1995

Nonlinear Robust Control of a Series DC Motor Utilizing the Recursive Design Approach

Michael J. Burridge
University of Central Florida

NONLINEAR ROBUST CONTROL OF A SERIES DC MOTOR UTILIZING THE RECURSIVE DESIGN APPROACH

by

MICHAEL J. BURRIDGE
B.S.E.E., University of Central Florida, 1992

THESIS
Submitted in partial fulfillment of the requirements
for the degree of
Master of Science in Electrical Engineering
College of Engineering
University of Central Florida
Orlando, Florida

Fall Term
1995

ABSTRACT

In this thesis, the investigation of asymptotic stability of the series DC motor with unknown load-torque and unknown armature inductance is considered. The control technique of recursive, or backstepping, design is employed. Three cases are considered. In the first case, the system is assumed to be perfectly known. In the second case, the load torque is assumed to be unknown and a proportional-integral controller is developed to compensate for this unknown quantity. In the final case, it is assumed that two system parameters, load torque and armature inductance, are not known exactly, but vary from expected nominal values within a specified range. A robust control is designed to handle this case. The Lyapunov stability criterion is applied in all three cases to prove the stability of the system under the developed control. The results are then verified through the use of computer simulation.

ACKNOWLEDGEMENTS

I wish to thank my thesis advisor, Professor Zhihua Qu, for all of the help he has provided me over the past few years. Thanks, too, to committee members Dr. Michael Haralambous and Dr. Takis Kasparis for their time and participation. Finally, I would like to express my thanks to former UCF professor, Dr. Mario Sznaier, who first introduced me to control theory in the undergraduate linear controls class.

TABLE OF CONTENTS

LIST OF FIGURES

NOMENCLATURE

Symbol	Quantity	Units
	Letter Symbols	
B	friction	Nm/rad/s
G	gain	–
i	current	A
J	moment of inertia	$K_g m^2$
K_m	torque/back-emf	Nm/(Wb A)
k_0, k_1, k_2	gain constants	–
L	inductance	H
R	resistance	Ω
t	time	s
u	control	V
V	voltage	V
	Greek Letters	
ϵ	measure of error	–
ρ	bounding function	–
ϕ	flux	Wb
τ	torque	Nm
ω	angular velocity	rad/s
	Subscripts	
a	armature	–
f	field	–
p	parallel	–
L	load	–
R	robust	–
1,2	case	–

CHAPTER 1

INTRODUCTION

For this paper, a robust control law is developed for the series DC motor using the recursive design, or backstepping approach. Initially the system is examined under the assumption that all system parameters, variables, and states are known explicitly. This is, admittedly, an unrealistic view, but it is quite useful since it provides a baseline for further analysis and serves to confirm the validity of the design approach. This analysis is then followed by a more practical one in which it is assumed that certain variables associated with the motor are unknown. However, it is also assumed that these unknown variables have known bounding functions. A suitable robust control is then designed. As an additional point of interest, a control is developed utilizing the PI approach for comparison when it is assumed that the load torque is unknown.

Motors

Motors are devices which convert electrical energy into mechanical energy. In its most basic form, a motor consists of a loop of wire in a magnetic field to which current is applied. The torque acting on the current carrying loop causes it to rotate. Useful mechanical work can be done by attaching the rotating armature to some external devices. A DC motor is one in which the armature windings are on the rotor with current conducted from it by means of carbon brushes. The rotor of a DC machine is often referred to as the armature. The field winding is on the stator and is excited by direct current. DC motors are the most common choice when a controlled electrical drive operating over a wide speed range is specified [13]. They have excellent operational properties and control characteristics [13].

DC motors are classified as shunt, series, or compound according to the method of field connection. A discussion of these motors and their system models may be

found in variety of sources, including [8, 12, 17, 25]. In a series motor, the field circuit is connected in series with the armature circuit, while in the shunt motor, the two circuits are connected in parallel. One of the major differences between the two motors is that the shunt motor is wound with a large number of turns which makes the resistance quite high. The fewer number of turns found in a series motor minimizes the voltage drop across it. In some cases the two configurations are combined to produce the compound motor. For the no load condition, this motor behaves much like shunt motor. At higher loads, the characteristics more resemble the series motor. Elaborate circuits are required to control compound motors [16].

Due to the configuration of the series DC motor, the electromagnetic torque produced by this motor is proportional to the square of the current. The flux in a series DC motor depends on the armature current, and thus varies with the load. As a result, the series-connected DC motor produces more torque per Ampere of current than any other DC motor [6]. Therefore, the series motor is used in applications where high starting torque is required and an appreciable load torque exists under normal operation [11]. Such applications include locomotives, trolley buses, cranes, and hoists [27]. In fact, the series motor is the most widely used DC motor for electric traction applications [6]. The DC motor provides easily adjusted speed, high efficiency, and great flexibility [27]. On the other hand, the mechanical commutator which restricts the power and speed of the motor, increases the inertia and the axial length and requires periodical maintenance [13].

The DC machine is the most straightforward to analyze of all electric machines [11]. However, the mathematical model of the series DC motor is nonlinear. As with all physical systems, the modeling of the series DC motor for feedback control invariably involves a trade-off between the simplicity of the model and its accuracy in matching the behavior of the physical motor [14]. Usually, the model obtained is close to describing the actual system, but some error will always exist.

The motor examined is that which was presented in a paper by J. Chiasson [6]. The motor equations used in the paper may also be found in the text by Leonhard [13]. The analysis of the motor is broken into two cases based upon the motor's speed: above base speed and below base speed. When the motor is above base speed,

it operates in the field-weakening region. The field current is less than the armature current, and thus the flux is less than it would be with full armature current. The purpose of field weakening is to raise the speed at reduced loads [13]. It is a valuable means of increasing the speed in the low torque region. Below the base speed, field weakening is not present, and the field current equals the armature current.

Various control laws have been developed over the years for the series DC motor, although it would appear that few new results have been presented recently, especially in the area of nonlinear control. Still, there are several papers worth noting. In [19], the author uses feedback linearization to develop a control law valid for most operating points. In particular he finds that the series DC motor is input-to-state linearizable and input-output linearizable at all points except when the armature current is zero. In [2], an adaptive controller is developed for a DC drive operating under varying load conditions. The authors successfully develop a robust self-tuning controller with an adaptive integral-proportional structure.

In several papers, Chiasson has studied both the series DC motor and the shunt motor. For the shunt motor [5], he considers feedback linearization, generalized controller canonical forms, and input-output linearization. His results indicate that input-output linearization is the simplest and least restrictive method for developing a nonlinear control. The series motor is treated in [6], in which the nonlinear differential-geometric technique is employed. With the use of an observer to estimate the speed and load torque based on current measurements, his results are quite good when all other system parameters are assumed to be known.

The approach in this paper is to utilize the recursive design approach to design a nonlinear robust control law. Such an approach allows the design of a control law capable of handling significant variations in system parameters. Although only two parameters are assumed to be uncertain, the method could be easily extended to handle additional uncertain terms.

Robust Control

In this paper a robust control law for a series DC motor is developed using the robust control methodology. The robust control problem is to design a fixed control system which guarantees the design requirements in the presence of significant, bounded uncertainties. Robust control design is divided into two stages. First, one of the nonlinear design methods is employed to stabilize the nominal system, the known part of the dynamic system, and to achieve the prescribed performance. Second, a robust control law is developed which maintains the prescribed goal for all uncertainties under a given bound. A controller satisfying these requirements is said to be robust with respect to the prescribed class of uncertainties.

Robust design can utilize either frequency domain or time domain approaches. As to be discussed, use of the time domain approach leads naturally to the use of Lyapunov's direct method. This can also be seen from the procedure from which a robust control is designed. First, stability analysis is done with respect to the nominal system by setting all uncertainties in the system to zero. With the nominal system now perfectly known, its stability can be determined. If it stabilizable under conventional control, the existence of a Lyapunov function is guaranteed by the converse theorem. Second, a robust control is designed by using the same Lyapunov function for the uncertain system.

Robust control is currently a very popular topic in the literature and many recent articles may be found covering a wide variety of topics. Qu has investigated robust control for nonlinear systems which satisfy the Generalized Matching Conditions [23] and for nonlinear systems which do not satisfy the conditions [24]. Bonivento, et. al. [3] have investigated robust control and the problems associated with its synthesis as applied to uncertain dynamical systems. Wu and Willgoss [29] have also addressed the problem of robust stabilization for a class of uncertain nonlinear dynamical systems. In [7], Dote discusses some of the applications of robust control theory to motor control.

General background information on robust control is presented in [9]. A math-

ematical description of robust control is presented later in the section where robust control is applied to the series DC motor problem.

CHAPTER 2

MATHEMATICAL PRELIMINARY

We cover some of the basic mathematical tools required to develop nonlinear control laws. As a first point, the various definitions of stability for nonlinear systems are presented and contrasted to the definitions applicable to linear systems. We then present some information concerning matrices, their properties, and commonly applied functions. Finally, the powerful and versatile Lyapunov Theory is presented.

Stability Theory

The definitions, lemmas, and theorems presented here are adapted from class notes [21, 22] and a text on nonlinear systems by Khalil [10].

As stated, the goal of this paper is to design a control law to stabilize a particular system. The concept of stability, while seemingly straightforward, does require some explanation and analysis. In fact, stability theory plays a central role in systems theory and engineering and there are different kinds of stability problems that arise in the study of dynamical systems [10]. For linear systems, stability may be classified as either stable, unstable, or marginally stable. For nonlinear systems, however, these three terms alone are inadequate to describe the stability possibilities. More specific descriptions such as asymptotic stability or exponential stability are needed. Furthermore, these descriptions may apply either locally or globally.

The choice of stability utilized in a design depends upon the requirements of the design and the amount of information available on the system to be stabilized. For the sake of convenience, all definitions and theorems of stability may be stated for the case when the equilibrium point is at the origin. There is no loss of generality in doing this because any equilibrium point can be shifted to the origin through a change of variables. Stability definitions related to the simpler case of autonomous systems are considered first, and then we extend the concepts presented to the nonautonomous case.

Consider the autonomous system

$$\dot{x} = f(x) \tag{2.1}$$

where $f : D \rightarrow R^n$ is a locally Lipschitz[1] map from a domain $D \subset R^n$ into R^n. Suppose $\bar{x} \in D$ is an equilibrium point of (2.1); that is

$$f(\bar{x}) = 0.$$

Then, the following definition may be stated.

Definition 1:

The equilibrium point $x = 0$ of the system (2.1) is

- *stable*, if for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that

$$||x(0)|| < \delta \Rightarrow ||x(t)|| < \epsilon, \quad \forall\, t \geq 0$$

- *unstable*, if not stable
- *asymptotically stable*, if it is stable and δ can be chosen such that

$$||x(0)|| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0$$

□

The concept of a "hyper-ball" is sometimes used to describe stability graphically. The following equation shows the basic form:

$$||x(t)|| \leq \epsilon \Rightarrow x(t) \in B(0, \epsilon).$$

The equation $B(0, \epsilon)$ represents the "hyper-ball" with a center at 0 and radius ϵ. If δ can be chosen arbitrarily large for an arbitrary value of ϵ then the system is globally stable. Otherwise the system is locally stable.

This initial definition of stability may be further extended to provide additional classifications of stability. For example, when the origin is asymptotically stable, we are often interested in determining how far from the origin the trajectory can be

[1] The Lipschitz condition is used to show existence and uniqueness and may be stated as follows: $||f(t, x) - f(t, y)|| \leq L||x - y||$

perturbed and still converge to the origin as $t \to \infty$; that is, how large can ϵ and δ become? A discussion of nonlinear systems and their sensitivity to such perturbations may be found in [26, 28].

For now, consider the nonautonomous system,

$$\dot{x} = f(t, x) \tag{2.2}$$

where $f : [0, \infty) \times D \to R^n$ is piecewise continuous in t and locally Lipschitz in x on $[0, \infty) \times D$, and $D \subset R^n$ is a neighborhood of the origin $x = 0$. Then the origin is an equilibrium point for (2.2) at 0 if

$$f(t, 0) = 0, \quad \forall\, t \geq 0.$$

It should be noted that while the solution of an autonomous system depends only on $(t - t_0)$, the solution of a nonautonomous system may depend on both t and t_0. Therefore, the stability of the equilibrium point will, in general, be dependent on t_0. The origin $x = 0$ is a stable equilibrium point for (2.2) if for each $\epsilon > 0$ and any $t_0 \geq 0$ there is $\delta = \delta(\epsilon, t_0) > 0$ such that

$$||x(t_0)|| < \delta \Rightarrow ||x(t)|| < \epsilon,\ \forall\, t \geq t_0.$$

The constant δ is, in general, dependent upon the initial time t_0.

Before we introduce additional stability definitions for the nonautonomous case, we present several special scalar functions which will help us characterize and study the stability behavior of nonautonomous systems.

Definition 2:

A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$. □

Definition 3:

A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if for each fixed s the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r, and for each fixed r the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$. □

Lemma 1:

Let $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ be class $\mathcal{K}$ functions on $[0, a)$, $\alpha_3(\cdot)$ and $\alpha_4(\cdot)$ be class $\mathcal{K}_\infty$ functions, and $\beta(\cdot, \cdot)$ be a class $\mathcal{KL}$ function. Denote the inverse of $\alpha_i(\cdot)$ by $\alpha_i^{-1}(\cdot)$. Then,

- α_1^{-1} is defined on $[0, \alpha_1(a))$ and belongs to class $\mathcal{K}$.
- α_3^{-1} is defined on $[0, \infty)$ and belongs to class $\mathcal{K}_\infty$.
- $\alpha_1 \circ \alpha_2$ belongs to class $\mathcal{K}$.
- $\alpha_3 \circ \alpha_4$ belongs to class $\mathcal{K}_\infty$.
- $\sigma(r, s) = \alpha_1(\beta(\alpha_2(r), s))$ belongs to class $\mathcal{KL}$. □

Now we may present the definitions of stability for a nonautonomous system.

