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Motion of a helical vortex filament in superfluid ^4He under the extrinsic form of the local induction approximation

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Very recently, Shivamoggi [“Vortex motion in superfluid ^4He : Reformulation in the extrinsic vortex-filament coordinate space,” *Phys. Rev. B* **84**, 012506 (2011)] studied the extrinsic form of the local induction approximation (LIA) for the motion of a Kelvin wave on a vortex filament in superfluid ^4He , and obtained some results in a cubic approximation. Presently, we study the motion of helical vortex filaments in superfluid ^4He under the exact fully nonlinear LIA considered in potential form by Van Gorder [“Fully nonlinear local induction equation describing the motion of a vortex filament in superfluid ^4He ,” *J. Fluid Mech.* **707**, 585 (2012)] and obtained from the Biot-Savart law through the equations of Hall and Vinen [“The rotation of liquid helium II. I. Experiments on the propagation of second sound in uniformly rotating helium II,” *Proc. R. Soc. London, Ser. A* **238**, 204 (1956)] including superfluid friction terms. Nonlinear dispersion relations governing the helical Kelvin wave on such a vortex filament are derived in exact form, from which we may exactly calculate the phase and group velocity of the Kelvin wave. With this, we classify the motion of a helical Kelvin wave on a vortex filament under the LIA. The dispersion relations and results, which follow are exact in nature, in contrast to most results in the literature, which are usually numerical approximations. As such, our results accurately capture the qualitative behavior of the Kelvin waves under the LIA. Extensions to other frameworks are discussed. © 2013 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4816639>]

I. INTRODUCTION

The self-induced velocity of a vortex filament has been described by the approximation $\mathbf{v} = \gamma \kappa \mathbf{t} \times \mathbf{n}$ (Da Rios,¹ Arms and Hama²), where $\mathbf{t}$ and $\mathbf{n}$ are unit tangent and unit normal vectors to the vortex filament, respectively, κ is the curvature, and γ is the strength of the vortex filament. The Da Rios equations have an interesting history stretching back over the last century; for an interesting account of the history of the Da Rios equations, see Ricca.³ This approximation is called the local induction approximation (LIA), and is useful in rather tame fluids without vortex interactions and the like. For more complicated regimes, the approximation naturally breaks down.

The superfluid regime is one in which the LIA can usefully be applied as a reasonable approximation due to zero viscosity, which allows the persistence of vortex filament structures.^{4,5} In this setting, the LIA can be used to model Kelvin waves on a vortex filament in superfluid ^4He . In the present paper, we study vortex filaments under the Cartesian reference frame (the extrinsic coordinate system⁶) for the LIA. Recently, a cubic approximation to this model was considered.⁷ The fully nonlinear form of the LIA, however, is required in order to pick up the full qualitative features of the solutions under the LIA.

Kelvin waves have previously been considered in a number of settings⁷ in order to study superfluid turbulence. That said, exact solutions for either the fully nonlinear or the approximate

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forms of the Hall-Vinen form of the LIA including superfluid friction parameters are rare. However, classification of solutions to such models is useful, as they set the foundation for numerical solutions under more complicated conditions, such as reconnection. The approximate (some terms with α , α' were dropped) solution presented by Shivamoggi⁷ was that of a propagating damped Kelvin kink on a vortex, which is reminiscent of the soliton on a vortex filament studied by Hasimoto.⁸ Some results for Hasimoto-type⁹ planar vortex filaments have also been given.⁶ Helical waves are a useful type of solution, and have been considered for the standard LIA¹⁰ (corresponding to setting the superfluid friction parameters to zero).

It should be noted, however, that for non-zero values of the superfluid friction parameters, there are no helical solutions studied for any of the models in the literature, full or approximate. Upon writing the LIA for a superfluid in potential form, we are able to construct helical Kelvin wave solutions, and accordingly determine the relevant exact dispersion relations between the frequency ω and wave number k . In the case where the friction parameters are non-zero, the amplitude A of a helical Kelvin wave is bounded and depends non-linearly on the wave number k . A degenerate case is also mentioned, which highlights a qualitative difference in the two friction parameters introduced by Hall and Vinen.⁵ Comparisons of the present results with those for approximating models^{7,11} related to the fully nonlinear LIA are made; while such models agree with the fully nonlinear LIA for small wave number solutions, the agreement breaks down for larger values of the wave number, both quantitatively and qualitatively. We are able to determine the helical solutions exactly in terms of the model parameters, and for each case we are able to determine the phase and group velocity of the solutions in closed form. Therefore, we are able to completely classify the motion of a helical filament in superfluid ⁴He under the LIA. For the Laurie *et al.*¹¹ model, we also give results for the planar vortex filament.

