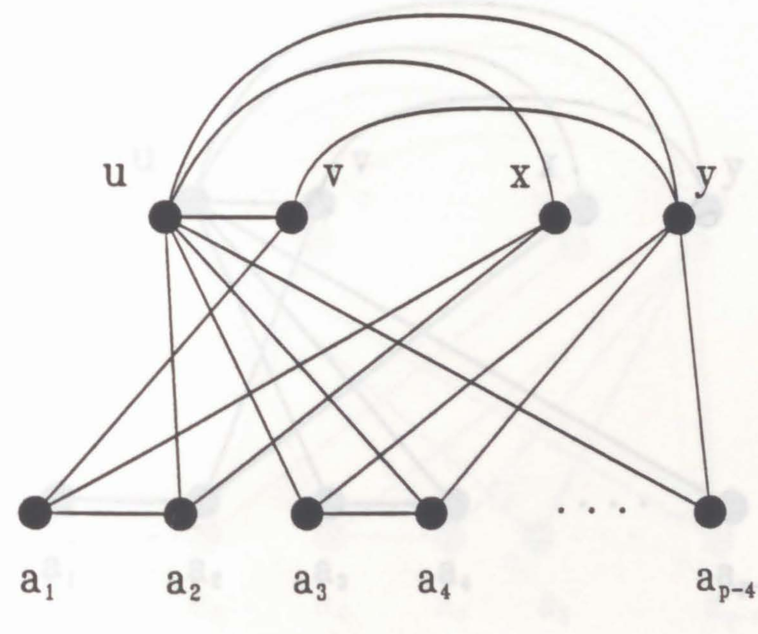


Figure 5.16(d)

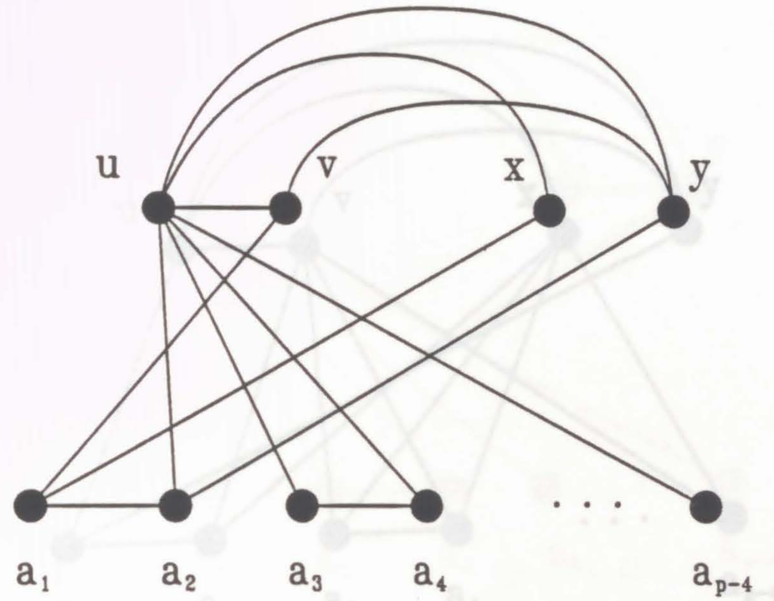
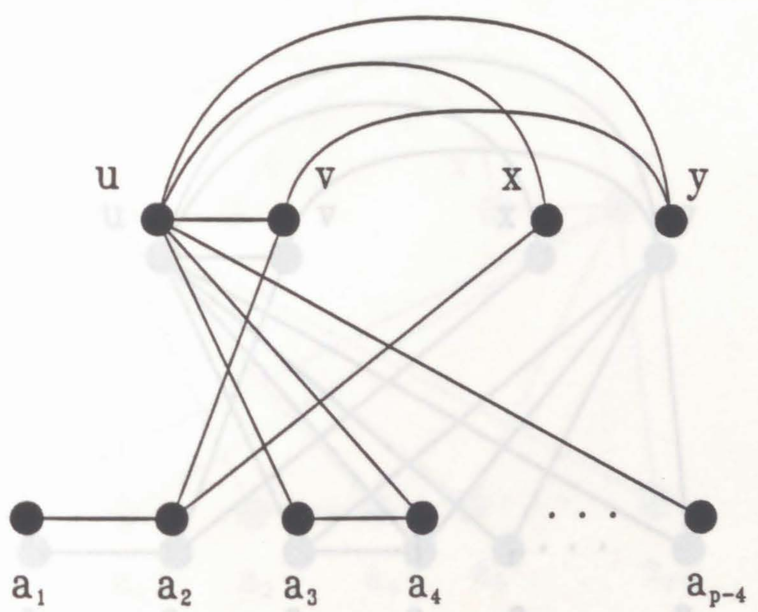


Figure 5.17



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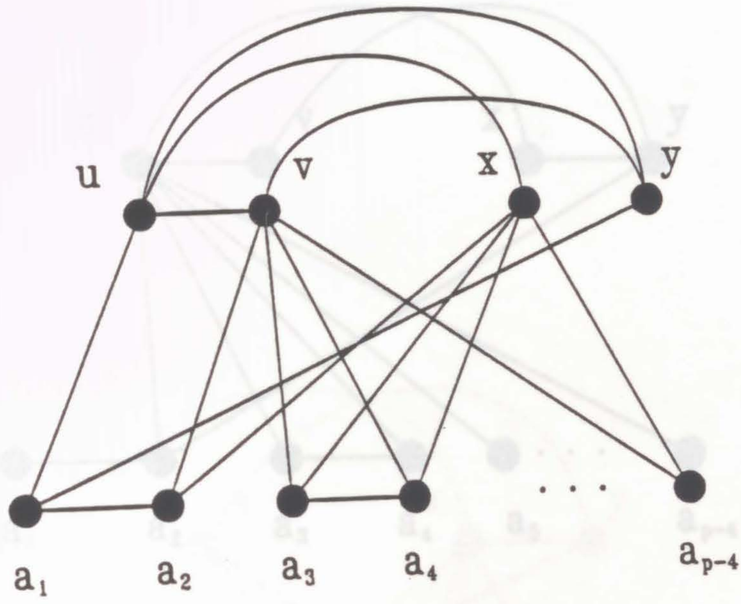


Figure 5.18(b)

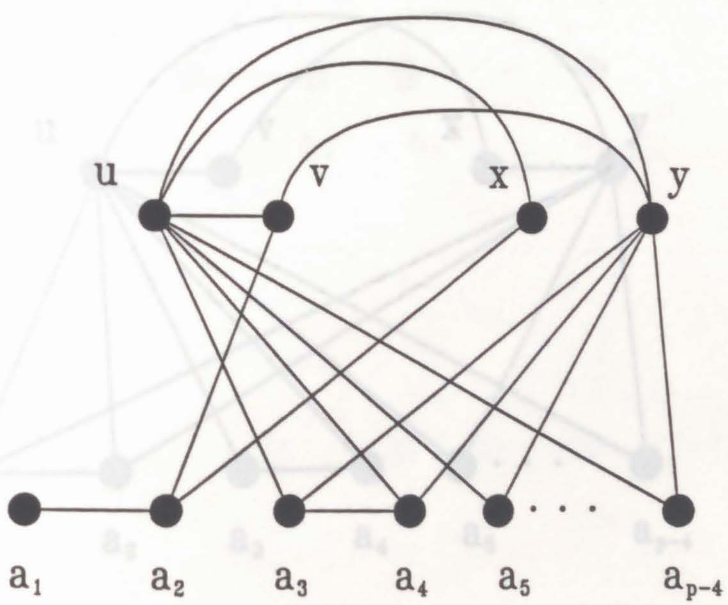


Figure 5.18(c)

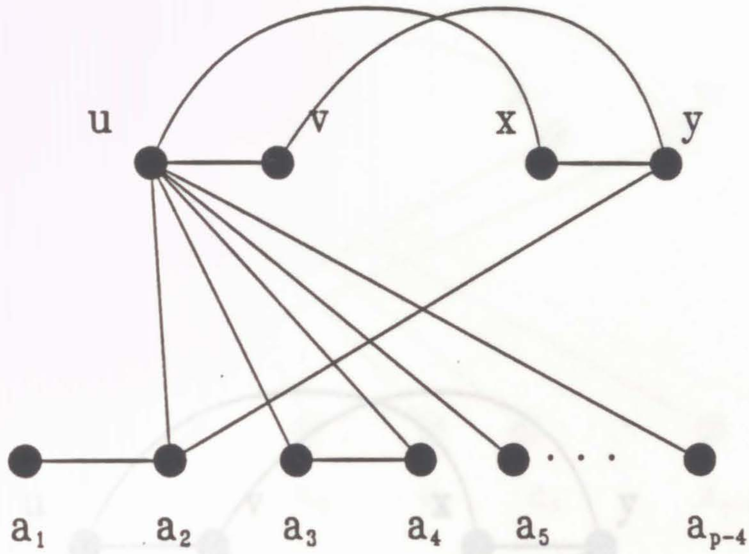


Figure 5.19(a)

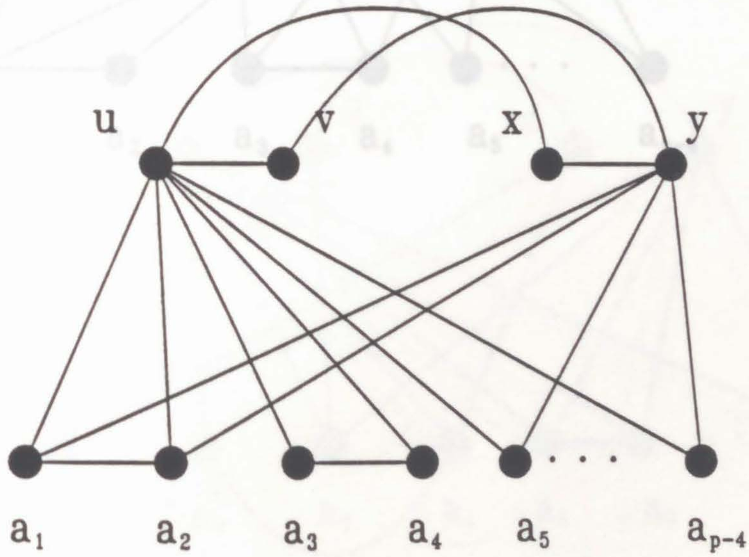
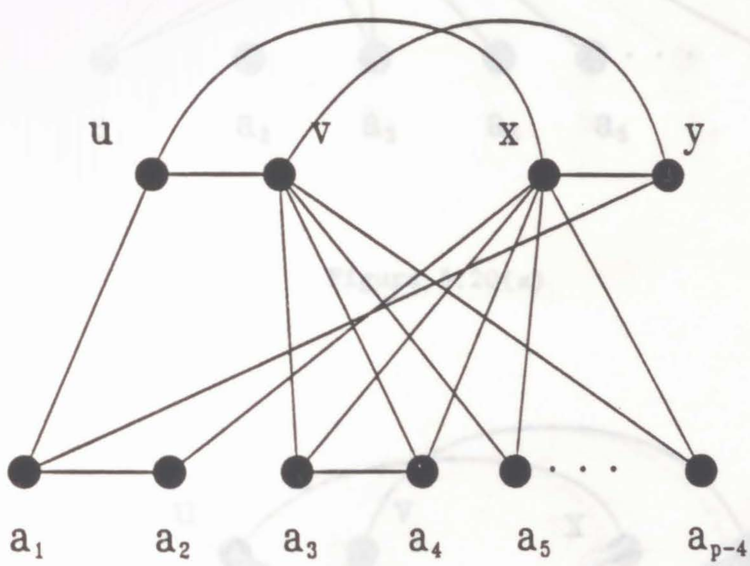


Figure 5.19(b)



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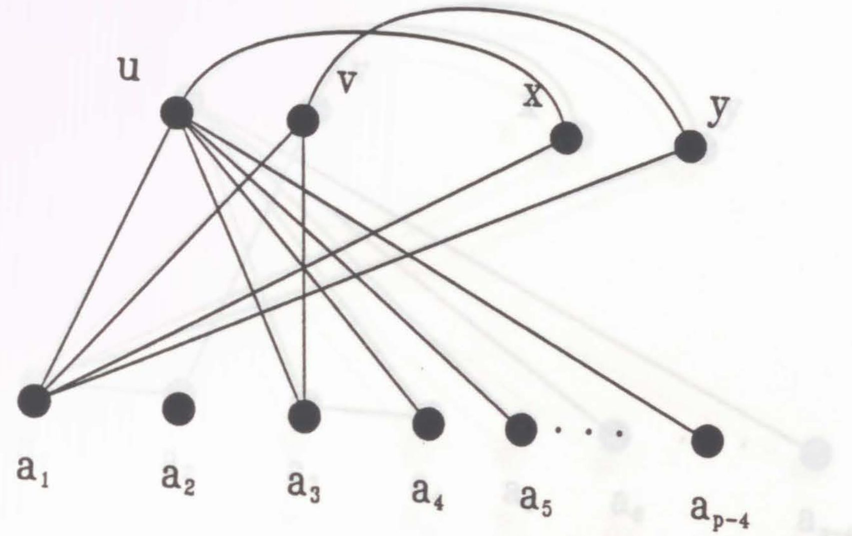

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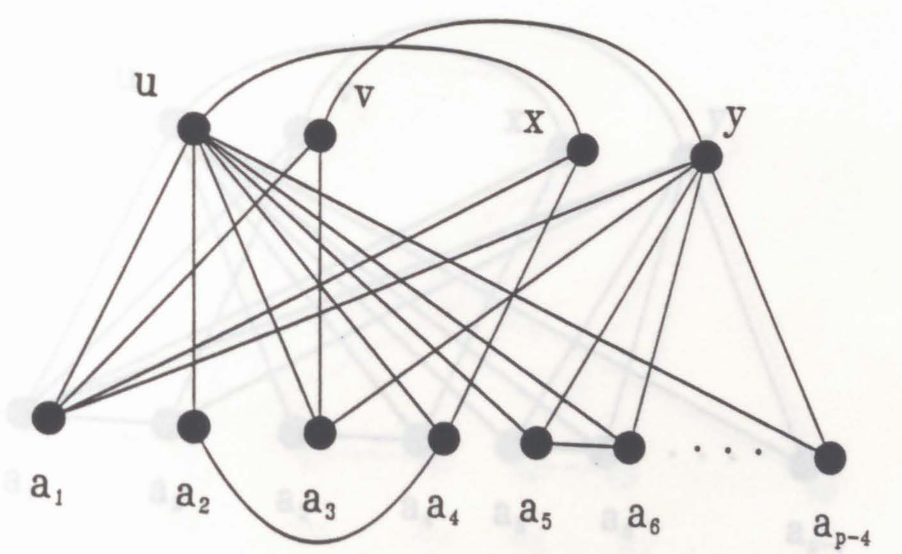

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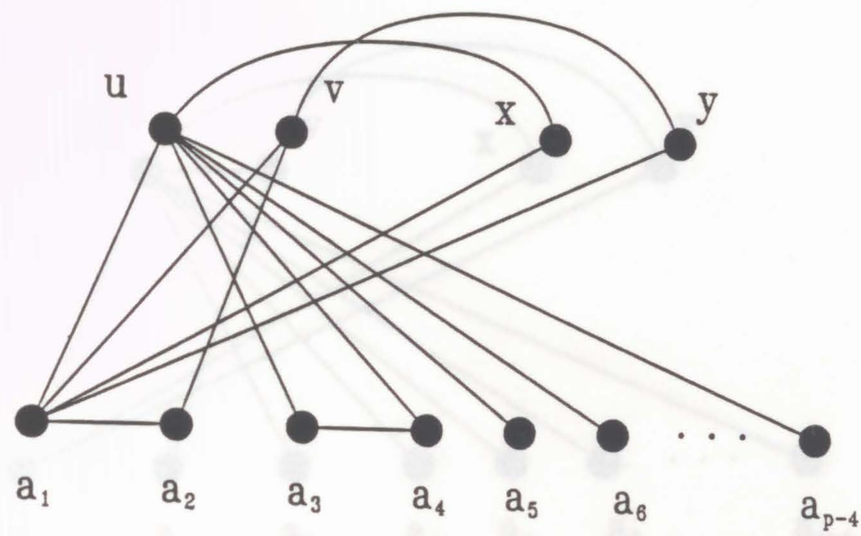

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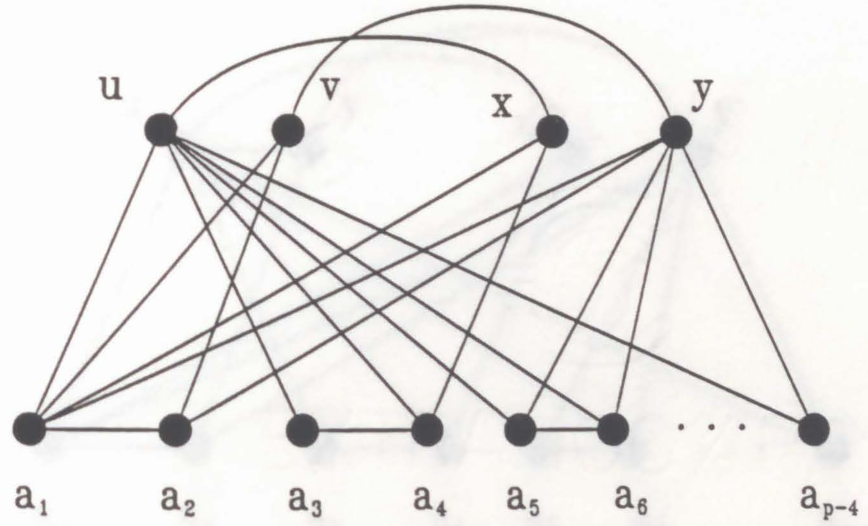

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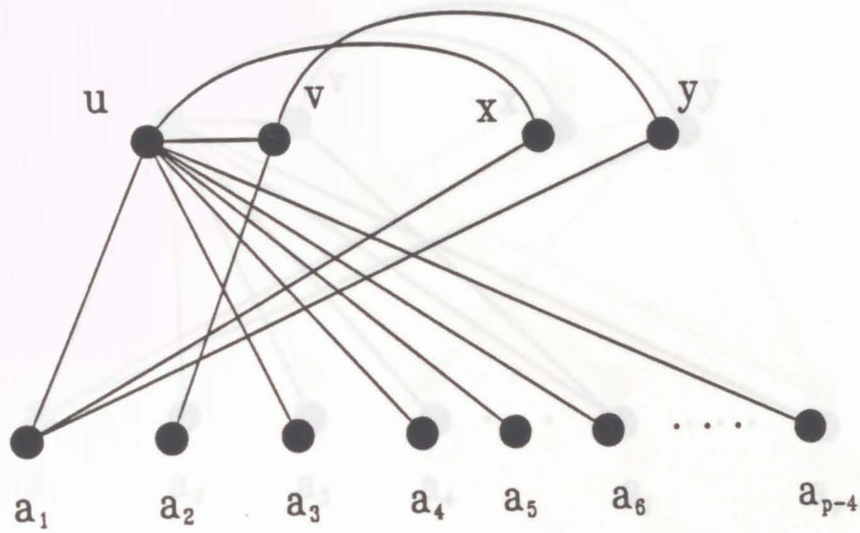

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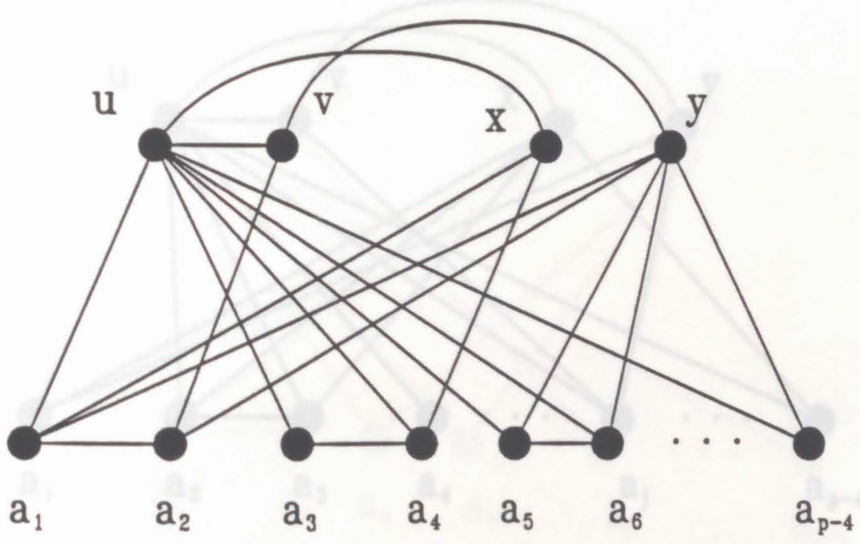

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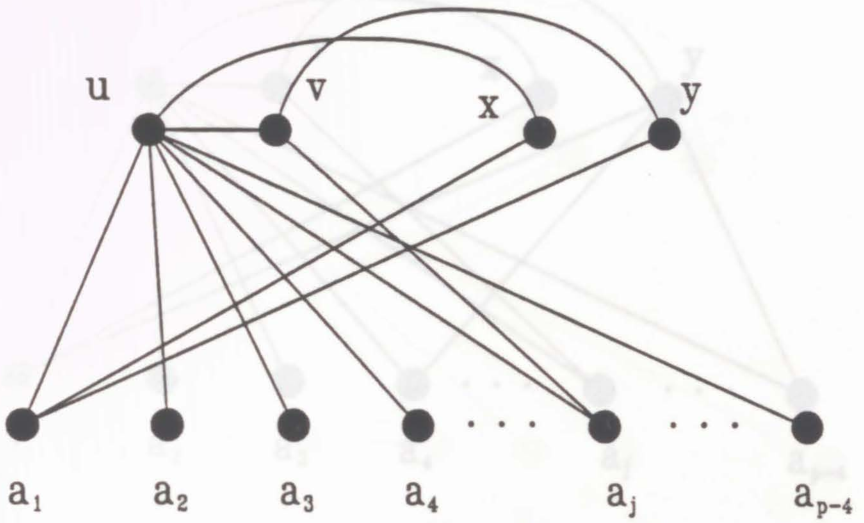

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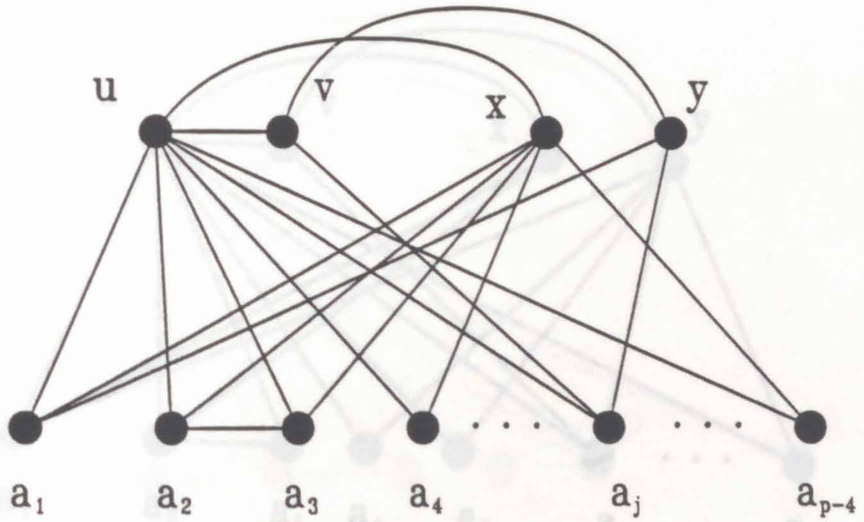

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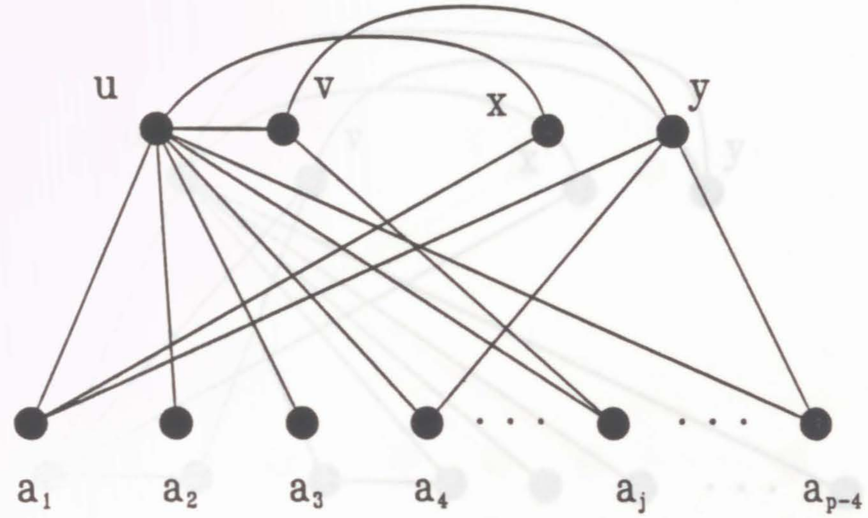

