Stability of Steel Structures

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STABILITY OF STEEL STRUCTURES

BY

ABBAS KARIMI HOSSEINI
B.S.E. University of South Florida, 1978

THESIS

Submitted in partial fulfillment of the requirements for the degree of Master of Science in the Graduate Studies Program of the College of Engineering University of Central Florida Orlando, Florida

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ABSTRACT

In the design of an elastic structure, a major consideration is the overall stability of the system under axial load or a combination of an axial load and bending moment. In first-order analysis the effects of axial forces on the stiffness formulation and their influence on deformations of the members are usually neglected. If the axial forces are sufficiently large, a better indication of the distribution of the forces and moments in the structure is obtained by using a second-order analysis.

In this type of analysis the moment equilibrium equations are based on the deformed shape of the structure. The secondary moments produced by axial forces acting through the deformed structure are considered in the computation of stiffness term. The secondary moments are called P-Delta effects. In this paper, by using a finite element computer program, the effects of axial loads on a plane frame structure are investigated. The method of analysis, based on the critical values of applied lateral forces on the elastic stability of the frame, will be referred to in this paper as a second order analysis. Example problems are included to serve to document this analytical approach.
DEDICATION

To my parents, whose love and understanding made this possible.
# TABLE OF CONTENTS

LIST OF TABLES .................................................. v
LIST OF FIGURES .................................................. vi
LIST OF SYMBOLS .................................................. viii

Chapter

I. INTRODUCTION .................................................. 1
II. REVIEW OF CURRENT DESIGN PRACTICE ......................... 4
III. METHOD OF ANALYSIS ........................................... 11
IV. PROGRAM IMPLEMENTATION ..................................... 20
    Example 1 .................................................. 26
    Example 2 .................................................. 31
    Example 3 .................................................. 35
V. CONCLUSIONS AND COMMENTS ................................... 40

APPENDICES .......................................................... 42

I. STABILITY FUNCTION ............................................. 43
    Rotational Function ......................................... 43
    Translation Function ......................................... 50
    Fixed End Action ............................................. 53

II. LIST OF FINITE ELEMENT FORTRAN PROGRAM FOR THE
    SECOND-ORDER ANALYSIS OF RIGID FRAMES .................... 60

III. LIST OF INPUT AND OUTPUT RESULTS OF FIRST-ORDER AND
     THREE CYCLES OF SECOND-ORDER ANALYSIS FOR EXAMPLE
     1 .......................................................... 78

REFERENCES ....................................................... 83
LIST OF TABLES

1. Coefficients relating the effect of axial force on stiffness ........................................... 16
2. Coefficients relating the effect of axial force on fixed end action ..................................... 18
3. Second order analysis of frame ....................................................................................... 28
4. Comparison of second-order analysis results with reference (8) ......................................... 28
5. Comparison of second-order analysis results from this paper with reference (6) ..................... 33
6. Results of first and second order analysis for all three load cases ....................................... 38
LIST OF FIGURES

1. Alignment chart for effective length of column in continuous frames . . . . . . . . . . . . 8
2. Frame with three different axial loadings . . . . 9
3. Free body diagram of element i . . . . . . . . 13
4. Free body diagram, beam with concentrated lateral load . . . . . . . . . . . . . . 17
5. Flow chart diagram of main program . . . . . . 21
6. Flow chart of subroutine ASSEM . . . . . . . . 23
7. Frame geometry and loading condition . . . . . . 26
8. Finite element mesh used for frame shown in Figure 7 . 27
9. Free body diagram of final second analysis . . . 29
10. Final moment diagram . . . . . . . . . . . . . . 30
11. Frame geometry and loading condition . . . . . . 31
12. Finite element mesh used for frame shown in Figure 11 . 32
13. Final second-order results . . . . . . . . . . . 32
14. Final moment diagram . . . . . . . . . . . . . . 34
15. Frame geometry and loading condition used in examples . 35
16. Finite element mesh used for frame shown in Figure 15 . 36
17. Free body diagram of final second order analysis for Case I (P=120 lb) . . . . . . . . . . . . 37
18. Free body diagram of final second order analysis for Case II (P=240 lb) . . . . . . . . . . . . 37
19. Free body diagram of final second order analysis for Case III (P=360 lb) . . . . . . . . . . . . 38
20. Beam element with axial force $P_i$ .................................. 44
21. Free body diagram of beam element $i$ .................................. 44
22. Free body diagram, beam element $i$ with unit translation of end 1 ................................................................. 51
23. Rigid body rotation of beam element $i$ .................................. 51
24. Prismatic beam element $i$ with concentrated load $W$ .......... 53
25. Free body diagram, x distance from end 1 when $x < a$ .......... 54
26. Free body diagram, x distance from end 1 when $x > a$ .......... 56
LIST OF SYMBOLS

The following symbols are used in this paper:

\begin{align*}
A &= \text{cross section area} \\
C_m &= \text{equivalent moment factor} \\
E &= \text{modulus of elasticity} \\
f_a &= \text{computed axial stress} \\
f_{bx} &= \text{computed bending stress at member end about x axis} \\
f_{by} &= \text{computed bending stress at member end about y axis} \\
F_{bx} &= \text{bending stress permitted of bending moment about x axis existed} \\
F_{by} &= \text{bending stress permitted of bending moment about y axis existed} \\
F_a &= \text{axial stress permitted if axial force existed} \\
F_e &= \frac{14900}{(KL/r_x)^2}, \text{in which } KL/r_x \text{ is the effective slenderness ratio in plane of bending} \\
F_{xj} &= \text{axial load at node } j \\
F_{xk} &= \text{axial load at node } k \\
F_y &= \text{minimum yield stress of the type of steel being used} \\
F_{yj} &= \text{vertical load at node } j \\
F_{yk} &= \text{vertical load at node } k \\
I &= \text{moment of inertia} \\
K_{11} &= \text{restraining moment at end 1} \\
K_{12} &= \text{shear force at end 1} \\
K_{21} &= \text{restraining moment at end 2}
\end{align*}
$K_{22} = \text{shear force at end 2}$

$L_i = \text{length of member } i$

$M_j = \text{moment at node } j$

$M_k = \text{moment at node } k$

$P_a = \text{axial force for member}$

$P_E = \text{buckling load for member}$

$P.E. = \text{potential energy}$

$S.E. = \text{strain energy}$

$S_i = \text{stiffness matrix for element } i$

$T_i = \text{coefficient relating the effect of axial force on stiffness}$

$TK_{11} = \text{moment coefficient for fixed end action at node 1}$

$TK_{12} = \text{shear coefficient for fixed end action at node 1}$

$TK_{21} = \text{moment coefficient for fixed end action at node 2}$

$TK_{22} = \text{shear coefficient for fixed end action at node 2}$

$\{U_i\} = \text{displacement matrix}$

$\{U_i\}^T = \text{transpose of displacement matrix}$

$U_{xj} = \text{displacement at } x \text{ direction for node } j$

$U_{xk} = \text{displacement at } x \text{ direction for node } k$

$U_{yj} = \text{displacement at } y \text{ direction for node } j$

$U_{yk} = \text{displacement at } y \text{ direction for node } k$

$W = \text{concentrated load}$

$\phi_n = \text{shape function}$

$\Theta_j = \text{rotation at node } j$

$\Theta_k = \text{rotation at node } k$
CHAPTER I
INTRODUCTION

In general, the analysis of elastic structural systems are based on first order analysis. In this type of analysis, the distribution of forces and moments throughout the structure are developed by using undeformed geometry of the structure. Therefore, the effect of the axial forces on deformed structure are completely ignored. If the magnitudes of the axial forces are sufficiently large, their influence on deformed members will produce additional moments which can be noticeably large and will intend to decrease the overall strength of the structure. Thus, a better indication of the distribution of the forces and moments in the structure is obtained by using second order analysis. In the latter, the moment equilibrium equations are based on the deformed shape of the structure and the secondary moments produced by axial forces acting through the deformed structure are considered in the computation of member stiffness term.

Today, the design of most multistory plane frame structures is based on the allowable stress technique. This type of analysis uses an elastic first-order procedure. The bending moments calculated by a first order approach are used directly in the design of the girders; the moments in the column are then calculated, employing the moment amplification equations.² For this type of design procedure a factor is used in the moment amplification equation to adjust for the absence
of secondary moments in the design. In the next section, we will show the allowable stress technique that is the first-order procedure in more detail. Later an analysis of a plane frame with members subjected to lateral forces combined with bending moments will be made. The method of analysis is based on the critical values of applied lateral forces and their influence on the elastic stability of frame.

It is clear that a compressive axial load reduces a member's overall effective bending resistance and thereby causes a greater deformation. Tensile forces, on the other hand, reduce deformations. In both cases, it is possible to determine the interrelationship among the loading parameters, the unit bending stiffness coefficient and the resulting deformations for a constant value of axial load. Then, assuming that buckling does not occur, one can define the stiffness coefficient and flexibility including the effects of axial load. The derivations and application of the stability approach will be shown in Appendix I.

The analysis is not straightforward due to the fact that the axial forces in the members are related to the joint displacements. Therefore, an analysis must be conducted in a cyclic fashion. In the first cycle of analysis, the stiffness matrix of a member is based on a first-order analysis. In the second cycle the lateral forces in the members resulting from the first cycle are used in determining the modified member stiffness matrix and also the modified fixed-end moments and shears; in this manner, new values are obtained for the lateral forces.
After each solution is completed, the results are compared with the previous solution. If the difference is small, the cyclic process is terminated and the last solution is taken as being correct.

The finite element program of frame analysis⁶ has been modified to accomplish the objective of second-order analysis. This program is written in fortran language and is listed under Appendix II. Example problems are included to illustrate the accuracy of the results. These analyses were performed with a 64 bit (cyber 176 mainframe) computer and are presented in a later chapter of this report.
At the present time, most steel structures are designed using the allowable stress technique. In this technique, the structure is analyzed under various loading combinations, using an elastic first order procedure. The bending moments are used directly in the design of the girder, while the moments and forces in the columns are checked against the moment amplification equations to arrive at suitable column sections. In this design procedure, terms in the interaction equation are adjusted to compensate for the neglect of secondary moment effects on individual moments. Members subjected to both axial compression and bending stresses will be proportioned at all points along their length to satisfy the following requirements:

\[
\frac{f_a}{F_a} + \frac{C_{mx} f_{bx}}{(1-f_a)F_{bx}} + \frac{C_{my} f_{by}}{(1-f_a)F_{by}} \leq 1.0 \quad (1a)
\]

or:

\[
\frac{f_a}{0.60F_y} + \frac{f_{bx}}{F_{bx}} + \frac{f_{by}}{F_{by}} \leq 1.0 \quad (1b)
\]

where:

- \( f_a \) = computed axial stress
- \( F_a \) = axial stress permitted if axial force existed
\( f_{bx} = \) computed bending stress at member end about x axis
\( f_{by} = \) computed bending stress at member end about y axis
\( F_{bx} = \) bending stress permitted of bending moment about x axis existed
\( F_{by} = \) bending stress permitted of bending moment about y axis existed
\( F_e = \frac{14900}{(KL/r_x)^2} \), in which KL/r_x is the effective slenderness ratio in plane of bending
\( F_y = \) minimum yield stress of the type of steel being used

In Equations (1a) and (1b), the subscripts x and y combined with subscripts b, m and e indicate the axis of bending about which a particular stress or design property applies.\(^2\)

The expressions

\[
\left(\frac{1}{1 - \frac{f_e}{F_{ex}}}\right) \quad \text{and} \quad \left(\frac{1}{1 - \frac{f_e}{F_{ey}}}\right)
\]

in Equation (1a) are called the amplification factors. It is intended through their existence to account for secondary moments produced by the lateral load, acting on the deformed member. Depending on the shape of the applied moment diagram (the critical location and magnitude of the induced eccentricity), the amplification factor may overestimate the extent of the secondary moment. To allow for this, the amplification factor is modified, as required, by a moment factor \( C_m \). When bending occurs about the x and y axis, the bending stress calculated about each of these axes is adjusted by the value of \( C_m \) and \( F_e \) corresponding to the distribution and the
slenderness ratio in its plane of bending. The bending stress is then taken as a fraction of that permitted for bending about that axis, with due regard to the unbraced length of compression flange when this is a factor.