The present results allow us to see exactly how the inclusion of the superfluid friction parameters influences the qualitative behavior of filament solutions in general, and the quantitative behavior of helical structures in particular. Such a solution is fundamental and serves as a baseline of comparison with other analytical solutions, and also for more complicated solutions which must be found numerically. For these reasons, the paper furthers our understanding of superfluid turbulence.

II. FORMULATION OF THE LIA

Including the effect of frictional force exerted by the normal fluid on a vortex, the self-induced velocity of the vortex in the reference frame moving with the superfluid according to the local induction approximation is given in non-dimensional form by Refs. 4 and 5,

$$\mathbf{v} = \gamma \kappa \mathbf{t} \times \mathbf{n} + \alpha \mathbf{t} \times (\mathbf{U} - \gamma \kappa \mathbf{t} \times \mathbf{n}) - \alpha' \mathbf{t} \times (\mathbf{t} \times (\mathbf{U} - \gamma \kappa \mathbf{t} \times \mathbf{n})),$$

where $\mathbf{U}$ is the dimensionless normal fluid velocity, $\mathbf{t}$ and $\mathbf{n}$ are the unit tangent and unit normal vectors to the vortex filament, κ is the dimensionless average curvature, $\gamma = \Gamma \ln(c/\kappa a_0)$ is a dimensionless composite parameter (Γ is the dimensionless quantum of circulation, c is a scaling factor of order unity, $a_0 \approx 1.3 \times 10^{-8}$ cm is the effective core radius of the vortex), α and α' are dimensionless friction coefficients which are small (except near the λ -point; for reference, the λ -point is the temperature (≈ 2.17 K) below which normal fluid helium transitions to superfluid helium¹²). Regarding reasonable values of α and α' , Table 1 of Schwarz¹³ shows that at temperature $T = 1$ K we have $\alpha = 0.005$ and $\alpha' = 0.003$, while at temperature $T = 1.5$ K we have $\alpha = 0.073$ and $\alpha' = 0.018$. Thus, it makes sense to consider these friction terms as small parameters. As we shall later see, the parameters may be small, but under the LIA they are not negligible, and will influence the qualitative structure of the helical vortex filaments when the LIA is used.

If we consider the vortex to be aligned along the x axis (without loss of generality) to correspond with a normal fluid velocity aligned like $\mathbf{U} = U \mathbf{i}_x$, the LIA reduces to⁷

$$\mathbf{v} = (1 - \alpha') \gamma \kappa \mathbf{t} \times \mathbf{n} + \alpha \mathbf{t} \times \mathbf{U} + \alpha \gamma \kappa \mathbf{n} - \alpha' U \mathbf{t} + \alpha' U \mathbf{i}_x. \quad (1)$$

Assuming the deviations along the x axis to be sufficiently bounded in variation, we consider solutions $\mathbf{r} = x \mathbf{i}_x + y(x, t) \mathbf{i}_y + z(x, t) \mathbf{i}_z$. We do not allow vortex filaments to self-intersect, form, and shed rings, or any such exotic behaviors which would yield non-uniqueness in the mathematical framework

of the LIA. Such behaviours are best modelled through direct simulation of the governing Biot-Savart law, which is best done numerically for a fixed situation. By contrast, we shall be interested in exact analytical results, within the confines of the LIA. Note that this restriction also prevents the strong bending or twisting of a vortex back onto itself (so, the vortex filament cannot cross or intersect itself, as mentioned above). Since we are interested in helical filaments, which never exhibit these poor behaviors, the present formulation will be sufficient.