Definition 4:

The equilibrium point $x = 0$ of (2.2) is

• *uniformly stable*, if there exist a class $\mathcal{K}$ function $\alpha(\cdot)$ and a positive constant c, independent of t_0, such that

$$||x(t)|| \leq \alpha(||x(t_0)||), \quad \forall\, t \geq t_0 \geq 0, \quad \forall\, ||x(t_0)|| < c$$

• *uniformly asymptotically stable*, if there exist a class $\mathcal{KL}$ function $\beta(\cdot, \cdot)$ and a positive constant c, independent of t_0, such that

$$||x(t)|| \leq \beta(||x(t_0)||, t - t_0), \quad \forall\, t \geq t_0 \geq 0, \quad \forall\, ||x(t_0)|| < c \tag{2.3}$$

• *globally uniformly asymptotically stable*, if inequality (2.3) is satisfied for any initial state $x(t_0)$.

• *exponentially stable*, if inequality (2.3) is satisfied with

$$\beta(r, s) = kre^{-\gamma s}, \quad k > 0, \quad \gamma > 0$$

□

When a system contains a nonvanishing perturbation, the origin $x = 0$ may no longer be an equilibrium point of the perturbed system. In that case, we may need to use the concept of boundedness rather than that of stability.

Definition 5:

The solutions of $\dot{x} = f(t, x)$ are said to be *uniformly ultimately bounded* if there exist constants b and c, and for every $\alpha \in (0, c)$ there is a constant $T = T(\alpha)$ such that

$$||x(t_0)|| < \alpha \Rightarrow ||x(t)|| < b, \quad \forall\, t > t_0 + T.$$

They are said to be globally uniformly ultimately bounded if the equation holds for arbitrarily large α. □

There are several relations between the various forms of stability which are worth noting. For example, asymptotic stability implies stability which implies boundedness. Stability implies ultimate boundedness if $\epsilon = ||x(t = t_0)||$. Finally, exponential stability implies asymptotic stability.

Now we present the definition of the region of attraction.

Definition 6:

Let $\phi(t; x)$ be the solution of (2.1) that starts at initial state x at time $t = 0$. Then, the region of attraction is defined as the set of all points x such that

$$lim_{t \to \infty} \phi(t; x) = 0.$$

□

In practice, finding the exact region of attraction analytically might be difficult or even impossible. However, Lyapunov functions, to be discussed shortly, can be used to estimate the region of attraction. With the region of attraction now defined, we may present a stronger definition of stability.

Definition 7:

Let $x = 0$ be an equilibrium point for (2.1). Let $V : R^n \to R$ be a continuously differentiable function such that

$$\begin{aligned} V(0) = 0 \quad \text{and} \quad V(x) > 0, &\quad \forall\, x \neq 0 \\ ||x|| \to \infty \Rightarrow V(x) &\to \infty \\ \dot{V}(x) < 0, &\quad \forall\, x \neq 0 \end{aligned}$$

then $x = 0$ is globally asymptotically stable. □

Matrix Theory

It is often convenient to rewrite system equations in a matrix form. To that end, some basic information on matrices is presented here. A good reference for this material is found in the linear systems text by Chen [4].

Definition 8:

Let $\mathbf{A}$ be a linear operator that maps (C^n, C) into itself. Then a scalar λ in C is called an *eigenvalue* of $\mathbf{A}$ if there exists a nonzero vector $\mathbf{x}$ in C^n such that $\mathbf{Ax} = \lambda\mathbf{x}$. Any nonzero vector $\mathbf{x}$ satisfying $\mathbf{Ax} = \lambda\mathbf{x}$ is called an *eigenvector* of $\mathbf{A}$ associated with the eigenvalue λ. □

In order to find an eigenvalue of $\mathbf{A}$, we write $\mathbf{Ax} = \lambda\mathbf{x}$ as

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$$

where I is the unit matrix of order n. The equation has a nontrivial solution if and only if $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. It follows that a scalar λ is an eigenvalue of $\mathbf{A}$ if and only if it is a solution of $\Delta(\lambda) \triangleq \det\,(\lambda\mathbf{I} - \mathbf{A}) = 0$. $\Delta(\lambda)$ is a polynomial of degree n in λ and is called the *characteristic polynomial* of $\mathbf{A}$. In other words, the eigenvalues of $\mathbf{A}$ are the roots of the characteristic polynomial of $\mathbf{A}$.

As an example, consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}.$$

The eigenvalues may be found as

$$\det\,(\lambda\mathbf{I} - \mathbf{A}) = \det \begin{bmatrix} \lambda - 1 & 1 \\ -2 & \lambda + 1 \end{bmatrix} = \lambda^2 + 1.$$

The eigenvalues are imaginary, namely $\lambda = \pm i$.

Another important concept in matrix theory is the norm of a matrix. The concept of norm can be extended to linear operators that map (C^n, C) into itself, or equivalently, to square matrices with complex coefficients. The norm of $\mathbf{A}$ is defined in terms of the norm of $\mathbf{x}$. For example, if $||\mathbf{x}||_2$ is used, then

$$||\mathbf{A}||_2 = (\lambda_{max}(\mathbf{A}^*\mathbf{A}))^{1/2}$$

where $\mathbf{A}^*$ is the complex conjugate transpose of $\mathbf{A}$ and $\lambda_{max}(\mathbf{A}^*\mathbf{A})$ denotes the largest eigenvalue of $\mathbf{A}^*\mathbf{A}$.

Lyapunov Theory

The analytical method used in nonlinear robust control is the direct method of Lyapunov. Of the different analysis and design approaches for robust control, the direct method of Lyapunov is of central importance. The reasons are twofold. First, time varying or nonlinear uncertainties can be easily bounded in the time domain. Second, time varying and nonlinear uncertain systems can be treated by Lyapunov's direct method.

Lyapunov was the Russian mathematician and engineer who first developed the approach which now bears his name. One of the most important aspects of Lyapunov's approach is that the stability of a system may be determined without explicitly finding the solution of the system equations. This is achieved through the use of the Lyapunov function, which often takes the form of an energy function. Finding such a function, however, is usually quite difficult.

There is no systematic method for finding a Lyapunov function. In many cases, finding an appropriate function is a matter of trial and error. One helpful approach is to search backward for a Lyapunov function. That is, the derivative of the Lyapunov function is chosen first, and then the function itself is chosen to achieve the desired dissipative property. The function under consideration is referred to as a Lyapunov function candidate if for a given system the time derivative of the candidate along the trajectory of the system has a certain type of dissipative property.

There are several important features of Lyapunov's method. One is that Lyapunov stability implies uniform boundedness. Another is that the Lyapunov theorem's conditions are only sufficient, not necessary. In fact, Lyapunov's method sometimes provides very conservative stability conditions [15]. Thus, failure of a Lyapunov function candidate to satisfy the conditions for stability or asymptotic stability does not mean that the equilibrium point chosen for study is unstable. It only means that the chosen function can not be used to establish the stability property.

Lyapunov's Theorem for Nonautonomous systems may now be stated. The proofs of Lyapunov's various methods are presented in Khalil [10] and elsewhere.

Theorem 1:

Let $x = 0$ be an equilibrium point for (2.2) and $D = \{x \in R^n \mid ||x|| < r\}$. Let $V : [0, \infty) \times D \to R$ be a continuously differentiable function such that

$$\alpha_1(||x||) \leq V(t,x) \leq \alpha_2(||x||) \tag{2.4}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -\alpha_3(||x||) \tag{2.5}$$

$\forall\, t \geq 0$, $\forall\, x \in D$, where $\alpha_1(\cdot), \alpha_2(\cdot)$, and $\alpha_3(\cdot)$ are class $\mathcal{K}$ functions defined on $[0, r)$. Then $x = 0$ is uniformly asymptotically stable. □

A function $V(t,x)$ satisfying the left inequality of (2.4) is said to be positive definite. A function satisfying the right inequality of (2.4) is said to be decresent. A function $V(t,x)$ is said to be negative definite if $-V(t,x)$ is positive definite. With the use of these terms, we may state that Lyapunov's theorem proves the origin is uniformly asymptotically stable if there is a continuously differentiable, positive definite, decresent function $V(t,x)$ whose derivative along the trajectories of the system is negative definite. Lyapunov's theorem may be expanded to two global versions.

Corollary 1:

Suppose that all the assumptions of the theorem are satisfied globally (for all $x \in R^n$) and $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ belong to class $\mathcal{K}_\infty$. Then $x = 0$ is globally uniformly asymptotically stable. □

Corollary 2:

Suppose that all the assumptions of the theorem are satisfied with $\alpha_i(r) = k_i r^c$, for some positive constants k_i and c. Then $x = 0$ is exponentially stable. Moreover, if the assumptions hold globally, then $x = 0$ is globally exponentially stable. □

As mentioned, Lyapunov's theorem can be described using sign definiteness. A class of functions for which sign definiteness can be easily determined is the class of functions of the quadratic form

$$V(x) = x^T P x = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} x_i x_j$$

where P is a real symmetric matrix. In this case, $V(x)$ is positive definite (positive semidefinite) if and only if all the eigenvalues of P are positive (nonnegative), which is true if and only if all the leading principle minors of P are positive (nonnegative).

One application of the properties of quadratic functions is the so called Lyapunov equation. Consider the linear time-invariant system

$$\dot{x} = Ax.$$

The derivative of V along the trajectories of the linear system is given by

$$\begin{aligned} \dot{V}(x) &= x^T P\dot{x} + \dot{x}^T Px \\ &= x^T(PA + A^T P)x \\ &= -x^T Qx \end{aligned}$$

where Q is a symmetric matrix defined by

$$PA + A^T P = -Q. \tag{2.6}$$

If Q is positive definite, we can conclude by the Lyapunov theorem that the origin is asymptotically stable, that is $Re\lambda_i < 0$, for all eigenvalues of A. Stability in terms of the solution of the Lyapunov equation may be stated in the following theorem.

Theorem 2:

A matrix A is a stability matrix, that is, $Re\lambda_i < 0$ for all eigenvalues of A, if and only if for any given positive definite symmetric matrix Q there exists a positive definite symmetric matrix P that satisfies the Lyapunov equation (2.6). Moreover, if A is a stability matrix, then P is the unique solution of (2.6). $\square$

Lyapunov's theorem may be restated in an inverse form known as the converse theorem. The converse theorem takes two forms, one for when the origin is an exponentially stable equilibrium and one when the origin is uniformly asymptotically stable.

Theorem 3:

Let $x = 0$ be an equilibrium point for the nonlinear system

$$\dot{x} = f(t, x)$$

where $f\,:\,[0,\infty) \times D \rightarrow R^n$ is continuously differentiable, $D = \{x \in R^n \mid ||x||_2 < r\}$, and the Jacobian matrix $[\partial f/\partial x]$ is bounded on D, uniformly in t. Let k, γ, and r_0

be positive constants with $r_0 < r/k$. Let $D_0 = \{x \in R^n \mid ||x||_2 < r_0\}$. Assume that the trajectories of the system satisfy

$$||x(t)||_2 \leq k||x(t_0)||_2 e^{-\gamma(t-t_0)},\ \forall\, x(t_0) \in D_0,\ \forall\, t \geq t_0 \geq 0.$$

Then, there is a function $V : [0, \infty) \times D_0 \to R$ that satisfies the inequalities:

$$\begin{aligned} c_1||x||_2^2 \leq V(t,x) &\leq c_2||x||_2^2 \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) &\leq -c_3||x||_2^2 \\ \left|\left|\frac{\partial V}{\partial x}\right|\right|_2 &\leq c_4||x||_2 \end{aligned}$$

for some positive constants c_1, c_2, c_3, and c_4. Moreover, if $r = \infty$ and the origin is globally exponentially stable, then $V(t,x)$ is defined and satisfies the above inequalities on R^n. Furthermore, if the system is autonomous, V can be chosen independent of t. □

Theorem 4:

Let $x = 0$ be an equilibrium point for the nonlinear system

$$\dot{x} = f(t,x)$$

where $f\ :\ [0,\infty) \times D \to R^n$ is continuously differentiable, $D = \{x \in R^n \mid ||x|| < r\}$, and the Jacobian matrix $[\partial f/\partial x]$ is bounded on D, uniformly in t. Let $\beta(\cdot,\cdot)$ be a class $\mathcal{KL}$ function and r_0 be a positive constant such that $\beta(r_0, 0) < r$. Let $D_0 = \{x \in R^n \mid ||x|| < r_0\}$. Assume that the trajectory of the system satisfies

$$||x(t)|| \leq \beta(||x(t_0)||, t - t_0),\ \forall\, x(t_0) \in D_0,\ \forall\, t \geq t_0 \geq 0.$$

Then, there is a function $V : [0, \infty) \times D_0 \to R$ that satisfies the inequalities:

$$\begin{aligned} \alpha_1(||x||) \leq V(t,x) &\leq \alpha_2(||x||) \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) &\leq \alpha_3(||x||) \\ \left|\left|\frac{\partial V}{\partial x}\right|\right| &\leq \alpha_4(||x||) \end{aligned}$$

where $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\alpha_3(\cdot)$, and $\alpha_4(\cdot)$ and class $\mathcal{K}$ functions defined on $[0, r_0]$. If the system is autonomous, V can be chosen independent of t. □

These theorems prove that, if the origin is asymptotically or exponentially stable, then there exists a Lyapunov function which satisfies the conditions of the Lyapunov theorem. Although these theorems do not help in the practical search for an auxiliary function, they at least provide the knowledge that a function exists. The theorems are also helpful in using Lyapunov theory to draw conceptual conclusions about the behavior of dynamical systems.

