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Filter Settle Time for Signal Processing Applications

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FILTER SETTLE TIME FOR SIGNAL PROCESSING APPLICATIONS

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ABSTRACT

In this paper, the step modulated sine wave [$u(t)A\sin 2\pi ft$] response of the low-pass, band-pass, and high-pass filters are evaluated. Butterworth filters from the first order on up to the ideal filter are analyzed, and expressions for the settle times developed. The longest settle time occurs for the ideal filter, with all other filters taking progressively less time to settle as the order decreases. A significant point is that the transient settle time for a filter depends on the difference in applied signal frequency and the filter cut off frequency. The set of expressions developed in this report are primarily intended to be used in selecting programming time delays in computer based signal measurement and processing systems.

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Introduction

One of the characteristics of a processor controlled measurement system that must be accounted for is the stability of the signal being sampled. A finite time interval exists between the time an input signal is applied to a circuit and the time at which the circuit output signal has stabilized. Therefore a time delay must be built into the measurement algorithm to allow all transients to settle within allowable error limits before collecting waveform data. This research is directed toward deriving expressions to describe the minimum delay time necessary in order to collect valid waveform data.

Many electrical circuits operate practically instantaneously, but circuits with reactive components, such as filters, require a short, measurable time before their output signals approach steady state value. It is well known that filters contribute to this settling time delay in two ways: phase delay and transient decay time. Phase delay is usually well defined by the filter type (Bessel, Butterworth, etc.), and can be derived using Laplace transform techniques or obtained from tables on filter functions. Derivation of the decay time, or settle time, is a more challenging problem because it involves a rigorous frequency domain analysis and subsequent conversion to the time domain. If the filter is treated as ideal (rectangular function of frequency), a general approximation may be developed which is useful in computing the maximum settling time. For non-ideal filters, it will be shown that the settle time gets progressively faster as the filter order

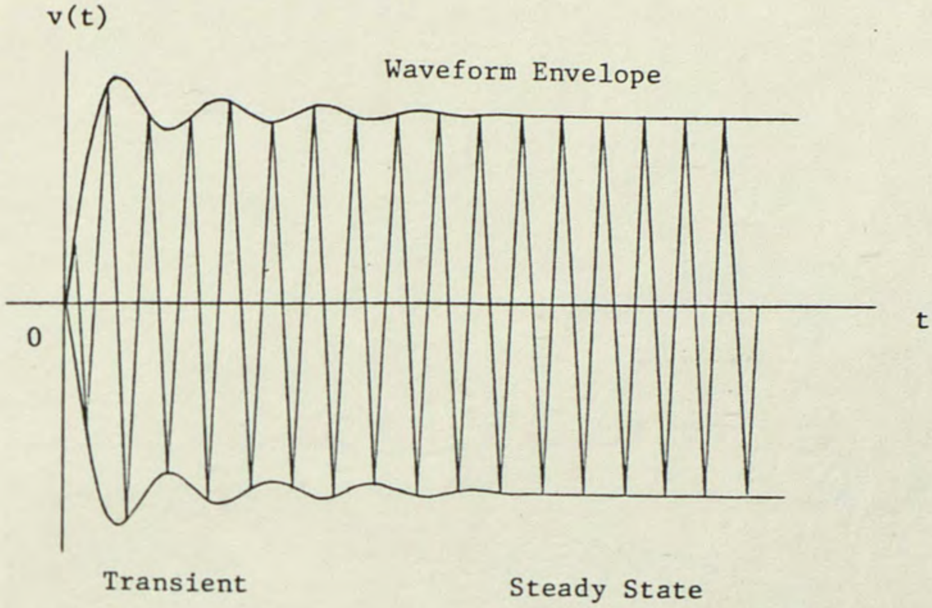


Figure 1. Ideal Filter Response to a Step Modulated Sine Wave.

decreases. Approximations for their settle times may be useful as an improvement in efficiency over the general approximation developed for the ideal filter.

The expressions developed in this report are useful approximations of the settle times required for low-pass, band-pass and high-pass filters with varying degrees of order. It will be shown that the step modulated sine wave causes the filter to react with an amplitude modulation similar to that shown in Figure 1. As the filter order increases, the modulation index increases, and in the limiting case of an ideal filter the envelope assumes the form of a sine integral. The settle time is defined as the delay time necessary for the transient waveform to settle to within a given percentage of steady state. In general the expressions for low-pass, band-pass and high-pass filter settle times are similar and the derivations will show how they are related. The conclusions of this research can be summed up as follows:

1. The band-pass filter takes more time to settle than the low-pass or high-pass filters.
2. A lower acceptable error requires longer settling time.
3. As the difference between the applied signal frequency and the filter cut off frequency decreases, the settling time increases.

The technical analysis presented in this report will begin with Chapter One which covers the Butterworth low-pass filter settle time derivation. It will be shown that a key quadratic filter section dominates the transfer function, and that it can be used to approximate the settle time expression. This key filter section principle can be extended to apply to many other filter types. It will also be shown that as the filter order increases, a trend will become apparent which will, in the limit, lead to a settle time function for a very high order filter. Chapter Two continues this trend with the analysis of an ideal filter which has, in theory, an order of infinity.

The analytical results presented in this report are all derived using Laplace and Fourier transforms and well known mathematical techniques. The derivation of settle time expressions is done slightly differently in each chapter for ease of obtaining a solution and presentation of the results. In Chapter One, Laplace transforms are used to derive the transfer functions for low order filters.

In Chapter Two, the ideal filter characterization is much easier to solve using Fourier transforms due to the double sided frequency arguments associated with the exponential integral.

Several low-pass filter circuits were designed and analyzed using SPICE, a circuit analysis program, on a Digital Equipment Corporation VAX 11/780 computer. One of the circuits, an 8-pole low-pass Butterworth filter, was constructed to verify the analytical and computer analysis. For the convenience of oscilloscope display a special module was designed and constructed to produce the modulating function $u(t)A\sin 2\pi ft$. These results are presented in Chapter Three with the comparisons, results, and conclusions.

CHAPTER 1

DERIVATION OF SETTLE TIME EXPRESSIONS FOR LOW ORDER FILTERS

In this chapter, several common filter transfer functions will be evaluated and approximations to the settle time expressions made. The unilateral, or one-sided, Laplace transforms and inverse transform techniques (Van Valkenburg 1974) will be used to solve for the steady state and the transient expressions. It will be shown that the settle time expression is a function of the transient portion of the inverse Laplace transform. The relationship between filter transfer functions and the transient response will be developed and reduced to algebraic approximations which can be used to describe the settle time.

In theory, the filter is always approaching steady state but never quite gets there. For practical analysis, a given error term (which will be referred to as ϵ in this report) will be used in the settle time expressions to designate the maximum difference that can be tolerated between the steady state response and the transient response. For a typical value of $\epsilon = 0.01$, the transient waveform at that particular time cannot contribute greater than a 1% error in the total amplitude measurement.

The Butterworth filter transfer function was selected for this analysis because it is one of the most basic of all filter types available. Other filter transfer functions may be analyzed using the same techniques that will be presented for the Butterworth filter. However, it will be shown that for filters with two or more quadratic sections, one of the sections will dominate the transient solution. This particular quadratic section will be referred to as the key section, which almost all higher order filter types will have in common with the Butterworth analysis. Therefore, at least to a good approximation, this analysis will be general enough to apply to most filter types.

Low-Pass Butterworth Filter Transfer Functions

A normalized n-order Butterworth filter is characterized in the s-plane by 2n equally spaced poles on a radius about the origin. The poles that lie in the left half s-plane are mirror images of the poles that lie in the right half s-plane. Even order filters do not have poles on the σ-axis, so with the exception of the first order filter, it will be convenient to work exclusively with even ordered filter transfer functions. The 8-pole filter s-plane map shown in Figure 2 depicts pole pairs symmetrical about the + and − σ-axis. Each pair can be characterized with a quadratic expression as follows:

$$[1] \quad TFq(s) = \frac{1}{s^2 + s/Q + 1}$$

where $Q = 1/(2\cos\Theta)$

and Θ can take on one of the angles: 11.25°, 33.75°, 56.25°, or 78.75°

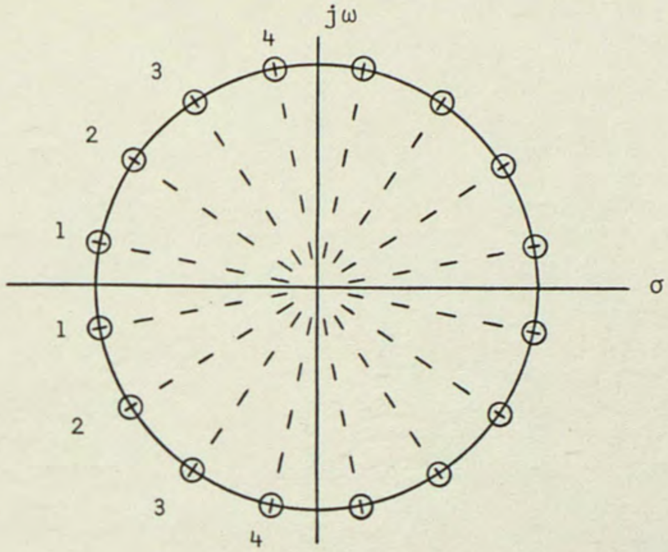


Figure 2. S-Plane Map of an 8-Pole Butterworth Filter.

The complete normalized transfer function is made up of the product of the four sections, each using a different value for Θ. Since Θ is a function of filter order and section number, a general equation can be given to describe the transfer function of any even ordered Butterworth filter:

$$[2] \qquad TF(s) = \prod_{i=1}^{n/2} \left[s^2 + 2s\cos\pi\frac{2i-1}{2n} + 1 \right]^{-1}$$

From this equation, it can be seen that the coefficients of s are the variables that characterize the Butterworth filter. For the normalized transfer function, these coefficients are the reciprocal of the Q's, which establish each filter section quality factor. The trend from low order to higher order filter section Q's can be observed by inspection of the calculated values for each quadratic section shown in Table 1.

TABLE 1

BUTTERWORTH FILTER QUALITY FACTORS

Filter Order	Q	Q	Q	Q
2-Pole	0.70711			
4-Pole	0.54120	1.30656		
6-Pole	0.51763	0.70711	1.93185	
8-Pole	0.50980	0.60135	0.89998	2.56292
16-Pole	0.50242	0.52250	0.56694	0.64682
	0.78816	1.06068	1.72245	5.10115
32-Pole	0.50060	0.50547	0.51545	0.53104
	0.55310	0.58293	0.62250	0.67481
	0.74454	0.85935	0.97257	1.16944
	1.48416	2.05778	3.40761	10.19000

For higher order filters, the first quadratic section Q approaches 0.5, and the last quadratic section Q approaches n/π.