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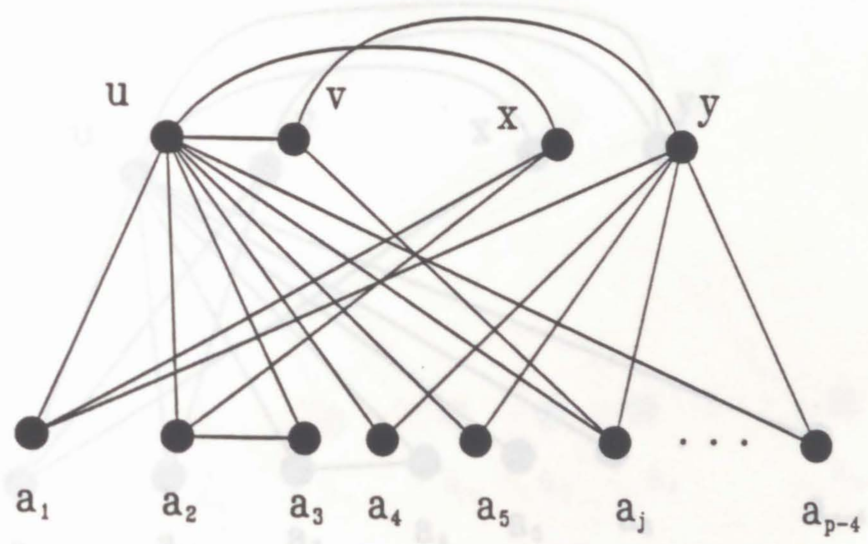

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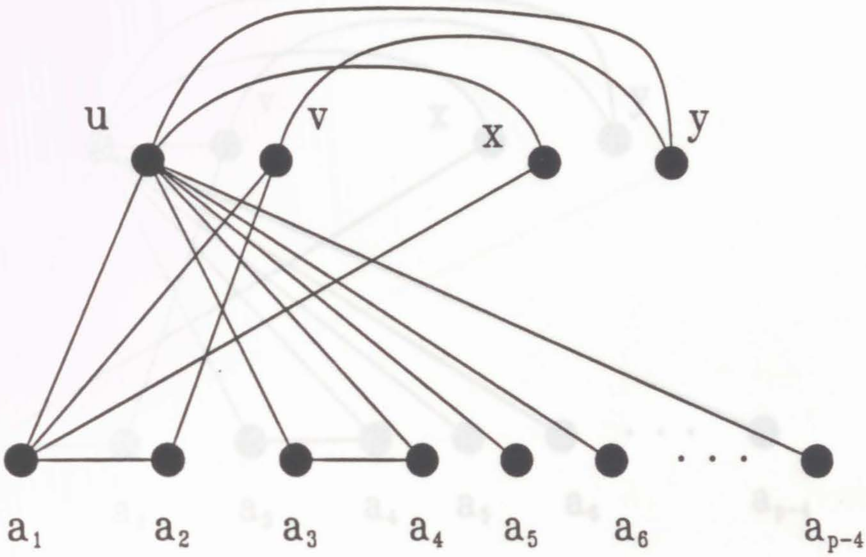

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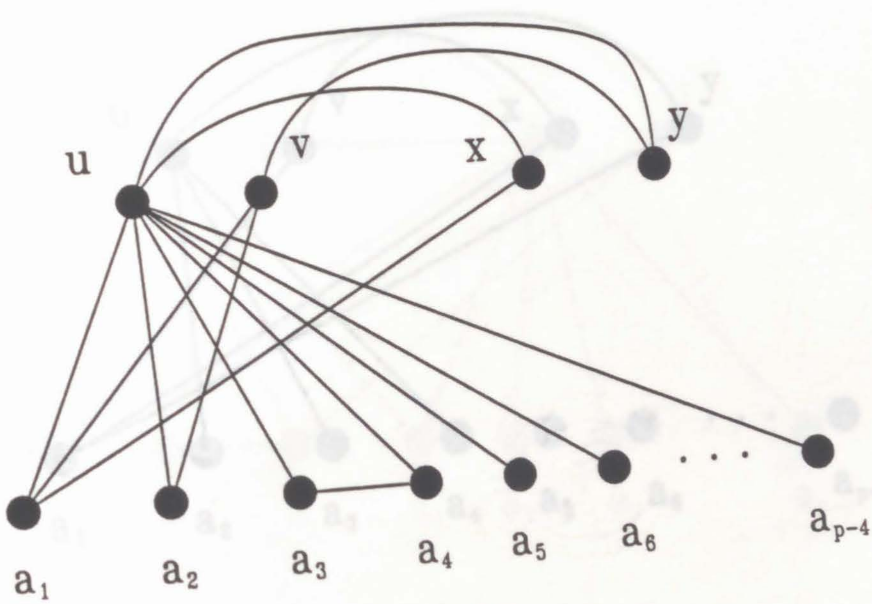

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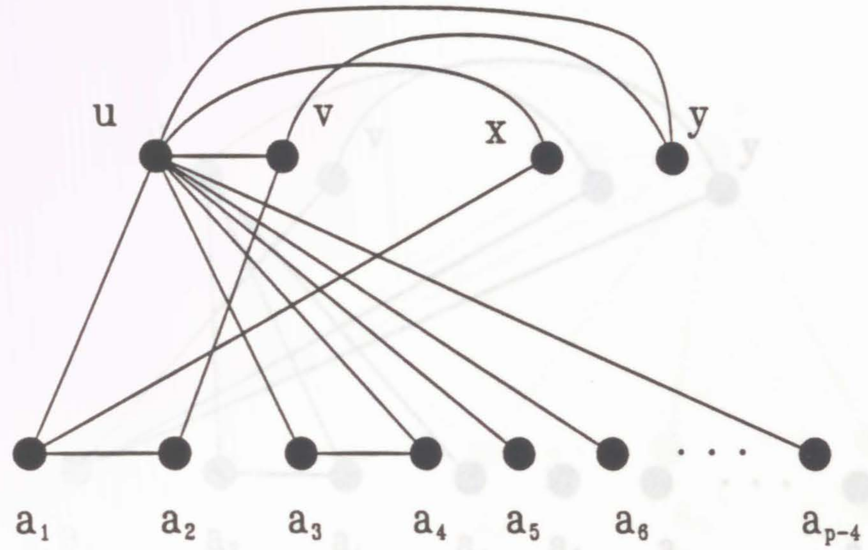

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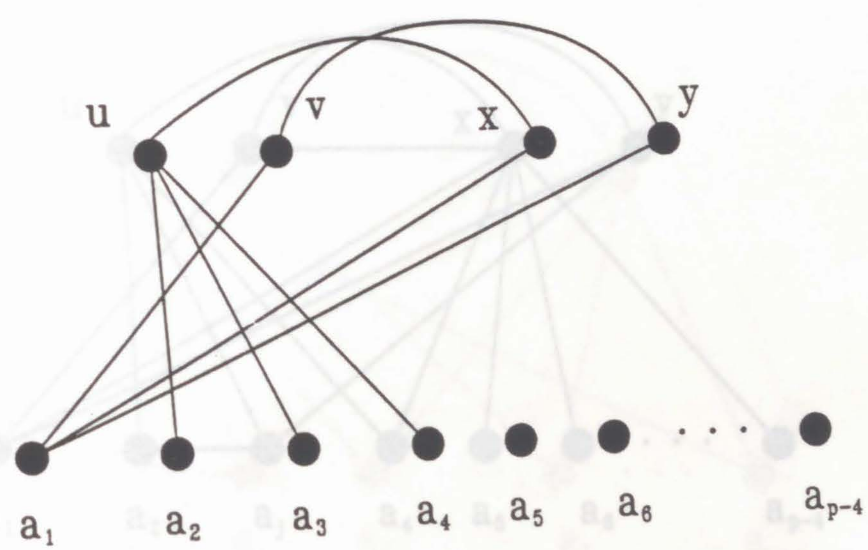

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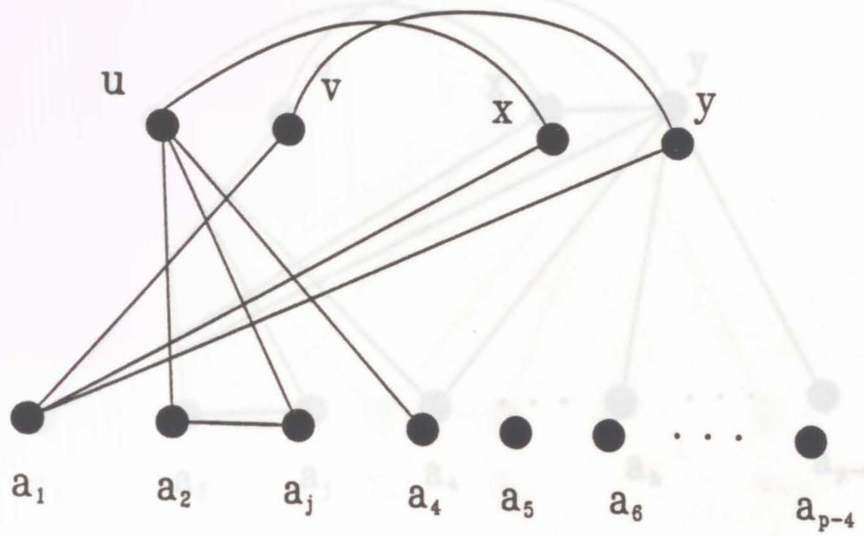

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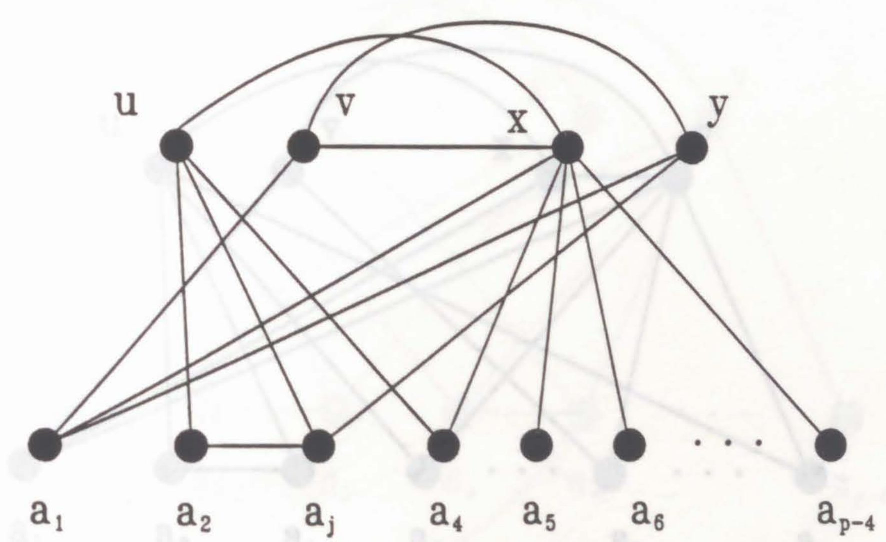

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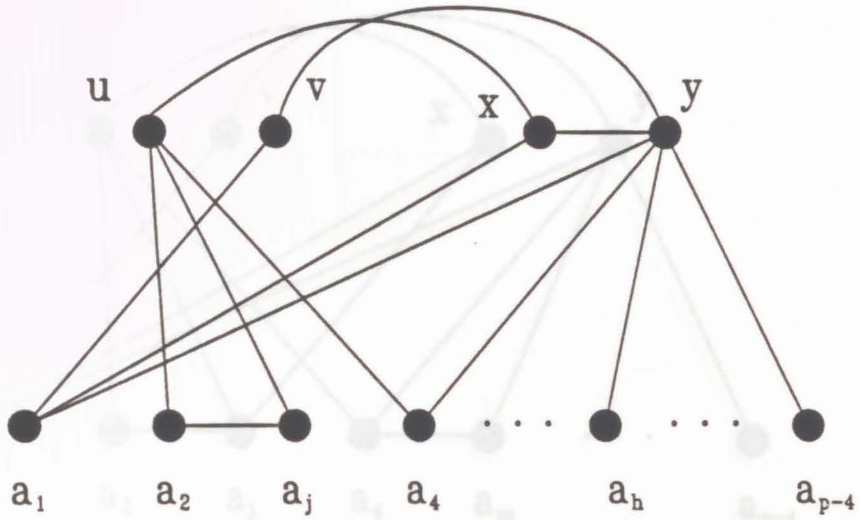

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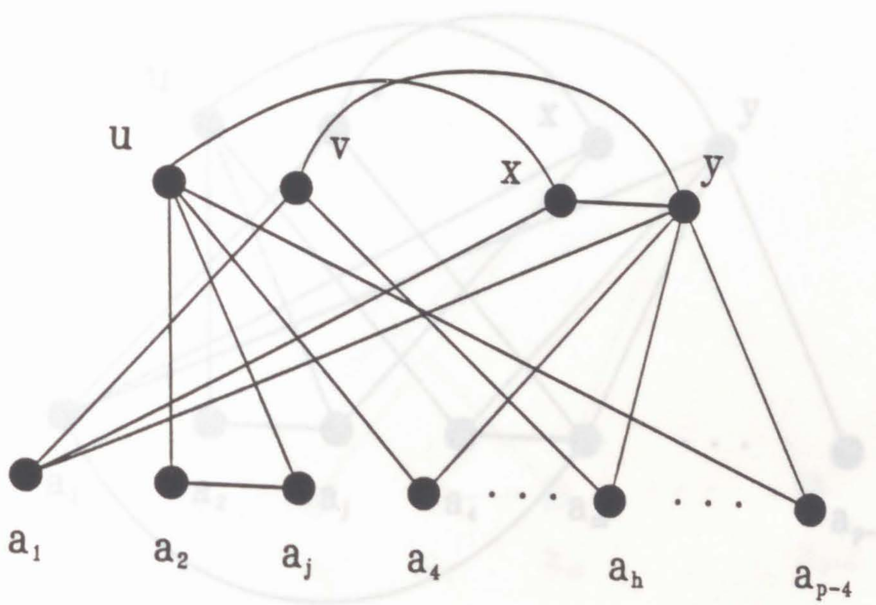

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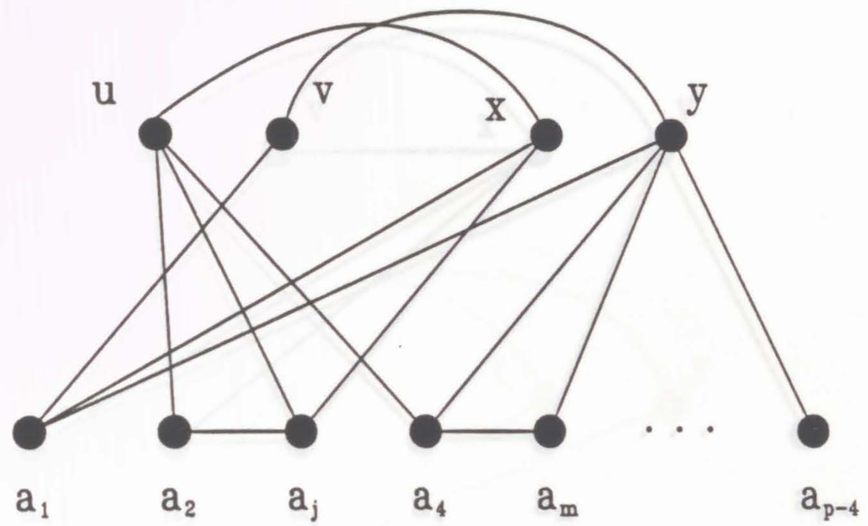

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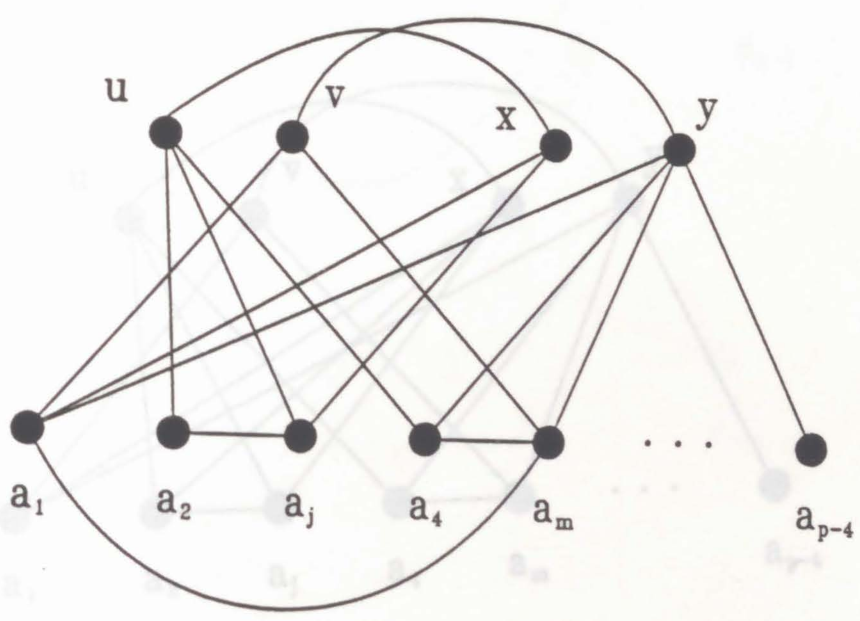

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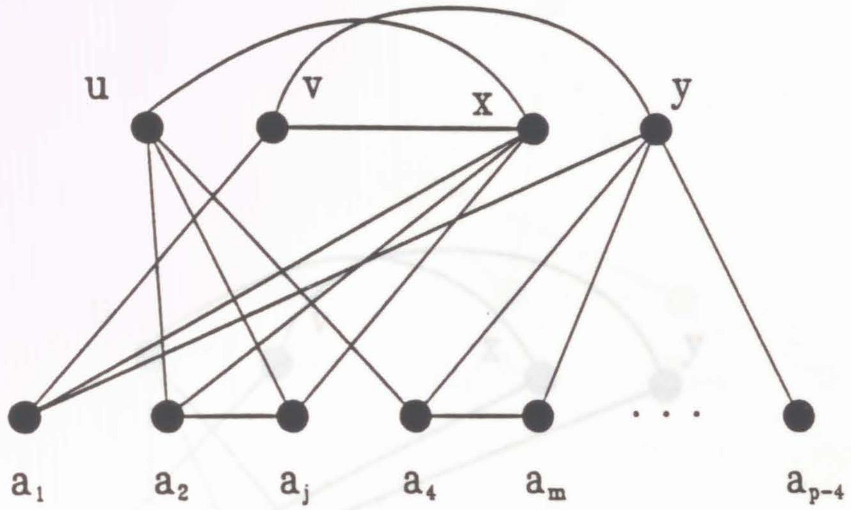

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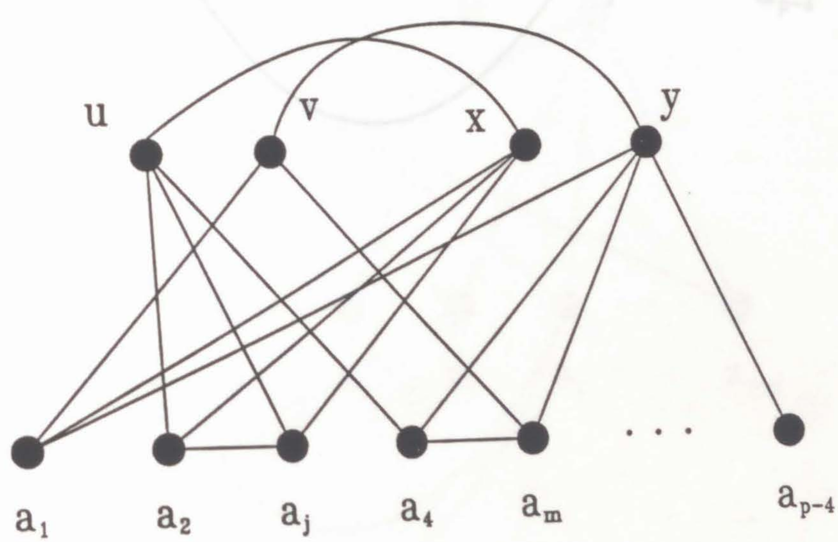

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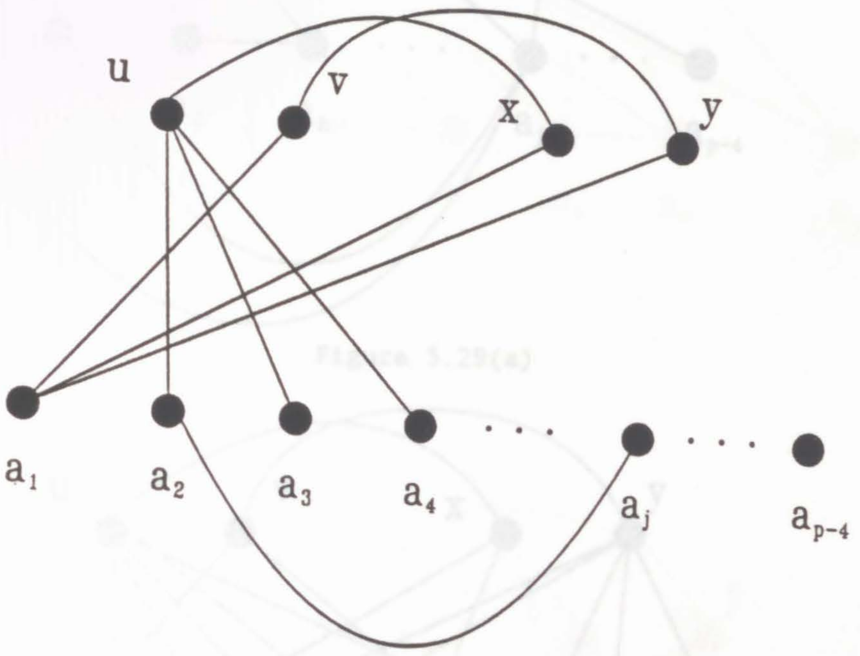

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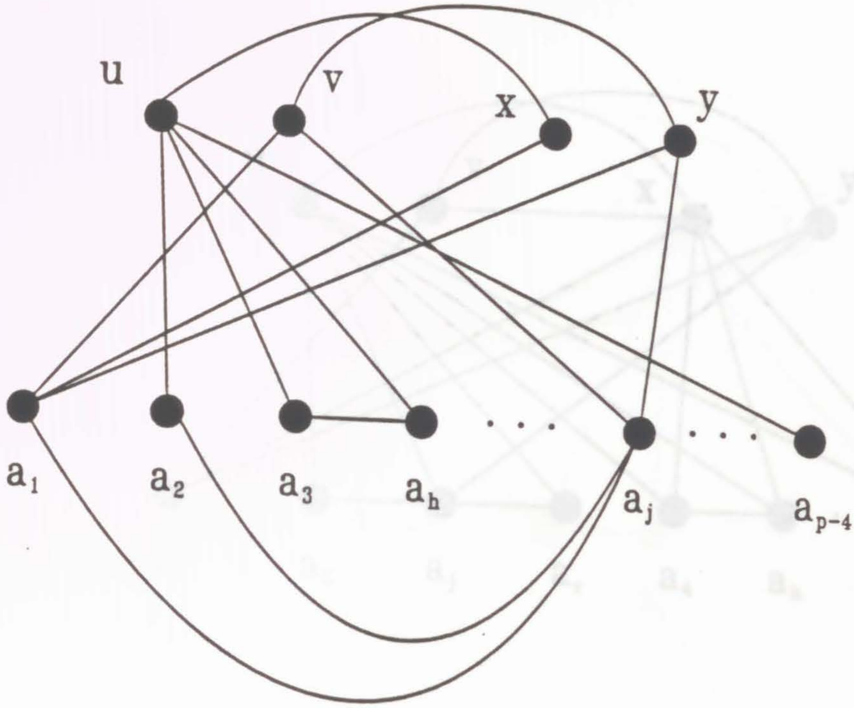