Van Gorder⁶ determined the derivative nonlinear Schrödinger (NLS) equation resulting from the local induction approximation (1). This model is true to the nonlinear form of (1), thereby preserving the inherent nonlinear effects on the vortex filament solution. From (1), we employ the relations

$$\begin{aligned}\mathbf{v} = \frac{d\mathbf{r}}{dt} &= y_t \mathbf{i}_y + z_t \mathbf{i}_z, \quad \mathbf{t} = \frac{d\mathbf{r}}{ds} = (\mathbf{i}_x + y_x \mathbf{i}_y + z_x \mathbf{i}_z) \frac{dx}{ds}, \quad \frac{dx}{ds} = \frac{1}{\sqrt{1 + y_x^2 + z_x^2}}, \\ \kappa \mathbf{n} &= -(y_x y_{xx} + z_x z_{xx}) \frac{dx}{ds} \mathbf{i}_x + \left(y_{xx} \frac{dx}{ds} - (y_x^2 y_{xx} + y_x z_x z_{xx}) \right) \frac{dx}{ds} \mathbf{i}_y \\ &\quad + \left(z_{xx} \frac{dx}{ds} - (z_x y_x y_{xx} + z_x^2 z_{xx}) \right) \frac{dx}{ds} \mathbf{i}_z,\end{aligned}$$

and make the change of dependent variable in terms of the complex potential function

$$\Phi(x, t) = y(x, t) + iz(x, t),$$

we find that the potential form of the LIA is given by

$$i\Phi_t + \frac{\alpha' U i \Phi_x}{(1 + |\Phi_x|^2)^{1/2}} - \frac{\alpha \gamma i \Phi_{xx}}{(1 + |\Phi_x|^2)^2} - \alpha U \Phi_x + \frac{(1 - \alpha') \gamma \Phi_{xx}}{(1 + |\Phi_x|^2)^{3/2}} = 0. \quad (2)$$

This describes the motion of a type of vortex filament which moves about the x -axis. Propagation of a wave along the x -axis is made possible as the filament rotates in a corkscrew manner: the filament rotates with an excitation propagating in the x -direction. Again, this formulation is useful if the filament is aligned along the x -axis, for which the above relations are well-defined. If the filament alignment deviates too much from this (for instance, if the filament bends back on itself), then the formulation breaks down, and a full three-dimensional simulation may be needed. Still, in the regime where this form of the LIA is valid, exact solutions are possible, which are of course the most preferred.

III. HELICAL VORTEX FILAMENT

In the present paper, we shall primarily be interested in helix solutions to the LIA, of the form $\Phi(x, t) = A \exp(i(kx - \omega t + x_0))$. These can describe the propagation of a Kelvin wave on a helical vortex filament under the LIA. Mapping the potential form of a solution into the Cartesian representation, one obtains $\mathbf{r} = x \mathbf{i}_x + A \cos(kx - \omega t + x_0) \mathbf{i}_y + A \sin(kx - \omega t + x_0) \mathbf{i}_z$. Such a solution takes the form of a helix oriented along the x -axis which rotates in time; see Fig. 1. Assuming such a helical solution, we arrive at necessary and sufficient conditions (separating real and imaginary parts) on the model parameters:

$$-\omega + \frac{\alpha' U k}{(1 + A^2 k^2)^{1/2}} + \frac{\gamma(1 - \alpha') k^2}{(1 + A^2 k^2)^{3/2}} = 0, \quad (3)$$

$$\alpha i \left(-U k + \frac{\gamma k^2}{(1 + A^2 k^2)^2} \right) = 0. \quad (4)$$

When these conditions are satisfied, a helical solution exists. In order to determine the precise motion, one needs to determine the angular frequency as a function of the wave number and the physical parameters. With this information, the position of the helix structure can be determined at any time t .

We have two cases, one degenerate and one non-degenerate, depending on α . In the degenerate case where $\alpha = 0$, it is condition (4) that degenerates, leaving the wave number k and the amplitude

FIG. 1. Schematic of the problem geometry for a prototypical helical vortex filament. The helical vortex filament is oriented along the x -axis, with amplitude A and wave-number k . The angular frequency, ω , will dictate the motion of this helical vortex filament.

A arbitrary (unlike in the $\alpha \neq 0$ case, where they are related in a fixed manner). However, since physically we should have $\alpha > \alpha'$, then both friction parameters should vanish in this case. Hence, we reduce to the standard LIA, which has been studied elsewhere.