Analyses of General Low-Pass Filters

In this section, the frequency domain expressions will be developed for 1, 2, and 4 pole low-pass filters. Using Laplace transform techniques, the time domain and settle time expressions will be derived. Since the time domain solutions quickly become very complex, only filters of order 4 and less will be evaluated.

General First Order Low-Pass Filter

The Laplace transform of the filter transfer function is given as

$$H(s) = \frac{\alpha}{s + \alpha}$$

The Laplace transform of the step modulated sine wave is given as

$$Vi(s) = L\{u(t)\sin\beta t\} = \frac{\beta}{s^2 + \beta^2}$$

The frequency domain output function is therefore given as

$$[3] \qquad Vo(s) = Vi(s)H(s) = \frac{\beta}{s^2 + \beta^2} \cdot \frac{\alpha}{s + \alpha}$$

Where α = filter cut off frequency

and β = sine wave frequency

Using partial fraction expansions

$$Vo(s) = \alpha\beta \left[\frac{1}{s + j\beta} \cdot \frac{1}{s - j\beta} \cdot \frac{1}{s + \alpha} \right]$$

The residues may be expressed as follows:

$$Vo(s) = \frac{R1}{s + j\beta} + \frac{R1^*}{s - j\beta} + \frac{R2}{s + \alpha}$$

Where

$$R1 = -\frac{1}{2} \cdot \frac{\alpha}{\beta + j\alpha}$$

$$R1^* = -\frac{1}{2} \cdot \frac{\alpha}{\beta - j\alpha}$$

$$R2 = \frac{\alpha\beta}{\alpha^2 + \beta^2}$$

Transforming to the time domain,

$$vo(t) = R1\ \exp(-j\beta) + R1^{*}\ \exp(j\beta) + R2\ \exp(-\alpha)$$

Which gives the time domain solution

$$[4] \quad vo(t) = \left[1 + \frac{\beta^2}{\alpha^2} \right]^{-\frac{1}{2}} \sin(\beta t + \Theta) + \frac{\alpha\beta}{\alpha^2 + \beta^2} \exp(-\alpha t)$$

Where $\Theta = -\arctan(\beta/\alpha)$

The steady state and transient solutions for vo(t) can be expressed as

$$vo(t) = S \sin\beta t + T \exp(-\alpha t)$$

Which, for the convenience of future analyses, will be referred to as

$$vo(t) = vo_1(t) + vo_2(t)$$

The error term $\epsilon(t)$ is defined as the transient solution $vo_2(t)$. Therefore, whenever a transient condition occurs, $\epsilon(t)$ describes the behavior of the output signal which decays to zero shortly after the input signal achieves a steady value. Note that whenever the sine wave is abruptly changed, another transient will be generated. The reactive nature of the filter is such that whenever the amplitude or frequency changes from one steady state value to another, a transient waveform is generated which will require a short time to settle.

An expression for the error versus time is given as

$$\epsilon(t) = \frac{\alpha\beta}{\alpha^2 + \beta^2} \exp(-\alpha t)$$

Rearranging, and solving for the settle time ts

$$[5] \qquad ts = -\frac{1}{\alpha} \ln \epsilon\left[\frac{\alpha}{\beta} + \frac{\beta}{\alpha} \right] \qquad ; \beta < \alpha$$

Where ts is the time required for the filter output waveform to settle to a value within the error term ϵ of steady state.

Equation 5 shows that the settle time for a 1-pole filter is a natural log function of the maximum error allowed and the input sine wave frequency β.

General Second Order Low-Pass Filter

The Laplace transform of the filter transfer function is given as

$$H(s) = \frac{\alpha^2}{s^2 + s(\alpha/Q) + \alpha^2}$$

The frequency domain output function is therefore given as

$$[6] \quad Vo(s) = \frac{\beta}{s^2 + \beta^2} \cdot \frac{\alpha^2}{s^2 + s(\alpha/Q) + \alpha^2}$$

Where α = filter cut off frequency

and β = sine wave frequency

The residues to be solved for are

$$Vo(s) = \frac{R1}{s + j\beta} + \frac{R1^*}{s - j\beta} + \frac{R2}{s + \frac{\alpha}{2Q} + j\alpha\left[1 - \frac{1}{4Q^2}\right]^{\frac{1}{2}}}$$

$$+ \frac{R2^*}{s + \frac{\alpha}{2Q} - j\alpha\left[1 - \frac{1}{4Q^2}\right]^{\frac{1}{2}}}$$

R1 and R2 are

$$R1 = -\frac{1}{2} \cdot \frac{\alpha^2 Q}{\alpha\beta + jQ(\alpha^2 - \beta^2)}$$

$$R2 = \frac{1}{2} \cdot \left[\frac{\alpha}{Q\beta}\left[1 - \frac{1}{4Q^2}\right] + j\frac{\alpha}{\beta}\left[1 - \frac{1}{2Q^2} - \frac{\beta^2}{\alpha^2}\right]\left[1 - \frac{1}{4Q^2}\right]^{\frac{1}{2}}\right]^{-1}$$

R1* and R2* are the complex conjugates of R1 and R2 which can be determined by inspection.

Transforming to the time domain and letting $vo(t) = vo_1(t) + vo_2(t)$, the steady state solution is given as $vo_1(t)$.

$$[7] \quad vo_1(t) = \left[\frac{\beta^2}{Q^2\alpha^2} + \left[\frac{\alpha^2 - \beta^2}{\alpha^2} \right]^2 \right]^{-\frac{1}{2}} \sin(\beta t + \Theta)$$

Where $\Theta = \arctan \dfrac{\alpha\beta}{Q(\alpha^2 - \beta^2)}$

The transient solution is given as $vo_2(t)$.

$$[8] \quad vo_2(t) = T \exp\left[- \frac{\alpha t}{2Q} \right] \sin\left[\alpha\left[1 - \frac{1}{4Q^2} \right]^{\frac{1}{2}} t + \Phi \right]$$

Where $T = \left[\dfrac{\alpha^2}{Q^2\beta^2}\left[1 - \dfrac{1}{4Q^2} \right]^2 + \dfrac{\alpha^2}{\beta^2}\left[1 - \dfrac{1}{2Q^2} - \dfrac{\beta^2}{\alpha^2} \right]^2 \left[1 - \dfrac{1}{4Q^2} \right] \right]^{-\frac{1}{2}}$

and $\Phi = \arctan \left[\dfrac{\left[1 - \dfrac{1}{4Q^2} \right]^{\frac{1}{2}}}{\dfrac{\beta^2}{\alpha^2} - \dfrac{1}{2Q^2} - 1} \right]$

From equation 8, it is evident that the transient portion of the solution is an exponentially decaying sine wave. The rate of decay and the frequency of the sine wave are both functions of the filter section Q. For the Butterworth filter, the filter section natural frequency, ωn, is equal to 0.7071α, where α is the filter cut off frequency.

The error term is approximated by the transient solution, where $\epsilon(t) = vo_2(t)$, which drives toward zero shortly after application of the sine wave. The maximum error may be expressed under the condition that the sine term is set equal to one. With this assumption, the settle time, ts, can be expressed as

$$[9] \quad ts = -\frac{2Q}{\alpha} \ln \frac{\epsilon}{T} \qquad ; \beta < \alpha$$

Where ts is the time required for the filter output waveform to settle to a value within the error term ϵ of steady state value.

This result, which is similar to the simple low-pass filter, shows that the settle time is a log function of the maximum error term ϵ and the input frequency. $T(max) = 1$ for $\alpha = \beta$, but T decreases toward zero as the applied frequency increases toward or decreases away from the cut off frequency.

The trigonometric relationship

$$[10] \quad \sin(x) + \sin(y) = 2 \sin(\tfrac{1}{2}(x + y))\cos(\tfrac{1}{2}(x - y))$$

can be used to illustrate the amplitude modulation that takes place while the transient is large enough to be observed. For the 2-pole, 1-KHz Butterworth filter, note that when driven at the cut off frequency the "sum" frequency is 854 Hz and the "difference" frequency is 146 Hz. If the same filter is driven at 707 Hz, which is $\omega n/2\pi$, the "sum" frequency is 707 Hz, and the "difference" frequency is zero. For this latter case in which β is equal to the quadratic filter section natural frequency, there will be no apparent amplitude modulation. The transient response will be an exponentially decaying level.

General Fourth Order Low-Pass Filter

The Laplace transform of the filter transfer function is given as

$$H(s) = \frac{\alpha_1^2}{s^2 + s(\alpha_1/Q_1) + \alpha_1^2} \cdot \frac{\alpha_2^2}{s^2 + s(\alpha_2/Q_2) + \alpha_2^2}$$

The frequency domain output function is therefore given as

$$[12] \quad Vo(s) = \frac{\beta}{s^2 + \beta^2} \cdot \frac{\alpha_1^2}{s^2 + s(\alpha_1/Q_1) + \alpha_1^2} \cdot \frac{\alpha_2^2}{s^2 + s(\alpha_2/Q_2) + \alpha_2^2}$$

Where α_1 = First filter section cut off frequency

α_2 = Second filter section cut off frequency

β = sine wave frequency

The residues to be solved for are

$$Vo(s) = \frac{R1}{s + \frac{\alpha_1}{2Q_1} + j\alpha_1\left[1 - \frac{1}{4Q_1^2}\right]^{\frac{1}{2}}} + \frac{R1^*}{s + \frac{\alpha_1}{2Q_1} - j\alpha_1\left[1 - \frac{1}{4Q_1^2}\right]^{\frac{1}{2}}}$$

$$+ \frac{R2}{s + \frac{\alpha_2}{2Q_2} + j\alpha_2\left[1 - \frac{1}{4Q_2^2}\right]^{\frac{1}{2}}} + \frac{R2^*}{s + \frac{\alpha_2}{2Q_2} - j\alpha_2\left[1 - \frac{1}{4Q_2^2}\right]^{\frac{1}{2}}}$$

$$+ \frac{R3}{s + j\beta} + \frac{R3^*}{s - j\beta}$$

R1, R2, and R3 for the general 4-pole filter are

$$R1 = \frac{\beta}{2} \cdot \frac{\alpha_2^2}{\alpha_1^2}\left[\frac{\alpha_1}{Q_1}\left[1 - \frac{1}{4Q_1^2}\right] + j\alpha_1\left[1 - \frac{1}{4Q_1^2}\right]^{\frac{1}{2}}\left[1 - \frac{1}{2Q_1^2} - \frac{\beta^2}{\alpha_1^2}\right]\right]^{-1}$$

$$\cdot \left[\frac{1}{2Q_1}\left[\frac{1}{Q_1} - \frac{1}{Q_2}\right] + \frac{j}{Q_1}\left[1 - \frac{1}{4Q_1^2}\right]^{\frac{1}{2}} - \frac{j}{Q_2}\left[1 - \frac{1}{4Q_2^2}\right]^{\frac{1}{2}}\right]^{-1}$$

$$R2 = \frac{\beta}{2} \cdot \frac{\alpha_1^2}{\alpha_2^2} \left[\frac{\alpha_2}{Q_2}\left[1 - \frac{1}{4Q_2^2} \right] + j\alpha_2\left[1 - \frac{1}{4Q_2^2} \right]^{\frac{1}{2}} \left[1 - \frac{1}{2Q_2^2} - \frac{\beta^2}{\alpha_2^2} \right] \right]^{-1}$$

$$\cdot \left[\frac{1}{2Q_1}\left[\frac{1}{Q_1} - \frac{1}{Q_2} \right] + \frac{j}{Q_2}\left[1 - \frac{1}{4Q_2^2} \right]^{\frac{1}{2}} - \frac{j}{Q_1}\left[1 - \frac{1}{4Q_1^2} \right]^{\frac{1}{2}} \right]^{-1}$$

$$R3 = -\frac{1}{2} \cdot \left[\left[1 - \frac{\beta^2}{\alpha_1^2} \right] - j\frac{\beta}{2Q_1} \right]^{-1} \cdot \left[\left[1 - \frac{\beta^2}{\alpha_2^2} \right] - j\frac{\beta}{2Q_2} \right]^{-1}$$

R1*, R2*, and R3* are the complex conjugates of R1, R2, and R3, which can be determined by inspection.