When the computed axial stress is no greater than 15 percent of the allowable axial stress, the influence of

\[
\frac{C_m}{(1 - \frac{f_a}{F_e})}
\]

is generally small and may be neglected, and Equation (1a) could reduce to:

\[
\frac{f_a + \frac{f_{bx}}{F_a} + \frac{f_{by}}{F_{by}}}{1.0} \leq 1.0
\]

(2)

However, its use in Equation (1a) is not intended to permit a value of \(f_b\) greater than \(F_b\) when the value of \(C_m\) and \(f_a\) are both small. Depending upon the unbraced length of a member in the plane of bending, the combined stress computed at one or both ends of this length may exceed the combined stress at all intermediate points where lateral displacement is created by the applied moments.

The limiting value of the combined stress ratios in this case is pointed out in Equation (1b).

\(C_m\) in Equation (1a) will increase if the primary bending moment is not uniform over the member length. The strength of the column will also be increased, and because \(C_m\) is directly related to the
strength of the column, it will also increase. So the equivalent moment factor $C_m$ would be:

$$C_m = 0.60 - 0.40 \frac{M_1}{M_2} \quad \text{with} \quad C_m \geq 0.40$$

where $\frac{M_1}{M_2}$ is the ratio of smaller to larger moments at the ends of that portion of the member unbraced in the plane of bending under consideration.

The ratio of $\frac{M_1}{M_2}$ is positive when the member is bent in a reversed curvature, and it has a negative magnitude when bent in a single curvature. The value of $C_m$ can differ for the following conditions:

1. For compression members in frames subject to joint translation (sideway). $C_m = 0.85$.
2. For compression members in frames braced against joint translation in the plane of loading and subjected to transverse loading between their supports, the value of $C_m$ may be determined by a rational analysis. However, instead of such an analysis, the following values may be used:
   a) For members with both ends restrained, $C_m = 0.85$
   b) For members with both ends not restrained, $C_m = 1.0$

Furthermore, considerable attention has also been given to another technique called the effective column length factor. It is believed in using the effective column length factor, $K$, that an adequate allowance is made for the secondary moment. The value of $K$ is
obtained by constructing a straight line between $G_A$ and $G_B$ from the alignment chart shown in Figure 1.

$$G_A$$ and $G_B$ are determined from $G = \frac{I_c}{\Sigma L_c} - \frac{I}{\Sigma L_g}$
The subscripts A and B represent the two ends of the column section and \( \sum \) indicates a summation of all members rigidly connected to that join. \( I_c \) and \( L_c \) are the moment of inertia and unbraced length of the column selection and \( I \) and \( L \) the corresponding unbraced length or other restraining member respectively. Although the present K-factor approach is not always satisfactory for columns, neither from a safety or economic point of view, its concept can be easily demonstrated by considering the three frames shown in Figure 2.

![Figure 2. Frame with three different axial loadings.](image-url)
Column AB, which can resist overturning moments and lateral loads (as well as instability effects), has the same cross section in the three frames shown. Since $K$, as obtained from the alignment chart, is the same in all three cases, it is clear that if the design obtained is adequate for frame (a), it must then be under designed for frame (c) in which the secondary moment effect is more significant. Or if it is adequate for frame (c), it must be over designed for frame (a), in which the instability effect is smaller. Ideally, there should be three different designs for the three frames of Figure 2, each reflecting the degree of severity of the secondary effect that exists. 3

Although the $K$ factor provides a means of accounting for the secondary effect in a column, it cannot ensure overall frame stability since it cannot control the instability effect in the compressive beam column. The beam in frame (a), for example, should not only be dimensioned based on a bending moment to lateral load $P_a$, but also it should consider the effect of the vertical force on sway deformation. In general, because no provision similar to the $K$ factor has been made for the beams, these members will be under designed and may well lead to overall frame distress. An essential step for an improved design, therefore, must be the direct consideration of the secondary overturning moment in the beams. 3
CHAPTER III

METHOD OF ANALYSIS

The finite element technique has been found to be a very powerful tool for solving structural problems. The concept of the finite element method consists of subdividing the structure into many sub-regions which are referred to as elements. The intersections of the sides of the elements are called nodes or grids. The nodes are used to specify the geometry of individual elements. There is no fixed rule that can be used to determine the number of nodes and elements needed in a finite element mesh. In general, the finer the mesh, the better will be the numerical solution that is obtained. As more computer time is needed as the number of nodes and elements increase, the process eventually leads to excessive costs. Therefore, at the start of an analysis, it is up to engineering judgement and previous experience to find the coarsest possible mesh, which will achieve a numerical solution within the desired precision.

Once a finite element mesh is selected, nodes and elements are numbered for reference purposes in the remaining steps. The nodal points of the element provide mathematical expression for defining the shape as an unknown quantity over the domain of the element. A set of mathematical functions such as polynomials or trigonometric series can be used for this purpose in the finite element formulation.
If $U$ is denoted as the unknown quantity for element, then the polynominal series can be written as:

$$U = \phi_1 U_1 + \phi_2 U_2 + \ldots + \phi_n U_n$$  \hspace{1cm} (3)

where $U_1, U_2, \ldots, U_n$ are the unknown values at the nodal points and $\phi_1, \phi_2, \ldots, \phi_n$ are the interpolation functions.

For example, in the case of displacement, these interpolation functions represent the deformed shape of an element under given loading conditions. Also, $U_1, U_2, \ldots, U_n$ are the unknown values of displacements at each node, that are also called degrees of freedom.

The interpolation functions must also satisfy the compatibility requirements for displacements along the boundaries with adjacent elements. There are many different types of approximate methods. The two most important methods for engineering applications are the Rayleigh-Ritz and Galerkin method. The concept of minimum potential energy of Rayleigh-Ritz applied for computing the stiffness matrix is used in the Finite Element Program. The potential energy is obtained as the summation of the internal strain energy minus the potential of the external loads.  

The equilibrium relation for elements consists of the stiffness matrix $[S_i]$, the displacement vector $\{U_i\}$, and the nodal force vector $\{F_i\}$. These relations are defined as a set of simultaneous linear equations, which evolve from a process of minimizing the potential energy in accordance with the associated conditions.

Potential Energy (P.E.) = Strain Energy (S.E.) - Work
After various steps of derivation are completed, the potential energy in effect for element \( i \) can be written as:

\[
(P.E.)_i = \frac{1}{2} \{U_i\}^T [S_i] \{U_i\} - \{U_i\}^T \{F_i\}
\]

The potential energy \( P.E. \) is then minimized by taking its derivative with respect to the various displacements and setting these qualities to zero.

\[
\frac{\partial (P.E.)_i}{\partial U_i} = 0 \quad i = 1, 2 \ldots, m
\]

or:

\[
\frac{\partial}{\partial U_i} \left[ \frac{1}{2} \{U_i\}^T [S_i] \{U_i\} - \{U_i\}^T \{F_i\} \right] = 0
\]

Thus:

\[
[S_i] \{U_i\} - \{F_i\} = 0
\]

or:

\[
[S_i] \{U_i\} = \{F_i\} \tag{4}
\]

For a simple one dimensional, two noded element \( i \) shown in Figure 3, the displacement and force vector can be written knowing that each node includes two translational and one rotational degrees of freedom.

**Figure 3. Free body diagram of element \( i \).**
The translational and rotational shape functions for the element shown in Figure 3 are:

\[ \phi_1 = \left[ 1 - 3 \left( \frac{x}{L_i} \right)^2 + 2 \left( \frac{x}{L_i} \right)^3 \right] \]
\[ \phi_2 = \left[ x - \frac{2x^2}{L_i} + \frac{x^3}{L_i^2} \right] \]
\[ \phi_3 = \left[ 3 \left( \frac{x}{L_i} \right)^2 - 2 \left( \frac{x}{L_i} \right)^3 \right] \]
\[ \phi_4 = \left[ -\frac{x^2}{L_i} + \frac{x^3}{L_i^2} \right] \]

and

\[ \{ F_i \} = \begin{bmatrix} F_{xj} \\ F_{yj} \\ M_j \\ F_{xk} \\ F_{yk} \\ M_k \end{bmatrix} \quad \text{and} \quad \{ U_i \} = \begin{bmatrix} U_{xj} \\ U_{yj} \\ \theta_j \\ U_{xk} \\ U_{yk} \\ \theta_k \end{bmatrix} \]

And stiffness matrix can be obtained from:

\[ S_{jk} = \int_0^L \frac{EI}{L_i} \frac{\partial^2 \phi_j}{\partial x^2} \frac{\partial^2 \phi_k}{\partial x^2} \, dx \]

where: E and I represent the modulus of elasticity and moment of inertia of element i respectively; \( \phi_j \) and \( \phi_k \) represent the interpolation or shape functions at node j and k. By using the above equation, the stiffness matrix can be easily derived and will lead to:
where \( T_1 = T_2 = T_3 = T_4 = 1 \), in a first-order analysis.

The total potential energy of the entire structure is the summation of potential energy of each individual element. Thus,

\[
\sum_{i=1}^{m} \text{P.E.} = \sum_{i=1}^{m} \frac{1}{2} \{U_i\}^T [S_i] \{U_i\} - \sum_{i=1}^{m} \{U_i\}^T \{F_i\} \tag{6}
\]

where \( m \) is the total number of elements of the structure.

A solution cannot be obtained until the geometric boundary conditions are taken into account. Boundary conditions are the physical constraints or supports that must exist so that the structure can stand in space uniquely.

After the boundary conditions are introduced, the above set of equations are solved for the unknown displacements. Equation (5) shows the general stiffness matrix of a member. For the first cycle, the values of \( T_1, T_2, T_3, \) and \( T_4 \) which are the coefficients
relating the effect of axial force on bending stiffness terms, are one. And the axial stiffness is assumed to be unaffected by moments.