If $\alpha \neq 0$, condition (4) implies $k(1 + A^2 k^2)^{-2} = U/\gamma$, a condition fixing A and k in terms of one another. It is simpler to solve for amplitude in terms of wave number, and thus this condition implies that the amplitude must satisfy the condition

$$A = \frac{1}{k} \sqrt{\sqrt{\gamma k/U} - 1}, \quad (5)$$

provided $k > U/\gamma$; such solutions do not exist for $k \leq U/\gamma$. Placing (5) into (3), we find

$$\omega = U^{5/4} \gamma^{-1/4} k^{5/4} \left(\frac{\alpha'}{\sqrt{k}} + (1 - \alpha') \sqrt{\frac{\gamma}{U}} \right). \quad (6)$$

We then find that the phase velocity is

$$v_p = \frac{\omega}{k} = U^{5/4} \gamma^{-1/4} k^{1/4} \left(\frac{\alpha'}{\sqrt{k}} + (1 - \alpha') \sqrt{\frac{\gamma}{U}} \right),$$

while the group velocity is

$$v_g = \frac{\partial \omega}{\partial k} = U^{5/4} \gamma^{-1/4} k^{1/4} \left(\frac{3\alpha'}{4\sqrt{k}} + \frac{5}{4}(1 - \alpha') \sqrt{\frac{\gamma}{U}} \right).$$

From condition (5), we see that for large wave numbers, the amplitude must be small. Meanwhile, for large normal velocity flows, the wave number, obeying $k > U/\gamma$ must be sufficiently large, leading to solutions of sufficiently small amplitude.

We plot the amplitude of a helical solution in the $\alpha \neq 0$ regime in Fig. 2 for various values of normal fluid velocity (normalizing $\gamma = 1$). As expected, the increased instability due to higher normal fluid velocities decreases the permitted size of the helical solutions (an increase in U decreases the amplitude A). Interestingly, a maximal amplitude solution exists for wave number k_A , where $k_A = \frac{16U}{9\gamma}$. This maximal value of A is given by $A_{\max} = \frac{3\sqrt{3}\gamma}{16U}$.

The dispersion relation $\omega(k)$ for the angular frequency given by (6) scales as $k^{5/4}$ for large k , so the phase and group velocities scale as $k^{1/4}$. An increase in the superfluid friction parameter α serves to increase the angular frequency along with the phase and group velocities. The dispersion relation $\omega(k)$ is plotted in Fig. 3.

FIG. 2. Plot of the nonlinear dependence of the amplitude A given in Eq. (5) on the wave number k and the normal fluid velocity U . The permissible wave numbers satisfy $k > U/\gamma$, and for the sake of demonstration we normalize $\gamma = 1$. As the normal fluid velocity increases, the permitted amplitude values decrease, owing to the added instability induced by the normal fluid. We observe a unique peak value in amplitude at some wave number k_A for each fixed value of U .

We should remark that, in the case where $\alpha \neq 0$ yet $U = 0$, condition (4) implies $k = 0$, so the only helical solution is a constant. In this case, the solution simply corresponds to a line vortex filament $\mathbf{r} = (x, 0, 0)$ (a zero-curvature solution). This is reasonable: the normal fluid velocity is driving the helical structure through the friction parameters, so without the normal flow, a non-zero friction parameter makes no sense.

FIG. 3. Plots of the nonlinear dispersion relation Eq. (6) for $\omega(k)$ in the non-degenerate case $\alpha \neq 0$. When $k = 0$, $\omega(0) = 0$, while $\omega > 0$ for $k > 0$.

IV. APPROXIMATIONS TO THE LIA

In the present section, we shall consider two approximations to the fully nonlinear form of the LIA used above. We highlight the utility and shortcomings of each reduced model.

A. The cubic model of Shivamoggi

Taking an approximation to (1), Shivamoggi⁷ obtained a linearized model, which preserved up to cubic nonlinearity affiliated with the curvature of a vortex filament while linearizing the superfluid friction terms α and α' . An approximate locally stationary wave solution was then presented.

If one takes a cubic approximation of the nonlinear contribution due to arclength, and drops nonlinear terms which are multiplied with superfluid friction parameters, we arrive at the cubic derivative NLS equation:

$$i\Phi_t + (\alpha'i - \alpha)U\Phi_x + (1 - \alpha' - \alpha i)\gamma\Phi_{xx} - \frac{3\gamma}{2}|\Phi_x|^2\Phi_{xx} = 0. \quad (7)$$

Assuming a helical Kelvin wave, we obtain the conditions:

$$\omega - \alpha'Uk - (1 - \alpha')\gamma k^2 + \frac{3\gamma}{2}A^2k^4 = 0, \quad (8)$$

$$\alpha i(-Uk + \gamma k^2) = 0. \quad (9)$$

Here, a few features of the solutions are apparent. First, if $\alpha \neq 0$, the wave number depends only on U (the normal velocity) and γ (the strength of the filament). If $\alpha = 0$, the wave number is arbitrarily determined. In either case, α does not enter into the helical solution. Now, in the $\alpha \neq 0$ case, $k = U/\gamma$, so the angular frequency is

$$\omega = \frac{U^2}{\gamma} \left\{ 1 - \frac{3A^2U^2}{2\gamma^2} \right\}. \quad (10)$$

Some observations are in order. Note that there is no amplitude restriction on the solution. Also, when $U = 0$ and $\alpha \neq 0$, note that $k = 0$ necessarily, in which case $\Phi = \text{constant}$.

