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SPECTRAL ANALYSIS OF THE TRUNCATED HILBERT TRANSFORM WITH OVERLAP*

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Abstract. We study a restriction of the Hilbert transform as an operator H_T from $L^2(a_2, a_4)$ to $L^2(a_1, a_3)$ for real numbers $a_1 < a_2 < a_3 < a_4$. The operator H_T arises in tomographic reconstruction from limited data, more precisely in the method of differentiated back-projection. There, the reconstruction requires recovering a family of one-dimensional functions f supported on compact intervals $[a_2, a_4]$ from its Hilbert transform measured on intervals $[a_1, a_3]$ that might only overlap, but not cover $[a_2, a_4]$. We show that the inversion of H_T is ill-posed, which is why we investigate the spectral properties of H_T. We relate the operator H_T to a self-adjoint *two-interval Sturm–Liouville problem*, for which we prove that the spectrum is discrete. The Sturm–Liouville operator is found to commute with H_T, which then implies that the spectrum of $H_T^* H_T$ is discrete. Furthermore, we express the singular value decomposition of H_T in terms of the solutions to the Sturm–Liouville problem. The singular values of H_T accumulate at both 0 and 1, implying that H_T is not a compact operator. We conclude by illustrating the properties obtained for H_T numerically.

Key words. Hilbert transform, spectrum, Sturm–Liouville, limited data, tomography, inverse problems

AMS subject classifications. 45Q05, 47A10, 47A75, 34B24, 34B27

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1. Introduction.

In tomographic imaging, which is widely used for medical applications, a two-dimensional (2D) or three-dimensional (3D) object is illuminated by a penetrating beam (usually X-rays) from multiple directions, and the projections of the object are recorded by a detector. Then one seeks to reconstruct the full 2D or 3D structure from this collection of projections. When the beams are sufficiently wide to fully embrace the object and when the beams from a sufficiently dense set of directions around the object can be used, this problem and its solution are well understood [16]. When the data are more limited, e.g., when only a reduced range of directions can be used or only a part of the object can be illuminated, the image reconstruction problem becomes much more challenging.

Reconstruction from limited data requires the identification of specific subsets of line integrals that allow for an exact and stable reconstruction. One class of such configurations that have already been identified, relies on the reduction of the 2D and 3D reconstruction problem to a family of one-dimensioal (1D) problems. The Radon transform can be related to the 1D Hilbert transform along certain lines by differentiation and back-projection of the Radon transform data (*differentiated back-projection* or DBP). Inversion of the Hilbert transform along a family of lines covering a subregion of the object (*region of interest* or ROI) then allows for the reconstruction within the ROI.

This method goes back to a result by Gelfand and Graev [6]. Its application to tomography was formulated by Finch [4] and was later made explicit for two dimen-

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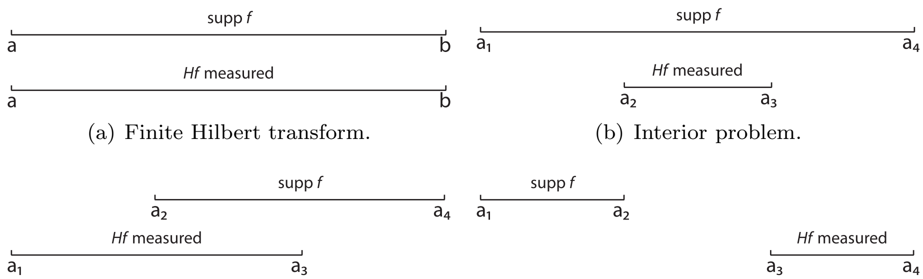


(a) Finite Hilbert transform. (b) Interior problem.

(c) Truncated Hilbert transform *with overlap*. (d) Truncated Hilbert transform *with a gap*.

FIG. 1.1. *Different setups for $\mathcal{P}_{\Omega_1} H \mathcal{P}_{\Omega_2}$. The upper interval shows the support Ω_2 of the function f to be reconstructed. The lower interval is the interval Ω_1 where measurements of the Hilbert transform Hf are taken. This paper investigates case* (c).

sions in [17, 24, 29] and for three dimensions in [18, 25, 27, 28]. To reconstruct from data obtained by the DBP method, it is necessary to solve a family of 1D problems which consist of inverting the Hilbert transform data on a finite segment of the line. If the Hilbert transform Hf of a 1D function f was given on all of $\mathbb{R}$, then the inversion would be trivial, since $H^{-1} = -H$. In case f is compactly supported, it can be reconstructed even if Hf is not known on all of $\mathbb{R}$. Due to an explicit reconstruction formula by Tricomi [22], f can be found from measuring Hf only on an interval that covers the support of f. However, a limited field of view might result in configurations in which the Hilbert transform is known only on a segment that does not completely cover the object support. One example of such a configuration is known as the interior problem [1, 10, 12, 23]. Given real numbers $a_1 < a_2 < a_3 < a_4$, the interior problem corresponds to the case in which the Hilbert transform of a function supported on $[a_1, a_4]$ is measured on the smaller interval $[a_2, a_3]$.

In this paper, we study a different configuration, namely, supp $f = [a_2, a_4]$ and the Hilbert transform is measured on $[a_1, a_3]$. We will refer to this configuration as the truncated problem *with overlap*: the operator H_T we consider is given by $\mathcal{P}_{[a_1,a_3]} H \mathcal{P}_{[a_2,a_4]}$, where H is the usual Hilbert transform acting on $\mathcal{L}^2(\mathbb{R})$, and $\mathcal{P}_\Omega$ stands for the projection operator $(\mathcal{P}_\Omega f)(x) = f(x)$ if $x \in \Omega$, $(\mathcal{P}_\Omega f)(x) = 0$ otherwise. For finite intervals Ω_1, Ω_2 on $\mathbb{R}$, the *interior* problem corresponds to $\mathcal{P}_{\Omega_1} H \mathcal{P}_{\Omega_2}$ for $\Omega_1 \subset \Omega_2$. The truncated Hilbert transform *with a gap* occurs when the intervals Ω_1 and Ω_2 are separated by a gap, as in [8]. Figure 1.1 shows the different setups. Examples of configurations in which the truncated Hilbert transform *with overlap* and the interior problem occur are given in Figures 1.2 and 1.3. The truncated problem *with overlap* arises for example in the "missing arm" problem. This is the case where the field of view is large enough to measure the torso but not the arms.

DEFINITION 1.1. *Fix any four real numbers $a_1 < a_2 < a_3 < a_4$. We define the truncated Hilbert transform with overlap as the operator*

$$(H_T f)(x) := \frac{1}{\pi} p.v. \int_{a_2}^{a_4} \frac{f(y)}{y-x} dy, \quad x \in (a_1, a_3), \tag{1.1}$$

where p.v. stands for the principal value. In short,

$$H_T := \mathcal{P}_{[a_1,a_3]} H \mathcal{P}_{[a_2,a_4]},$$

where H is the ordinary Hilbert transform on $\mathcal{L}^2(\mathbb{R})$.

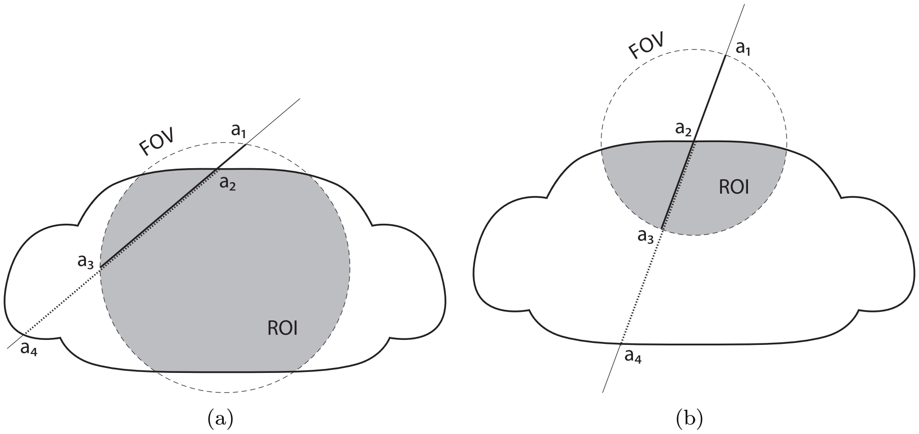


Fig. 1.2. *Two examples of the truncated problem with overlap. Figure* 1.2(a) *shows the missing arm problem. In both cases, the field of view (FOV) does not cover the object support. On the line intersecting the object, measurements can only be taken within the FOV, i.e., from a_1 to a_3. The Hilbert transform is not measured on $[a_3, a_4]$. Consequently, a reconstruction can only be aimed at in the gray-shaded intersection of the FOV with the object support, called the ROI.*

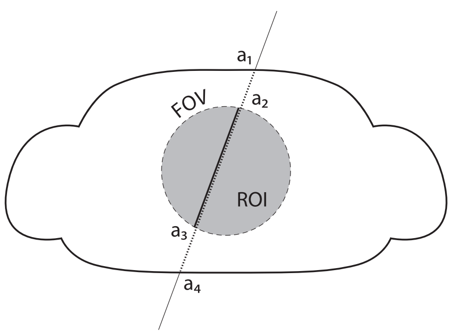


Fig. 1.3. *The interior problem. Here, the FOV also does not cover the object support. The line intersecting the object is such that the Hilbert transform is only measured in a subinterval $[a_2, a_3]$ of the intersection $[a_1, a_4]$ of the line with the object support. The ROI is the gray-shaded intersection of the FOV with the object support. In this case stable reconstruction of the shaded ROI is impossible unless additional information is available.*

As we will prove in what follows, the inversion of H_T is an ill-posed problem in the sense of Hadamard [3]. In order to find suitable regularization methods for its inversion, it is crucial to study the nature of the underlying ill-posedness, and therefore the spectrum $\sigma(H_T^* H_T)$. An important question that arises here is whether the spectrum is purely discrete. This question has been answered for similar operators before, but with two very different answers. In [11], it was shown that the finite Hilbert transform defined as $H_F = \mathcal{P}_{[a,b]} H \mathcal{P}_{[a,b]}$ has a continuous spectrum $\sigma(H_F) = [-i, i]$. On the other hand, in [9], we find the result that for the interior problem $H_I = \mathcal{P}_{[a_2,a_3]} H \mathcal{P}_{[a_1,a_4]}$, the spectrum $\sigma(H_I^* H_I)$ is purely discrete.