To account for secondary moments, the stiffness matrix has been modified based on the critical values of applied lateral forces on each individual member. Therefore, by using certain stability functions which are related to the equilibrium conditions of a deformed structure, the stiffness of each member can be modified to present its true strength.

The analysis is conducted in a cyclic fashion. After the first cycle is completed, the lateral forces obtained from the first cycle are used to evaluate the $T_i$ coefficients of each member throughout the entire structure for the next cycle. The cyclic process continues until two successive analyses yield approximately the same results. Values of $T_i$ are shown in Table 1.

**TABLE 1**

**COEFFICIENTS RELATING THE EFFECT OF AXIAL FORCE ON STIFFNESS (Ref. 5)**

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Condition of Axial Force</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_1$</td>
<td>Compression $(KL)^3 \sin KL/12 , fc$</td>
</tr>
<tr>
<td>$T_2$</td>
<td>$(KL)^2 (1-\cos KL)/6 , fc$</td>
</tr>
<tr>
<td>$T_3$</td>
<td>$KL(\sin KL- KL \cos KL)/4 , fc$</td>
</tr>
<tr>
<td>$T_4$</td>
<td>$KL(KL- \sin KL)/2 , fc$</td>
</tr>
</tbody>
</table>
where: \[ fc = 2 - 2 \cos \, KL - KL \sin \, KL \]
\[ ft = 2 - 2 \cosh \, KL + KL \sinh \, KL \]
\[ KL = \pi \sqrt{\frac{P_a}{P_E}} \]

\( P_E \) is the Euler Buckling load for a pin-ended column with no transverse load and is evaluated:

\[ P_E = \frac{\pi^2 EI}{L^2} \]

If the axial force in any member equals the load where buckling occurs, a singular matrix occurs and results are not obtained.

Since fixed-end action values are affected by the axial load, therefore fixed-end action as well as stiffness in members should be recomputed for each cycle.\(^5\) Table 2 shows the value of coefficients for evaluating the fixed-end action for the concentrated load members shown in Figure 4.

![Free body diagram, beam with concentrated lateral load.](image-url)
# TABLE 2

**COEFFICIENTS RELATING THE EFFECT OF AXIAL FORCE ON FIXED END ACTION** (Ref. 5)

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>CONDITION OF AXIAL FORCE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Compression</td>
</tr>
<tr>
<td><strong>TK(_{12})</strong></td>
<td>(\frac{1}{f_c}(\cos KL - \cos Ka + \cos Kb + Kb \sin KL - 1))</td>
</tr>
<tr>
<td><strong>TK(_{11})</strong></td>
<td>(\frac{1}{Kf_c}(\sin KL - \sin Ka - \sin Kb - Kb \cos KL + KL \cos Kb - Ka))</td>
</tr>
<tr>
<td><strong>TK(_{22})</strong></td>
<td>(\frac{1}{f_c}(\cos KL - \cos Kb + \cos Ka + Ka \sin KL - 1))</td>
</tr>
<tr>
<td><strong>TK(_{21})</strong></td>
<td>(\frac{1}{Kf_c}(-\sin KL + \sin Kb + \sin Ka + Ka \cos KL - KL \cos Ka + Kb))</td>
</tr>
</tbody>
</table>

where:  
\(f_c = 2 - 2 \cos KL - KL \sin KL\)  
\(f_t = 2 - 2 \cosh KL + KL \sinh KL\)  
\(K = \sqrt{\frac{P}{EI}}\)
In case of uniform loaded member, similar terminology applied and the uniform member sub-divided into small segments and each segment was treated as a member with concentrated load at the center of each portion.

The cyclic fashion of analysis described in this chapter may be used to evaluate the buckling load for a frame. The critical load on the member may be found by increasing the load in increments until the stiffness matrix becomes singular. This singularity is the basis for obtaining the magnitude of loading which causes elastic instability in the fundamental buckling mode.5
CHAPTER IV
PROGRAM IMPLEMENTATION

This chapter will briefly discuss the program's operations and capabilities.

The program is capable of solving two-dimensional frame problems with concentrated and uniform load conditions. The computations are conducted in a cyclic fashion. After each solution is completed, the axial force of each member is compared with the previous solution. If the absolute difference of the axial force between two successive cycles is less than .0001, the computation process is terminated and the last solution is taken as being the final result.

The main program will initialize the program's parameters and then call on a set of six different subroutines. The sequence of operations is shown in Figure 5, where:

NRMX = Row dimension for the stiffness matrix
MCMX = Column dimension for the stiffness matrix or maximum half band width allowed
NDF = Number of degrees of freedom per node
NNE = Number of node per element
NDFEL = Total number of degrees of freedom for one element
FORC((I-1)*6+1) = Axial force in local coordinate system from previous cycle
PF((I-1)*6+1) = Axial force in local coordinate system
Figure 5. Flow chart diagram of main program.
The fundamental scheme of the program, which leads to the problem's solution, is as follows:

1 - Structural identification
2 - Computation of the element stiffness matrix
3 - Assembling of total stiffness matrix
4 - Applying boundary condition
5 - Solution of the system of equations
6 - Evaluation of the elemental forces

The input data, consisting of five sets of data cards, is formatted in the subroutine INPUTT which is sequenced in the following manner:

1 - Material properties of the structure
2 - Nodal locations with respect to global coordinate system
3 - Element properties and connectivity table
4 - Description of the applied load
5 - Boundary or support conditions

The total element stiffness matrix is evaluated in subroutine ASSEM and a brief flow chart of this subroutine is shown in Figure 6. This subroutine first calls the subroutine STIFF to compute the individual element stiffness matrix, and then by calling subroutine ELASS it will assemble the total stiffness matrix. However, it should be pointed out that the stiffness values used in each cycle have been modified using subroutine PDELTA. Calculating adjusting coefficients, shown in Table 1, the PDELTA subroutine accounts for the stability of the structure. At the beginning of subroutine STIFF the element
Figure 6. Flow chart of subroutine ASSEM.
rotation matrix is computed and placed in Array ROT. Then the terms of the local member stiffness matrix are computed according to Equation (5) and stored in Array ELST. Finally, this matrix is transferred to a global coordinate system. For this computation, subroutine ETAB3 is used. Notice that the global member stiffness matrix is also stored in Array ELST.

Once all the element matrices have been assembled into the global stiffness matrix, we introduce the boundary conditions and store them in subroutine BOUND.

The element nodal forces will be computed in subroutine FORCE. The computation of the member end forces is done within a loop from one to the total number of elements. These forces are stored in Array F and assembled into total member end forces in Array FORC. These member end forces refer to the local system, and then they are rotated to the global coordinated system, which are then stored in Array FG. Finally, these member contributions to the nodal resultants are added to the Array REAC.

The uniformly loaded members are integrated into fifty sub-elements and apply equivalent concentrated load at the center of each element and then, fixed end moments and shears for each element are calculated in subroutine FEM and stored in Array SUM and SUE, respectively. After the first cycle, their values are adjusted based on coefficients shown in Table 2.

It should be pointed out that the solution will be the nodal displacement at the node points. Final solution is a combination
of solutions in each element lumped together at the common boundaries.

There are many methods of solving simultaneous equations, but the method adopted in this program is by the GAUSS elimination method, taking advantage of symmetric banded matrices. This is shown in subroutine SLBSI.

Finally, the subroutine OUTPT prints the nodal displacements and forces in X and Y coordinates the moment about the Z axis in both local and global coordinate systems.
Example 1

As an illustration of the application of second-order analysis, let us consider the case of the frame shown in Figure 7 which indicates the overall dimensions of all members.

![Frame geometry and loading condition.](image)

For all members:
- Cross sectional area = 3.174 in$^2$
- Modulus of elasticity = 29000 Ksi
- Moment of inertia = 7.233 in$^4$
Figure 8 indicates the node and element numbering used in the Finite Element Analysis. The results of first order and two cycles of a second order analysis are listed in Table 3.

The input and output results of this example are shown under Appendix III as a sample model.
The results of the 3rd cycle of second-order analysis are compared with the reference (8) and tabulated in Table 4.
The free body diagram for each member relating to the results of second order analysis is given in Figure 9.

Figure 9. Free body diagram of final second analysis.
Final moment diagrams with and without inclusion of axial load effects is shown in Figure 10.

Figure 10. Final moment diagram.
Example 2

This problem is chosen from Reference (6) to compare the results for the frame shown in Figure 11, assuming that the effect of axial load upon its flexure is neglected.

\[ I = 250 \text{ in}^4 \]
\[ E = 30000 \text{ Ksi} \]

Figure 11. Frame geometry and loading condition.

The axial forces resulted from each cycle are used to modify the fixed-end moments and shears as well as stiffnesses for the next cycle.
Figure 12 indicates the node and element numbering used in Finite Element Analysis.

Figure 12. Finite element mesh used for frame shown in Figure 11.

The free body diagram for each member relating to the results of the second-order analysis is given in Figure 13.

Figure 13. Final second-order results.
The results of the second-order analysis using the finite element computer program listed in Appendix II versus Reference (6) are shown in Table 5.

**TABLE 5**

COMPARISON OF SECOND-ORDER ANALYSIS RESULTS FROM THIS PAPER WITH REFERENCE (6)

<table>
<thead>
<tr>
<th>Moment</th>
<th>Results From this Paper K-ft</th>
<th>Results From Ref. (6) K-ft</th>
<th>% Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>39.98</td>
<td>40.02</td>
<td>0.100</td>
</tr>
<tr>
<td>$M_2$</td>
<td>9.20</td>
<td>9.14</td>
<td>0.656</td>
</tr>
<tr>
<td>$M_3$</td>
<td>50.83</td>
<td>50.86</td>
<td>0.059</td>
</tr>
<tr>
<td>$M_4$</td>
<td>96.10</td>
<td>96.11</td>
<td>0.010</td>
</tr>
<tr>
<td>$M_5$</td>
<td>35.47</td>
<td>35.44</td>
<td>0.084</td>
</tr>
<tr>
<td>$M_6$</td>
<td>54.45</td>
<td>54.45</td>
<td>0.000</td>
</tr>
</tbody>
</table>
Figure 14 shows the final moment diagram with and without the effect of axial force.

Figure 14. Final moment diagram.
Example 3

An illustration of the effect of secondary moments on the frame is shown in Figure 15 with three different axial forces.

Case I; $P = 120 \text{ lb}$
Case II; $P = 240 \text{ lb}$
Case III; $P = 360 \text{ lb}$

Figure 15. Frame geometry and loading condition used in examples.

For all members:

Cross sectional area = 30 inch$^2$
Modulus of elasticity = 29000 Ksi
Moment of inertia = 2.25 in$^4$

Figure 16 indicates the node and element numbering used in Finite Element Analysis for all three cases.
The final free body diagram of each member relating to the results of second-order analysis for Case I, Case II and Case III are shown in Figures 17, 18, and 19, respectively.
Figure 17. Free body diagram of final second order analysis for Case I (P = 120 lb).