Since the Lyapunov equation will be used later in the control design of the series DC motor, a simple example of its use is presented here.

Example 1:

Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

where, due to symmetry, $p_{12} = p_{21}$. The Lyapunov equation (2.6) can be written as

$$PA + A^T P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}.$$

So,

$$\begin{aligned} 2p_{12} &= -1 \\ -p_{11} - p_{12} + p_{22} &= 0 \\ -2p_{12} - 2p_{22} &= -1 \end{aligned}$$

or

$$\begin{bmatrix} 0 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \\ p_{22} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}.$$

The unique solution of this equation is given by

$$\begin{bmatrix} p_{11} \\ p_{12} \\ p_{22} \end{bmatrix} = \begin{bmatrix} 1.5 \\ -0.5 \\ 1.0 \end{bmatrix}.$$

The matrix

$$P = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1.0 \end{bmatrix}$$

is positive definite since its leading principal minors (1.5 and 1.25) are positive. Hence, all eigenvalues of A are in the open left-half complex plane. □

As another example, the use of the Lyapunov theorem in proving stability is given below.

Example 2:

The linear time-varying (i.e. nonautonomous) system

$$\dot{x} = A(t)x \tag{2.7}$$

has an equilibrium point at $x = 0$. Let $A(t)$ be piecewise continuous for all $t \geq 0$. Suppose there is a piecewise continuously differentiable, symmetric, bounded, positive definite matrix $P(t)$, that is,

$$0 < c_1 I \leq P(t) \leq c_2 I, \quad \forall\, t \geq 0$$

which satisfies the matrix differential equation

$$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + Q(t)$$

where $Q(t)$ is continuous, symmetric, and positive definite; that is

$$Q(t) \geq c_3 I > 0, \quad \forall\, t \geq 0.$$

Notice the slightly different form of the Lyapunov equation for the nonautonomous case.

Consider a Lyapunov function candidate

$$V(t,x) = x^T P(t) x.$$

The function $V(t,x)$ is positive definite and decresent since

$$c_1 ||x||_2^2 \leq V(t,x) \leq c_2 ||x||_2^2.$$

Moreover, it is radially unbounded[2] since the function $c_1||x||_2^2$ belongs to class $\mathcal{K}_\infty$.

The derivative of $V(t,x)$ along the trajectories of the system (2.7) is given by

$$
\begin{aligned}
V(t,x) &= x^T\dot{P}(t)x + x^TP(t)\dot{x} + \dot{x}^TP(t)x \\
&= x^T[\dot{P}(t) + P(t)A(t) + A^T(t)P(t)]x \\
&= -x^TQ(t)x \leq -c_3||x||_2^2.
\end{aligned}
$$

Hence, $\dot{V}(t,x)$ is negative definite. All the assumptions of theorem 1 are satisfied globally with $\alpha_i = c_ir^2$. Therefore, the origin is globally exponentially stable. □

---

[2] $V(x) \to \infty$ as $||x|| \to \infty$

CHAPTER 3

DESIGN OF CONTROL LAW FOR SERIES DC MOTOR

Recursive Design

The design methodology we chose to be applied to the problem of controlling a series DC motor is the backstepping approach. This approach, developed in the sixties, works systematically for multiple-integrator systems. Extension of this method to nonlinear control, adaptive control, and robust control has only been accomplished in the past several years [22]. Mathematically, the design procedure can be generalized and applied to nonlinear systems because it basically forms a sequence of state transformations, that is, a recursive mapping [22]. A recursive nonlinear mapping involving norms and differentiation operators is required for robust control design and is referred to as recursive design.

For the purposes of increased readability, many of the intervening steps in the derivation of equations have been omitted from the body of the thesis. Instead, these steps are included in separate appendices located at the end of the thesis.

Background

Recursive design may be applied to cascaded systems. A system in the following form:

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1,t) + \Delta f_1(x_1,v_1,t) + g_1(x_1,x_2,v_1,t) \\
\dot{x}_i &= f_i(x_1,...,x_i,t) + \Delta f_i(x_1,...,x_i,v_i,t) + g_i(x_1,...,x_i,x_{i+1},v_i,t) \\
\dot{x}_m &= f_m(x_1,...,x_m,t) + \Delta f_m(x_1,...,x_m,v_m,t) + g_m(x_1,...,x_m,u,v_m,t)
\end{aligned}
$$

where u is the control and v_i are the time-varying uncertainties, is said to be cascaded if these conditions are true:

$$
\begin{aligned}
f_i(x_1,...,x_i,t) &= f_i(x_i,t) \\
\Delta f_i(x_1,...,x_i,v_i,t) &= \Delta f_i(x_i,v_i,t) \\
g_i(x_1,...,x_i,x_{i+1},v_i,t) &= g_i(x_i,x_{i+1},v_i,t).
\end{aligned}
$$

A cascaded system consists of a sequence of cascaded nonlinear uncertain subsystems. With such a system, the recursive approach may be used to design a robust control. The recursive approach can also be applied to feedback linearizable systems to design adaptive control and robust control. As will be discussed later, cascaded systems are actually a special case of the generalized matching conditions.

At each step of the recursive approach, the design contains a change of coordinates and the construction of a fictitious robust control law. Based on the structure, the state variable x_1 is the system output. The variable x_1^d represents the desired output trajectory of the system. The objective in every step is to define a new state $z_i = x_i - x_i^d$, choose the bounding function ρ and the Lyapunov function V_i, justify the choice of x_{i+1}^d and derive the expression for $\dot{V_i}$. A fictitious control is designed such that it is differentiable.

A simple example of the backstepping design approach is presented below.

Example 3:

Consider the second order system,

$$\dot{x_1} = x_2 \qquad \dot{x_2} = u.$$

This system consists of two cascaded integrators. We can see that u can control x_2 to anywhere. We also see that if x_2 were a control variable, then we could control x_1 to anywhere. Such a control, for example $x_2 = -x_1$, also written as $x_2^d = -x_1$, is called a fictitious control. Let us rewrite the first equation as $\dot{x_1} = -x_1 + (x_2 + x_1)$. If the term $(x_2 + x_1)$ can be made to go to zero, then $\dot{x_1} = -x_1$ is stable.

This can be done by introducing the new variable z_2 as $z_2 = x_2 + x_1$. Then $\dot{z_2} = \dot{x_2} + \dot{x_1} = u + x_2$. If we choose the Lyapunov function $V = x_1^2 + z_2^2$ and the control $u = -x_2 - (x_2 + x_1)$, then we can show that the origin of the system is globally asymptotically stable. □

Application of Recursive Design to the Problem

We begin our analysis of the design of a control law for the series DC motor by assuming that all variables and quantities are known. We use the recursive design approach discussed above with the exception that, since all values are assumed to be

known, we do not need to develop a bounding function, ρ. Our examination of the motor is split into two cases, due to the fact the equations describing the system are slightly different when the motor operates above base speed than when it operates below base speed.

We follow the pattern used by Chiasson in his paper [6], that is the first case examined is the motor above base speed and the second is the motor below base speed. The equations as presented by Chiasson for the first case (i.e. the motor above base speed with $R_p < \infty$) are:

$$\begin{aligned} L_a di_a/dt &= V - R_a i_a - R_p(i_a - i_f) - K_m \phi_f(i_f)\omega && (3.1) \\ d\phi_f/dt &= -R_f i_f + R_p(i_a - i_f) && (3.2) \\ J d\omega/dt &= K_m \phi_f(i_f) i_a - B\omega - \tau_L. && (3.3) \end{aligned}$$

As mentioned, these equations are valid for the series-wound DC motor in the *field-weakening* region. That is, at high speeds (above the so-called *base speed*) the switch is closed ($R_p < \infty$) so that the field current i_f is less than the armature current i_a.

Before applying the recursive design approach, the system must be transformed into the cascaded form. This may be accomplished by making the following variable transformation:

$$\lambda = \phi_f(i_f) L_a i_a. \qquad (3.4)$$

Taking the derivative yields:

$$\begin{aligned} d\lambda/dt &= \dot{\lambda} = \dot{\phi_f}(i_f) L_a i_a + \frac{d}{dt}[L_a i_a]\phi_f(i_f) \\ &= -L_a R_f i_a i_f + R_p L_a(i_a^2 - i_f i_a) + V\phi_f(i_f) - R_a i_a \phi_f(i_f) \\ &\quad -R_p(i_a - i_f)\phi_f(i_f) - K_m \phi_f^2(i_f)\omega. \end{aligned}$$

The system equations are then:

$$\begin{aligned} J d\omega/dt &= K_m \phi_f(i_f) i_a - B\omega - \tau_L && (3.5) \\ d\lambda/dt &= -L_a R_f i_a i_f + R_p L_a(i_a^2 - i_f i_a) + V\phi_f(i_f) \\ &\quad -\frac{R_a + R_p}{L_a}\lambda + R_p i_f \phi_f(i_f) - K_m \phi_f^2(i_f)\omega. && (3.6) \end{aligned}$$

Let $x_1 = \omega$, $x_2 = \lambda$, and $u = V$. Then,

$$\begin{aligned} \dot{x_1} &= \frac{K_m}{JL_a}x_2 - \frac{B}{J}x_1 - \frac{\tau_L}{J} && (3.7) \\ \dot{x_2} &= -L_aR_fi_ai_f + R_pL_a(i_a^2 - i_fi_a) - K_m\phi_f^2(i_f)x_1 \\ &\quad +R_p\phi_f(i_f)i_f - \frac{R_a+R_p}{L_a}x_2 + \phi_f(i_f)u. && (3.8) \end{aligned}$$

The system is now in the proper cascaded form required for recursive design.

Suppose we wish to control x_1 to the speed ω_0. We may introduce the new variable:

$$z_1 = x_1 - x_1^d \tag{3.9}$$

with $x_1^d = \omega_0$, $\dot{z_1} = \dot{x_1}$, and $x_1 = z_1 + x_1^d$. The first system equation in the new variable is then:

$$\dot{z_1} = -\frac{B}{J}z_1 - \frac{B}{J}\omega_0 + \frac{K_m}{JL_a}x_2 - \frac{\tau_L}{J}.$$

We now wish to select the appropriate value for x_2^d such that the following terms go to zero:

$$-\frac{B}{J}\omega_0 + \frac{K_m}{JL_a}x_2^d - \frac{\tau_L}{J} = 0,$$

which yields

$$x_2^d = \frac{L_a}{K_m}(\tau_L + B\omega_0). \tag{3.10}$$

Introduce the new variable z_2:

$$z_2 = x_2 - x_2^d. \tag{3.11}$$

Thus, $\dot{z_2} = \dot{x_2}$ and $x_2 = z_2 + x_2^d$. The second system equation in the new variable is then:

$$\begin{aligned} \dot{z_2} &= -K_m\phi_f^2(i_f)z_1 - K_m\phi_f^2(i_f)\omega_0 - \frac{R_a+R_p}{L_a}z_2 \\ &\quad -\frac{R_a+R_p}{L_a}x_2^d - L_aR_fi_ai_f + R_pL_a(i_a^2 - i_fi_a) \\ &\quad +R_p\phi_f(i_f)i_f + \phi_f(i_f)u. && (3.12) \end{aligned}$$

Replacing x_2 in the the first system equation and rewriting yields:

$$\dot{z_1} = -\frac{B}{J}z_1 - \frac{B}{J}\omega_0 + \frac{K_m}{JL_a}x_2^d + \frac{K_m}{JL_a}z_2 - \frac{\tau_L}{J}. \tag{3.13}$$

To design the control, choose the following Lyapunov function:

$$V(z) = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2. \tag{3.14}$$

Then,

$$\begin{aligned}
\dot{V}(z) &= z_1\dot{z}_1 + z_2\dot{z}_2 \\
\dot{V}(z) &= -\frac{B}{J}z_1^2 - \frac{B}{J}\omega_0 z_1 + \frac{K_m}{JL_a}z_1 z_2 + \frac{K_m}{JL_a}x_2^d z_1 - \frac{\tau_L}{J}z_1 \\
&\quad -K_m\phi_f^2(i_f)z_1 z_2 - K_m\phi_f^2(i_f)\omega_0 z_2 - \frac{R_a + R_p}{L_a}z_2^2 \\
&\quad -\frac{R_a + R_p}{L_a}x_2^d z_2 - L_a R_f i_a i_f z_2 + R_p L_a(i_a^2 - i_f i_a)z_2 \\
&\quad +R_p\phi_f(i_f)i_f z_2 + \phi_f(i_f)z_2 u
\end{aligned}$$

Grouping terms and substituting for the value of x_2^d yields,

$$\begin{aligned}
&= -\frac{B}{J}z_1^2 - \frac{R_a + R_p}{L_a}z_2^2 + z_2[-K_m\phi_f^2(i_f)\omega_0 \\
&\quad -\frac{R_a + R_p}{K_m}(\tau_L + B\omega_0) - L_a R_f i_a i_f + R_p L_a(i_a^2 - i_f i_a) \\
&\quad +R_p\phi_f(i_f)i_f + \frac{K_m}{JL_a}z_1 - K_m\phi_f^2(i_f)z_1 + \phi_f(i_f)u].
\end{aligned}$$

To cancel terms, choose the following control:

$$\begin{aligned}
u &= \frac{1}{\phi_f(i_f)}[K_m\phi_f^2(i_f)\omega_0 + \frac{R_a + R_p}{K_m}(\tau_L + B\omega_0) + L_a R_f i_a i_f \\
&\quad -R_p L_a(i_a^2 - i_f i_a) - R_p\phi_f(i_f)i_f - \frac{K_m}{JL_a}z_1 + K_m\phi_f^2(i_f)z_1].
\end{aligned}$$

Such a choice gives:

$$\begin{aligned}
\dot{V}(z) &= -\frac{B}{J}z_1^2 - \frac{R_a + R_p}{L_a}z_2^2 \\
&\leq 0
\end{aligned}$$

which shows that the derivative of the Lyapunov equation is negative definite; thus the system is globally uniformly asymptotically stable.