The main result of this paper is that $H_T^* H_T$ has only a discrete spectrum. In addition, we obtain that 0 and 1 are accumulation points of the spectrum. Furthermore, we find that the singular value decomposition (SVD) of the operator H_T can be related to the solutions of a Sturm–Liouville (S-L) problem. For the actual reconstruction, one would aim at finding f in (1.1) only within an ROI, i.e., on $[a_2, a_3]$. A stability estimate as well as a uniqueness result for this setup were obtained by Defrise et al. in [2]. A possible method for ROI reconstruction is the truncated SVD. Thus, it is of interest to study the SVD of H_T also for the development of reconstruction algorithms.

In [8] and [9], SVDs are obtained for the truncated Hilbert transform *with a gap* $\mathcal{P}_{[a_3,a_4]} H \mathcal{P}_{[a_1,a_2]}$ and for H_I. This is done by relating the Hilbert transforms to differential operators that have discrete spectra. We follow this procedure, but obtain a differential operator that is different in nature. In [8] and [9] the discreteness of the spectra follows from standard results of singular S-L theory (see, e.g., [26]). In the case of the truncated Hilbert transform (1.1) we have to investigate the discreteness of the spectrum of the related differential operator explicitly.

The idea is to find a differential operator for which the eigenfunctions are the singular functions of H_T on (a_2, a_4). We define the differential operator similarly to the one in [8, 9], but then the question is which boundary conditions to choose in order to relate the differential operator to H_T. To answer this question we first develop an intuition about the singular functions of H_T.

Let $\{\sigma_n; f_n, g_n\}$ denote the singular system of H_T that we want to find. The problem can be formulated as finding a complete orthonormal system $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{L}^2(a_2, a_4)$ and an orthonormal system $\{g_n\}_{n\in\mathbb{N}}$ in $\mathcal{L}^2(a_1, a_3)$ such that there exist real numbers σ_n for which

$$\begin{aligned} H_T f_n &= \sigma_n g_n, \\ H_T^* g_n &= \sigma_n f_n. \end{aligned}$$

At the moment, the g_n's only have to be complete in $\mathrm{Ran}(H_T)$, but as we will see in section 5, $\mathrm{Ran}(H_T)$ is dense in $\mathcal{L}^2(a_1, a_3)$.

As will be shown in section 4, the functions f_n and g_n

(a) can only be bounded or of logarithmic singularity at the points a_i,

(b) do not vanish at the edges of their supports (a_2^+, a_4^- for f_n, and a_1^+, a_3^- for g_n).

We will now make use of the following results from [5, sections 8.2 and 8.5].

Lemma 1.2 (local properties of the Hilbert transform). *Let f be a function with support $[b, d] \subset \mathbb{R}$. And let c be in the interior of $[b, d]$.*

1. *If f is Hölder continuous (for some Hölder index α) on $[b, d]$, then close to b the Hilbert transform of f is given by*

$$(Hf)(x) = -\frac{1}{\pi} f(b^+) \ln|x - b| + H_0(x), \tag{1.2}$$

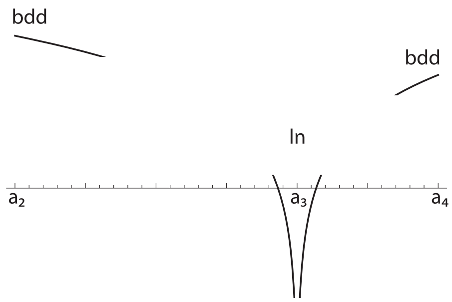
where H_0 is bounded and continuous in a neighborhood of b.

2. *If in a neighborhood of c, the function f is of the form $f(x) = \tilde{f}(x) \ln|x - c|$ for Hölder continuous $\tilde{f}$, then close to the point c its Hilbert transform is of the form*

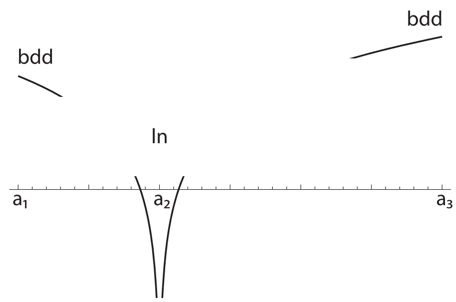
$$(Hf)(x) = H_0(x),$$

where H_0 is bounded with a possible finite jump discontinuity at c.

3. *If f is of the form $f(x) = \tilde{f}(x) \ln|x-b|$ on $[b, c]$, where $\tilde{f}$ is Hölder continuous, then its Hilbert transform at b has a singularity of the order $\ln^2|x - b|$ if $\tilde{f}(b) \neq 0$.*



(a) Sketch of f_n's.



(b) Sketch of g_n's.

FIG. 1.4. *Intuition about the singular functions of H_T.*

Suppose f_n has a logarithmic singularity at a_2^+. Since H_T integrates over $[a_2, a_4]$, the function $H_T f_n$ would have a singularity at a_2 of order $\ln^2|x - a_2|$. Hence, this would violate the property of g_n at a_2. Therefore, f_n has to be bounded at a_2^+. If f_n does not vanish at a_2^+, this leads to logarithmic singularities of $H_T f_n$ and g_n at a_2. Using the same argument we conclude that g_n is bounded at a_3^- and f_n has a logarithmic singularity at a_3.

On the other hand, since g_n is bounded at a_3^-, $H_T f_n$ is also bounded there. This requires that close to a_3, $f_n = f_{n,1} + f_{n,2}\ln|x-a_3|$ for functions $f_{n,i}$ continuous at a_3. A similar argument holds for g_n at a_2. Close to that point, $g_n = g_{n,1} + g_{n,2}\ln|x - a_2|$ for functions $g_{n,i}$ continuous at a_2.

Clearly, $H_T f_n$ is bounded at a_1^+ and $H_T g_n$ is bounded at a_4^-. Therefore, f_n has to be bounded at a_4^- and g_n must be bounded at a_1^+.

Thus, if we want to show the commutation of H_T with a differential operator that acts on $f_n(x)$, $x \in (a_2, a_4)$, we need to impose boundary conditions at a_2^+ and a_4^- that require boundedness and some transmission conditions at a_3 that make the bounded term and the term in front of the logarithm in f_n continuous at a_3.

Having found these properties of the singular functions of H_T (in case the SVD for H_T exists), in section 2 we introduce a differential operator and find a self-adjoint extension for this operator. We then show in section 3 that this self-adjoint differential operator L_S has a discrete spectrum. In section 4 we establish that L_S commutes with the operator H_T. This allows us to find the SVD of H_T. In section 5 we then study the accumulation points of the singular values of H_T. In particular, we find that H_T is not a compact operator. Finally, we conclude by showing numerical examples in section 6.

2. Introducing a differential operator.

In this section, we find two differential operators L_S and $\tilde{L}_S$ that will turn out to have a commutation property of the form

$$H_T L_S = \tilde{L}_S H_T. \tag{2.1}$$

In order to find the SVD of H_T, we will be interested in finding L_S and $\tilde{L}_S$ with simple discrete spectra. Initially, it is not apparent whether differential operators with such properties exist and if so, how to find them. We do not know of a coherent theory that relates certain integral operators to differential operators via a commutation property as the above. However, there have been examples of integral operators for which—by what seems to be a lucky accident—such differential operators exist.

One instance is the well-known Landau–Pollak–Slepian (LPS) operator that arises in signal processing in the study of time- and band-limited representations of signals [20, 13, 14]. There, it is of interest to find the largest eigenvalue of the LPS operator $\mathcal{P}_{[-T,T]}\mathcal{F}^{-1}\mathcal{P}_{[-W,W]}\mathcal{F}\mathcal{P}_{[-T,T]}$. Here, $\mathcal{F}$ is the Fourier transform, and T and W are some positive numbers. This operator happens to commute with a second order differential operator, of which the eigenfunctions and eigenvalues had been studied long before its connection to the LPS operator was known. The eigenfunctions of this differential operator are the so-called prolate spheroidal wave functions and they turn out to be the eigenfunctions of the LPS operator as well. The work of Landau, Pollak, and Slepian has been generalized and extended by Grünbaum, Longhi, and Perlstadt [7].

More recent examples of integral operators with commuting differential operators are the interior Radon transform [15] and two instances of the truncated Hilbert transform mentioned earlier [8, 9].

To start our search for L_S and $\tilde{L}_S$, we follow the procedure in [8, 9] and define a differential operator.