Figure 18. Free body diagram of final second order analysis for Case II (P = 240 lb).
Figure 19. Free body diagram of final second order analysis for Case III (P = 360 lb).

The results of first order and second order analysis for all three cases are summarized in Table 6.

TABLE 6
RESULTS OF FIRST AND SECOND ORDER ANALYSIS FOR ALL THREE LOAD CASES

<table>
<thead>
<tr>
<th>Moment (lb-inch)</th>
<th>First Order Analysis</th>
<th>Second Order Analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>For All Three Cases</td>
<td>Case I P=120 lb</td>
</tr>
<tr>
<td>M₁</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>M₂</td>
<td>3636.37</td>
<td>3684.75</td>
</tr>
<tr>
<td>M₃</td>
<td>5090.92</td>
<td>5166.17</td>
</tr>
<tr>
<td>M₄</td>
<td>7272.70</td>
<td>7364.79</td>
</tr>
</tbody>
</table>
As the results of second-order analysis indicate, the moments in column AB for the three cases are different, which can yield to three different column selections. However, the results of first-order analysis show the same moments for their load cases, and using the present K method, which was presented in an earlier section of this report, the value of K, as obtained from the alignment chart, will be the same for all cases:

\[ G_A = 1.0 \text{ for fixed-end condition} \]

\[ G_B = \frac{I_c/L_c}{I_g/L_g} = \frac{2.25/200}{2.25/200} = 1.0 \]

Therefore, \( K = 1.3 \) which will lead to the same column selection for column AB in all three load cases.

It is clear then if the column is adequate for Case II, it must be overdesigned for Case I and underdesigned for Case III. Thus by using the second-order analysis, a more efficient design can be obtained.
CHAPTER V
CONCLUSIONS AND COMMENTS

In general, neglecting the effect of axial force on deformed structure (first-order analysis) overestimates the stiffness and the overall strength of the structure. These effects are compensated for in the current allowable-stress technique by using the effective column length and \( C_m \) factors in the design of members. As it was demonstrated in an earlier section of this report, the effective length factor can lead to an over or underestimate of the member size.

A more reliable technique for calculating these secondary moments is introduced in this paper along with a finite element computer program. The derivation of adjusting coefficients for secondary effects is based on stability function and critical load of each member. The computer program will provide nodal displacements and element forces for first- and second-order analysis of the frame structure. Generally, the compressive axial load tends to increase the deformation and moment which is tantamount to a reduction in the strength of the member; on the other hand, tensile axial load will slightly increase the strength of the member.

The final solution obtained by cyclic fashion and the convergence is extremely fast and normally the third and fourth iteration produces acceptable results (if it does not converge within seven or eight trials, there is a strong possibility that the structure is unstable).
The example problems in this paper were performed with a 64 bit of accuracy. However, the author would caution a 32 bit computer can cause losing the accuracy of results because the coefficients used to modify the stiffness matrix were calculated in terms of sinh and cosh functions and have a very small magnitude. Thus, a double precision processor is recommended.

Also, by using a mini computer, the execution time can easily be noticeable due to cyclic techniques used to calculate the secondary moments.

At the present time, the computer program is capable of solving two dimensional frame problems with concentrated and uniform load conditions. However, it can easily be extended to account for members with trapezoid type loading. Also, if the axial force in any member equals the load where buckling occurs, a singular matrix occurs and the results cannot be obtained. This can be a base for further improvement of the program to calculate the buckling load of each member.
APPENDIX I

STABILITY FUNCTION
The coefficients involved in the effects of secondary moments on the stiffness and fixed end action shown in Tables 1 and 2 were developed based on stability function and critical load of each member. The development of some of these coefficients are introduced in the following three parts which include A) Rotational functions; B) Translational function; and C) Fixed end action.

**Rotational Functions**

In order to develop the stiffness coefficients, consider prismatic or uniform beam element i which has no deformation when it is unloaded with length $L_i$.

At the joint 1, moment $K_{11}$ and at the joint 2, restraining moment $K_{21}$ act. The shear forces at end of the beam shown in Figure 20 would be:

$$K_{12} = K_{22} = -\frac{K_{11} + K_{21}}{L_i}$$ (7)
By applying the analysis of buckling load based on the differential equation of an elastic curve, we can write the following equation for the free body diagram shown in Figure 21:

\[ EI \frac{d^2y}{dx^2} = M = -(P_a^i y + K_{11} - K_{12} x) \]  

(8)
where $P_a^i$ is the internal force applied to element $i$. By replacing the value of $K_{12}$ from Equation (7), we can rewrite Equation (8) as follows:

$$EI \frac{d^2y}{dx^2} = -P_a^i y - K_{11} + (K_{11} + K_{21}) \frac{x}{L_i}$$

(9)

The lateral load $P_a^i$ can be expressed as a scalar multiple of $P_i^E$, the pinned end Euler load for beam element $i$ for the buckling in the plane of the applied unit rotation.

$$P_a^i = \lambda_i P_i^E$$

(10)

Since the critical or buckling load for a pinned end uniform member is $P_i^E = \frac{\pi^2 EI}{L_i^2}$, therefore, Equation (10) becomes:

$$P_a^i = \lambda_i \frac{\pi^2 (EI)}{L_i^2}$$

(11)

Substituting Equation (11) into Equation (9) and rearranging gives:

$$\frac{d^2y}{dx^2} + \frac{2}{L_i} \frac{\lambda_i^2}{\pi^2} Y = \frac{(K_{11} + K_{21})}{EI} \frac{x}{L_i} - K_{11}$$

(12)

The general solution of Equation (12) can be expressed as:

$$Y = A \sin\left(\frac{K_{11}}{L_i} X\right) + B \cos\left(\frac{K_{11}}{L_i} X\right) + \frac{L_i^2}{(K_{11} + K_{21})^2} \left[\left(K_{11} + K_{21}\right) x \frac{1}{L_i} - K_{11}\right]$$

(13)
where: \[
KL_i = \sqrt[\pi]{\lambda_i}
\] (14)

Substituting the boundary conditions \(Y = 0\) at \(X = 0\) and \(Y = 0\) at \(X = L_i\) yields values for the constants \(A\) and \(B\):

\[
A = -\frac{L_i}{(KL_i)^2 EI} \left[ K_{11} \cot(KL_i) + K_{21} \csc(KL_i) \right]
\] (15)

and

\[
B = \frac{L_i^2}{(KL_i)^2 EI} K_{11}
\] (16)

Substituting these values of \(A\) and \(B\) into Equation (13) yields:

\[
\frac{(KL_i)^2 EI}{L_i^2} Y = -\left[ K_{11} \cot(KL_i) + K_{21} \csc(KL_i) \right] \sin \left( KL_i \frac{X}{L_i} \right)
+ K_{11} \cos \left( KL_i \frac{X}{L_i} \right) + (K_{11} + K_{21}) \frac{X}{L_i} - K_{11}
\] (17)

Differentiating Equation (17) with respect to \(X\) in order to obtain the slope of the elastic curve:

\[
\frac{(KL_i)^2 EI}{L_i} \frac{dy}{dx} = K_{11} \left[ 1 - (KL_i) \sin \left( KL_i \frac{X}{L_i} \right) - (KL_i) \cot \left( KL_i \right) \right]
\cos \left( KL_i \frac{X}{L_i} \right) + K_{21} \left[ 1 - (KL_i) \csc \left( KL_i \right) \cos \left( KL_i \frac{X}{L_i} \right) \right]
\] (18)

Substituting the boundary condition \(\frac{dy}{dx} = 0\) at \(X = L_i\) into Equation (18), the relation between end moments would be:
Now by considering \( \frac{dy}{dx} = 1 \) at \( X = 0 \) along with Equation (19) into Equation (18), the value of the restraining moment at left end of Figure 2 is given by:

\[
K_{21} = \frac{(KL_i) - \sin(KL_i)}{\sin(KL_i) - (KL_i) \cos(KL_i)} K_{11}
\]  
(19)

Now letting

\[
f_c = 2(1 - \cos(KL_i)) - (KL_i) \sin(KL_i)
\]  
(20)

Equation (20) can be rewritten as:

\[
K_{11} = \frac{KL_i (\sin(KL_i) - KL_i \cos KL_i)}{f_c} \left( \frac{EI}{L_i} \right)
\]  
(21)

By substituting Equation (22) into (19)

\[
K_{21} = \frac{(KL_i) - \sin(KL_i)}{\sin(KL_i) - (KL_i) \cos(KL_i)} \cdot \frac{KL_i (\sin(KL_i) - KL_i \cos KL_i)}{f_c} \left( \frac{EI}{L_i} \right)
\]  
(22)

Thus:

\[
K_{21} = \frac{KL_i (KL_i - \sin KL_i)}{f_c} \left( \frac{EI}{L_i} \right)
\]  
(23)

For zero axial force, \( KL_i \) becomes zero, by applying La'Hospital's rule, which states that in attempting to obtain the limit of the quotient

\[
\frac{f(KL_i)}{g(KL_i)}
\]
Where both functions approach zero as \( KL_i \) approaches zero, one may instead consider the quotient

\[
\frac{f'(KL_i)}{g'(KL_i)}
\]

where \( f'(KL_i) \) and \( g'(KL_i) \) are first derivatives of \( f(KL_i) \) and \( g(KL_i) \) with respect to \( KL_i \).

\[
\frac{KL_i (\sin KL_i - KL_i \cos KL_i)}{2(1 - \cos KL_i) - KL_i \sin KL_i} \quad \lambda = 0
\]

and

\[
\frac{KL_i - \sin (KL_i)}{\sin KL_i - KL_i \cos KL_i} \quad \lambda = 0
\]

By using the above set of formulas, Equations (22) and (24) can reduce to:

\[
K_{11} = \frac{4 \, EI}{L_i}
\]

\[
K_{21} = \frac{2 \, EI}{L_i}
\]

Thus, it is seen for the case of zero axial force, Equations (24) and (25) represent the first-order analysis bending stiffness for a prismatic member. The shear forces at the end of the member can also be represented by substituting the value of \( K_{11} \) and \( K_{21} \) into Equation (7) which leads to:
\[ K_{12} = -K_{22} = \frac{K_{11} + K_{21}}{L_i} \]

or

\[ K_{12} = -K_{22} = \frac{EI}{L_i} \left[ \frac{KL_i (\sin KL_i - KL_i \cos KL_i)}{fc} + \frac{KL_i (KL_i - \sin KL_i)}{fc} \right] \]  \hfill (27)

\[ K_{12} = -K_{22} = \frac{EI}{L_i} \left( KL_i \right)^2 (1 - \cos KL_i) \]  \hfill (28)

A unit rotation at node right and of Figure 2 could lead to the same stiffness values of the left end.