Rewriting u for $z_1 = x_1 - \omega_0 = \omega - \omega_0$ yields:

$$\begin{aligned}
u &= \frac{1}{\phi_f(i_f)}\Big[\frac{R_a + R_p}{K_m}(\tau_L + B\omega_0) + L_a R_f i_a i_f \\
&\quad -R_p L_a(i_a^2 - i_f i_a) - R_p\phi_f(i_f)i_f \\
&\quad -\frac{K_m}{JL_a}(\omega - \omega_0) + K_m\phi_f^2(i_f)\omega].
\end{aligned} \tag{3.15}$$

This is the final form of the control for the motor when it operates above base speed. We now turn our attention to the second case, when the motor is operating below base speed.

In this case the switch is open, i.e. $R_p \to \infty$, and field weakening is not present. Therefore $i_f = i_a = i$, and the equations are

$$L_a di/dt = V - R_a i - K_m \phi_f(i)\omega \quad (3.16)$$

$$d\phi_f/dt = -R_f i \quad (3.17)$$

$$J d\omega/dt = K_m \phi_f(i) i - B\omega - \tau_L. \quad (3.18)$$

We again make the following variable substitution:

$$\lambda = \phi_f(i) L_a i.$$

Taking the derivative yields:

$$\begin{aligned} d\lambda/dt &= \frac{\partial}{\partial i}[\phi_f(i) i] L_a \frac{di}{dt} \\ &= F(-R_a i - K_m \phi_f(i)\omega + V), \end{aligned}$$

where

$$F(i, \phi_f(i), \partial\phi_f(i)/\partial i) = \frac{\partial \phi_f(i)}{\partial i} i + \phi_f(i). \quad (3.19)$$

The system equations are then:

$$J d\omega/dt = K_m \phi_f(i) i - B\omega - \tau_L \quad (3.20)$$

$$d\lambda/dt = F[-R_a i - K_m \phi_f(i)\omega + V]. \quad (3.21)$$

Let $x_1 = \omega$, $x_2 = \lambda$, and $u = V$. Then,

$$\dot{x_1} = \frac{K_m}{J L_a} x_2 - \frac{B}{J} x_1 - \frac{\tau_L}{J} \quad (3.22)$$

$$\dot{x_2} = F[-R_a i - K_m \phi_f(i) x_1 + u]. \quad (3.23)$$

This system is now in the proper cascaded form for recursive design.

As in the first case, we wish to control x_1 to the speed ω_0. We introduce z_1 and z_2 as before. The first system equation in the new variable is then:

$$\dot{z_1} = -\frac{B}{J} z_1 - \frac{B}{J}\omega_0 + \frac{K_m}{J L_a} x_2 - \frac{\tau_L}{J}.$$

The term x_2^d is the same as was found before, namely

$$x_2^d = \frac{L_a}{K_m}(\tau_L + B\omega_0).$$

The second system equation in the variable z_2 is then:

$$\dot{z}_2 = F[-R_a i - K_m \phi_f(i) z_1 - K_m \phi_f(i)\omega_0 + u]. \tag{3.24}$$

Replacing x_2 in the the first system equation and rewriting yields:

$$\dot{z}_1 = -\frac{B}{J}z_1 - \frac{B}{J}\omega_0 + \frac{K_m}{JL_a}x_2^d + \frac{K_m}{JL_a}z_2 - \frac{\tau_L}{J}. \tag{3.25}$$

To design the control, we might choose the same Lyapunov function as for the case of the motor above base speed, namely,

$$V(z) = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2.$$

However, after attempting several simulations through a trial and error approach, it was discovered that a better choice of Lyapunov function is

$$V(z) = \frac{1}{2}z_1^2 + \frac{L_a}{2}z_2^2. \tag{3.26}$$

Then,

$$\begin{aligned}
\dot{V}(z) &= z_1\dot{z}_1 + L_a z_2 \dot{z}_2 \\
\dot{V}(z) &= -\frac{B}{J}z_1^2 - \frac{B}{J}\omega_0 z_1 + \frac{K_m}{JL_a}z_1 z_2 + \frac{K_m}{JL_a}x_2^d z_1 \\
&\quad -\frac{\tau_L}{J}z_1 - FL_aR_a i z_2 - FL_a K_m \phi_f(i) z_1 z_2 \\
&\quad -FL_a K_m \phi_f(i)\omega_0 z_2 + FL_a z_2 u.
\end{aligned}$$

Grouping terms and substituting for x_2^d yields,

$$\begin{aligned}
\dot{V}(z) &= -\frac{B}{J}z_1^2 - \frac{B}{J}\omega_0 z_1 - \frac{\tau_L}{J}z_1 + \frac{B}{J}\omega_0 z_1 \\
&\quad +\frac{\tau_L}{J}z_1 + z_2[\frac{K_m}{JL_a}z_1 - FL_a K_m \phi_f(i) z_1 \\
&\quad -FL_a R_a i - FL_a K_m \phi_f(i)\omega_0 + FL_a u].
\end{aligned}$$

Rewrite the equation by factoring out the coefficients F and L_a :

$$\begin{aligned}\dot{V}(z) &= -\frac{B}{J}z_1^2 + FL_a z_2[\frac{K_m}{FJL_a^2}z_1 - K_m\phi_f(i)z_1 \\ &\quad -R_a i - K_m\phi_f(i)\omega_0 + u].\end{aligned}$$

To cancel terms, choose the following control:

$$u = -\frac{K_m}{FJL_a^2}z_1 + K_m\phi_f(i)z_1 + R_a i + K_m\phi_f(i)\omega_0 - \frac{G}{FL_a}z_2.$$

We introduce the term $\frac{G}{FL_a}z_2$ in order to generate a negative definite term in the second variable, z_2, for the derivative of the Lyapunov function. Without such a term, the control may not be characterized as being asymptotically stable. We choose to let $G = G_1\frac{B}{J}$ where G_1 is a gain we may vary in the simulation to produce the best results. Therefore, such a choice gives:

$$\begin{aligned}\dot{V}(z) &= -\frac{B}{J}z_1^2 - Gz_2^2 \\ &\leq 0.\end{aligned}$$

Rewriting u for $z_1 = x_1 - \omega_0 = \omega - \omega_0$, $z_2 = x_2 - x_2^d$, and $G = G_1\frac{B}{J}$, yields:

$$u = -\frac{K_m}{FJL_a}(\omega - \omega_0) + K_m\phi_f(i)\omega + R_a i - \frac{BG_1}{FJL_a}(\phi_f(i)L_a i - x_2^d). \tag{3.27}$$

The final control law is now completely known.

Simulation

The results from both case 1 and case 2 were combined to simulate the DC motor under the control law when all quantities are assumed to be known. The control law changes as the motor moves from below base speed to above base speed. Base speed was chosen as $\omega_{base} = 200.0$ rad/s.

Below base speed (case 2), the control law is

$$u = \frac{K_m}{FJL_a}(\omega - \omega_0) + K_m\phi_f(i)\omega + R_a i - \frac{BG_1}{FJL_a}(\phi_f(i)L_a i - x_2^d)$$

with

$$F = \frac{\partial\phi_f(i)}{\partial i}i + \phi_f(i) \quad \text{and} \quad x_2^d = \frac{L_a}{K_m}(\tau_L + B\omega_0)$$

and above base speed (case 1), the control law is

$$\begin{aligned} u &= \frac{1}{\phi_f(i_f)}\left[\frac{R_a+R_p}{K_m}(\tau_L+B\omega_0)+L_aR_fi_ai_f-R_pL_a(i_a^2-i_fi_a)\right. \\ &\quad \left. -R_p\phi_f(i_f)i_f+\frac{K_m}{JL_a}(\omega_0-\omega)+K_m\phi_f^2(i_f)\omega\right]. \end{aligned}$$

The load torque, τ_L, was given in Chiasson [6] as

$$\tau_L = \begin{cases} 0 & Nm \quad 0 \le t \le 5 \\ 1250(t-5)/5 & Nm \quad 5 \le t \le 10 \\ 1250 & Nm \quad 10 \le t \end{cases}.$$

The parameters related to this motor are the armature inductance (L_a), the resistance of the field windings (R_f), the parallel resistance of field weakening (R_p), the resistance of the armature windings (R_a), the viscous friction (B), the torque/back-emf (K_m), and the moment of inertia (J). The values of these parameters are

$$\begin{aligned} L_a &= 0.0014\ H \\ R_f &= 0.01485\ \Omega \\ R_p &= 0.01696\ \Omega \\ R_a &= 0.00989\ \Omega \\ B &= 0.1\ Nm/rd/s \\ K_m &= 0.04329\ (Nm)/(Wb\cdot A) \\ J &= 3.0\ K_gm^2 \end{aligned}$$

The reference speed was chosen to start from 0 and go up to 520 rad/s in 20 seconds. It is simulated as a hyperbolic function:

$$\omega_0 = \begin{cases} 3.2t^2, & t<5 \\ 34.0\cdot(t-5)+80.0, & 5\le t<15 \\ -4.0\cdot(t-20.0)^2+520.0, & 15\le t<20 \\ 520.0, & t\ge 20 \end{cases}.$$

The flux, $\phi_f(i_f)$, was derived from figure 4 of Chiasson.

The system was simulated using SIMNON. Several different simulations were attempted by varying the value of the control gain constant, G_1. As G_1 is increased, the error during the first few seconds settles down and the control law becomes smoother. Past a certain value, however, the error begins to increase during the first few seconds without any improvement in the control law. The effect of varying G_1 on the error and control law is presented in the figures in the appendix. For the best choice of G_1,

figures of various system parameters are presented below. An additional figure is included which shows the effect of a 10% perturbation in the load torque and armature inductance on the steady state error.