Definition 2.1.

$$L(x, d_x)\psi(x) := (P(x)\psi'(x))' + 2(x-\sigma)^2\psi(x), \tag{2.2}$$

where

$$P(x) = \prod_{i=1}^{4}(x-a_i), \quad \sigma = \frac{1}{4}\sum_{i=1}^{4} a_i. \tag{2.3}$$

The four points a_i are all *regular singular*, and in a complex neighborhood of each a_i the functions $(x-a_i)\cdot P'(x)/P(x)$ and $(x-a_i)^2 \cdot 2(x-\sigma)^2/P(x)$ are complex analytic. The term *regular singular point* is standard in the general theory of differential equations and, as such, is also used in the theory of S-L equations; see, e.g., [21] for this and other terminology and basic properties of S-L equations. Consequently, by the method of Fuchs–Frobenius it follows that for $\lambda \in \mathbb{C}$ any solution of $L\psi = \lambda\psi$ is either bounded or of logarithmic singularity close to any of the points a_i; see [21]. Away from the singular points a_i the analyticity of the solutions follows from the analyticity of the coefficients of the differential operator L. More precisely, in a left and a right neighborhood of each regular singular point a_i, there exist two linearly independent solutions of the form

$$\psi_1(x) = |x-a_i|^{\alpha_1}\sum_{n=0}^{\infty} b_n (x-a_i)^n, \tag{2.4}$$

$$\psi_2(x) = |x-a_i|^{\alpha_2}\sum_{n=0}^{\infty} d_n (x-a_i)^n + k\ln|x-a_i|\psi_1(x), \tag{2.5}$$

where without loss of generality we can assume $b_0 = d_0 = 1$. The exponents α_1 and α_2 are the solutions of the indicial equation

$$\alpha^2 + (p_0 - 1)\alpha + q_0 = 0,$$

where

$$p_0 = \lim_{x\to a_i}(x-a_i)P'(x)/P(x), \tag{2.6}$$

$$q_0 = \lim_{x\to a_i}(x-a_i)^2[2(x-\sigma)^2 - \lambda]/P(x). \tag{2.7}$$

With our choice of P, this gives $\alpha_1 = \alpha_2 = 0$ which implies $k \neq 0$. For the bounded solution in (2.4), $\alpha_1 = 0$ results in $\psi_1(a_i) \neq 0$. The radius of convergence of the series in (2.4) and (2.5) is the distance to the closest singular point different from a_i. In a left and in a right neighborhood of a_i, the general form of the solutions of $(L-\lambda)\psi = 0$ is

$$\psi_1(x) = \ell_0 \sum_{n=0}^{\infty} b_n (x-a_i)^n, \tag{2.8}$$

$$\psi_2(x) = \ell_1 \sum_{n=0}^{\infty} d_n (x-a_i)^n + \ell_2 \ln|x-a_i| \sum_{n=0}^{\infty} b_n (x-a_i)^n \tag{2.9}$$

for some constants ℓ_j. Hence we have one degree of freedom for the bounded solution, and two for the unbounded solution. Clearly, for the bounded solutions (2.8), the coefficients b_n are the same on both sides of a_i, since we have assumed $b_0 = 1$. However, the bounded part of the unbounded solutions (2.9) may have different coefficients d_n^- and d_n^+ to the left and to the right of a_i, respectively.

2.1. The maximal and minimal domains and self-adjoint realizations. Since we are interested in a differential operator that commutes (on some set to be defined) with H_T, we want to consider L on the interval (a_2, a_4). Due to the regular singular point a_3 in the *interior* of the interval, standard techniques for singular S-L problems are not applicable. It is crucial for our application that we identify a commuting *self-adjoint* operator, for which the spectral theorem can be applied. We therefore wish to study all self-adjoint realizations; we follow the treatment in Chapter 13 in [26] which gives a characterization of all self-adjoint realizations for *two-interval problems*, of which problems with an interior singular point are a special case.

First of all, one needs to define the maximal and minimal domains on $I_j = (a_j, a_{j+1})$ (see Chapter 9 in [26]). Let $AC_{loc}(I)$ be the set of all functions that are absolutely continuous on all compact subintervals of the open interval I. Then,

$$D_{j,\max} := \{\psi : I_j \to \mathbb{C} : \psi, P\psi' \in AC_{loc}(I_j); \psi, L\psi \in \mathcal{L}^2(I_j)\}, \tag{2.10}$$

$$D_{j,\min} := \{\psi \in D_{j,\max} : \text{supp } \psi \subset (a_j, a_{j+1})\}, \tag{2.11}$$

and the related maximal and minimal operators are defined as follows:

$$L_{j,\max} := L(D_{j,\max}) : D_{j,\max} \to \mathcal{L}^2(I_j), \tag{2.12}$$

$$L_{j,\min} := L(D_{j,\min}) : D_{j,\min} \to \mathcal{L}^2(I_j). \tag{2.13}$$

We shall follow essentially the procedure in Chapter 13 in [26], to which we refer for more detail. On (a_2, a_4), the maximal and minimal domains and the corresponding operators are defined as the direct sums.

Definition 2.2. *The maximal and minimal domains $D_{\max}$, $D_{\min} \subset \mathcal{L}^2(a_2, a_4)$ and the operators $L_{\max}$, $L_{\min}$ are defined as*

$$D_{\max} := D_{2,\max} + D_{3,\max}, \qquad D_{\min} := D_{2,\min} + D_{3,\min},$$
$$L_{\max} := L_{2,\max} + L_{3,\max}, \qquad L_{\min} := L_{2,\min} + L_{3,\min},$$

and, therefore,

$$L_{\max} : D_{\max} \to \mathcal{L}^2(a_2, a_4), \tag{2.14}$$

$$L_{\min} : D_{\min} \to \mathcal{L}^2(a_2, a_4). \tag{2.15}$$

The operator $L_{\min}$ is a closed, symmetric, densely defined operator in $\mathcal{L}^2(a_2, a_4)$ and $L_{\max}$, $L_{\min}$ form an adjoint pair, i.e., $L^*_{\max} = L_{\min}$ and $L^*_{\min} = L_{\max}$. In order to define a self-adjoint extension of $L_{\min}$, we need to introduce the notion of the Lagrange sesquilinear form

$$[u, v] := uP\overline{v}' - \overline{v}Pu', \tag{2.16}$$

where, at the singular points,

$$[u, v](a_i^+) := \lim_{\alpha \to a_i^+} [u, v](\alpha), \tag{2.17}$$

$$[u, v](a_i^-) := \lim_{\alpha \to a_i^-} [u, v](\alpha). \tag{2.18}$$

These limits exist and are finite for all u, $v \in D_{\max}$. If we choose u, $v \in D_{\max}$ such that $[u, v](a_i) = 1$ for all the singular points $(a_2^+, a_3^-, a_3^+, a_4^-)$, then the extension of $L_{\min}$ defined by the following conditions

$$[\psi, u](a_2^+) = 0 = [\psi, u](a_4^-), \tag{2.19}$$

$$[\psi, u](a_3^-) = [\psi, u](a_3^+), \tag{2.20}$$

$$[\psi, v](a_3^-) = [\psi, v](a_3^+), \tag{2.21}$$

is self-adjoint. We refer to (2.19) as boundary conditions, and to (2.20) and (2.21) as transmission conditions. The latter connect the two subintervals (a_2, a_3) and (a_3, a_4). Motivated by the conditions mentioned in section 1, we define a self-adjoint extension of $L_{\min}$.

LEMMA 2.3. *The extension $L_S : D(L_S) \to \mathcal{L}^2(a_2, a_4)$ of $L_{\min}$ to the domain*

$$\begin{aligned} D(L_S) := \{\psi \in D_{\max} : &[\psi, u](a_2^+) = [\psi, u](a_4^-) = 0, \\ &[\psi, u](a_3^-) = [\psi, u](a_3^+), [\psi, v](a_3^-) = [\psi, v](a_3^+)\} \end{aligned} \tag{2.22}$$

with the following choice of maximal domain functions $u, v \in D_{\max}$,

$$u(y) := 1, \tag{2.23}$$

$$v(y) := \sum_{i=1}^{4} \prod_{\substack{j \neq i \\ j \in \{1,\ldots,4\}}} \frac{1}{a_i - a_j} \ln |y - a_i|, \tag{2.24}$$

is self-adjoint.

This choice of maximal domain functions gives $[u, v](a_i) = 1$ for $i = 1, \ldots, 4$. The boundary conditions simplify to

$$\lim_{y \to a_2^+} P(y)\psi'(y) = \lim_{y \to a_4^-} P(y)\psi'(y) = 0. \tag{2.25}$$

For an eigenfunction ψ of L_S this is equivalent to ψ being bounded at a_2^+ and a_4^- (because the only possible singularity is of logarithmic type). Let ϕ_1 and ϕ_2 be the restrictions of ψ to the intervals (a_2, a_3) and (a_3, a_4), respectively. Since ψ is an eigenfunction, on the corresponding intervals, ϕ_1 and ϕ_2 are of the form $\phi_i(y) = \phi_{i1}(y) + \phi_{i2}(y) \ln |y - a_3|$. Here, the functions ϕ_{ij} are analytic on (a_2, a_3) for $i = 1$

and on (a_3, a_4) for $i = 2$. Having this, the transmission conditions can be simplified as follows:

$$\begin{aligned} [\psi, u](a_3^+) &= [\psi, u](a_3^-), \\ \lim_{y\to a_3^+} P(y)\psi'(y) &= \lim_{y\to a_3^-} P(y)\psi'(y), \\ \lim_{y\to a_3^-} \phi_{12}(y) &= \lim_{y\to a_3^+} \phi_{22}(y). \end{aligned} \tag{2.26}$$

The condition involving v yields

$$\begin{aligned} [\psi, v](a_3^-) &= [\psi, v](a_3^+), \\ \lim_{y\to a_3^-} [\psi(y) - v(y)(P\psi')(y)] &= \lim_{y\to a_3^+} [\psi(y) - v(y)(P\psi')(y)], \end{aligned} \tag{2.27}$$

$$\lim_{y\to a_3^-} \phi_{11}(y) = \lim_{y\to a_3^+} \phi_{21}(y). \tag{2.28}$$

Note that on each side of (2.27) the logarithmic terms in ϕ_{i2} cancel because of the choice of the constants in v. The properties (2.25), (2.26), and (2.28) are the same as the ones found for f_n in section 1. Thus, we have constructed an operator L_S for which, close to the points a_2, a_3 and a_4, the eigenfunctions behave in the same way that is expected for the f_n's.