For tensile axial forces, \( \lambda_i \) would have negative value and variable \( KL_i \) can be defined as:

\[ KL_i = \frac{\pi \sqrt{\lambda_i}}{i} = i \pi \sqrt{|\lambda_i|} \]  \hfill (29)

where \( i \) represents the imaginary unit. Since \( KL_i \) is a complex variable, the following functions are defined:

\[ \sin(KL_i) = \frac{e^{i(KL_i)} - e^{-i(KL_i)}}{2i} \]  \hfill (30a)

and

\[ \cos(KL_i) = \frac{e^{i(KL_i)} + e^{-i(KL_i)}}{2} \]  \hfill (30b)

\[ \sinh(KL_i) = \frac{e^{KL_i} + e^{-KL_i}}{2} \]  \hfill (30c)

\[ \cosh(KL_i) = \frac{e^{KL_i} + e^{-KL_i}}{2} \]  \hfill (30d)
By using the above relations into Equations (22), (24), and (28) and after simplification, the following relationships could be written for stiffness matrix with tensile force:

\[
K_{11} = \frac{KL_1 [\sinh (KL_1) - KL_1]}{2-2 \cosh KL_1 + KL_1 \sinh KL_1} \frac{EI}{L_1} \tag{31}
\]

\[
K_{21} = \frac{KL_1 (KL_1 \cosh KL_1 - \sinh KL_1)}{2-2 \cosh KL_1 + KL_1 \sinh KL_1} \frac{EI}{L_1} \tag{32}
\]

\[
K_{12} = \frac{(KL_1)^2 (\cosh KL_1 - 1)}{2-2 \cosh KL_1 + KL_1 \sinh KL_1} \frac{EI}{L_1} \tag{33}
\]

**Translation Function**

The rotation functions are related with a node rotation. Another operation is a node translation, Figure 22. Node 1 is translated through a unit distance relative to Node 2. The translation operation may be alternatively regarded as the rotation of the 1 and 2 through angles of \(\frac{1}{L_1}\) during which the restraining moments \(K_{11}\) and \(K_{21}\) remain unchanged. The lateral force in Figure 23 differs somewhat from the lateral force in Figure 22. But for a small angle of translation the difference may be neglected.
Figure 22. Free body diagram, beam element $i$ with unit translation of end $1$.

Figure 23. Rigid body rotation of beam element $i$. 
Taking moments about Node 2 in Figure 22:

\[ K_{12} = \frac{K_{11} + K_{21} - P_a^i(1)}{L_i} \]  \hspace{1cm} (34a)

From equilibrium condition it is also clear to show:

\[ K_{21} = -K_{22} \]

or

\[ K_{21} = \frac{+P_a^i(1) - (K_{11} + K_{21})}{L_i} \]  \hspace{1cm} (34b)

The rotational relation for members shown in Figure 23 are:

\[ K_{11} = K_{21} = \frac{(K_{L_i})^2(1 - \cos K_{L_i})}{fc} \frac{EI}{L_i} \]  \hspace{1cm} (35)

Replacing Equation (35) into (34a):

\[ K_{12} = -K_{22} = 2\left[\frac{(K_{L_i})^2(1 - \cos K_{L_i})}{fc} \frac{EI}{L_i}\right] - \frac{P_a^i}{L_i^2} \]  \hspace{1cm} (36)

By following the procedure shown in rotational function, it could be shown:

for compressive axial force:

\[ K_{12} = -K_{22} = \frac{(K_{L_i})^3 \sin K_{L_i}}{12(2 - 2 \cos K_{L_i} - K_{L_i} \sin K_{L_i})L_i^3} \frac{EI}{L_i^3} \]  \hspace{1cm} (37)
and for tensile axial force:

\[
K_{12} = -K_{22} = \left(\frac{(KL_i)^3 \sinh KL_i}{2-2 \cosh KL_i + KL_i \sinh KL_i}\right) \frac{EI}{L_i^3}
\]  

(38)

The member stiffness matrix as presented by Equation (5) was defined by rotational and translational function which was shown in previous sections. Axial deformation of the member has been neglected.

**Fixed End Actions**

Let's look at a concentrated load shown in Figure 24. Member i is subjected to compressive axial force \( P_a^i \). The ends 1 and 2 are fixed against rotation and translation with moments \( K_{11} \) and \( K_{21} \) and shear forces \( K_{12} \) and \( K_{22} \).

![Figure 24. Prismatic beam element i with concentrated load W.](image)
Applying the Equation of elasticity, the free body diagram is shown in Figure 25.

Figure 25. Free body diagram, x distance from end 1 when $x < a$.

$$EI \frac{d^2 y}{dx^2} = M = - [K_{11} - P_a (-Y) - K_{12}(X)]$$ (39)

The general solution of Equation (39) would be:

$$Y = A \sin \left( \frac{K L_i}{L_i} X \right) + B \cos \left( \frac{K L_i}{L_i} X \right) - \frac{L_i^2}{(K L_i)^2 \ EI} [K_{11} - K_{12}(X)]$$ (40)

Take a derivative respect to $X$ to yield:

$$\frac{dy}{dx} = \frac{K L_i}{L_i} A \cos \left( \frac{K L_i}{L_i} X \right) - \frac{K L_i}{L_i} B \sin \left( \frac{K L_i}{L_i} X \right) + \frac{L_i^2}{(K L_i)^2 \ EI} K_{12}$$ (41)
By considering the boundary condition:

At $X = 0$

$Y = 0$

$\frac{dy}{dx} = 0$

Therefore:

$$A = \frac{L_i^3}{(KL_i)^3 EI} K_{12} \quad (42a)$$

and

$$B = \frac{L_i^2}{(KL_i)^2 EI} K_{11} \quad (42b)$$

By substituting the values of $A$ and $B$ into Equation (40) and (41) and rearranging, the following expression can be obtained:

$$\frac{(KL_i)^2}{L_i^2 EI} Y = K_{11} \cos \left( \frac{KL_i}{L_i} X \right) - \frac{L_i(K_{12})}{KL_i} \sin \left( \frac{KL_i}{L_i} X \right)$$

$$+ K_{12} (X) - K_{11} \quad 0 \leq X \leq a \quad (43)$$

and

$$\frac{(KL_i)^2}{L_i^2 EI} \frac{dy}{dx} = - \frac{KL_i}{L_i} K_{11} \sin \left( \frac{KL_i}{L_i} X \right) -$$

$$K_{12} \cos \left( \frac{KL_i}{L_i} X \right) + K_{12} \quad 0 \leq X \leq a \quad (44)$$
The free body diagram for the same member, when distance $X$ from end 1 would be greater than $a$, is shown in Figure 26.

![Free body diagram](image)

Figure 26. Free body diagram, $x$ distance from end 1 when $x > a$

By applying the differential equation elasticity for the free body diagram shown in Figure 26 yields:

$$EI \frac{d^2Y}{dx^2} = M = -[K_{12} - P_a(-Y) - K_{12}(x) + (-W)(x-a)] \quad x > a \quad (45)$$

And the general solution of Equation (45) can be shown as:

$$Y = A \sin \left(\frac{K_1}{L_1} X\right) + B \cos \left(\frac{K_1}{L_1} X\right) - \frac{L_1^2}{(KL_1)^2 EI} \left[K_{11} + W(a)ight] - (K_{12} + W)X \quad x > a \quad (46)$$
Differentiating Equation (47) with respect to $x$ yields:

$$\frac{dy}{dx} = \frac{KL_i}{L_i} A \cos \left( \frac{KL_i}{L_i} x \right) - \frac{KL_i}{L_i} B \sin \left( \frac{KL_i}{L_i} x \right) + \frac{L_i^2}{(KL_i)^2 EI}$$

$$(K_{12} + W)$$

Equating the impressions for $\frac{dy}{dx}$ from Equations (44) and (47) and the expression for $Y$ from Equations (43) and (46) for the two segment lengths $a$ and $b$ at $x = a$ yields:

$$A = \frac{-L_i^3}{(KL_i)^3 EI} (K_{12} + W \cos \alpha_k)$$  \hspace{1cm} (48a)

and

$$B = \frac{-L_i^2}{(KL_i)^2 EI} (K_{11} + \frac{LW}{K_i} \sin \alpha_k)$$  \hspace{1cm} (48b)

By substituting these values of $A$ and $B$ into Equations (46) and (47) and rearranging, we get:

$$\frac{KLEI}{L_i} \frac{dy}{dx} = \left( K_{11} + \frac{LW}{K_i} \sin \alpha_k \right) \cos \left( \frac{KL_i}{L_i} x \right) - \frac{KL_i}{K_i} (K_{12} + W \cos \alpha_k)$$

$$\sin \left( \frac{KL_i}{L_i} x \right) + (K_{12} + W)x - K_{11} - W_a \hspace{1cm} a < x < L_i$$  \hspace{1cm} (49)

and

$$\frac{(KL_i)^2}{L_i^2} \frac{dy}{dx} = \frac{KL_i}{L_i \frac{KL_i}{L_i}} \left( K_{11} + \frac{LW}{K_i} \sin \alpha_k \right) \sin \left( \frac{KL_i}{L_i} x \right) - (K_{12} +$$

$$W \cos \alpha_k) \ast \cos \left( \frac{KL_i}{L_i} x \right) + K_{12} + W \hspace{1cm} a \leq x \leq L_i$$  \hspace{1cm} (50)
Finally, by knowing the boundary conditions $Y = 0$ and $\frac{dy}{dx} = 0$ at $x = L_i$, and eliminating $K_{12}$ between Equations (49) and (50), the following expression for $K_{11}$ is obtained:

$$K_{11} = -WL \frac{a}{L} \frac{(1 - \cos K L_i) - (1 - \cos ak)(\frac{\sin K L_i}{K L_i} - \cos K L_i)}{2(1 - \cos K L_i) - K L_i \sin K L_i}$$

$$+ \frac{1 - \cos K L_i}{KL_i} - \sin K L_i)(\sin ak)$$

$$\frac{1}{2(1 - \cos K L_i) - K L_i \sin K L_i}$$

(51)

and also knowing

$$\cos ak \cos K L_i = \frac{1}{2} [\cos (ak + K L_i) + \cos (ak - K L_i)]$$

$$\sin ak \sin K L_i = \frac{1}{2} [\cos (ak - K L_i) - \cos (ak + K L_i)]$$

and total length of member $i$ is $L_i = a + b$, by substituting for $L_i - a = b$, Equation (51) will lead to:

$$K_{11} = -\frac{WL}{K} \frac{\sin K L_i - \sin Ka - \sin Kb - Kb \cos K L_i + KL_i \cos Kb - Ka}{f_c}$$

(52)

For tensile axial load, Equation (52) may be expressed as:

$$K_{11} = -WL \frac{a(1 - \cosh K L_i) - (1 - \cosh ak)(\frac{\sinh K L_i}{K L_i} - \cosh K L_i)}{2(1 - \cosh K L_i) + K L_i \sinh K L_i}$$

$$+ [\frac{1 - \cosh K L_i + \sinh K L_i}{2(1 - \cosh K L_i) + K L_i \sinh K L_i} \sinh ak]$$