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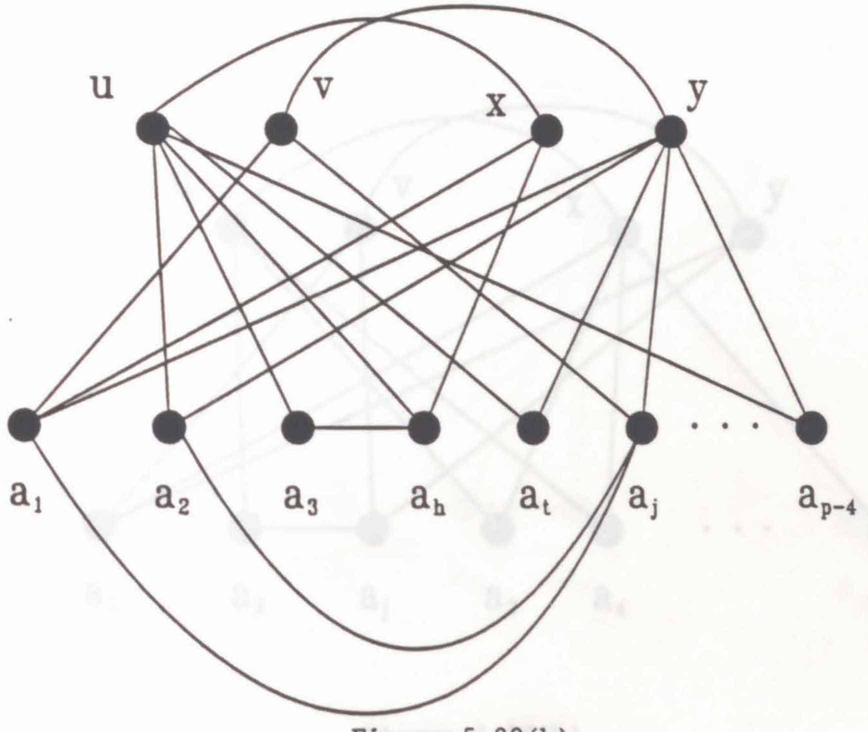

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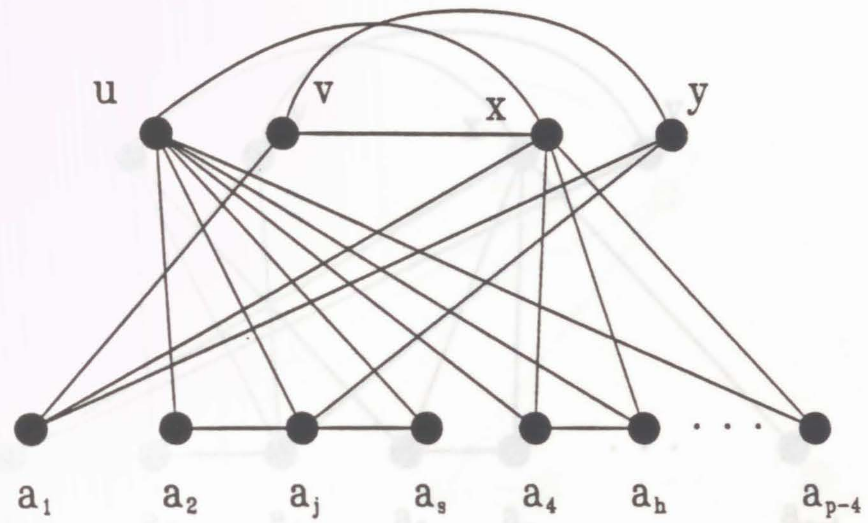

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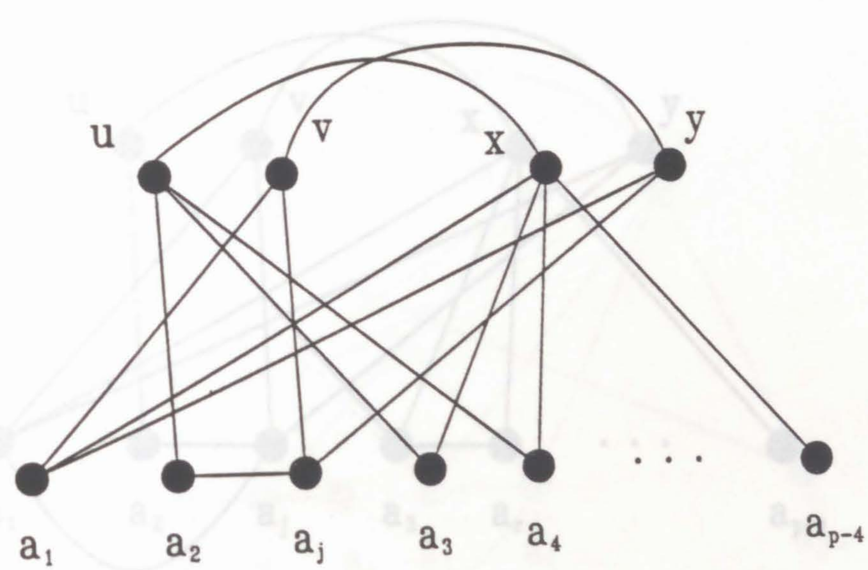

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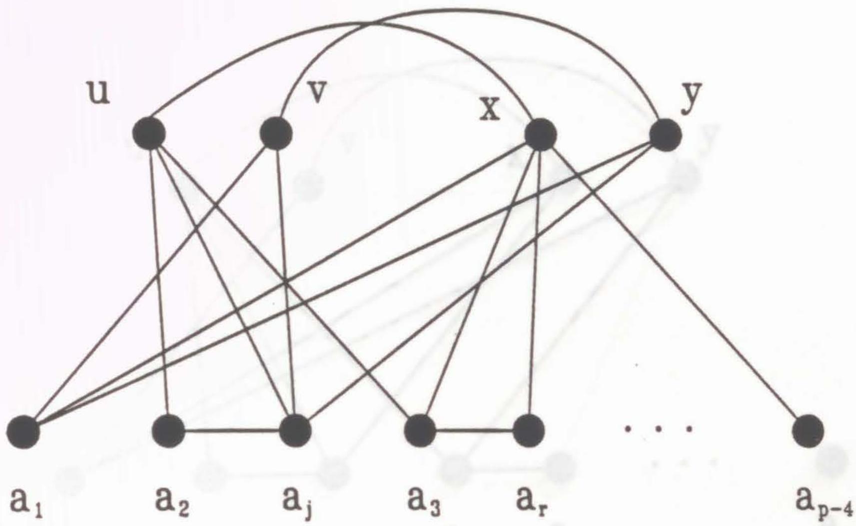

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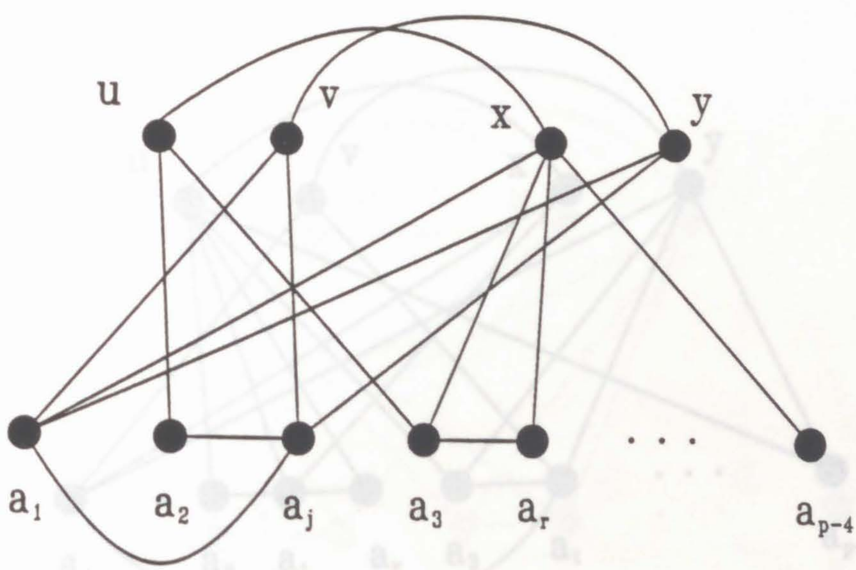

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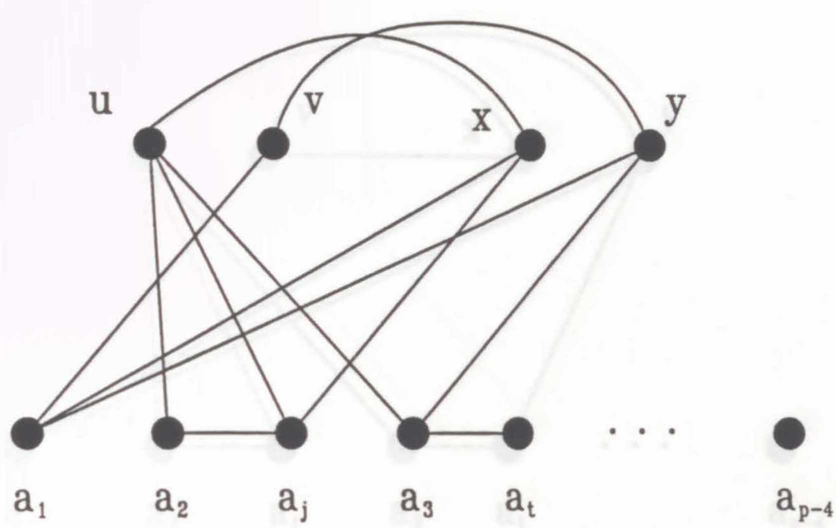

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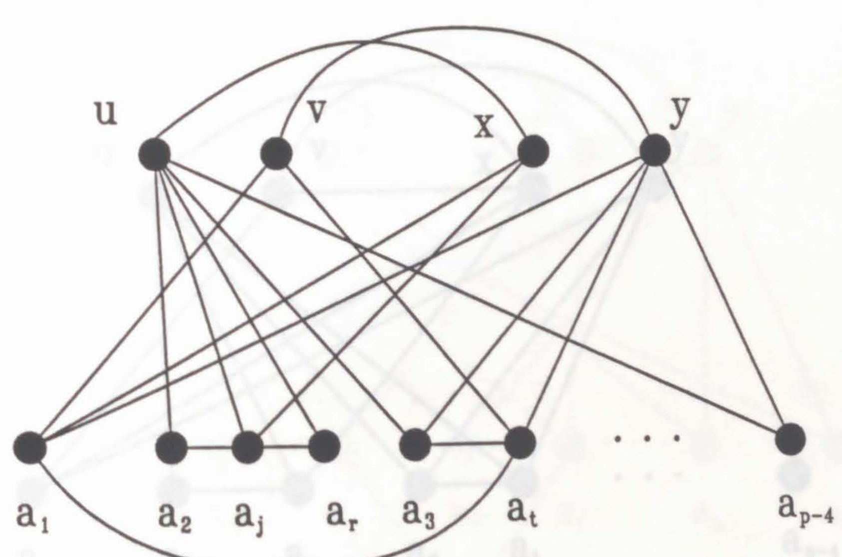

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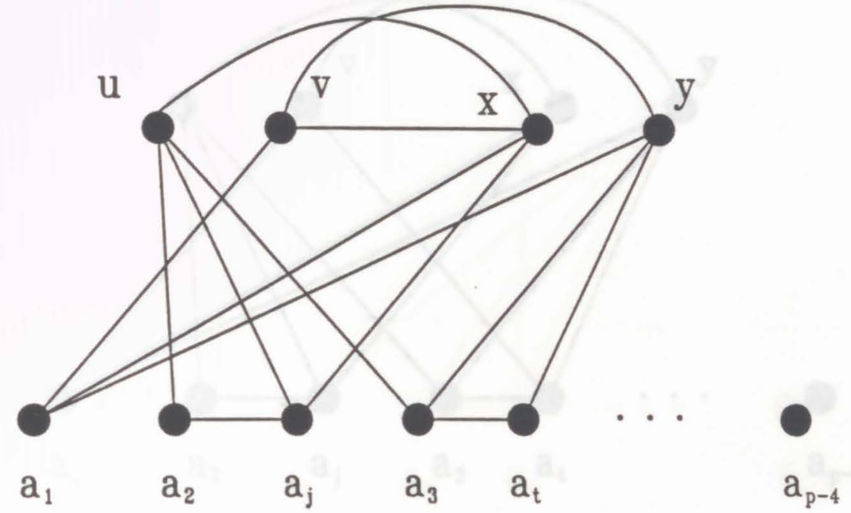

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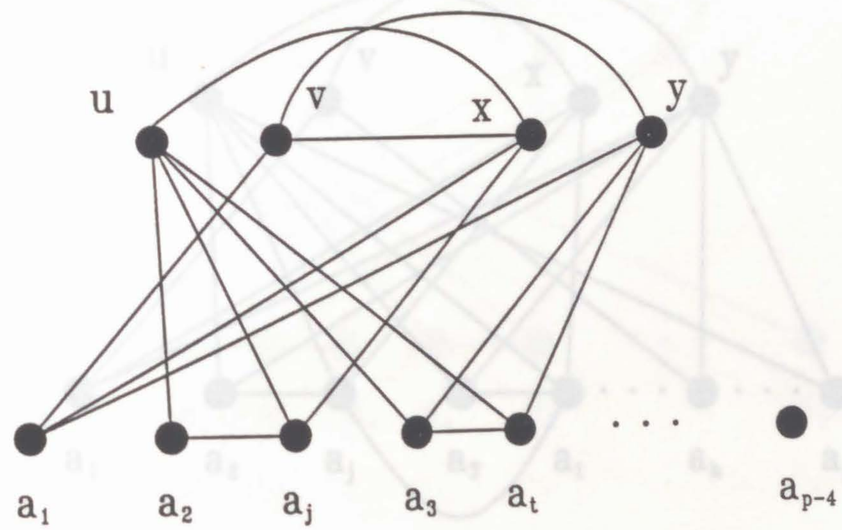

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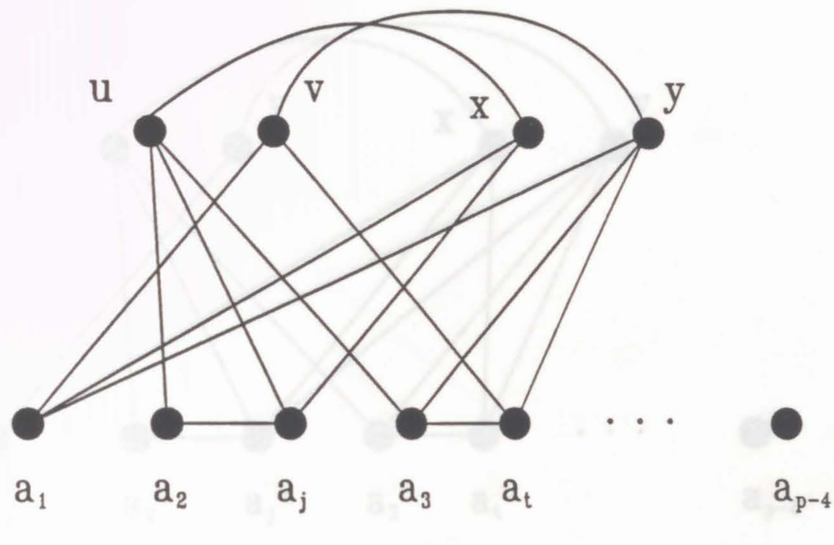

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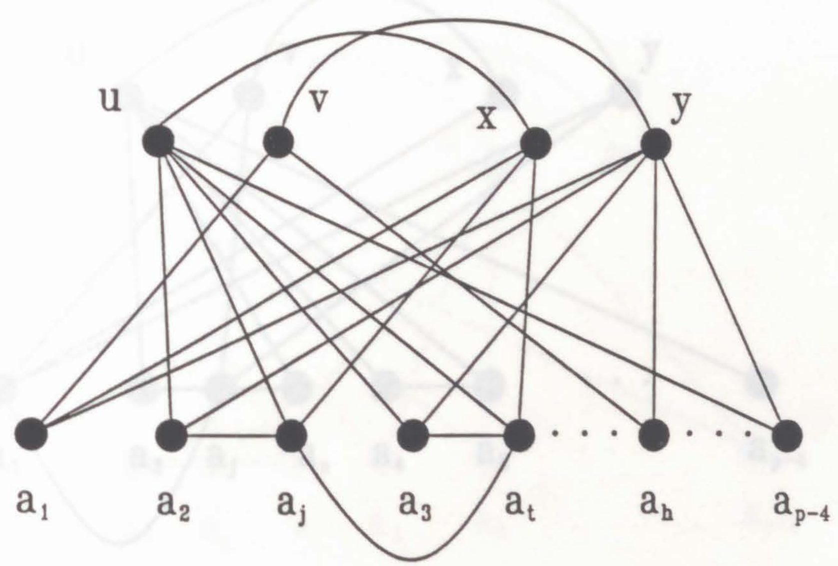

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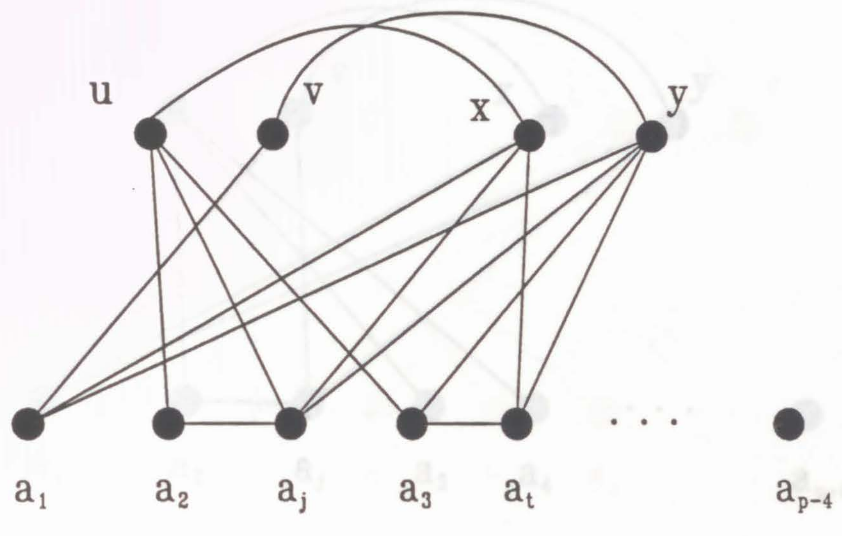

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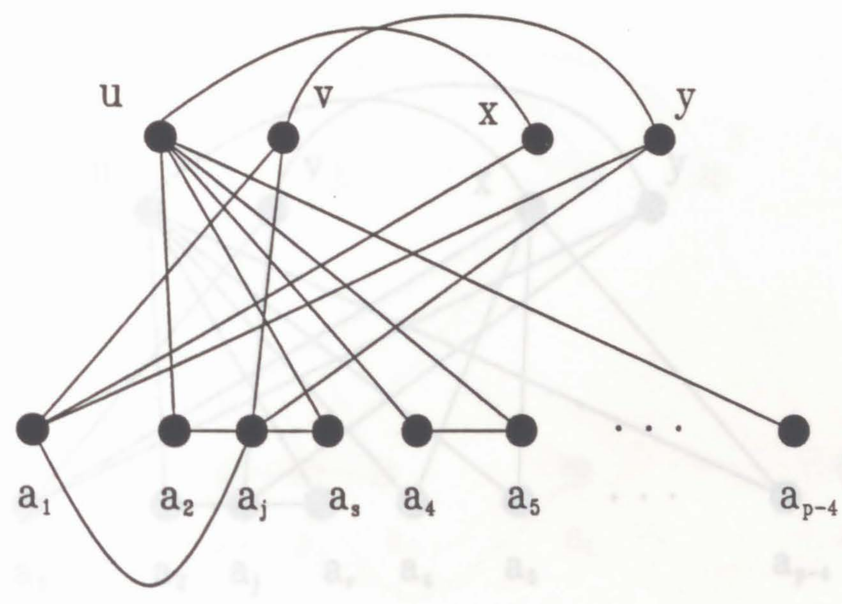

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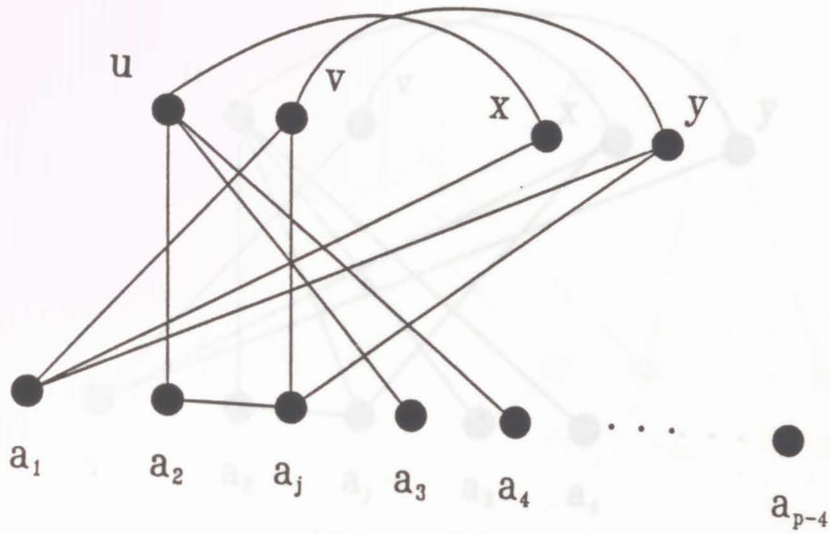

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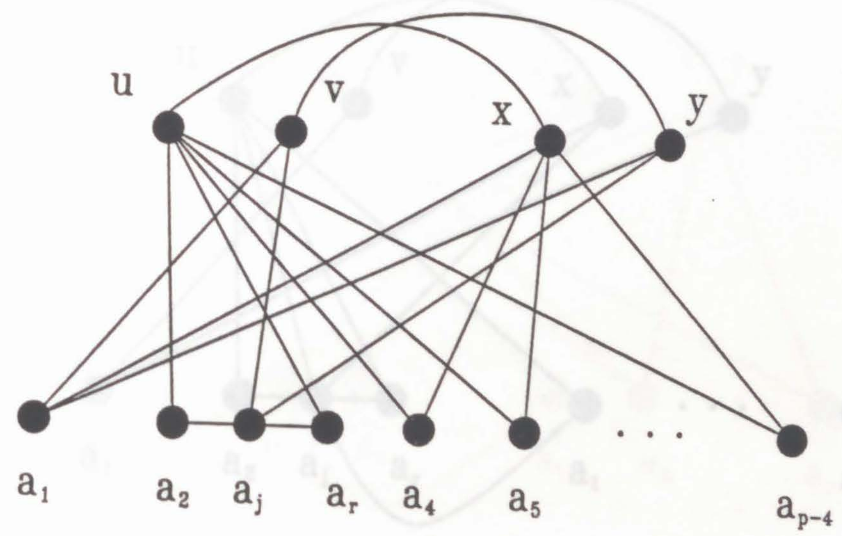

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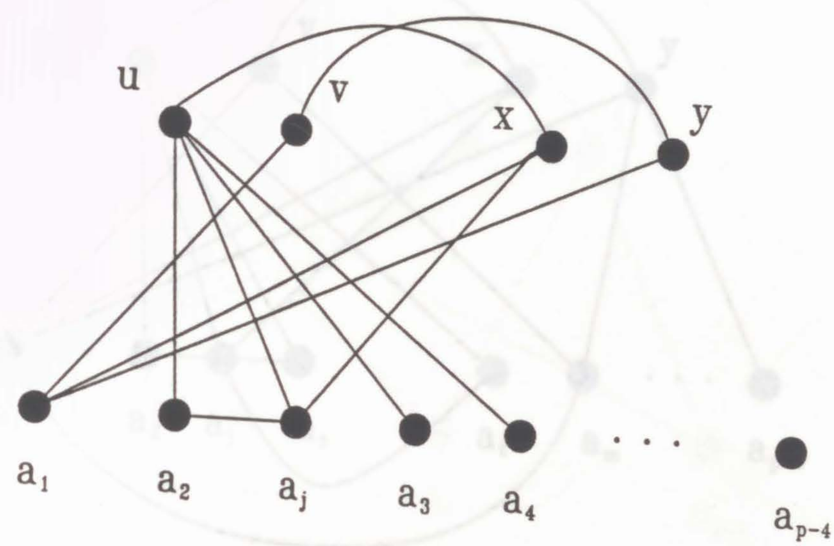