B. Local nonlinear equation of Laurie *et al.*

Laurie *et al.*¹¹ derived a local nonlinear equation (LNE) from the Biot-Savart law, which was proposed as a simple model for the motion of Kelvin waves. The LNE is simpler than the fully nonlinear LIA, yet permits energy exchange between Kelvin waves, unlike the linearized LIA. The LNE of Laurie *et al.*¹¹ reads

$$i\Psi_t + \hat{\gamma} \frac{\partial}{\partial x} \left\{ \left(\Lambda - \frac{1}{4}|\Psi_x|^4 \right) \Psi_x \right\} = 0. \quad (11)$$

Here, $\hat{\gamma} = \kappa/(4\pi) > 0$ (κ is the quantum of velocity circulation) and $\Lambda > 0$ scale as the natural logarithm of the mean intervortex distance over the vortex core radius. Nonlinearity in this equation is responsible for the energy transfer among the Kelvin wave modes. Making the change of variable $\Phi(x, t) = (4\Lambda/5)^{-1/4}\Psi(x, t)$, and letting $\gamma = \hat{\gamma}\Lambda$, we obtain the scaled equation:

$$i\Phi_t + \gamma (1 - |\Phi_x|^4) \Phi_{xx} = 0. \quad (12)$$

Assuming a helical solution, $\omega = \gamma k^2(1 - A^4k^4)$. Like in the case of the cubic approximation to the LIA, the angular frequency is unbounded for the helix solution to the LNE. However, like in the case of the full nonlinear LIA, there exists a maximum value of ω (beyond this value, ω decays to $-\infty$). The phase velocity is $v_p = \gamma k(1 - A^4k^4)$, while the group velocity is $v_g = 2\gamma k(1 - 3A^4k^4)$. Setting the group velocity to zero, we find that ω attains a unique maximum value at $k^* = 3^{-1/4}/A$, which is approximately half the value of k^* found for the fully nonlinear LIA. Thus, while the LNE picks up the qualitative maximum, the quantitative agreement is quite off.

C. Rotating planar filaments

For a standard fluid, Hasimoto⁹ considered a planar vortex filament. This was explored for the LIA formulation of a standard fluid in the Cartesian frame by Van Gorder,¹⁶ where locally stationary solutions describing the planar vortex filament were obtained. In Van Gorder,⁶ it was shown that simple stationary solutions of the form $\Phi(x, t) = e^{-i\gamma t}\phi(x)$ do not exist for real-valued $\phi(x)$ for the superfluid case when such solutions do exist as $\alpha = 0$ and $\alpha' = 0$, as this corresponds to the standard fluid case. For the standard fluid case, the planar vortex filament was described in terms of elliptic functions for the present Cartesian coordinate system¹⁶ as well as for the arclength system.¹⁵ Under a transformation, solutions like these were shown to exist, but they were for a complex-valued function $\phi(x)$, which complicates the analysis. For small-amplitude space-periodic planar vortex filaments, we shall now show that the LNE (12) provides an accurate approximation.

Consider the stationary solution

$$\Phi(x, t) = Ae^{-i\gamma t}\phi(x), \quad (13)$$

for real-valued $\phi(x)$, which describes a distinct structure from that of the helical Kelvin waves discussed above, namely, it gives a rotating planar vortex filament; see Fig. 4 for the problem geometry. In particular, from the form of the time-dependence, Eq. (13) describes a rotating planar vortex filament, which moves around the x -axis as time progresses. Both of the LIA models (2) and (7) do not permit real-fields ϕ due to the appearance of α and α' , as previously discussed by Van Gorder.⁶ Since the LNE was derived differently than the LIA models, it can support such structures. With an assumption (13), the planar vortex filament can be described in Cartesian coordinates as

$$\mathbf{r} = x\mathbf{i}_x + A\cos(\gamma t)\phi(x)\mathbf{i}_y - A\sin(\gamma t)\phi(x)\mathbf{i}_z. \quad (14)$$

