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# A Computer Simulation of an Incoherent Optical Processor for Spatial Frequency Analysis

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# A COMPUTER SIMULATION OF AN INCOHERENT OPTICAL PROCESSOR FOR SPATIAL FREQUENCY ANALYSIS

BY

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## ABSTRACT

A computer simulation demonstrates the feasibility of a technique for performing spatial frequency analysis on an incoherent scene in one dimension using an electronically addressable spatial light modulator (SLM). By applying a biased sinusoidal variation to the SLM transmittance, the real and imaginary parts of the Fourier transform are available and readily yield both magnitude and phase of the Fourier transform.

In addition to the Fourier transform this simulation investigates the use of a simple binary pulse train transform for the sake of reducing system complexity and increasing speed of operation.

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# INTRODUCTION

Although coherent systems of Fourier analysis based on Fraunhofer diffraction have produced many useful results in the laboratory, there is room for further development of incoherent optical processors capable of Fourier analysis in less artificial environments, where coherent illumination is impossible or impractical.

This study will consist of a computer simulation of a particularly straightforward approach to measuring the spatial frequency content of an incoherently illuminated scene. The basic approach employed is motivated by viewing an integral transform as the area under a product of two functions; namely the function to be transformed and the kernel function which characterizes the transform. The quantity to be transformed in this study represents the luminous intensity of an incoherent scene. The transform kernel is represented by the spatial variation in transmittance of a spatial light modulator (SLM). One type of spatial light modulator appropriate for implementing this approach would be a high resolution liquid crystal SLM consisting of an array of individually addressable pixels with gray scale capability (Boreman and Raudenbush 1986). The amount of light from the scene transmitted by the SLM is equal to the product of the luminous intensity of the image and the transmittance function of the SLM. The integral of the product,

(image intensity) x (transmittance of SLM)

is obtained by collecting all the light transmitted by the SLM with a photocell or CCD array. This approach is equivalent to the correlation of an image with successive targets of increasing spatial frequency in order to obtain the Fourier coefficients (Rogers 1977, p. 33; Armitage and Lohman 1965). The proposed implementation is seen schematically in Figure 1.

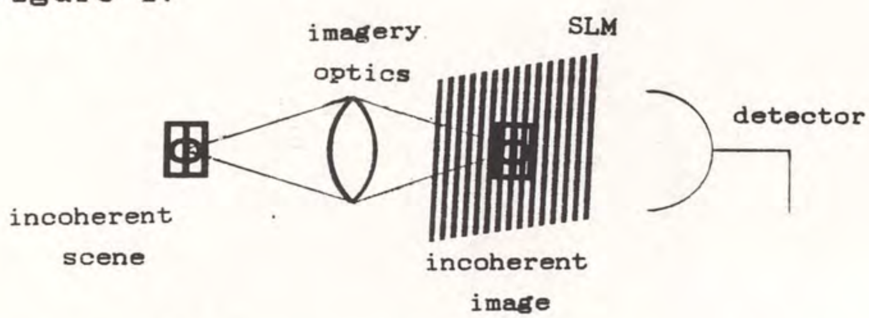



Figure 1: Schematic diagram of incoherent spatial frequency analyzer

Since the transmittance function of the SLM cannot represent negative quantities it is necessary to add a constant bias to the transform kernel to ensure that it is always positive. For example the one-dimensional Fourier transform can be implemented by a SLM with transmittance functions of the form

$$T(x;u) = B + M\cos(2\pi ux)$$

$$\text{and} \quad T(x;u) = B + M\sin(2\pi ux)$$

where u represents the spatial frequency variable. The result of adding the constant bias term to the transform kernel is a reduction in contrast.

The idea of performing Fourier analysis using an incoherent optical processor was tested by H.C. Montgomery of the Bell Laboratories as early as 1938. The apparatus tested by Montgomery measured the magnitude and phase of 30 spectral components of a function f(x) recorded on a photographic transparency as a gradation in the density of the film. The film containing f(x) was illuminated by an incandescent lamp and condensing lens system. An image of the function to be analyzed was focused on a screen with sinusoidal variation in density of the form $B[1 + M\cos(nx)]$. The total light transmitted by the cosine screen was measured by a photocell which in effect performed the operation of outputting the integral of the product of $f(x)\cos(nx)$. A different screen was required for each frequency to be measured. This process was automated with a mechanical arrangement of cams and levers which loaded the cosine screens from a drum-shaped magazine.

The magnitude and phase of the spectral components were obtained by moving the cosine screen through one complete period and recording the position of the cosine screen where the maximum transmitted light is received. The magnitude of the maximum is proportional to the magnitude of that particular frequency component and the position indicates the phase of that component. An alternative method is to shift the cosine screen by a quarter cycle and thereby measure the integral of the product $f(x)\sin(nx)$. The two measurements yield the real and imaginary parts of the spectral component which is readily converted to the polar form.

Montgomery's optical harmonic analyzer was used for speech sound analysis taken from the sound track on motion picture film.

The analyzer implemented by Montgomery was limited by the technology available in 1938. Producing high quality cosine transparencies with uniform modulation and average transmission was one difficulty. The mechanically complicated nature of the analyzer and necessity for accurate alignment were also limiting factors.

These limitations are largely mitigated by the use of an electronically addressable SLM, and a number of processor architectures which might have been otherwise too complex mechanically, now become feasible.

## THEORETICAL BACKGROUND FOR THE FOURIER TRANSFORM

In the context of incoherent spatial frequency processing with an SLM, it is advantageous to express the Fourier transform, F(u), in rectangular form:

$$F(u) = \int_{-\infty}^{\infty} f(x)\cos(2\pi ux)dx - j\int_{-\infty}^{\infty} f(x)\sin(2\pi ux)dx$$

where f(x) represents a real valued one dimensional scene function and u represents spatial frequency in cycles per unit distance of x. Throughout this paper f(x) will denote a real valued function of x. It will be convenient to define the symbols $F_c(u)$ and $F_s(u)$ to be the first and second integrals respectively in the above equation. The Fourier transform can then be written in the compact form:

$$F(u) = F_c(u) - jF_s(u).$$

$F_c(u)$ is the real part of the Fourier transform and it is an even function of u, i.e., $F_c(-u) = F_c(u)$, provided that f(x) is a real valued function. $F_s(u)$ is the negative of the imaginary part of F(u) and it is an odd function of u, i.e., $F_s(-u) = -F_s(u)$, if f(x) is real valued. In this paper $F_c(u)$ and $F_s(u)$ will be referred to as the cosine transform and sine transform respectively of the function f(x). Note, however, that $F_c(u)$ as defined is not equal to

$2\int_0^\infty f(x)\cos(2\pi ux)dx$ unless f(x) is an even function nor is $F_s(u)$ equal to $2\int_0^\infty f(x)\sin(2\pi ux)dx$ unless f(x) is an odd function.

The magnitude of the Fourier transform, $|F(u)|$, can be expressed in terms of $F_c(u)$ and $F_s(u)$ as:

$$|F(u)| = \sqrt{F_c(u)^2 + F_s(u)^2}$$

$|F(u)|$ is an even function of u provided that f(x) and therefore $F_c(u)$ and $F_s(u)$ are real functions.

The phase of the Fourier transform, arg[F(u)], can also be expressed in terms of $F_c(u)$ and $F_s(u)$ as follows:

$$\arg[F(u)] = -\arctan[F_s(u)/F_c(u)], \text{ if } F_c(u)>0$$
$$\arg[F(u)] = \pi - \arctan[F_s(u)/F_c(u)], \text{ if } F_c(u)<0 \text{ and } F_s(u)<0$$
$$\arg[F(u)] = -\pi - \arctan[F_s(u)/F_c(u)], \text{ if } F_c(u)<0 \text{ and } F_s(u)>0$$

The phase of the Fourier transform is an odd function of u.

It is interesting to note that the cosine transform as defined above only "sees" the even part of f(x) and the sine transform only "sees" the odd part of f(x). This property is evident if f(x) is expressed as the sum of its even part, e(x), and odd part, o(x), where $e(x) = 0.5[f(x) + f(-x)]$ and $o(x) = 0.5[f(x) - f(-x)]$. $F_c(u)$ and $F_s(u)$ simplify as shown below.

$$F_c(u) = \int_{-\infty}^{\infty} e(x)\cos(2\pi ux)dx + \cancelto{0}{\int_{-\infty}^{\infty} o(x)\cos(2\pi ux)dx}$$

$$= 2\int_{0}^{\infty} e(x)\cos(2\pi ux)dx$$

$$F_s(u) = \cancelto{0}{\int_{-\infty}^{\infty} e(x)\sin(2\pi ux)dx} + \int_{-\infty}^{\infty} o(x)\sin(2\pi ux)dx$$

$$=2\int_{0}^{\infty} o(x)\sin(2\pi ux)dx$$

Therefore only $e(x)$ is seen by $F_c(u)$ and only $o(x)$ is seen by $F_s(u)$.

## Implementing The Fourier Transform

By superimposing a sinusoidal variation on a d-c bias in the transmittance function of a spatial light modulator, it should be possible to implement the two integrals in the Fourier transform optically. If the transmittance of the SLM is of the form $B + M\cos(2\pi ux)$, then the total luminous intensity emanating from an illuminated image and transmitted through the SLM is given by the product $f(x)[B+M\cos(2\pi ux)]$. If all the light transmitted by the SLM is collected by a photocell or CCD array, the output of the detector will be the integral of the above product, i.e.,

$$\int_{-\infty}^{\infty} f(x)[B + M\cos(2\pi ux)]dx.$$

By interpreting $f(x)$ to be zero outside the limits of the image, the above integral can be expressed as:

$$B\int_{-\infty}^{\infty} f(x)dx + M\int_{-\infty}^{\infty} f(x)\cos(2\pi ux)dx = BF(0) + MF_c(u)$$

which contains the desired quantity, $F_c(u)$, plus a bias term, $BF(0)$. The d-c bias term, $BF(0)$, can be measured before applying the cosinusoidal variation to the transmittance of the SLM, stored, and then subtracted from the measurement taken with modulation (assuming a stationary scene). The modulation depth, M, is a known constant. Therefore it is possible to determine $F_c(u)$ using an SLM having transmittance with a cosinusoidal variation of frequency in the x direction (or any direction of interest) and a light detector to obtain the integral of the product $f(x)\cos(2\pi ux)$. By shifting the cosinusoid a quarter cycle it is possible to obtain $F_s(u)$ in a similar fashion. Thus, the complete Fourier transform, magnitude and phase or real and imaginary parts, can be obtained from two measurements of the incoherent image at each spatial frequency of interest.

The advantage of the SLM is the relative speed and ease with which the spatial frequency of the imposed sinusoidal variation can be changed, allowing the measurement of a large number of frequency components as fast as the trigonometric function values can be generated and applied to the SLM. The loss in contrast caused by the addition of the d-c bias level can be readily compensated for digitally before the processed image is displayed.

Accurate representation of a sinusoidal variation in the SLM transmittance places an upper limit on the frequencies that can be measured. As shown below, even a crude representation using only four quantization levels would require six columns of pixels per cycle, thereby limiting the number of cycles representable per unit distance x.

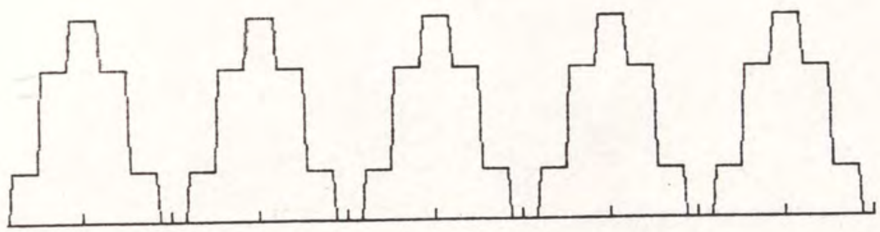


Figure 2: Approximation to cosinusoid using only six columns of pixels per cycle.

Also, in a near real time application, the time required for the calculation or look up of the trigonometric function values may be significant. Therefore, for the sake of increased frequency range, speed, and simplicity, it is of interest to investigate the use of a raised square pulse train in place of the trigonometric functions.

A square pulse train can be represented with as few as two pixels per cycle thus increasing the upper frequency limit by a factor of three over the sine and cosine transforms. Generation of the binary transmittance function on the SLM can be implemented at higher speeds than attainable with a scheme involving the calculation or look-up of trigonometric function values.

In place of the cosine function appearing in the real part of the Fourier transform, a raised square pulse train which is symmetric about the origin could be used to extract frequency information from the even part of f(x). In a similar manner the sine function can be replaced with a square wave having odd symmetry plus a d-c bias to extract frequency information from the odd part of f(x). The following chapter will explore the relationship of such pulse train transforms to the Fourier transform and provide examples illustrating some of the properties of square pulse train transforms.

## THE EVEN AND ODD PULSE TRAIN TRANSFORMS

The even pulse train transform will be defined as:

$$\text{EPTX}[f(x)] = \int_{-\infty}^{\infty} \text{Ke}(x;u)\ f(x)\ dx$$

where Ke(x;u) is the even square pulse train shown below.

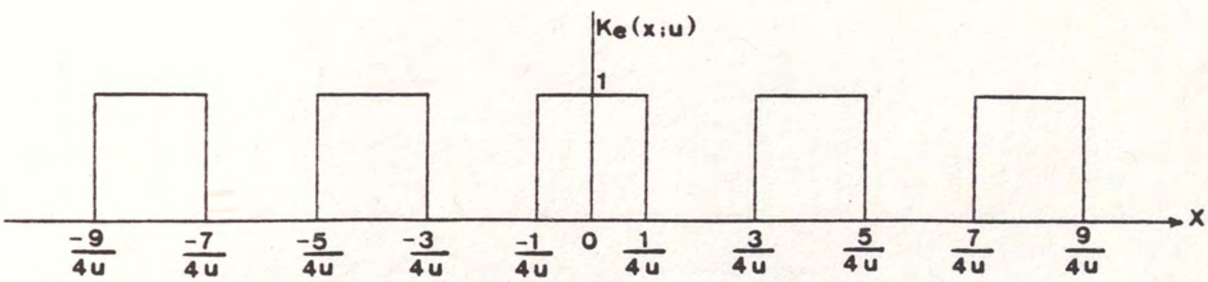



Figure 3: The kernel function for the even pulse train transform.

The parameter u indicates the fundamental frequency in the Fourier series expansion of Ke(x;u). A closed form expression for Ke(x;u) is given by:

$$\text{Ke}(x;u) = 0.5 + 0.5\ \text{sgn}[\cos(2\pi ux)].$$

The functional notation for the even pulse train transform will be Pe(u) = EPTX[f(x)].

In a manner analogous to the cosine Fourier transform, the even pulse train transform only "sees" the even part of the scene function. The above characteristic becomes apparent upon expressing Ke(x;u) as a d-c bias plus an even square wave ESW(x;u) which oscillates between +1 and -1, with fundamental frequency u, i.e.,

$$Ke(x;u) = 0.5 + 0.5\, ESW(x;u).$$

Substituting f(x)= e(x) + o(x) into the expression for the even pulse train transform yields:

$$\begin{aligned} EPTX[f(x)] &= \int_{-\infty}^{\infty} [0.5 + 0.5ESW(x;u)][e(x) + o(x)]dx \\ &= 0.5\int_{-\infty}^{\infty} e(x)dx + 0.5 \underbrace{\cancel{\int_{-\infty}^{\infty} o(x)dx}}_{0} \\ &\quad +0.5\int_{-\infty}^{\infty} ESW(x;u)\, e(x)dx \\ &\quad +0.5 \underbrace{\cancel{\int_{-\infty}^{\infty} ESW(x;u)\, o(x)dx}}_{0} \end{aligned}$$

Therefore the even pulse train transform only reflects characteristics of the even part of f(x), e(x).

In order to see the relationship between the even pulse train transform and the Fourier cosine transform it is necessary to substitute the Fourier series expression for

Ke(x;u) into the equation for EPTX[f(x)]. The Fourier series expression for Ke(x;u) is:

$$\begin{aligned} Ke(x;u) = 1/2 + (2/\pi)\{&\cos(2\pi ux) \\ &- (1/3)\cos[2\pi(3u)x] \\ &+ (1/5)\cos[2\pi(5u)x] \\ &- (1/7)\cos[2\pi(7u)x] + \ldots\} \end{aligned}$$

Therefore, the even pulse train transform can be written as:

$$\begin{aligned} EPTX[f(x)] = Pe(u) = (&F_c(0)/2) \\ &+ (2/\pi)\{F_c(u) - (1/3)F_c(3u) \\ &+ (1/5)F_c(5u) - (1/7)F_c(7u) + \ldots\} \end{aligned}$$

From the above expression it is apparent that the even pulse train transform is made up of a d-c bias term, F(0)/2, plus an alternating series of compressed or rescaled Fourier cosine transforms.

The odd pulse train transform will be defined as:

$$OPTX[f(x)] = \int_{-\infty}^{\infty} Ko(x;u)f(x)dx$$

where Ko(x;u) is the odd square wave plus d-c bias shown below.

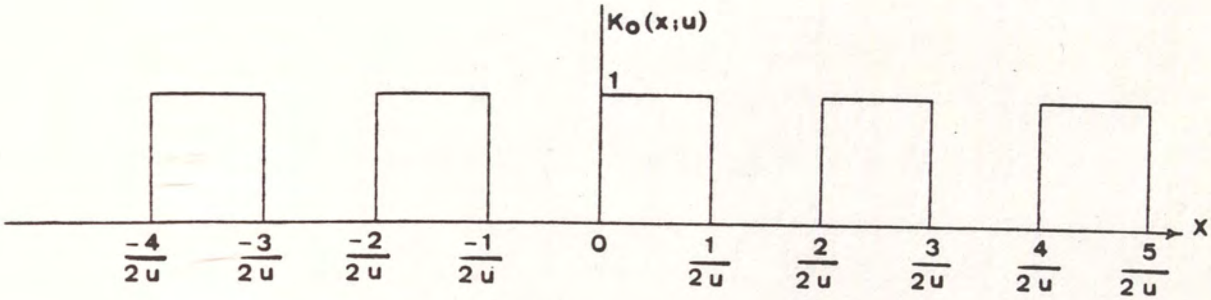



Figure 4: The kernel function for the odd pulse train transform.

The functional notation for the odd pulse train transform will be Po(u) = OPTX[f(x)]. Although Ko(x;u) is not, strictly speaking, an odd function, it will serve the purpose of extracting spatial frequency information from the odd component of the scene function f(x). This can be seen by writing Ko(x;u) as an odd square wave OSW(x;u) which oscillates between +1 and -1 with fundamental frequency u plus a d-c bias, Ko(x;u) = 0.5 + 0.5 OSW(x;u), and writing f(x) = e(x) + o(x).

$$\begin{aligned}
\mathrm{OPTX}[f(x)] &= \int_{-\infty}^{\infty} [0.5 + 0.5\ \mathrm{OSW}(x;u)][e(x) + o(x)]dx \\
&= .5\int_{-\infty}^{\infty} e(x)dx + .5\int_{-\infty}^{\infty} \mathrm{OSW}(x;u)e(x)dx \\
&\quad + .5\int_{-\infty}^{\infty} o(x)dx + .5\int_{-\infty}^{\infty} \mathrm{OSW}(x;u)o(x)dx \\
&= 0.5\ F(0) + 0.5\int_{-\infty}^{\infty} \mathrm{OSW}(x;u)o(x)dx
\end{aligned}$$

where F(0) = Fourier transform evaluated at u=0. Therefore the odd pulse train transform only "sees" the odd part of the scene function.

In order to see the relationship between the Fourier sine transform and the odd pulse train transform of f(x), it is necessary to replace Ko(x;u) with its Fourier series expression.

$$\begin{aligned} Ko(x;u) = 1/2 + (2/\pi)\{&\sin(2\pi ux) + (1/3)\sin[2\pi(3u)x] \\ &+ (1/5)\sin[2\pi(5u)x] \\ &+ (1/7)\sin[2\pi(7u)x] + \ldots\} \end{aligned}$$

The odd pulse train transform of f(x) can then be expressed as a sum of Fourier sine transforms.

$$\begin{aligned} OPTX[f(x)] = 0.5F(0) + (2/\pi)\{&F_s(u) + (1/3)F_s(3u) \\ &+ (1/5)F_s(5u) + \ldots\} \end{aligned}$$

In order to gain some insight into the significance of the higher order terms in the series expressions for the even and odd pulse train transforms, a computer program was written in BASIC which computes and graphs EPTX[f(x)] and OPTX[f(x)] as a function of spatial frequency, u. The program, PTX, calculates the area under the product K(x;u)f(x) for both the even and odd pulse trains for a range of spatial frequency values entered by the user. The program was modified to also plot the raised cosine and sine Fourier transforms to model the result of a sinusoidal variation in the SLM transmittance function for comparison.

Consider the simple example of a single square pulse of unit width centered about the origin.

$$f(x) = 10\mathrm{rect}(x) = \begin{cases} 10 & -1/2 \leq x \leq 1/2 \\ 0 & \text{otherwise} \end{cases}$$

Figure 5(b) shows the graph of the raised Fourier cosine transform of the scene with luminous intensity f(x), 5(a), that would be measured if the SLM were given a cosinusoidal variation in transmittance of the form

$$T(x;u) = 0.5 + 0.5\cos(2\pi ux).$$

The result is proportional to the Fourier cosine transform $F_c(u)$ plus a d-c bias level.

Figure 5(c) shows the graph of the even pulse train transform, EPTX[f(x)]. Figure 5(d) shows the raised Fourier cosine transform superimposed on the graph of the even pulse train transform for comparison.