Close to a_3, an eigenfunction ψ is given by

$$\psi(y) = \begin{cases} \ell_{11} \sum_{m=0}^{\infty} d_m^-(y-a_3)^m + \ell_{21} \ln|y-a_3| \sum_{m=0}^{\infty} b_m(y-a_3)^m, & y < a_3, \\ \ell_{12} \sum_{m=0}^{\infty} d_m^+(y-a_3)^m + \ell_{22} \ln|y-a_3| \sum_{m=0}^{\infty} b_m(y-a_3)^m, & y > a_3, \end{cases} \tag{2.29}$$

where similarly to (2.5), we assume $d_0^- = d_0^+ = 1$ and $b_0 = 1$. The transmission conditions require that

$$\ell_{11} = \ell_{12}, \tag{2.30}$$

$$\ell_{21} = \ell_{22}. \tag{2.31}$$

We can thus express ψ in a sufficiently small neighborhood of a_3 as

$$\psi(y) = \ell_{11} + \ell_{21} \ln|y-a_3| \sum_{m=0}^{\infty} b_m(y-a_3)^m + \sum_{m=1}^{\infty} \ell_m^{\pm}(y-a_3)^m, \tag{2.32}$$

where $\ell_m^{\pm}$ stands for $\ell_m^+ = \ell_{11}d_m^+$, when $y > a_3$ and for $\ell_m^- = \ell_{11}d_m^-$, when $y < a_3$.

3. The spectrum of L_S.

In order to prove that the spectrum of the differential self-adjoint operator L_S introduced in Lemma 2.3 is discrete, we need to show that for some z in the resolvent set, $(L_S - zI)^{-1}$ is a compact operator. To do so, it is sufficient to prove that the Green's function G of $L_S - zI$, which for z in the resolvent set exists and is unique, is a function in $\mathcal{L}^2((a_2, a_4)^2)$. This would allow us to conclude that the integral operator T_G with G as its integral kernel is a compact operator from $\mathcal{L}^2(a_2, a_4)$ to $\mathcal{L}^2(a_2, a_4)$, where T_G is equivalent to the inversion of $L_S - zI$.

LEMMA 3.1. *The Green's function $G(x, \xi)$ associated with $L_S - i$ is in $\mathcal{L}^2((a_2, a_4)^2)$ and consequently, $(L_S - i)^{-1} : \mathcal{L}^2(a_2, a_4) \to D(L_S) \subset \mathcal{L}^2(a_2, a_4)$ is a compact operator.*

Proof. The self-adjointness of L_S is equivalent to $L_S - i$ being one-to-one and onto (Theorem VIII.3 in [19]). Moreover, the a_i's are limit-circle points and thus, the deficiency index d equals 4 (Theorem 13.3.1 in [26]). This means that if we do not impose boundary and transmission conditions, there are two linearly independent solutions p_1 and p_2 of $(L-i)p = 0$ on (a_2, a_3) as well as two linearly independent solutions q_1 and q_2 of $(L-i)q = 0$ on (a_3, a_4). Note that none of these four solutions can be bounded at both of its endpoints because i is not an eigenvalue of the self-adjoint operator $L_{j,S} : D(L_{j,S}) \to \mathcal{L}^2(I_j)$ with $D(L_{j,S}) = \{\psi \in D_{j,\max} : \lim_{y\to a_j^+} P(y)\psi'(y) = \lim_{y\to a_{j+1}^-} P(y)\psi'(y) = 0\}$. By taking appropriate combinations, if necessary, we can eliminate the logarithmic singularity at a_2^+ of one of the solutions, and at a_4^- of another solution. We can thus assume that

- on (a_2, a_3): p_1 is bounded at a_2^+ and logarithmic at a_3^-, p_2 is logarithmic at both endpoints;
- on (a_3, a_4): q_1 is logarithmic at a_3^+ and bounded at a_4^-, q_2 is logarithmic at both endpoints.

We next check the restrictions imposed by the transmission conditions at a_3. Close to a_3, both functions p_1 and q_2 are of the form (2.9). Let ℓ_{11}, ℓ_{21} denote the free parameters in the expression for p_1 and ℓ_{12}, ℓ_{22} the ones in q_2. These can be chosen such that they satisfy (2.30) and (2.31). Thus, there exists a solution $h_1(x)$ on (a_2, a_4) given by

$$h_1(x) = \begin{cases} p_1(x) & \text{for } x \in (a_2, a_3), \\ q_2(x) & \text{for } x \in (a_3, a_4), \end{cases}$$

that is bounded at a_2^+ and logarithmic at a_4^-. In addition, it is of the form (2.32) close to a_3, i.e., it is logarithmic at a_3 and satisfies the transmission conditions (2.26), (2.28) there. Similarly, with p_2 and q_1 we can obtain a solution h_2 on (a_2, a_4) that satisfies the transmission conditions at a_3 and is of ln-ln-bounded-type. Thus, imposing only the transmission conditions, we obtain two linearly independent solutions of $(L-i)h = 0$ on $(a_2, a_3) \cup (a_3, a_4)$. One of them, h_1, is of a bounded-ln-ln-type, and the other one, h_2, is of a ln-ln-bounded-type, at the points a_2^+, a_3, a_4^-, respectively. We are now in a position to consider the Green's function $G(x, \xi)$ of $L_S - i$. Close to a_3, we can write the two functions as $h_j(x) = h_{j1}(x) + \ln|x - a_3| h_{j2}(x)$ with continuous functions h_{j1} and h_{j2}. By rescaling if necessary, we can assume $h_{12}(a_3) = h_{22}(a_3)$. We construct G from h_1 and h_2 as follows:

$$G(x, \xi) = \begin{cases} c_1(\xi) h_1(x) & \text{for } x < \xi, \\ c_2(\xi) h_2(x) & \text{for } x > \xi, \end{cases} \tag{3.1}$$

where $\xi \in (a_2, a_3) \cup (a_3, a_4)$ and the functions $c_1(\xi)$ and $c_2(\xi)$ are chosen such that G is continuous at $x = \xi$ and $\partial G/\partial x$ has a jump discontinuity of $1/P(\xi)$ at $x = \xi$:

$$c_1(\xi) h_1(\xi) - c_2(\xi) h_2(\xi) = 0, \tag{3.2}$$

$$c_1(\xi) h_1'(\xi) - c_2(\xi) h_2'(\xi) = -\frac{1}{P(\xi)}. \tag{3.3}$$

In other words, G is the solution of $(L-i)G = \delta$, where δ is the Dirac delta function. For ξ away from a_3, $G(x, \xi)$ is continuous in ξ but with logarithmic singularities at a_2^+ and a_4^-. This can be seen as follows. Consider ξ close to a_2. There, we can write

$$h_2(\xi) = \tilde{h}_{21}(\xi) + \tilde{h}_{22}(\xi) \ln|\xi - a_2|$$

and, since h_1 is bounded close to a_2^+, it is of the form (2.8), i.e., $h_1(a_2^+) \neq 0$. Let W_{h_1,h_2} denote the Wronskian of h_1 and h_2, i.e., $W_{h_1,h_2} = h_1 h_2' - h_1' h_2$. For c_1 and c_2 we obtain

$$c_1(\xi) = \frac{h_2(\xi)}{P(\xi) W_{h_1,h_2}(\xi)}, \tag{3.4}$$

$$c_2(\xi) = \frac{h_1(\xi)}{P(\xi) W_{h_1,h_2}(\xi)}. \tag{3.5}$$

The denominator in the above expressions is bounded by

$$P(\xi)(h_1(\xi)h_2'(\xi) - h_2(\xi)h_1'(\xi)) = \mathcal{O}((\xi - a_2)\ln|\xi - a_2|) + h_1(\xi)\tilde{h}_{22}(\xi)p(\xi),$$

where $p(\xi) = P(\xi)/(\xi - a_2)$ and $h_1(a_2^+)\tilde{h}_{22}(a_2)p(a_2) \neq 0$. Thus, in a neighborhood of a_2^+,

$$c_1(\xi) = \mathcal{O}(\ln|\xi - a_2|), \tag{3.6}$$

$$c_2(\xi) = \mathcal{O}(1). \tag{3.7}$$

Similarly, since $h_2(a_4^-) \neq 0$, close to a_4^-

$$c_1(\xi) = \mathcal{O}(1), \tag{3.8}$$

$$c_2(\xi) = \mathcal{O}(\ln|\xi - a_4|). \tag{3.9}$$

For each fixed $\xi \in (a_2, a_3) \cup (a_3, a_4)$, $G(x, \xi)$ as a function in x is continuous on $[a_2, a_3) \cup (a_3, a_4]$ and has a logarithmic singularity at a_3, due to the singularities in $h_1(x)$ and $h_2(x)$. It remains to check what happens as $\xi \to a_3$. We need to make sure that the functions $c_1(\xi)$ and $c_2(\xi)$ behave in such a way that $G \in \mathcal{L}^2((a_2, a_4)^2)$. Therefore, we derive the asymptotics of $c_1(\xi)$ and $c_2(\xi)$ as $\xi \to a_3^-$. For $\xi = a_3 - \epsilon$ and small $\epsilon > 0$, (3.2) becomes

$$c_1(a_3 - \epsilon)h_1(a_3 - \epsilon) - c_2(a_3 - \epsilon)h_2(a_3 - \epsilon) = 0.$$

Since close to a_3, $h_i = h_{i1} + h_{i2}\ln(\epsilon)$ and the h_{ij} are continuous, the ratio c_1/c_2 is of the form

$$\frac{a + b\ln(\epsilon)}{c + d\ln(\epsilon)},$$

where b and d are nonzero (because the logarithmic singularity is present). Thus, the ratio tends to the finite limit b/d as $\epsilon \to 0$. Conditions (3.2) and (3.3) together imply

$$\begin{aligned} c_2(a_3 - \epsilon) &= \frac{h_1(a_3 - \epsilon)}{P(a_3 - \epsilon) W_{h_1,h_2}(a_3 - \epsilon)} \\ &= \frac{h_{11}(a_3 - \epsilon) + h_{12}(a_3 - \epsilon)\ln(\epsilon)}{r_1(\epsilon) + \epsilon \cdot r_2(\epsilon)}, \end{aligned}$$

where

$$\begin{aligned} r_1(\epsilon) &= -\frac{1}{\epsilon} P(a_3 - \epsilon)\big(h_{21}h_{12} - h_{22}h_{11}\big)(a_3 - \epsilon), \\ r_2(\epsilon) &= \mathcal{O}(1), \end{aligned}$$

and $h_{12}(a_3) \neq 0$. If $r_1(0) \neq 0$, then c_2 is of order $\mathcal{O}(\ln(\epsilon))$, removing a possible obstruction to square integrability of G.