Under the assumption of such a stationary solution (13), (12) gives

$$\phi + \left(1 - A^4\phi'^4\right)\phi'' = 0. \quad (15)$$

Without loss of generality, consider ϕ normalized such that $\phi(0) = 1$, $\phi'(0) = 0$. A first integral for (15) is then

$$\phi^2 + \phi'^2 - \frac{A^4}{3}\phi'^6 = 1. \quad (16)$$

In order to have a helical solution of the form (13), we need ϕ to be periodic and continuous. Then, there must exist some $x \in \mathbb{R}$ at which $\phi = 0$. At such a point, $\phi'^2 - \frac{A^4}{3}\phi'^6 = 1$, which is an algebraic equation for the derivative of ϕ at such a point. This condition is equivalent to the existence of a positive root to $\frac{A^4}{3}\mu^3 - \mu + 1 = 0$, or, in transformed coordinates, $v^3 - v + \frac{A^2}{\sqrt{3}} = 0$. We find that a necessary and sufficient condition for the existence of a positive root $v > 0$ is $A \leq \sqrt{2/3}$. Hence, $A \leq \sqrt{2/3}$ is a necessary condition for the existence of a periodic solution ϕ for (15).

In order to directly solve (16), we would need to separate variables, which involves solving a cubic for ϕ'^2 , and then performing an integration over the positive branch. To make matters worse, the integral cannot be evaluated in closed form (which is expected from the non-integrability of the LNE (12); see Ref. 11). We therefore bypass such an exact method, and we instead seek a perturbation solution. Primitive perturbation theory does not adequately capture the periodicity of the true solution, so we consider the method of multiple scales. To do so, let us assume a solution of the form

$$\phi(x) = \hat{\phi}(\eta, A^4) = \hat{\phi}_0(\eta) + A^4\hat{\phi}_1(\eta) + O(A^8),$$

where

$$\eta = \delta(A^4)x \quad \text{and} \quad \delta(A^4) = \delta_0 + A^4\delta_1 + O(A^8).$$

Equation (15) is then reduced to

$$\hat{\phi} + \delta^2(A^4)(1 - \delta^4(A^4)A^4\hat{\phi}_\eta^4)\hat{\phi}_{\eta\eta} = 0,$$

FIG. 4. Schematic of the problem geometry for a planar vortex filament. The planar vortex filament is oriented along the x -axis, with radius A and x -period $T(A)$. The angular frequency, $\omega = \gamma$ (by assumption (13)), determined the motion of the vortex filament with time. The temporal period is then $2\pi/\gamma$.

which results in the terms

$$\begin{aligned} \delta_0^2(\hat{\phi}_0)_{\eta\eta} + \hat{\phi}_0 &= 0, \\ \hat{\phi}_0(0) &= 1, (\hat{\phi}_0)_\eta(0) = 0, \hat{\phi}_0(\eta + 2\pi) = \hat{\phi}_0(\eta), \end{aligned} \quad (17)$$

$$\begin{aligned} \delta_0^2(\hat{\phi}_1)_{\eta\eta} + \hat{\phi}_1 &= (\delta_0^6(\hat{\phi}_0)_\eta^4 - 2\delta_0\delta_1)(\hat{\phi}_0)_{\eta\eta}, \\ \hat{\phi}_1(0) &= 0, (\hat{\phi}_1)_\eta(0) = 0, \hat{\phi}_1(\eta + 2\pi) = \hat{\phi}_1(\eta), \end{aligned} \quad (18)$$

where the periodic condition is to ensure that the solutions are 2π -periodic in η (and hence, $2\pi/\delta(A^4)$ -periodic in x). The periodicity requirement forces $\delta_0 = 1$, and thus the zeroth order solution is found to be $\hat{\phi}_0(\eta) = \cos(\eta)$.