The residues can be simplified by letting some of the filter related variables take on the values for a Butterworth filter. The following residue equations are valid for the Butterworth filter only.

$$R1 = \frac{\beta}{2} \cdot \left[\frac{\alpha}{Q_1}\left[1 - \frac{1}{4Q_1^2} \right] + j\alpha\left[1 - \frac{1}{4Q_1^2} \right]^{\frac{1}{2}} \left[1 - \frac{1}{2Q_1^2} - \frac{\beta^2}{\alpha^2} \right] \right]^{-1}$$

$$R2 = \frac{\beta}{2} \cdot \left[\frac{\alpha}{Q_2}\left[1 - \frac{1}{4Q_2^2} \right] + j\alpha\left[1 - \frac{1}{4Q_2^2} \right]^{\frac{1}{2}} \left[1 - \frac{1}{2Q_2^2} - \frac{\beta^2}{\alpha^2} \right] \right]^{-1}$$

$$R3 = -\frac{1}{2} \cdot \frac{\alpha^2 Q_1}{\alpha\beta + jQ_1(\alpha^2 - \beta^2)} \cdot \frac{\alpha^2 Q_2}{\alpha\beta + jQ_2(\alpha^2 - \beta^2)}$$

Transforming to the time domain and letting $vo(t) = vo_1(t) + vo_2(t) + vo_3(t)$, the steady state solution is given as $vo_3(t)$.

[13]

$$vo_3(t) = \left[\frac{\beta^2}{Q_1^2\alpha^2} + \left[\frac{\alpha^2 - \beta^2}{\alpha^2} \right]^2 \right]^{-\frac{1}{2}} \left[\frac{\beta^2}{Q_2^2\alpha^2} + \left[\frac{\alpha^2 - \beta^2}{\alpha^2} \right]^2 \right]^{-\frac{1}{2}} \sin(\beta t + \Theta)$$

Where

$$\Theta = \arctan\left[\frac{\alpha\beta}{\alpha^2 - \beta^2}\left[\frac{1}{Q_1} - \frac{1}{Q_2} \right] \right]$$

The transient solutions are given as $v_{o1}(t)$ and $v_{o2}(t)$.

$$[14] \quad v_{o1}(t) = T_1 \exp\left[-\frac{\alpha t}{2Q_1} \right] \sin\left[\alpha\left[1 - \frac{1}{4Q_1^2} \right]^{\frac{1}{2}} t + \Phi_1 \right]$$

$$[15] \quad v_{o2}(t) = T_2 \exp\left[-\frac{\alpha t}{2Q_2} \right] \sin\left[\alpha\left[1 - \frac{1}{4Q_2^2} \right]^{\frac{1}{2}} t + \Phi_2 \right]$$

Where

$$T_1 = \left[\frac{\alpha^2}{Q_1^2\beta^2}\left[1 - \frac{1}{4Q_1^2} \right]^2 + \frac{\alpha^2}{\beta^2}\left[1 - \frac{1}{2Q_1^2} - \frac{\beta^2}{\alpha^2} \right]^2 \left[1 - \frac{1}{4Q_1^2} \right] \right]^{-\frac{1}{2}}$$

$$\Phi_1 = \arctan\left[\frac{\left[1 - \frac{1}{4Q_1^2} \right]^{\frac{1}{2}}}{\frac{\beta^2}{\alpha^2} - \frac{1}{2Q_1^2} - 1} \right]$$

$$T_2 = \left[\frac{\alpha^2}{Q_2^2\beta^2}\left[1 - \frac{1}{4Q_2^2} \right]^2 + \frac{\alpha^2}{\beta^2}\left[1 - \frac{1}{2Q_2^2} - \frac{\beta^2}{\alpha^2} \right]^2 \left[1 - \frac{1}{4Q_2^2} \right] \right]^{-\frac{1}{2}}$$

$$\Phi_2 = \arctan\left[\frac{\left[1 - \frac{1}{4Q_2^2} \right]^{\frac{1}{2}}}{\frac{\beta^2}{\alpha^2} - \frac{1}{2Q_2^2} - 1} \right]$$

From equations 14 and 15, it is evident that the transient portion of the solution is an exponentially decaying sine wave. The rate of decay and the frequency of the sine wave are both functions of the filter section Q. For the Butterworth filter the natural frequency, ωn, is equal to 0.381α for the first filter section and 0.924α for the second filter section (α is the filter cut off frequency).

The error term is approximated by the transient solution, where $\epsilon(t) = vo_1(t) + vo_2(t)$, which drives toward zero shortly after application of the sine wave. The maximum error may be expressed under the condition that the sine terms are set equal to one. With this assumption the settle time, ts, can be expressed as

$$[16] \quad \epsilon(t) = T_1 \exp\left[- \frac{\alpha t}{2Q_1} \right] + T_2 \exp\left[- \frac{\alpha t}{2Q_2} \right]$$

For the 4-pole Butterworth filter, $vo_2(t)$ will take 2.414 times longer to settle than $vo_1(t)$ because of the difference in Q's. A comparison of T_1 and T_2 at $\alpha = \beta$ shows that $T_1 = T_2 = 1.4142$. Therefore, the dominant term in the settle time solution is $vo_2(t)$, which will be used to approximate the settle time ts.

$$[17] \quad ts = - \frac{2Q_2}{\alpha} \ln \frac{\epsilon}{T_2} \qquad ; \beta < \alpha$$

Where ts is the time required for the filter output waveform to settle to a value within the error term ϵ of steady state value.

This result is similar to the 1 and 2 pole filter settle time expressions, and it can be shown that the approximation holds true for higher order filters as well. In general, the highest Q filter section can be used to approximate the settle time.

The trigonometric identity (equation 10) can be used to illustrate the amplitude modulation that takes place while the transient is large enough to be observed. Note that when the

Butterworth filter is driven at the cut off frequency, the "sum" frequency is 962 Hz and the "difference" frequency is 38 Hz. If the same filter is driven at 962 Hz, which is $\omega n/2\pi$, the "sum" frequency is 962 Hz and the "difference" frequency is zero. For this latter case in which β is equal to the key quadratic filter section natural frequency, there will be no apparent amplitude modulation. The transient response will be an exponentially decaying level.

High Order Low-Pass Filters

Higher order filters can be analyzed using the same techniques as those shown for the second and the fourth order filters. Due to the symmetrical pole pattern of the Butterworth filter, several terms cancel and the Laplace analysis can be carried out with less difficulty. Further simplification can be made by observing the trend of the pole pattern on the σ - $j\omega$ axes.

As the filter order increases, the highest Q pole pair approaches the $j\omega$-axis. In reference to figure 2, the pole pair labeled 4 (closest to the $j\omega$-axis) is the highest Q pole pair for the 8-pole low-pass Butterworth filter. This pair is implemented in the filter with a quadratic transfer function. Due to its high Q, this key section dominates the settle time expression and the other three sections become relatively insignificant. Using this principle, high order filter settle times may be approximated by considering the filter transfer function characteristics of the highest Q section only. With slight modification, equation 17 is adequate to describe the settle time for most filters.

Another interesting characteristic of higher order filters is that as the filter order increases, the natural frequency of the highest Q section tends toward the cut off frequency. This should be expected as it leads in the limit to the ideal filter characteristics.

$$\lim_{n \to \infty} \alpha \left[1 - \frac{1}{4Q_m^2} \right]^{\frac{1}{2}} = \alpha$$

Summary

The results of this chapter are engineering approximations which do not include phase information, filter delay, or lower order effects. These results are intended to be used for approximating the settle time required for signal processing applications. Equations 18, 19 and 20 sum up the results for the low-pass filters evaluated in this chapter.

$$[18] \quad ts = -\frac{2Qm}{\alpha} \ln \frac{\epsilon}{T} \qquad ; \beta < \alpha$$

$$[19] \quad T = \left[\frac{\alpha^2}{Qm^2 \beta^2} \left[1 - \frac{1}{4Qm^2} \right]^2 + \frac{\alpha^2}{\beta^2} \left[1 - \frac{1}{2Qm^2} - \frac{\beta^2}{\alpha^2} \right]^2 \left[1 - \frac{1}{4Qm^2} \right] \right]^{-\frac{1}{2}}$$

$$[20] \quad \omega n = \alpha \left[1 - \frac{1}{4Qm^2} \right]^{\frac{1}{2}}$$

Where α = Filter section cut off frequency

ωn = Filter section natural frequency

Qm = Highest filter section quality factor

Settle time expressions can be developed for the band-pass and high-pass filters using a similar technique. Equations 21 to 23 were developed for Butterworth filters of second order and higher and are presented to show how the different filter types are related. Equation 21 must be used in place of equation 19 to describe the settle time required for a band-pass filter. Note that the differences between equations 19 and 21 are the factor 4 in both terms and the factor Q^2 in the second term. The Q^2 term establishes the bandwidth dependency of the T-term for the band-pass filter.

$$[21] \quad T = \left[\frac{4\alpha^2}{Qm^2\beta^2}\left[1 - \frac{1}{4Qm^2} \right]^2 + \frac{4Q^2\alpha^2}{\beta^2}\left[1 - \frac{1}{2Qm^2} - \frac{\beta^2}{\alpha^2} \right]^2 \left[1 - \frac{1}{4Qm^2} \right]\right]^{-\frac{1}{2}}$$

Equation 18 may be used for the high-pass filter, but equations 19 and 20 must be replaced with equations 22 and 23 to accurately describe the settle time. Equation 18 is shown again for convenience. High-pass filter settle time expression:

$$[18] \quad ts = -\frac{2Qm}{\alpha} \ln \frac{\epsilon}{T} \qquad ; \text{ valid for } \beta \text{ within the passband.}$$

$$[22] \quad T = \left[\frac{\beta^2}{Qm^2\alpha^2}\left[1 - \frac{1}{4Qm^2} \right]^2 + \frac{\beta^2}{\alpha^2}\left[1 - \frac{1}{2Qm^2} - \frac{\alpha^2}{\beta^2} \right]^2 \left[1 - \frac{1}{4Qm^2} \right]\right]^{-\frac{1}{2}}$$

$$[23] \quad \omega n = \alpha\left[1 - \frac{1}{4Qm^2} \right]^{-\frac{1}{2}}$$

The maximum value of T, Tmax, occurs when the filter is driven at its cut off frequency, where $\alpha = \beta$, and is observed to approach Qm for high orders of Butterworth filters. T rapidly drops off on either side of the filter cut off frequency and approaches zero for very high and very low frequencies. Table 2 shows some representative values of Qmax, ωn, and Tmax for several different orders of filters.