Suppose $r_1(0) = 0$, i.e.,

$$h_{21}(a_3)h_{12}(a_3) - h_{22}(a_3)h_{11}(a_3) = 0.$$

This would imply

$$h_{11}(a_3) = C \cdot h_{21}(a_3), \tag{3.10}$$

$$h_{12}(a_3) = C \cdot h_{22}(a_3) \tag{3.11}$$

for some constant C. By assumption, $h_{12}(a_3) = h_{22}(a_3)$, so that $C = 1$. Now if both (3.10) and (3.11) hold for $C = 1$, the function defined by

$$h(x) = \begin{cases} h_1(x) & \text{for } x \in (a_2, a_3), \\ h_2(x) & \text{for } x \in (a_3, a_4) \end{cases}$$

would be a nontrivial solution of $(L_S - i)h = 0$ (fulfilling both boundary and transmission conditions), i.e., i would be an eigenvalue of L_S. But this contradicts the self-adjointness of L_S. We can thus conclude that $r_1(0) \neq 0$. This shows that $c_2(a_3 - \epsilon)$ is of order $\mathcal{O}(\ln(\epsilon))$ and, therefore, also $c_2 \cdot \frac{c_1}{c_2} = c_1 = \mathcal{O}(\ln(\epsilon))$.

Analogously, we can find the same asymptotics of $c_1(\xi)$ and $c_2(\xi)$ as $\xi \to a_3^+$.

Therefore, the properties of the Green's function $G(x, \xi)$ can be summarized as follows:

- $G(\cdot, \xi)$ has logarithmic singularities at a_2^+, a_3, and a_4^-;
- $G(x, \cdot)$ is of logarithmic singularity at a_3;
- away from these singularities $G(x, \xi)$ is continuous in x and ξ.

Thus, G is in $\mathcal{L}^2((a_2, a_4)^2)$. Hence, $T_G : \mathcal{L}^2(a_2, a_4) \to \mathcal{L}^2(a_2, a_4)$ is a compact Fredholm integral operator. ∎

From this we conclude the following.

PROPOSITION 3.2. *The operator L_S has only a discrete spectrum, and the associated eigenfunctions are complete in $\mathcal{L}^2(a_2, a_4)$.*

Proof. By Theorem VIII.3 in [19], the self-adjointness of L_S implies that for the operator $(L_S - i) : D(L_S) \to \mathcal{L}^2(a_2, a_4)$ we have

$$\operatorname{Ker}(L_S - i) = \{0\}, \tag{3.12}$$

$$\operatorname{Ran}(L_S - i) = \mathcal{L}^2(a_2, a_4). \tag{3.13}$$

Consequently, $(L_S - i)^{-1} : \mathcal{L}^2(a_2, a_4) \to D(L_S)$ is one-to-one and onto. Moreover, it is a normal compact operator and thus we get the spectral representation

$$(L_S - i)^{-1} f = \sum_{n=0}^{\infty} \lambda_n \langle f, f_n \rangle f_n, \tag{3.14}$$

where $\{f_n\}_{n\in\mathbb{N}}$ is a complete orthonormal system in $\mathcal{L}^2(a_2, a_4)$. This can be transformed into the spectral representation for L_S:

$$L_S f = \sum_{n=0}^{\infty} \left(\frac{1}{\lambda_n} + i \right) \langle f, f_n \rangle f_n. \qquad ∎ \tag{3.15}$$

Clearly, the eigenfunctions f_n of L_S can be chosen to be real valued. The completeness of $\{f_n\}_{n\in\mathbb{N}}$ is essential for finding the SVD of H_T. Another property that will be needed for the SVD is that the spectrum of L_S is simple, i.e., that each eigenvalue has multiplicity 1.

PROPOSITION 3.3. *The spectrum of L_S is simple.*

Proof. From the compactness of $(L_S - i)^{-1}$, we know that each eigenvalue has finite multiplicity. Suppose f_1 and f_2 are linearly independent eigenfunctions of L_S corresponding to the same eigenvalue $\lambda \in \mathbb{R}$. Then, on all of $(a_2, a_3) \cup (a_3, a_4)$ the following holds:

$$f_1 L f_2 - f_2 L f_1 = 0. \tag{3.16}$$

Consequently,

$$\begin{aligned} 0 = f_1 L f_2 - f_2 L f_1 &= f_1 (P f_2')' - f_2 (P f_1')' \\ &= [f_1, f_2]'. \end{aligned}$$

Thus, $[f_1, f_2]$ is constant on both (a_2, a_3) and (a_3, a_4). From the boundary conditions that f_1 and f_2 satisfy, we find that $[f_1, f_2](a_2^+) = 0 = [f_1, f_2](a_4^-)$, which implies $[f_1, f_2] = 0$ on $(a_2, a_3) \cup (a_3, a_4)$. Since $[f_1, f_2] = P(f_1' f_2 - f_1 f_2')$, we get that

$$f_1' f_2 - f_1 f_2' = 0 \text{ on } (a_2, a_3) \cup (a_3, a_4). \tag{3.17}$$

The functions f_1 and f_2 satisfy the transmission conditions at a_3. Consequently, they can be written as

$$\begin{aligned} f_1(x) &= f_{11}(x) + f_{12}(x) \ln |x - a_3|, \\ f_2(x) &= f_{21}(x) + f_{22}(x) \ln |x - a_3| \end{aligned}$$

in a neighborhood of a_3, where f_{ij} are continuous. Since the one-sided derivatives f_{ij}' are bounded at a_3, (3.17) implies

$$\frac{(f_{12} f_{21} - f_{11} f_{22})(x)}{x - a_3} + \mathcal{O}(\ln^2 |x - a_3|) = 0. \tag{3.18}$$

Note that the terms containing $\ln |x - a_3| / (x - a_3)$ cancel. Taking the limit $x \to a_3$ in (3.18), we obtain

$$f_{12}(a_3) f_{21}(a_3) - f_{11}(a_3) f_{22}(a_3) = 0.$$

Thus, for some constant C,

$$\begin{pmatrix} f_{11}(a_3) \\ f_{12}(a_3) \end{pmatrix} = C \begin{pmatrix} f_{21}(a_3) \\ f_{22}(a_3) \end{pmatrix}.$$

If we take f_1 on (a_2, a_3), then $f_{11}(a_3)$ and $f_{12}(a_3)$ define a singular initial value problem on (a_3, a_4) that is uniquely solvable (Theorem 8.4.1 in [26]). Thus, $f_1 = C \cdot f_2$ on (a_3, a_4). Now, on the other hand, by considering f_1 on (a_3, a_4), the values $f_{11}(a_3)$

and $f_{12}(a_3)$ define a singular initial value problem on (a_2, a_3) which has a unique solution. Hence, $f_1 = C \cdot f_2$ on $(a_2, a_3) \cup (a_3, a_4)$ in contradiction to our assumption. ∎

4. Singular value decomposition of H_T.

Having introduced the differential operator L_S, we now want to relate it to the truncated Hilbert transform H_T. The main result of this section is that the eigenfunctions of L_S fully determine the two families of singular functions of H_T. We start by stating the following.

PROPOSITION 4.1. *On the set of eigenfunctions $\{f_n\}_{n\in\mathbb{N}}$ of L_S, the following commutation relation holds:*

$$(H_T L(y, d_y) f_n)(x) = L(x, d_x)(H_T f_n)(x) \quad \text{for } x \in (a_1, a_2) \cup (a_2, a_3). \tag{4.1}$$

Sketch of proof. This proof follows the same general idea as the proof of Proposition 2.1 in [9]. We therefore provide full details only for those steps where additional care needs to be taken because of the singularity at a_3. The steps that are completely analogous to those in the proof of Proposition 2.1 in [9] are only sketched here.

Let $\psi \in \{f_n\}_{n\in\mathbb{N}}$. The boundedness of ψ at a_2^+ and a_4^- implies that $P\psi' \to 0$ and $P\psi \to 0$ there. Moreover, the transmission conditions at a_3 guarantee that $P\psi'$ is continuous at a_3. With these properties, the commutation relation for $x \in (a_1, a_2)$, i.e., where the Hilbert kernel is not singular, can be shown similarly to the proof of Proposition 2.1 in [8].