Using this solution in Eq. (18), we have

$$(\hat{\phi}_1)_{\eta\eta} + \hat{\phi}_1 = \left(2\delta_1 - \frac{1}{8}\right)\cos(\eta) + \frac{3}{16}\cos(3\eta) - \frac{1}{16}\cos(5\eta). \quad (19)$$

In order to remove secular contributions, we set $\delta_1 = 1/16$, so that $\delta(A^4) = 1 + A^4/16 + O(A^8)$. Then, using the initial conditions in (18), we obtain

$$\begin{aligned} \hat{\phi}_1(\eta) &= \frac{1}{48}\cos(\eta) - \frac{3}{128}\cos(3\eta) + \frac{1}{384}\cos(5\eta) \\ &= \frac{1}{48}\sin^2(\eta)(5 - 2\cos^2(\eta))\cos(\eta). \end{aligned} \quad (20)$$

From here, the perturbation solution reads

$$\begin{aligned} \phi(x) &= \left\{1 + \frac{A^4}{48}\sin^2\left(\left[1 + \frac{A^4}{16}\right]x\right)\right. \\ &\quad \left.\times \left(5 - 2\cos^2\left(\left[1 + \frac{A^4}{16}\right]x\right)\right)\right\}\cos\left(\left[1 + \frac{A^4}{16}\right]x\right). \end{aligned} \quad (21)$$

The period $T(A)$ of the solution is then approximated as

$$T(A) \approx 2\pi \left(1 + \frac{A^4}{16}\right)^{-1} = 2\pi - \frac{\pi}{8}A^4 + \frac{\pi}{128}A^8 + O(A^{12}). \quad (22)$$

Comparing the period $T(A)$ computed in Eq. (22) with an exact value is useful in determining the accuracy of the approximation (21). From Eq. (16), we find

$$\phi' = \sqrt{\mu(\phi^2 - 1, A)}, \quad (23)$$

where

$$\mu(\zeta, A) = \frac{4 + \left\{ 12A^2\zeta + 4\sqrt{9A^4\zeta^2 - 4} \right\}^{2/3}}{2A^2 \left\{ 12A^2\zeta + 4\sqrt{9A^4\zeta^2 - 4} \right\}^{1/3}} \quad (24)$$

is the branch of the cubic $\zeta + \mu - A^4\mu^3/3 = 1$ giving real solutions for small A . Separating variables in (23) and integrating, we obtain

$$x = \int_{\phi}^1 \frac{d\xi}{\sqrt{\mu(\xi^2 - 1, A)}}. \quad (25)$$

Now, integrating over one quarter-period,

$$T(A) = 4 \int_0^1 \frac{d\xi}{\sqrt{\mu(\xi^2 - 1, A)}}. \quad (26)$$

The integral (26) is too complicated to evaluate exactly. However, it may certainly be plotted numerically as a function of A . In Fig. 5, we plot the exact relation for the period $T(A)$ given in (26) along with the approximate value obtained by multiple scales analysis in Eq. (22). As expected, for small A , the agreement between the approximate solution and the exact relation is good. With this, we have enough information to accurately determine the motion of a planar vortex filament. In Fig. 6, we plot the planar part ($\phi(x)$) of such a vortex filament, for various values of A . To recover the three-dimensional structure, one would use ϕ in (14) to recover a structure like that shown in Fig. 4.

FIG. 5. Plot of the approximation (22) to the space period $T(A)$ of the planar vortex filament given by the method of multiple scales. Also plotted are the exact numerical values for $T(A)$ that may be found by numerically integrating (15) in order to construct a solution. Clearly, the multiple scales approximation is a very good fit for small-amplitude solutions. Periodic solutions exist for $A < \sqrt{2/3} \approx 0.81649$.

FIG. 6. Plot of the space periodic part of the planar vortex filament for various values of A . Solid lines denote the perturbation approximation (21), while the dashed lines denote numerical simulations via RKF45 method of numerical integration. We see remarkable agreement for small amplitude solutions. For the large amplitude solutions, the numerics and analytics begin to go out of phase for large x , due to the small error in the approximate period (22) and the true period.

We have demonstrated that the LNE permits a planar vortex filament solution of the Hasimoto type, in addition to the class of helical vortex filaments presented in Sec. IV B.

V. CONCLUSIONS

Under the assumption of a helical vortex filament, we have determined the dispersion relations describing the motion of this helical vortex filament in superfluid ^4He under the fully nonlinear form of the LIA (which is the most true form of the LIA to the Biot-Savart formulation) corresponding to the Hall and Vinen^{4,5} formulation. With this, we have been able to determine the phase and group velocities, effectively classifying all possible motions of a helical vortex filament under the LIA. Our results include the effect of both superfluid friction parameters, as well as normal fluid velocity, along the lines of the Hall and Vinen^{4,5} framework.