TABLE 2

BUTTERWORTH FILTER VALUES FOR Qmax, ωn, and Tmax

Filter Order	Qmax	ωn	Tmax
2	0.70711	0.707α	1.0000
4	1.30656	0.924α	1.4142
6	1.93185	0.966α	2.0000
8	2.56292	0.981α	2.6131
16	5.10115	0.995α	5.1258
32	10.1900	0.999α	10.202

If the assumptions under which equations 18 to 23 were developed are satisfied, it can be seen that the settle time is a logarithmic function the error term ϵ and T. The T term is dependent on the difference of frequency between the applied frequency β and the filter cut off frequency α. Since the band-pass filter effectively has two cut off frequencies, the settle time will be longer than that for a low-pass or high-pass filter. This increase in ts is dependent on the key filter section Q which determines the bandwidth. Table 3 lists some representative settle times for 1-KHz low-pass Butterworth filters from order 2 to order 16.

TABLE 3

1-KHZ LOW-PASS BUTTERWORTH FILTER SETTLE TIMES FOR ϵ = 1%

Frequency	2-Pole	4-Pole	8-Pole	16-Pole
20 Hz	0.234 ms	0.321 ms	0.58 ms	1.14 ms
50	0.441	0.703	1.33	2.62
100	0.597	0.993	1.90	3.76
200	0.753	1.291	2.49	4.94
500	0.952	1.732	3.42	6.81
900	1.035	2.044	4.44	9.51
1000	1.036	2.060	4.54	10.13

Note: The settle times for very high order filters are best approximated near the cut off frequency. Negative settle times can result for very low frequencies, which is invalid information. There is a transition region at moderately low frequencies at which the settle time expression for the ideal low-pass filter will yield more accurate results. This error results from use of the formulas for conditions that are not within the assumptions made for the approximations. Phase information was neglected in order to make the engineering simplifications necessary to develop the settle time formulas. Therefore, for very low frequencies, the use of the ideal settle time equations in Chapter Two are recommended.

CHAPTER 2

DERIVATION OF SETTLE TIME EXPRESSIONS FOR IDEAL FILTERS

In Chapter 2, low order low-pass filters were analyzed and generalizations made to approximate the settle time behavior as the filter order was increased to high values. In this chapter the trend is continued to the limiting condition of an ideal filter. Conversion of low-pass filter results to band-pass and high-pass filter results is easily accomplished during the analysis by changing the limits of integration. Therefore, the final results of this chapter will include settle time expressions for the band-pass and high-pass filters. A comparison of these three filter types will verify that the low-pass filter analysis can be used to approximate the settle times for band-pass and high-pass filters.

The analysis of the ideal low-pass filter was accomplished using Fourier transforms and inverse transforms (Van Valkenburg 1974) in order to preserve the two-sided frequency spectrum. The reason for this will become evident in the section dealing with the evaluation of the exponential integral. Fourier transformation shows more clearly that the cosine component of the exponential integral cancels and that it therefore simplifies to the sine integral.

A brief review of the concepts of this problem can be summed up as follows. Upon application of a step modulated sine wave $u(t)A\sin\omega t$, if ω is within the filter passband, the filter output will be an amplitude modulated version of the input signal delayed in time by some filter factor $\tau(\omega)$. (For the theoretical results presented in this chapter $\tau(\omega)$ is set equal to zero). This amplitude modulation is described by the sine integral, which decays after a short time to a steady state value. In theory, the filter is always approaching steady state but never quite gets there. For practical analysis, a given error term ϵ will be used in the settle time expressions to designate the maximum difference that can be tolerated between the steady state response and the transient response.

Fourier Transform of the Step Modulated Sine Wave

The input signal can be expressed as $vi(t) = u(t)A\sin\beta t$, where β will be used to replace ω for this analysis. Since the filtering function is more easily represented in the frequency domain, $Vi(\omega)$ will be derived. As shown in the development of equation 21, $Vi(\omega)$ is the convolution of the Fourier transforms of the unit step function $u(t)$ and the sine function $\sin\beta t$. The amplitude A of the sine wave will be set equal to one for the analyses in this chapter. The resulting expression for $Vi(\omega)$ will be used as an input signal to the filters.

$$Vi(\omega) = F\{u(t)\} * F\{\sin(\beta t)\}$$

After performing the convolution, the resulting expression for $Vi(\omega)$ is

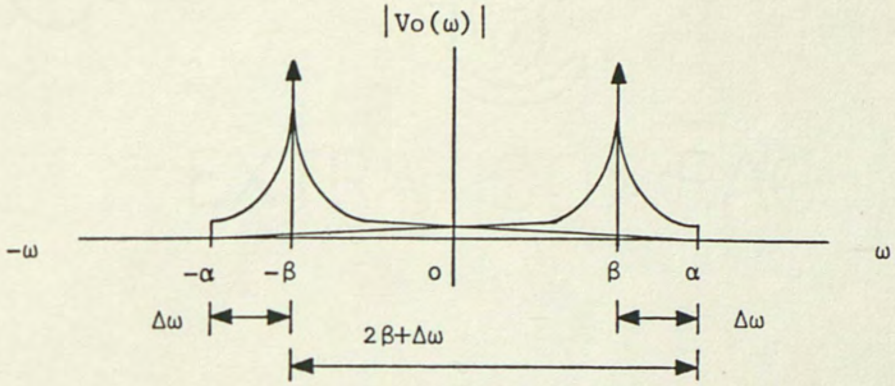
$$[24] \quad Vi(\omega) = j\frac{\pi}{2}\left[\ \delta(\omega+\beta) - \delta(\omega-\beta)\ \right] + \frac{1}{2}\left[\ \frac{1}{\omega+\beta} - \frac{1}{\omega-\beta}\ \right]$$

<u>Filter Output Expressions</u>

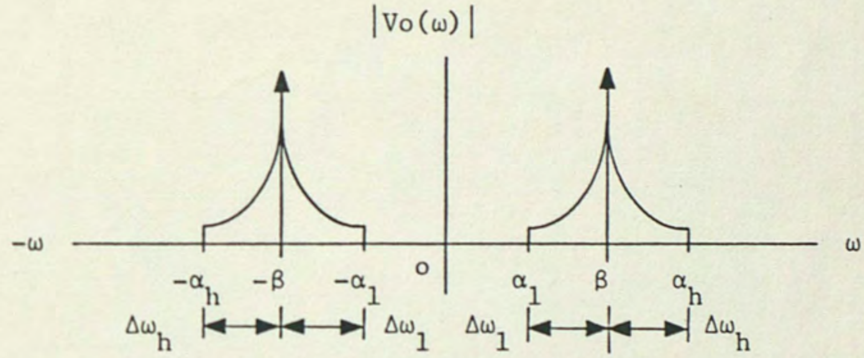
The low-pass filter (LPF) frequency domain output signal $Vo(\omega)$ is a band limited form of $Vi(\omega)$. The output signal in the time domain is the inverse Fourier Transform of $Vo(\omega)$, which is easily carried out using the transform integral. It will be shown that for the low-pass filter, $vo_l(t)$ is an amplitude modulated version of the input signal $vi(t)$. Similarly, for the band-pass and high-pass filters, $vo_b(t)$ and $vo_h(t)$ are also amplitude modulated versions of the input signal.

In order to develop the output signal expressions for the band-pass (BPF) and high-pass filters (HPF), the limits of integration for the transform integral must be changed. The spectral response of the three filter types is shown graphically in Figure 3. For the derivation of equations 22, 23 and 24, refer to Figure 3 which shows how the filter cut off frequencies correspond to the integration limits.

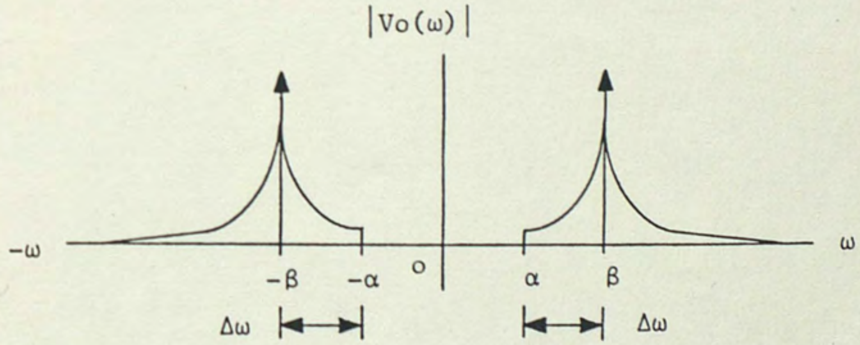
The upper limit $\Delta\omega t$ of the sine integral determines the settle time, where $\Delta\omega$ represents the difference in frequency between $\sin\beta t$ and the filter cut off frequency α. For the band-pass filter, the limits split into two parts, one corresponding to the high frequency $\Delta\omega_h t$ and the other corresponding to the low frequency $\Delta\omega_l t$.



Low-Pass Filter



Band-Pass Filter



High-Pass Filter

Figure 3. Filter Spectral Response Functions to u(t)sinβt.