It is interesting to note that the raised Fourier cosine transform and the even pulse train transform coincide

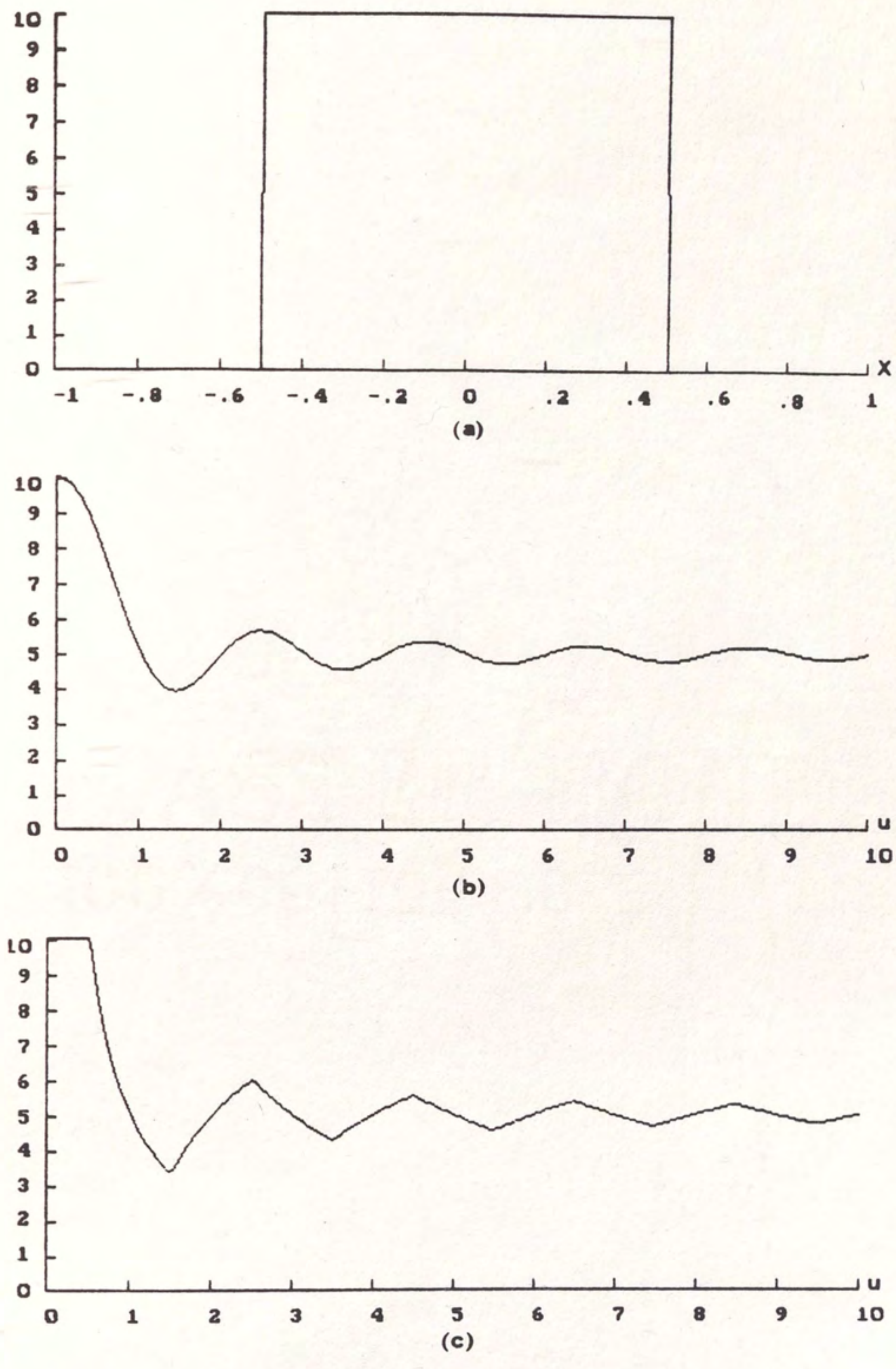
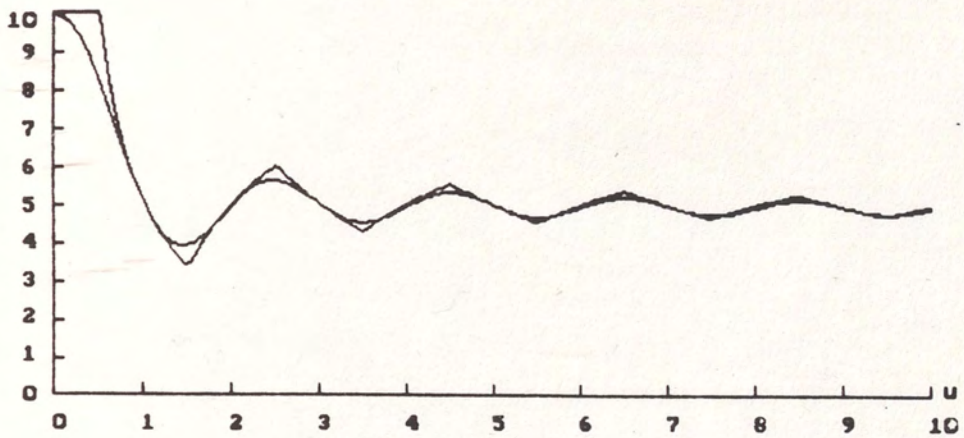
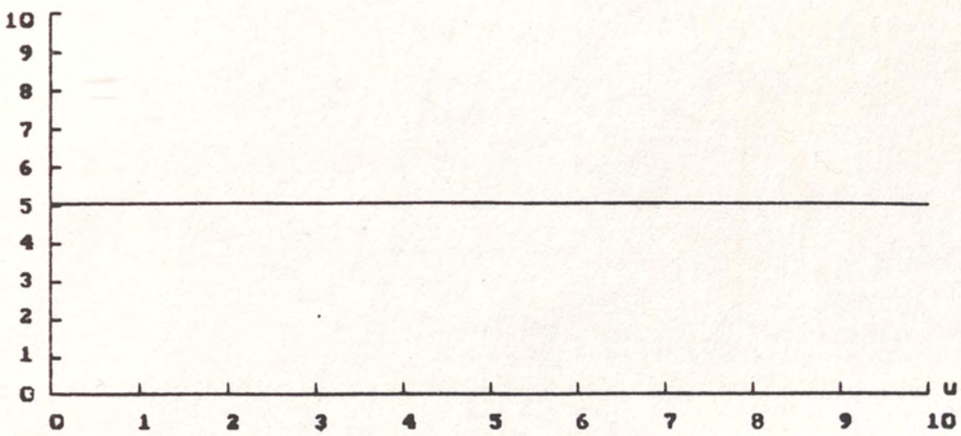



Figure 5: (a) $f(x) = 10\mathrm{rect}(x)$.
(b) Raised Fourier cosine transform of $f(x)$.
(c) Even pulse train transform of $f(x)$.




(d)




(e)

Figure 5: (d) Raised Fourier cosine transform superimposed on EPTX[f(x)] for comparison.

(e) Odd pulse train transform of f(x).

for $u = 0,1,2,3,4,\ldots$ in this example. The maximum deviation of EPTX[f(x)] from the raised cosine transform occurs at spatial frequencies $u = .5, 1.5, 2.5, \ldots$. The even pulse train transform is qualitatively very similar in shape to the raised cosine transform.

The odd pulse train transform and the raised Fourier sine transform are shown in Figure 5(e). Both OPTX[f(x)] and the raised sine transform are equal to the constant value $F(0)/2$ since $F\ (u) = 0$ for an even scene function.

## PROPERTIES OF EPTX[f(x)] AND OPTX[f(x)]

Given that the even pulse train transform and odd pulse train transform are linear integral transforms, i.e.,

$$EPTX[af(x) + bg(x)] = a\ EPTX[f(x)] + b\ EPTX[g(x)]$$

$$OPTX[af(x) + bg(x)] = a\ OPTX[f(x)] + b\ OPTX[g(x)]$$

it is of interest to investigate other properties of the pulse train transforms for comparison with properties of the Fourier transform.

### Change of Scale Property

If the Fourier transform of f(x) is F(u) then the Fourier transform of f(ax) is $(1/|a|)F(u/a)$ or $(1/|a|)[F_c(u/a)-jF_s(u/a)]$. This result can be substituted directly into the expressions for EPTX[f(ax)] and OPTX[f(ax)].

$$\begin{aligned} EPTX[f(ax)] &= F_c(0)/(2|a|) + (2/\pi|a|)\{F_c(u/a) \\ &\quad -(1/3)F_c(3u/a)+ (1/5)F_c(5u/a) +\dots\} \\ &=( 1/|a|)Pe(u/a) \end{aligned}$$

$$\mathrm{OPTX}[f(ax)] = F(0)/(2|a|) + (2/\pi|a|)\{F_s(u/a) + (1/3)F_s(3u/a) + (1/5)F_s(5u/a) + \ldots\}$$
$$= (1/|a|)\mathrm{Po}(u/a)$$

The change of scale property for the even and odd pulse train transforms is identical to the Fourier transform property.

Figure 6 shows the raised Fourier cosine transform, 6(b), and the even pulse train transform, 6(c), of $f(x) = 10\mathrm{rect}(5x)$. As $f(x)$ is compressed, the spatial frequency content spreads out to higher frequencies. Comparison with Figure 5 reveals that the maxima and minima have indeed moved from 1.5, 2.5, 3.5, etc. in Figure 5 to five times those values in Figure 6: 7.5, 12.5, 17.5, etc.

The maximum value of both the raised cosine transform and the even pulse train transform has been scaled to 1/5 of the value in Figure 5, as expected, since the total area under $f(x)$ in Figure 6 is 1/5 of the total area in Figure 5.

The raised Fourier sine transform and odd pulse train transform (not shown) are again constant at one-half the area of the scene function i.e., $F(0)/2 = 1$, since $f(x)$ is still an even function and $F_s(u) = 0$.

Figures 7 and 8 show another example of the change of scale property. Figure 7 shows the raised Fourier cosine transform, 7(b), and the even pulse train transform, 7(c), of

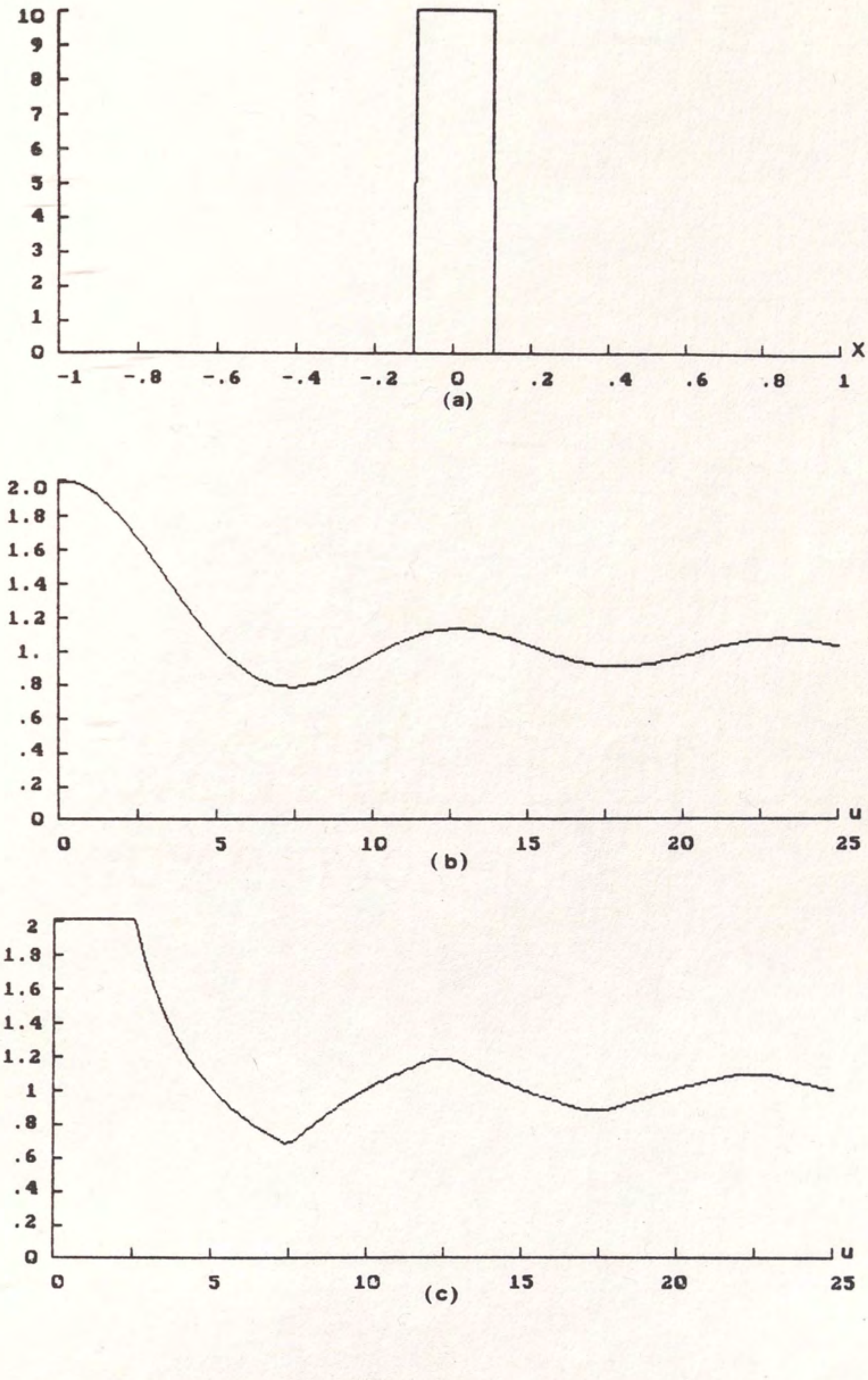



Figure 6: (a) $f(x) = 10\text{rect}(5x)$.
(b) Raised Fourier cosine transform of $f(x)$.
(c) Even pulse train transform of $f(x)$.

a gaussian pulse, $f(x) = 10\exp(-\pi x^2)$. The raised Fourier cosine transform yields a gaussian pulse plus d-c bias equal to F(0)/2. The even pulse train transform in Figure 7(c) differs from the raised Fourier cosine transform primarily at very low frequencies but it is qualitatively very similar otherwise. The raised Fourier sine transform and odd pulse train transform (not shown) are both constant and equal to $F(0)/2 = 5$ since $F_s(u) = 0$.

Figure 8 shows the raised Fourier cosine transform and even pulse train transform of the narrow gaussian pulse $f(x) = 10\exp(-25\pi x^2)$ which has a total area of 2. Comparing the raised Fourier cosine transform, 8(b), and even pulse train transform, 8(c), with the corresponding plots in Figure 7, clearly shows the effect of the change in scale, i.e. as the scene function is compressed in the spatial domain, the transform spreads out in the frequency domain.

## Shifting Property

If the Fourier transform of f(x) is $F(u)=F_c(u)-jF_s(u)$ it is well known that the Fourier transform of f(x-a) is $F(u)\exp[-j2\pi ua]$. The real part of $F(u)\exp[-j2\pi ua]$ is the Fourier cosine transform of f(x-a) and the negative imaginary part of $F(u)\exp[-j2\pi ua]$ is the Fourier sine transform of f(x-a), for f(x) real valued.

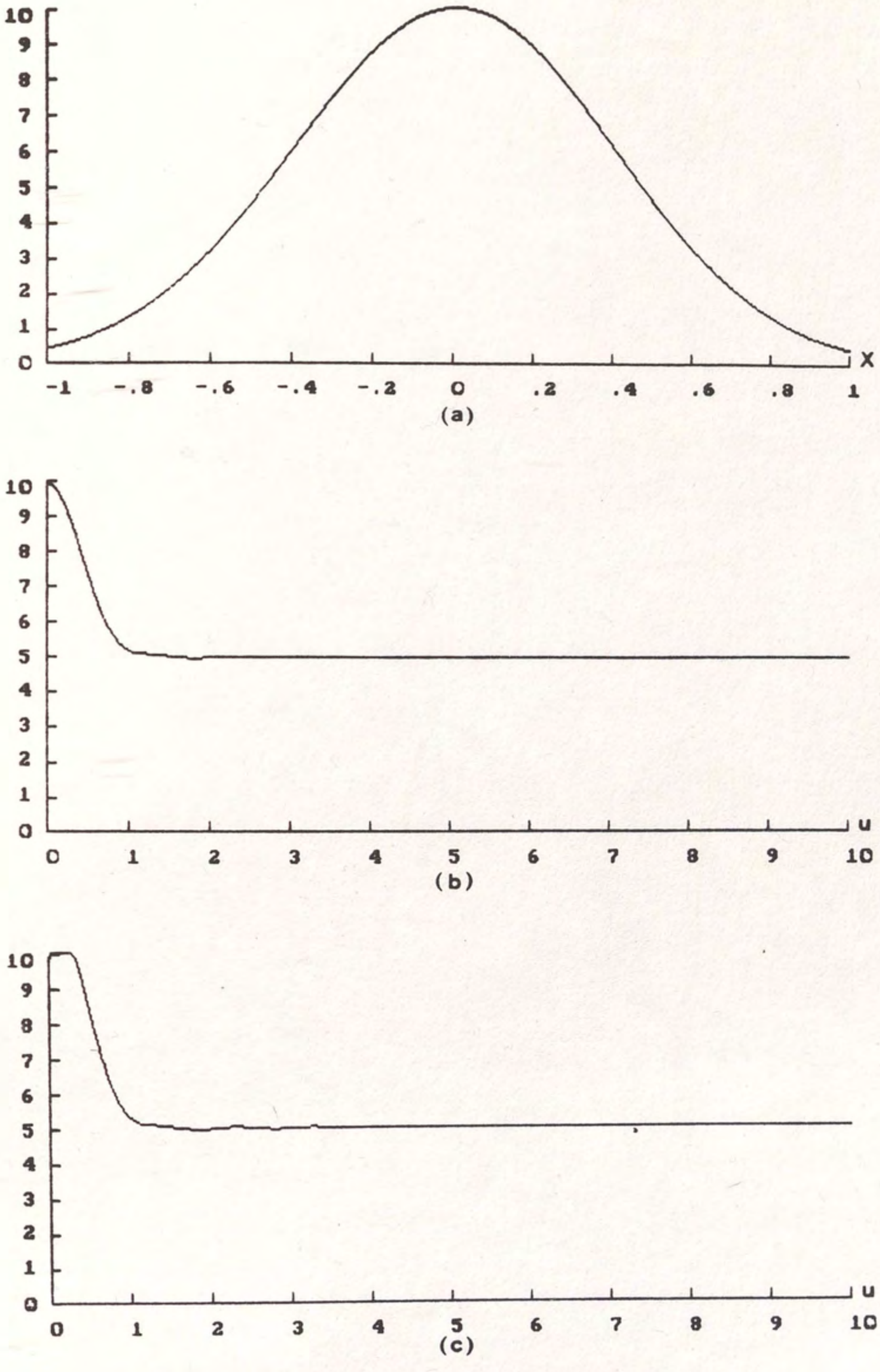



Figure 7: (a) $f(x) = 10\exp(-\pi x^2)$.
(b) Raised Fourier cosine transform of f(x).
(c) Even pulse train transform of f(x).