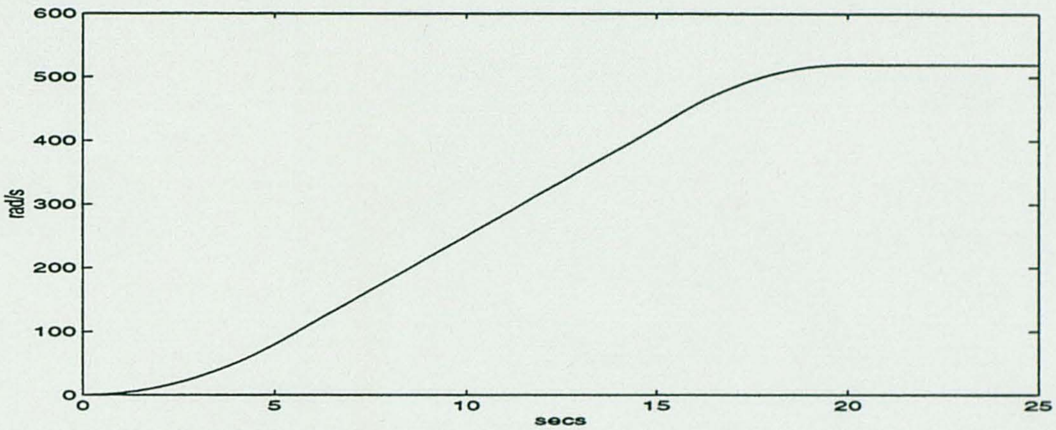

Figure 3.1: Plot of reference speed for the motor

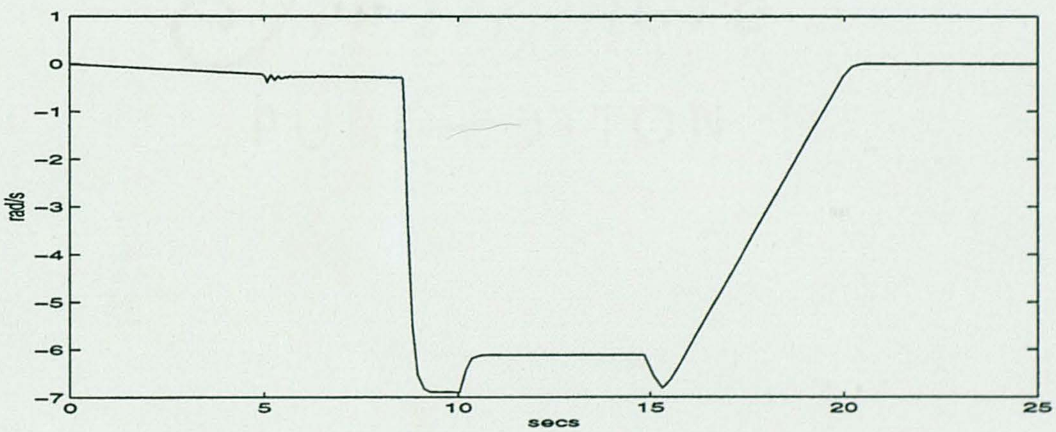

Figure 3.2: Plot of error for $G_1 = 20.0$

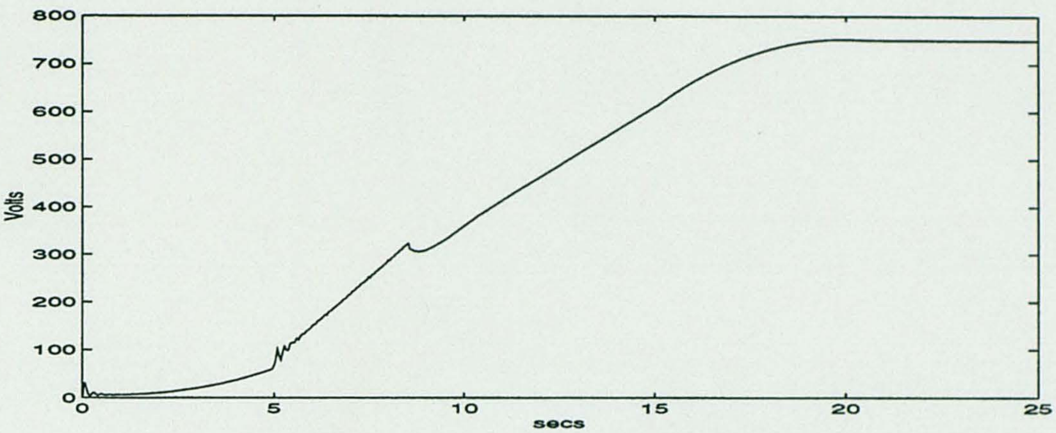

Figure 3.3: Plot of the combined control law for $G_1 = 20.0$

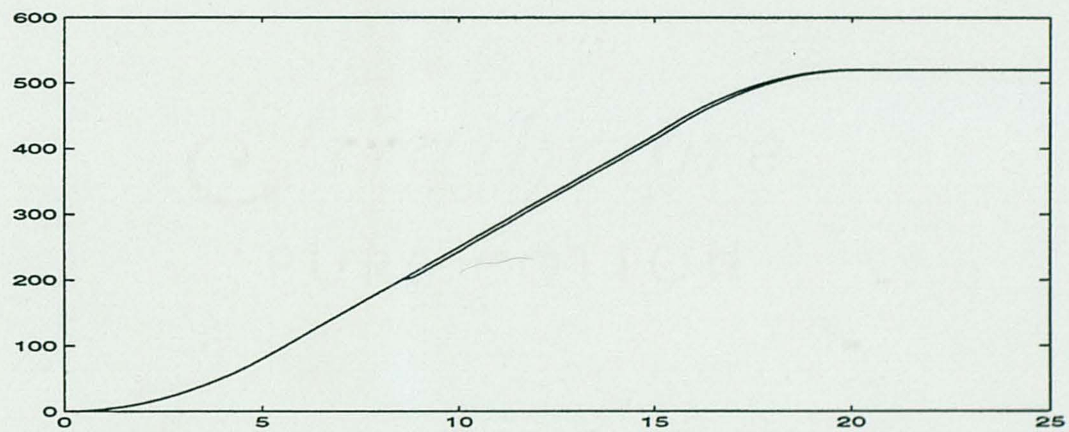

Figure 3.4: Plot of both actual motor speed and reference speed for $G_1 = 20.0$

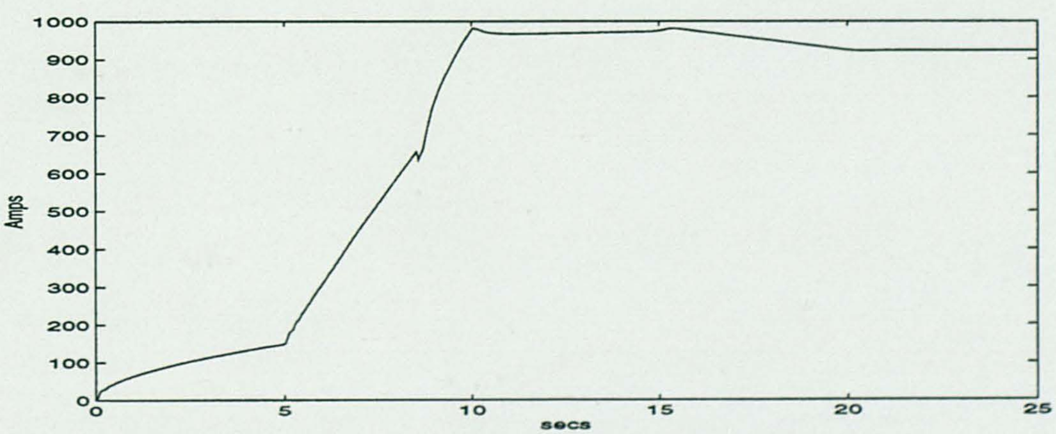

Figure 3.5: Plot of armature current for $G_1 = 20.0$

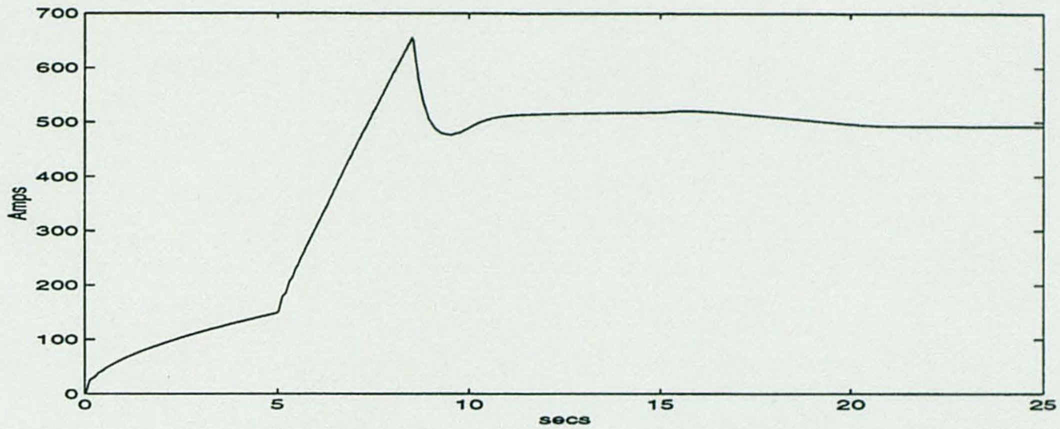

Figure 3.6: Plot of field current for $G_1 = 20.0$

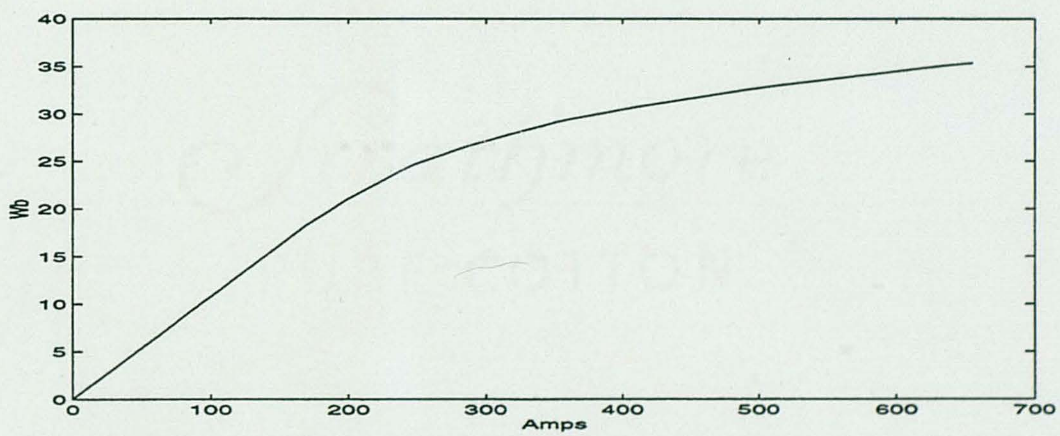

Figure 3.7: Plot of flux versus field current for $G_1 = 20.0$

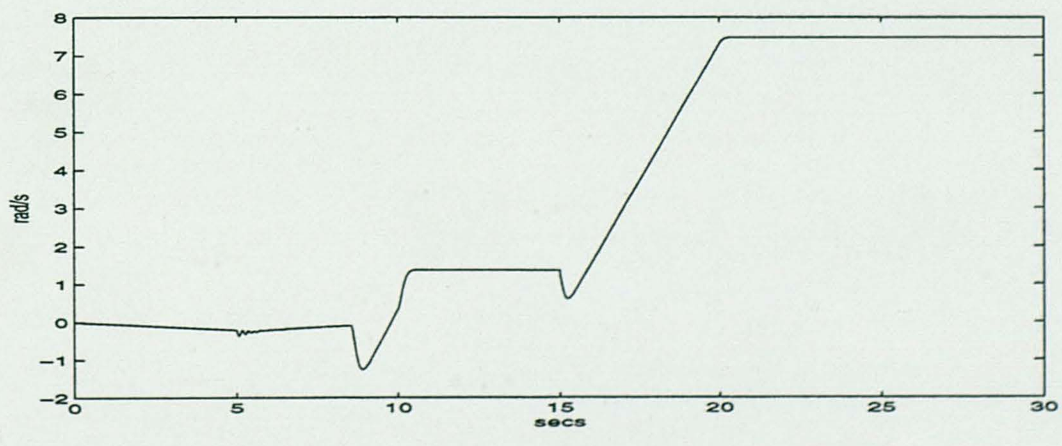

Figure 3.8: Plot of error for 10% perturbation in τ_L and L_a

PI Control

We now consider the case when the load torque in the equation for the DC motor is unknown. By using a PI control, we eliminate the need to know the load torque explicitly, and thus overcome this problem. However, since PI control can be somewhat destabilizing, we extend the control development to also include a PD term. Such a term helps to reduce the destabilizing effect of the PI term. Due to the nonlinear nature of the system under study, the resulting control is also nonlinear. We first present background information on the PI control approach, then address the issue of designing a PI control for the series DC motor, and finally present some simulation results.

Background

The background information on PI controllers presented here was taken from a text on feedback control systems [20]. Another source of information on PID controllers is available in [1]. Examples of the application of PI control to DC machines may be found in [2].

The PID controller is probably the most commonly used compensator in feedback control systems. The proportional term gives the controller output a component that is a function of the present state of the system. The integrator term provides an output which is determined by the past state of the system. The differentiation term provides a prediction of the future state of the system.

One or more of these terms, P, I, and D, are inserted into the feedback loop and their values adjusted to provide the best control. Each term affects the system in a slightly different way.

The PI controller introduces phase lag. It has the following properties:

1. The system low frequency characteristics are maintained or improved.
2. Stability margins are maintained or improved.
3. High frequency noise response is reduced.
4. The system type increases by one.
5. The system response slows down and the settling time increases.
6. Some systems can not be stabilized using this control.

The PD controller introduces phase lead. It has the following properties:

1. Stability margins are improved.
2. High frequency performance is improved.
3. It is the only control applicable to certain systems.
4. Rate feedback is easy to implement in some systems.
5. May accentuate high frequency noise problems.
6. May generate large signals at the plant input.

Combining the two yields a PID controller. If the gains are chosen properly, then the stabilizing properties of both can be maintained while the destabilizing properties are decreased. Generally speaking, PD control improves the transient response of the system while PI control improves the steady state response. Usually, the gain for I control is chosen to be smaller than the P and D gains. For motors, controls are implemented electrically, so gains as high as 50 or so are not a problem.

Application of PI Control to the Problem

We first consider the system when the motor operates above base speed, that is $R_p < \infty$. The initial steps involved in this case are similar to those for the case without PI control. Recall, the system equations in terms of ω and λ for the motor above base speed are:

$$\begin{aligned}
\frac{d\omega}{dt} &= \frac{K_m}{J}\phi_f(i_f)i_a - \frac{B}{J}\omega - \frac{\tau_L}{J} \\
\frac{d\lambda}{dt} &= -L_a R_a i_a i_f + R_p L_a(i_a^2 - i_f i_a) + u\phi_f(i_f) \\
&\quad -\frac{R_a + R_p}{L_a}\lambda + R_p i_f \phi_f(i_f) - K_m \phi_f^2(i_f)\omega.
\end{aligned}$$

For this case, x_1 will be defined differently than in the previous case. Let

$$\begin{aligned}
x_1 &= \omega - \omega_0 \\
\omega &= x_1 + \omega_0 \\
\dot{x_1} &= \dot{\omega} - \dot{\omega_0}.
\end{aligned}$$

x_2 will be the same as before, namely $x_2 = \lambda$. Thus, the systems equations are now

$$\dot{x_1} = -\frac{B}{J}x_1 - \frac{B}{J}\omega_0 + \frac{K_m}{JL_a}x_2 - \frac{\tau_L}{J} - \dot{\omega_0} \tag{3.28}$$

$$\begin{aligned}\dot{x_2} &= -L_aR_fi_ai_f + R_pL_a(i_a^2 - i_fi_a) + R_p\phi_f(i_f)i_f \\ &\quad -K_m\phi_f^2(i_f)x_1 - K_m\phi_f^2(i_f)\omega_0 - \frac{R_a+R_p}{L_a}x_2 + \phi_f(i_f)u. \end{aligned} \tag{3.29}$$

For the additional step of designing a PI control we will introduce the following equation:

$$\dot{x_0} = x_1. \tag{3.30}$$

The PI control can be inserted into equation (3.28) by adding and subtracting the terms k_0x_0 (integral part) and k_1x_1 (proportional part). By doing so, the need to know τ_L is eliminated:

$$\begin{aligned}\dot{x_1} &= -k_0x_0 - k_1x_1 - \frac{B}{J}\omega_0 - \frac{\tau_L}{J} - \dot{\omega_0} \\ &\quad + \frac{K_m}{JL_a}\left[x_2 + \frac{JL_a}{K_m}(k_0x_0 + k_1x_1 - \frac{B}{J}x_1)\right].\end{aligned}$$

Thus,

$$\begin{aligned}\dot{x_1} &= -k_0[x_0 + (1/k_0)(\tau_L/J)] - k_1x_1 + (K_m/JL_a)z_2 - \frac{B}{J}\omega_0 - \dot{\omega_0} \\ &= -k_0z_0 - k_1x_1 + (K_m/JL_a)z_2 - \frac{B}{J}\omega_0 - \dot{\omega_0}\end{aligned}$$

where $z_0 = [x_0 + (1/k_0)(\tau_L/J)]$, and $z_2 = x_2 + (JL_a/K_m)(k_0x_0 + k_1x_1 - (B/J)x_1)$. Thus, if $z_2 \to 0$, then $\dot{x_1}$ is stable. Note, although the equation $\dot{z_0}$ is only marginally stable, the system $[z_0\ x_1]^T$ is stabilizable.