The filter transfer functions are given as

$$\text{LPF} \quad H_l(\omega) = u(\omega+\beta+\Delta\omega) - u(\omega-\beta-\Delta\omega)$$

$$\text{BPF} \quad H_b(\omega) = u(\omega+\beta+\Delta\omega_h) - u(\omega+\beta-\Delta\omega_l) + u(\omega-\beta+\Delta\omega_l) - u(\omega-\beta-\Delta\omega_h)$$

$$\text{HPF} \quad H_h(\omega) = u(-\omega+\beta-\Delta\omega) + u(\omega-\beta+\Delta\omega)$$

Low-Pass Filter Time Domain Solution

Using $Vo(\omega) = H(\omega)\ Vi(\omega)$

$$Vo_l(\omega) = j\frac{\pi}{2}\left[\ \delta(\omega+\beta) - \delta(\omega-\beta)\ \right] + \frac{1}{2}\left[\ \frac{1}{\omega+\beta} - \frac{1}{\omega-\beta}\ \right]_{-\beta-\Delta\omega\ <\ \omega\ <\ \beta+\Delta\omega}$$

Then, transforming to the time domain and using appropriate substitution of variables, $vo_l(t)$ is expressed as follows:

$$vo_l(t) = \sin\beta t\left[\ \frac{1}{2} + \frac{1}{2\pi}\cdot\frac{1}{j2}\int_{-\Delta\omega}^{\Delta\omega}\frac{e^{j\omega t}}{\omega}\,d\omega\ +\ \frac{1}{2\pi}\cdot\frac{1}{j2}\int_{-2\beta-\Delta\omega}^{2\beta+\Delta\omega}\frac{e^{j\omega t}}{\omega}\,d\omega\ \right]$$

The exponential integral can be simplified and evaluated by letting $\exp(x) = \cos x + j\sin x$. As can be seen in Figure 4 the evaluation of every set of points on the cosine integral curve cancels when the limits of integration are equal and opposite in sign. Following the same logic, it can be seen that the evaluation of the sine integral is simplified by doubling the integral and setting the lower limit equal to zero.

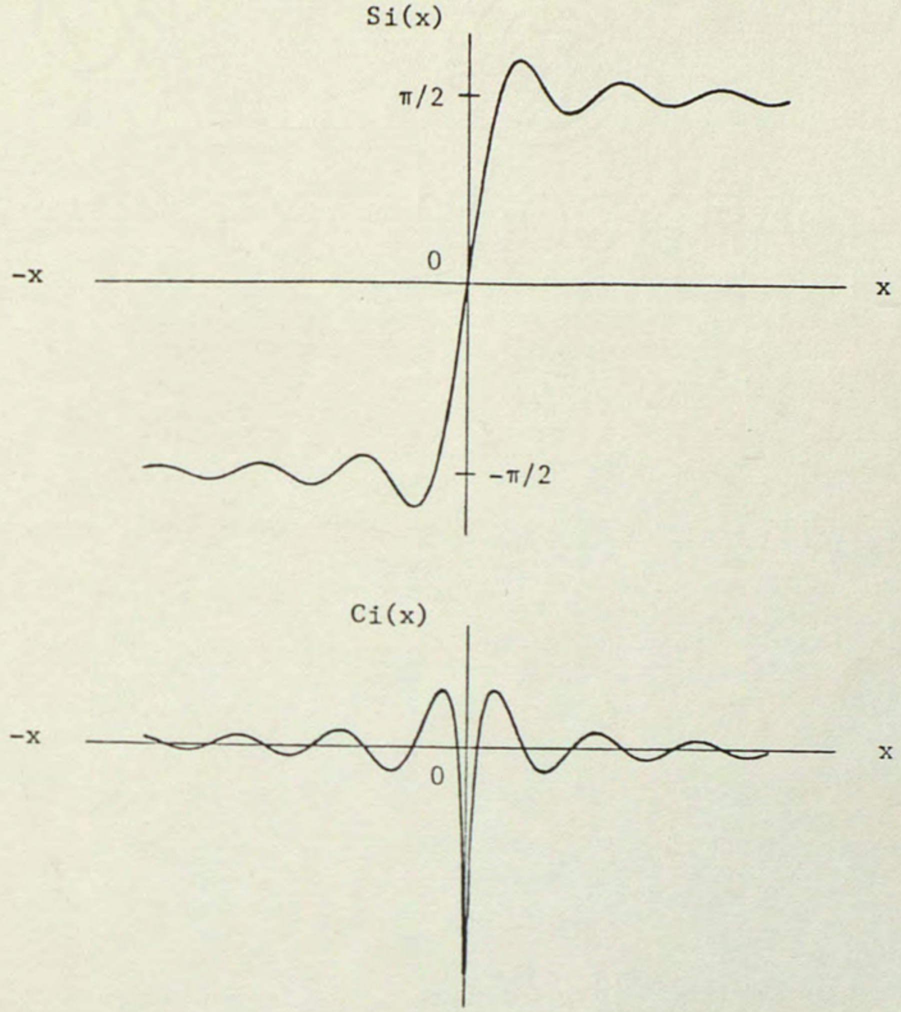


Figure 4. Evaluation of the Sine and Cosine Integrals.

The resulting expression for the output equation is

$$[25] \quad vo_1(t) = \sin\beta t \left[\frac{1}{2} + \frac{1}{2\pi} \int_0^{\Delta\omega t} \frac{\sin\omega t}{\omega}\, d\omega \; + \; j \int_0^{(2\beta+\Delta\omega)t} \frac{\sin\omega t}{\omega}\, d\omega \right]$$

Band-Pass and High-Pass Time Domain Solutions

A derivation similar to the one just presented for the low-pass solution can be used to solve for $vo_b(t)$ and $vo_h(t)$. The results are given as equations 26 and 27.

$$[26]\quad vo_b(t) = \sin\beta t \left[\frac{1}{2} + \frac{1}{2\pi} \int_0^{\Delta\omega_1 t} \frac{\sin\omega t}{\omega} d\omega \; + \; j \int_0^{\Delta\omega_h t} \frac{\sin\omega t}{\omega} d\omega \right]$$

$$[27]\quad vo_h(t) = \sin\beta t \left[\frac{1}{2} + \frac{1}{2\pi} \int_0^{\Delta\omega t} \frac{\sin\omega t}{\omega} d\omega \; + \; j \int_0^{\infty} \frac{\sin\omega t}{\omega} d\omega \right]$$

Each filter output expression may be written as the sum of the steady state solution and the transient solution: $vo(t) = S\ vo_1(t) + T\ vo_2(t)$. The time dependence of the transient solution $T\ vo_2(t)$ is seen in the sine integrals where the upper limit drives the evaluation to a steady state value as t goes to ∞. It will be shown that in the limit as $t \to \infty$ the evaluation of the sine integral approaches $\pi/2$.

Sine Integral Evaluation

Evaluation of the sine integral, Si(x), may be carried out by numerical methods or reference to tabulated values in a book of integrals, such as Abramowitz and Stegun in Handbook of Mathematical Functions, National Bureau of Standards, Applied Math Series 55 (1968). The evaluation in either case does not lead to a readily usable solution, so an effort was made to find an engineering approximation of the sine integral. A close examination of the maxima and minima points on the evaluation curve led to the discovery that the envelope of the evaluated sine integral could be closely approximated by a hyperbolic function. Using this function in equations 25, 26 and 27 led to time domain expressions that can be used for describing the settle time.

A numerical technique was used to evaluate the sine integral so its characterization would not be limited by tabulated values. This was accomplished by using the infinite series representation of $\sin\omega t$ to break down Si(x) into a series of integrals that could easily be evaluated. Then a program was written for the Hewlett Packard 41CV calculator to perform the evaluation. This program is listed for reference in Appendix 1. A summary of evaluated points for Si(x) is given is Table 4, and a graph, which was constructed from a larger set of points, is given in Figure 4.

First, the sine integral is evaluated as a series of integrals:

$$[28] \quad \mathrm{Si}(x) = \int_0^x \frac{\sin\omega}{\omega}\, d\omega = \int_0^x \frac{\omega}{\omega}\, d\omega - \int_0^x \frac{\omega^3}{\omega(3!)}\, d\omega + \int_0^x \frac{\omega^5}{\omega(5!)}\, d\omega - \ldots$$

The evaluation of Si(x) can then be expressed in an equivalent form by reducing the series of integrals in equation 28 to the summation in equation 29 (Abramowitz and Stegun 1968).

$$[29] \quad \mathrm{Si}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)[(2n+1)!]}$$

Numerical evaluation of Si(x) can then be approximated by summation of a significant number of terms in equation 29. Up to 33 terms can be used in the calculator program for the HP41CV which was used to evaluate each limit point x. The graph of the evaluated sine integral presented in Figure 4 was developed from a large number of points collected using this program. (Another calculator program was developed using a similar technique to generate the graph of the cosine integral which was also presented in Figure 4).

The tabulated results listed in Table 4 show the values of the evaluated function at the maxima and minima points. These points occur at integer multiples of π, which can easily be shown by taking the derivative of the sine integral, and solving for sinx equal to zero.

TABLE 4

TABULATED VALUES OF Si(x)

x	Si(x)	x	Si(x)
0	0	6π (18.85)	1.5180339
π (3.14)	1.8519371	7π (21.99)	1.6160855
2π (6.28)	1.4181516	8π (25.12)	1.5311313
3π (9.42)	1.6747617	9π (28.27)	1.6060769
4π (12.57)	1.4921612	10π (31.41)	1.5390291
5π (15.71)	1.6339648	∞	$\pi/2$

The maxima and minima values of the evaluated sine integral represent the maximum amplitude error. Therefore, a smooth curve faired through the absolute value of these points represents the boundary limits of error. In other words, the error in amplitude will always be less than or equal to the envelope of the damped sinusoid.

An expression was then developed for approximating the envelope of the evaluated function Si(x). By plotting x vs $|Si(x) - \pi/2|$ on log-log graph paper it became evident that the function was asymptotically approaching a straight line. The equation of that line corresponded to the hyperbolic function 1/x. Thus, the following expression led to the approximation function shown in equation 30.

$$-\log x = \log\left|Si(x) - \frac{\pi}{2}\right|$$

Therefore

$$[30] \qquad \left|Si(x) - \frac{\pi}{2}\right| \sim \frac{1}{x}$$

The approximation function, given in equation 30, was compared with the evaluated sine integral values given in Table 4. The error analysis results are shown in Table 5 where it can be seen that for large values of x, the error decreases toward zero.