The helical vortex filaments are free of α (yet depend on α') due to the orientation of the filaments. However, as we have shown, there is a qualitative disconnect between the $\alpha = 0$ and $\alpha \neq 0$ solutions. This is in agreement with a finding of Shivamoggi,⁷ where in the case of a Hasimoto-type 2D planar vortex solution,⁹ “the friction term associated with α appears again to be able to change the vortex motion aspects in a qualitative way.” The $\alpha = 0$ (zero temperature limit) has already been considered by Sonin.¹⁰ While α' enters into the helix solution explicitly, its effects are relatively minor. In the $\alpha \neq 0$ and $\alpha' \rightarrow 0$ limit, we have $\omega = U^{3/4} \gamma^{1/2} k^{5/4}$, and for small non-zero α' we simply have a linear perturbation of this quantity.

There are two sensible reductions of the fully nonlinear form of the LIA: (i) an integrable cubic reduction preserving lowest order nonlinearity;⁷ (ii) a non-integrable quintic reduction useful for studying Kelvin wave interactions.¹¹ For sake of comparison with the fully nonlinear LIA, we may determine the dispersion relations (and other associated quantities) for these two reduced models. The three models considered become exactly comparable in the limit $\alpha = \alpha' = U = 0$, which corresponds to the standard LIA. For small values of the wave number k the agreement between

the angular frequency, phase velocity, and group velocity all agree qualitatively in the case of helical Kelvin waves. However, for the cubic approximating model, there is a fixed wave number $k = U/\gamma$, while for the other models there is a continuous spectrum. For large values of the wave number k , both approximating models break down, as they fail to capture enough of the nonlinearity governing the vortex filament motion. For the two approximating models, the angular frequency becomes unbounded for large k in the $\alpha = 0$ degenerate case, while the fully nonlinear LIA (2) ensures bounded angular frequency for all k in the degenerate case. For very small k , the cubic approximation to the LIA is the best fit. However, for intermediate k , the angular frequency from the local nonlinear equation produces the qualitative peak observed in the full nonlinear LIA (2). Hence, there is utility in each model, though one must be mindful of the parameter regimes for which each simplified model is used.

In summary, some qualitative results are:

- the fully nonlinear LIA demonstrates that the amplitude A of a helical wave is strongly related to the wave number k ;
- the approximating models lose the dependence of the amplitude A on the wave number k ;
- the cubic approximation is valid for the unique wave number $k = U/\gamma$, while the fully nonlinear LIA is valid for an unbounded continuous spectra of wave numbers k ;
- in the $k \rightarrow 0$, $U \rightarrow 0$ limit, the full and approximating models are in qualitative agreement, while for $k \rightarrow \infty$, only the fully nonlinear model is physically consistent;
- the amplitude of a helical wave under the fully nonlinear model is always bounded, and assumes a maximal value of $A_{\max} = \frac{3\sqrt{3}\gamma}{16U}$ at wave number $k_A = \frac{16U}{9\gamma}$.

Rotating planar Kelvin waves have also been shown to exist for the LNE, and explicit perturbation solutions have been developed for the small amplitude regime. This formulation is much simpler than that given previously for the fully nonlinear LIA.⁶ In particular, by considering a multiple scales analysis, we have been able to accurately construct perturbation solutions for these planar vortex filaments. Using the results, we can determine the nonlinear dependence of the spatial period of such planar filaments on their amplitude. In the $\alpha, \alpha' \rightarrow 0$ regime (i.e., the low-temperature regime), we have therefore obtained accurate planar filaments. Unlike the helical filaments, the planar filaments lie on a plane which rotates about the x -axis, along the same lines as the Hasimoto filament which was related to elastica.⁹

The Cartesian formulation discussed here (the extrinsic coordinate system (x, y, z) is used) is one possibility. Umeki¹⁴ presented the fully nonlinear LIA for the standard fluid case. This formulation was later shown to admit stationary solutions.¹⁵ It would be interesting to reformulate the superfluid form of the LIA in terms of the intrinsic arclength formulation. While an implicit exact relation for the planar vortex filament in the Cartesian framework was obtained in Van Gorder,^{16,17} an explicit exact solution for the planar vortex filament in the arclength representation was derived in Van Gorder.¹⁵ So, perhaps it is possible that the arclength formulation of the LIA for a superfluid might permit planar vortex filament solutions, or other equally interesting solutions, in exact form. Self-similar solutions, distinct from the types of solutions considered here, we discussed in the curvature-torsion frame by Lipniacki;¹⁸ the construction of self-similar solutions for the Cartesian form of the LIA in the case of superfluid ^4He will be the topic of a forthcoming work.

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