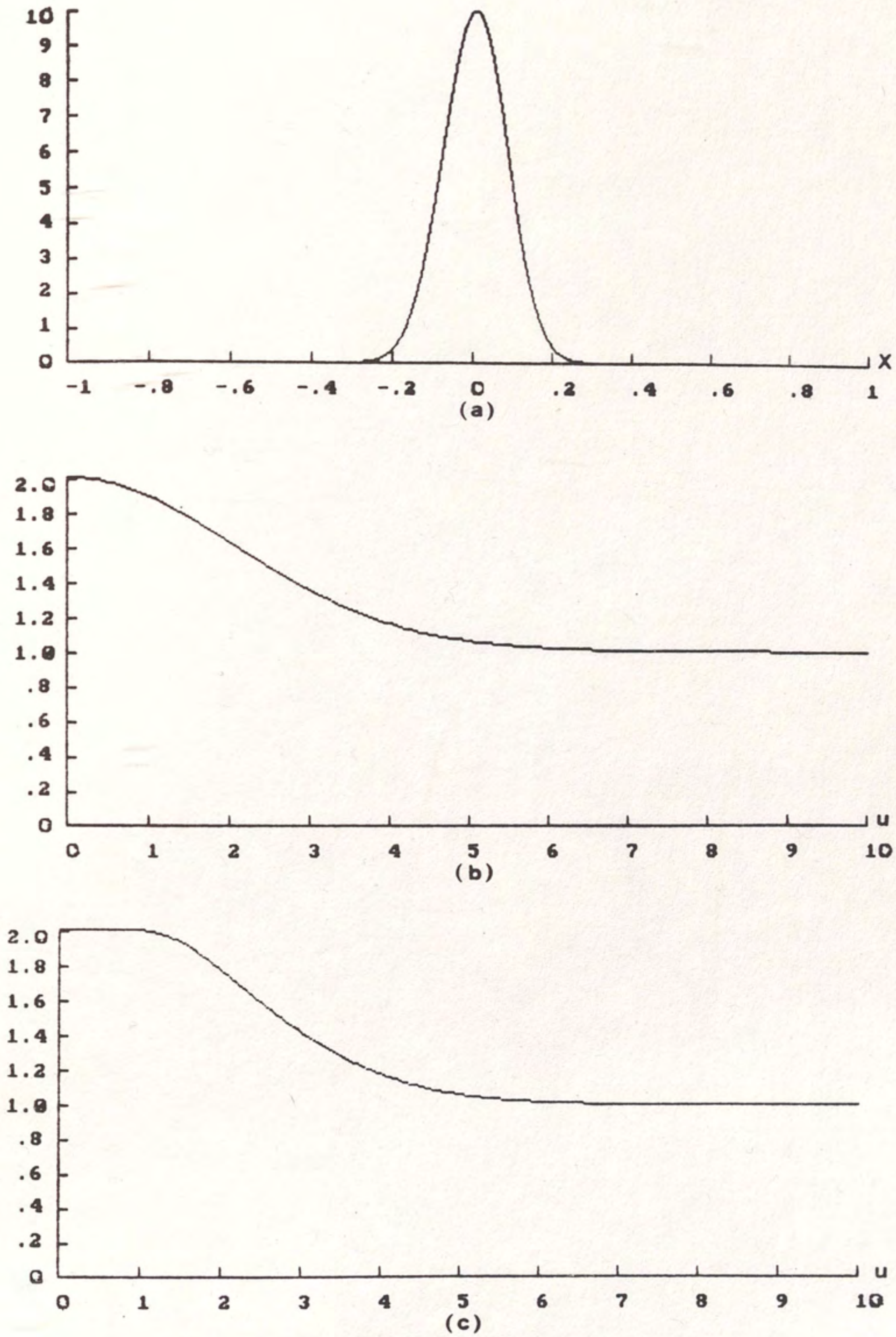



Figure 8: (a) $f(x) = 10 \exp(-25\pi x^2)$.
(b) Raised Fourier cosine transform of $f(x)$.
(c) Even pulse train transform of $f(x)$.

$$f(x-a) \longleftarrow \text{cosine transform} \longrightarrow F_c(u)\cos(2\pi ua)-F_s(u)\sin(2\pi ua)$$

$$f(x-a) \longleftarrow \text{sine transform} \longrightarrow F_c(u)\sin(2\pi ua)+F_s(u)\cos(2\pi ua)$$

The above results can be substituted directly into the expressions for the even and odd pulse train transforms of a shifted function.

$$\begin{aligned}
EPTX[f(x-a)] = (F(0)/2)+ (2/\pi)\{&F_c(u)\cos(2\pi ua)-F_s(u)\sin(2\pi ua)\\
&-(1/3)[F_c(3u)\cos(2\pi(3u)a)-F_s(3u)\sin(2\pi(3u)a)]\\
&+(1/5)[F_c(5u)\cos(2\pi(5u)a)-F_s(5u)\sin(2\pi(5u)a)]\ldots\}
\end{aligned}$$

$$\begin{aligned}
OPTX[f(x-a)] = (F(0)/2)+ (2/\pi)\{&F_c(u)\sin(2\pi ua)+F_s(u)\cos(2\pi ua)\\
&+(1/3)[F_c(3u)\sin(2\pi(3u)a)+F_s(3u)\cos(2\pi(3u)a)]\\
&+(1/5)[F_c(5u)\sin(2\pi(5u)a)+F_s(5u)\cos(2\pi(5u)a)]\ldots\}
\end{aligned}$$

It is instructive to consider the special case of an even scene function f(x). If f(x) is an even function then the Fourier sine transform of f(x) is zero, i.e., $F_s(u)=0$, and the Fourier transform shift theorem takes a particularly simple form.

$$f(x-a) \longleftarrow \text{cosine transform} \longrightarrow F_c(u)\cos(2\pi ua)$$

$$f(x-a) \longleftarrow \text{sine transform} \longrightarrow F_c(u)\sin(2\pi ua)$$

From the above expressions it is apparent that if one starts with an even real function f(x) (which has the purely real

Fourier transform $F_c(u)$), and then shifts it by a distance "a", the result is a complex Fourier transform of the form $F_c(u)\cos(2\pi ua)-jF_c(u)\sin(2\pi ua)$ in which the real and imaginary parts are oscillatory within an envelope, which is the transform of the original unshifted function.

For the case of the even and odd pulse train transforms of an even function f(x), the shifting property can be simplified to the form shown below.

$$\begin{aligned}
\mathrm{EPTX}[f(x-a)] = (F(0)/2) + (2/\pi)\{&F_c(u)\cos(2\pi ua) \\
&-(1/3)F_c(3u)\cos(2\pi(3u)a) \\
&+(1/5)F_c(5u)\cos(2\pi(5u)a)\ldots\} \\
\mathrm{OPTX}[f(x-a)] = (F(0)/2) + (2/\pi)\{&F_c(u)\sin(2\pi ua) \\
&+(1/3)F_c(3u)\sin(2\pi(3u)a) \\
&+(1/5)F_c(5u)\sin(2\pi(5u)a)\ldots\}
\end{aligned}$$

Comparison of the above equations with the formulae for the even and odd pulse train transforms of an unshifted scene reveals that each term in the unshifted EPTX and OPTX expressions is multiplied by a cosine or sine coefficient when the scene is shifted. As a result of multiplying the terms of the unshifted transform by coefficients which oscillate between +1 and -1, EPTX[f(x-a)] and OPTX[f(x-a)] oscillate within an envelope determined by EPTX[f(x)] and

OPTX[f(x)] respectively. This property is illustrated by the following examples.

Figure 9 shows the raised Fourier cosine (b) and sine (c) transforms of a shifted square pulse (a). As expected, the Fourier sine and cosine transforms of the shifted pulse oscillate within an envelope equal to the transform of the unshifted square pulse (see Figure 6).

Figure 10 shows the even pulse train transform (b) and odd pulse train transform (c) of the same shifted square pulse as Figure 9. The dotted line is the even pulse train transform of the unshifted square pulse (see Figure 6). Again it is apparent that the transform of the shifted scene function oscillates within an envelope determined by the even pulse train transform of the unshifted function.

Figures 11 and 12 show similar results for the square pulse originally shown in Figure 5 but shifted 0.5 units to the right.

An interesting variation is obtained when a sort of "symmetric shifting" is introduced, i.e., starting with an even, real valued function f(x) define a new function g(x) by the equation:

$$g(x) = 0.5[f(x + a) + f(x - a)].$$

The new function g(x) will be an even function also since

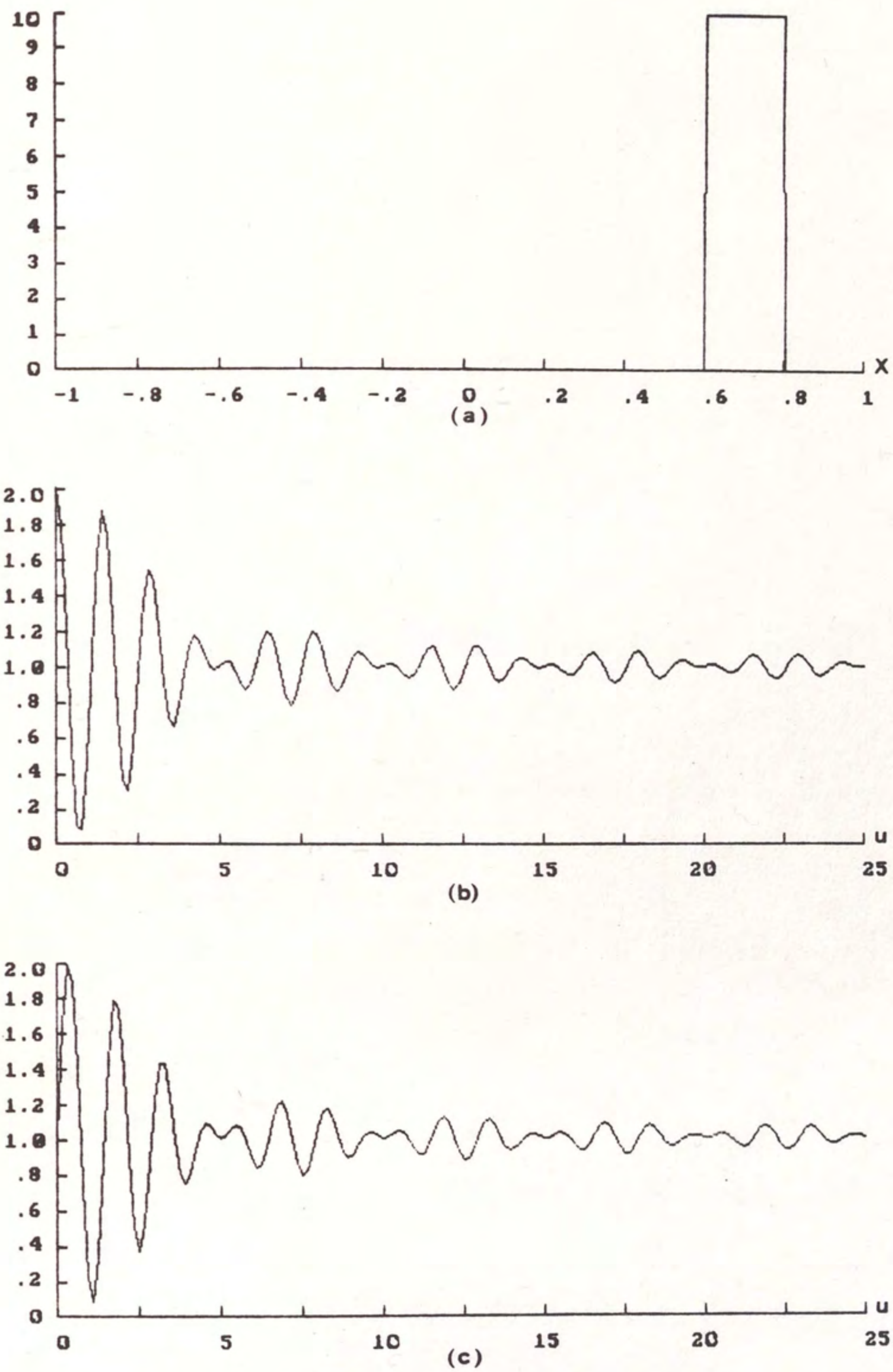



Figure 9: (a) $f(x) = 10\mathrm{rect}[5(x-.7)]$.
(b) Raised Fourier cosine transform of $f(x)$
(c) Raised Fourier sine transform of $f(x)$

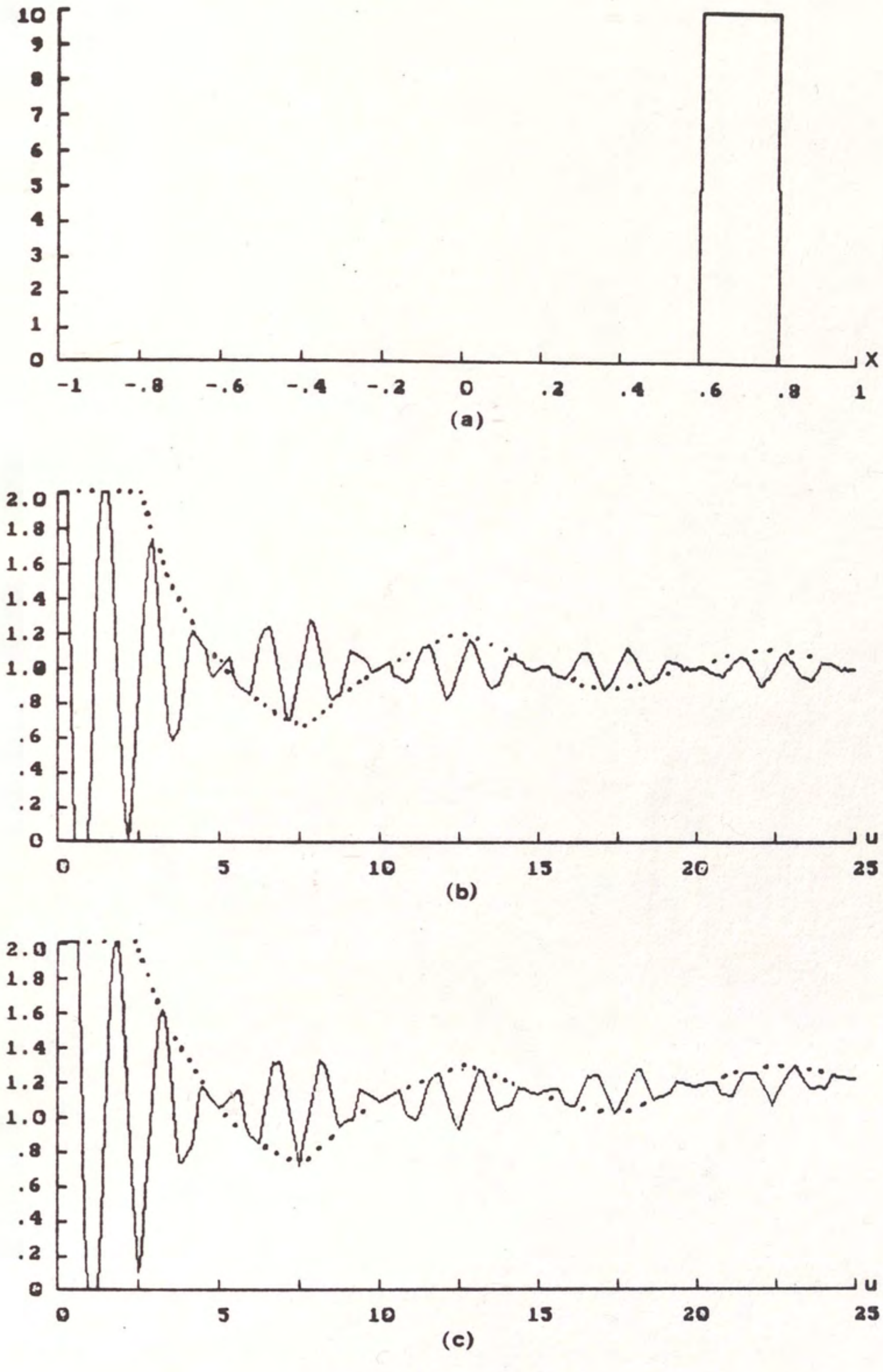



Figure 10: (a) $f(x) = 10\mathrm{rect}[5(x-.7)]$.
(b) Even pulse train transform of $f(x)$.
(c) Odd pulse train transform of $f(x)$.

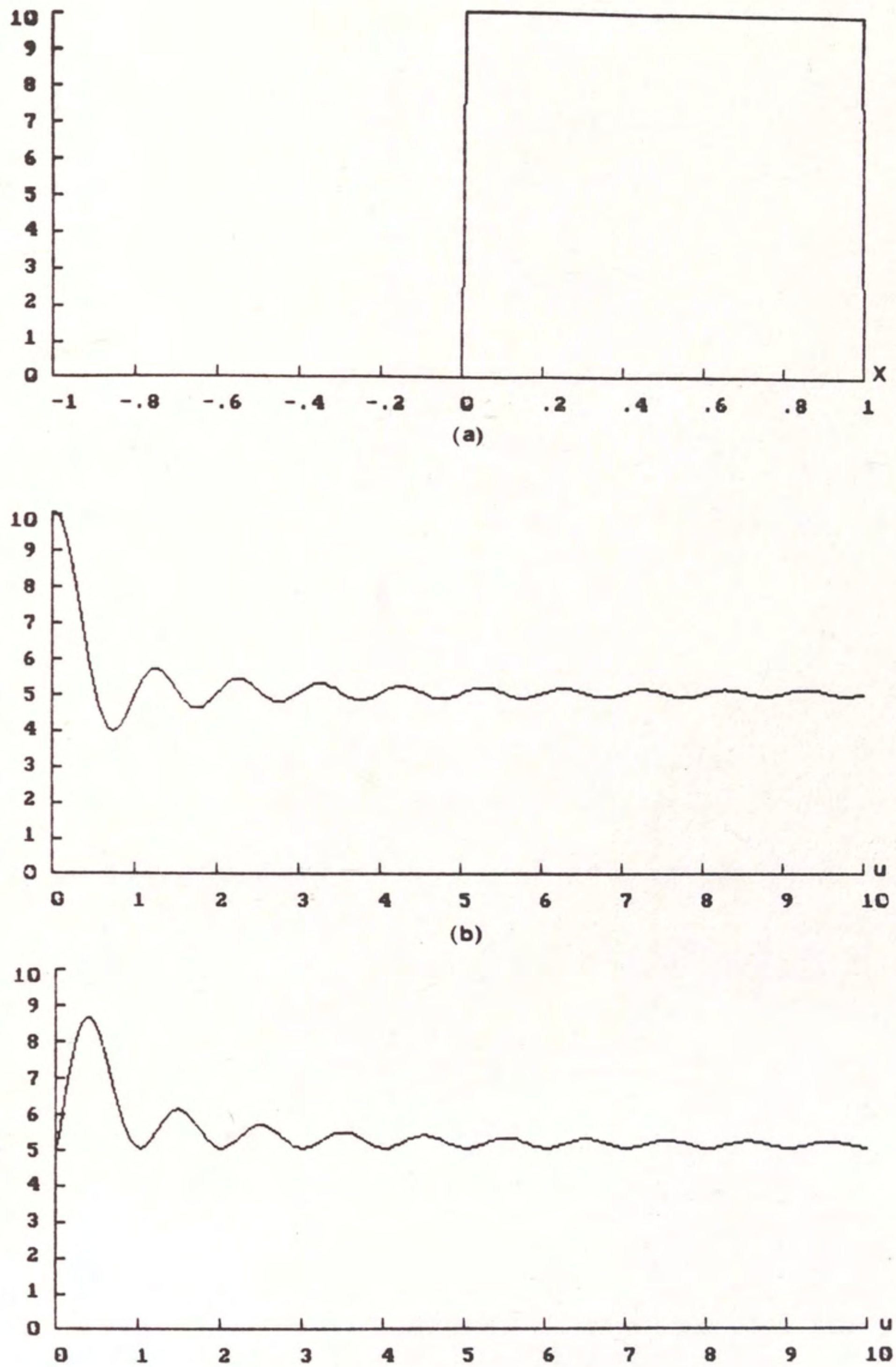



Figure 11: (a) $f(x) = 10\text{rect}(x - 0.5)$.
(b) Raised Fourier cosine transform of $f(x)$.
(c) Raised Fourier sine transform of $f(x)$.

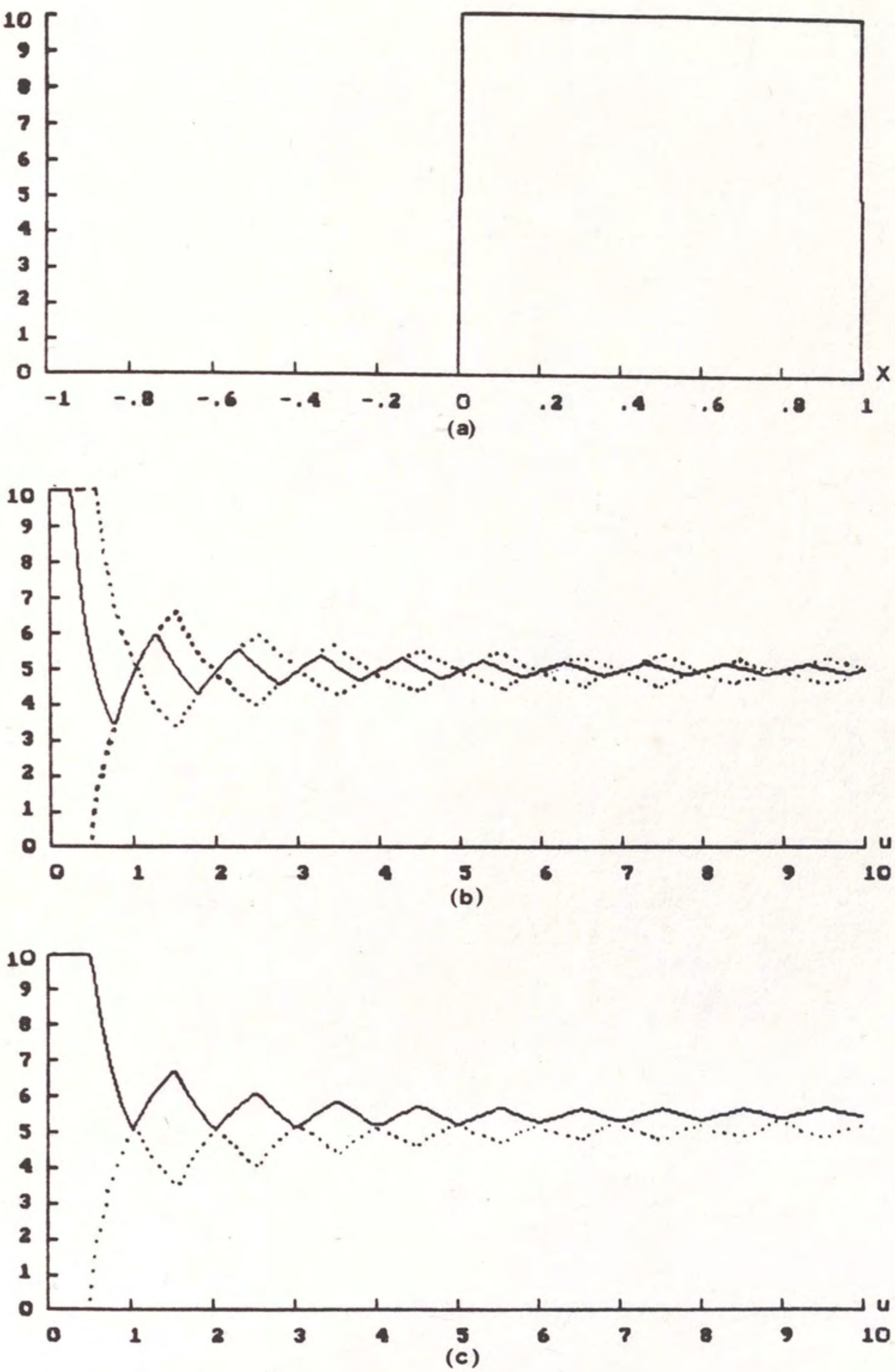



Figure 12: (a) $f(x) = 10\mathrm{rect}(x - 0.5)$.
(b) Even pulse train transform of $f(x)$ with envelope from fig. 5c.
(c) Odd pulse train transform of $f(x)$ with envelope from fig. 5c.