TABLE 5

ERROR ANALYSIS OF THE APPROXIMATION FUNCTION FOR Si(x)

x	$\left\|Si(x) - \frac{\pi}{2}\right\|$	$\frac{1}{x}$	Δ	% error
π	0.2811407	0.3183099	0.0371690	13.22
2π	0.1526447	0.1591550	0.0065102	4.26
3π	0.1039655	0.1061033	0.0021378	2.06
4π	0.0786351	0.0795775	0.0009424	1.20
5π	0.0631685	0.0636620	0.0004935	0.781
6π	0.0527624	0.0530517	0.00028925	0.548
7π	0.0452892	0.0454730	0.00018364	0.405
8π	0.0396650	0.0397887	0.00012374	0.312
9π	0.0352806	0.0353677	0.00008717	0.247
10π	0.0317672	0.0318310	0.00006379	0.201

For an arbitrarily small error ϵ, the function $Si(x) - \pi/2$ may be set equal to $1/x$ to yield the following expression.

$$[31] \quad Si(x) = \frac{\pi}{2} - \frac{1}{x}$$

Filter Settle Time Expressions

Combining equation 31 with equations 25, 26 and 27 yields the following expressions which approximate the ideal filter responses.

$$[32] \quad vo_l(t) = \sin\beta t \left[\frac{1}{2} + \frac{1}{2\pi}\left[\frac{\pi}{2} - \frac{1}{\Delta\omega t} \right] + \frac{1}{2\pi}\left[\frac{\pi}{2} - \frac{1}{2\beta+\Delta\omega t} \right] \right]$$

$$[33] \quad vo_b(t) = \sin\beta t \left[\frac{1}{2} + \frac{1}{2\pi}\left[\frac{\pi}{2} - \frac{1}{\Delta\omega_h t} \right] + \frac{1}{2\pi}\left[\frac{\pi}{2} - \frac{1}{\Delta\omega_l t} \right] \right]$$

$$[34] \quad vo_h(t) = \sin\beta t \left[\frac{1}{2} + \frac{1}{2\pi}\left[\frac{\pi}{2} - \frac{1}{\Delta\omega t} \right] + \frac{1}{2\pi}\left[\frac{\pi}{2} - \frac{1}{\infty} \right] \right]$$

The envelope, $E(t)$, of equations 32, 33 and 34 can be expressed as follows:

$$[35] \quad \text{LPF} \qquad E_l(t) = 1 - \epsilon(t) = 1 - \frac{1}{2\pi\Delta\omega t} - \frac{1}{2\pi(2\beta+\Delta\omega t)}$$

$$[36] \quad \text{BPF} \qquad E_b(t) = 1 - \epsilon(t) = 1 - \frac{1}{2\pi\Delta\omega_h t} - \frac{1}{2\pi\Delta\omega_l t}$$

$$[37] \quad \text{HPF} \qquad E_h(t) = 1 - \epsilon(t) = 1 - \frac{1}{2\pi\Delta\omega t}$$

The error expressions are simply the envelope expressions which can be rearranged as follows:

$$[38] \quad \text{LPF} \qquad \epsilon(t) = \frac{1}{2\pi\Delta\omega t} + \frac{1}{2\pi(2\beta+\Delta\omega t)}$$

$$[39] \quad \text{BPF} \qquad \epsilon(t) = \frac{1}{2\pi\Delta\omega_h t} + \frac{1}{2\pi\Delta\omega_l t}$$

$$[40] \quad \text{HPF} \qquad \epsilon(t) = \frac{1}{2\pi\Delta\omega t}$$

Where $\epsilon(t)$ = the maximum absolute difference between the steady state response and the transient response of the ideal filters.

$\Delta\omega_h$ = filter high frequency cut off - input frequency ($\omega c_h - \beta$)

$\Delta\omega_l$ = input frequency - filter low frequency cut off ($\beta - \omega c_l$)

and $\Delta\omega = |\omega c - \beta|$ for the LPF or the HPF.

Equations 38, 39 and 40 may be used to solve for the settle time of any ideal filter. A close comparison with equations 18 to 23 from Chapter 1 would show that the ideal filter takes longer to settle. In fact, the ideal filter will never settle if the input signal is at the exact same frequency as the filter cut off frequency.

If the applied frequency is near the middle of the passband for the low-pass or the band-pass filter, the following engineering approximations can be made:

$\Delta\omega t \ll 2\beta + \Delta\omega t$

$\Delta\omega_h = \Delta\omega_l$

The settle time equations may then be expressed as

$$[41] \quad \text{LPF and HPF} \qquad ts = \frac{1}{2\pi\epsilon\Delta\omega}$$

$$[42] \quad \text{BPF} \qquad ts = \frac{1}{\pi\epsilon\Delta\omega}$$

If the assumptions under which these approximations were made are satisfied, equations 41 and 42 show that the settle time is a hyperbolic function of $\Delta\omega$. Since the band-pass filter effectively has two cut off frequencies, the settle time is twice as long as that of a high-pass or low-pass filter. Table 6 lists some representative settle times for the ideal low-pass filter.

TABLE 6

IDEAL 1-KHZ LOW-PASS FILTER SETTLE TIMES FOR $\epsilon = 1\%$

Frequency	ts
100 Hz	2.81 ms
200	3.17
500	5.07
900	25.3
1000	∞
1100	25.3
1500	5.07
2000	2.53

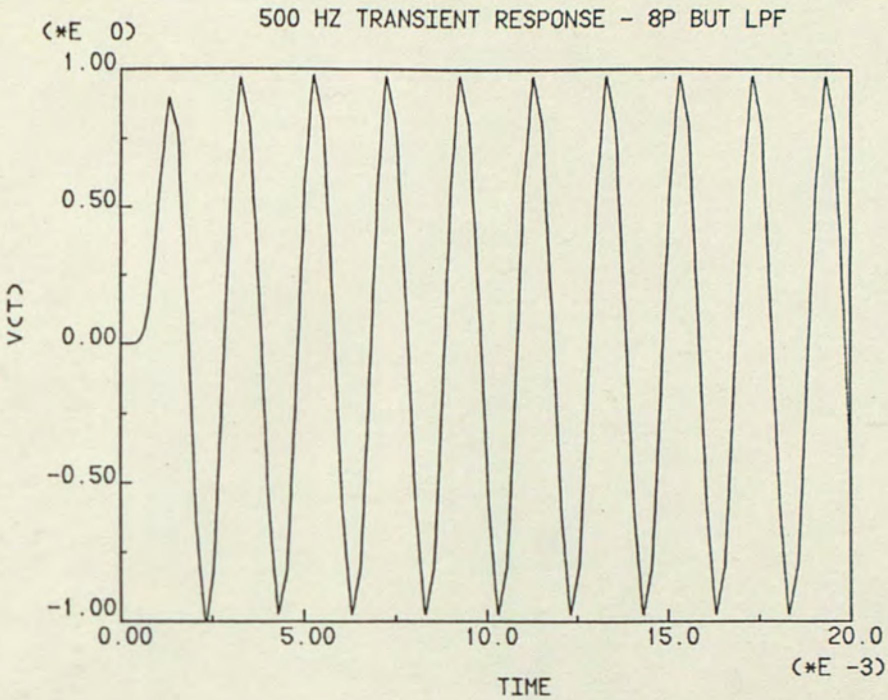
CHAPTER 3

COMPARISON OF THEORETICAL AND MEASURED DATA

The theoretical solutions for the transient settle times of filters has been presented in Chapters One and Two. For most applications, the settle time and the phase delay will not take longer than 100 milliseconds. However, for signal measurement and processing applications, an efficient algorithm can save time and maximize the accuracy of the system being measured. For example, if several hundred measurements of a pulsed system are to be made using an analog to digital converter under computer control, the programmer would have to examine the trade-off between speed and accuracy. The formulas presented in this paper can provide a reasonable guideline for selection of program delays to allow the system to settle to within prescribed limits of error.

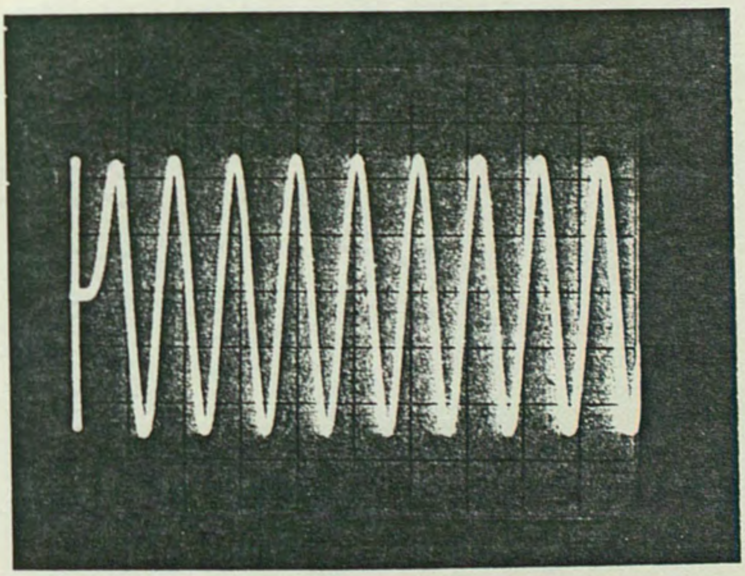
Computer Circuit Analysis Using SPICE

A circuit analysis of an 8-pole Butterworth filter was performed using the SPICE 2G program originally developed by Nagel (1975). This program was used on a Digital Equipment Corporation VAX 11/780 computer and the tabulated transient results recorded using a Tektronix 4662 Digital Plotter. Each graph shown has a 1000 point resolution.