Next, let $x \in (a_2, a_3)$. The main difference from the proof of Proposition 2.1 in [9] is that now the eigenfunctions are not in $C^\infty([a_2, a_4])$, but are singular at a_3. However, the fact that we exclude the point $x = a_3$ allows us to always have a neighborhood of x away from a_3 on which ψ is bounded. We further note that $\psi \in C^\infty([a_2, a_3) \cup (a_3, a_4])$. Since the Hilbert kernel is singular, we need to use principal value integration and introduce the following notation: $I_\epsilon(x) := [a_2, x - \epsilon] \cup [x + \epsilon, a_4]$. Here $\epsilon > 0$ is so small that $(x - \epsilon, x + 2\epsilon) \subset (a_2, a_3)$, i.e., the ϵ-neighborhood of x is well separated from a_3. Then,

$$\pi(H_T L(y, d_y)\psi)(x) = \lim_{\epsilon\to 0^+} \int_{I_\epsilon(x)} \left[\frac{(P(y)\psi'(y))'}{y - x} + \frac{2(y - \sigma)^2 \psi(y)}{y - x} \right] dy.$$

For the first term under the integral, we integrate by parts twice and plug in the boundary conditions. Again, we use that $P\psi' \to 0$ and $P\psi \to 0$ at a_2^+ and a_4^-:

$$\begin{aligned} \int_{I_\epsilon(x)} \frac{(P(y)\psi'(y))'}{y - x} dy = &- \frac{(P\psi')(x - \epsilon) + (P\psi')(x + \epsilon)}{\epsilon} + \frac{(P\psi)(x - \epsilon) - (P\psi)(x + \epsilon)}{\epsilon^2} \\ &+ \int_{I_\epsilon(x)} \psi(y) \frac{2P(y) - P'(y)(y - x)}{(y - x)^3} dy. \end{aligned} \tag{4.2}$$

The integral on the right-hand side of (4.2) can be related to the derivatives of $\int \psi(y)/(y - x)dy$. In [9] similar relations (cf. (2.7)) were obtained from the Leibniz integral rule, using explicitly that the integrand was continuous. In our case, the function ψ is no longer continuous because of the singularity at a_3. We can generalize the argument of [9] by invoking the dominated convergence theorem and rewrite the last term in (4.2) as follows:

$$\begin{aligned}
&\int_{I_\epsilon(x)} \psi(y)\frac{2P(y)-P'(y)(y-x)}{(y-x)^3}dy \\
&\quad = P(x)\left[\frac{d^2}{dx^2}\int_{I_\epsilon(x)}\frac{\psi(y)}{y-x}dy + \frac{\psi'(x-\epsilon)+\psi'(x+\epsilon)}{\epsilon} - \frac{\psi(x-\epsilon)-\psi(x+\epsilon)}{\epsilon^2}\right] \\
&\qquad + P'(x)\left[\frac{d}{dx}\int_{I_\epsilon(x)}\frac{\psi(y)}{y-x}dy + \frac{\psi(x-\epsilon)+\psi(x+\epsilon)}{\epsilon}\right] \\
&\qquad - \int_{I_\epsilon(x)} 2\psi(y)\frac{(y-\sigma)^2-(x-\sigma)^2}{y-x}dy.
\end{aligned}$$

Putting all the pieces together, we obtain

$$\begin{aligned}
&\pi(H_T L(y,d_y)\psi)(x) \\
&\quad = \lim_{\epsilon\to 0^+}\left\{-\frac{(P\psi')(x-\epsilon)+(P\psi')(x+\epsilon)}{\epsilon} + \frac{(P\psi)(x-\epsilon)-(P\psi)(x+\epsilon)}{\epsilon^2}\right. \\
&\qquad + P(x)\left[\frac{\psi'(x-\epsilon)+\psi'(x+\epsilon)}{\epsilon} - \frac{\psi(x-\epsilon)-\psi(x+\epsilon)}{\epsilon^2}\right] \\
&\qquad \left. + P'(x)\frac{\psi(x-\epsilon)+\psi(x+\epsilon)}{\epsilon} + L(x,d_x)\int_{I_\epsilon(x)}\frac{\psi(y)}{y-x}dy\right\}.
\end{aligned}$$

The eigenfunction ψ is in $C^\infty[a_2, x+2\epsilon)$. Following [9], we can thus express the boundary terms in the above equation by Taylor expansions around x and make use of the fact that the boundary terms consist only of odd functions in ϵ. The boundary terms are then of the order $\mathcal{O}(\epsilon)$. We thus have

$$\pi(H_T L(y,d_y)\psi)(x) = \lim_{\epsilon\to 0^+} L(x,d_x)\int_{I_\epsilon(x)}\frac{\psi(y)}{y-x}dy. \tag{4.3}$$

Since for $\epsilon > 0$ sufficiently small, $\psi \in C^\infty([x-\epsilon, x+\epsilon])$, one can interchange the limit with $L(x,d_x)$ as in [9]. ∎

Because the spectrum of L_S is purely discrete, we have thus found an orthonormal basis (the eigenfunctions of L_S) $\{f_n\}_{n\in\mathbb{N}}$ of $\mathcal{L}^2(a_2,a_4)$ for which (4.1) holds. Let us define $g_n := H_T f_n/\|H_T f_n\|_{\mathcal{L}^2(a_1,a_3)}$. Then, in order to obtain the SVD for H_T (with singular functions f_n and g_n), it is sufficient to prove that the g_n's form an orthonormal system of $\mathcal{L}^2(a_1,a_3)$ (they will then consequently form an orthonormal basis of $\mathcal{L}^2(a_1,a_3)$; see Proposition 5.2).

The orthogonality of the g_n's will follow from the commutation relation. Since f_n is an eigenfunction of L_S for some eigenvalue λ_n, we obtain

$$L(x,d_x)g_n(x) = \lambda_n g_n(x), \quad x \in (a_1,a_2)\cup(a_2,a_3).$$

Similarly to L_S, we define a new self-adjoint operator that acts on functions supported on $[a_1,a_3]$.

Definition 4.2. *Let* $\tilde{D}_{\max} := D_{1,\max} + D_{2,\max}$ *and* $\tilde{L}_{\min} := L_{1,\min} + L_{2,\min}$*. The operator* $\tilde{L}_S : D(\tilde{L}_S) \to \mathcal{L}^2(a_1,a_3)$ *is defined as the self-adjoint extension of* $\tilde{L}_{\min}$*, where*

$$\begin{aligned}
D(\tilde{L}_S) := \{\psi \in \tilde{D}_{\max} : {}&[\psi,u](a_1^+) = [\psi,u](a_3^-) = 0, \\
&[\psi,u](a_2^-) = [\psi,u](a_2^+), [\psi,v](a_2^-) = [\psi,v](a_2^+)\}
\end{aligned} \tag{4.4}$$

with the maximal domain functions $u, v \in \tilde{D}_{\max}$ *as in* (2.23), (2.24).

The intuition then is the following. The function f_n is bounded at a_2^+ and logarithmic at a_3, where it satisfies the transmission conditions. Consequently, as will be shown below, g_n is bounded at a_3^-, logarithmic at a_2, and satisfies the corresponding transmission conditions at a_2. Clearly, it is also bounded at a_1^+. Thus, g_n is an eigenfunction of the self-adjoint operator $\tilde{L}_S$. As a consequence, the g_n's form an orthonormal system.

PROPOSITION 4.3. *If $L_S f_n = \lambda_n f_n$, then $g_n := H_T f_n / \|H_T f_n\|_{\mathcal{L}^2(a_1,a_3)}$ is an eigenfunction of $\tilde{L}_S$ corresponding to the same eigenvalue*

$$\tilde{L}_S g_n = \lambda_n g_n. \tag{4.5}$$

Proof. First of all, the commutation relation for f_n yields

$$\begin{aligned} L(x,d_x)(H_T f_n)(x) &= (H_T L(y,d_y) f_n)(x), \\ L(x,d_x) g_n(x) &= \lambda_n g_n(x), \quad x \in (a_1,a_2) \cup (a_2,a_3). \end{aligned}$$

What remains to be shown is that g_n satisfies the boundary and transmission conditions. Therefore, we consider $p.v. \int_{a_2}^{a_4} f_n(y)/(y-x)dy$ for x close to a_1, a_2, and a_3. In a neighborhood of a_1 away from $[a_2,a_4]$ this function is clearly analytic. Next, let x be confined to a small neighborhood of a_2. Since the discontinuity of f_n is away from a_2, we can split the above integral into two, one that integrates over a right neighborhood of a_2 and another one that is an analytic function. The first item in Lemma 1.2 then implies that

$$p.v. \int_{a_2}^{a_4} \frac{f_n(y)}{y-x} dy = \tilde{g}_{n,1}(x) - f_n(a_2^+) \ln|x-a_2|, \tag{4.6}$$

where $\tilde{g}_{n,1}(x)$ is continuous in a neighborhood of $x = a_2$. Thus, g_n satisfies the transmission conditions (2.26), (2.28).

It remains to check the behavior of g_n close to a_3^-. We first express f_n as

$$f_n(y) = f_{n,1}(y) + f_{n,2}(y) \ln|y-a_3|,$$

where both $f_{n,1}$ and $f_{n,2}$ are Lipschitz continuous. Then, in view of Lemma 1.2, both summands on the right-hand side of the equation

$$p.v. \int_{a_2}^{a_4} \frac{f_n(y)}{y-x} dy = p.v. \int_{a_2}^{a_4} \frac{f_{n,1}(y)}{y-x} dy + p.v. \int_{a_2}^{a_4} \frac{f_{n,2}(y) \ln|y-a_3|}{y-x} dy$$

remain bounded as x tends to a_3. ∎

Since the spectrum of L_S is simple, we can conclude that the g_n's form an orthonormal system and thus the following holds.

THEOREM 4.4. *The eigenfunctions f_n of L_S, together with $g_n := H_T f_n / \|H_T f_n\|_{\mathcal{L}^2(a_1,a_3)}$ and $\sigma_n := \|H_T f_n\|_{\mathcal{L}^2(a_1,a_3)}$ form the SVD for H_T:*

$$H_T f_n = \sigma_n g_n, \tag{4.7}$$
$$H_T^* g_n = \sigma_n f_n. \tag{4.8}$$

5. Accumulation points of the singular values of H_T. The main result of this section is that 0 and 1 are accumulation points of the singular values of H_T. To find this, we first analyze the null space and range of H_T, which will also prove the ill-posedness of the inversion of H_T. First, we need to state the following.