To show this, we need to choose k_0 and k_1 such that the following system is stable:

$$\begin{bmatrix} \dot{z_0} \\ \dot{x_1} \end{bmatrix} = A\begin{bmatrix} z_0 \\ x_1 \end{bmatrix} + Bz_2$$

with

$$A = \begin{bmatrix} 0 & 1 \\ -k_0 & -k_1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 \\ \frac{K_m}{JL_a} \end{bmatrix}.$$

To show stability, choose

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

where, by symmetry, $p_{12} = p_{21}$, and solve $PA + A^T P = -Q$:

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -k_0 & -k_1 \end{bmatrix} + \begin{bmatrix} 0 & -k_0 \\ 1 & -k_1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The following four equations result:

$$\begin{aligned} -k_0 p_{12} - k_0 p_{21} &= -1 \\ p_{11} - k_1 p_{21} - k_0 p_{22} &= 0 \\ p_{11} - k_1 p_{12} - k_0 p_{22} &= 0 \\ p_{12} + p_{21} - 2k_1 p_{22} &= -1. \end{aligned}$$

From this we obtain

$$P = \begin{bmatrix} \frac{k_0^2 + k_0 + k_1^2}{2k_0 k_1} & \frac{1}{2k_0} \\ \frac{1}{2k_0} & \frac{k_0 + 1}{2k_0 k_1} \end{bmatrix}.$$

If k_0 and k_1 are chosen both greater than zero, then P is positive definite.

When a system is rewritten in a form which includes the matrix P, it provides the following advantages [21]:

1. A closed form solution for either analytical proof or analysis;
2. Ability to use the Lyapunov approach which is applicable to linear time-varying systems; the eigenvalue test is not;
3. Ability to use the Lyapunov approach to analyze or design control for nonlinear systems with a linear part.

Choose $V_1(x) = x^T P x$. Then $\dot{V_1} = -x^T Q x$, and the first system is stable:

$$\begin{aligned} V_1 &= [z_0 \; x_1] P \begin{bmatrix} z_0 \\ x_1 \end{bmatrix} \\ \dot{V_1} &= -[z_0 \; x_1] Q \begin{bmatrix} z_0 \\ x_1 \end{bmatrix} + 2[z_0 \; x_1] P B z_2. \end{aligned}$$

Now we wish to find $V_2(z_2)$ such that z_2 is stable. That is, by finding a proper choice for u, we can force $z_2 \to 0$. Then, from these equations, we may derive suitable values of k_0, k_1, and k_2, where k_2 is the D gain term of the PID controller. One choice is the function $V_2(z_2) = \frac{1}{2}z_2^2$. Then $\dot{V}_2(z_2) = z_2\dot{z}_2$. Since the derivation is complicated, it will be presented step by step.

We begin with the z_2 equation and its derivative:

$$\begin{aligned} z_2 &= x_2 + \frac{JL_a}{K_m}[k_0 x_0 + (k_1 - B/J)x_1] \\ \dot{z}_2 &= \dot{x}_2 + \frac{JL_a}{K_m}[k_0\dot{x}_0 + (k_1 - B/J)\dot{x}_1]. \end{aligned}$$

We must rewrite $\dot{x}_0$, $\dot{x}_1$, and $\dot{x}_2$ in terms of z. Recall, $\dot{x}_0 = x_1$. Let $z_1 = x_1$. Then,

$$\begin{aligned} \dot{x}_1 &= -k_0 z_0 - k_1 x_1 + \frac{K_m}{JL_a}z_2 - \frac{B}{J}\omega_0 - \dot{\omega}_0 \\ &= -k_0 z_0 - k_1 z_1 + \frac{K_m}{JL_a}z_2 - \frac{B}{J}\omega_0 - \dot{\omega}_0 \\ \dot{x}_2 &= -L_a R_f i_a i_f + R_p L_a(i_a^2 - i_f i_a) + R_p\phi_f(i_f)i_f + \phi_f(i_f)u \\ &\quad -K_m\phi_f^2(i_f)z_1 - K_m\phi_f^2(i_f)\omega_0 - \frac{R_a + R_p}{L_a}x_2. \end{aligned}$$

Recall,

$$\begin{aligned} x_2 &= z_2 - \frac{JL_a}{K_m}[k_0 x_0 + (k_1 - B/J)x_1] \\ x_1 &= z_1 \end{aligned}$$

and note that we do not rewrite x_0 in terms of z_0. So,

$$x_2 = z_2 - \frac{JL_a}{K_m}k_0 x_0 - \frac{JL_a}{K_m}k_1 z_1 + \frac{BL_a}{K_m}z_1,$$

and

$$\begin{aligned} -\frac{R_a + R_p}{L_a}x_2 &= -\frac{R_a + R_p}{L_a}z_2 + \frac{J}{K_m}(R_a + R_p)k_0 x_0 \\ &\quad + \frac{J}{K_m}(R_a + R_p)k_1 z_1 - \frac{B}{K_m}(R_a + R_p)z_1. \end{aligned}$$

The expression for $\dot{x}_2$ is then

$$\begin{aligned} \dot{x}_2 &= -L_a R_f i_a i_f + R_p L_a(i_a^2 - i_f i_a) + R_p\phi_f(i_f)i_f + \phi_f(i_f)u - K_m\phi_f^2(i_f)z_1 \\ &\quad -K_m\phi_f^2(i_f)\omega_0 - \frac{R_a + R_p}{L_a}z_2 + \frac{J}{K_m}(R_a + R_p)k_0 x_0 \\ &\quad + \frac{J}{K_m}(R_a + R_p)k_1 z_1 - \frac{B}{K_m}(R_a + R_p)z_1. \end{aligned}$$

Upon substituting these expressions for $\dot{x_0}$, $\dot{x_1}$, and $\dot{x_2}$ into the following equation

$$\dot{z_2} = \dot{x_2} + \frac{JL_a}{K_m}[k_0\dot{x_0} + (k_1 - B/J)\dot{x_1}]$$

and multiplying by z_2, the result is

$$\begin{aligned}
z_2\dot{z_2} \;=\; & -L_aR_fi_ai_fz_2 + R_pL_a(i_a^2 - i_fi_a)z_2 + R_p\phi_f(i_f)i_fz_2 \\
& +\phi_f(i_f)uz_2 - K_m\phi_f^2(i_f)z_1z_2 - K_m\phi_f^2(i_f)\omega_0z_2 - \frac{R_a+R_p}{L_a}z_2^2 \\
& +\frac{J}{K_m}(R_a+R_p)k_0x_0z_2 + \frac{J}{K_m}(R_a+R_p)k_1z_1z_2 \\
& -\frac{B}{K_m}(R_a+R_p)z_1z_2 + \frac{JL_a}{K_m}k_0z_1z_2 - \frac{JL_a}{K_m}k_0k_1z_0z_2 \\
& -\frac{JL_a}{K_m}k_1^2z_1z_2 + k_1z_2^2 - \frac{BL_a}{K_m}k_1\omega_0 - \frac{JL_a}{K_m}k_1\dot{\omega_0} \\
& +\frac{BL_a}{K_m}k_0z_0z_2 + \frac{BL_a}{K_m}k_1z_1z_2 - (B/J)z_2^2 \\
& +\frac{B^2L_a}{JK_m}\omega_0 + \frac{BL_a}{K_m}\dot{\omega_0}.
\end{aligned}$$

Grouping terms and rewriting for the Lyapunov function yields,

$$\begin{aligned}
\dot{V_2}(z_2) \;=\; & -\left(\frac{R_a+R_p}{L_a} + \frac{B}{J}\right)z_2^2 \\
& +z_2\Bigg[k_1z_2 + \left(\frac{BL_a}{K_m}k_0 - \frac{JL_a}{K_m}k_0k_1\right)z_0 \\
& +\Bigg(\frac{J}{K_m}(R_a+R_p)k_1 + \frac{JL_a}{K_m}k_0 + \frac{BL_a}{K_m}k_1 \\
& -K_m\phi_f^2(i_f) - \frac{B}{K_m}(R_a+R_p) - \frac{JL_a}{K_m}k_1^2\Bigg)z_1 \\
& +\left(\frac{B^2L_a}{JK_m} - \frac{BL_a}{K_m}k_1 - K_m\phi_f^2(i_f)\right)\omega_0 \\
& +R_pL_a(i_a^2 - i_fi_a) + R_p\phi_f(i_f)i_f - L_aR_fi_ai_f + \frac{J}{K_m}(R_a+R_p)k_0x_0 \\
& +\left(\frac{BL_a}{K_m} - \frac{JL_a}{K_m}k_1\right)\dot{\omega_0} + \phi_f(i_f)u\Bigg].
\end{aligned}$$

The term $k_2z_2^2$ is added and subtracted to the expression for $\dot{V_2}(z_2)$.

$$\dot{V_2}(z_2) \;=\; -\left(\frac{R_a+R_p}{L_a} + \frac{B}{J}\right)z_2^2 - k_2z_2^2$$

$$
\begin{aligned}
&+z_2\left[k_1z_2+k_2z_2+\left(\frac{BL_a}{K_m}k_0-\frac{JL_a}{K_m}k_0k_1\right)z_0\right.\\
&+\left(\frac{J}{K_m}(R_a+R_p)k_1+\frac{JL_a}{K_m}k_0+\frac{BL_a}{K_m}k_1\right.\\
&\left.-K_m\phi_f^2(i_f)-\frac{B}{K_m}(R_a+R_p)-\frac{JL_a}{K_m}k_1^2\right)z_1\\
&+\left(\frac{B^2L_a}{JK_m}-\frac{BL_a}{K_m}k_1-K_m\phi_f^2(i_f)\right)\omega_0\\
&+R_pL_a(i_a^2-i_fi_a)+R_p\phi_f(i_f)i_f-L_aR_fi_ai_f+\frac{J}{K_m}(R_a+R_p)k_0x_0\\
&\left.+\left(\frac{BL_a}{K_m}-\frac{JL_a}{K_m}k_1\right)\dot{\omega}_0+\phi_f(i_f)u\right].
\end{aligned}
$$

The method of compensating the z_0 terms is presented later. For now, put those terms aside and cancel the other terms by choosing an appropriate u. As a further step for clarification, rewrite $\dot{V}_2$ as

$$
\begin{aligned}
\dot{V}_2(z_2) \;=\; &-\left(\frac{R_a+R_p}{L_a}+\frac{B}{J}\right)z_2^2-k_2z_2^2+\left(\frac{BL_a}{K_m}k_0-\frac{JL_a}{K_m}k_0k_1\right)z_0z_2\\
&+z_2\left[k_1z_2+k_2z_2+\left(\frac{J}{K_m}(R_a+R_p)k_1+\frac{JL_a}{K_m}k_0+\frac{BL_a}{K_m}k_1\right.\right.\\
&\left.-K_m\phi_f^2(i_f)-\frac{B}{K_m}(R_a+R_p)-\frac{JL_a}{K_m}k_1^2\right)z_1\\
&+\left(\frac{B^2L_a}{JK_m}-\frac{BL_a}{K_m}k_1-K_m\phi_f^2(i_f)\right)\omega_0\\
&+\left(\frac{BL_a}{K_m}-\frac{JL_a}{K_m}k_1\right)\dot{\omega}_0\\
&+R_pL_a(i_a^2-i_fi_a)+R_p\phi_f(i_f)i_f-L_aR_fi_ai_f\\
&\left.+\frac{J}{K_m}(R_a+R_p)k_0x_0+\phi_f(i_f)u\right].
\end{aligned}
$$

Now, choose u to cancel the terms within the square brackets above:

$$
\begin{aligned}
u \;=\; &\frac{1}{\phi_f(i_f)}[-k_1z_2-k_2z_2\\
&+\left(K_m\phi_f^2(i_f)+\frac{B}{K_m}(R_a+R_p)+\frac{JL_a}{K_m}k_1^2\right.\\
&\left.-\frac{J}{K_m}(R_a+R_p)k_1-\frac{JL_a}{K_m}k_0-\frac{BL_a}{K_m}k_1\right)z_1\\
&+\left(\frac{BL_a}{K_m}k_1+K_m\phi_f^2(i_f)-\frac{B^2L_a}{JK_m}\right)\omega_0
\end{aligned}
$$

$$+L_aR_fi_ai_f - R_pL_a(i_a^2 - i_fi_a) - R_p\phi_f(i_f)i_f$$
$$+\left(\frac{JL_a}{K_m}k_1 - \frac{BL_a}{K_m}\right)\dot{\omega}_0 - \frac{J}{K_m}(R_a+R_p)k_0x_0\Bigg].$$

Then

$$\dot{V}_2(z_2) = -\left(\frac{R_a+R_p}{L_a} + \frac{B}{J}\right)z_2^2 + \left(\frac{BL_a}{K_m}k_0 - \frac{JL_a}{K_m}k_0k_1\right)z_0z_2$$

which is stable for $z_2 \geq 0$, assuming that the z_0 terms can be compensated. Rewriting u in terms of the original variables,

$$\begin{aligned}
u \;=\; & \frac{1}{\phi_f(i_f)}\Big[-(k_1+k_2)L_ai_a\phi_f(i_f) + L_aR_fi_ai_f - R_pL_a(i_a^2 - i_fi_a) \\
& -R_p\phi_f(i_f)i_f - k_0\frac{J}{K_m}(R_a+R_p)x_0 - k_0(k_1+k_2)\frac{JL_a}{K_m}x_0 \\
& +\Big(k_2\frac{BL_a}{K_m} + \frac{B}{K_m}(R_a+R_p) - k_1k_2\frac{JL_a}{K_m} \\
& -k_1\frac{J}{K_m}(R_a+R_p) - k_0\frac{JL_a}{K_m}\Big)(\omega-\omega_0) \\
& +\left(\frac{JL_a}{K_m}k_1 - \frac{BL_a}{K_m}\right)\dot{\omega}_0 \\
& +K_m\phi_f^2(i_f)\omega + \left(k_1\frac{BL_a}{K_m} - \frac{B^2L_a}{JK_m}\right)\omega_0\Big].
\end{aligned} \tag{3.31}$$