SPICE

Analysis



Oscilloscope

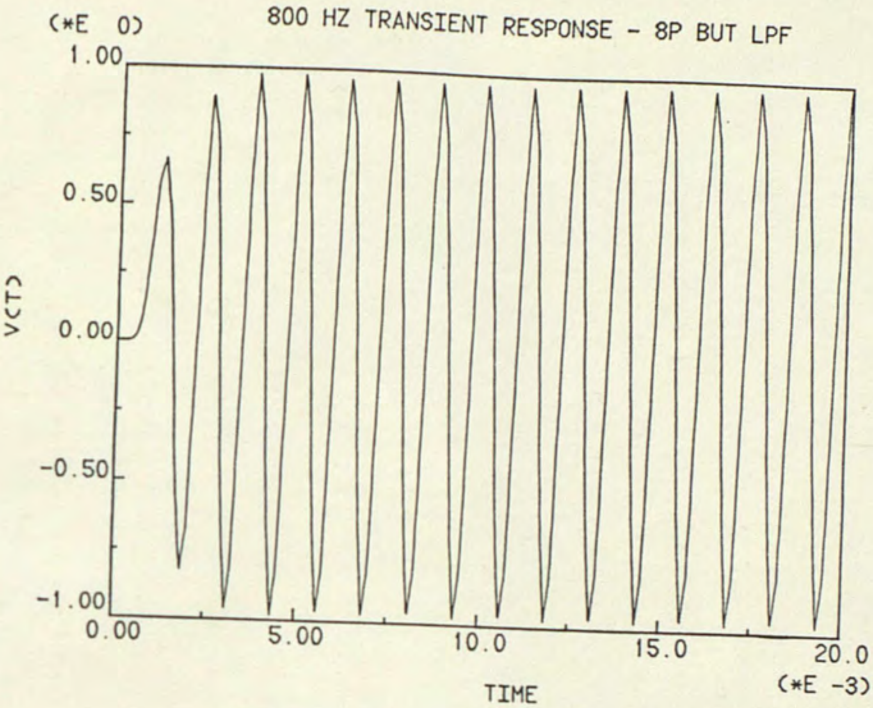
Photograph

f = 500 Hz

HOR = 2ms/cm

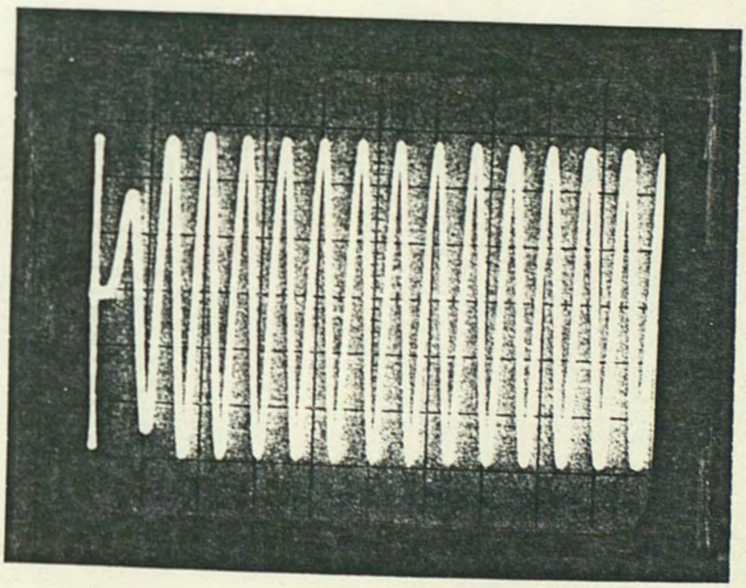
VER = 0.5V/cm

Figure 5. Comparison of SPICE Transient Analysis with Actual Circuit Transient Response at 500 Hz.



SPICE

Analysis



Oscilloscope

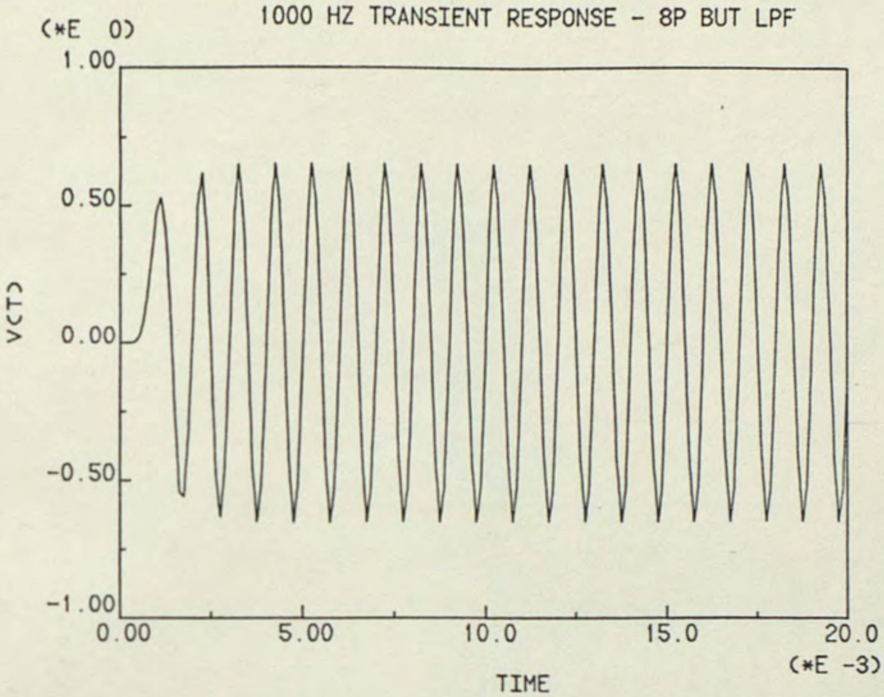
Photograph

f = 800 Hz

HOR = 2ms/cm

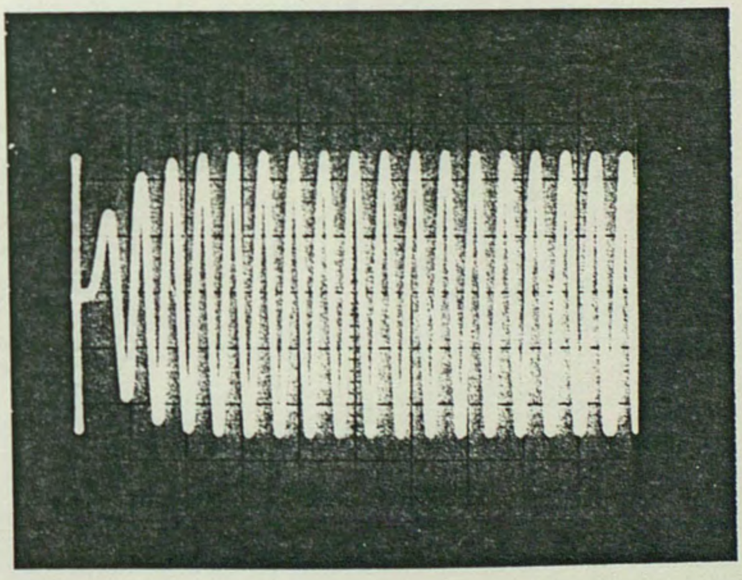
VER = 0.5V/cm

Figure 6. Comparison of SPICE Transient Analysis with Actual Circuit Transient Response at 800 Hz.



SPICE

Analysis



Oscilloscope

Photograph

f = 1000 Hz

HOR = 2ms/cm

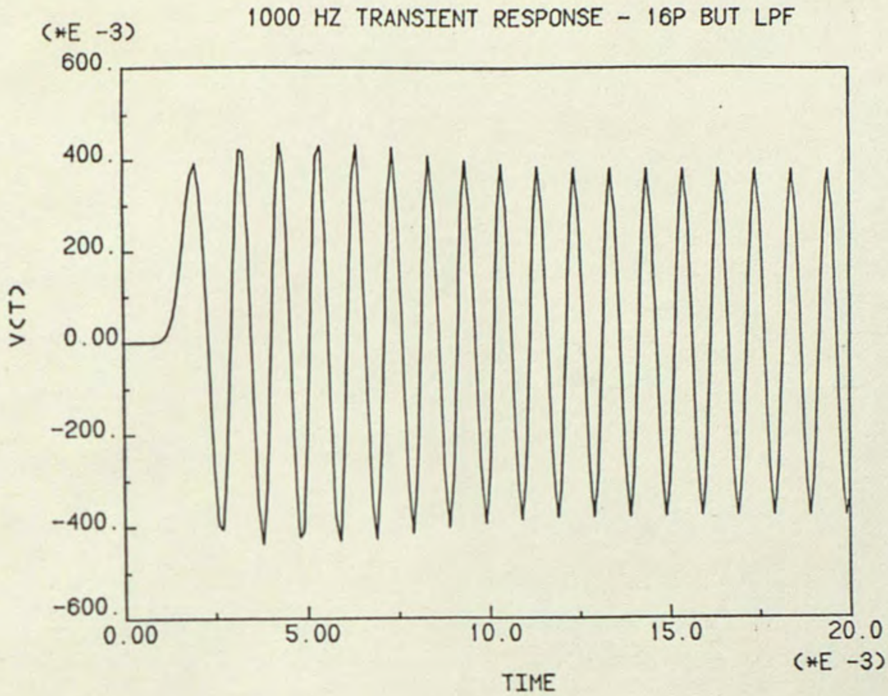
VER = 0.5V/cm

Figure 7. Comparison of SPICE Transient Analysis with Actual Circuit Transient Response at 1000 Hz.

Figures 5, 6 and 7 show the transient response functions of an 8-pole Butterworth low-pass filter with a design cut off frequency of 1032 Hz. A comparison of the computer generated graphs with the oscilloscope photographs shows that close agreement was obtained between the computer analysis and the breadboard circuit analysis. Therefore, it can be shown without a supporting breadboard analysis that the 16 and 32-pole Butterworth filters may be expected to respond to a 1000 Hz step modulated input as shown in Figure 8.

Note that the frequencies compared were 500, 800 and 1000 Hz, but it was found that some intermediate frequencies displayed more amplitude variations. This is the result of phasing relationships between the filter sections which go through ranges of cancellation and reinforcement depending on frequency.

Amplitude modulation is much more visible in Figure 8 and the settle time can be seen to increase for higher order filters. A comparison of Figures 7 and 8 shows that the settle time increases from approximately three milliseconds for the 8-pole filter to approximately 20 milliseconds for the 32-pole filter. The noise visible on the 32-pole filter transient analysis graph indicates the SPICE program numerical technique has reached the computer limit for digital precision. If higher order filters were evaluated using the techniques developed in Chapters One and Two, it would be shown that the settle time would increase without bound. In the limit, equation 41 would apply, where it can be seen that an ideal filter driven at its cut off frequency with a step modulated input signal would never settle.



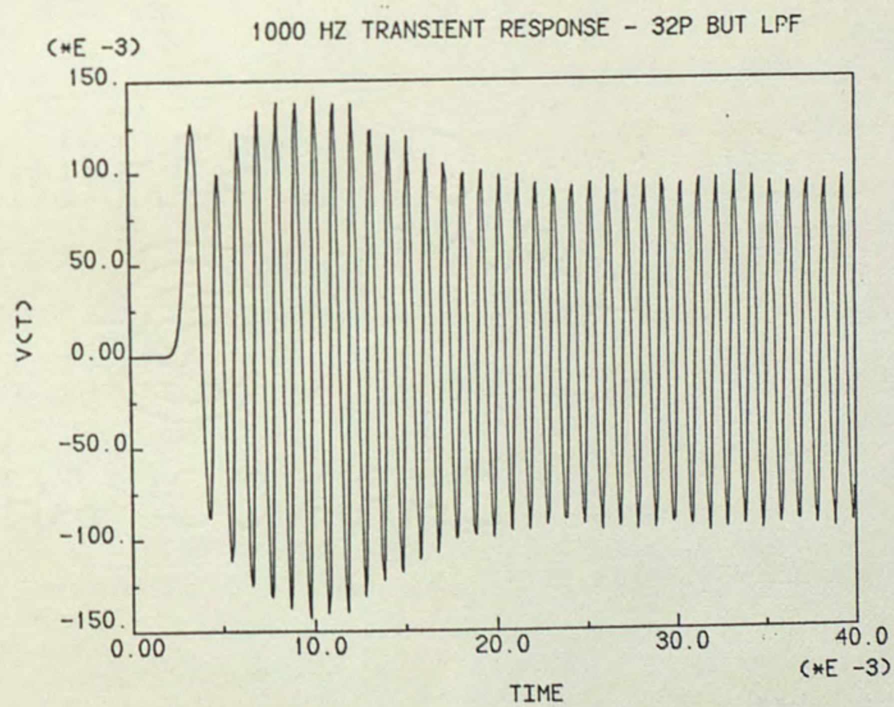


Figure 8. SPICE Predicted Transient Behavior for Higher Order Filters.