$f(x)$ is even. Therefore the Fourier transform of $g(x)$ will be an even, real valued function of the spatial frequency, u. This permits the study of shifting effects without generating a complex valued Fourier transform. The odd pulse train transform of $g(x)$ will be a constant equal to $G(0)/2$ since $G_s(u) = 0$.

Figure 13 shows the raised Fourier cosine transform (b) and the even pulse train transform (c) of a pair of gaussian pulses that would be generated from "symmetric shifting" of the scene function $f(x) = 20\exp[-36\pi x^2]$ which yields:

$$g(x) = 10\exp[-36\pi(x + 0.7)^2] + 10\exp[-36\pi(x - 0.7)^2].$$

The raised Fourier cosine transform (b) and even pulse train transform (c) of the unshifted scene function, $f(x)$, are shown in Figure 14 for comparison. The raised Fourier sine transform and odd pulse train transform (not shown) are constant for both $f(x)$ and $g(x)$ and equal to one-half the total area under $f(x)$, i.e., $F(0)/2 = G(0)/2 = 10/6$.

Figure 15 shows the raised Fourier cosine transform (b) and even pulse train transform (c) of a narrow even pulse pair (a):

$$g(x) = 10\mathrm{rect}[20(x+0.8)] + 10\mathrm{rect}[20(x-0.8)].$$

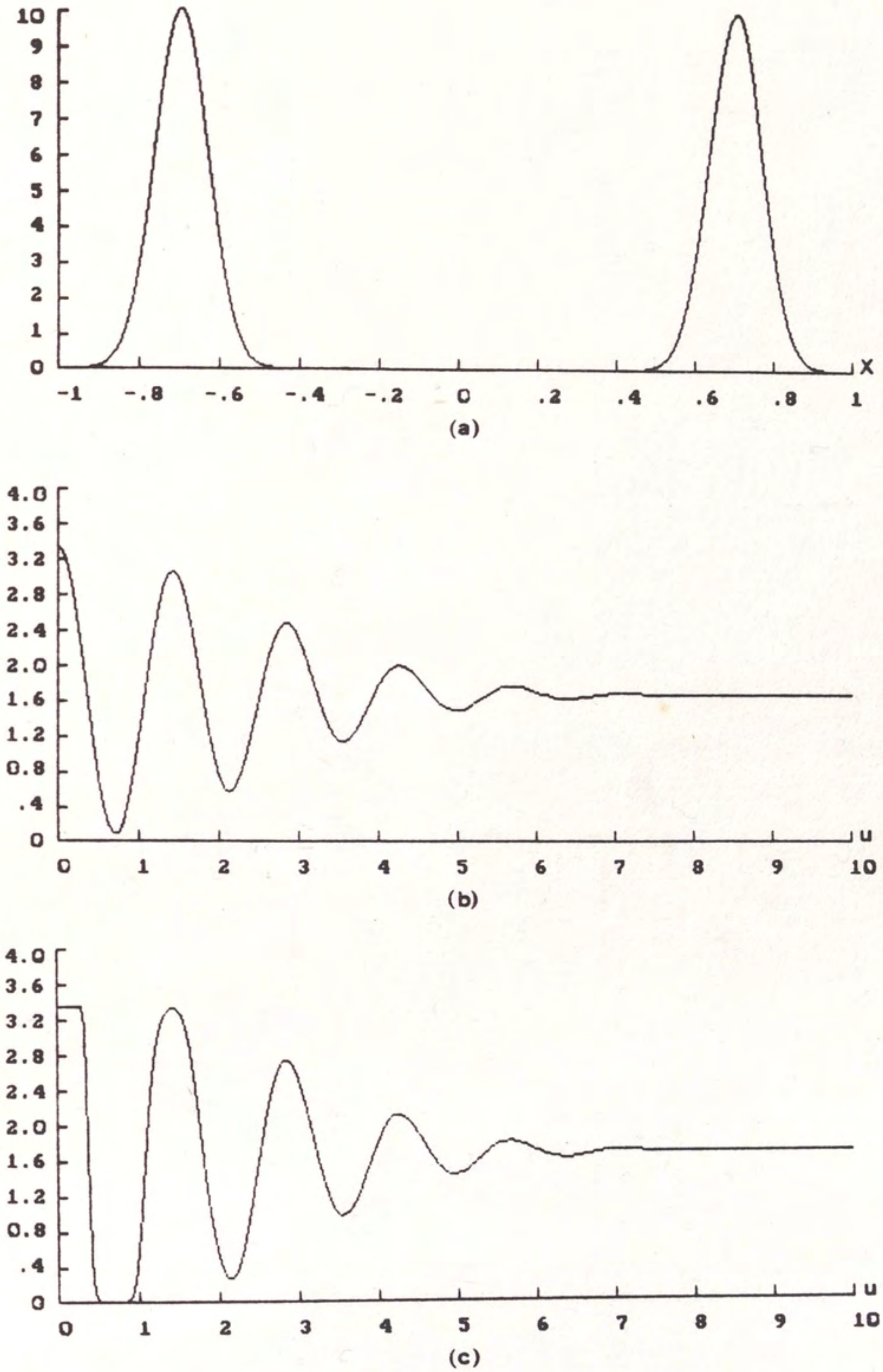



Figure 13: (a) $g(x) = 10\exp[-36\pi(x + 0.7)^2] + 10\exp[-36\pi(x-.07)^2]$.
(b) Raised Fourier cosine transform of $g(x)$.
(c) Even pulse train transform of $g(x)$.

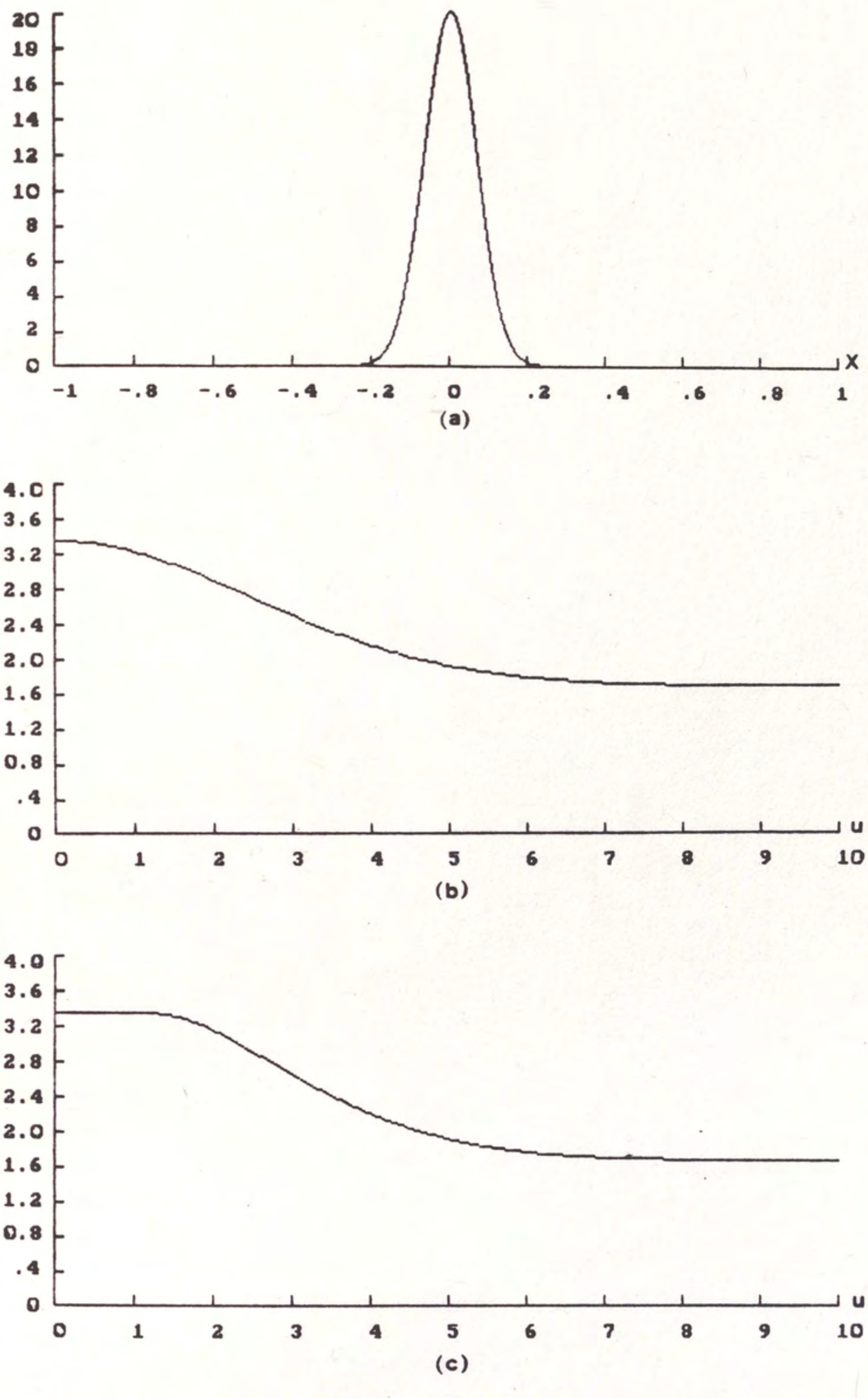



Figure 14: (a) $f(x) = 20\exp[-36\pi x^2]$
(b) Raised Fourier cosine transform of $f(x)$.
(c) Even pulse train transform of $f(x)$.

The even pulse pair is generated by the "symmetric shifting" of the single narrow pulse $f(x) = 20\mathrm{rect}[20x]$ shown in Figure 16 with its raised Fourier cosine transform (b) and even pulse train transform (c).

Comparison of figures 15 and 16 once again illustrates that the transform of the shifted scene function oscillates within an envelope determined by the transform of the unshifted function for both the even pulse train transform as well as the raised Fourier cosine transform.

The raised Fourier sine transform and odd pulse train transform of $f(x)$ and $g(x)$ are constant and equal to one-half the total area under the functions.

It is of interest to compare the even pulse train transforms of the narrow square pulse of Figure 16 and the not so narrow square pulse of Figure 5 from the standpoint of using a narrow pulse to approximate an impulse. If the highest spatial frequency of interest in a particular application happens to be ten, then the narrow square pulse of Figure 16 is indistinguishable from an ideal impulse as far as the even pulse train transform is concerned since the EPTX shown in Figure 16(c) is flat up to $u = 10$.

Figure 17 illustrates the same shift properties as Figure 15 but the change in scale makes it possible to get a detailed view of the even pulse train transform of the pulse

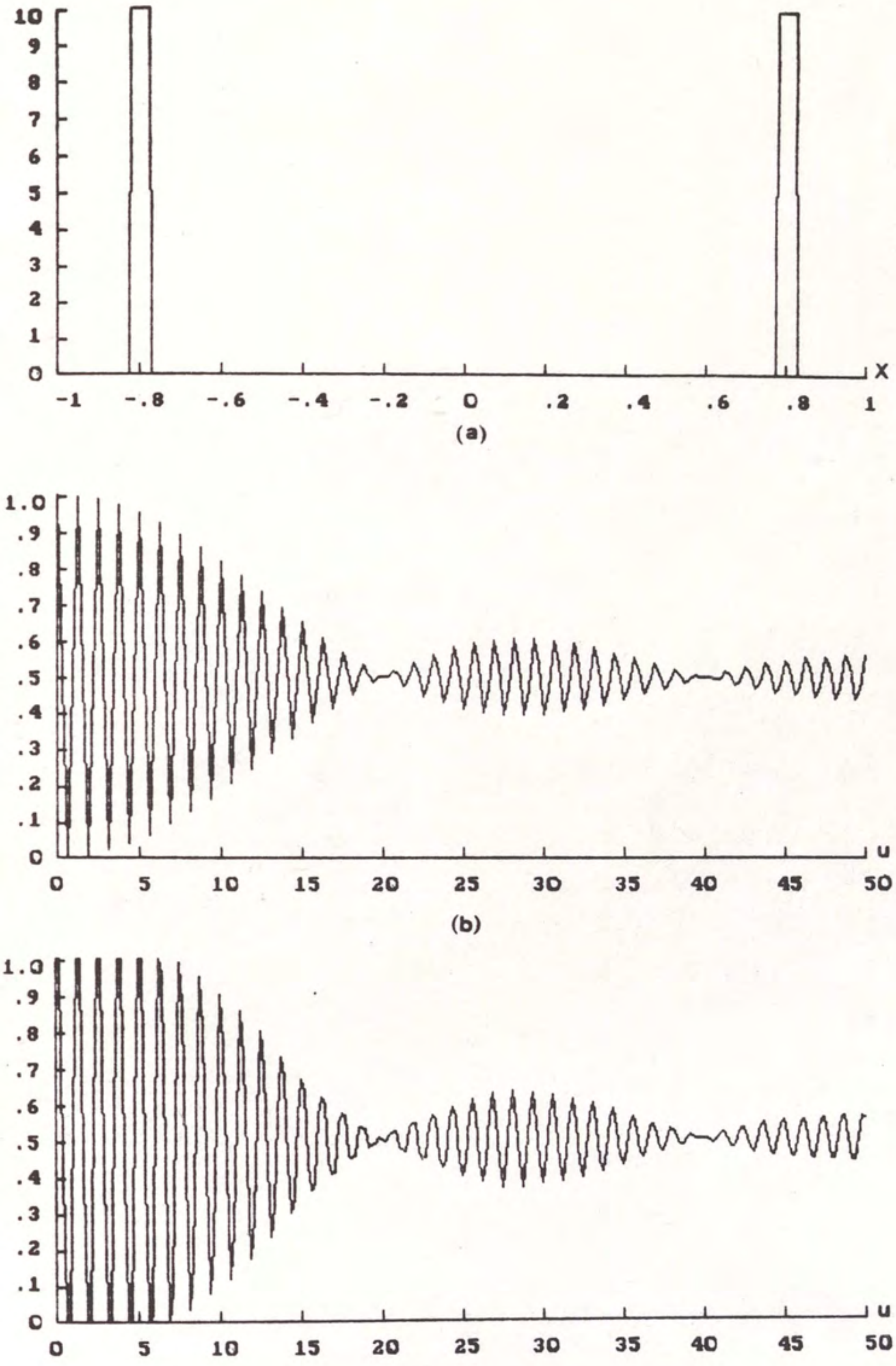



Figure 15: (a) $g(x) = 10\text{rect}[20(x + 0.8)] + 10\text{rect}[20(x-0.8)]$.
(b) Raised Fourier cosine transform of $g(x)$.
(c) Even pulse train transform of $g(x)$

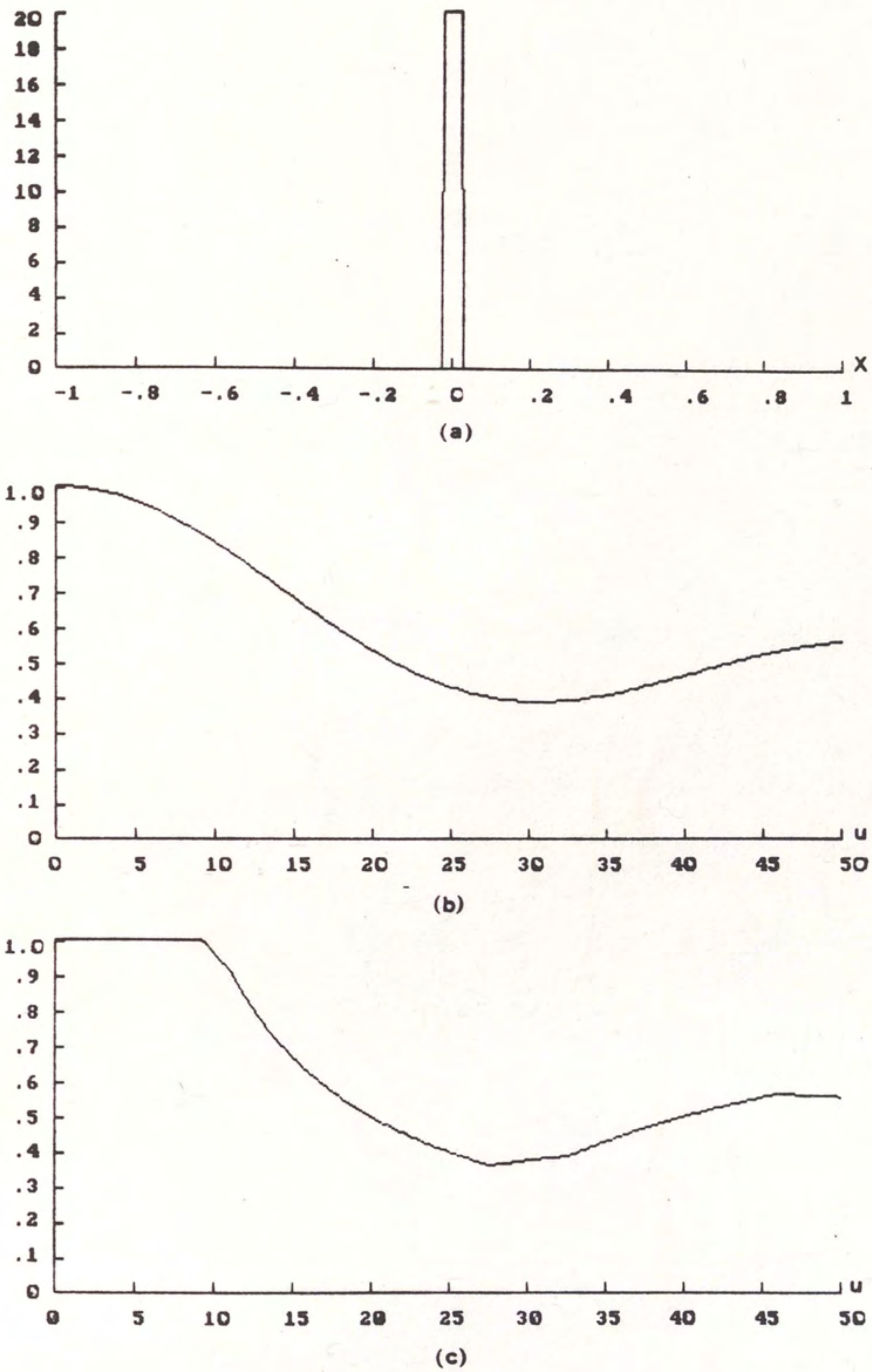



Figure 16: (a) $f(x) = 20\text{rect}(20x)$.
(b) Raised Fourier cosine transform of $f(x)$.
(c) Even pulse train transform of $f(x)$.

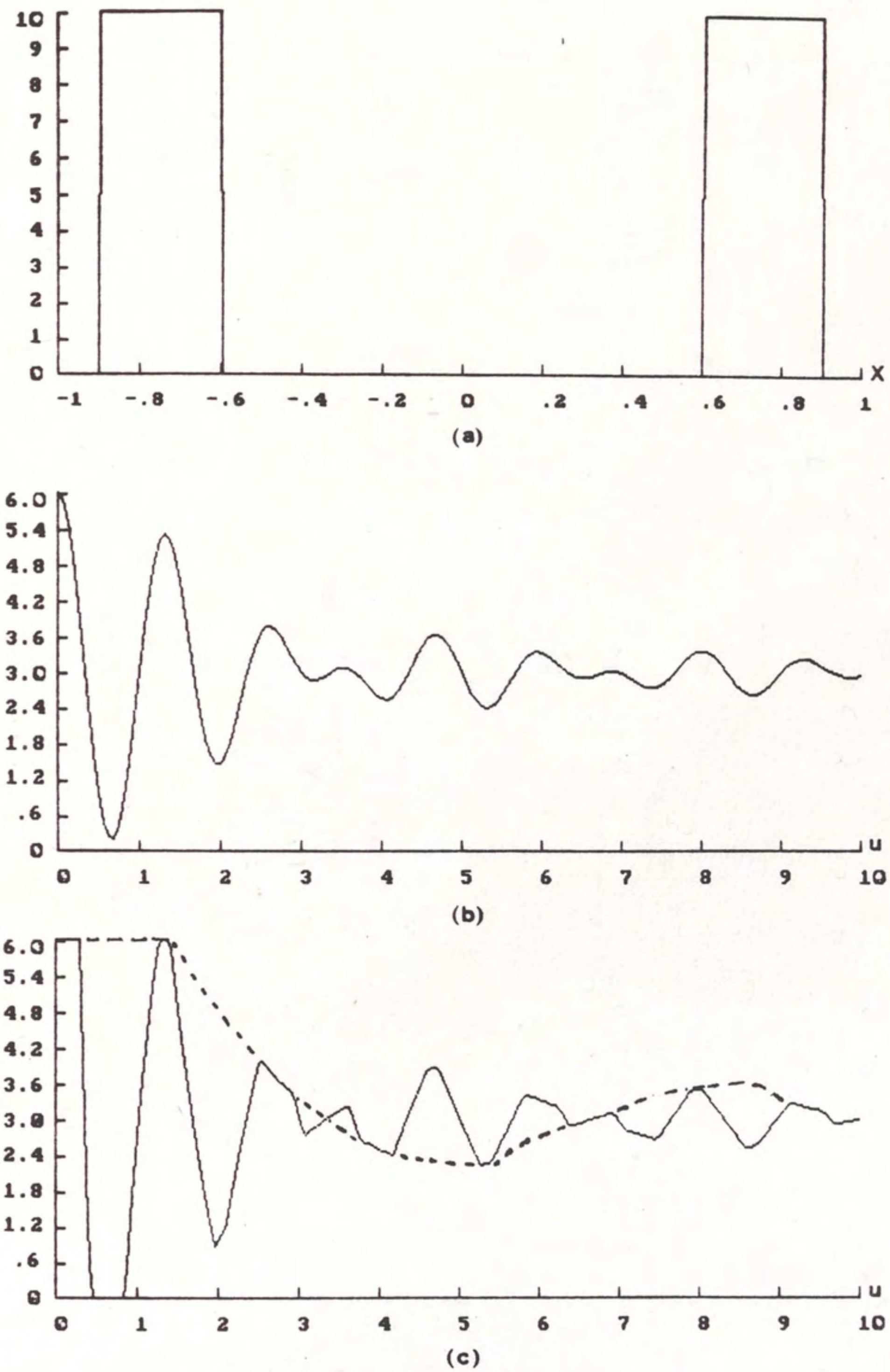



Figure 17: (a) $g(x) = 10\text{rect}[3.33(x + .75)] + 10\text{rect}[3.33(x-.75)]$.
(b) Raised Fourier cosine transform of $g(x)$.
(c) EPTX[$g(x)$] with EPTX of unshifted pulse $f(x) = 20\text{rect}(3.33x)$ superimposed (dotted line).

pair. The dotted line on Figure 17(c) is the even pulse train transform of an unshifted pulse, $f(x) = 20\mathrm{rect}(3.33x)$.