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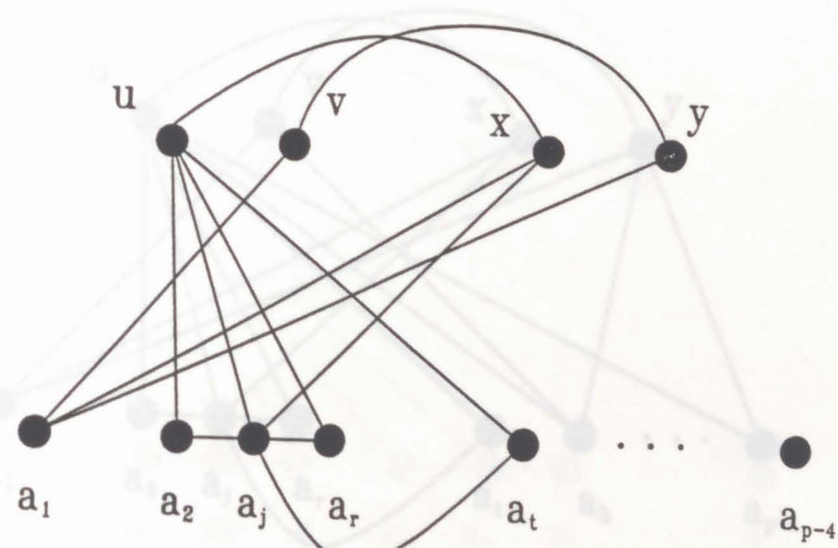

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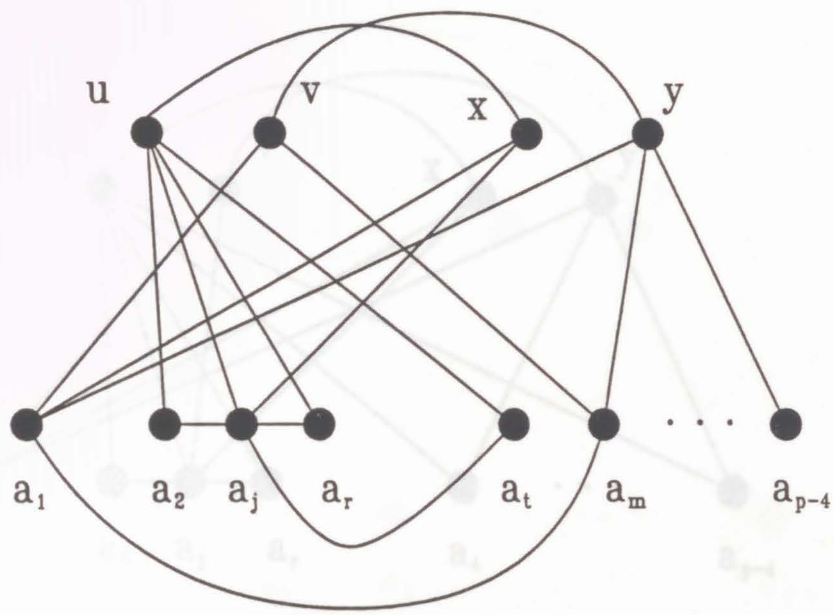

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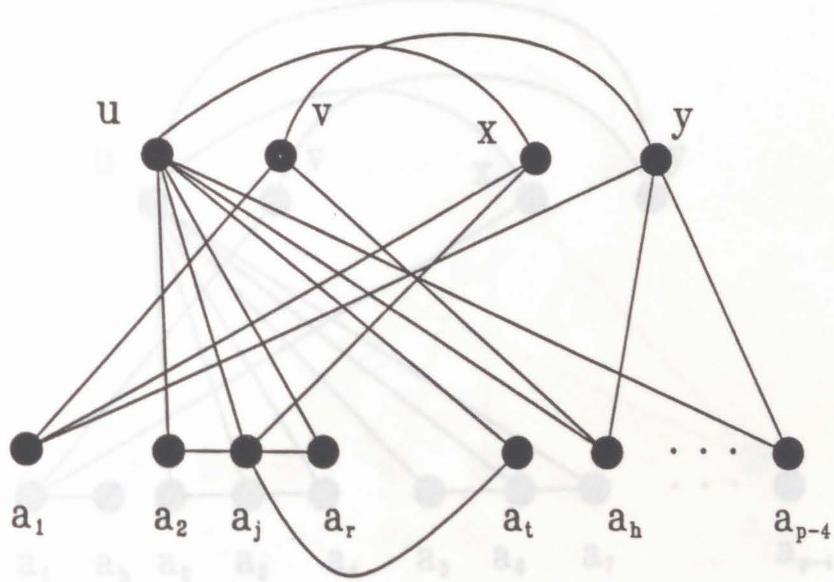

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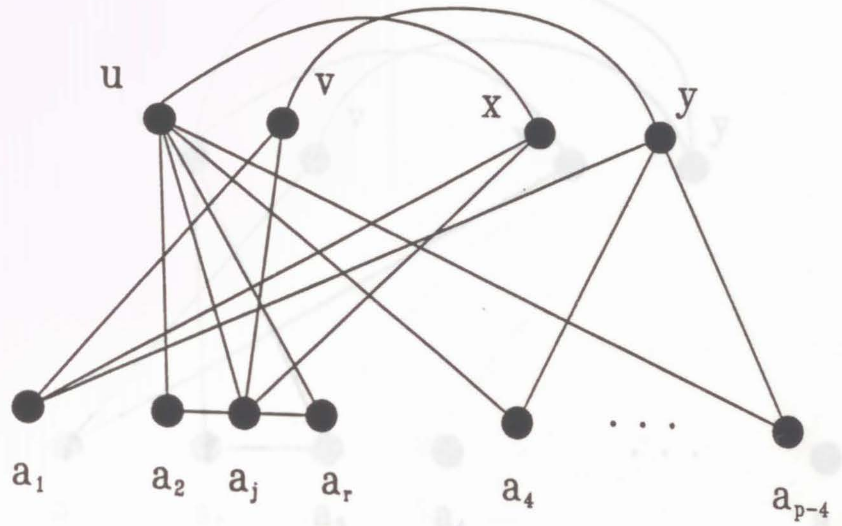

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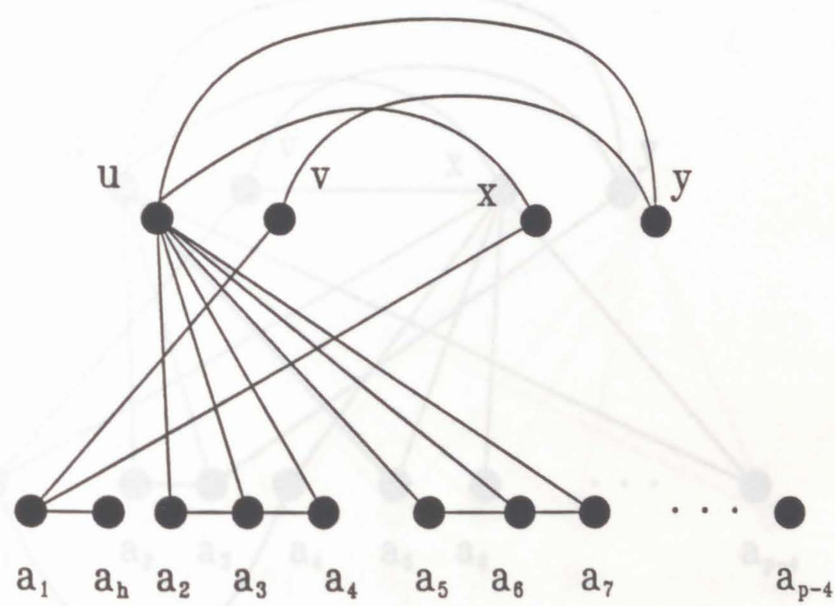

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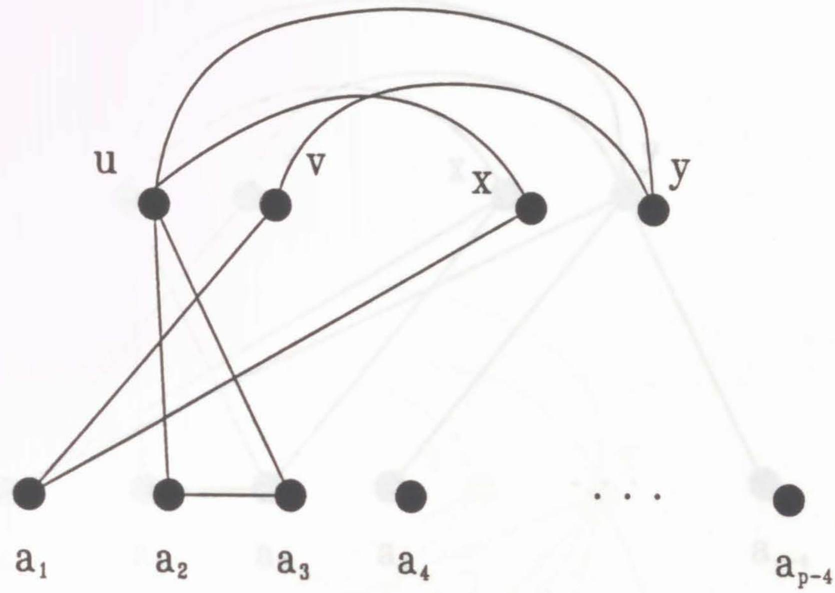

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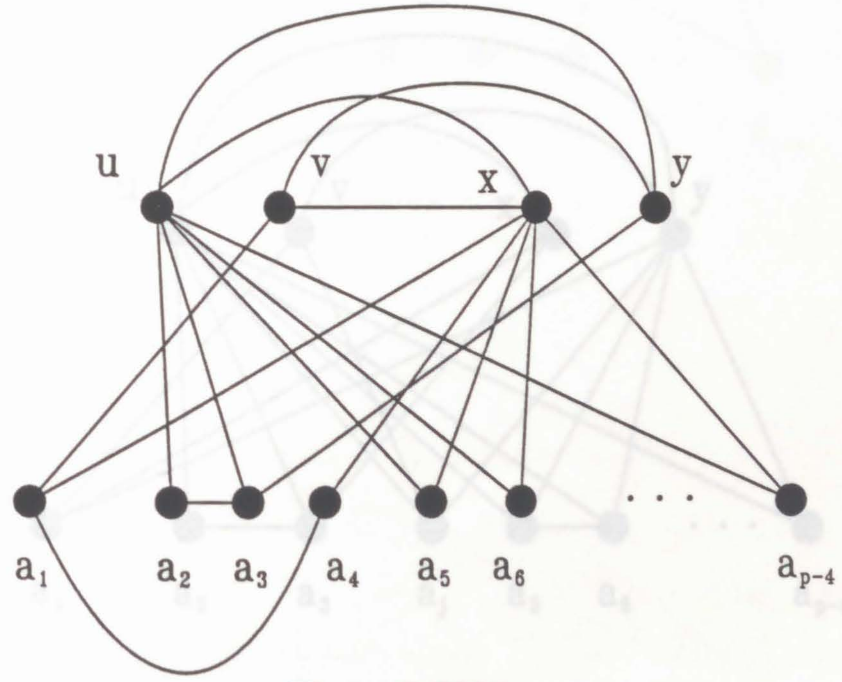

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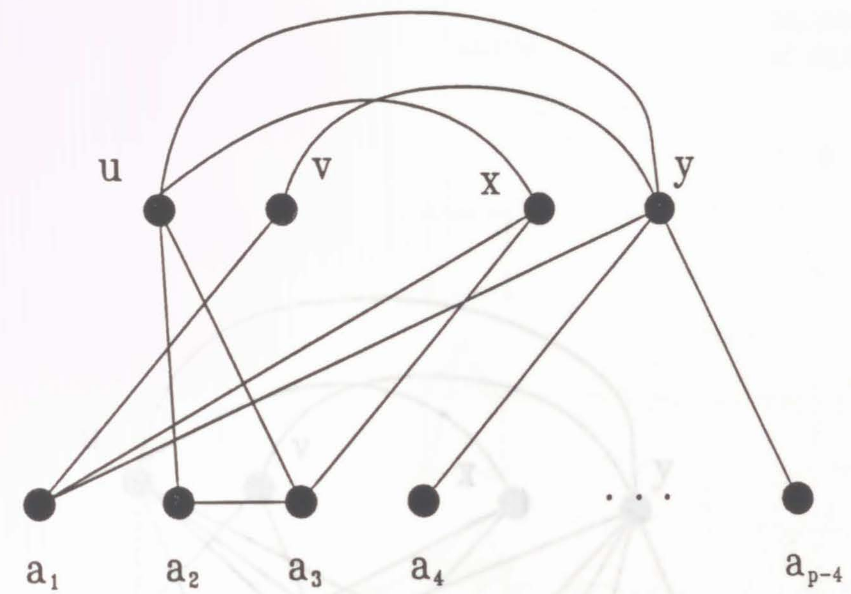

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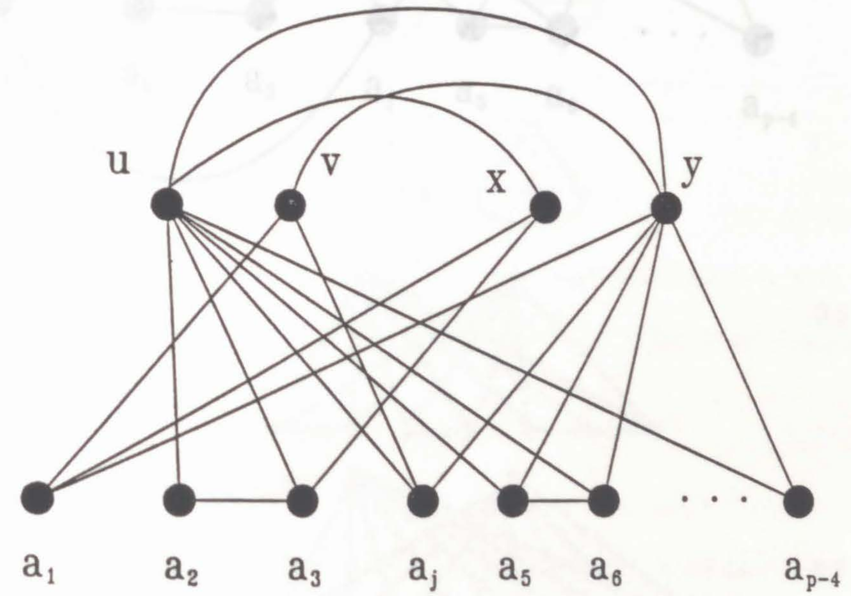

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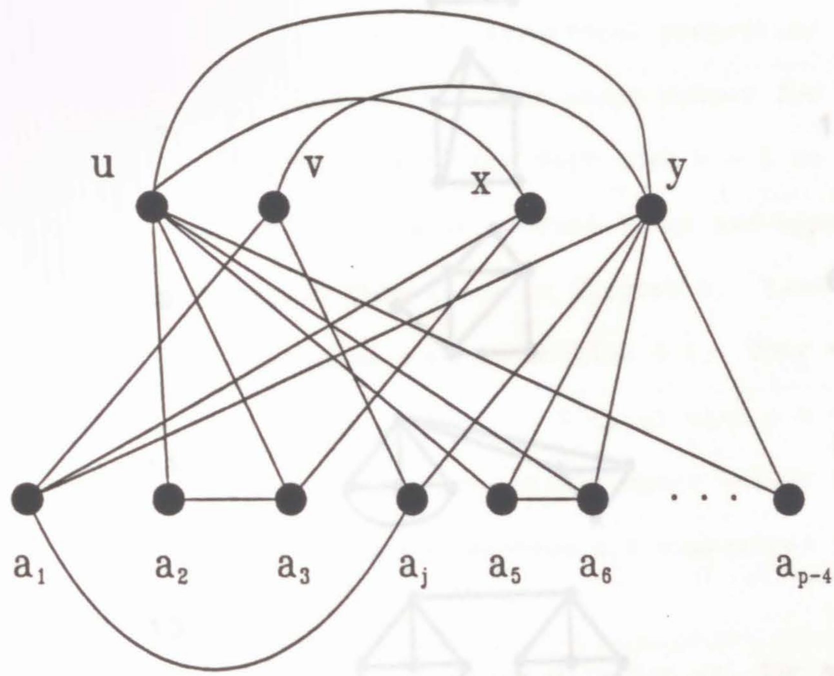

Figure 5.36(e)

Program Results

p	E^2 (p,2)	Example	Number of Graphs
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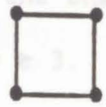
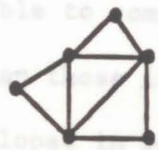
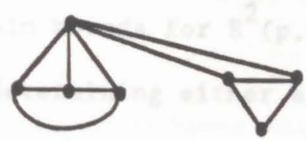
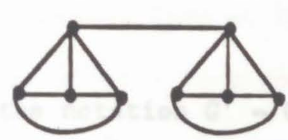
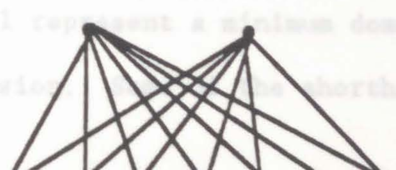
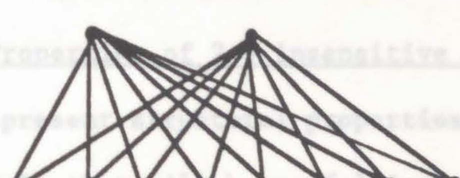
4	4		1
5	7		1
6	9		6
7	11		3
8	13		1
9	17		33
10	20		57

Figure 5.37

6. BOUNDS ON $E^2(p,\gamma)$ WHEN $\gamma \geq 3$

In this chapter we improve the Chapter 4 bounds on $E^k(p,\gamma)$ for the special case when $k = 2$ and $\gamma \geq 3$. In Chapter 5 we saw how restrictions imposed by the specific structural properties of 2-2-insensitive graphs permitted us to obtain exact values for $E^2(p,2)$. Here we similarly take advantage of the fact that $k = 2$ to develop properties which make it possible to compute lower and upper bounds on $E^2(p,\gamma)$ which are better than those in Chapter 4. Some of these structural properties are developed in Section 6.1. They will be used in Section 6.2 to obtain bounds for $E^2(p,\gamma)$ when $p \geq \gamma^2$ and in Section 6.3 to facilitate determining either exact values or bounds for $E^2(p,\gamma)$ when $p < \gamma^2$. Finally, Section 6.4 summarizes the known results for all p.

Throughout we employ the notation $G' = G-e_1-e_2$ for arbitrary edges e_1 and e_2 and D will represent a minimum dominating set for the graph G′ under discussion. Some of the shorthand of Chapter 5 also will be employed.

6.1. Structural Properties of 2-γ-insensitive Graphs

In this section we present structural properties of 2-γ-insensitive graphs $G = (V,E)$ that will be useful in establishing values for $E^2(p,\gamma)$ in the remaining sections. Let $G = (V,E)$ be a

2-γ-insensitive graph. Then the following lemmas apply.

Lemma 6.1

If the two edges incident to a degree two node x are removed, then xϵD and neither of its neighbors is in D.

Proof

Let x be a degree two node with neighbors u and v. R(xu,xv). Then x is isolated, so xϵD. If either u or v is in D, then D-{x} is a dominating set for G having size γ-1, a contradiction. ■

Lemma 6.2

A node vϵV can have at most one degree one neighbor.

Proof

Suppose a and b are degree one nodes with common neighbor v, as shown in Figure 6.1. R(va,vb). Then a, b, and γ-2 nodes form a dominating set D. But (D-{a, b})∪{v} is a dominating set for G with size γ-1, a contradiction. ■

Corollary 6.1

There are at most γ degree one nodes.

Proof

By Lemma 6.2 each node of a dominating set dominates at most one node of degree one. ■

<u>Lemma 6.3</u>

No node is adjacent to both a degree one node and a degree two node.

<u>Proof</u>

Suppose x is adjacent to degree one node a and degree two node b, as shown in Figure 6.2. Let y be the other node adjacent to b. R(ax,by). Then $a \epsilon D$ and either x or b must be in D to dominate b and, since x dominates at least the nodes b does, we may as well assume $x \epsilon D$. Then D-{a} is a dominating set for G of size γ-1, a contradiction. ■

<u>Lemma 6.4</u>

A degree one node cannot be adjacent to a degree two node.

<u>Proof</u>

Suppose xy is an edge with $d_x = 1$ and $d_y = 2$. R(xy,yz) where z is y's other neighbor. Then x and y must be in D. But D-{x} is a dominating set for G of size γ-1, a contradiction. ■

<u>Corollary 6.2</u>

If y is adjacent to degree one node x, then $d_y \geq 3$ and $v \epsilon N(y)$-{x} implies $d_v \geq 3$.

<u>Proof</u>

Lemmas 6.2, 6.3 and 6.4. ■

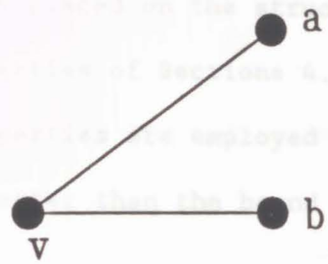


Figure 6.1

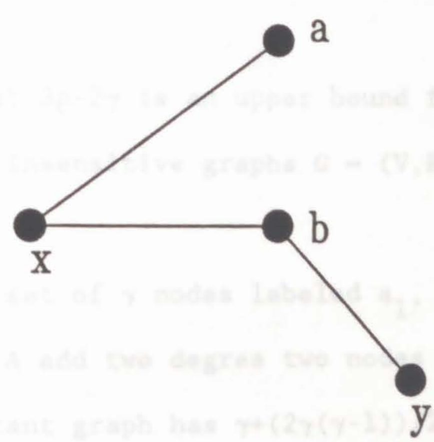


Figure 6.2

6.2. Bounds When $p \geq \gamma^2$

In this section we present upper and lower bounds for $E^2(p,\gamma)$ when $p \geq \gamma^2$ and $\gamma \geq 3$. Section 6.2.1 demonstrates upper bounds by constructing appropriate graphs. These graphs evolved from a study of the restrictions placed on the structure of 2-γ-insensitive graphs by the properties of Sections 4.1 and 6.1. In Section 6.2.2 the structural properties are employed directly to develop lower bounds which are better than the bound in Chapter 4.

6.2.1. Upper Bounds

The first theorem provides an upper bound for $E^2(p,\gamma)$ when $p = \gamma^2$ and $\gamma \geq 3$.

Theorem 6.1

$E^2(p,\gamma) \leq 2p-2\gamma$ if $p = \gamma^2$ and $\gamma \geq 2$.

Proof

We show that $2p-2\gamma$ is an upper bound for $E^2(p,\gamma)$ by constructing 2-γ-insensitive graphs $G = (V,E)$ with $2p-2\gamma$ edges as follows.

Let A be a set of γ nodes labeled $a_1, a_2, \ldots, a_\gamma$. For each pair of nodes in A add two degree two nodes adjacent to both of them. The resultant graph has $\gamma+(2\gamma(\gamma-1))/2 = \gamma^2$ nodes and $2((2\gamma(\gamma-1)/2) = 2p-2\gamma$ edges. Figure 6.3 illustrates this construction for a graph having $\gamma = 4$ and $p = 16$.

We now show that G has domination number γ. Certainly A

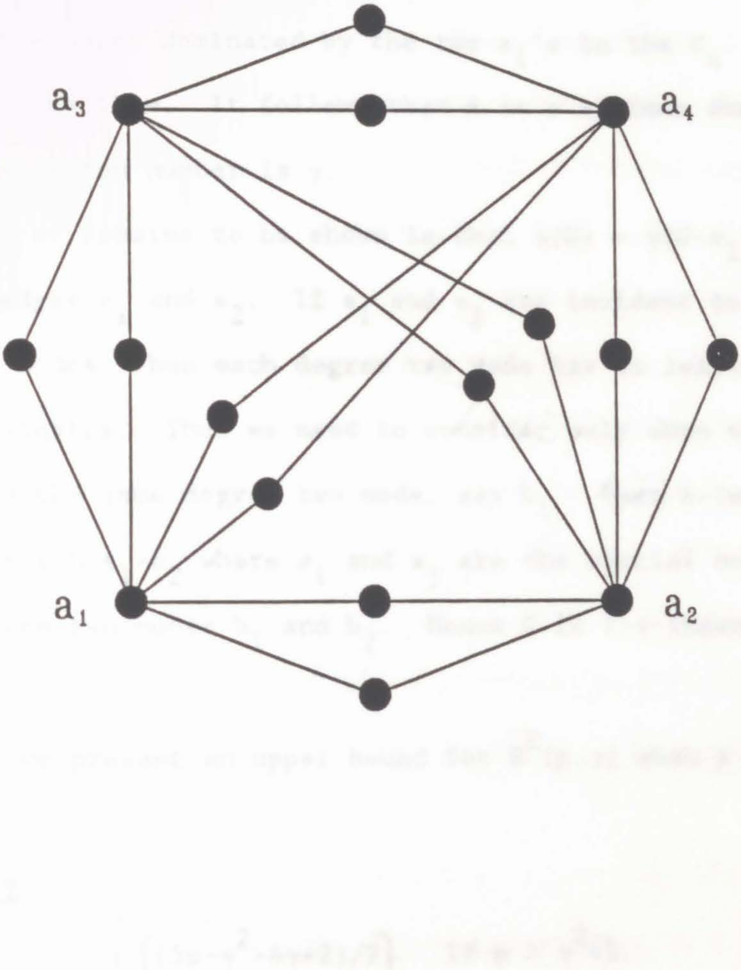


Figure 6.3

dominates G, so the domination number is at most γ. To see that any dominating set D contains at least γ nodes, observe that each pair of nodes of A and the two degree two nodes between them induce a C_4. Furthermore, at least two nodes are required in the C_4 itself to dominate the degree two nodes, and any such two nodes dominate a subset of the nodes dominated by the two a_i's in the C_4. Thus we may as well use them. It follows that A is a minimum dominating set and the domination number is γ.