Breadboard Circuit Analysis

A final analysis of the filter transient settle times was made using an actual circuit. The filter selected for construction was the Butterworth 8-pole low-pass filter with a cut-off frequency of 1032 Hz. This circuit is identical to the model used in the SPICE analysis, and its behavior was observed to closely match the graphical results. The design technique was based on Daryanani (1976) with a voltage divider added to provide unity gain. A schematic diagram of the circuit is presented in Appendix 2.

The step modulated sine wave input to the filter was obtained by using a circuit designed for and constructed with commercially available integrated circuits. A schematic diagram of the circuit is presented in Appendix 3.

The Butterworth filter transient response was observed at different input frequencies on an oscilloscope and the settle times were noted to fall within the range of two to five milliseconds. The settle time formulas predicted worst case transient settle times and the breadboard analysis always took less time to settle. Note that the settle time was shown to increase as the step modulated input frequency approached the filter cut off frequency. At frequencies beyond the cut off, the steady state signal decreased to such low levels that the transient signal could be monitored more accurately. The transient waveform was then observed to be an exponentially decaying sinusoid. Settle time continued to increase as the input frequency was increased beyond the filter cut off.

Conclusions

Several useful properties of filters have been presented with the intention of predicting filter response times for signal processing applications. The time response of filters to step modulated sine waves is an amplitude modulation of the input signal. Settle time is determined by establishing a desired error limit ϵ at which the envelope must be dampened. The formulas presented show that the settle time is a function of ϵ, the filter order, and the difference in frequency between the sine wave input signal and the filter cut off. The actual transient settle time will be less than the predicted value due to phasing relationships not accounted for, and an engineering approximation made that assumed a value of 1 for the sine function. So with an understanding of these limitations, the low order settle time expressions can be useful for describing the delay time necessary for Butterworth or similar filters. Filters with zeros in the stop-band, such as Elliptic filters, have a cut off slope which is not accurately represented by the denominator quadratic filter sections described in Chapter One. For these filters, the analysis technique for low order filters can be used, or the ideal filter expressions can be used. In the limit, the settle time expressions for the ideal filters predict the worst case time delays and are valid for all filter types.

APPENDIX 1. PROGRAMS TO SOLVE Si(x) AND Ci(x).

These programs evaluate the sine and cosine integrals by numerical approximation. The infinite series equivalent expression is presented first, followed by the program listing that is used to solve the evaluation. Both programs are written for a Hewlett Packard 41C series calculator. The operator is prompted by the calculator to enter the argument x, which is the limit to be evaluated, and n, the number of terms to be used from the series. Large values of x will yield poor results due to limitations in the range of the calculator. Best results will be obtained by evaluating values of x less than 25 and using int(2x) terms for n; n < 33.

$$Si(x) \sim \sum_{n=0}^{n<33} \frac{(-1)^n x^{2n+1}}{(2n+1)[(2n+1)!]}$$

$$Ci(x) \sim \gamma + \ln x + \sum_{n=1}^{n<33} \frac{(-1)^n x^{2n}}{(2n)(2n!)}$$

$$\gamma = 0.577215664$$

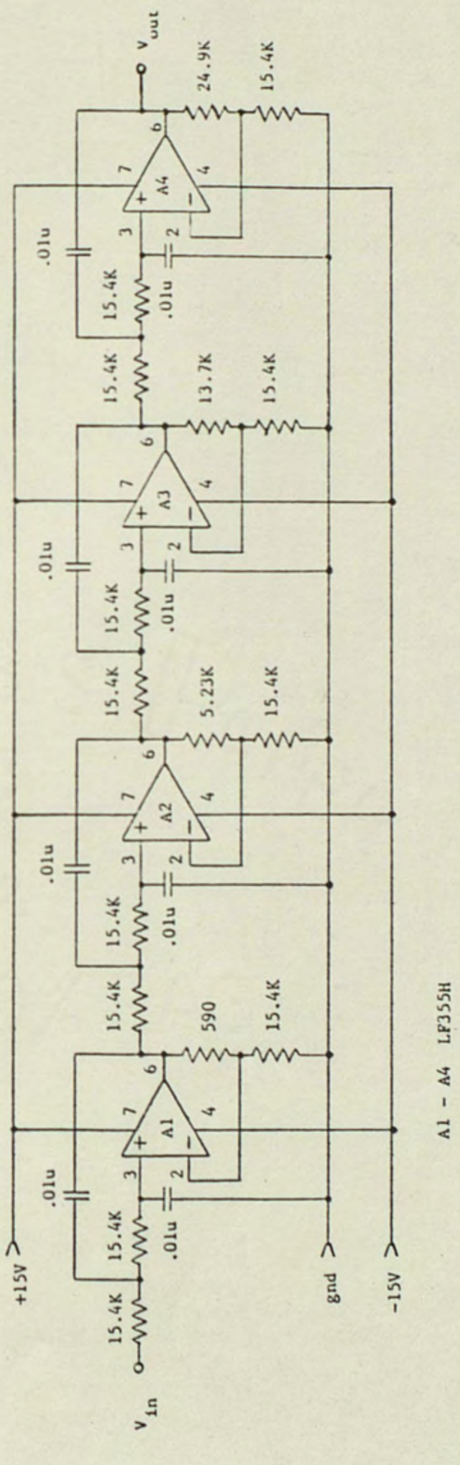
Si(x) LISTING

```
LBL SI
LBL A
αARG Xα
PROMPT
ABS
STO 00
STO 01
αN?α
PROMPT
INT
STO 02
34
X<=Y?
GTO 02
1
STO 03
LBL 01
RCL 03
2
*
1
+
STO 04
RCL 00
X<->Y
Y↑X
RCL 04
FACT
/
RCL 04
/
-1
RCL 03
Y↑X
*
ST+ 01
1
ST+ 03
DSE 02
GTO 01
RCL 01
STOP
GTO SI
LBL 02
αN>33α
AVIEW
END
```

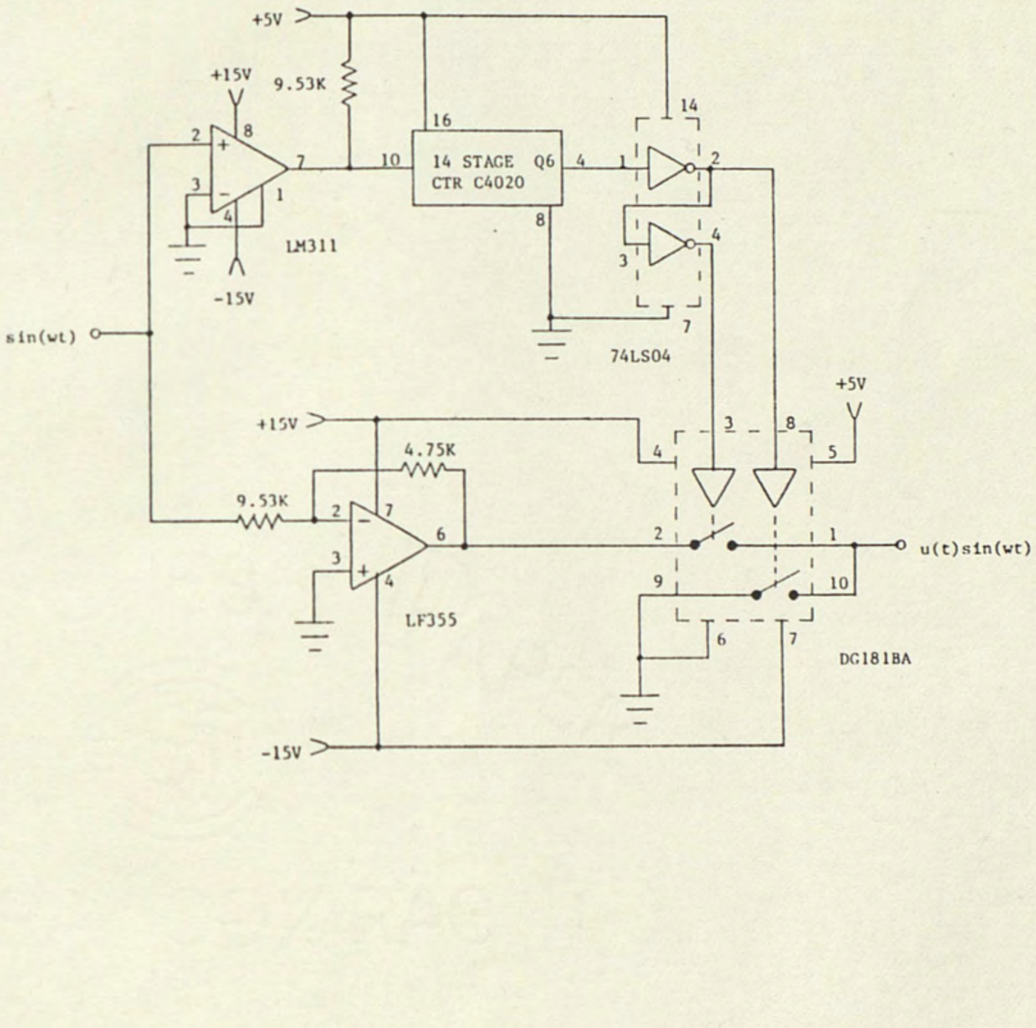
Ci(x) LISTING

```
LBL CI
LBL A
αARG Xα
PROMPT
ABS
STO 00
αN?α
PROMPT
INT
STO 02
34
X<=Y?
GTO 02
1
STO 03
0.577215664
STO 01
RCL 00
LN
ST+ 01
LBL 01
RCL 03
2
*
STO 04
RCL 00
X<->Y
Y↑X
RCL 04
FACT
/
RCL 04
/
-1
RCL 03
Y↑X
*
ST+ 01
1
ST+ 03
DSE 02
GTO 01
RCL 01
STOP
GTO CI
LBL 02
αN>33α
AVIEW
END
```

APPENDIX 2. SCHEMATIC DIAGRAM OF AN 8-POLE LOW-PASS BUTTERWORTH FILTER.



APPENDIX 3. SCHEMATIC DIAGRAM OF A STEP MODULATOR.



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