## Convolution Property

If the Fourier cosine and sine transforms of the real valued functions $f(x)$ and $g(x)$ are $F_c(u)$, $F_s(u)$, $G_c(u)$ and $G_s(u)$ respectively, then the following convolution property holds for the Fourier cosine and sine transforms:

$$f(x) * g(x) \xleftarrow{\quad} \text{cosine trans.} \xrightarrow{\quad} F_c(u)G_c(u) - F_s(u)G_s(u)$$

$$f(x) * g(x) \xleftarrow{\quad} \text{sine trans.} \xrightarrow{\quad} F_c(u)G_s(u) + F_s(u)G_c(u)$$

where $f(x) * g(x) = \int_{-\infty}^{\infty} f(r)g(x-r)dr$.

Substituting the above results into the expressions for the even and odd pulse train transforms yields the following result.

$$\begin{aligned}
\mathrm{EPTX}[f(x) * g(x)] = {} & (F(0)G(0)/2) \\
& + (2/\pi)\{[F_c(u)G_c(u) - F_s(u)G_s(u)] \\
& -(1/3)[F_c(3u)G_c(3u) - F_s(3u)G_s(3u)] \\
& +(1/5)[F_c(5u)G_c(5u) - F_s(5u)G_s(5u)]\ldots\}
\end{aligned}$$

$$\begin{aligned}
\mathrm{OPTX}[f(x) * g(x)] = {} & (F(0)G(0)/2) \\
& +(2/\pi)\{[F_c(u)G_s(u) + F_s(u)G_c(u)] \\
& +(1/3)[F_c(3u)G_s(3u) + F_s(3u)G_c(3u)] \\
& +(1/5)[F_c(5u)G_s(5u) + F_s(5u)G_c(5u)]\ldots\}
\end{aligned}$$

In order to gain some insight into the even and odd pulse train transform convolution properties consider the special case of $f(x)$ and $g(x)$ even so that $F_s(u) = 0$ and $G_s(u) = 0$. If $f(x)$ and $g(x)$ are even functions, then the cosine transform of $f(x) * g(x)$ simplifies to $F_c(u)G_c(u)$ and the sine transform of $f(x) * g(x)$ is zero. The even and odd pulse train transforms of $f(x) * g(x)$ in this special case simplify as shown below.

$$
\begin{aligned}
\mathrm{EPTX}[f(x) * g(x)] = {} & (F(0)G(0)/2) \\
& + (2/\pi)\{F_c(u)G_c(u) - (1/3)F_c(3u)G_c(3u) \\
& + (1/5)\, F_c(5u)G_c(5u)\ldots\} \\
\mathrm{OPTX}[f(x) * g(x)] = {} & F(0)G(0)/2
\end{aligned}
$$

Now compare the above expression to EPTX[f(x)] EPTX[g(x)] shown below.

$$
\begin{aligned}
& \mathrm{EPTX}[f(x)]\ \mathrm{EPTX}[g(x)] = (F(0)G(0)/4) \\
& +(F(0)/\pi)\{G_c(u) - (1/3)G_c(3u) + \ldots\} \\
& +(G(0)/\pi)\{F_c(u) - (1/3)F_c(3u) + \ldots\} \\
& +(2/\pi)\ \{F_c(u) - (1/3)F_c(3u) + \ldots\}\{G_c(u) - (1/3)G_c(3u) + \ldots\}
\end{aligned}
$$

The d-c bias terms F(0) and G(0) generate unwanted multiples of EPTX[f(x)] and EPTX[g(x)] in the above expression. Since F(0) and G(0) are readily measured, these terms can be suppressed electronically. Suppressing F(0) and G(0) yields the following expression:

$$\{EPTX[f(x)] - F(0)\}\{EPTX[g(x)] - G(0)\} =$$
$$(2/\pi)^2\{F_c(u) - (1/3)F_c(3u) + \ldots\}\{G_c(u) - (1/3)G_c(3u) + \ldots\}$$
$$= (2/\pi)^2\{F_c(u)G_c(u) + \text{higher order terms}\}$$

In order to see the significance of the higher order terms, consider the simple example of f(x) = 10rect(5x), shown in Figure 18(a). Let h(x), Figure 19(a), be equal to the convolution of f(x) with itself, i.e.:

$$h(x) = f(x) * f(x) = \begin{cases} 20 - 100\,x\,, & x < 0.2 \\ 0 & \text{otherwise} \end{cases}$$

Since f(x) and h(x) are even, real functions of x, their Fourier transforms will be even, real functions of u, equivalent to the Fourier cosine transforms.

$$F(u) = F_c(u) = 2\mathrm{sinc}(u/5)$$
$$H(u) = H_c(u) = F(u)\times F(u) = 4\mathrm{sinc}^2(u/5)$$

The Fourier sine transforms of f(x) and h(x) are equal to zero. Therefore OPTX[f(x)] = F(0)/2 = 1.0 and OPTX[h(x)] = H(0)/2 = 2.0. Substituting these expressions for $F_c(u)$ and $H_c(u)$ into the equations for EPTX[f(x)] and EPTX[h(x)] yields the following result.

$$\begin{aligned} \mathrm{EPTX}[f(x)] &= 1.0 + (2/\pi)\{2\mathrm{sinc}(u/5) \\ &\quad -(1/3)[2\mathrm{sinc}(3u/5)] \\ &\quad +(1/5)[2\mathrm{sinc}(5u/5)]\ldots\} \\ \mathrm{EPTX}[h(x)] &= 2.0 + (2/\pi)\{4\mathrm{sinc}\ (u/5) \\ &\quad -(1/3)[4\mathrm{sinc}\ (3u/5)] \\ &\quad +(1/5)[4\mathrm{sinc}\ (5u/5)]\ldots\} \end{aligned}$$

Dropping the d-c term from EPTX[f(x)] and squaring does not yield the transform of h(x) exactly due to the higher order terms and the factor of $(2/\pi)$.

Figure 18(c) shows the result of first subtracting the d-c term from EPTX[f(x)], squaring, and then restoring the d-c level for comparison with EPTX[h(x)] shown in Figure 19(b). Figure 19(c) shows Figure 18(c) superimposed on the graph of EPTX[h(x)] for comparison.

The convolution properties of the even and odd pulse train transforms are not directly analogous to the convolution properties of the Fourier transform due to the presence of the higher order terms. If the approximation obtained by suppressing the d-c terms is not accurate enough for a given application then it would be necessary to use the sinusoidal variation in SLM transmittance.

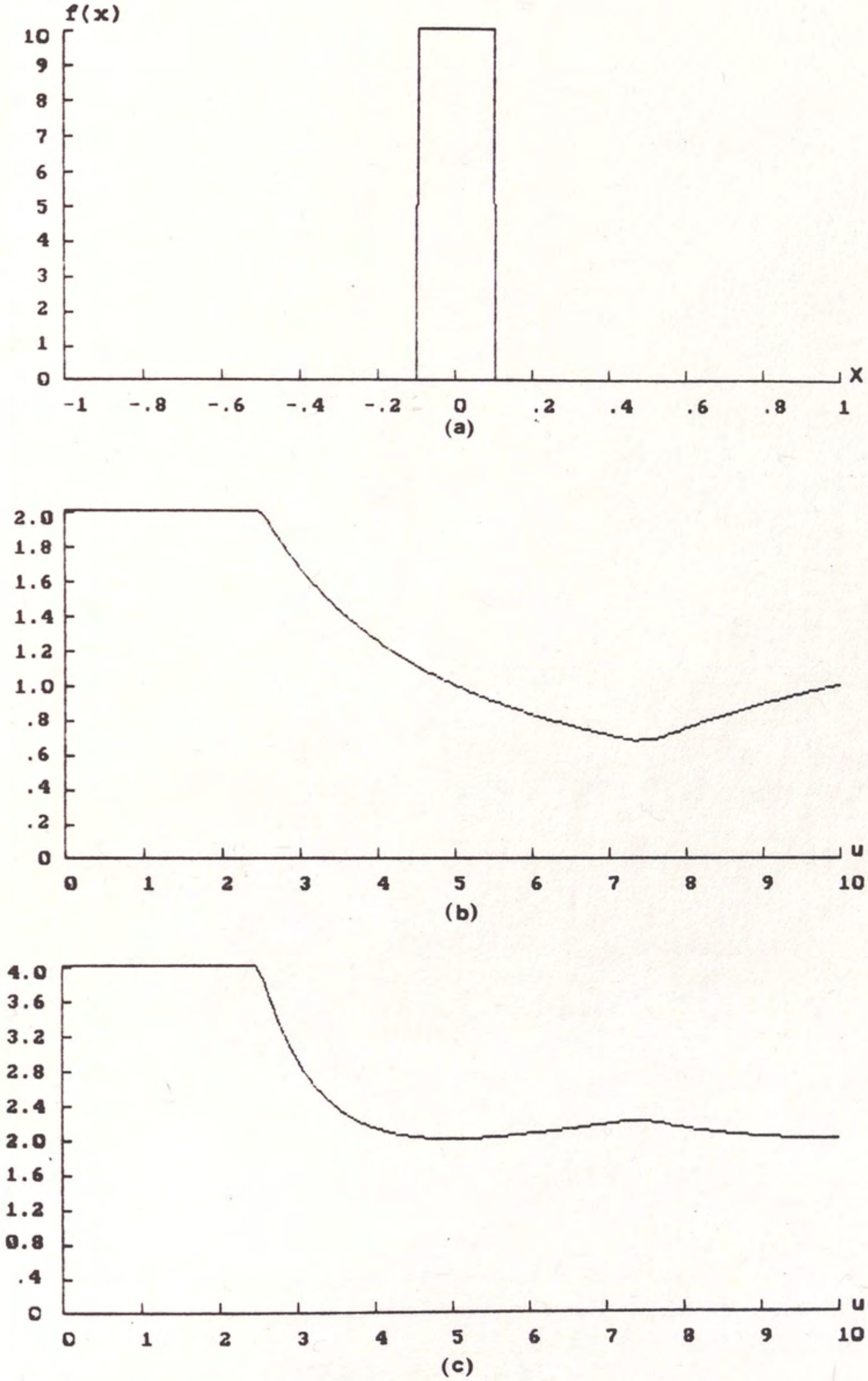



Figure 18: (a) f(x) = 10rect(5x).
(b) Even pulse train transform of f(x).
(c) EPTX[f(x)] minus d-c term, squared, and then d-c term restored.

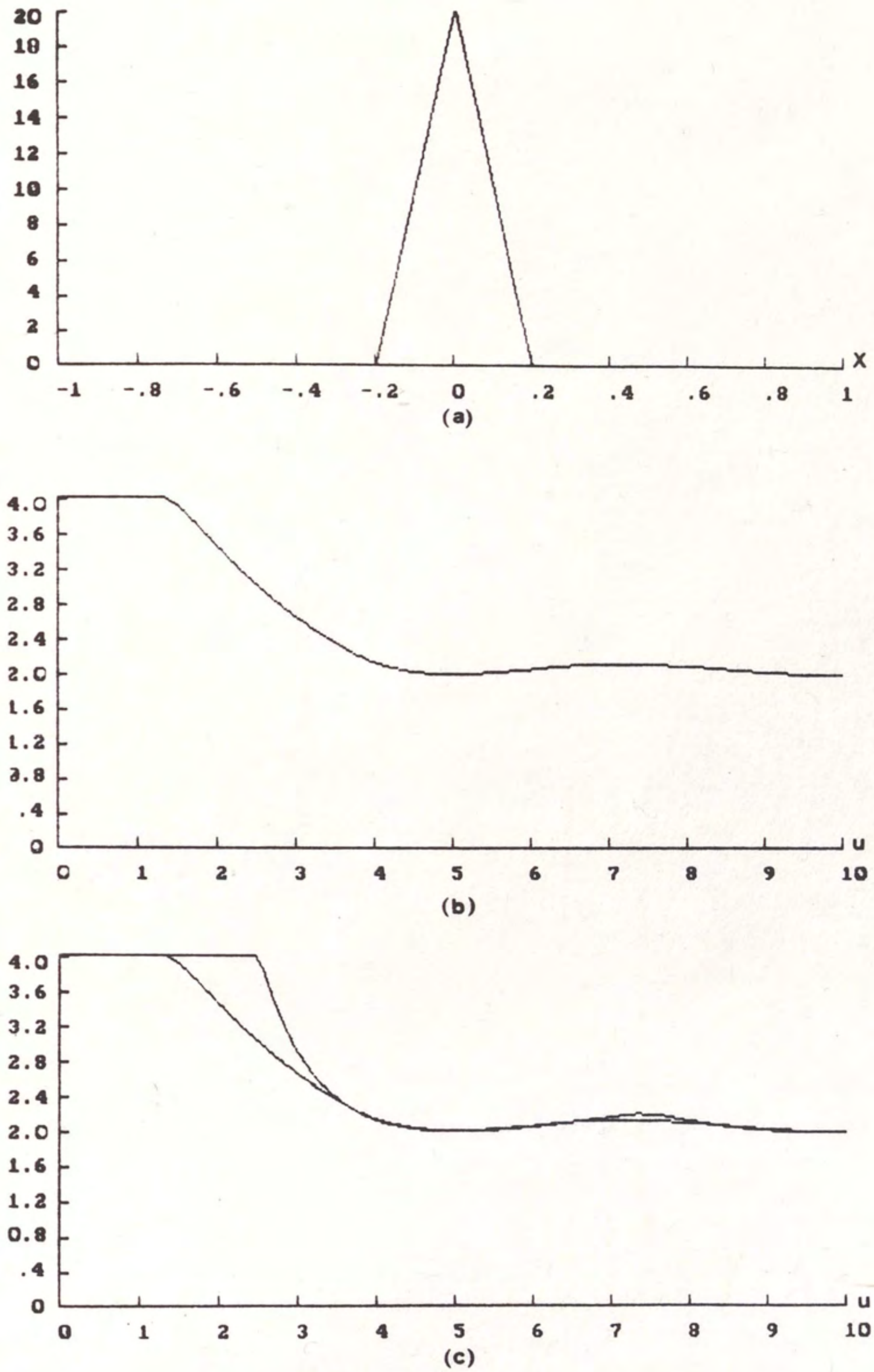



Figure 19: (a) $h(x) = f(x) * f(x) = 20-100|x|$, $|x| < 0.2$.
(b) Even pulse train transform of $h(x)$.
(c) Figure 18(c) superimposed on EPTX[$h(x)$].

After suppressing the d-c term in the raised sine and cosine transforms, the application of the convolution property is exact. The raised cosine transform of $f(x) * g(x)$ is given by:

$$\int_{-\infty}^{\infty}[f(x) * g(x)][B + M\cos(2\pi ux)]dx$$

$$= B\int_{-\infty}^{\infty} f(x) * g(x)dx + M\int_{-\infty}^{\infty}[f(x) * g(x)]\cos(2\pi ux)dx$$

$$= BF(0)G(0) + M\{F_c(u)G_c(u) - F_s(u)G_s(u)\}$$

Similarly the raised sine transform of $f(x) * g(x)$ is given by:

$$BF(0)G(0) + M\{F_c(u)G_s(u) + F_s(u)G_c(u)\}.$$

If it were necessary for example to obtain the magnitude of the Fourier transform of $f(x) * g(x)$ by multiplying the magnitudes of the separate Fourier transforms it would be necessary to perform the following steps:

(1). Measure and store F(0) and G(0) by making the SLM as transparent as possible and multiply by the appropriate scaling factor.

(2). Measure the raised sine and cosine transforms of $f(x)$ and $g(x)$ separately.

(3). Subtract the d-c bias level from the result of step (2) and multiply by the appropriate scaling factor.

(4). Obtain $|F(u)|^2|G(u)|^2$ by adding the squares of the cosine transform and the sine transform obtained in step (3).

## Edge Effects

By assuming that the image luminous intensity is zero beyond the portion seen by the spatial light modulator, the scene function is in effect being multiplied by a rectangular window, $w(x)$. The Fourier transform of the product $f(x)w(x)$ is equal to the convolution of $F(u)$ with $W(u)$ where $W(u)$ is the Fourier transform of the rectangular window. If the left and right edges of the window function are at $\pm k$ then $W(u) = 2k\mathrm{sinc}(2ku)$ (where $\mathrm{sinc}(Q) = \sin(\pi Q)/(\pi Q)$).

Figure 20 shows the effect of rectangular windowing on the Fourier cosine transform of the raised cosine scene function $f(x) = 5 + 5\cos(9\pi x)$. If not for the windowing, the Fourier transform would consist of one impulse at the origin for the d-c component and an even impulse pair at $u = \pm 4.5$. As a result of convolving the Fourier transform of

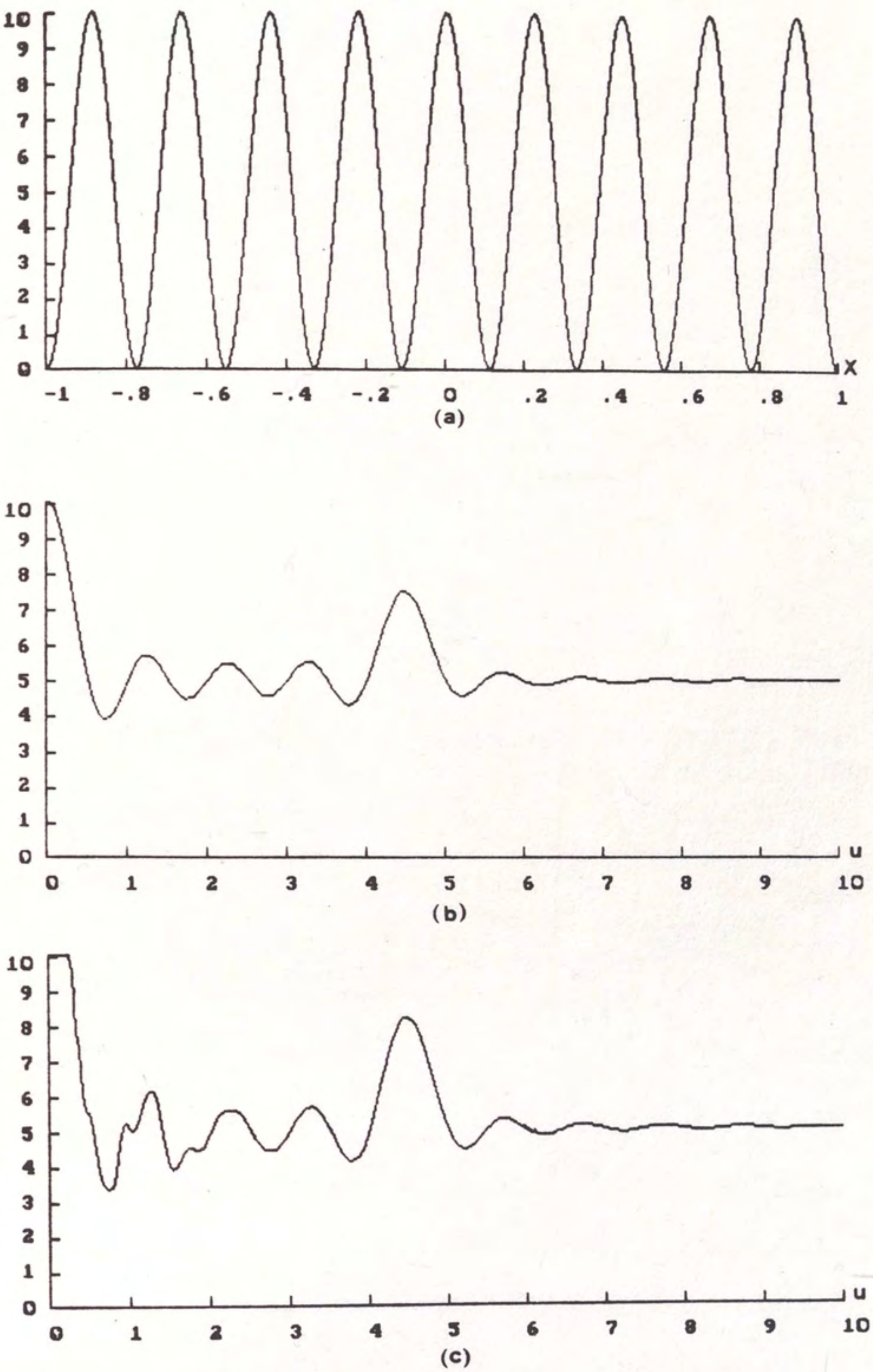



Figure 20: (a) $f(x) = [5 + 5\cos(9\pi x)]\mathrm{rect}(x/2)$.
(b) Raised Fourier cosine transform of $f(x)$.
(c) Even pulse train transform of $f(x)$.

the window, $W(u) = 2\text{sinc}(2u)$, with the three impulses making up the transform of the unwindowed scene, the sinc(2u) function is replicated and shifted to the location of each impulse as seen in Figure 20(b).