LEMMA 5.1. *If the Hilbert transform of a compactly supported* $f \in \mathcal{L}^2(a,b)$ *vanishes on an open interval* (c,d) *disjoint from the object support, then* $f=0$ *on all of* $\mathbb{R}$.

Sketch of proof. A similar statement (and proof) can be found in [1]. The main difference is that here we consider a more general class of functions f. By dominated convergence, $f \in L^1(a,b)$ implies that for any $z \in \Omega = \mathbb{C}\backslash((-\infty,a) \cup (b,\infty))$, the function $g(z) = \int_a^b f(x)/(x-z)dx$ is differentiable in a neighborhood of z. Thus, g is analytic on Ω. The statement then follows in the same way as Lemma 2.1 in [1]. ∎

With this property of the Hilbert transform, we can obtain results on the null space and the range of H_T.

PROPOSITION 5.2. *The operator* $H_T : \mathcal{L}^2(a_2,a_4) \to \mathcal{L}^2(a_1,a_3)$ *has a trivial null space and dense range that is not all of* $\mathcal{L}^2(a_1,a_3)$, *i.e.,*

$$Ker(H_T) = \{0\}, \tag{5.1}$$

$$Ran(H_T) \neq \mathcal{L}^2(a_1,a_3), \tag{5.2}$$

$$\overline{Ran(H_T)} = \mathcal{L}^2(a_1,a_3). \tag{5.3}$$

Proof of (5.1). Suppose $H_T f = 0$. Then

$$H\chi_{[a_2,a_4]}f = 0 \text{ on } (a_1,a_2),$$

and by Lemma 5.1, $f=0$ on all of $[a_2,a_4]$. Thus, $f \in \mathcal{L}^2(a_2,a_4)$ can always be uniquely determined from $H_T f$.

Proof of (5.2). Take any $g \in \mathcal{L}^2(a_1,a_3)$ that vanishes on (a_1,a_2) and such that $\|g\|_{\mathcal{L}^2(a_1,a_3)} \neq 0$. Suppose $g \in \mathrm{Ran}(H_T)$. By Lemma 5.1, if $f \in \mathcal{L}^2(a_2,a_4)$ and $H_T f = g$, then f is zero on $[a_2,a_4]$. This implies that $g=0$ on (a_1,a_3), which contradicts the assumption $\|g\| \neq 0$.

Proof of (5.3). The operator H_T^* is also a truncated Hilbert transform with the same general properties. By the above argument, $\mathrm{Ker}(H_T^*) = \{0\}$. Thus, $\mathrm{Ran}(H_T)^\perp = \{0\}$. ∎

Equation (5.2) shows the ill-posedness of the problem. It is not true that for every $g \in \mathcal{L}^2(a_1,a_3)$ there is a solution f to the equation $H_T f = g$. Since $\mathrm{Ran}(H_T)$ is dense, the solution need not depend continuously on the data. Thus, our problem violates two properties of Hadamard's well-posedness criteria [3]. These are the existence of solutions for all data and the continuous dependence of the solution on the data. We now turn to the spectrum of $H_T^* H_T$. In what follows, $\|\cdot\|$ denotes the norm associated with $\mathcal{L}^2(\mathbb{R})$, and $\langle\cdot,\cdot\rangle$ denotes the $\mathcal{L}^2(\mathbb{R})$ inner product. We begin with proving the following.

LEMMA 5.3. *The operator* $H_T^* H_T$ *has norm equal to* 1.

Proof. From $\|H\| = 1$, we know that $\|H_T^* H_T\| \leq 1$. Since

$$\|H_T^* H_T\| = \sup_{\|\psi\|=1} \|H_T^* H_T \psi\|,$$

finding a sequence ψ_n with $\|\psi_n\| = 1$ and $\|H_T^* H_T \psi_n\| \to 1$ would prove the assertion.

Take a compactly supported function $\psi \in \mathcal{L}^2([-1,1])$ with $\|\psi\| = 1$ and two vanishing moments, $\int_{-1}^{1} \psi(x)dx = 0 = \int_{-1}^{1} x \cdot \psi(x)dx$. From this, we define a family of functions, such that the norm is preserved but the supports decrease. More precisely, for $a > 2/(a_3 - a_2)$, we set

$$\psi_a(x) = \sqrt{a}\psi\left(a\left(x - \frac{a_2 + a_3}{2}\right)\right). \tag{5.4}$$

These functions satisfy $\|\psi_a\| = 1$ and supp $\psi_a = [\frac{a_2+a_3}{2} - \frac{1}{a}, \frac{a_2+a_3}{2} + \frac{1}{a}] \subset [a_2, a_3]$. For their Hilbert transforms we obtain

$$(H\psi_a)(x) = \sqrt{a}(H\psi)\left(a\left(x - \frac{a_2 + a_3}{2}\right)\right). \tag{5.5}$$

We can write

$$\begin{aligned}
H_T^* H_T \psi_a &= -\chi_{[a_2,a_4]} H \chi_{[a_1,a_3]} H \chi_{[a_2,a_4]} \psi_a \\
&= -\chi_{[a_2,a_4]} H (I - (I - \chi_{[a_1,a_3]})) H \chi_{[a_2,a_4]} \psi_a \\
&= \psi_a + \chi_{[a_2,a_4]} H (I - \chi_{[a_1,a_3]}) H \chi_{[a_2,a_4]} \psi_a, \\
(I - H_T^* H_T)\psi_a &= -\chi_{[a_2,a_4]} H (I - \chi_{[a_1,a_3]}) H \chi_{[a_2,a_4]} \psi_a. \qquad (5.6)
\end{aligned}$$

Consider the $\mathcal{L}^2$-norm of the last expression

$$\begin{aligned}
\|(I - H_T^* H_T)\psi_a\|^2 &= \|\chi_{[a_2,a_4]} H (I - \chi_{[a_1,a_3]}) H \chi_{[a_2,a_4]} \psi_a\|^2 \\
&\leq \|(I - \chi_{[a_1,a_3]}) H \chi_{[a_2,a_4]} \psi_a\|^2 \\
&= \int_{(-\infty,a_1)\cup(a_3,\infty)} |(H\psi_a)(x)|^2 dx \\
&= a \int_{(-\infty,a_1)\cup(a_3,\infty)} \left|(H\psi)\left(a\left(x - \frac{a_2+a_3}{2}\right)\right)\right|^2 dx \\
&= \int_{-\infty}^{a\cdot(a_1 - \frac{a_2+a_3}{2})} |H\psi|^2 dy + \int_{a\cdot(a_3 - \frac{a_2+a_3}{2})}^{\infty} |H\psi|^2 dy. \qquad (5.7)
\end{aligned}$$

Because of the ordering of the a_i's, we have that $a_1 - \frac{a_2+a_3}{2} < 0$ and $a_3 - \frac{a_2+a_3}{2} > 0$. Since ψ has two vanishing moments, $H\psi$ asymptotically behaves like $1/|y|^3$ and hence, both integrals in (5.7) are of the order $\mathcal{O}(a^{-5})$. Thus, given any $\epsilon > 0$, one can find $a > 2/(a_3 - a_2)$ such that

$$\|(I - H_T^* H_T)\psi_a\| < \epsilon.$$

Consequently,

$$\|H_T^* H_T \psi_a\| \geq \|\psi_a\| - \|(I - H_T^* H_T)\psi_a\| > 1 - \epsilon.$$

Therefore

$$\|H_T^* H_T \psi_a\| \to 1 \text{ as } a \to \infty,$$

which implies that $\|H_T^* H_T\| = 1$. ∎

We are now in a position to prove the following theorem.

Theorem 5.4. *The values* 0 *and* 1 *are accumulation points of the singular values of* H_T.

Proof. First of all, 0 and 1 are both elements of the spectrum $\sigma(H_T^* H_T)$. For the value 0, this follows from $\mathrm{Ran}(H_T^* H_T) \subset \mathrm{Ran}(H_T^*) \neq \mathcal{L}^2(a_2, a_4)$. Moreover, since $\|H_T^* H_T\| = 1$ and $H_T^* H_T$ is self-adjoint, the spectral radius is equal to 1. Thus, $1 \in \sigma(H_T^* H_T)$.

The second step is to show that 0 and 1 are not eigenvalues of $H_T^* H_T$.

0 *is not an eigenvalue:* If $H_T^* H_T f = 0$, then $\|H_T f\|^2 = \langle f, H_T^* H_T f\rangle = 0$. Since $\mathrm{Ker}(H_T) = 0$, this implies $f = 0$. Thus, $\mathrm{Ker}(H_T^* H_T) = \{0\}$.

1 *is not an eigenvalue:* Suppose there exists a nonvanishing function $f \in \mathcal{L}^2(a_2, a_4)$, such that

$$-\chi_{[a_2,a_4]} H \chi_{[a_1,a_3]} H \chi_{[a_2,a_4]} f = f.$$

Then,

$$\begin{aligned}
\|H\chi_{[a_2,a_4]} f\|^2 = \|f\|^2 &= -\langle \chi_{[a_2,a_4]} H \chi_{[a_1,a_3]} H \chi_{[a_2,a_4]} f, f\rangle \\
&= \langle \chi_{[a_1,a_3]} H \chi_{[a_2,a_4]} f, H\chi_{[a_2,a_4]} f\rangle \\
&= \|\chi_{[a_1,a_3]} H \chi_{[a_2,a_4]} f\|^2.
\end{aligned}$$

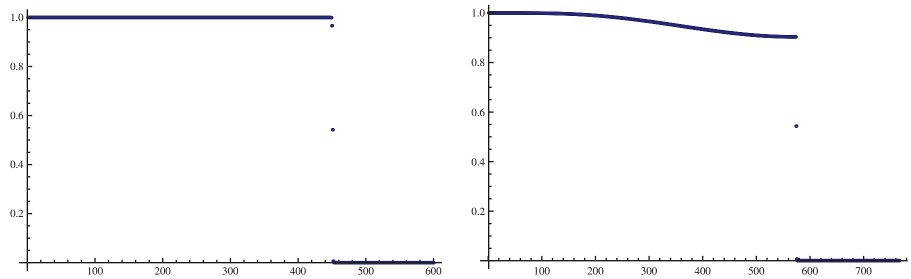
This implies that $H\chi_{[a_2,a_4]} f$ is identically zero outside $[a_1, a_3]$. By Lemma 5.1, this implies $f = \chi_{[a_2,a_4]} f = 0$, contradicting the assumption $f \not\equiv 0$. Therefore, 0 and 1 are accumulation points of the eigenvalues of $H_T^* H_T$ and consequently, of the singular values of H_T. ∎

Since the singular values of H_T also accumulate at a point other than zero, the operator H_T is not compact.