All that remains to be shown is that $\gamma(G) = \gamma(G-e_1-e_2)$ for arbitrary edges e_1 and e_2. If e_1 and e_2 are incident to different degree two nodes, then each degree two node has at least one edge to A and A dominates. Thus we need to consider only when e_1 and e_2 are incident to the same degree two node, say b_1. Then $A-\{a_i, a_j\}\cup\{b_1, b_2\}$ dominates $G-e_1-e_2$ where a_i and a_j are the special nodes adjacent to the degree two nodes b_1 and b_2. Hence G is 2-γ-insensitive. ∎

Next we present an upper bound for $E^2(p,\gamma)$ when $p > \gamma^2$.

<u>Theorem 6.2</u>

$$E^2(p,\gamma) \le \begin{cases} \lceil (5p-\gamma^2-4\gamma+2)/2 \rceil & \text{if } p > \gamma^2+1. \\ \lfloor (5p-\gamma^2-4\gamma+2)/2 \rfloor & \text{if } p = \gamma^2+1. \end{cases}$$

<u>Proof</u>

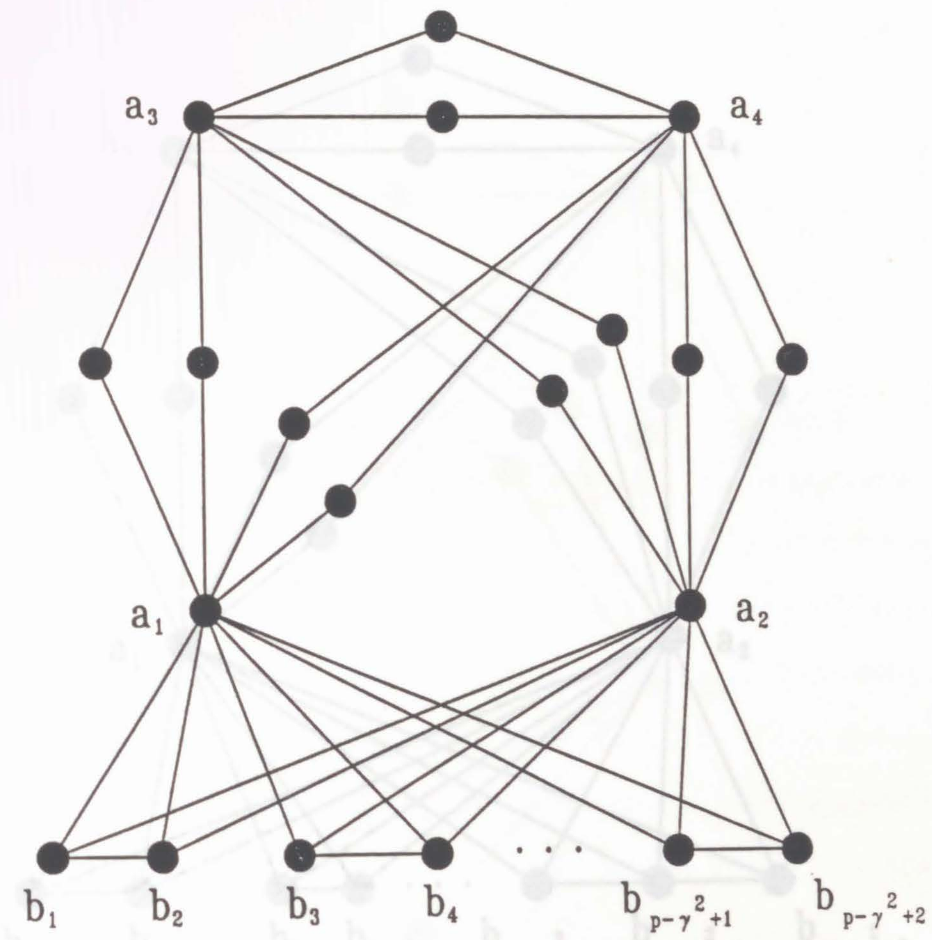
Again we construct an appropriate family of graphs $G = (V,E)$. Begin with the graph described in Theorem 6.1, with the two degree two nodes between a_1 and a_2 removed. At this point, we have γ^2-2

nodes and $2\gamma^2-2\gamma-4$ edges. Label the remaining nodes b_i and add edges a_1b_i, a_2b_i for $1 \le i \le p-\gamma^2+2$. First consider $p-\gamma^2 > 1$. When $p-\gamma^2$ is even add edges b_ib_{i+1} for $i = 1, 3, 5, \ldots, p-\gamma^2+1$ and when $p-\gamma^2$ is odd add edges b_ib_{i+1} for $i = 1, 3, \ldots, p-\gamma^2$ and b_rb_{r+1} where $r = p-\gamma^2+1$. This gives a total of $2\gamma^2-2\gamma-4+2(p-\gamma^2+2)+\lceil(p-\gamma^2+2)/2\rceil$ edges, which reduces to the bound of the theorem. Figures 6.4(a) and 6.4(b) illustrate the construction for $\gamma = 4$. For the special case of $p-\gamma^2 = 1$, the only remaining nodes are b_1, b_2 and b_3 and the only additional edge is b_1b_2. Thus b_3 has degree two and G has $\lfloor(5p-\gamma^2-4\gamma+2)/2\rfloor$ edges. Figure 6.5 gives an example of this construction for $p = 17$ and $\gamma = 4$. In the remainder of this proof we shall refer to this case as the exceptional case.

Next we show that graphs constructed according to the above specifications have domination number γ. Clearly, the set A of size γ dominates G. An argument identical to that in the proof to Theorem 6.1 shows that A is a minimum dominating set of the a_i's and their connecting degree two nodes, and hence it is a minimum dominating set of all of G.

It remains to be shown that $\gamma(G-e_1-e_2) = \gamma(G)$ for arbitary edges e_1 and e_2. If each node in V-A has an edge to a node in A, then A dominates and the result follows. Thus we need to consider only cases where a node x has both its edges to A removed.

<u>Case 1</u> Suppose x is a degree two node with neighbors a_i and a_j, and $x \ne b_3$ in the exceptional case. Then $D = A-\{a_i, a_j\}\cup\{x, y\}$ dominates where y is the other degree two node adjacent to both a_i



$p-\gamma^2$ even

Figure 6.4(a)

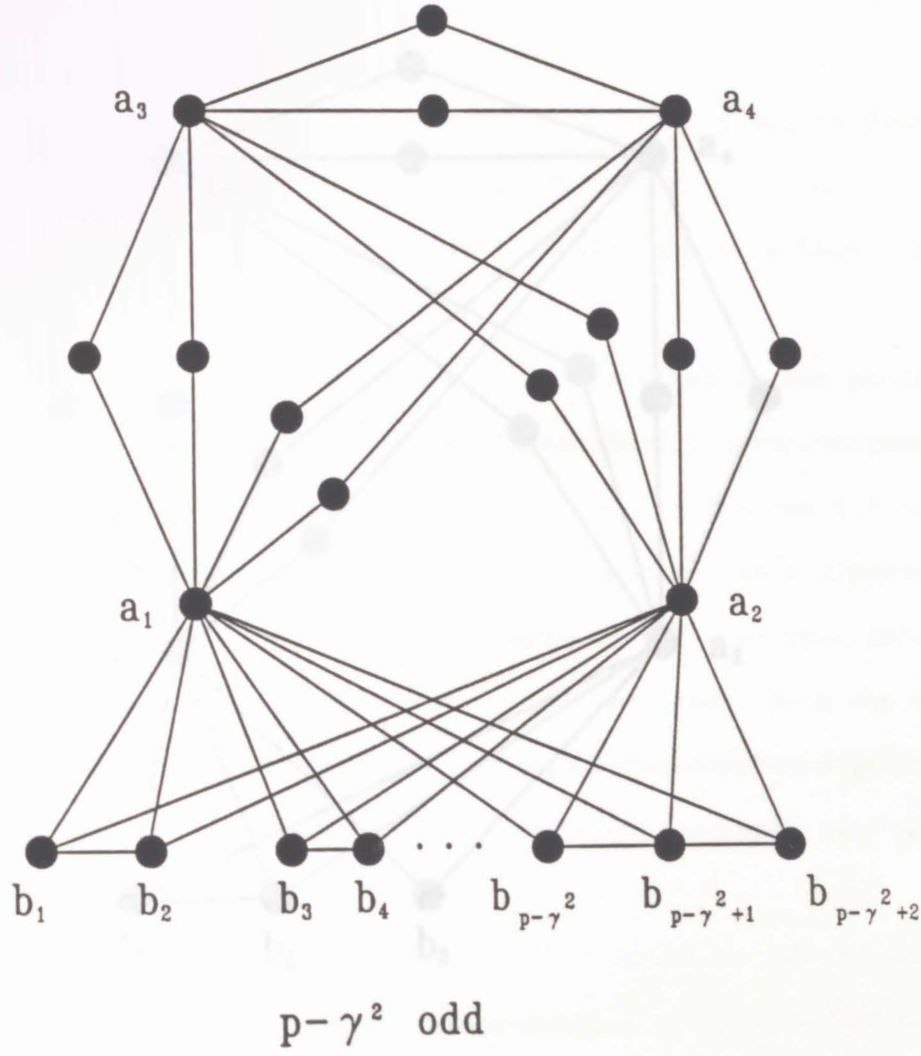

a_3
a_4
a_1
a_2
b_1
b_2
b_3
b_4
. . .
$b_{p-\gamma^2}$
$b_{p-\gamma^2+1}$
$b_{p-\gamma^2+2}$
$p-\gamma^2$ odd

Figure 6.4(b)

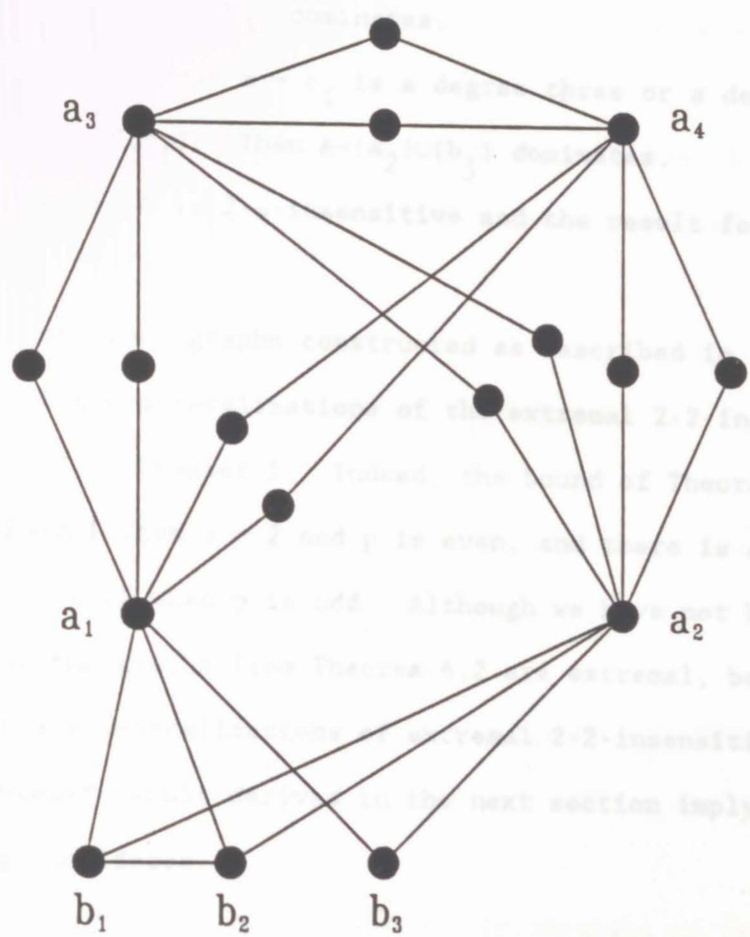


Figure 6.5

and a_j. Notice that D dominates all the b_i's because at least one of a_1 and a_2 will be in D since there are no degree two nodes adjacent to both a_1 and a_2, except b_3 in the exceptional case.

Suppose $x = b_3$ and we are discussing the exceptional case. Then $A-\{a_1, a_2\} \cup \{b_1, b_3\}$ dominates.

Case 2 Suppose $x = b_i$ is a degree three or a degree four node. Let $b_j \epsilon N(b_i)$. Then $A-\{a_2\} \cup \{b_j\}$ dominates.

Therefore G is 2-γ-insensitive and the result follows. ∎

We note that graphs constructed as described in the proof to Theorem 6.2 are generalizations of the extremal 2-2-insensitive graphs found in Chapter 5. Indeed, the bound of Theorem 6.2 reduces to $\lfloor (5p-10)/2 \rfloor$ when $\gamma = 2$ and p is even, and there is a difference of only one edge when p is odd. Although we have not been able to prove that the graphs from Theorem 6.2 are extremal, both the fact that they are generalizations of extremal 2-2-insensitive graphs and the asymptotic result derived in the next section imply they are promising candidates.

6.2.2. Lower Bounds

Substituting k = 2 into the general lower bound of Theorem 4.6 gives $E^2(p,\gamma) \geq (5p-\gamma^2-8\gamma-f(\gamma,2))/2$ where $f(\gamma,2) = 2\gamma^2-3\gamma+2$ when $p \geq 3\gamma^2-\gamma+2$. Considering the upper bound from Section 6.2.1, we have $(5p-3\gamma^2-5\gamma-2)/2 \leq E^2(p,\gamma) \leq \lceil (5p-\gamma^2-4\gamma+2)/2 \rceil$ when $p \geq 3\gamma^2-\gamma+2$ for a difference of at most $\gamma^2+\gamma/2+2$. In this section we employ the structural properties from Sections 4.1 and 6.1, along with others obtained below, to derive better lower bounds and hence narrow the above gap.

First we develop the additional structural properties. Let N_i be the maximum number of nodes having degree at most 2 which have i common neighbors in the 2-γ-insensitive graph G, $1 \le i \le 2$. By Theorem 4.3, $N_2 \le 2$ and by Theorem 4.4, $N_1 \le (\gamma-1)N_2+1 \le 2\gamma-1$. Our next lemma shows that this bound for N_1 can be reduced by one.

Lemma 6.5

In any 2-γ-insensitive graph, $N_1 \le 2\gamma-2$.

Proof

Let v be adjacent to m nodes having degree at most two which are labeled $a_1, a_2, \ldots, a_m$ and let $b_1, b_2, \ldots, b_j$ be the other neighbors of the a_i's.

Case 1 Node v has a degree one neighbor a_1. Then by Corollary 6.2 a_1 is the only neighbor of v having degree at most two.

Case 2 Assume some b_i, say b_1, is adjacent to two degree two nodes a_1 and a_2 as shown in Figure 6.6(a). $R(a_1v, a_1b_1)$. By Lemma 6.1 $a_1 \epsilon D$ and neither v nor b_1 is in D. Hence $a_2 \epsilon D$. Furthermore, each a_j for $j \ne 1$, 2 or its neighboring b_i must be in D. Now D can include at most γ-2 such nodes. The situation which maximizes the number of degree two neighbors of v is when each of the γ-2 nodes is a b_i with two a_j neighbors. In this case v has at most $2+2(\gamma-2)$ =

2γ-2 degree two neighbors.

<u>Case 3</u> Each b_i has at most one degree two neighbor which is adjacent to v. See Figure 6.6(b). $R(a_1b_1,a_1v)$. Now $a_1 \epsilon D$ and so is either each a_j for $2 \le j \le m$ or its neighboring b_j. There can be at most γ-1 of these nodes implying at most γ degree two neighbors of v.

Since $\gamma \ge 3$, the greatest possible count occurs in Case 2 so $N_1 \le 2\gamma-2$. ■

We shall be discussing the situation described in Case 2 of Lemma 6.5, and it will be convenient to let $S = \{v, a_1, a_2, \ldots, a_{2\gamma-2}, b_1, b_2, \ldots, b_{\gamma-1}\}$. Observe that there are no degree one nodes in this situation.

<u>Lemma 6.6</u>

Let v have 2γ-2 degree two neighbors as described in Case 2 of Lemma 6.5. Then u is adjacent to at least two b_i's for all $u \epsilon V$-S.

<u>Proof</u>

Let v be adjacent to 2γ-2 degree two nodes as described in Case 2 of Lemma 6.5. See Figure 6.7. Assume there is an $x \epsilon V$-S which is adjacent to at most one b_i, say b_1 if any. $R(a_1v,a_1b_1)$. By Lemma 6.1 $a_1 \epsilon D$ and neither v nor b_1 is in D. Hence $D = \{a_1, a_2, b_2, b_3, \ldots, b_{\gamma-1}\}$. But then x is not dominated. Thus x is adjacent to at least two b_i's. ■

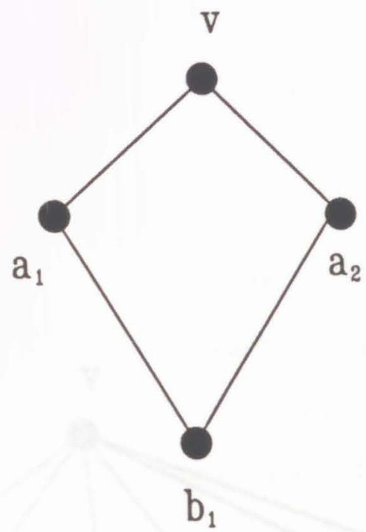


Figure 6.6(a)

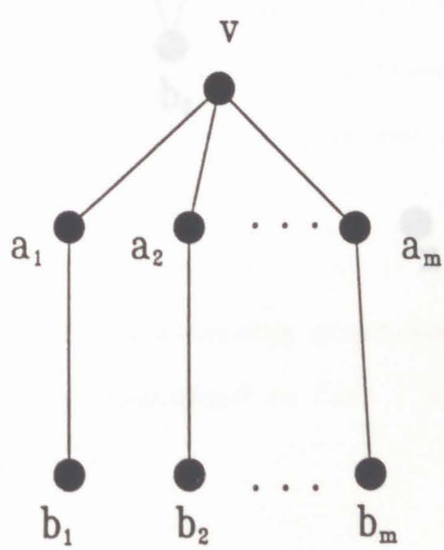


Figure 6.6(b)

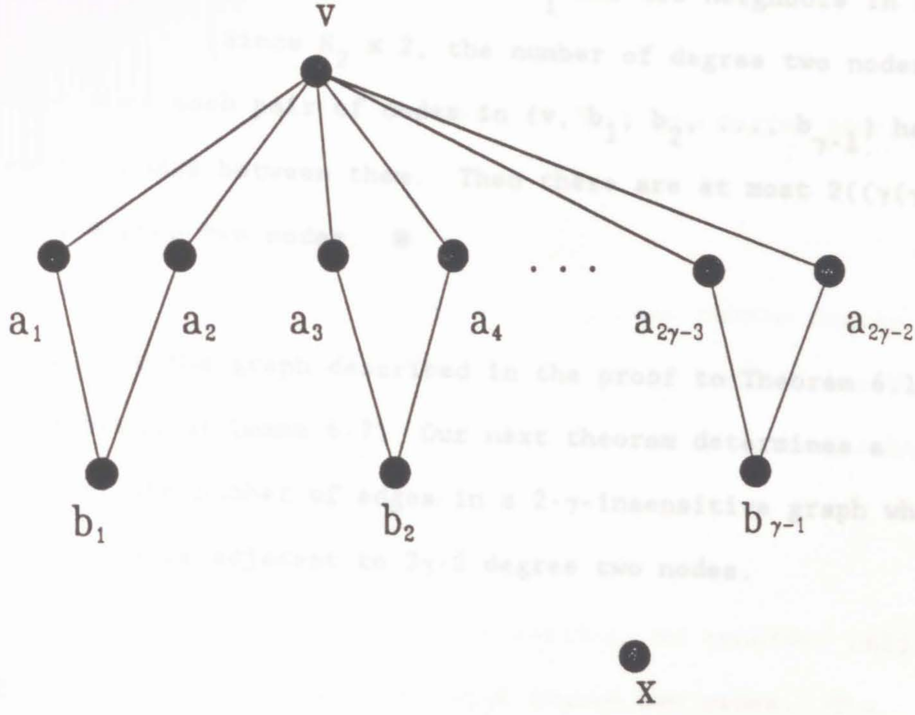


Figure 6.7

Lemma 6.7

Let v be adjacent to $2\gamma-2$ degree two nodes as described in Case 2 of Lemma 6.5. Then the maximum possible number of degree two nodes in the entire graph is $\gamma^2-\gamma$.

Proof

By Lemma 6.6 each degree two node in V-S has two neighbors in $\{b_1, b_2, \ldots, b_{\gamma-1}\}$. Furthermore, each a_i has two neighbors in $\{v, b_1, b_2, \ldots, b_{\gamma-1}\}$. Since $N_2 \leq 2$, the number of degree two nodes is maximized when each pair of nodes in $\{v, b_1, b_2, \ldots, b_{\gamma-1}\}$ has two degree two nodes between them. Then there are at most $2((\gamma(\gamma-1))/2 = \gamma^2-\gamma$ degree two nodes. ∎

Notice that the graph described in the proof to Theorem 6.1 achieves the bound of Lemma 6.7. Our next theorem determines a lower bound for the number of edges in a 2-γ-insensitive graph when at least one node is adjacent to $2\gamma-2$ degree two nodes.

Lemma 6.8

Let $G = (V,E)$ be a 2-γ-insensitive graph and let v be adjacent to $2\gamma-2$ degree two nodes as described in Case 2 of Lemma 6.5. Then $|E| \geq (5p-\gamma^2-4\gamma)/2$ for $p \geq \gamma^2$.

Proof

It follows from Lemma 6.6 that $D = \{v, b_1, b_2, \ldots, b_{\gamma-1}\}$ dominates G and each $u \in V-D$ has two edges into D. Lemma 6.7 implies there are at most $\gamma^2-\gamma$ nodes having degree at most two, so at least

$p-\gamma-(\gamma^2-\gamma) = p-\gamma^2$ nodes in V-D must have degree at least three. Thus $DS \geq 2(p-\gamma)+2(\gamma^2-\gamma)+3(p-\gamma^2) = 5p-\gamma^2-4\gamma$ implying $|E| \geq (5p-\gamma^2-4\gamma)/2$. ∎

Lemma 6.9 gives another useful property of 2-γ-insensitive graphs.

<u>Lemma 6.9</u>

A pair of adjacent nodes can have at most one degree two node as a common neighbor.

<u>Proof</u>

Suppose x is adjacent to y, and x and y have common degree two neighbors a and b. See Figure 6.8. R(ax,ay). By Lemma 6.1 $a \in D$ and neither x nor y is in D. Then $b \in D$ and $D-\{a,b\} \cup \{x\}$ is a dominating set of size $\gamma-1$, a contradiction. ∎

For the remaining portion of this section, we consider only graphs where no node is adjacent to $2\gamma-2$ degree two nodes. The next lemma considers the situation where some node is adjacent to $2\gamma-3$ degree two nodes.

<u>Lemma 6.10</u>

Let G = (V,E) be a 2-γ-insensitive graph in which no node has $2\gamma-2$ degree two neighbors but at least one node is adjacent to $2\gamma-3$ degree two nodes. Then $|E| \geq (5p-\gamma^2-7\gamma/2-2)/2$. Furthermore, this

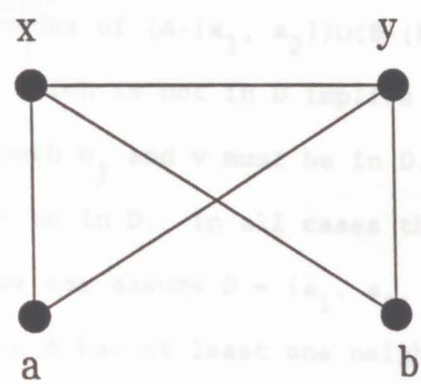


Figure 6.8

situation cannot exist in extremal graphs when $\gamma \geq 10$.