Figure 20(c) shows the even pulse train transform of the windowed cosine scene function. The even pulse train transform is qualitatively very similar to the raised Fourier cosine transform Figure 20(b), however it yields a slightly sharper peak at the waveform frequency $u = 4.5$ than the raised Fourier cosine transform.

Because the scene function is even $F_s(u) = 0$ and the odd pulse train transform and raised Fourier sine transform (not shown) are both constant and equal to $F(0)/2 = 5$.

Figure 21 shows the raised Fourier cosine transform (b) and raised Fourier sine transform (c) of the scene function, (a),

$$f(x) = [5 + 5\sin(9.5\pi x)]\text{rect}(x/2).$$

The even part of f(x) is $e(x) = 5\text{rect}(x/2)$ and the odd part of f(x) is $o(x) = 5\sin(9.5\pi x)\text{rect}(x/2)$.

The Fourier cosine transform only sees the even part of f(x) which is a square pulse of height 5 and width 2 centered at the origin. The expected result of the raised

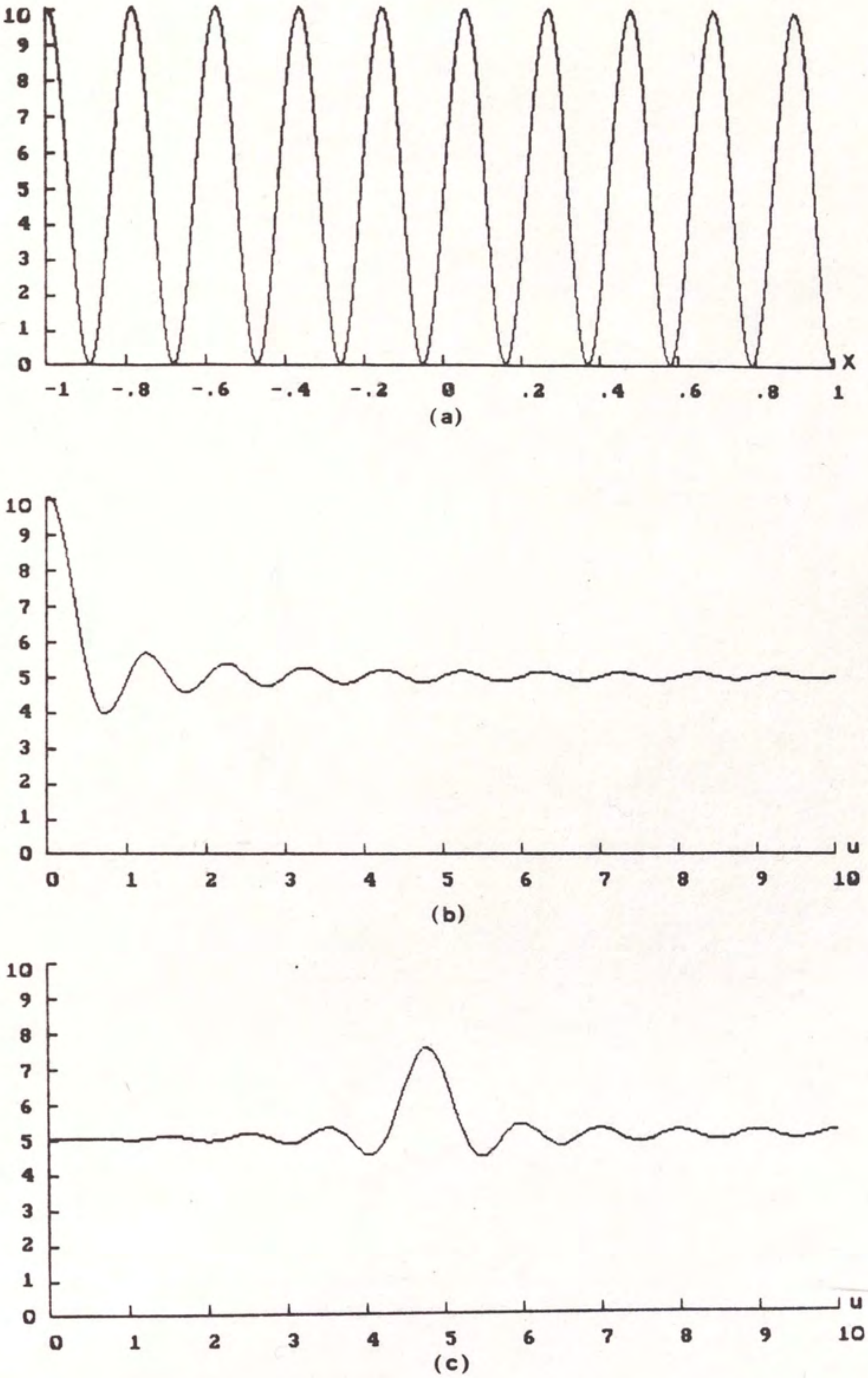



Figure 21: (a) $f(x) = [5 + 5\sin(9.5\pi x)]\mathrm{rect}(x/2)$.
(b) Raised Fourier cosine transform of $f(x)$.
(c) Raised Fourier sine transform of $f(x)$.

Fourier cosine transform is:

$$(F(0)/2) + 0.5F_c(u) = 5 + 5\text{sinc}(2x).$$

The Fourier sine transform only sees the odd part of f(x). As expected, Figure 21(c) shows an impulse at $u = 4.75$ convolved with $W(u) = 2\text{sinc}(2u)$ which in effect yields a shifted version of W(u).

Figure 22 shows the even pulse train transform (b), and odd pulse train transform (c) of the scene function (a),

$$f(x) = [5 + 5\sin(9.5\pi x)]\text{rect}(x/2)$$

for comparison with Figure 21. Because the even pulse train transform only sees the even part of the scene function, Figure 22(b) is identical to the even pulse train transform of the square pulse, 5rect(x/2). The odd pulse train transform shown in Figure 22(c) is similar to the raised Fourier sine transform of Figure 21(c), with a pronounced peak at the sinusoidal waveform frequency of $u = 4.75$.

Figure 23 shows the raised Fourier cosine transform (b), and the even pulse train transform (c), of an even pulse train (a),

$$f(x) = \{5 + 5\text{sgn}[\cos(10\pi x]\}\text{rect}(x/2).$$

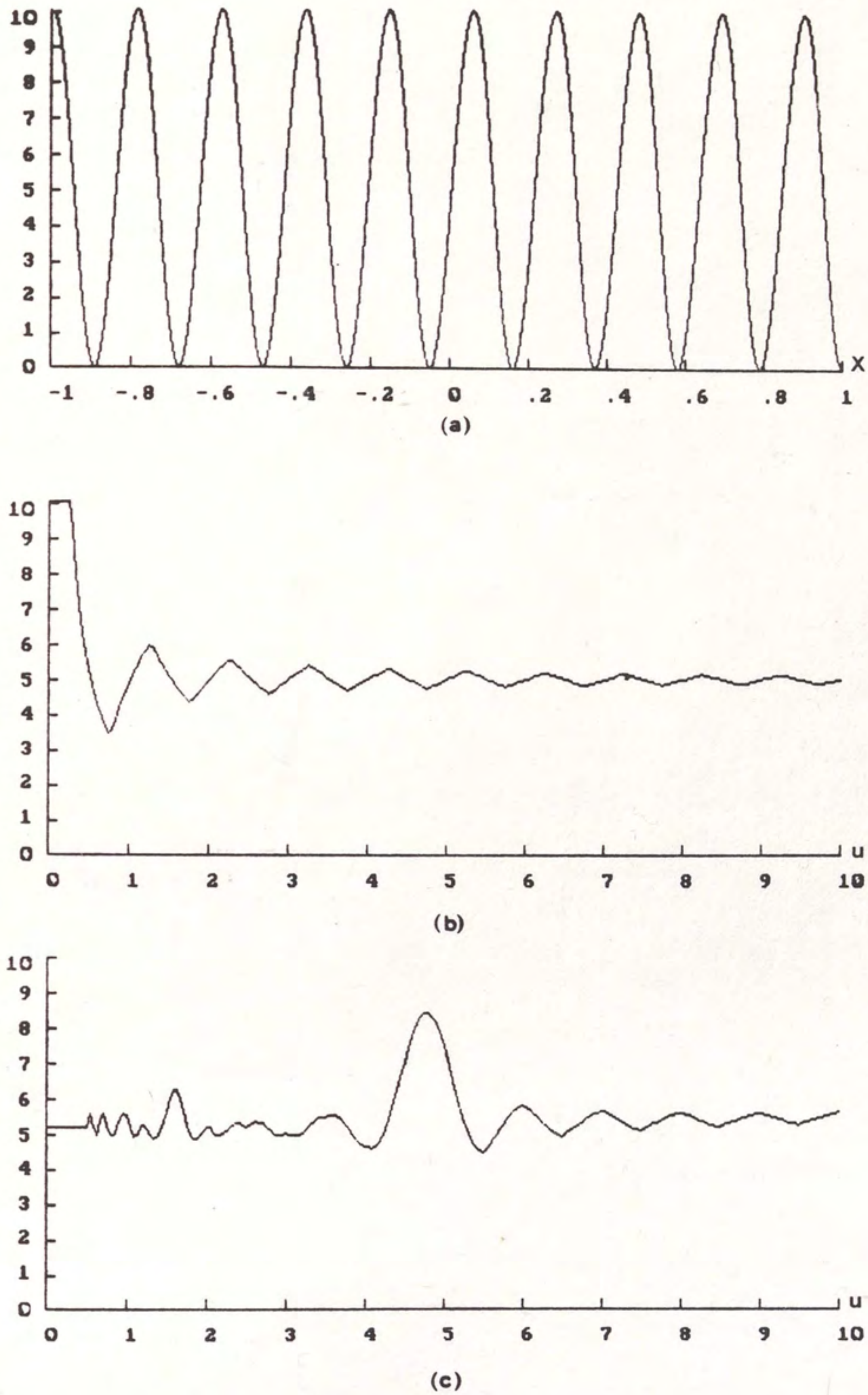



Figure 22: (a) $f(x) = [5 + 5\sin(9.5\pi x)]\mathrm{rect}(x/2)$.
(b) Even pulse train transform of $f(x)$.
(c) Odd pulse train transform of $f(x)$.

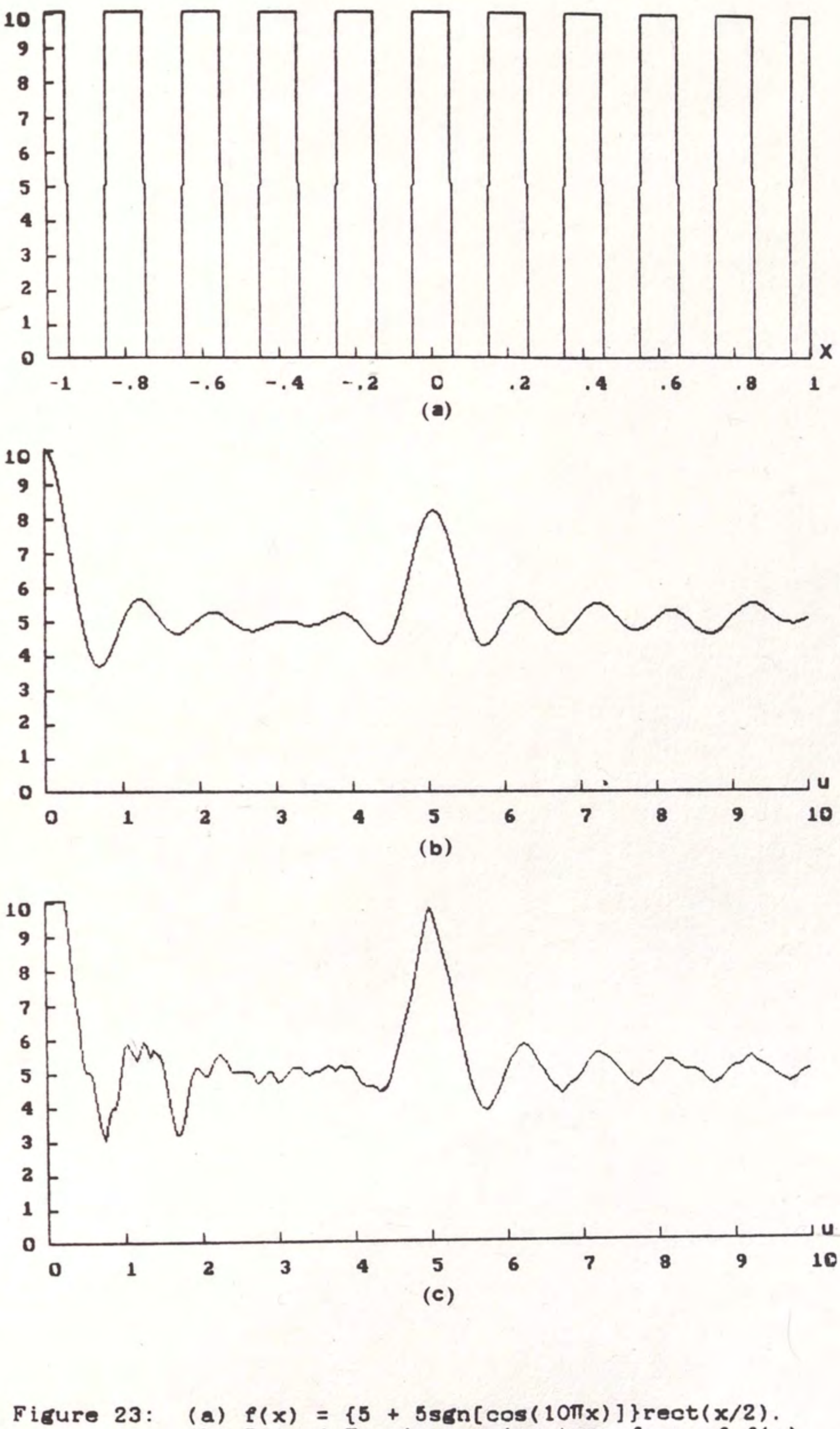



Figure 23: (a) f(x) = {5 + 5sgn[cos(10πx)]}rect(x/2).
(b) Raised Fourier cosine transform of f(x).
(c) Even pulse train transform of f(x).

The fundamental frequency of the even pulse train scene function is $u_o = 5$. Both the raised Fourier cosine transform and the even pulse train transform show pronounced peaks at u = 5 as expected. The even pulse train transform exhibits sharper peaks since the kernel function coincides with the scene function when u = 5.

Since the scene function f(x) is an even function, the Fourier sine transform is zero, i.e., $F_s(u) = 0$, and the raised sine transform and odd pulse train transforms (not shown) are constants equal to F(0)/2 = 5.

Figure 24 shows the raised Fourier cosine transform (b) and raised Fourier sine transform (c) of the scene function (a),

$$f(x) = \{5 + 5\mathrm{sgn}[\sin(10\pi x)]\}\mathrm{rect}(x/2)$$

which is an odd square wave plus d-c bias multiplied by a rectangular window function. The Fourier cosine transform only sees the even part of f(x) which is:

$$e(x) = 5\mathrm{rect}(x/2).$$

Therefore the Fourier cosine transform is:

$$(F(0)/2) + 0.5F_c(u) = 5 + 5\mathrm{sinc}(2x).$$

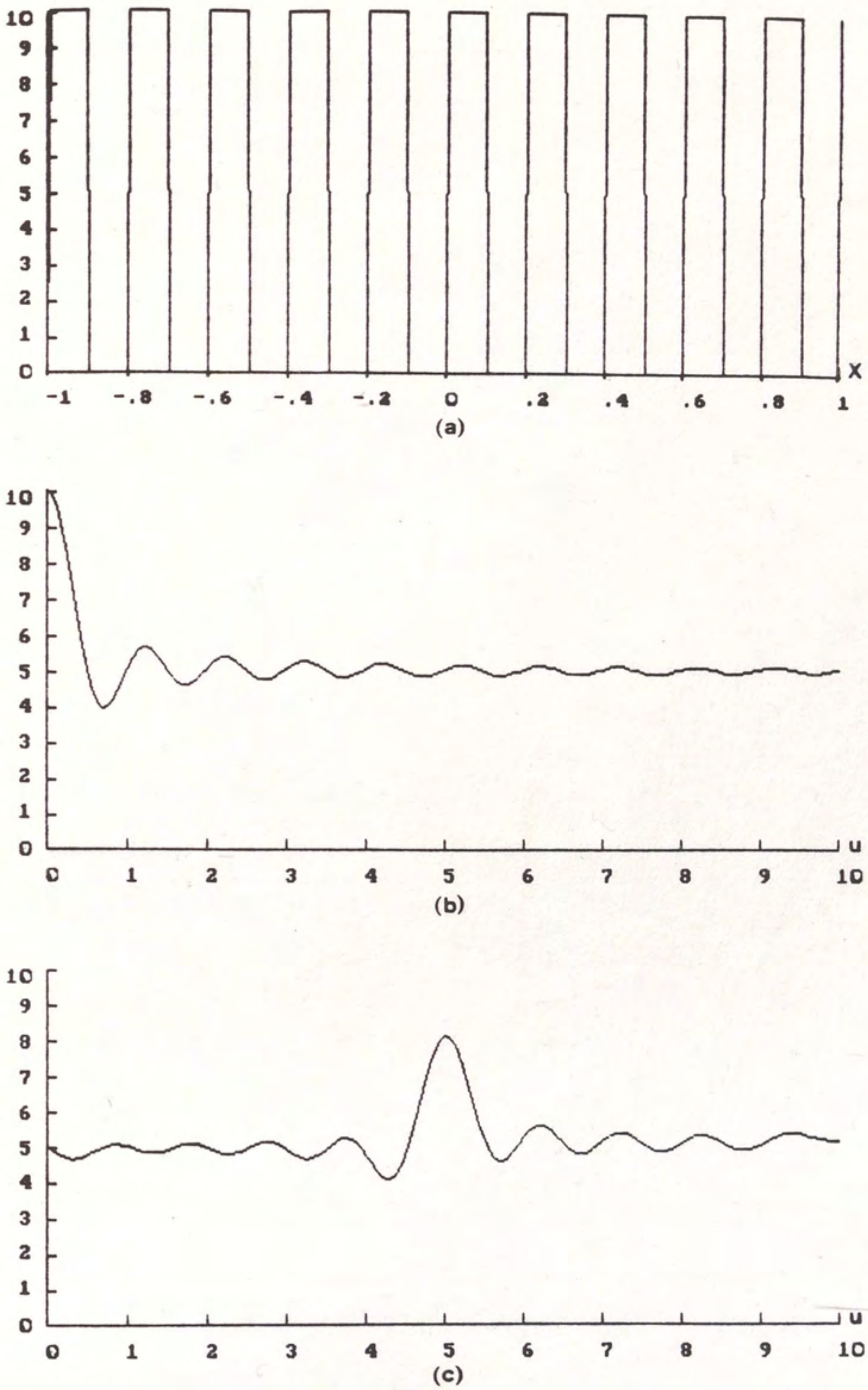



Figure 24: (a) f(x)={5 + 5sgn[sin(10πx)]}rect(x/2).
(b) Raised Fourier cosine transform of f(x).
(c) Raised Fourier sine transform of f(x).

The Fourier sine transform only sees the odd part of f(x) which is:

$$o(x) = 5\mathrm{sgn}[\sin(10\pi x)]\mathrm{rect}(x/2).$$

The Fourier sine transform shows a pronounced peak at the fundamental frequency of the scene function, $u_o = 5$.

Figure 25 shows the even pulse train transform (b), and odd pulse train transform (c), of the scene function (a),

$$f(x) = \{5 + 5\mathrm{sgn}[\sin(10\pi x)]\}\mathrm{rect}(x/2)$$

for comparison with Figure 24.

The even pulse train transform, (b), only responds to the even part of f(x) which is the square pulse,

$$e(x) = 5\mathrm{rect}(x/2).$$

Therefore Figure 25(b) is identical to EPTX[5rect(x/2)].