6. Numerical illustration. We want to illustrate the properties of the truncated Hilbert transform *with overlap* obtained above for a specific configuration. We choose $a_1 = 0$, $a_2 = 1.5$, $a_3 = 6$, and $a_4 = 7.5$. First, we consider two different discretizations of H_T and calculate the corresponding singular values. We choose the first discretization to be a uniform sampling with 601 partition points in each of the two intervals $[0, 6]$ and $[1.5, 7.5]$. Let vectors X and Y denote the partition points of $[0, 6]$ and $[1.5, 7.5]$, respectively. To overcome the singularity of the Hilbert kernel the vector X is shifted by half of the sample size. The ith components of the two vectors X and Y are given by $X_i = \frac{1}{100}(i + \frac{1}{2})$ and $Y_i = 1.5 + \frac{1}{100} i$; H_T is then discretized as $(H_T)_{i,j} = (1/\pi)(X_i - Y_j)$, $i, j = 0, \ldots, 600$. Figure 6.1(a) shows the singular values for the uniform discretization. We see a very sharp transition from 1 to 0.

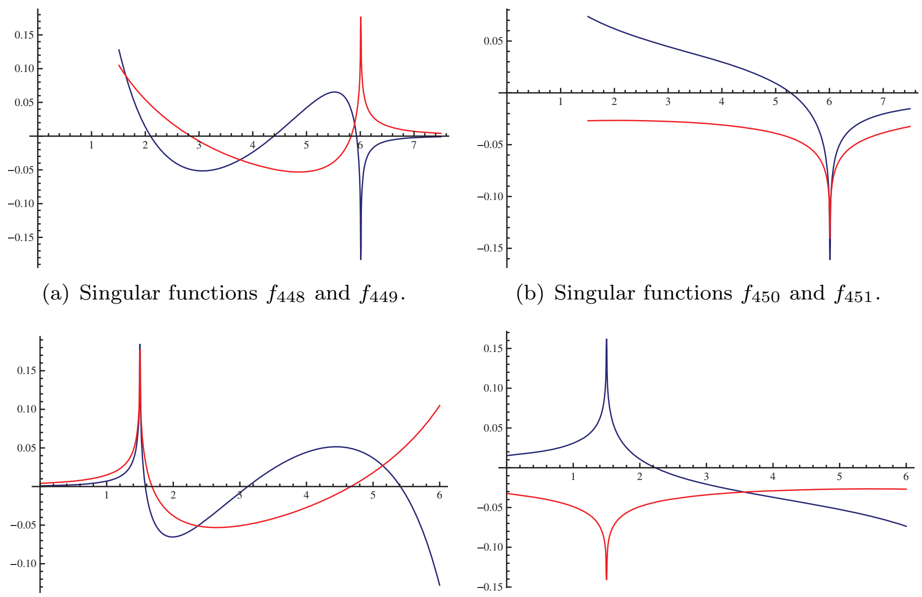
The second discretization uses orthonormal wavelets with two vanishing moments. Let ϕ denote the scaling function. For the discretization we define a finest scale $J = -7$. The scaling functions on $[1.5, 7.5]$ are taken to be $\phi_{-7,k}$ for integers $k = 192, \ldots, 957$, i.e., such that $\mathrm{supp}\, \phi_{-7,k} \subset [1.5, 7.5]$. On the interval $[0, 6]$ the scaling functions are shifted in the sense that we take them to be $\phi_{-7,\ell+\frac{1}{2}}$ for integers $\ell = 0, \ldots, 765$, i.e., such that $\mathrm{supp}\, \phi_{-7,\ell+\frac{1}{2}} \subset [0, 6]$. Figure 6.1(b) shows a plot of the singular values of this wavelet discretization of H_T. Although the transition is not as sharp as in 6.1(a), the singular values in both cases very clearly accumulate at 0 and 1.

Next, we consider the singular functions. Figure 6.2 shows the singular functions of the uniform discretization for singular values in the transmission region between 0 and 1. Figure 6.3 illustrates the behavior of singular functions for small singular values. As anticipated, they are bounded at the two endpoints and singular at the



(a) Uniform discretization of size 601×601. (b) Wavelet discretization of size 766×766.

FIG. 6.1. $a_1 = 0$, $a_2 = 1.5$, $a_3 = 6$, $a_4 = 7.5$. *Singular values of two discretizations of* H_T.



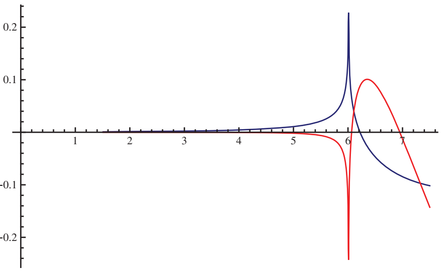
(a) Singular functions f_{448} and f_{449}. (b) Singular functions f_{450} and f_{451}.

(c) Singular functions g_{448} and g_{449}. (d) Singular functions g_{450} and g_{451}.

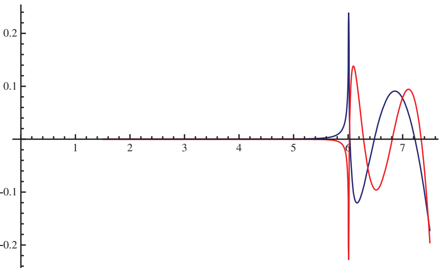
FIG. 6.2. *Consecutive singular functions for the uniform discretization with* 3, 2, 1, *and no zeros within the overlap region. The corresponding singular values are* $\sigma_{448} = 0.999963$, $\sigma_{449} = 0.998782$, $\sigma_{450} = 0.966192$, $\sigma_{451} = 0.542071$.

point of truncation. Figure 6.4 gives two examples of the close to linear behavior in a log-linear plot of the singular functions. In agreement with the theory in section 4, these plots confirm that the singularities are of logarithmic kind.

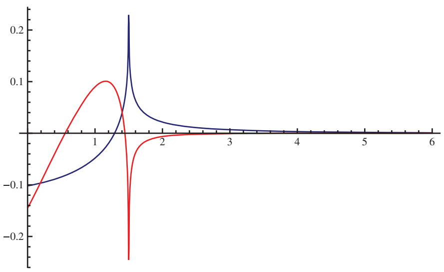
Based on the numerical experiments conducted, we make the following observations on the behavior of the singular functions and singular values. First, the singular functions in Figures 6.2 and 6.3 have the property that two functions with consecutive indices have their number of zeros differing by 1. Moreover, the zeros are located only inside one subinterval I_j. Furthermore, the plots show that singular functions with zeros within the overlap region correspond to significant singular values, whereas those which have zeros outside the overlap region correspond to small singular values. Finally, we remark that singular functions for small singular values are concentrated *outside* the ROI $I_2 = [a_2, a_3]$.



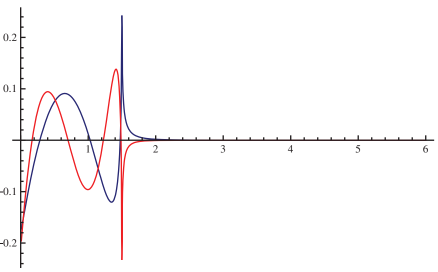
(a) Singular functions f_{452} and f_{453}.



(b) Singular functions f_{454} and f_{455}.

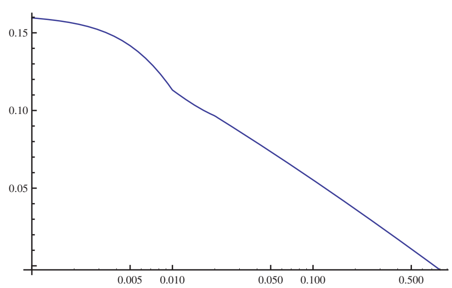


(c) Singular functions g_{452} and g_{453}.

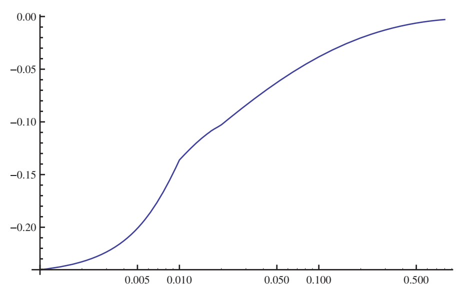


(d) Singular functions g_{454} and g_{455}.

FIG. 6.3. *Consecutive singular functions for the uniform discretization with* 1*,* 2*,* 3*, and* 4 *zeros outside the overlap region. The corresponding singular values are* $\sigma_{452} = 6.29189 \cdot 10^{-3}$*,* $\sigma_{453} = 2.83533 \cdot 10^{-5}$*,* $\sigma_{454} = 1.18274 \cdot 10^{-7}$*,* $\sigma_{455} = 4.83357 \cdot 10^{-10}$.



(a)



(b)

FIG. 6.4. *Log-linear plot, i.e., with a logarithmic x scale of the singular functions* g_{450} *(left) and* g_{453} *(right) on the interval* $[1.5, 2.3]$.

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