(53)
Equation (54) will reduce to:

\[
K_{11} = \frac{-WL}{K} \left[ \frac{\text{Sinh } KL_1 \cdot \text{Sinh } Ka \cdot \text{Sinh } Kb \cdot \text{Cosh } KL_1 + KL_1 \cdot \text{Cosh } Kb \cdot Ka}{2-2 \cdot \text{Cosh } KL_1 + KL_1 \cdot \text{Sinh } KL_1} \right] \quad (54)
\]

The shear forces \(K_{12}\) and \(K_{22}\) can also be obtained by considering equilibrium conditions of the deformed beam element. Their values were shown in Table 2.
APPENDIX II

THE FINITE ELEMENT FORTRAN PROGRAM FOR THE SECOND-ORDER ANALYSIS OF RIGID FRAMES
This program will calculate the effect of secondary moment in cycle fashion and is written in Fortran language. The main program will initialize the Program's Parameters and then call on a set of six different subroutines. The flow chart of the Main Program is shown in Figure 5.
PROGRAM FRAME(INPUT,OUTPUT,DATAF,OUTF,TAPE5=DATAF,TAPE6=OUTF)

PROGRAM 34 - MAIN PROGRAM

STATIC ANALYSIS FOR PLANE FRAME SYSTEMS

COMMON NRMX,NCMX,NDFEL,NN,NE,NLN,NBN,NDF,NNE,NNS,IN,IO,E,G,NLE

DIMENSION X(100),Y(100),COM(200),PROF(200),IB(80),TK(300,30),
*AL(300),REAC(300),FORC(600),ELST(6,6),V(30),PF(600),SUM(200),
*EL(300),SUE(200),PFORC(600)

INITIALIZATION OF PROGRAM PARAMETERS

NRMX = ROW DIMENSION FOR THE STIFFNESS

CMATRIX

NCMX = COLUMN DIMENSION FOR THE STIFFNESS

CMATRIX, ON MAXIMUM HALF BAND WIDTH

ALLOWED

NDF = NUMBER OF DEGREES OF FREEDOM PER

NODE

NNE = NUMBER OF NODES PER ELEMENT, EQUAL

TO 2 FOR BAR SYSTEMS

NDFEL = TOTAL NUMBER OF DEGREES OF FREEDOM

FOR ONE ELEMENT

NRMX=300
NCMX=300
NIF=3
NNE=5
NDFEL=NDF*NNE

ASSIGN DATA SET NUMBERS TO IN, FOR INPUT,

AND IO, FOR OUTPUT

IN=5
IO=6

DO 1001 I=1,NE
FORC((I-1)*6+1)=0
PFORC((I-1)*6+1)=0
1001 FF((I-1)*6+1)=0

APPLY THE ANALYSIS STEPS

DO 310 IEND=1,11
DO 410 IH=1,NE
PFORC((IH-1)*6+1)=FORC((IH-1)*6+1)
410 CONTINUE

DO 801 II=1,NE
DO 800 12=2,6
REAC((I1-1)*6+1)=0
REAC((I1-1)*6+12)=0
800 CONTINUE
801 CONTINUE
C INPUT
   CALL INPUTT(X,Y,CON,PROP,AL,IB,REAC,FORC,SUM,BL,SUE)
C ASSEMBLING OF TOTAL STIFFNESS MATRIX
   PFORC((I-1)*6+1)=FORC((I-1)*6+1)
C
   CALL ASSEXY,Y,CON,PROP,TK,ELST,AL,FORC,PFORC)
C
C INTRODUCTION OF BOUNDARY CONDITIONS
C
   CALL BOUND(TK,AL,REAC,IB)
C
C SOLUTION OF THE SYSTEM OF EQUATIONS
C
   CALL SLBS1(TK,AL,V,N,MS,NRMX,NCMX)
C
C COMPUTATION OF MEMBER FORCES
C
   CALL FORCE(CON,PROP,FORC,REAC,X,Y,AL,PFORC)
C
C OUTPUT
C
   CALL OUTPT(CON,AL,FORC,REAC)
C
   DO 1002 I=1,NE
   IFABS(FORC((I-1)*6+1)-PF((I-1)*6+1)),GE.0.0001) GO TO 1005
1002 CONTINUE

1005 DO 1003 I=1,NE
1003 PF((I-1)*6+1)=FORC((I-1)*6+1)
310 CONTINUE
END

SUBROUTINE FEM (IEL,X,Y,CON,FORCE,YI,SUM,G,IS,IE,SUE)

C
C COMPUTATION OF FIXED-END ACTION
C
   COMMON NRMX,NCMX,NDFEL,NN,NE,NLN,NBN,NDF,NNE,NS,IN,IO,E,GNLE
   DIMENSION X(1),Y(1),CON(1),FORC(300),SUM(6),SUE(6)
C
C IEL = CURRENT ELEMENT
C N1 = NUMBER OF START NODE
C N2 = NUMBER OF END NODE
C
C PB = BUCKLING LOAD
C SUM = END ACTION FOR START NODE
C SUE = END ACTION FOR END NODE
??
C SARM = LENGTH OF PARTIAL ELEMENT
C A = DISTANCE FROM CENTER OF ELEMENT TO START NODE
C B = DISTANCE FROM CENTER OF ELEMENT TO END NODE
C
LNKE = (NL-1)
N1 = COM(L+1)
N2 = COM(L+2)
DX = X(N2) - X(N1)
DY = Y(N2) - Y(N1)
D = SQRT(DX**2 + DY**2)
SARM = D / 50
A = SARM / 2
B = A**3
M = 3**NN
DO 50 II = 1, M
SUM(II) = 0
SUE(II) = 0
50 CONTINUE
IF (FORC((IS-1)*&+1)) 101, 111, 121
C COMPUTE FIXED-END ACTION FOR TENSILE MEMBER
C 101 PB = ((3.1415*2)*(E*Y2))/(D**2)
VB = (-FORC((IS+1)*&+1)/PB)**.5
BK = (-FORC((IS-1)*&+1)/(E*Y2))**.5
FT = 2.2*COSH(BK*D) + (BK*D)*Sinh(BK*D)
DO 103 KK = 1, 50
JJ = NDF*(IS-1)
SUM(JJ+1) = 0
SUM(JJ+2) = SUM(JJ+2) + SARM*Q*((1/FT)*COSH(BK*D) - COSH(BK*A) + COSH(BK* B) + (BK*B)*SINH(BK*D)-1))
SUM(JJ+3) = SUM(JJ+3) + SARM*Q*((1/FT)*SINH(BK*D) - SINH(BK*A) - SIN
*H(BK*B) - (BK*B)*COSH(BK*D) + (BK*D)*COSH(BK*B) - BK*A))
JJ = NDF*(IE-1)
SUM(JJ+1) = 0
SUE(JJ+2) = SUE(JJ+2) + SARM*Q*((1/FT)*COSH(BK*D) - COSH(BK*B) + COSH(BK
**A) - (BK*A)*SINH(BK*D)-1))
SUE(JJ+3) = SUE(JJ+3) + SARM*Q*((1/FT)*-SINH(BK*D) + SINH(BK*B) + SIN
*H(BK*A) + (BK*A)*COSH(BK*D) - (BK*D)*COSH(BK*A) + (BK*B)))
A = A + SARM
B = B - A
103 CONTINUE
RETURN
C COMPUTE FIXED-END ACTION FOR NONZERO FORCE MEMBER
C 111 DO 104 KK = 1, 50
JJ = NDF*(IS-1)
SUM(JJ+1) = 0
SUM(JJ+2) = SUM(JJ+2) + SARM*Q*(-B**2*(3*A+B)/D**3)
??
SUM(J+3) = SUM(J+3) + SARM*C*(A+2)*E/D+2
JU = NDF*(IE-1)
SUE(J+1) = 0
SUE(J+2) = SUE(J+2) + SARM*C*(-A+2)*(A+3*E)/D+2
SUE(J+3) = SUE(J+3) + SARM*C*(A+2)*E/D+2
A = A+SARM
B = D-A
104 CONTINUE
RETURN
C COMPUTE FIXED-END ACTION FOR COMPRESSION MEMBER
C
121 PB=((3.1415*2)*(E+Y))/D+2
VH=(FORC((IS-1)*6+1)/PB)**.5
Bi=(FORC((IS-1)*6+1)/PB)**.5
FC=2*COS(Bi*D)-SIN(Bi*D)
SUM(J+1)=SUM(J+1) + A*SARM*(1/FC)*COS(Bi*D)*SIN(Bi*A)+COS(Bi*E)+*SIN(Bi*D-1)
SUM(J+2)=SUM(J+2) + A*SARM*(1/FC)*COS(Bi*D)*SIN(Bi*A)+COS(Bi*E)+*SIN(Bi*D)
SUM(J+3)=SUM(J+3) + A*SARM*(1/FC)*COS(Bi*D)*SIN(Bi*A)+COS(Bi*E)+*SIN(Bi*D-1)
SUM(J+4)=SUM(J+4) + A*SARM*(1/FC)*COS(Bi*D)*SIN(Bi*A)+COS(Bi*E)+*SIN(Bi*D)
SUM(J+5)=SUM(J+5) + A*SARM*(1/FC)*COS(Bi*D)*SIN(Bi*A)+COS(Bi*E)+*SIN(Bi*D)
SUM(J+6)=SUM(J+6) + A*SARM*(1/FC)*COS(Bi*D)*SIN(Bi*A)+COS(Bi*E)+*SIN(Bi*D)
A = A+SARM
C
C B = D-A
105 CONTINUE
RETURN
END
C
PROGRAM 35 - INPUTT PROGRAM
C SUBROUTINE INPUTT(X,Y,CON,PROP,AL,IB,REAC,FORC,SUM,BL,SUE)
C
COMMON NRMX,NCMX,NDFEL,NN,NE,NLN,NN,MS,IN,IO,E,G,NLE
DIMENSION X(1),Y(1),CON(1),PROP(1),AL(1),IB(1),REAC(1),W(2),IC(3),
*FORC(600),SUM(300),BL(1),SUE(200)
C ELEMENT PROPERTIES, NODAL LOADS, AND PRESCRIBED UNKNOWN VALUES
C IC = AUXILIARY ARRAY TO STORE TEMPORARLY THE CONNECTIVITY
C INDICATORS
C READ BASIC PARAMETERS
C
C NN = NUMBER OF NODES
C NE = NUMBER OF ELEMENTS
C NLN = NUMBER OF LOADED NODES
C NBN = NUMBER OF BOUNDARY NODES
C NLE = NUMBER OF LOADED ELEMENT
C E = MODULUS OF ELASTICITY
C
WRITE(10,20)
20 FORMAT("\",72("*"))
READ(IN,1) NN,NE,NLN,NBN,E,NLE
WRITE(10,21) NN,NE,NLN,NBN,E,NLE
21 FORMAT("\"INTERNAL DATA\"\" NUMBER OF NODES :","15/" NUMBE
*R OF ELEMENTS :",15/" NUMBER OF LOADED NODES :","15/" NUMBE
*OF SUPPORT NODES :","15/" MODULUS OF ELASTICITY :",F10.0/" NUMBER
* OF LOADED ELEMENTS :",I4//\"NODAL COORDINATES\"/7X,"NODE",6X,"Y",
*9X,"Y")
1 FORMAT(4110,1F10.0,110)
C REAL NODAL COORDINATES IN ARRAY X AND Y
C
READ(IN,2) (I,X(I),Y(I), J=1,NN)
WRITE(10,2) (I,X(I),Y(I), I=1,NN)
2 FORMAT(110,2F10.4)
C READ ELEMENT CONNECTIVITY IN ARRAY CON
C AND ELEMENT PROPERTIES IN ARRAY PROP
C
WRITE(10,22)
22 FORMAT("\"ELEMENT CONNECTIVITY AND PROPERTIES\"/4X,"ELEMENT",5X,"ST
*ART NODE END NODE",5X,"AREA", 5X, "M. OF INERTIA")
DO 3 I=1,NE
READ(IN,4) ITEP,IC(1),IC(2),W(1),W(2)
WRITE(10,24) ITEP,IC(1),IC(2),W(1),W(2)
34 FORMAT(3110,2F15.5)
4 FORMAT(3110,2F10.5)
N1=NN*1(I-1)
PROP(N1+1)=W(1)
PROP(N1+2)=W(2)
CON(N1+1)=IC(1)
3 CON(N1+2)=IC(2)
C COMPUTE N, ACTUAL NUMBER OF UNKNOWNS, AND CLEAR THE LOAD
C VECTOR
C
WRITE(10,23)
N=NN*NDF
??
DO 5 I=1,N
5 BL(I)=0.
C READ THE NODAL LOADS AND STORE THEM IN ARRAY AL
C
IF (NLN.EQ.0.) GO TO 166
DO 6 I=1,NLN
READ (IN,8) J, (W(K),K=1,NDF)
WRITE(10,8) J, (W(K),K=1,NDF)
8 FORMAT(I10,3F10.2)
DO 6 K=1,NDF
L=NDF*(J-1)+K
BL(L)=W(K)
6 CONTINUE
166 L=NLN+NDF
DO 180 JJ=1,K
AL(JJ)=0
SUE(JJ)=0
180 SUM(JJ)=0
C IF (NLN.EQ.0.) GO TO 80
WRITE(10,30)
DO 44 MM=1,NLF
30 FORMAT("NUMBER OF",5X,"START",8X,"END",8X,"UNIFORM"/1,"LOAD"
*O ELEMENT",4X,"NODE",3X,"NODE",8X,"LOAD")
READ (IN,88) IEL,IS,IE,0
88 FORMAT(2I10,F15.5)
WRITE(10,99) IEL,IS,IE,0
99 FORMAT(116,113,17,F10.5)
L3=NNE*(IEL-1)
Y2=FF0P(L3+2)
CALL FEM (IEL,1,Y,CON,FORC,YZ,SUM,O,IS,IE,SUE)
44 CONTINUE
C READ BOUNDARY NODES DATA, STORE UNKNOWNS STATUS
C INDICATORS IN ARRAY IB, AND PRESCRIBED UNKNOWNS:
C VALUES IN ARRAY REAC
C
26 WRITE(10,24)
24 FORMAT("BOUNDARY CONDITION DATA"/27X,"STATUS",24X,"PRESCRIBED "
+,16X,"U",9X,"V",8X,"W")
DO 7 I=1,NBN
READ(IN,9) J, (IC(K),K=1,NDF), (W(K),K=1,NDF)
WRITE(10,10) J, (IC(K),K=1,NDF), (W(K),K=1,NDF)
??
L1=(NDF*1)*(I-1)+1
L2=NDF*(J-1)
I1=L1+1
I2=L2+1
IF(N1)=IC(I)
7 FORMAT(4110,3F10,4)
9 FORMAT(4110,10X,3F10,4)
RETURN
END
C