The odd pulse train transform only responds to the odd part of f(x) which is:

$$o(x) = 5\mathrm{sgn}[\sin(10\pi x)]\mathrm{rect}(x/2).$$

When the odd pulse train transform kernel lines up perfectly with the scene function (which occurs when u = 5) the odd

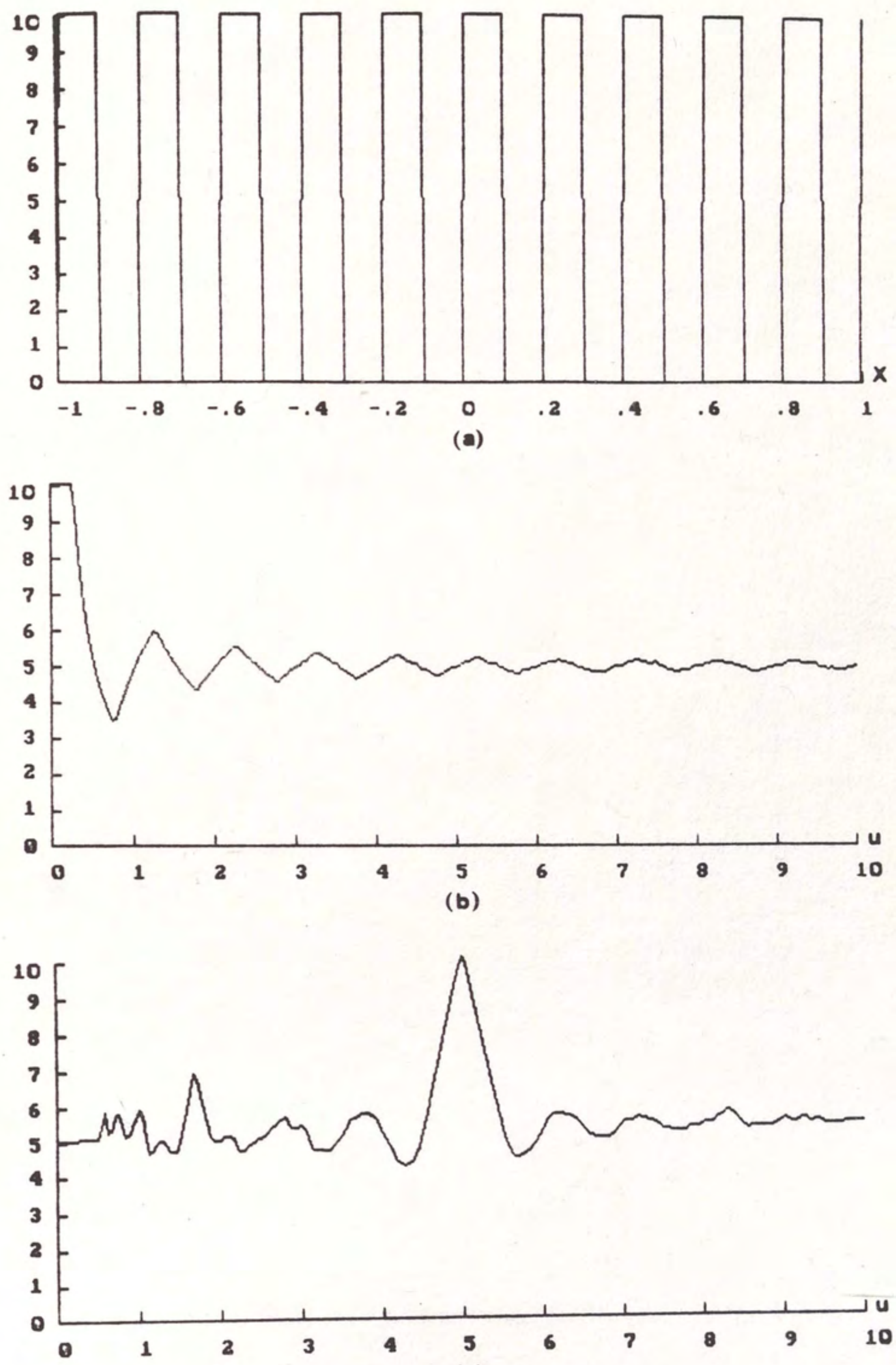



Figure 25: (a) $f(x) = \{5 + 5\text{sgn}[\sin(10\pi x)]\}\text{rect}(x/2)$.
(b) Even pulse train transform of $f(x)$.
(c) Odd pulse train transform of $f(x)$.

pulse train transform is equal to the total area under the scene function, which causes the sharp peak at u = 5.

## Summary Of EPTX and OPTX Properties

The superposition and scaling properties of the even and odd pulse train transforms are directly analogous to the properties of the Fourier transform.

The shifting property of the pulse train transforms is very similar to the shifting property of the Fourier transform in that the transform of the shifted function oscillates within an envelope determined by the transform of the unshifted function. Although no simple closed form expression for the oscillating coefficient has been obtained, it is apparent from the simulation that the fundamental frequency of the oscillation, $u_o$, is the same as that for the Fourier sine and cosine transforms, namely $u = a$ where "a" is the distance of the shift. It is also apparent that the oscillating coefficient, which multiplies the transform of the unshifted function in order to generate the transform of the shifted function, oscillates between +1 and -1.

When the series expressions for the pulse train transforms of f(x) and g(x) are multiplied in order to obtain an approximation to the transform of f(x) * g(x) it is necessary to first suppress the d-c bias level. The error in the approximation results from the cross multiplication of the higher order terms in the series. If a given application required use of the Fourier convolution property it would be necessary to apply a sinusiodal variation to the transmittance of the SLM.

## SEQUENCY ANALYSIS AND THE WALSH TRANSFORM

Another form of optical processing related to spatial frequency analysis is the more general sequency analysis employing the Walsh transform (Beauchamp 1984). Since frequency analysis refers specifically to the resolution of a signal into sinusoidal components, it is necessary to generalize the concept of frequency analysis when discussing the resolution of a function with respect to a non-sinusoidal basis set.

Sequency is defined to be one half the average number of zero crossings per unit time or distance (Harmuth 1972). In the context of the non-negative quantities considered in this study, sequency would represent one half the average number of crossings of the d.c. bias level. Thus it is clear that frequency can be viewed as a special case of sequency.

The Walsh functions are a set of orthogonal rectangular waveforms taking only the values +1 and -1. The Walsh functions are defined to be zero outside of the limited time base T or distance base L. The first six Walsh functions plotted over the spatial interval, $-L/2 \leq x \leq L/2$, are shown below. The first argument in the notation WAL(n,x) indicates the number of zero crossings per unit distance x.

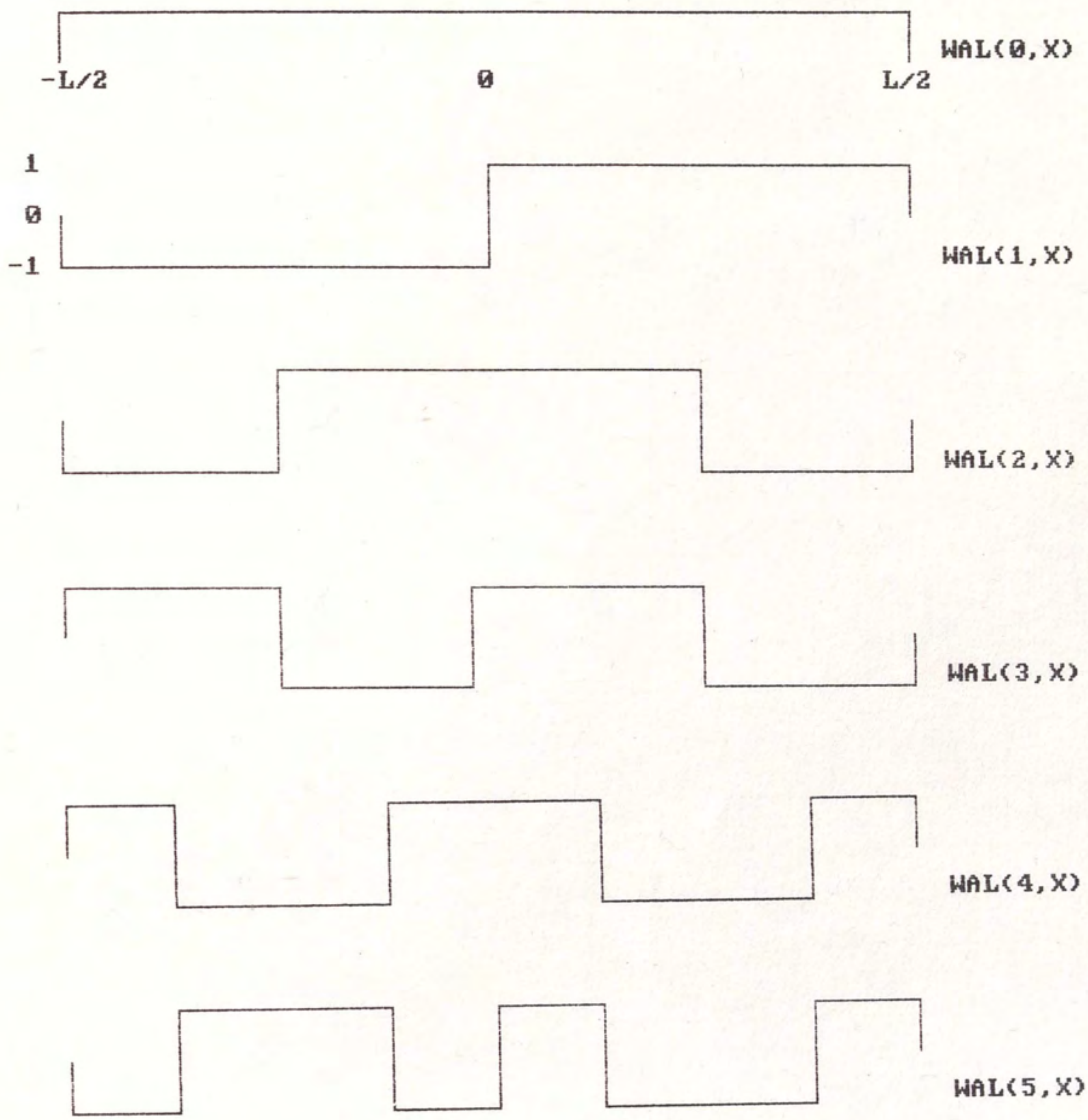



Figure 26: The First Six Walsh Functions

An alternate notation introduced to reflect even and odd symmetry in the Walsh functions is the CAL and SAL symbolism shown below.

$$\mathrm{CAL}(k,x) = \mathrm{WAL}(2k,x)$$

$$\mathrm{SAL}(k,x) = \mathrm{WAL}(2k-1,x)$$

In the above notation k represents the sequency of the CAL or SAL function. The CAL functions are even and the SAL functions are odd.

In a manner analogous to the Fourier Series, a function which is zero outside of the interval $-L/2 \leq x \leq L/2$ can be represented as a sum of CAL and SAL functions as shown below.

$$f(x) = \sum_{k=0}^{\infty} c_k \,\mathrm{CAL}(k,x) + d_k \,\mathrm{SAL}(k,x)$$

Where $d_0 = 0$ and $c_k$ and $d_k$ are given by the expressions below.

$$c_k = \frac{1}{L} \int_{-L/2}^{L/2} f(x)\ \mathrm{CAL}(k,x)\ dx$$

$$d_k = \frac{1}{L} \int_{-L/2}^{L/2} f(x)\ \mathrm{SAL}(k,x)\ dx$$

One practical drawback to the use of Walsh transforms is the awkwardness encountered in the mathematical representation of the Walsh functions. One method for generating the Walsh functions recursively starting with WAL(0,x)=1, is the difference equation given below (Harmuth 1972).

$$\mathrm{WAL}(2r+s,x) = (-1)^{A}\{\mathrm{WAL}(r,2x+0.5) + (-1)^{B}\ \mathrm{WAL}(r,2x-0.5)\}$$

Where $-0.5 \leq x \leq 0.5$ , $r = 0,1,2 \ldots$ , $s = 0$ or $1$ and $A = s + \mathrm{INT}(r/2)$ and $B = r + s$. Here INT(r/2) represents the largest integer smaller or equal to (r/2).

The relationship between the Fourier and Walsh series can be obtained by substituting the Walsh function expansion of f(x) into the formula for the Fourier Series coefficients. This exercise reveals the lack of any particularly compact relationship between the Fourier frequency domain and the Walsh sequency domain. An infinite number of Walsh terms are required to construct a single sinusoidal frequency.

One of the distinctive and important properties of frequency domain analysis is that the magnitudes of the spectral components do not change when the function analyzed is shifted, i.e. frequency content is invariant to shifting in the time or space domain. The Walsh transform does not exhibit the property of invariance of sequency content with respect to shifting.

The lack of a shift theorem for the Walsh transform also leads to another important contrast with Fourier analysis, the lack of a convolution theorem. When working with frequency analysis in the Fourier domain, the inverse transform of the product $F(u)G(u)$ is equal to the convolution of $f(x)$ and $g(x)$. The lack of a similar relationship between the products of Walsh transforms and convolution in the time or spatial domain is a major drawback to sequency analysis. One consequence of the above characteristic is that no simple expression for the correlation of two functions in terms of their Walsh transform coefficients exists.

Despite the inherent limitations of sequency analysis, there have been some applications in which Walsh transforms have been used to advantage. Waveform synthesis of certain discontinuous functions is sometimes more efficiently handled with Walsh functions. Walsh coded masks have been successfully used in a multi-slit spectrometer to improve the signal-to-noise ratio of the measurements (Harwit and Sloane 1979).

# CONCLUSION

This computer simulation demonstrates the feasibility of obtaining the real and imaginary part of the Fourier transform at each spatial frequency of interest from two measurements on an incoherent scene using a spatial light modulator with a biased sinusoidal transmittance function. The complete magnitude and phase of the Fourier transform are readily calculated from the real and imaginary parts.

A reduction in system complexity can be realized with a binary pulse train as the transform kernel at the expense of some accuracy. However, the salient features of the transform are intact to such an extent that the apparatus would be useful in applications where a pulse train transform signature is measured, stored and then tracked.

Other advantages of the pulse train transforms include increased speed of operation resulting from the simple binary nature of the kernel and the ability to look at higher frequency components for a given SLM resolution. All of these analytical operations can be carried out on incoherent scenes, at high resolution, without moving parts and without the necessity of bipolar or complex filter functions.

The pulse train transforms exhibited sharper features than the corresponding raised Fourier sine and cosine transform as a result of the well-defined edges of the kernel function, which in effect sweep across the scene being analyzed as the frequency of the pulse train increases.

These considerations make the SLM based processor an ideal candidate for practical application to robotic vision and inspection systems, where some limited information on Fourier frequency content is desired from an incoherent object scene.

Apart from the fact that Fourier frequency analysis and Walsh sequency analysis both represent the resolution of a signal into orthogonal components, the interpretation of the two domains is quite different. The great differences that exist between the Fourier frequency domain and the Walsh sequency domain are the direct result of the differences in the fundamental properties of the respective basis function sets. In the case of Fourier frequency analysis, we are resolving the complex signal into a set of elementary functions which happen to be the eigenfunctions of the most widely used system model, namely linear shift invariant (LSIV) systems. The trigonometric functions possess a shifting property that gives rise to the useful convolution properties of the Fourier transform. Walsh functions on the other hand do not possess a similar shifting property and are not elementary in that their analytical representation is rather complex. Walsh functions have found certain applications of a data reduction and transmission nature in the digital realm because of their binary nature and some efficient software and hardware realizations. In contrast to the Even and Odd Pulse Train transforms, which are more like binary approximations to the Fourier cosine and sine transforms, the Walsh transform takes the signal into a totally different domain with a totally different interpretation from the frequency domain.

## APPENDIX

The BASIC program PTX plots the scene function, $f(x)$, and then calculates and plots the even and odd pulse train transforms for up to 480 equally spaced frequency values over the range entered by the user.

The first step in evaluating the even and odd pulse train transforms is the calculation and storage of the indefinite integral of the scene function at 480 equally spaced points along the x axis. PTX achieves this by applying the trapezoid rule at each step in the original evaluation of $f(x)$. PTX evaluates $f(x)$ at 480 points and stores these values in the array H(I). At each point PTX also calculates the incremental area $0.5(\Delta x)[H(I) + H(I-1)]$ and stores the cumulative area in the array G(I) so that

$$G(I) = G(I-1) + 0.5(\Delta x)[H(I) + H(I-1)].$$

The even pulse train transform is calculated by summing the area under each part of the (pulse train)x(image function) product.

$$\int_{-\infty}^{\infty} Ke(x;u)\ f(x)\ dx = \sum_{N=LN}^{UN} G\left[\frac{4N+1}{4u}\right] - G\left[\frac{4N-1}{4u}\right]$$

where $\frac{4N + 1}{4u}$ is the leading edge of the Nth pulse and $\frac{4N - 1}{4u}$ is the trailing edge of the Nth pulse of an even pulse train of fundamental frequency u. LN and UN are the lower and upper limits of the summation determined by the boundaries of the scene. In theory LN and UN could be replaced by $-\infty$ and $+\infty$ respectively since the image intensity is assumed to be zero for $x < XL$ and $x > XU$.

The odd pulse train transform is calculated in a similar manner with the exception that the coordinates of the leading and trailing edges of the Nth pulse are $\frac{2N + 1}{2u}$ and $\frac{N}{u}$ respectively. Therefore OPTX[f(x)] is given by:

$$\int_{-\infty}^{\infty} Ko(x;u)f(x)dx = \sum_{N=LN}^{UN} G\left[\frac{2N + 1}{2u}\right] - G\left[\frac{N}{u}\right]$$