Proof

Suppose v is adjacent to $2\gamma-3$ degree two nodes. Since $N_2 = 2$, the only possibility is illustrated in Figure 6.9 where the set of degree two nodes adjacent to v is $A = \{a_1, a_2, \ldots, a_{2\gamma-3}\}$ and the other neighbors of the nodes in A are $B = \{b_1, b_2, \ldots, b_{\gamma-1}\}$. $R(a_1b_1, a_1v)$. Then $a_1, a_2 \in D$ and $v, b_1 \notin D$. Now the $\gamma-2$ other nodes in D must dominate the nodes of $(A-\{a_1, a_2\}) \cup (B-\{b_1\})$. Since $v \notin D$, any $b_j \in B$, $2 \leq j \leq \gamma-2$, which is not in D implies that the two degree two nodes adjacent to both b_j and v must be in D. Furthermore, one of $a_{2\gamma-3}$ and $b_{\gamma-1}$ must be in D. In all cases there is no advantage in using the b_i's so we may assume $D = \{a_1, a_2, b_2, \ldots, b_{\gamma-1}\}$. Thus every node $x \in V-\{v\}-A-B$ has at least one neighbor in $\{b_2, \ldots, b_{\gamma-1}\}$. Notice that if x's neighbor is in $\{b_2, \ldots, b_{\gamma-2}\}$, say b_2, we can remove two edges to a_3 and by an analogous argument to the one above, x must have a neighbor in $B-\{b_2\}$. Thus any such x having a neighbor in $\{b_1, b_2, \ldots, b_{\gamma-2}\}$ must have at least two neighbors in B.

Now we derive lower bounds for $|E|$. First assume that each $x \in V-\{v\}-A-B$ has a neighbor in $B-\{b_{\gamma-1}\}$, implying that every such x has at least two neighbors in B. Let $D' = B \cup \{v\}$. Then each node in $V-D'$ has at least two edges to D'. Since $N_2 = 2$ and by assumption a node is adjacent to at most $2\gamma-3$ degree two nodes, there are at most $\gamma(2\gamma-3)/2$ degree two nodes in $V-D'$. Then at least $(p-\gamma-\gamma^2+3\gamma/2) = p-\gamma^2+\gamma/2$ nodes in $V-D'$ must have degree at least three. Thus DS $\geq$

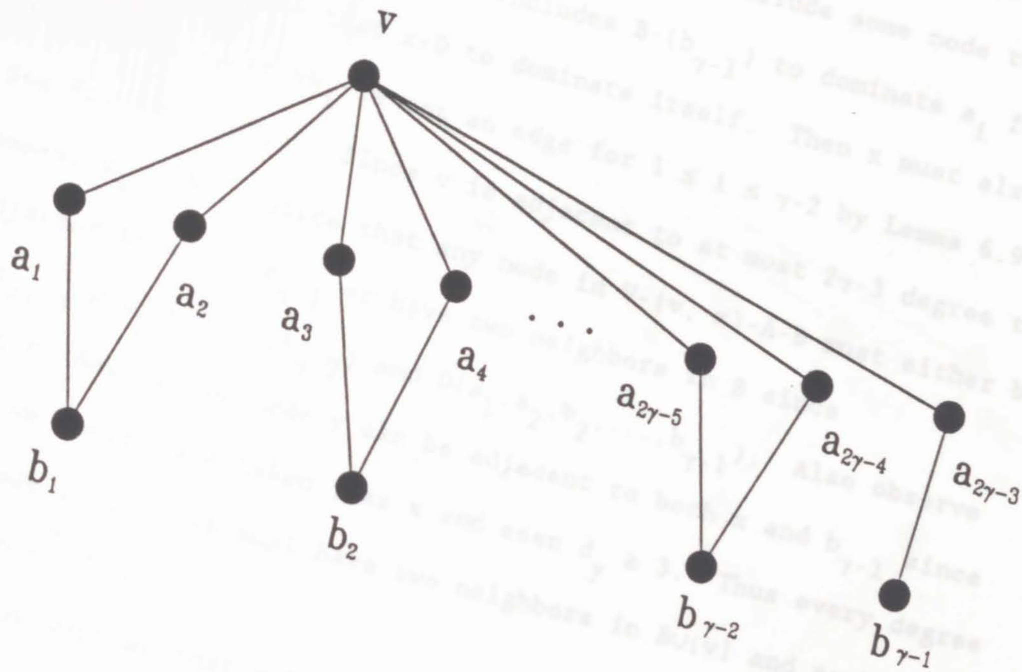


Figure 6.9

$2(p-\gamma)+2(\gamma^2-3\gamma/2)+3p-3\gamma^2+3\gamma/2 = 5p-\gamma^2-7\gamma/2$ implying $|E| \geq (5p-\gamma^2-7\gamma/2)/2$.

Next assume there exists at least one $x \in V-\{v\}-A-B$ adjacent to $b_{\gamma-1}$ and not adjacent to any other node of B. See Figure 6.10(a). $R(a_{2\gamma-3}b_{\gamma-1}, a_{2\gamma-3}v)$. Then $a_{2\gamma-3} \in D$ and $v, b_{\gamma-1} \notin D$. Since x is not adjacent to a b_i for $i \neq \gamma-1$, D also must include some node to dominate x. Furthermore, D includes $B-\{b_{\gamma-1}\}$ to dominate a_i for $1 \leq i \leq 2\gamma-4$. Assume that $x \in D$ to dominate itself. Then x must also dominate v since vb_i is not an edge for $1 \leq i \leq \gamma-2$ by Lemma 6.9. See Figure 6.10(b). Since v is adjacent to at most $2\gamma-3$ degree two nodes, $d_x \geq 3$. Notice that any node in $V-\{v, x\}-A-B$ must either be adjacent to x and $b_{\gamma-1}$ or have two neighbors in B since $D(a_{2\gamma-3}, x, b_1, b_2, \ldots, b_{\gamma-2})$ and $D(a_1, a_2, b_2, \ldots, b_{\gamma-1})$. Also observe that no degree two node y can be adjacent to both x and $b_{\gamma-1}$ since then we could have taken y as x and seen $d_y \geq 3$. Thus every degree two node in $V-\{v\}-B$ must have two neighbors in $B \cup \{v\}$ and as before $DS \geq 5p-\gamma^2-7\gamma/2$.

Now suppose that $y \in N(x)$ dominates x. Then $D = \{a_{2\gamma-3}, y, b_1, b_2, \ldots, b_{\gamma-2}\}$ dominates $G-a_{2\gamma-3}v-a_{2\gamma-3}b_{\gamma-1}$ and yv is an edge to dominate v, again because of Lemma 6.9. See Figure 6.10(c). By reasoning similar to above every node of $V-\{v\}-A-B$ must either be adjacent to both $b_{\gamma-1}$ and y or be adjacent to two nodes in B. Thus every degree two node has two neighbors in $B \cup \{v,y\}$, and there are no degree two nodes adjacent to both y and b_i for $1 \leq i \leq \gamma-2$. First assume y is adjacent to $b_{\gamma-1}$. Then Lemma 6.9 implies there

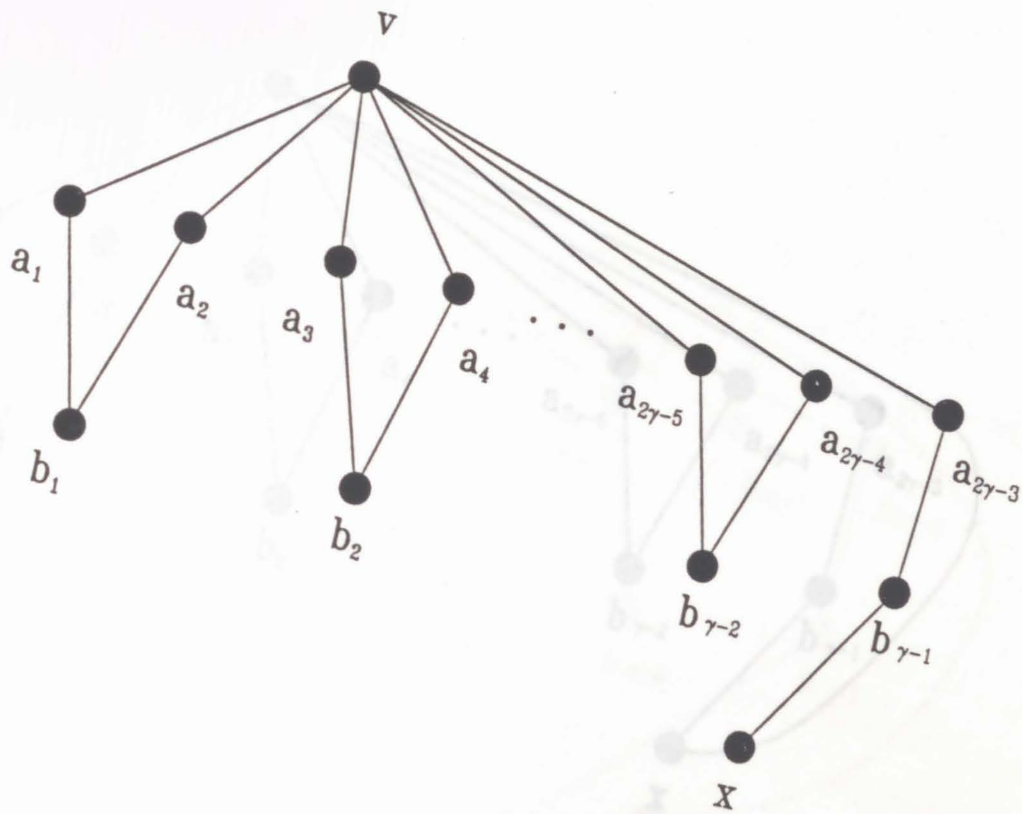

v
a_1
a_2
a_3
a_4
...
$a_{2\gamma-5}$
$a_{2\gamma-4}$
$a_{2\gamma-3}$
b_1
b_2
$b_{\gamma-2}$
$b_{\gamma-1}$
x

Figure 6.10(a)

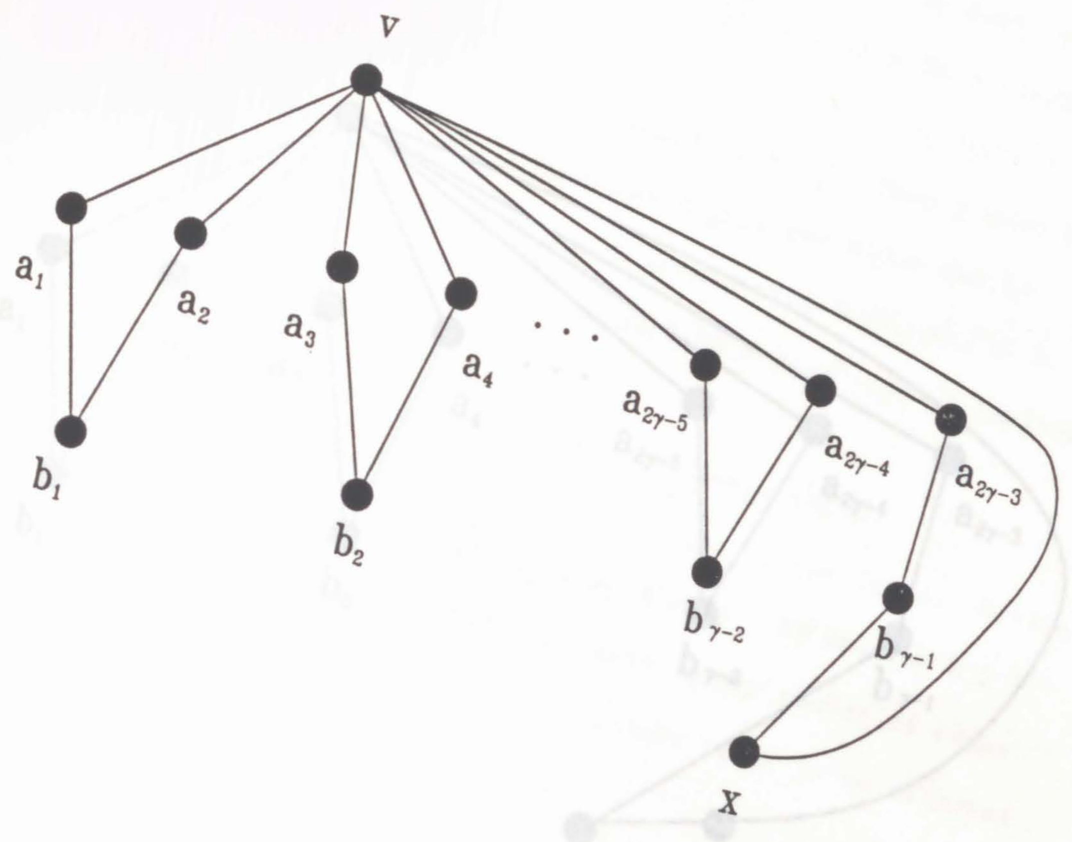

v
a_1
a_2
a_3
a_4
$a_{2\gamma-5}$
$a_{2\gamma-4}$
$a_{2\gamma-3}$
b_1
b_2
$b_{\gamma-2}$
$b_{\gamma-1}$
x

Figure 6.10(b)

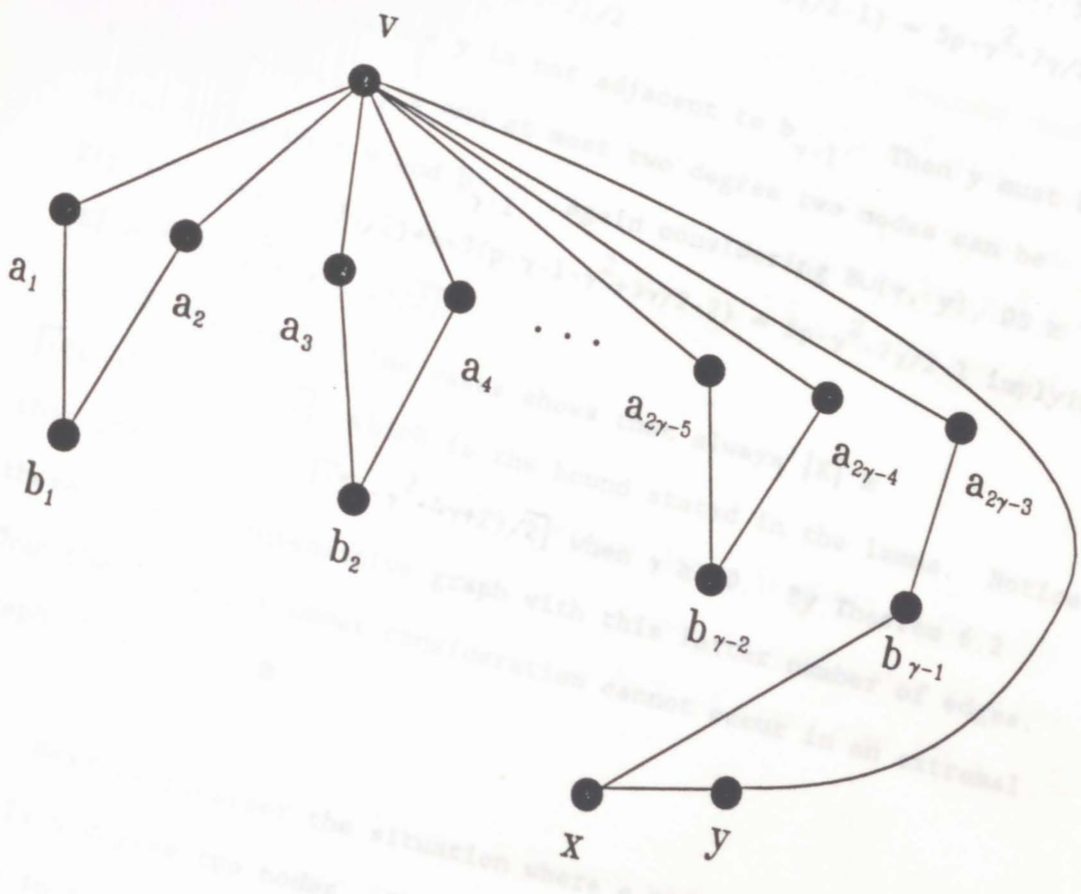


Figure 6.10(c)

is at most one degree two node adjacent to both y and $b_{\gamma-1}$. Since no node is adjacent to $2\gamma-2$ degree two nodes, there are at most $\gamma^2-3\gamma/2+1$ degree two nodes in $V-(B\cup\{v, y\})$. As before, we shall compute the degree sum. Notice that it must include a count of 4 in consideration of the edges vy and $yb_{\gamma-1}$ internal to $B\cup\{v, y\}$. Thus $DS \geq 2(p-\gamma-1)+4+2(\gamma^2-3\gamma/2)+2+3(p-\gamma-1-\gamma^2+3\gamma/2-1) = 5p-\gamma^2-7\gamma/2-2$ implying $|E| \geq (5p-\gamma^2-7\gamma/2-2)/2$.

Finally assume y is not adjacent to $b_{\gamma-1}$. Then y must be adjacent to two b_i's and at most two degree two nodes can be adjacent to both y and $b_{\gamma-1}$. Again considering $B\cup\{v, y\}$, $DS \geq 2(p-\gamma-1)+6+2(\gamma^2-3\gamma/2)+4+3(p-\gamma-1-\gamma^2+3\gamma/2-2) = 5p-\gamma^2-7\gamma/2-1$ implying $|E| \geq \lceil(5p-\gamma^2-7\gamma/2-1)/2\rceil$.

Examining all the cases shows that always $|E| \geq \lceil(5p-\gamma^2-7\gamma/2-2)/2\rceil$, which is the bound stated in the lemma. Notice that this exceeds $\lceil(5p-\gamma^2-4\gamma+2)/2\rceil$ when $\gamma \geq 10$. By Theorem 6.2 there is a 2-γ-insensitive graph with this latter number of edges. Thus the situation under consideration cannot occur in an extremal graph if $\gamma \geq 10$. ■

Next we consider the situation where a node is adjacent to at most $2\gamma-4$ degree two nodes. The next theorem establishes lower bounds in this situation.

Lemma 6.11

Let $G = (V,E)$ be a 2-γ-insensitive graph in which no node has more than 2γ-4 degree two neighbors. If any pair of nodes of G has two degree two nodes adjacent to both of them, then $|E| \geq (5p-3\gamma^2-11)/2$ for $p \geq 3\gamma^2-10\gamma+10$.

Proof

Suppose a_1 and a_2 are degree two nodes with common neighbors x and y. $R(a_1x, a_1y)$. By Lemma 6.1 $a_1 \epsilon D$ and neither x nor y is in D. Thus $a_2 \epsilon D$ along with γ-2 other nodes. Hence the maximum number of nodes of degree at most two is $1+2+(\gamma-2)(2\gamma-4) = 2\gamma^2-8\gamma+11$ since each degree two node must have a neighbor in D and at most one of x and y has degree two since G is connected. Substituting $2\gamma^2-8\gamma+11$ for $f(\gamma,k)$ in Theorem 4.6 and using $|E|$ for $E^2(p,\gamma)$, we get $|E| \geq (5p-3\gamma^2-11)/2$ when $p \geq 3\gamma^2-10\gamma+10$. ■

Lemma 6.12

Let $G = (V,E)$ be a 2-γ-insensitive graph in which no node has more than 2γ-4 degree two neighbors. If each pair of nodes of G has at most one degree two node adjacent to both of them, then $|E| \geq (5p-2\gamma^2-6\gamma-2)/2$ when $p \geq 2\gamma^2+2$.

Proof

Case 3 in the proof of Lemma 6.5 shows that any node can have at most γ degree two neighbors. Suppose node v has γ degree two neighbors $a_1, a_2, \ldots, a_\gamma$ and these neighbors are also adjacent to $b_1, b_2, \ldots, b_\gamma$, respectively. Suppose further that $x \epsilon V-\{v, a_1, \ldots, a_\gamma, b_1, b_2, \ldots, b_\gamma\}$ is adjacent to only one b_i, say b_1. See Figure 6.11. $R(a_1v, a_1b_1)$. Then $a_1 \epsilon D$ and neither v nor b_1 is in D

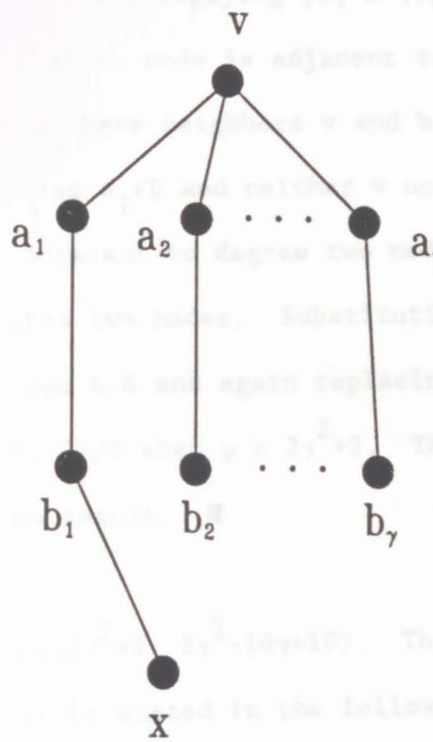


Figure 6.11

implying that a_i or b_i is in D for $2 \le i \le \gamma$. But then x is not dominated, a contradiction. Thus all nodes in $V-\{v, b_1, \ldots, b_\gamma\}$ have at least two edges to $\{v, b_1, \ldots, b_\gamma\}$, so there are at most $(\gamma^2+\gamma)/2$ degree two nodes. Then $DS \ge 2(p-\gamma-1)+2(\gamma^2+\gamma)/2)+3(p-\gamma-1-(\gamma^2+\gamma)/2) = 5p-\gamma^2/2-11\gamma/2-5$ implying $|E| \ge (5p-\gamma^2/2-11\gamma/2-5)/2$.

Suppose next that no node is adjacent to γ degree two nodes. Let degree two node a_1 have neighbors v and b_1. $R(a_1b_1,a_1v)$. Again Lemma 6.1 implies $a_1 \epsilon D$ and neither v nor b_1 is in D. Thus at most γ-1 nodes are adjacent to degree two nodes in V-D, so there are at most $\gamma^2-2\gamma+2$ degree two nodes. Substituting for $f(\gamma,2)$ into the lower bound of Theorem 4.6 and again replacing $E^2(p,\gamma)$ by $|E|$, we get $|E| \ge (5p-2\gamma^2-6\gamma-2)/2$ when $p \ge 2\gamma^2+2$. This is the smallest value and yields the result. ∎

Let $g(\gamma) = \max(2\gamma^2+2,\ 3\gamma^2-10\gamma+10)$. The conclusion drawn from the preceding lemmas is stated in the following theorem.

<u>Theorem 6.3</u>

Let $p \ge g(\gamma)$. Then $E^2(p,\gamma) \ge (5p-3\gamma^2-11)/2$.

<u>Proof</u>

Let G = (V,E) be a 2-γ-insensitive graph. Examination of Lemmas 6.8, 6.10, 6.11 and 6.12 shows that in any situation $|E| \ge (5p-3\gamma^2-11)/2$ for all $\gamma \ge 3$. However, the different bounds can be compared for only those values of p which enabled their calculation. Thus we must use the largest such value which is $2\gamma^2+2$ if $3 \le \gamma \le 9$

and $3\gamma^2-10\gamma+10$ if $\gamma \geq 10$. ■

We now investigate the situation when $p < g(\gamma)$. The next two theorems provide a general although undoubtedly loose lower bound for $E^2(p,\gamma)$ for all p. This bound will be employed in this section for the interval $\gamma^2 \leq p < g(\gamma)$ and in Section 6.3 when $p < \gamma^2$.

Theorem 6.4

A tree cannot be 2-γ-insensitive for $\gamma \geq 3$.

Proof

Suppose tree T is a 2-γ-insensitive graph rooted at r. Consider leaf node v on the bottom level of T. Let u be v's parent. By Corollary 6.2 $d_u \geq 3$ and each of u's neighbors except v must have degree at least three, a contradiction since v is on the bottom level of T. ■

Corollary 6.3

Any connected 2-γ-insensitive graph has at least p edges when $\gamma \geq 3$.