SUBROUTINE PIDELTA(FORC, YZ, D, AK1, AK2, AK3, AK4, I)
C
C COMPUTATION OF COEF. USED IN STIFFNESS MATRIX
C
COMMON NRMY, NZMY, NDFEL, NN, NE, NLN, NEN, NDF, NNF, K, MS, IN, IO, E, G, NLE
DIMENSION FORC(600)
C
YZ = MOMENT OF INERTIA OF ELEMENT CROSS-SECTION
PB = BUCKLING LOAD
C AK1, AK2, AK3, AK4 ARE COEF USED IN STIFFNESS MATRIX
C
IF(FORC((I-1)*6+1),LE.0.) GO TO 500
C
COMPUTE THE COEF. FOR COMPRESSION MEMBER
C
B1=(FORC((I-1)*6+1)/(F*YZ))**.5
FC=2-2*COS(B1*DI)-(B1*DI)*((SIN(B1*DI))
A1=(((B1*DI)**3)*SIN(D1*DI))/(12*FC)
A2=(((B1*DI)**2)*(1-COS(B1*DI)))/(6*FC)
A3=((B1*DI)*(SIN(B1*DI)-(B1*DI)*COS(B1*DI)))/(4*FC)
GO TO 1004
C
COMPUTE THE COEF. FOR TENSION MEMBER
C
500 IF(FORC((I-1)*6+1),LE.0.) GO TO 600
B1=(-FORC((I-1)*6+1)/(E*YZ))**.5
FT=2-2*COSH(B1*DI)+(B1*DI)*SINH(B1*DI)
A1=((((B1*DI)**3)*SINH(B1*DI))/(12*FT)
A2=(((B1*DI)**2)*(COSH(B1*DI)-1))/(6*FT)
A3=2*(B1*DI)*((COSH(B1*DI)-1))/(6*FT)
GO TO 1004
C
COMPUTE THE COEF. FOR NONZERO MEMBER
C
??
C PROGRAM 36
C COMPUTATION OF ELEMENT STIFFNESS MATRIX FOR THE CURRENT ELEMENT
C COMMON NRMX, NCMX, NDFEL, N1, NC, NLN, NEN, NDF, NINE, NME, IN, ID, IE, NLE
DIMENSION Y(1), Y(1), CON(1), PROF(1), ELST(6, 6), AL(1)
* ROT(6, 6), V(6), FORC(600)
C NEL = CURRENT ELEMENT NUMBER
C N1 = NUMBER OF START NODE
C N2 = NUMBER OF END NODE
C AX = AREA OF ELEMENT CROSS-SECTION
C YZ = MOMENT OF INERTIA OF ELEMENT CROSS-SECTION
C L=NLE+(NEL-1)
N1=CON(L+1)
N2=CON(L+2)
AX=PROF(L+1)
YZ=PROF(L+2)
C X AXIS, AND STORE IN D, SI, AND CO, RESPECTIVELY
C
DX=Y(N2)-Y(N1)
DY=Y(N2)-Y(N1)
I=SORT(DX**2+DY**2)
CALL PDELTAT(FORC,YZ,D,AX1,AX2,AX3,AX4,1)
CO=DX/D
SI=DX/D
DO 1 I=1,6
DO 1 J=1,6
ELST(I,J)=0.
C FORM ELEMENT ROTATION MATRIX
C
1 ROT(1,J)=0.
ROT(1,1)=CO
ROT(1,2)=SI
ROT(2,1)=-SI
ROT(2,2)=CO
ROT(3,3)=1.
DO 2 I=1,3
DO 2 J=1,3
??
2 ROT(I+3*J+3)=ROT(I,J)
C
C COMPUTE ELEMENT LOCAL STIFFNESS MATRIX
C
ELST(1,1)=E*AY/D
ELST(1,4)=-ELST(1,1)
ELST(2,2)=(12*E*YZ/D*3)*AY1
ELST(2,3)=(6*E*YZ/(D*D))*AY2
ELST(2,5)=-ELST(2,2)
ELST(2,6)=ELST(2,3)
ELST(3,2)=ELST(2,3)
ELST(3,3)=(4*E*YZ/D)*AY3
ELST(3,5)=-ELST(2,3)
ELST(4,6)=(2*E*YZ/D)*AY4
ELST(4,1)=ELST(1,4)
ELST(4,4)=ELST(1,1)
ELST(5,2)=ELST(2,5)
ELST(5,3)=ELST(3,5)
ELST(5,5)=ELST(2,2)
ELST(5,6)=ELST(3,5)
ELST(6,2)=ELST(2,6)
ELST(6,3)=ELST(3,6)
ELST(6,5)=ELST(5,6)
ELST(6,6)=ELST(3,3)
C
C ROTATE ELEMENT STIFFNESS MATRIX TO GLOBAL COORDINATES
C
CALL BTA3(ELST,ROT,V,NDFEL,NDFEL)
RETURN
END
C
SUBROUTINE FORCE(CON,PROP,FORC,REAC,Y,Y,AL,FFORC)

C
PROGRAM 37

C COMPUTATION OF ELEMENT FORCES
C
COMMON NRMX,NCMX,NDFEL,NN,NE,NN,NBN,NDF,NNE,N.MS,IN,IO,E,ENL
DIMENSION CON(1),PROP(1),FORC(1),REAC(1),Y(1),Y(1),AL(1),ROT(3,3),
*U(6),F(6),UL(6),FG(6),FFORC(600)
DO 1 I=1,N
1 REAC(I)=0.
DO 100 NEL=1,NE
C
NEL = CURRENT ELEMENT NUMBER
C N1 = NUMBER OF START NODE
C N2 = NUMBER OF END NODE
C AX = AREA OF ELEMENT CROSS-SECTION
C YZ = MOMENT OF INERTIA OF ELEMENT CROSS-SECTION
C
L=NNE*(NEL-1)
N1=CON(L+1)
N2=CON(L+2)
AX=PROF(L+1)
Y2=PROF(L+2)

C COMPUTE LENGTH OF ELEMENT, AND SINE AND COSINE OF ITS LOCAL X AXIS, AND STORE IN D, S1, AND CO, RESPECTIVELY
DX=X(N2)-X(N1)
DY=Y(N2)-Y(N1)
D=SQR(DX**2+DY**2)

C CO=DX/D
$1=DY/D

C FORM ELEMENT ROTATION MATRIX

C ROT(1,1)=CO
ROT(1,2)=SI
ROT(1,3)=0.
ROT(2,1)=SI
ROT(2,2)=-CO
ROT(2,3)=0.
ROT(3,1)=C.
ROT(3,2)=0.
ROT(3,3)=1.