PTX plots f(x) and the pulse train transforms on axes scaled according to the subroutines at lines 9000 and 10000 respectively. The BASIC program listing for PTX is given on the following pages.

```
1 'THE PROGRAM PTX PLOTS THE SCENE FUNCTION ,FNSN,
2 'DEFINED BY THE USER IN LINE 190 WITH THE AXES
3 'SCALED AS SPECIFIED IN LINES 9000-9450.
4 '
5 'PTX THEN CALCULATES THE EVEN PULSE TRAIN
6 'TRANSFORM (EPTX) OF THE SCENE FUNCTION
7 'AT 480/XR EQUALLY SPACED FREQUENCIES,
8 'PLOTS THE EVEN PULSE TRAIN TRANSFORM
9 '(WITH AXES SCALED PER LINES 10000-10450),
10 'AND STORES THE DATA IN THE FILE SPECI-
11 'FIED BY THE USER IN LINE 200.
12 '
13 'PTX THEN CALCULATES THE ODD PULSE TRAIN
14 'TRANSFORM (OPTX) OF FNSN AT 480/XR EQUALLY
15 'SPACED FREQUENCIES,PLOTS THE OPTX WITH
16 'THE SAME SCALE AS USED FOR THE EPTX AND
17 'STORES THE DATA IN THE FILE SPECIFIED BY THE
18 'USER IN LINE 201.
19 '
20 'SUMMARY OF USER ADJUSTABLE PARAMETERS
21 '(1) LEFT & RIGHT EDGE OF SCENE (XL,XU)
22 '(2) LOWER & UPPER FREQ RANGE (LF,UF)
23 '(3) RESOLUTION PARAMETER XR:PTX EVALUATES
24 '     EPTX AND OPTX AT 480/XR EQUALLY SPACED
25 '     FREQUENCIES BETWEEN LF AND UF.
26 '
27 'PROCEDURE FOR USE OF PTX
28 '(1) DEFINE SCENE FUNCTION OVER THE RANGE
29 '     (XL,XU) IN LINE 190.
30 '(2) NAME FILE TO RECEIVE OUTPUT DATA IN
31 '     LINES 200-201.  FILES WILL CONTAIN
32 '     480/XR DATA POINTS.
33 '(3) SET PARAMETER XR IN LINE 200
34 '(4) IF SCALED AXES ARE DESIRED SEE LINES
35 '     9000 & 10000.
36 '(5) RUN PTX AND PROGRAM WILL PROMPT FOR
37 '     COORDINATES OF SCENE EDGES (XL,XU)
38 '     AND FREQUENCY LIMITS (LF,UF)
40 '
140 LPRINT CHR$(27)"N"CHR$(3);
141 REM Skips over perf, leaving 3 spaces
150  PI=3.141592654#:P4=4*PI
160 TO$=TIME$
170 SW=0
180 DIM F(480) :DIM G(480):DIM H(480)
190 DEF FNSN(X)=5*(SGN(X+.1)-SGN(X-.1))
200 OPEN "O",1,"RECT-E3.DAT":XR=2!
201 OPEN "O",2,"RECT-O3.DAT"
202 REM FILES WILL CONTAIN 480/XR VALUES
205 TXR=XR:LPRINT:LPRINT
```

```
210 INPUT "ENTER THE LOWER FREQ AND UPPER FREQ";LF,UF
211 IF LF>UF THEN TEMP=LF:LF=UF:UF=TEMP ELSE 215
215 INPUT "ENTER XL AND XU";XL,XU:DSX=ABS(XU-XL)/479
220 CLS
230 YMAX=0:YMIN=0!
240 REM
250 XMIN=LF :XMAX=UF:IF SW=0 THEN XMIN=XL:XMAX=XU ELSE 260
260 IF SW=0 THEN XR=1 ELSE XR=TXR
280 DX=ABS(XMAX-XMIN)/479
290 US=INT(100*XMAX+.5)/100:LS=INT(100*XMIN+.5)/100
295 OUTLIM=ABS(XU): IF OUTLIM<ABS(XL) THEN OUTLIM=ABS(XL)
300 CLS
310 FOR I=1 TO 480
320 IF I=1 THEN 340
330 IF (I MOD XR)<>0 THEN GOTO 510
340 V=XMIN+DX*(I-1)
350 IF SW=0 THEN A=FNSN(V):H(I)=A:GOTO 460  ELSE A=0
360 V4=4*V
362 IF OUTLIM*V4<1 THEN A=G(480):GOTO 460 ELSE A=0
364 IV4=1/V4
370 UN=INT(XU*V+.25):LN=INT(XL*V+.75)
380 A=0
390 FOR N=LN TO UN
400 WT1=(4*N-1)*IV4:WT2=(4*N+1)*IV4
410 IF WT1 < XL THEN W1=XL ELSE W1=WT1
420 IF WT2 >XU THEN W2=XU ELSE W2=WT2
425 IF W2<XL THEN LPRINT V4;WT1;W1;WT2;W2
430 GOSUB 770
440 A=A+AREA
450 NEXT N
460 F(I)=A: IF SW=1 THEN WRITE #1,F(I):GOTO 480  ELSE 470
470 IF I=1 THEN G(I)=0 ELSE G(I)=G(I-1)+.5*DSX*(H(I)+H(I-1))
480 REM G(I) CONTAINS THE CUMULATIVE AREA OF SCENE
490 IF YMIN >F(I) THEN YMIN=F(I)
500 IF YMAX < F(I) THEN YMAX =F(I)
510 NEXT I
520 CLS
525 LPRINT:LPRINT
530 IF SW=0 THEN GOSUB 9000
540 IF SW=1 THEN GOSUB 10000
550 SW=SW+1:IF SW=2 THEN CLOSE #1 ELSE 220
560 PRINT "YMAX=";YMAX,"YMIN=";YMIN
570 GOTO 1000
760 END
770 I1=1+(W1-XL)/DSX:J1=INT(I1):P1=I1-J1
772 F1=H(J1):G1=G(J1+1)
780 I2=1+(W2-XL)/DSX:J2=INT(I2):DW=(W2-W1)
782 P2=I2-J2:F2=H(J2):G2=G(J2)
785 AA=F1*DSX*(1-P1)+(G2-G1)+F2*DSX*P2
790 IF DW<DSX THEN AREA=DW*F1 ELSE AREA=AA
800 RETURN
```

```
810 END
1000 REM LINES 1000-1320 COMPUTE THE OPTX OF FNSN
1010 OUTLIM=ABS(XU): IF OUTLIM<ABS(XL) THEN OUTLIM=ABS(XL)
1020 CLS
1030 IF XL>0 THEN IO=1 ELSE IO=1+INT(1-XL/DSX)
1040 FOR I=1 TO 480
1050 IF I=1 THEN 1070
1060 IF (I MOD XR)<>0 THEN GOTO 1240
1070 V=XMIN+DX*(I-1)
1080 A=0
1090 T=INT(XL*2*V):U=INT(XU*2*V)
1095 T1=(ABS(T))MOD 2:U1=(ABS(U))MOD 2
1097 MTEST=(ABS(XL)+ABS(XU))*2*V
1100 IF MTEST<1 THEN A=G(480)-G(IO):GOTO 1210 ELSE A=0
1110 IV2=1/(2*V)
1120 UN=U-U1:LN=T+T1
1130 A=0
1140 FOR N=LN TO UN STEP 2
1150 WT1=N*IV2:WT2=WT1+IV2
1160 IF WT1 < XL THEN W1=XL ELSE W1=WT1
1170 IF WT2 >XU THEN W2=XU ELSE W2=WT2
1180 GOSUB 1290
1190 A=A+AREA
1200 NEXT N
1210 F(I)=A:  WRITE #2,F(I)
1240 NEXT I
1250 CLS
1260 CLOSE #2
1265 GOSUB 10000
1280 END
1290 I1=1+(W1-XL)/DSX:J1=INT(I1):P1=I1-J1
1295 F1=H(J1):G1=G(J1)
1300 I2=1+(W2-XL)/DSX:J2=INT(I2):DW=(W2-W1):P2=I2-J2
1305 F2=H(J2):G2=G(J2)
1307 AA=F1*DSX*(1-P1)+(G2-G1)+F2*DSX*P2
1310 IF DW<DSX THEN AREA=DW*F1 ELSE AREA=AA
1320 RETURN
1330 END
9000 'THIS SUBROUTINE SCALES THE AXES
9001 'FOR THE SCENE FUNCTION FNSN.
9002 'LINE 9240 SPECIFIES THE LOWER LIMIT
9003 'OF THE X AXIS AND LINES 9250-9340
9004 'LABEL THE X AXIS AT 10 EQUALLY
9005 'SPACED POINTS
9006 'LINE 9360 SPECIFIES THE LOWER LIMIT
9007 'OF THE Y AXIS AND LINES 9370-9450
9008 'LABEL THE Y AXIS AT 10 EQUALLY
9009 'SPACED POINTS.
9020 REM FINAL SCALING FOR X=-1 TO +1 Y=0-10
9090 CLS
9230 FOR I=130 TO 610 STEP 48:LINE (I,175)-(I,180):NEXT I
```

```
9240 SYMBOL (130,190),"-1",1,1,3:REM X-AXIS SCALE
9250 SYMBOL (170,190),"-.8",1,1,3
9260 SYMBOL (218,190),"-.6",1,1,3
9270 SYMBOL (266,190),"-.4",1,1,3
9280 SYMBOL (314,190),"-.2",1,1,3
9290 SYMBOL (370,190),CHR$(48),1,1,3
9300 SYMBOL (418,190),".2",1,1,3
9310 SYMBOL (466,190),".4",1,1,3
9320 SYMBOL (514,190),".6",1,1,3
9330 SYMBOL (562,190),".8",1,1,3
9340 SYMBOL (610,190),"1",1,1,3
9360 SYMBOL (114,177),CHR$(48),1,1,3:REM Y-AXIS SCALE
9370 SYMBOL (114,158),CHR$(49),1,1,3
9380 SYMBOL (114,140),CHR$(50),1,1,3
9390 SYMBOL (114,123),CHR$(51),1,1,3
9400 SYMBOL (114,105),CHR$(52),1,1,3
9410 SYMBOL (114,88),CHR$(53),1,1,3
9420 SYMBOL (114,70),CHR$(54),1,1,3
9430 SYMBOL (114,52),CHR$(55),1,1,3
9440 SYMBOL (114,34),CHR$(56),1,1,3
9450 SYMBOL (114,16),CHR$(57),1,1,3:SYMBOL(106,1),"10",1,1,3
9460 FOR I=1 TO 180 STEP 18:LINE (130,I)-(135,I):NEXT I
9470 LINE (130,1)-(130,180):LINE(130,180)-(610,180)
9480 RY=ABS(YMAX-YMIN):DY=180/RY:Y1=(YMAX-F(1))*DY
9490 PSET (130,Y1):P=130:Q=Y1
9500 FOR I=2 TO 480
9510 IF (I MOD XR)<>0 THEN GOTO 9550
9520 Y=(YMAX-F(I))*DY
9530 PSET(I+130,Y)
9540 LINE (P,Q)-(I+130,Y):P=I+130:Q=Y
9550 NEXT I
9560 MAX=INT(100*YMAX+.5)/100:MIN=INT(100*YMIN+.5)/100
9580 LOCATE 10,40,0
9590 RESTORE
9600 DEF SEG
9610 FOR ADRS=0 TO 4
9620 READ CODE
9630 POKE ADRS,CODE
9640 NEXT ADRS
9650 CALL 0
9660 DATA &H55,&HCD,&H05,&H5D,&HCB
9690 RETURN
9700 END
10000 REM   SCALES X AND Y AXIS FOR THE TRANSFORM.
10010 REM
10020 REM FINAL SCALING FOR X=1-10,Y=1-10
10030 LPRINT:LPRINT:LPRINT:LPRINT
10220 CLS
10230 FOR I=130 TO 610 STEP 48:LINE (I,175)-(I,180):NEXT I
10240 SYMBOL (130,190),CHR$(48),1,1,3
10250 SYMBOL (178.190).CHR$(49).1.1.3
```

```
10260 SYMBOL (226,190),CHR$(50),1,1,3
10270 SYMBOL (274,190),CHR$(51),1,1,3
10280 SYMBOL (322,190),CHR$(52),1,1,3
10290 SYMBOL (370,190),CHR$(53),1,1,3
10300 SYMBOL (418,190),CHR$(54),1,1,3
10310 SYMBOL (466,190),CHR$(55),1,1,3
10320 SYMBOL (514,190),CHR$(56),1,1,3
10330 SYMBOL (562,190),CHR$(57),1,1,3
10340 SYMBOL (610,190),"10",1,1,3
10360 SYMBOL (114,177),CHR$(48),1,1,3:REM Y-AXIS SCALE
10370 SYMBOL (106,158),".2",1,1,3
10380 SYMBOL (106,140),".4",1,1,3
10390 SYMBOL (106,123),".6",1,1,3
10400 SYMBOL (106,105),".8",1,1,3
10410 SYMBOL (114,88),CHR$(49),1,1,3
10420 SYMBOL (100,70),"1.2",1,1,3
10430 SYMBOL (100,52),"1.4",1,1,3
10440 SYMBOL (100,34),"1.6",1,1,3
10450 SYMBOL (100,16),"1.8",1,1,3:SYMBOL(114,1),"2",1,1,3
10460 FOR I=1 TO 180 STEP 18:LINE (130,I)-(135,I):NEXT I
10470 LINE (130,1)-(130,180):LINE(130,180)-(610,180)
10480 RY=ABS(YMAX-YMIN):DY=180/RY:Y1=(YMAX-F(1))*DY
10490 PSET (130,Y1):P=130:Q=Y1
10500 FOR I=2 TO 480
10510 IF (I MOD XR)<>0 THEN GOTO 10550
10520 Y=(YMAX-F(I))*DY
10530 PSET(I+130,Y)
10540 LINE (P,Q)-(I+130,Y):P=I+130:Q=Y
10550 NEXT I
10560 MAX=INT(100*YMAX+.5)/100:MIN=INT(100*YMIN+.5)/100
10580 LOCATE 10,40,0
10590 RESTORE
10600 DEF SEG
10610 FOR ADRS=0 TO 4
10620 READ CODE
10630 POKE ADRS,CODE
10640 NEXT ADRS
10650 CALL 0
10660 DATA &H55,&HCD,&H05,&H5D,&HCB
10665 RETURN
10700 END
```

The BASIC program, TRIGTX, plots the scene function, f(x), and calculates and plots the raised cosine and sine transforms for up to 480 equally spaced frequency values over the range entered by the user.

TRIGTX uses a Simpson's rule numerical integration to calculate the area under the product $f(x)[.5 + .5\cos(2\pi ux)]$ at each of 480/XR frequencies to obtain the raised cosine transform. XR is a resolution parameter which allows the user to set the number of frequencies to be evaluated. The output is stored in the file named by the user in line 200. The raised sine transform is evaluated in a similar manner by finding the area under the product $f(x)[.5 +.5\sin(2\pi ux)]$ and the output is stored in the file named in line 205.

TRIGTX plots the raised cosine and sine transforms on axes scaled according to the subroutine following line 10000.

TRIGTX can be used to plot the result of any integral transform with real valued kernel function by merely replacing the factors $[.5 + .5\cos(2\pi ux)]$ and $[.5 + .5\sin(2\pi ux)]$ in lines 822 and 824 with the new kernel functions of interest.

```
1 'THE PROGRAM TRIGTX PLOTS THE SCENE FUNCTION
2 'DEFINED BY THE USER IN LINE 190.
4 '
5 'TRIGTX THEN CALCULATES THE RAISED COSINE
6 'TRANSFORM (RCTX) OF THE SCENE FUNCTION
7 'AT 480/XR EQUALLY SPACED FREQUENCIES,
8 'PLOTS THE RAISED COSINE TRANSFORM
10 'AND STORES THE DATA IN THE FILE SPECI-
11 'FIED BY THE USER IN LINE 200.
12 '
13 'TRIGTX THEN CALCULATES THE RAISED SINE
14 'TRANSFORM (RSTX) OF FNSN AT 480/XR EQUALLY
15 'SPACED FREQUENCIES,PLOTS THE RSTX WITH
16 'THE SAME SCALE AS USED FOR THE RCTX AND
17 'STORES THE DATA IN THE FILE SPECIFIED BY THE
18 'USER IN LINE 205.
19 '
20 'SUMMARY OF USER ADJUSTABLE PARAMETERS
21 '(1) LEFT & RIGHT EDGE OF SCENE (XL,XU)
22 '(2) LOWER & UPPER FREQ RANGE (LF,UF)
23 '(3) RESOLUTION PARAMETER XR:TRIGTX EVAL-
24 '    UATES RCTX & RSTX AT 480/XR EQUALLY SPACED
25 '    FREQUENCIES BETWEEN LF AND UF.
26 '
27 'PROCEDURE FOR USE OF TRIGTX:
28 '(1) DEFINE SCENE FUNCTION OVER THE RANGE
29 '    (XL,XU) IN LINE 190.
30 '(2) NAME FILE TO RECEIVE OUTPUT DATA IN
31 '    LINES 200,205.  FILES WILL CONTAIN
32 '    480/XR DATA POINTS.
33 '(3) SET PARAMETER XR IN LINE 200
36 '(4) RUN TRIGTX AND PROGRAM WILL PROMPT FOR
37 '    COORDINATES OF SCENE EDGES (XL,XU)
38 '    AND FREQUENCY LIMITS (LF,UF)
125 '
140 LPRINT CHR$(27)"N"CHR$(3);
145 REM SKIPS OVER PERF, LEAVING 3 SPACES
150  PI=3.141592654#:PI2=2*PI
160 TO$=TIME$:SW=0
180 DIM F(480) :DIM G(480)
190 DEF FNSN(X)=10*(SGN(X-.8)-SGN(X-.9))
200 OPEN "O",1,"CTX.DAT":XR=2!
205 OPEN "O",2,"STX.DAT"
206 REM FILES WILL CONTAIN 480/XR VALUES
210 INPUT "ENTER THE LOWER FREQ AND UPPER FREQ";LF,UF
211 IF LF>UF THEN TEMP=LF:LF=UF:UF=TEMP ELSE 215
215 INPUT "ENTER XL AND XU";XL,XU
220 CLS
230 YMAX=0!:YMIN=0!
250 IF SW=0 THEN XMIN=XL:XMAX=XU ELSE XMIN=LF:XMAX=UF
280 DX=ABS(XMAX-XMIN)/479
```

```
290 US=INT(100*XMAX+.5)/100:LS=INT(100*XMIN+.5)/100
310 FOR I=1 TO 480
320 IF I=1 THEN 340
330 IF (I MOD XR)<>0 THEN GOTO 510
340 V=XMIN+DX*(I-1)
350 IF SW=0 THEN F(I)=FNSN(V):GOTO 490 ELSE GOSUB 800
360 F(I)=AREA
460 IF SW=1 THEN WRITE #1,F(I) ELSE WRITE #2,F(I)
490 IF YMIN >F(I) THEN YMIN=F(I)
500 IF YMAX < F(I) THEN YMAX =F(I)
510 NEXT I
520 CLS
530 LPRINT:LPRINT
540 IF SW=0 THEN GOSUB 9000
550 IF SW=1 THEN GOSUB 10000
560 IF SW=2 THEN GOSUB 10000
750 LPRINT: SW=SW+1: IF SW=2 THEN CLOSE #1
752 IF SW=3 THEN CLOSE #2
755 IF SW<3 THEN 220
760 END
800 KK=159
810 DELTA=(XU-XL)/(KK-1):X=XL
820 FOR J2=1 TO KK
822 IF SW=1 THEN G(J2)=FNSN(X)*(.5+.5*COS(PI2*V*X))
824 IF SW=2 THEN G(J2)=FNSN(X)*(.5+.5*SIN(PI2*V*X))
826 X=X+DELTA:NEXT J2
830 ODD=0
840 FOR J2=3 TO (KK-2) STEP 2:ODD=ODD+2*G(J2): NEXT J2
850 EVEN=0
860 FOR J2=2 TO KK-1 STEP 2:EVEN=EVEN+4*G(J2):NEXT J2
870 AREA=(DELTA/3)*(G(1)+EVEN+ODD+G(KK))
880 RETURN
890 END
9000 'THIS SUBROUTINE SCALES THE AXES
9001 'FOR THE SCENE FUNCTION FNSN.
9002 'LINE 9240 SPECIFIES THE LOWER LIMIT
9003 'OF THE X AXIS AND LINES 9250-9340
9004 'LABEL THE X AXIS AT 10 EQUALLY
9005 'SPACED POINTS
9006 'LINE 9360 SPECIFIES THE LOWER LIMIT
9007 'OF THE Y AXIS AND LINES 9370-9450
9008 'LABEL THE Y AXIS AT 10 EQUALLY
9009 'SPACED POINTS.
9020 REM FINAL SCALING FOR X=-1 TO +1 Y=0-10
9090 CLS
```

```
9230 FOR I=130 TO 610 STEP 48:LINE (I,175)-(I,180):NEXT I
9240 SYMBOL (130,190),"-1",1,1,3:REM X-AXIS SCALE
9250 SYMBOL (170,190),"-.8",1,1,3
9260 SYMBOL (218,190),"-.6",1,1,3
9270 SYMBOL (266,190),"-.4",1,1,3
9280 SYMBOL (314,190),"-.2",1,1,3
9290 SYMBOL (370,190),CHR$(48),1,1,3
9300 SYMBOL (418,190),".2",1,1,3
9310 SYMBOL (466,190),".4",1,1,3
9320 SYMBOL (514,190),".6",1,1,3
9330 SYMBOL (562,190),".8",1,1,3
9340 SYMBOL (610,190),"1",1,1,3
9360 SYMBOL (114,177),CHR$(48),1,1,3:REM Y-AXIS SCALE
9370 SYMBOL (114,158),CHR$(49),1,1,3
9380 SYMBOL (114,140),CHR$(50),1,1,3
9390 SYMBOL (114,123),CHR$(51),1,1,3
9400 SYMBOL (114,105),CHR$(52),1,1,3
9410 SYMBOL (114,88),CHR$(53),1,1,3
9420 SYMBOL (114,70),CHR$(54),1,1,3
9430 SYMBOL (114,52),CHR$(55),1,1,3
9440 SYMBOL (114,34),CHR$(56),1,1,3
9450 SYMBOL (114,16),CHR$(57),1,1,3:SYMBOL(106,1),"10",1,1,3
9460 FOR I=1 TO 180 STEP 18:LINE (130,I)-(135,I):NEXT I
9470 LINE (130,1)-(130,180):LINE(130,180)-(610,180)
9480 RY=ABS(YMAX-YMIN):DY=180/RY:Y1=(YMAX-F(1))*DY
9490 PSET (130,Y1):P=130:Q=Y1
9500 FOR I=2 TO 480
9510 IF (I MOD XR)<>0 THEN GOTO 9550
9520 Y=(YMAX-F(I))*DY
9530 PSET(I+130,Y)
9540 LINE (P,Q)-(I+130,Y):P=I+130:Q=Y
9550 NEXT I
9560 MAX=INT(100*YMAX+.5)/100:MIN=INT(100*YMIN+.5)/100
9580 LOCATE 10,40,0
9590 RESTORE
9600 DEF SEG
9610 FOR ADRS=0 TO 4
9620 READ CODE
9630 POKE ADRS,CODE
9640 NEXT ADRS
9650 CALL 0
9660 DATA &H55,&HCD,&H05,&H5D,&HCB
9690 RETURN
9700 END
10000 REM   SCALES X AND Y AXIS FOR THE TRANSFORM.
10010 REM
10020 REM FINAL SCALING FOR X=1-10,Y=1-10
10030 LPRINT:LPRINT:LPRINT:LPRINT
10220 CLS
```

```
10230 FOR I=130 TO 610 STEP 48:LINE (I,175)-(I,180):NEXT I
10240 SYMBOL (130,190),CHR$(48),1,1,3
10250 SYMBOL (178,190),CHR$(49),1,1,3
10260 SYMBOL (226,190),CHR$(50),1,1,3
10270 SYMBOL (274,190),CHR$(51),1,1,3
10280 SYMBOL (322,190),CHR$(52),1,1,3
10290 SYMBOL (370,190),CHR$(53),1,1,3
10300 SYMBOL (418,190),CHR$(54),1,1,3
10310 SYMBOL (466,190),CHR$(55),1,1,3
10320 SYMBOL (514,190),CHR$(56),1,1,3
10330 SYMBOL (562,190),CHR$(57),1,1,3
10340 SYMBOL (610,190),"10",1,1,3
10360 SYMBOL (114,177),CHR$(48),1,1,3:REM Y-AXIS SCALE
10370 SYMBOL (106,158),".2",1,1,3
10380 SYMBOL (106,140),".4",1,1,3
10390 SYMBOL (106,123),".6",1,1,3
10400 SYMBOL (106,105),".8",1,1,3
10410 SYMBOL (114,88),CHR$(49),1,1,3
10420 SYMBOL (100,70),"1.2",1,1,3
10430 SYMBOL (100,52),"1.4",1,1,3
10440 SYMBOL (100,34),"1.6",1,1,3
10450 SYMBOL (100,16),"1.8",1,1,3:SYMBOL(114,1),"2",1,1,3
10460 FOR I=1 TO 180 STEP 18:LINE (130,I)-(135,I):NEXT I
10470 LINE (130,1)-(130,180):LINE(130,180)-(610,180)
10480 RY=ABS(YMAX-YMIN):DY=180/RY:Y1=(YMAX-F(1))*DY
10490 PSET (130,Y1):P=130:Q=Y1
10500 FOR I=2 TO 480
10510 IF (I MOD XR)<>0 THEN GOTO 10550
10520 Y=(YMAX-F(I))*DY
10530 PSET(I+130,Y)
10540 LINE (P,Q)-(I+130,Y):P=I+130:Q=Y
10550 NEXT I
10560 MAX=INT(100*YMAX+.5)/100:MIN=INT(100*YMIN+.5)/100
10580 LOCATE 10,40,0
10590 RESTORE
10600 DEF SEG
10610 FOR ADRS=0 TO 4
10620 READ CODE
10630 POKE ADRS,CODE
10640 NEXT ADRS
10650 CALL 0
10660 DATA &H55,&HCD,&H05,&H5D,&HCB
10665 RETURN
10700 END
```

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