We now show that the terms associated with z_0 are compensated by the combined Lyapunov functions without including the terms in the equation for the control, u. Combining the derivatives of the two Lyapunov functions yields,

$$\begin{aligned}
\dot{V}_1 + \dot{V}_2 \;=\; & -[z_0\ x_1]Q\begin{bmatrix} z_0 \\ x_1 \end{bmatrix} + 2[z_0\ x_1]PBz_2 \\
& -\left(\frac{R_a+R_p}{L_a} + \frac{B}{J}\right)z_2^2 - k_2z_2^2 + \left(\frac{BL_a}{K_m}k_0 - \frac{JL_a}{K_m}k_0k_1\right)z_0z_2.
\end{aligned}$$

Rewrite as

$$\begin{aligned}
\dot{V}_1 + \dot{V}_2 \;=\; & -[z_0\ x_1]Q\begin{bmatrix} z_0 \\ x_1 \end{bmatrix} + 2[z_0\ x_1]PBz_2 \\
& +2[z_0\ x_1]\begin{bmatrix} \frac{1}{2}\left(\frac{BL_a}{K_m}k_0 - \frac{JL_a}{K_m}k_0k_1\right) \\ 0 \end{bmatrix}z_2 \\
& -\left(k_2 + \frac{R_a+R_p}{L_a} + \frac{B}{J}\right)z_2^2.
\end{aligned}$$

Take norms of the right hand side and rewrite as an inequality to yield:

$$\begin{aligned} \dot{V}_1 + \dot{V}_2 \;\; &\leq \;\; -||[z_0 \; x_1]||^2 \lambda_{min}(Q) \\ &\quad +2||[z_0 \; x_1]|| \, ||z_2|| \sigma_{max} \left(PB + \begin{bmatrix} \frac{1}{2}\left(\frac{BL_a}{K_m}k_0 - \frac{JL_a}{K_m}k_0k_1\right) \\ 0 \end{bmatrix} \right) \\ &\quad - \left(k_2 + \frac{R_a + R_p}{L_a} + \frac{B}{J}\right) ||z_2||^2, \end{aligned}$$

where we have taken the minimum value of the matrix Q since it is associated with a negative term and the maximum value of the other matrix since it is associated with a positive term. Note that

$$\lambda_{min}(Q) = 1$$

and

$$PB = \begin{bmatrix} \left(\frac{1}{2k_0}\frac{K_m}{JL_a}\right) \\ \left(\frac{k_0+1}{2k_0k_1}\frac{K_m}{JL_a}\right) \end{bmatrix}.$$

At this point, it is helpful to introduce the following relationship known as the triangular inequality:

$$a^2 + b^2 \geq 2ab.$$

Let

$$a^2 = \frac{1}{2}||[z_0 \; x_1]||^2 \lambda_{min}(Q)$$

and

$$b^2 = k_2||z_2||^2.$$

Solving this will give us a condition for k_2 such that the derivative of the Lyapunov function is negative definite

$$\begin{aligned} \dot{V}_1 + \dot{V}_2 \;\; &\leq \;\; -\frac{1}{2}||[z_0 \; x_1]||^2 \lambda_{min}(Q) - \left(\frac{R_a + R_p}{L_a} + \frac{B}{J}\right) ||z_2||^2 \\ &\quad + \Bigg\{ 2||[z_0 \; x_1]|| \cdot ||z_2|| \sigma_{max} \left(PB + \begin{bmatrix} \frac{1}{2}\left(\frac{BL_a}{K_m}k_0 - \frac{JL_a}{K_m}k_0k_1\right) \\ 0 \end{bmatrix} \right) \\ &\quad -\frac{1}{2}||[z_0 \; x_1]||^2 \lambda_{min}(Q) - k_2||z_2||^2 \Bigg\} \\ &\leq \;\; 0. \end{aligned}$$

Consider the terms

$$PB + \begin{bmatrix} \frac{1}{2}\left(\frac{BL_a}{K_m}k_0 - \frac{JL_a}{K_m}k_0k_1\right) \\ 0 \end{bmatrix}.$$

Denote this as the matrix W and write as

$$W = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(\frac{K_m}{JL_a k_0} + \frac{BL_a}{K_m}k_0 - \frac{JL_a}{K_m}k_0 k_1\right) \\ \frac{k_0+1}{2k_0 k_1}\frac{K_m}{JL_a} \end{bmatrix}.$$

The norm of this matrix is

$$\begin{aligned} \sigma_{max}\left\{\begin{bmatrix} W_1 \\ W_2 \end{bmatrix}\right\} &= \sqrt{\lambda_{max}\left([W_1\ W_2]\begin{bmatrix} W_1 \\ W_2 \end{bmatrix}\right)} \\ &= \sqrt{\lambda_{max}\begin{bmatrix} W_1^2 & W_1 W_2 \\ W_1 W_2 & W_2^2 \end{bmatrix}}. \end{aligned}$$

To find eigenvalues, take

$$det(\lambda I - W) = \begin{bmatrix} \lambda - W_1^2 & -W_1 W_2 \\ -W_1 W_2 & \lambda - W_2^2 \end{bmatrix}$$

$$\begin{aligned} det &= \lambda^2 - (W_1^2 + W_2^2)\lambda \\ \lambda_{max} &= \lambda(\lambda - W_1^2 - W_2^2) = 0 \end{aligned}$$

$$W_1^2 = \left[\frac{1}{2}\left(\frac{K_m}{JL_a k_0} + \frac{BL_a k_0}{K_m} - \frac{JL_a k_0 k_1}{K_m}\right)\right]^2$$

$$W_2^2 = \left(\frac{k_0+1}{2k_0 k_1}\frac{K_m}{JL_a}\right)^2.$$

Therefore,

$$\lambda_{max} = 0$$

or

$$\lambda_{max} = W_1^2 + W_2^2.$$

The result $\lambda_{max} = 0$ is meaningless. Instead we will use the second result $\lambda_{max} = W_1^2 + W_2^2$. Any choice of k_0 and k_1 will yield a value of λ_{max} such that $\lambda_{max} > 0$. The formula used to calculate λ_{max} is then

$$\lambda_{max} = \left[\frac{1}{2}\left(\frac{K_m}{JL_a k_0} + \frac{BL_a k_0 - JL_a k_0 k_1}{K_m}\right)\right]^2 + \left(\frac{k_0+1}{2k_0 k_1}\frac{K_m}{JL_a}\right)^2. \tag{3.32}$$

When choosing values for k_0, k_1, and k_2 it is important to recall that, in general, the contribution of a PD controller is stabilizing while the contribution of a PI controller

is destabilizing. As stated, k_0 is the integral gain, k_1 is the proportional gain, and k_2 is the derivative gain. The value of k_0 is chosen to be less than the values chosen for k_1 and k_2.

For example, for a choice of $k_0 = 5$ and $k_1 = 5$, $\lambda_{max} = 1.56007$ and

$$\sigma_{max} = \sqrt{\lambda_{max}} = 1.249.$$

By the triangular inequality,

$$a^2 + b^2 \geq 2ab$$

$$2ab = 2\frac{1}{\sqrt{2}}||[z_0\ x_1]||(1) \cdot \sqrt{k_2}||z_2||$$

$$\begin{aligned} 2\sqrt{\frac{k_2}{2}} &\geq 2\sigma_{max} \\ k_2 &\geq 2\sigma_{max}^2 \\ k_2 &\geq 3.12. \end{aligned}$$

Therefore, with $k_0 = 5$ and $k_1 = 5$, k_2 must be chosen to be greater than 3.12. Such a choice guarantees stability by the proof above.

With a control law designed for the case when the motor is operating above base speed, we now turn our attention to the case when the motor is operating below base speed, that is $R_p \to \infty$. Recall that for this case the system equations are:

$$\begin{aligned} \frac{d\omega}{dt} &= \frac{K_m}{J}\phi_f(i)i - \frac{B}{J}\omega - \frac{\tau_L}{J} \\ \frac{d\lambda}{dt} &= F[-R_a i - K_m\phi_f(i)\omega + u] \end{aligned}$$

and the system equations in terms of x_1 and x_2 are then

$$\begin{aligned} \dot{x_1} &= -\frac{B}{J}x_1 - \frac{B}{J}\omega_0 + \frac{K_m}{JL_a}x_2 - \frac{\tau_L}{J} - \dot{\omega_0} \\ \dot{x_2} &= F[-R_a i - K_m\phi_f(i)x_1 + u] \end{aligned}$$

where

$$F = \frac{\partial\phi_f(i)}{\partial i}i + \phi_f(i).$$

The steps of proving stability for the first system are exactly the same as before. We present the steps required to prove that the second system may be stabilized.

Rewriting $\dot{x_0}$, $\dot{x_1}$, and $\dot{x_2}$ in terms of z yields

$$\dot{x_0} = x_1 = z_1$$
$$\dot{x_1} = -k_0 z_0 - k_1 z_1 + \frac{K_m}{JL_a} z_2 - \frac{B}{J}\omega_0 - \dot{\omega_0}$$
$$\dot{x_2} = F[-R_a i - K_m \phi_f(i) z_1 - K_m \phi_f(i)\omega_0 + u].$$

Recall,

$$\begin{aligned} x_2 &= z_2 - \frac{JL_a}{K_m}[k_0 x_0 + (k_1 - B/J)x_1] \\ x_1 &= z_1. \end{aligned}$$

Upon substituting these expressions for $\dot{x_0}$, $\dot{x_1}$, and $\dot{x_2}$ into the following equation

$$\dot{z_2} = \dot{x_2} + \frac{JL_a}{K_m}[k_0 \dot{x_0} + (k_1 - B/J)\dot{x_1}]$$

and multiplying by z_2, the result is

$$\begin{aligned} z_2 \dot{z_2} &= -FR_a i z_2 - FK_m \phi_f(i) z_1 z_2 - FK_m \phi_f(i)\omega_0 z_2 \\ &\quad +Fuz_2 + \frac{JL_a}{K_m} k_0 z_1 z_2 - \frac{JL_a}{K_m} k_0 k_1 z_0 z_2 \\ &\quad -\frac{JL_a}{K_m} k_1^2 z_1 z_2 + k_1 z_2^2 - \frac{BL_a}{K_m} k_1 \omega_0 - \frac{JL_a}{K_m} k_1 \dot{\omega_0} \\ &\quad +\frac{BL_a}{K_m} k_0 z_0 z_2 + \frac{BL_a}{K_m} k_1 z_1 z_2 - \frac{B}{J} z_2^2 \\ &\quad +\frac{B^2 L_a}{JK_m}\omega_0 + \frac{BL_a}{K_m}\dot{\omega_0}. \end{aligned}$$

Grouping terms and rewriting for the Lyapunov function yields,

$$\begin{aligned} \dot{V_2}(z_2) &= -\frac{B}{J} z_2^2 \\ &\quad + z_2 \left[k_1 z_2 + \left(-\frac{JL_a}{K_m} k_0 k_1 + \frac{BL_a}{K_m} k_0 \right) z_0 \right. \\ &\quad + \left(\frac{JL_a}{K_m} k_0 + \frac{BL_a}{K_m} k_1 - FK_m \phi_f(i) - \frac{JL_a}{K_m} k_1^2 \right) z_1 \\ &\quad + \left(\frac{B^2 L_a}{JK_m} - \frac{BL_a}{K_m} k_1 - FK_m \phi_f(i) \right) \omega_0 \\ &\quad \left. + \left(\frac{BL_a}{K_m} - \frac{JL_a}{K_m} k_1 \right) \dot{\omega_0} - FR_a i + Fu \right]. \end{aligned}$$

We then add and subtract the term $k_2 z_2^2$,

$$\begin{aligned}
\dot{V}_2(z_2) &= -\frac{B}{J} z_2^2 - k_2 z_2^2 \\
&\quad + z_2 \left[k_1 z_2 + k_2 z_2 + \left(-\frac{J L_a}{K_m} k_0 k_1 + \frac{B L_a}{K_m} k_0 \right) z_0 \right. \\
&\quad + \left(\frac{J L_a}{K_m} k_0 + \frac{B L_a}{K_m} k_1 - F K_m \phi_f(i) - \frac{J L_a}{K_m} k_1^2 \right) z_1 \\
&\quad + \left(\frac{B^2 L_a}{J K_m} - \frac{B L_a}{K_m} k_1 - F K_m \phi_f(i) \right) \omega_0 \\
&\quad \left. + \left(\frac{B L_a}{K_m} - \frac{J L_a}{K_m} k_1 \right) \dot{\omega_0} - F R_a i + F u \right].
\end{aligned}$$

The method of compensating the terms associated with z_0 is presented later. For now, to cancel the other terms, choose

$$\begin{aligned}
u &= \frac{1}{F} [-k_1 z_2 - k_2 z_2 + F R_a i \\
&\quad + \left(F K_m \phi_f(i) + \frac{J L_a}{K_m} k_1^2 - \frac{J L_a}{K_m} k_0 - \frac{B L_a}{K_m} k_1 \right) z_1 \\
&\quad + \left(\frac{J L_a}{K_m} k_1 - \frac{B L_a}{K_m} \right) \dot{\omega_0} \\
&\quad + \left(\frac{B L_a}{K_m} k_1 + F K_m \phi_f(i) - \frac{B^2 L_a}{J K_m} \right) \omega_0].
\end{aligned}$$