Proof

A connected graph must have at least p-1 edges and one with exactly p-1 edges is a tree. By Theorem 6.4 no tree is 2-γ-insensitive, so at least p edges are necessary. ■

The next theorem uses Dutton and Brigham's (1988) result that $E(p,\gamma) = 2p-3\gamma$ when $p \geq 3\gamma$ to establish a lower bound for $E^2(p,\gamma)$.

Theorem 6.5

Let $p \geq 3\gamma$ and $\gamma \geq 3$. Then $E^2(p,\gamma) \geq 2p-3\gamma+1$.

Proof

Let G be a 2-γ-insensitive graph on $p \geq 3\gamma$ nodes. Note that G-e must be 1-γ-insensitive for an arbitrary edge e. If G is connected for some edge e, then G-e must be a connected 1-γ-insensitive graph. By Dutton and Brigham's result G-e has at least $2p-3\gamma$ edges, so G must have at least $2p-3\gamma+1$ edges.

If G is disconnected for all edges e, then G must be a tree. But no tree is 2-γ-insensitive, a contradiction. ∎

6.3. Bounds When $p < \gamma^2$

In this section we derive bounds for $E^2(p,\gamma)$ when $p < \gamma^2$. Since we require that a 2-γ-insensitive graph G be initially connected, $p \geq 2\gamma$, which implies that $E^2(p,\gamma)$ is undefined when $p < 2\gamma$. From Corollary 6.3 we know that any connected 2-γ-insensitive graph must have at least p edges. The next three theorems construct the only 2-γ-insensitive graphs having exactly p edges.

Theorem 6.6

Let $p = 2\gamma$ with $\gamma \geq 3$. Then $E^2(p,\gamma) = p$.

Proof

By Corollary 6.3 any connected 2-γ-insensitive graph must have at least p edges. It suffices to show such a graph having the insensitive property with p edges. Consider a cycle on γ nodes with a pendant edge incident to each node on the cycle, as shown in Figure 6.12. The domination number of G is γ since each degree one node or its neighbor must be in any dominating set. All that remains to be shown is that the domination number remains γ when arbitrary edges e_1 and e_2 are removed. In all situations a dominating set can be found using any isolated nodes along with the nodes on the cycle which, in G, are not adjacent to the nodes which have become isolated. ■

Theorem 6.7

Let $p = 3\gamma-2$ and $\gamma \geq 3$. Then $E^2(p,\gamma) = p$.

Proof

Again Corollary 6.3 guarantees $E^2(p,\gamma) \geq p$, so it suffices to demonstrate a 2-γ-insensitive graph having $p = 3\gamma-2$ nodes and p edges. Consider the cycle on $3\gamma-2$ nodes. Clearly G has p edges and domination number $\lceil(3\gamma-2)/3\rceil = \gamma$. Remove two arbitrary edges e_1 and e_2 to create disjoint paths P_i and P_{p-i} where $1 \leq i \leq \lfloor p/2 \rfloor$. Then $\lceil i/3 \rceil$ nodes dominate P_i and $\lceil (p-i)/3 \rceil$ nodes dominate P_{p-i}. It can be shown that $\lceil i/3 \rceil + \lceil (3\gamma-2-i)/3 \rceil = \gamma$. Therefore G is 2-$\gamma$-insensitive. ■

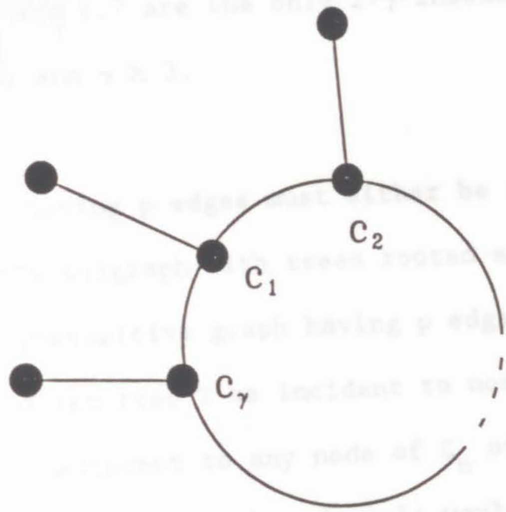


Figure 6.12

Next we show that the graphs described in Theorems 6.6 and 6.7 are the only 2-γ-insensitive graphs having p edges, and thereby establish that p+1 is a lower bound for $E^2(p,\gamma)$ when $p \neq 2\gamma$, $3\gamma-2$ and $\gamma \geq 3$.

Theorem 6.8

Graphs on $p = 2\gamma$ and $p = 3\gamma-2$ nodes as described in the proofs of Theorems 6.6 and 6.7 are the only 2-γ-insensitive graphs having p edges for $p \geq 2\gamma$ and $\gamma \geq 3$.

Proof

Any graph having p edges must either be a cycle or have exactly one cycle subgraph with trees rooted at nodes on the cycle. Let G be a 2-γ-insensitive graph having p edges and a cycle subgraph C_n, and let tree T be incident to node v on the cycle. No node of T can be adjacent to any node of C_n other than v or to another tree because then a second cycle would be formed. Suppose T has more than one level and consider a leaf node on the lowest level. An argument identical to that in the proof of Theorem 6.4 shows that this leads to a contradiction. Therefore T has at most one level, implying T is a pendant edge since Lemmma 6.2 shows a node can be adjacent to at most one degree one node.

By Corollary 6.2 any neighbor of a degree one node cannot be adjacent to a degree two node, so either every node on the cycle is adjacent to a degree one node or no node on the cycle is adjacent to a degree one node. Since at least one endpoint from each pendant edge must be in any minimum dominating set, the only situation where each node on the cycle is incident to a pendant edge is the graph on $p - 2\gamma$ nodes as described in Theorem 6.6. Therefore we consider only cycles C_p. If $p < 3\gamma-2$ or $p > 3\gamma$, the domination number is not γ, a contradiction. Suppose $p = 3\gamma-1$ or 3γ, and remove two adjacent edges to form $P_1 \cup P_{p-1}$. This has domination number $\lceil 1/3 \rceil + \lceil (p-\gamma)/3 \rceil = \gamma+1$, a contradiction. ∎

Corollary 6.4

Let $p \neq 2\gamma$, $3\gamma-2$. Then $E^2(p,\gamma) \geq p+1$.

Therefore $E^2(p,\gamma)$ is determined for $p \leq 2\gamma$ and $p = 3\gamma-2$. Next we consider the range $2\gamma+1 \leq p \leq 3\gamma-3$.

Theorem 6.9

Let $2\gamma+1 \leq p \leq 3\gamma-3$. Then $E^2(p,\gamma) \leq 2p-2\gamma$.

Proof

Construct a 2-γ-insensitive graph $G = (V,E)$ having $2p-2\gamma$ edges as follows. Let $V = B \cup C$ where $B = \{b_1, b_2, \ldots, b_\gamma\}$ and $C = \{c_1, c_2, \ldots, c_{p-\gamma}\}$. Form a cycle on the nodes in C and let each $b_i \in B$ be adjacent to one c_j or to two adjacent c_j's such that no two b_i's are adjacent to the same c_j and each c_j has a neighbor in B. Observe that this construction can always be carried out since $\gamma+1 \leq p-\gamma \leq 2\gamma-3$.

It remains to be shown that G is 2-γ-insensitive. Obviously B dominates and since either b_i or one of it's neighbor must be in any dominating set and each c_j dominates exactly one b_i, γ nodes are necessary. Now we show that $\gamma(G-e_1-e_2) = \gamma(G)$ for arbitrary edges e_1 and e_2.

<u>Case 1</u> If both e_1 and e_2 are on the cycle, then B dominates.

<u>Case 2</u> Suppose e_1 is on the cycle and e_2 is incident to b_i off the cycle. If $e_2 = b_i c_j$ where b_i is a degree one node, then c_j can be dominated by at least one neighbor in C. Hence c_j's neighbor on C and $B-\{b_k\}$ dominate, where b_k is adjacent to c_j. If $e_2 = b_i c_j$ and b_i has degree two, then γ nodes on the cycle dominate.

<u>Case 3</u> Both e_1 and e_2 are off the cycle. If e_1 and e_2 are incident to degree one nodes b_i and b_j, then b_i, b_j and $\gamma-2$ nodes on the cycle dominate. If e_1 and e_2 are both adjacent to b_i, then b_i and $\gamma-1$ nodes on the cycle dominate. If e_1 is adjacent to a degree one node b_i and e_2 is adjacent to a degree two node, then b_i and $\gamma-1$ nodes on the cycle will dominate. If e_1 and e_2 are adjacent to distinct degree two nodes, then γ nodes on the cycle dominate.

Thus G is 2-γ-insensitive and the result follows. ■

<u>Corollary 6.5</u>

Let $p = 2\gamma+1$ and $\gamma \geq 4$. Then $E^2(p,\gamma) = p+1$ when $p = 2\gamma+1$.

<u>Proof</u>

Theorem 6.9 implies $E^2(p,\gamma) \leq 2p-2\gamma = p+1$ and Corollary 6.4 implies $E^2(p,\gamma) \geq p+1$ if $\gamma \geq 4$. ■

Next we develop an upper bound for the interval $3\gamma-1 \leq p \leq \gamma^2-1$.

Theorem 6.10

Let $3\gamma-1 \leq p \leq \gamma^2-1$. Then $E^2(p,\gamma) \leq 3p-6\gamma+4$.

Proof

We construct 2-γ-insensitive graphs as follows. Form a cycle on $3\gamma-2$ nodes labeled $c_1, c_2, \ldots, c_{3\gamma-2}$ and from each node $b_i \epsilon V$ that is not on the cycle place edges to the same three consecutive nodes on the cycle, say c_1, c_2, and c_3. See Figure 6.13 for the constructions when $p = 3\gamma-1$ and $p = 3\gamma$. Then G has $3\gamma-2+3(p-3\gamma+2) = 3p-6\gamma+4$ edges. Obviously, γ nodes dominate G and γ nodes are necessary since any node dominates at most three nodes on the cycle. To show G is 2-γ-insensitive we consider the possible ways of removing two edges e_1 and e_2.

Case 1 Both e_1 and e_2 are on the cycle subgraph. By Theorem 6.7 $C_{3\gamma-2}$ is 2-γ-insensitive, so γ nodes, including at least one of c_1, c_2 and c_3, dominate G.

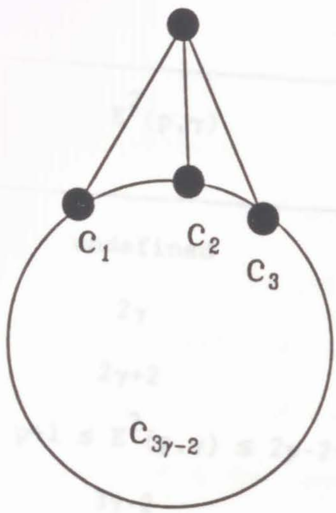
Case 2 Both e_1 and e_2 are incident to b_i's. Then at least one of c_1, c_2 and c_3 can still dominate all b_i's and it, along with γ-1 other nodes of the cycle, dominate $G-e_1-e_2$.

Case 3 Edge e_1 is incident to a b_i and edge e_2 is on the cycle. Then γ nodes on the cycle, including one of c_1, c_2 and c_3, dominate.

Therefore G is 2-γ-insensitive. ∎

6.4. Summary of Results for All p

Table 6.1 summarizes the results of this chapter for $E^2(p,\gamma)$ when $\gamma \geq 3$ and for all values of p.



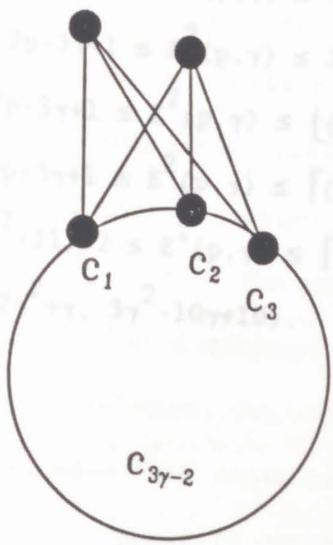


Figure 6.13

TABLE 6.1

p	$E^2(p,\gamma)$	Difference Between Lower and Upper Bounds
$< 2\gamma$	undefined	----
2γ	2γ	0
$2\gamma+1$	$2\gamma+2$	0
$2\gamma+2 \le p \le 3\gamma-3$	$p+1 \le E^2(p,\gamma) \le 2p-2\gamma$	$O(\gamma)$
$3\gamma-2$	$3\gamma-2$	0
$3\gamma-1$	$p+1 \le E^2(p,\gamma) \le p+2$	1
$3\gamma \le p \le \gamma^2-1$	$2p-3\gamma+1 \le E^2(p,\gamma) \le 3p-6\gamma+4$	$O(\gamma^2)$
γ^2	$2p-3\gamma+1 \le E^2(p,\gamma) \le 2p-2\gamma$	$O(\gamma)$
γ^2+1	$2p-3\gamma+1 \le E^2(p,\gamma) \le \lfloor(5p-\gamma^2-4\gamma+2)/2\rfloor$	$O(\gamma^2)$
$\gamma^2+1 < p < g(\gamma)$	$2p-3\gamma+1 \le E^2(p,\gamma) \le \lceil(5p-\gamma^2-4\gamma+2)/2\rceil$	$O(\gamma^2)$
$p \ge g(\gamma)$	$(5p-3\gamma^2-11)/2 \le E^2(p,\gamma) \le \lceil(5p-\gamma^2-4\gamma+2)/2\rceil$	$O(\gamma^2)$

Here $g(\gamma) = \max(2\gamma^2+\gamma,\ 3\gamma^2-10\gamma+10)$.

7. APPLICATIONS OF 2-γ-INSENSITIVE GRAPHS IN NETWORK DESIGN

A number of interesting network designs for large distributed/ multiprocessor systems have appeared in the literature (Akers 1987; Ciminiera and Serra 1986; Pradhan 1985; Yanney and Hayes 1986). Some of the key criteria considered in connecting the processors are connection complexity, multiple configuration flexibility, fault tolerance, cost, and simple yet fast routing (Pradhan 1985). Techniques from graph theory are useful in producing minimum cost network designs that meet specified requirements (Tannenbaum 1981).

Networks are represented by undirected graphs whose nodes denote processors and whose edges denote communication links. A graph G having p nodes with domination number γ corresponds to a network having p processors where a minimum number, γ, of them can communicate directly (in 1 hop) with the remaining p-γ processors. A network with this property has a minimum sized core group that could function in a variety of ways, for example, as "masters" or as repositories for a global data base essential to the other nodes. It may be desirable that this property not be lost when a component fails. In such situations it is worthwhile to establish redundant network links or otherwise construct a topological design which preserves this property when one or more links or nodes fail.

This chapter introduces a topological design which evolved from our search for extremal 2-γ-insensitive graphs. Its characteristics make it a suitable architecture for point-to-point communication networks and for interconnection networks. We shall discuss both applications. The features of this architecture are an integral part of the design and meet most of the key requirements mentioned above. The network, which we call the G-network (the G for "gamma"), maintains its properties in the presence of faults in the system. In this context, a fault is interpreted as the failure of a single node or one or two links.

The G-network is an extremal 2-γ-insensitive graph, so γ nodes will dominate G even after two links fail. Notice that the dominating set of γ nodes is not fixed and may change in a way which depends on which links fail. It is beyond the scope of this dissertation to discuss protocols to reconfigure the network to reflect the switch to a different set of dominating nodes. Instead we concentrate on the "nice" properties of the G-network that are inherent from its basic structure.

We begin by constructing the G-network and discussing its fault tolerant properties in a point-to-point network architecture. The design allows a core group of processors to communicate directly with the remaining processors. We treat the core group as file servers and analyze the performance of the network when a fault occurs.

Next we consider the G-network as a processor to processor interconnection and compare it to the mesh connected Illiac, Barrel Shifter, and Hypercube networks (Hwang and Briggs 1984). The G-network shows a significant improvement over these popular networks in the maximum number of routing steps required for any two nodes to communicate. Moreover, the G-network has a relatively small number of links in comparison to these networks.

Finally, we introduce a multi-layered G-network obtained by interconnecting copies of the G-network in parallel. This design is suitable for interconnection networks used in massively parallel computation.

7.1. The G-network and Its Properties

The G-network on $p = \gamma^2$ nodes is constructed as follows. Designate γ nodes as "special" nodes and label them a_1, a_2, ..., a_γ. For each pair $\{a_i, a_j\}$, $i \neq j$, add two degree two nodes adjacent to both a_i and a_j. Label arbitrarily the degree two nodes b_i, $1 \leq i \leq \gamma^2-\gamma$. Figure 7.1 illustrates the G-network for $\gamma = 3$ and Figure 7.2 shows it for $\gamma = 4$. Clearly the γ special nodes form a minimum dominating set for the graph. Notice that the G-network has $p = \gamma+2[\gamma(\gamma-1)/2] = \gamma^2$ nodes, γ special nodes with degree $2\gamma-2$, $\gamma^2-\gamma$ nodes with degree two, and $e = 2\gamma^2-2\gamma = 2p-2\gamma$ links. This is one of the graphs discussed in Chapter 6 where it is shown to be 2-γ-insensitive.

Even if the G-network suffers the loss of up to two links, the

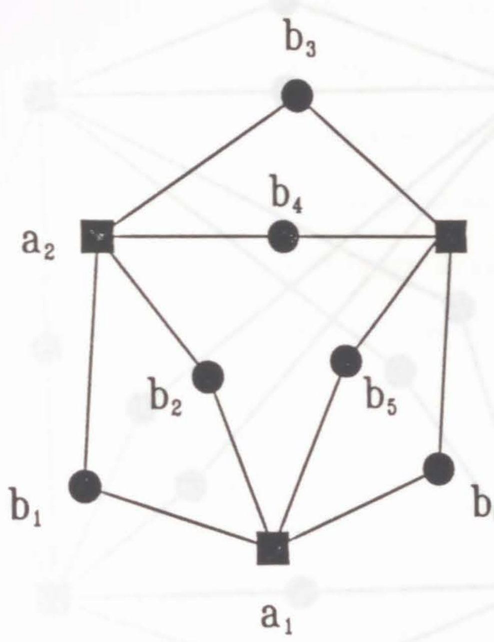


Figure 7.1

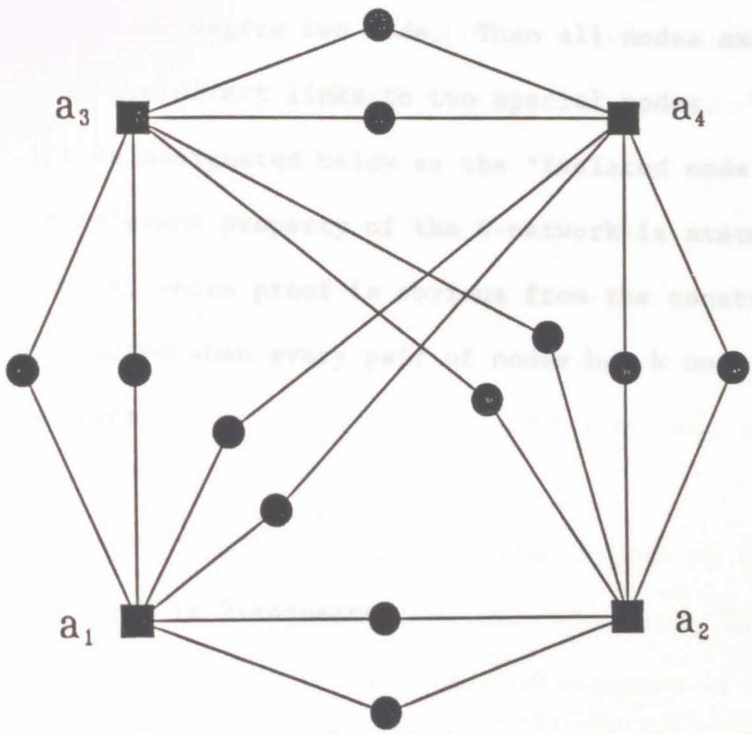


Figure 7.2

set of γ special nodes can still communicate directly with the remaining p-γ nodes, with the possible exception of a single isolated node. To see this consider the failure of two arbitrary links, e_1 and e_2. If e_1 and e_2 are incident to different degree two nodes, then each degree two node has at least one direct link to a special node. The only other situation is when e_1 and e_2 are incident to the same degree two node. Then all nodes except the isolated nodes have direct links to two special nodes. The last situation will be designated below as the "isolated node case."

Another relevant property of the G-network is stated in the following theorem, whose proof is obvious from the construction. A graph is k-connected when every pair of nodes has k node disjoint paths between them.

<u>Theorem 7.1</u>

The G-network is 2-connected.

7.1.1. The G-network: A Point-to-Point Network

We first consider the G-network as a point-to-point architecture where the γ special nodes are designated to serve the function of file servers. For discussion we refer to these special nodes as "file servers" and the others of degree two simply as "nodes." A file server is a processor that contains replicated software to provide a shared disk system. Access to the file servers in spite of link faults is important in order to ensure continuous service. Thus the property of the G-network discussed in the previous section is an attractive feature in this application. Each degree two node can communicate directly (in one hop) with at least one file server even if one link fails and, except in the isolated node case, each degree two node can communicate directly with at least one file server when two links fail. Even in the exceptional case, all nonisolated degree two nodes still have direct access to some file server. The following connectivity observations follow immediately:

(1) The G-network remains connected when any arbitrary link fails.

(2) The G-network remains connected when any two arbitrary links fail with only the isolated node case as an exception.

Consequently, the G-network has a high degree of fault tolerance with respect to at most two link failures. Theorem 7.2 considers the failure of a node and is a consequence of Theorem 7.1.

<u>Theorem 7.2</u>

Any single node or file server in the G-network can fail (the node and all links incident to it are removed) without disrupting the service to the remaining active nodes.

Proof

First suppose the node that fails is a node with degree two. Then all other nodes with degree two are unaffected and each can still access two file servers. On the other hand, suppose a file server fails. Each degree two node still can access directly at least one file server. ■

Each pair of nodes in the core group (file servers) are connected by two paths of length two. If files are duplicated in two or more servers, then at least one set of files will be available to all nonisolated active nodes in spite of the faults under consideration. Furthermore, each nonisolated active node can access the files in at most three hops.

7.1.2. The G-network: An Interconnection Network

We turn our attention to the G-network as a model for a processor-to-processor interconnection network. The properties dicussed in this section are applicable when the G-network is any point-to-point network, but we shall limit our discussion to interconnection networks. The following theorem states a routing property of this configuration.

Theorem 7.3

The maximum number of routing steps (hops) required between any two nodes in the G-network is four.

Proof

The worst case situation occurs when the source node and the destination node are both degree two nodes and are not adjacent to a common special node. The source node can reach either one of the two special nodes in its neighborhood with one hop. This special node can reach a special node adjacent to the destination node with two hops. One more hop reaches the destination. ■

Unlike other interconnection networks where routing steps are dependent on the number of processors, the G-network offers a constant maximum of four hops independent of p. This characteristic is an innate property of the design and is achieved without adding additional redundant links.

Table 7.1 shows comparisons between the G-network and the Illiac network when $p = \gamma^2$. The G-network has fewer links than the Illiac network while providing better routing. Furthermore, as γ approaches infinity, the speedup and the difference in the number of links approach infinity.

Table 7.2 illustrates the comparisons between the G-network and the Barrel Shifter interconnection network. There is a significant difference in the number of links. Moreover, the routing performance of the G-network is an improvement over the Barrel Shifter network for $p > 16$.

Table 7.3 gives the comparisons between the G-network and the Hypercube network with $p = \gamma^2$. Again for large networks as γ

TABLE 7.1

A COMPARISON OF THE ILLIAC NETWORK AND THE G-NETWORK

HERE $p = \gamma^2$.

γ	p	MAXIMUM NUMBER OF HOPS ILLIAC	G-NETWORK	SPEEDUP
4	16	3	4	3/4
8	64	7	4	7/4
16	256	15	4	15/4
32	1024	31	4	31/4
γ	γ^2	γ-1	4	(γ-1)/4

p	NUMBER OF LINKS ILLIAC	G-NETWORK	DIFFERENCE
16	32	24	8
64	128	112	16
256	512	480	32
1024	2048	1984	64
γ^2	$2\gamma^2 = 2p$	$2p-2\gamma = 2p-2\sqrt{p}$	$2\gamma = 2\sqrt{p}$

TABLE 7.2

A COMPARISON OF THE BARREL SHIFTER AND THE G-NETWORK

Here $p = \gamma^2 = 2^{2k}$, $\gamma = \sqrt{p} = 2^k$, $\log_2 p = 2k$.