C ROTATE ELEMENT NODAL DISPLACEMENTS TO ELEMENT LOCAL
C REFERENCE FRAME, AND STORE IN ARRAY UL

C K1=NDF*(N1-1)
K2=NDF*(N2-1)
DO 2 I=1,3
  J1=J+1
  J2=K+1
  U(J)=AL(J1)
2 U(I+3)=AL(J2)
  DO 3 I=1,3
    UL(I)=0.
    UL(I+3)=0.
  3 DO 3 J=1,3
    UL(I)=UL(I)*ROT(I,J)*U(J)
  3 UL(I+3)=UL(I+3)+ROT(I,J)*UL(J+3)

C COMPUTE MEMBER END FORCES IN LOCAL COORDINATES

C IPP=NEL
CALL FDELTAB(FFORC,YZ,D,AK1,AK2,AK3,AK4,IPP)
F(1)=E*AX/D*(UL(1)-UL(4))
F(2)=E*YZ*AK1/(D**3)+(UL(2)-UL(5))*6*E*YZ*AK2/(D*D)+/
(UL(3)+UL(6))
F(3)=6*E*YZ*AK2/(D*D)*(UL(2)-UL(5))+2*E*YZ/D*(...
C STORE MEMBER END FORCES IN ARRAY FORC
F(4)=-F(1)
F(5)=-F(2)
I1=I1+(NEL-1)
C
DO 4 I=1,6
   J2=I1+I
4 FORC(J2)=F(I)
C ROTATE MEMBER END FORCES TO THE GLOBAL REFERENCE FRAME
C AND STORE IN ARRAY FG
C
   DO 5 J=1,3
      FG(1)=-D
      FG(1+3)=0.
   DO 5 J=1,3
      FG(J)=FH(J)*COS(J-1)*F(J)
5 FG(1+3)=FG(I+3)*COS(J-1)*F(J+3)
C ADD ELEMENT CONTRIBUTION TO NOEI-L RESULTS.
C IN ARRAY REAC
   DO 6 I=1,3
      J1=I1+I
      J2=I2+I
      REAC(J1)=REAC(J1)+FG(I)
6 REAC(J2)=REAC(J2)+FG(I+3)
100 CONTINUE
RETURN
END
SUBROUTINE OUTPUT(CON,AL,FORC,REAC)
C
PROGRAM 38 - OUTPUT PROGRAM
C
COMMON NRMX,NCMX,NDL,E,N,NN,LNL,NBN,NDF,NNE,NMS,IN,IO,E,G,NE
DIMENSION CON(1),AL(1),FORC(1),REAC(1)
C C WRITE NODAL DISPLACEMENTS
C
WRITE(10,1)
1 FORMAT(/1X,72("*")//" RESULTS"//" NODAL DISPLACEMENTS"/7X,"NODE"
   DO 10 I=1,NN
      K1=NDF*(I-1)+1
      K2=K1+NDF-1
10 WRITE(10,2)I,AL(J),J=K1,K2
2 FORMAT(110,6F15.6)
WRITE NODAL REACTIONS
WRITE(10,5)
2 FORMAT(/" NODAL REACTIONS":/'X,"NODEN",10),"FX",13X,"FY",13X,"M2")
   DO 20 I=1,NN
   K1=NDF*(I-1)+1
   K2=K1+NDF-1
   20 WRITE(10,2) I,((REAC(J)),J=K1,K2)
WRITE MEMBER END FORCES
WRITE(10,4)
   DO 30 I=1,NE
   K1=6*(I-1)+1
   K2=K1+2
   30 WRITE(10,7) ICON,((FORC(J)),J=K1,K2)
WRITE(10,5)
5 FORMAT(//1X,72(" "))
RETURN
END
SUBROUTINE BOUND(TI,AL,REAC,II)

PROGRAM 31

INTRODUCTION OF THE BOUNDARY CONDITIONS
COMMON NRMX,NCMX,NDFEL,NN,NE,NLN,NBN,NDF,NNE,NS,E,IN,IO,E,NLE
DIMENSION AL(1),IB(1),REAC(1),TI(300,30)
DO 100 L=1,NBN
NO = NUMBER OF THE CURRENT BOUNDARY NODE
L1=(NDF+1)*(L-1)+1
NO=IB(L1)
K1=NDF*(NO-1)
DO 100 I=1,NDF
L2=L1+I
IF(IB(L2))100,10,100
C PRESCRIBED UNKNOWN TO BE CONSIDERED
C PLACE PRESCRIBED UNKNOWN VALUE IN AL C
C SET DIAGONAL COEFFICIENT OF Ti TO EQUAL TO 1 C
10 KR=1+1
   DO 50 J=2,MS
      IV=KR+J-1
      IF (N-IV) .GE. 20, 20
C MODIFY ROW OF Ti AND CORRESPONDING ELEMENTS IN AL C
20 AL (IV)=AL (IV) - Ti (IV,J) * REAC (Kr)
   Ti (IV,J)=0
30 IV=1F- J+1
      IF (IV) .LE. 50, 50, 40
C MODIFY COLUMN IN Ti AND CORRESPONDING ELEMENT IN AL C
40 AL (IV)=AL (IV) - Ti (IV,J) * REAC (Kr)
   Ti (IV,J)=0
50 CONTINUE
   Ti (IV,1)=1
   AL (IV)=REAC (Kr)
100 CONTINUE
   RETURN
END
SUBROUTINE ASSEMBLY,Y,CON,PROP,Ti,ELST,AL,FORC,FFORC C
C ASSEMBLING OF THE TOTAL MATRIX FOR THE PROBLEM C
COMMON NAXY,NMX,NDFEL,NN,NE,NLN,NBU,NDF,NUE,N,MX,IN,10,E,G,NLE
DIMENSION X(1),Y(1),CON(1),Ti(300,30),ELST(6,6),
*PROP(1),AL(1),FORC(600),FFORC(600) C
C COMPUTE HALF BAND WIDTH AND STORE IN MS C
N1=NNE-1
MS=0
DO 7 J=1,NE
   L1=NNE*(1-J)
7 DO 7 J=1,N1
   L2=L1+J
   J1=J+1
   DO 7 K=J1,NNE
      L3=ABS (CON (L2)-CON (L3))
   IF (MS-L) .LE. 7, 7
   MS=L
??
7 CONTINUE
MS=NSD * (MS+1)

CLEAR THE TOTAL STIFFNESS MATRIX

DO 10 I=1,N
DO 10 J=1,MS
10 TI(I,J)=0

LOOP ON THE ELEMENTS AND ASSEMBLE THE TOTAL STIFFNESS MATRIX

DO 20 NEL=1,NE
   I=NEL

STIFF COMPUTES THE STIFFNESS MATRIX FOR ELEMENT NEL
   CALL STIFF(NEL,X,Y,PROF,CON,ELST,AL,1,FORC)

ELASS PLACES THE STIFFNESS MATRIX OF ELEMENT NEL IN THE TOTAL
   STIFFNESS MATRIX

20 CALL ELASS(NEL,CON,Ti,ELST)
   RETURN
END
SUBROUTINE ELASS(NEL,CON,TM,ELMAT)

PROGRAM 30

THIS PROGRAM STORES THE ELEMENT MATRIX FOR ELEMENT NEL IN
   THE TOTAL MATRIX FOR THE PROBLEM

COMMON: NRMY,NCMX,NDFEL,NN,NE,NNN,NNN,NDF,NNE,M,M,S,NSC,EL,6,NLE
DIMENSION CON(1),TM(300,30),ELMAT(6,6)

NEL = CURRENT ELEMENT NUMBER
N1 = NUMBER OF START NODE
N2 = NUMBER OF END NODE

L1=NNE*(NEL-1)
   DO 50 I=1,NNE
      L2=L1+I
      N1=CON(L2)
      I1=NSD*(I-1)
      J1=NSD*(N1-1)
      DO 50 J=1,NNE
         L2=L1+J
         N2=CON(L2)
         I2=NSD*(J-1)
         J2=NSD*(N2-1)
         DO 50 K=1,NDF

??
K1=1
IF (N1-N2)20,10,30

C C STORE A DIAGONAL SUBMATRIX
C
10 K1=K:
C C STORE AN OFF DIAGONAL SUBMATRIX
C 20 KR=J1+K:
    IC=J1-KR+1
    K1=11+K:
    GO TO 40
C C STORE THE TRANSPOSE OF AN OFF DIAGONAL MATRIX
C 30 KR=J2+K:
    IC=J1-KR+1
    K2=12+K:
40 DO 50 L=K1,NDF
    KC=IC+L
    IF (N1-N2)45,45,46
45 K2=12+L
    GO TO 50
46 K1=11+L
50 TM(KR,KC)=TM(KR,KC)+ELMAT(K1,K2)
RETURN
END
SUBROUTINE SLISI(A,B,D,N,MS,NX,MY)

C C PROGRAM 15
C C SOLUTION OF LINEAR SYSTEM OF EQUATIONS
C BY THE GAUSS ELIMINATION METHOD, FOR
C SYMMETRIC BANDED SYSTEMS
C C A : ARRAY CONTAINING THE UPER TRIANGULAR
C PART OF THE SYSTEM MATRIX, STORED
C ACCORDING TO THE SYMMETRIC BANDED
C SCHEME
C B : ORIGINALLY IT CONTAINS THE INDEPENDENT
C COEFFICIENTS, AFTER SOLUTION IT CONTAINS
C THE VALUES OF THE SYSTEM UNKNOWNS.
C C N : ACTUAL NUMBER OF UNKNOWNS
C N : ACTUAL NUMBER OF UNKNOWNS
C MS : ACTUAL HALF BANDWIDTH
C NX : ROW DIMENSION OF A AND B
C MX : COLUMN DIMENSION OF A
C ??
C PROGRAM 9
C THIS PROGRAM COMPUTES THE MATRIX
C OPERATION A = TRANSPOSE(B) + A * B.
C WHERE A AND B ARE SQUARE MATRICES
C N : ACTUAL ORDER OF A AND B
C NX : ROW AND COLUMN DIMENSION OF A AND B
C V : AUXILIARY VECTOR
C DIMENSION A(NX,NX),B(NX,NX),V(NX)
C COMPUTE A * B AND STORE IN A
C DO 10 I=1,N
   DO 5 J=1,N
      V(J)=0.
   DO 5 J=1,N
      5 V(J)=V(J)+A(I,J)*B(I,J)
   DO 10 J=1,N
   10 A(I,J)=V(J)
C COMPUTE TRANSPOSE(B) * A AND STORE IN A
C DO 20 J=1,N
   DO 15 I=1,N
      V(I)=0.
   DO 15 I=1,N
      15 V(I)=V(I)+B(I,J)*A(I,J)
   DO 20 I=1,N
   20 A(I,J)=V(I)
RETURN
END
--END--
END OF FILE
APPENDIX III

INPUT AND OUTPUT RESULTS OF FIRST-ORDER AND THREE CYCLES OF SECOND-ORDER ANALYSIS FOR EXAMPLE 1
INTERNAL DATA

NUMBER OF NODES: 5
NUMBER OF ELEMENTS: 4
NUMBER OF LOADED NODES: 2
NUMBER OF SUPPORT NODES: 2
MODULUS OF ELASTICITY: 29000.
NUMBER OF LOADED ELEMENTS: 0

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ELEMENT CONNECTIVITY AND PROPERTIES

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BOUNDARY CONDITION DATA

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RESULTS

NODAL DISPLACEMENTS

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### Results

#### Nodal Displacements

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#### Nodal Reactions

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6 5 8.6980 2.5010 377.3274
7 5 -8.6980 -2.5010 311.4908

RESULTS

NODAL DISPLACEMENTS

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NODAL REACTIONS

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RESULTS

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REFERENCES


83