γ	p	MAXIMUM NUMBER OF HOPS BARREL SHIFTER	G-NETWORK	SPEEDUP
4	16	2	4	1/2
8	64	3	4	3/4
16	256	4	4	4/4
32	1024	5	4	5/4
γ	γ^2	$(\log_2 p)/2$	4	$(\log_2 p)/8$

p	NUMBER OF LINKS BARREL SHIFTER	G-NETWORK	DIFFERENCE
16	56	24	32
64	352	112	240
256	1920	480	1440
1024	9728	1984	7744
γ^2	$p(2\log_2 p-1)/2$	$2p-2\gamma=2p-2\sqrt{p}$	$O(p(\log_2 p))$

TABLE 7.3

A COMPARISON OF THE HYPERCUBE AND THE G-NETWORK

Here $p = \gamma^2 = 2^{2k}$, $\gamma = \sqrt{p} = 2^k$.

γ	p	k	MAXIMUM NUMBER OF HOPS HYPERCUBE	G-NETWORK	SPEEDUP
4	16	2	4	4	4/4
8	64	3	6	4	6/4
16	256	4	8	4	8/4
32	1024	5	10	4	10/4
γ	γ^2	k	$\log_2 p$	4	$(\log_2 p)/4$

p	NUMBER OF LINKS HYPERCUBE	G-NETWORK	DIFFERENCE
16	32	24	8
64	192	112	80
256	1024	480	544
1024	5120	1984	3136
γ^2	$p((\log_2 p)/2)$	$2p-2\gamma$	$O(p\log_2 p)$

approaches infinity, so do the speedup and the difference in the number of links.

In all these comparisons the G-network has fewer links while providing better routing.

Our next theorem reflects the fault tolerance of the G-network with respect to routing.

Theorem 7.4

Any single link can fail in the G-network and the maximum number of routing steps required will remain four.

Proof

Let an arbitrary link fail and consider the worst case where two nodes in the graph under discussion are not adjacent to a common special node. From previous observations any node can still communicate directly with at least one special node after any link fails. The special node still has a least one path of length two to every other special node. Finally, at least one path to the destination remains. Thus, the maximum number of routing steps stays four in spite of any link failure. ■

The following theorem considers the failures of two links.

Theorem 7.5

The failure of two links results in one of the following situations:

(i) Every pair of special nodes is still connected by at least one path of length two.

(ii) One pair of special nodes is not connected by a path of length two.

In both situations the maximum number of routing steps required between any two nonisolated nodes remains four, with one exception. The exception occurs in situation (ii) when one or both of the nodes have degree one, in which case the maximum number of steps is six.

<u>Proof</u>

If at least one path of length two exists between each pair of special nodes, routing can occur as before (except possibly for one node if it has become isolated). Assume then that there exists a pair of special nodes, say a_1 and a_2, no longer connected by a path of length two. Let b_1 and b_2 be the degree two nodes which were adjacent to both a_1 and a_2 prior to the link failures. By symmetry we need to consider only the following two possibilities: the faulty links are a_1b_1 and a_1b_2 or a_1b_1 and a_2b_2. If a_1b_1 and a_1b_2 fail, a maximum of five hops is needed for routing between either b_1 or b_2 and a_1. If a_1b_1 and a_2b_2 fail, the worst case requires six hops to route between b_1 and b_2. ∎

We now consider the tolerance of the G-network with respect to a faulty node.

Theorem 7.6

If any node in the G-network fails, the maximum number of routing steps required for any two active nodes to communicate remains four.

Proof

If a node with degree two fails, it is similar to case (i) of Theorem 7.5 where the node becomes isolated. If a special node fails, all degree two nodes can still reach at least one active special node in one hop and the result follows as before. ■

Theorems 7.4 and 7.6 show that this network can withstand a failure of either any single node or any single link and the maximum number of routing steps remains constant at four.

7.2. A Multi-layered G-network for Massively Parallel Computation

This section presents the Multi-layered G-network obtained by interconnecting copies of the G-network in parallel. The design is suitable for large interconnection networks and has the following desirable characteristics: efficient routing, small number of links and a high level of fault tolerance.

Each layer of this network is a copy of the G-network where the γ special nodes in copy i are labeled 1_i, 2_i, ..., γ_i. Here i, $0 \le i \le h-1$, is the layer number. Each special node a_i is connected to $a_{(i+1) \bmod h}$. Observe that the nodes a_0, a_1, ..., a_{h-1} induce a cycle subgraph. One layer of the Multi-layered G-network with $\gamma = 4$

is shown in Figure 7.3, with connections to other layers indicated by dotted lines. Each layer has γ^2 nodes and $2\gamma^2-2\gamma$ links. An additional γh links are required to connect the layers, giving a total of $2h\gamma^2-h\gamma$ links. There are $h(\gamma^2-\gamma)$ degree two nodes and $h\gamma$ degree 2γ nodes.

The maximum number of routing steps (hops) required between any two nodes in the same layer is four since each layer is a G-network. This fact is independent of the number of processors in the network.

Theorem 7.7

The maximum number of hops required between any two nodes in the Multi-layered G-network is $\lfloor h/2 \rfloor+4$.

Proof

In the worst case situation a source node can reach a special node in the same layer in one hop. From that special node it takes at most $\lfloor h/2 \rfloor$ hops to reach the layer containing the destination node. It then takes at most three additional hops to reach the destination node. ■

We note that the routing performance dependent only on the number of layers h. Given a fixed number of layers, h, the maximum number of routing steps required is constant. The following theorems demonstrate the network's fault tolerance by showing that

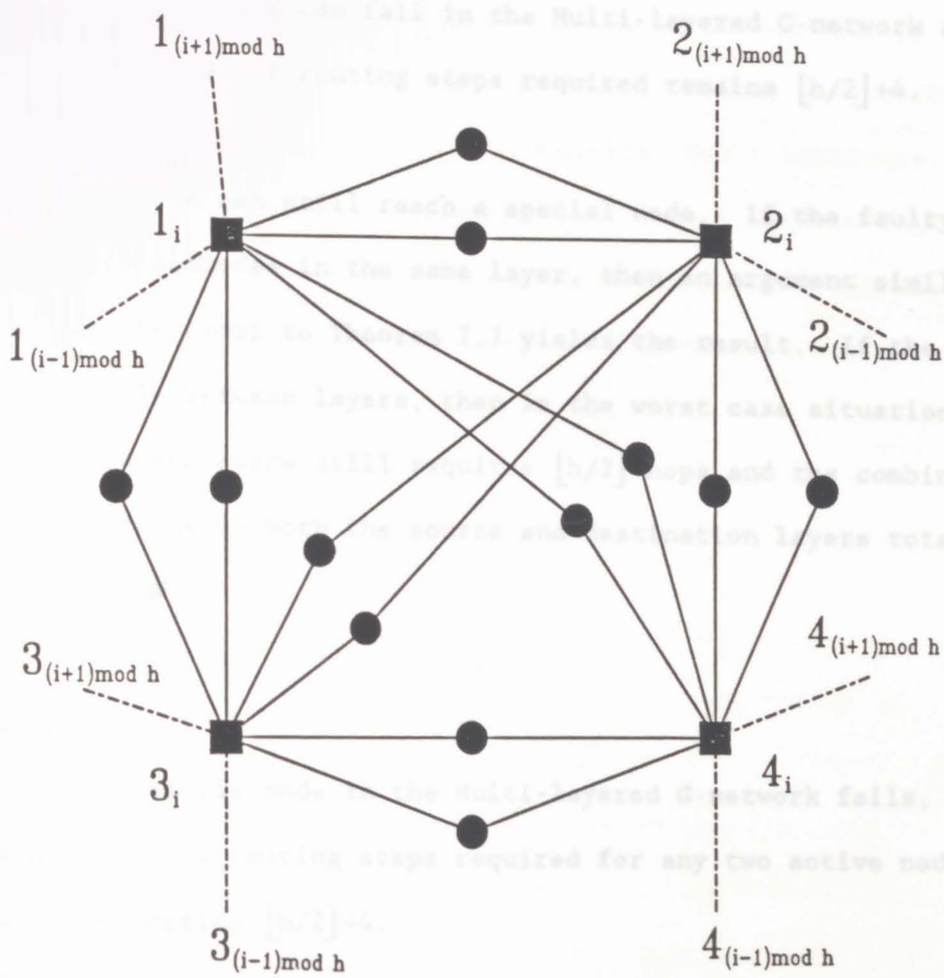

$1_{(i+1)\text{mod } h}$
$2_{(i+1)\text{mod } h}$
1_i
2_i
$1_{(i-1)\text{mod } h}$
$2_{(i-1)\text{mod } h}$
$4_{(i+1)\text{mod } h}$
$3_{(i+1)\text{mod } h}$
3_i
4_i
$3_{(i-1)\text{mod } h}$
$4_{(i-1)\text{mod } h}$

Figure 7.3

this characteristic remains unchanged when any single link or node fails.

Theorem 7.8

Any single link can fail in the Multi-layered G-network and the maximum number of routing steps required remains $\lfloor h/2 \rfloor+4$.

Proof

Every node can still reach a special node. If the faulty link has both its endnodes in the same layer, then an argument similar to the one in the proof to Theorem 7.7 yields the result. If the faulty link is between layers, then in the worst case situations routing between layers still requires $\lfloor h/2 \rfloor$ hops and the combination of routing steps on both the source and destination layers total at most four. ■

Theorem 7.9

If any single node in the Multi-layered G-network fails, the maximum number of routing steps required for any two active nodes to communicate remains $\lfloor h/2 \rfloor+4$.

Proof

If a degree two node fails, each active node is either a special node or still has two links to special nodes, and each pair of special nodes on the same layer is still joined by at least one path of length two. If a special nodes fails, each active degree two node can still reach a special node in one hop and each pair of active special nodes on the same layer is still joined by two paths of length two. Hence in either case routing between any two active nodes can be accomplished as before. ■

We note that the maximum degree of 2γ is independent of the total number of processors. Hence the Multi-layered G-network possesses the following desirable properties: fault tolerance, low cost, fast routing, and simple connections with a maximum degree dependent only on the number of processors per layer. Current research is comparing the performance of this network with other popular designs.

8. CHANGING AND UNCHANGING DOMINATION

In this chapter we present a problem posed by Frank Harary (1988) which is related to our research in insensitive domination. His general problem actually consists of six subproblems involving the changing or the unchanging of the domination number of a graph G in three different situations: when deleting a node, when deleting an edge and when adding an edge. Formally, the six subproblems are to characterize those graphs $G = (V,E)$ for which (1) $\gamma(G-v) \neq \gamma(G)$ for all $v \in V$; (2) $\gamma(G-v) = \gamma(G)$ for all $v \in V$; (3) $\gamma(G-e) \neq \gamma(G)$ for all $e \in E$; (4) $\gamma(G-e) = \gamma(G)$ for all $e \in E$; (5) $\gamma(G+e) \neq \gamma(G)$ for all $e \in \overline{E} = E(\overline{G})$; and (6) $\gamma(G+e) = \gamma(G)$ for all $e \in \overline{E} = E(\overline{G})$.

The insensitive domination problem is concerned with subproblem (4) where γ does not change when a single edge is removed, and we have extended this subproblem to the removal of $k > 1$ edges. Like our research, other reports in the literature have attacked the subproblems separately using varying terminology. The main objective of this chapter is to bring together current research in the form posed by Harary (1988). In addition, we include some new results.

We investigate changing and unchanging of domination for each of the three situations in the remaining sections of this chapter.

It will be useful to partition the nodes of graph G into three sets according to how their removal affects the domination number. Thus we define $V^0 = \{v \in V \mid \gamma(G-v) = \gamma(G)\}$, $V^+ = \{v \in V \mid \gamma(G-v) > \gamma(G)\}$ and $V^- = \{v \in V \mid \gamma(G-v) < \gamma(G)\}$. Throughout this chapter all the minimum dominating sets of G are labeled by $D_1, D_2, \ldots, D_n$.

8.1. Removal of a Node

8.1.1. The Domination Number Is Changed When Any Node Is Removed

Here the removal of any node from graph G changes the domination number, that is, $\gamma(G-v) \neq \gamma(G)$ for all $v \in V$. Clearly, the nodes of G in this case are partitioned into V^+ and V^-. We first state some elementary facts about these two sets.

Lemma 8.1

If $v \in V^+$, then v is in every minimum dominating set of G.

Proof

Suppose $v \in V^+$ and $v \notin D_i$ for some i. Then v is dominated by a neighbor $x \in D_i$. Now either x is necessary in D_i to dominate some node other than v in which case $\gamma(G-v) = \gamma(G)$, or x is required to dominate only v and $\gamma(G-v) = \gamma(G)-1$. In either case, we see $v \notin V^+$, a contradiction. ■

Corollary 8.1

$|V+| \leq \gamma(G)$.

The proof to the following lemma is obvious.

Lemma 8.2

Any isolated node is in V^-.

Lemma 8.3

If node $v \epsilon V^+$, then v is necessary in every D_i, for $1 \le i \le n$, in order to dominate at least two non-adjacent nodes, other than v itself, not dominated by D_i-{v}.

Proof

Suppose $v \epsilon V^+$. Then by Lemmas 8.1 and 8.2 v is in every minimum dominating set and is not isolated. Let S be those nodes of N(v) which are not dominated by D_i-{v} for an arbitrary minimum dominating set D_i. Suppose S forms a complete subgraph of G. Then D_i-{v} along with any node of S if S is nonempty or any neighbor of v otherwise is a minimum dominating set not containing v, a contradiction. Hence v is required in D_i to dominate at least two non-adjacent nodes. ■

Observe that $d_v \ge 2$ for $v \epsilon V^+$. We also note that the converse of Lemma 8.3 is not true. For a counterexample, consider graph G = (V, E) with V = {v, a, b, c, d, e, f} and E = {va, vb, vc, vd, ea, ed, ef, fb, fc}. Obviously, $\gamma(G) = 2$. Although the conditions of the converse of Lemma 8.3 are met, {e, f} dominates G-{v} implying that v is not in V^+.

Corollary 8.2

A node $v \in D_i$ for $1 \le i \le n$ if and only if $v \in V^+$ or v is isolated.

Proof

Sufficiency follows from Lemma 8.1 and the fact that an isolated node must be in every dominating set. To show necessity suppose non-isolated node v is in every minimum dominating set and consider the set S in the proof of Lemma 8.3. If S is empty or complete, v can be replaced by a neighbor to produce a minimum dominating set not containing v, a contradiction. Thus S must contain at least two non-adjacent nodes and the result follows from Lemma 8.3. ■

Theorem 8.1

If $x \in V^+$ and $y \in V^-$, then x is not adjacent to y.

Proof

Let D_y be a minimum dominating set of G-y of size $\gamma(G)$-1. If D_y contains x, D_y dominates G, a contradiction. On the other hand, if D_y does not contain x, then $D_y \cup \{y\}$ dominates G and does not contain x, violating Corollary 8.2. ■

By Lemma 8.3 each $v \in V^+$ has at least two neighbors in V-D_i which are not dominated by D_i-{v}, for $1 \le i \le n$. Since v is not adjacent to a node in V^- and no nodes of V^+ are in V-D_i, v has at least two neighbors in V^0 which are not dominated by D_i-{v}. Hence we have the following corollary.

Corollary 8.3

For any graph $|V^0| \geq 2|V^+|$.

Theorem 8.2

If graph G has the property that $\gamma(G-v) \neq \gamma(G)$ for all $v \epsilon V$, then $\gamma(G-v) < \gamma(G)$ for all v, implying $V = V^-$.

Proof

Suppose $\gamma(G-v) \neq \gamma(G)$ for all $v \epsilon V$. Then V^+ and V^- partition V, and from Theorem 8.1 we know that each component of G has node set $V' \subseteq V^+$ or $V' \subseteq V^-$. If $x \epsilon V^+$, then by the comment preceding Corollary 8.3 x has two neighbors in V^0, a contradiction. Hence no component has $V' \subseteq V^+$, so $V' \subseteq V^-$ for each component of G. ■

Thus in any graph G, such that $\gamma(G-v) \neq \gamma(G)$ for all $v \epsilon V$, it must be the case that $\gamma(G-v) < \gamma(G)$. These are precisely the graphs which Brigham, Chinn and Dutton (1988) call vertex critical or just γ-critical graphs. They state some basic properties of γ-critical graphs. We mention one of these which establishes when a graph G is not γ-critical.

Lemma 8.4 (Brigham, Chinn and Dutton 1988)

If G has a nonisolated node v such that N(v) is complete, then G is not γ-critical (changing in terms of domination).

According to Brigham, Chinn and Dutton attempts to characterize γ-critical graphs have been unsuccessful, and the problem remains unsolved. Furthermore, they show it is not possible to characterize these graphs in terms of forbidden subgraphs. On the other hand, they successfully characterize γ-critical graphs having $p = \gamma+\Delta$ nodes, the minimum number possible, in the following theorem.

Theorem 8.3 (Brigham, Chinn and Dutton 1988)

A graph G is γ-critical having $p = \gamma+\Delta$ if and only if G has the form shown in Figure 8.1 where B is the set of neighbors of a node of maximum degree Δ and

(1) the γ-1 nodes of C are independent,

(2) every B node is adjacent to exactly one C node,

(3) C forms the only set of γ-1 nodes which dominates $B \cup C$,

(4) for any $b \in B$ there is a set of γ-1 nodes of $B \cup C$ which dominates $B-\{b\}$ and includes a node $b' \in B$.

8.1.2. The Domination Number Is Unchanged When Any Node Is Removed

In this problem $\gamma(G-v) = \gamma(G)$ for all $v \in V$, so $V = V^0$. Graphs having this property have not been characterized. Although our current research on this subproblem seems promising, the problem remains open.

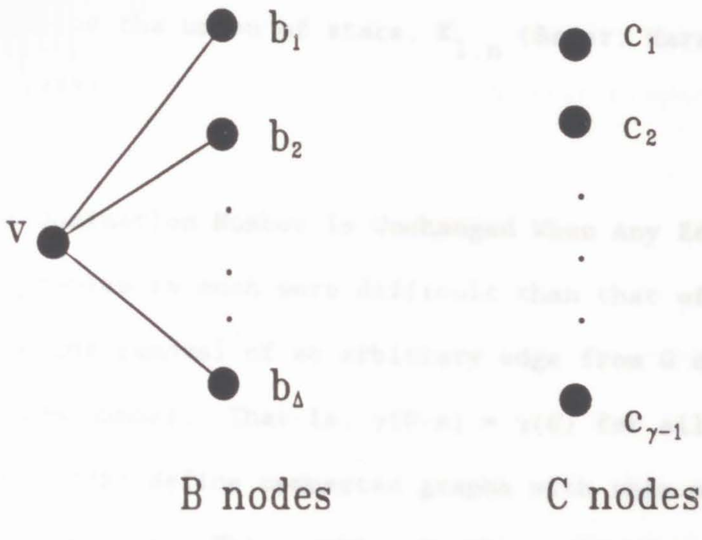


Figure 8.1

8.2. Removal of an Edge

8.2.1. The Domination Number Is Changed When Any Edge Is Removed

Here we treat the case where removal of any edge from G results in a change in the domination number, i.e., $\gamma(G\text{-}e) \neq \gamma(G)$ for all $e \in E$. Clearly, the removal of an edge cannot decrease the domination number, so such graphs have the property that $\gamma(G\text{-}e) > \gamma(G)$ for all $e \in E$. These graphs are easy to characterize and have been shown to be the union of stars, $K_{1,n}$ (Bauer, Harary, Nieminen and Suffel 1983).

8.2.2. The Domination Number Is Unchanged When Any Edge Is Removed

This problem is much more difficult than that of Section 8.2.1. Here the removal of an arbitrary edge from G does not change the domination number. That is, $\gamma(G\text{-}e) = \gamma(G)$ for all $e \in E$. Dutton and Brigham (1988) define connected graphs with this property to be γ-insensitive graphs. This problem is the topic of the research reported in this dissertation, which considers finding extremal graphs having the property.

In general, graphs having this unchanging property seem to be difficult to characterize, and the problem remains open.

8.3. Addition of an Edge

8.3.1. The Domination Number Is Changed When Any Edge Is Added

Just as the removal of an edge cannot decrease the domination number γ, the addition of an edge cannot increase γ. Thus, in this problem, we look at those graphs G which for each $u,v \epsilon V$, where uv is not an edge, $\gamma(G+uv) < \gamma(G)$.

This is a another hard problem. Sumner and Blitch (1983) studied such graphs. They were able to fully characterize them only in the cases where $\gamma(G) = 1$ or 2 and where $\gamma(G) = 3$ when $p \leq 8$.

8.3.2. The Domination Number Is Unchanged When Any Edge Is Added

In this problem $\gamma(G+e) = \gamma(G)$ for all $u,v \epsilon V$ where uv is not an edge. A characterization of graphs having this property is straightforward.

Theorem 8.5

$\gamma(G+e) = \gamma(G)$ for all $e \epsilon E(\bar{G})$ if and only if G contains no nodes in V^-.

Proof

Suppose G is unchanging with respect to domination when any edge is added, and G contains a node $x \epsilon V^-$. Thus $\gamma(G-x) < \gamma(G)$. Let D_x be a $(\gamma-1)$-dominating set of G-x. Then adding an edge e between x and any node of D_x shows $\gamma(G+e) = \gamma(G)-1$, a contradiction.

Suppose G has no nodes in V^- and $\gamma(G+uv) = \gamma(G)-1$ for some pair of nodes u and v. Then any $(\gamma-1)$-dominating set D of G+uv must include exactly one of u or v, say u, and furthermore D must dominate G-v. Thus $v \epsilon V^-$ which is a contradiction. ■

Theorem 8.5 relates this subproblem to the subproblem in Section 8.1.1 where $\gamma(G-v) < \gamma(G)$.

8.4. Remarks

It is interesting to note that straightforward characterizations were possible for subproblems (3) and (6). On the other hand, their counterparts (4) and (5), respectively, do not seem to lend themselves to useful characterizations, and much work remains in studying them. Furthermore, current research seems to indicate that subproblem (2) will be easier to solve than its counterpart (1).

9. CONCLUDING REMARKS

In Chapter 1 we introduced the problem of finding extremal k-γ-insensitive graphs and noted that according to Bollobás (1978) problems in extremal graph theory tend to be difficult. Although the problem in k-γ-insensitivity proved to be no exception, in this dissertation we have been able to attain exact values for $E^k(p,\gamma)$ in some cases and asymptotically correct values in general. Specifically, we solved the first of the two subproblems presented in the introduction which requires that the same fixed set of nodes dominate G when any set of k edges is removed. Significant results for the second of the two subproblems, where the only restriction is the initial connectedness of G, yielded asymptotically correct bounds for $E^k(p,\gamma)$ for all $k \geq 2$. Moreover, structural properties of extremal graphs allowed us to obtain exact values for $E^2(p,2)$ and to improve the upper and lower bounds for $E^2(p,\gamma)$ when $\gamma \geq 3$.

Applications in network design have been shown. We introduced a new network design, called the G-network, and demonstrated that the G-network has desirable characteristics for both point-to-point and interconnection networks while maintaining a high degree of fault tolerance.

Many ideas for future research have arisen during the course of this study and we conclude by listing some of them.

1. Find exact values for $E^2(p,\gamma)$ with $\gamma \geq 3$.
2. Find exact values for $E^k(p,\gamma)$ when $k \geq 3$.
3. Extend the definition of k-γ-insensitivity by adding the restriction that the graph remains connected after any k edges are removed and find extremal graphs having this property.
4. Study other applications of k-γ-insensitive graphs in network design.
5. Investigate the relationship between k-γ-insensitivity and fault tolerance in networks.
6. Determine routing algorithms for the G-network.
7. Determine protocols to provide a method for the G-network to dynamically switch to a new dominating set after k links fail.
8. Complete research on the Multi-layered G-network.
9. Study popular network topologies to determine whether or not they are k-γ-insensitive. If not, determine the the minimum number of additional links required to make them k-γ-insensitive.
10. Characterize graphs for the changing and unchanging domination subproblems (1), (4) and (5) which are described in Chapter 8.
11. Study the changing and unchanging problem posed by Harary (1988) with respect to other graphical invariants.

APPENDIX

A CATALOGUE OF ALL 2-2-INSENSITIVE GRAPHS ON $p \leq 10$ NODES

p = 4

0	0	1	1
0	0	1	1
1	1	0	0
1	1	0	0

p = 5

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0	0	1	1	1
0	1	0	1	1
1	1	1	0	0
1	1	1	0	0

p = 6

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1	1	0	1	1	0

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0	1	1	1	1	0

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1	0	0	0	1	1
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1	0	1	1	1	0

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p = 7

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p = 8

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p = 9

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