Rewrite u in terms of the original variables

$$\begin{aligned}
u &= \frac{1}{F} [-(k_1 + k_2) \phi_f(i) L_a i - (k_1 + k_2) \frac{J L_a}{K_m} [k_0 x_0 + (k_1 - \frac{B}{J})(\omega - \omega_0)] \\
&\quad + F R_a i + \left(F K_m \phi_f(i) + \frac{J L_a}{K_m} k_1^2 - \frac{J L_a}{K_m} k_0 - \frac{B L_a}{K_m} k_1 \right) (\omega - \omega_0) \\
&\quad + \left(\frac{J L_a}{K_m} k_1 - \frac{B L_a}{K_m} \right) \dot{\omega_0} \\
&\quad + \left(\frac{B L_a}{K_m} k_1 + F K_m \phi_f(i) - \frac{B^2 L_a}{J K_m} \right) \omega_0]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{F} [-(k_1 + k_2) \phi_f(i) L_a i - k_0 (k_1 + k_2) \frac{J L_a}{K_m} x_0 + F R_a i \\
&\quad - k_1^2 \frac{J L_a}{K_m} (\omega - \omega_0) - k_1 k_2 \frac{J L_a}{K_m} (\omega - \omega_0) + \frac{B L_a}{K_m} k_1 (\omega - \omega_0) \\
&\quad + \frac{B La}{K_m} k_2 (\omega - \omega_0) + k_1^2 \frac{J L_a}{K_m} (\omega - \omega_0) + F K_m \phi_f(i) (\omega - \omega_0)
\end{aligned}$$

$$-\frac{BL_a}{K_m}k_1(\omega-\omega_0)-\frac{JL_a}{K_m}k_0(\omega-\omega_0)+\frac{BL_a}{K_m}k_1\omega_0$$
$$+FK_m\phi_f(i)\omega_0-\frac{B^2L_a}{JK_m}\omega_0+\left(\frac{JL_a}{K_m}k_1-\frac{BL_a}{K_m}\right)\dot{\omega_0}$$

which reduces to

$$\begin{aligned}u &= \frac{1}{F}[-(k_1+k_2)\phi_f(i)L_a i-k_0(k_1+k_2)\frac{JL_a}{K_m}x_0+FR_a i\\ &\left(-k_1k_2\frac{JL_a}{K_m}+\frac{BL_a}{K_m}k_2+\frac{JLa}{K_m}k_0\right)(\omega-\omega_0)\\ &+FK_m\phi_f(i)\omega+\left(\frac{JL_a}{K_m}k_1-\frac{BL_a}{K_m}\right)\dot{\omega_0}\\ &+\left(\frac{BL_a}{K_m}k_1-\frac{B^2L_a}{JK_m}\right)\omega_0.\end{aligned} \tag{3.33}$$

Inserting this control law into the Lyapunov function then yields

$$\dot{V_2}(z_2)=-\frac{B}{J}z_2^2-k_2z_2^2+\left(\frac{BL_a}{K_m}k_0-\frac{JL_a}{K_m}k_0k_1\right)z_0z_2$$

which is stable for $z_2\geq 0$, assuming that the z_0 terms can be compensated.

The proof of the compensation of the z_0 terms for this case is similar to that presented for the case when the motor operates above base speed. Writing the combined Lyapunov function derivatives yields

$$\begin{aligned}\dot{V_1}+\dot{V_2} &= -[z_0\ x_1]Q\begin{bmatrix} z_0\\ x_1\end{bmatrix}+2[z_0\ x_1]PBz_2\\ &\quad -(\frac{B}{J}+k_2)z_2^2+\left(\frac{BL_a}{K_m}k_0-\frac{JL_a}{K_m}k_0k_1\right)z_0z_2,\end{aligned}$$

which we may then rewrite as

$$\begin{aligned}\dot{V_1}+\dot{V_2} &= -[z_0\ x_1]Q\begin{bmatrix} z_0\\ x_1\end{bmatrix}+2[z_0\ x_1]PBz_2\\ &\quad +2[z_0\ x_1]\begin{bmatrix}\frac{1}{2}\left(\frac{BL_a}{K_m}k_0-\frac{JL_a}{K_m}k_0k_1\right)\\ 0\end{bmatrix}z_2\\ &\quad -\left(k_2+\frac{B}{J}\right)z_2^2.\end{aligned}$$

Taking norms yields

$$\dot{V_1}+\dot{V_2}\ \leq\ -\|[z_0\ x_1]\|^2\lambda_{min}(Q)$$

$$+2||[z_0\ x_1]||||z_2||\sigma_{max}\left(PB+\left[\begin{array}{c}\frac{1}{2}\left(\frac{BL_a}{K_m}k_0-\frac{JL_a}{K_m}k_0k_1\right)\\ 0\end{array}\right]\right)$$
$$-\left(k_2+\frac{B}{J}\right)||z_2||^2,$$

where as before we have taken the minimum value of the matrix Q since it is associated with a negative term and the maximum value of the other matrix since it is associated with a positive term.

Again let

$$a^2=\frac{1}{2}||[z_0\ x_1]||^2\lambda_{min}(Q)$$

and

$$b^2=k_2||z_2||^2.$$

Solving this will give us a condition for k_2 such that the derivative of the Lyapunov function is negative definite:

$$\begin{aligned}\dot{V}_1+\dot{V}_2 &\le -\frac{1}{2}||[z_0\ x_1]||^2\lambda_{min}(Q)-\frac{B}{J}||z_2||^2\\ &\quad+\left\{2||[z_0\ x_1]||\cdot||z_2||\sigma_{max}\left(PB+\left[\begin{array}{c}\frac{1}{2}\left(\frac{BL_a}{K_m}k_0-\frac{JL_a}{K_m}k_0k_1\right)\\ 0\end{array}\right]\right)\right.\\ &\quad\left.-\frac{1}{2}||[z_0\ x_1]||^2\lambda_{min}(Q)-k_2||z_2||^2\right\}\\ &\le 0.\end{aligned}$$

Values for the control gains, k_0, k_1, and k_2, similar to the first case, may now be found.

Simulation

The results for the two different cases were combined to simulate the DC motor under the PID control law when all quantities are assumed to be known except for the load torque. The control law changes as the motor moves from below base speed to above base speed. As before base speed was chosen as $\omega_{base}=200.0$ rad/s.

Below base speed (case 2), the control law is

$$u \;=\; \frac{1}{F}[-(k_1+k_2)\phi_f(i)L_ai-k_0(k_1+k_2)\frac{JL_a}{K_m}x_0+FR_ai$$

$$
\begin{aligned}
&\left(-k_1 k_2 \frac{JL_a}{K_m} + \frac{BL_a}{K_m} k_2 + \frac{JLa}{K_m} k_0\right)(\omega - \omega_0) \\
&+ F K_m \phi_f(i)\omega + \left(\frac{JL_a}{K_m} k_1 - \frac{BL_a}{K_m}\right)\dot{\omega_0} \\
&+ \left(\frac{BL_a}{K_m} k_1 - \frac{B^2 L_a}{JK_m}\right)\omega_0
\end{aligned}
$$

with

$$
F = \frac{\partial \phi_f(i)}{\partial i} i + \phi_f(i),
$$

and above base speed (case 1), the control law is

$$
\begin{aligned}
u \;=\; & \frac{1}{\phi_f(i_f)}\Big[-(k_1 + k_2) L_a i_a \phi_f(i_f) + L_a R_f i_a i_f - R_p L_a (i_a^2 - i_f i_a) \\
& - R_p \phi_f(i_f) i_f - k_0 \frac{J}{K_m}(R_a + R_p) x_0 - k_0 (k_1 + k_2) \frac{JL_a}{K_m} x_0 \\
& + \left(k_2 \frac{BL_a}{K_m} + \frac{B}{K_m}(R_a + R_p) - k_1 k_2 \frac{JL_a}{K_m}\right. \\
& \left. - k_1 \frac{J}{K_m}(R_a + R_p) - k_0 \frac{JL_a}{K_m}\right)(\omega - \omega_0) \\
& + \left(\frac{JL_a}{K_m} k_1 - \frac{BL_a}{K_m}\right)\dot{\omega_0} \\
& + K_m \phi_f^2(i_f)\omega + \left(k_1 \frac{BL_a}{K_m} - \frac{B^2 L_a}{JK_m}\right)\omega_0\Big].
\end{aligned}
$$

Using the relationships developed previously, values of k_0 and k_1 were chosen and then the appropriate range of values for k_2 was calculated. For example, for the choices of $k_0 = 7$ and $k_1 = 16$, we find that k_2 must be chosen greater than 44.

Simulations were attempted for several different values of k_0, k_1, and k_2 and the results are presented in an appendix to provide an indication of the effect of the variation of the three gains on the stability of the system. Generally, the gains should be chosen in the range of 1 to 50. However, we examined some cases for choices of k_2 up to 200. These larger values produced better simulation results, but are more difficult to physically implement. For the best values of the gain constants, figures of various system parameters are presented below.

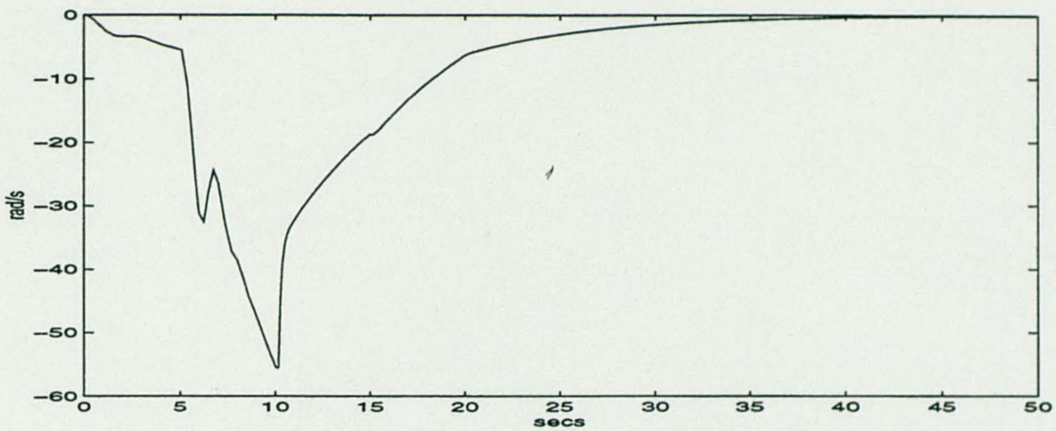

Figure 3.9: Error plot for k0 = 1.0, k1 = 8.0, k2 = 50.0

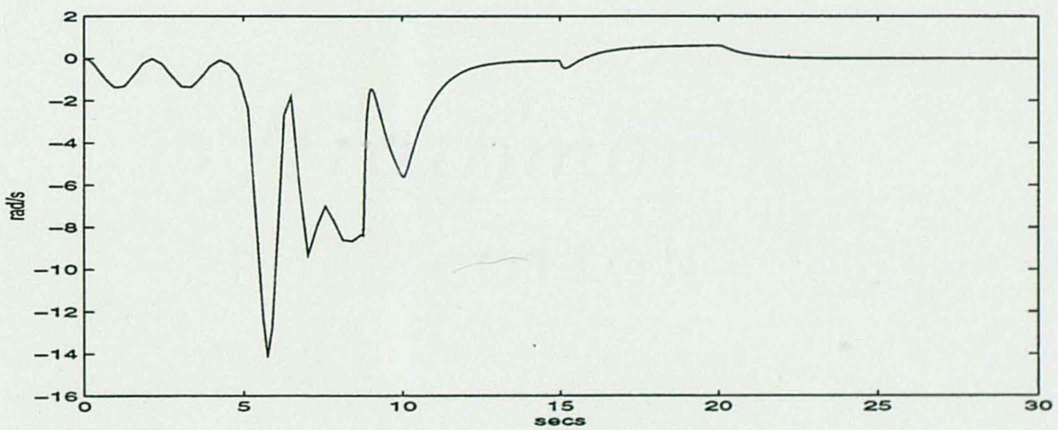

Figure 3.10: Error plot for k0 = 11.0, k1 = 11.0, k2 = 60.0

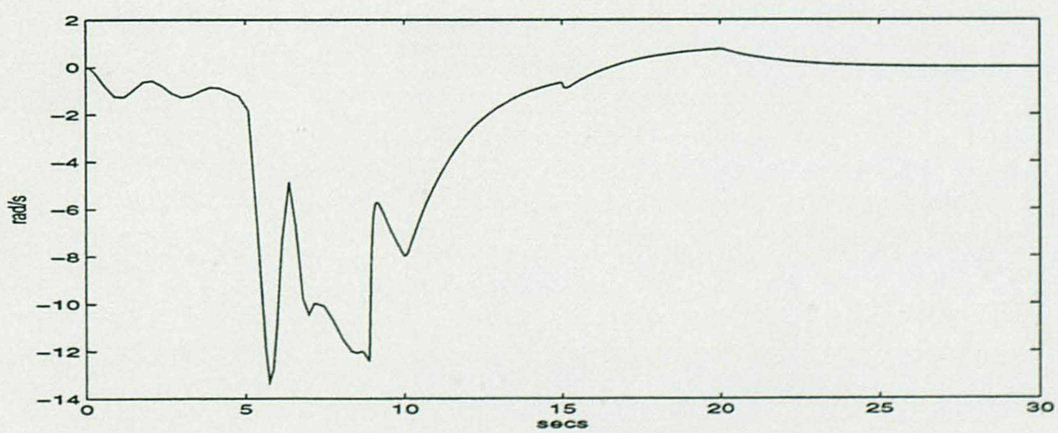

Figure 3.11: Error plot for k0 = 7.0, k1 = 16.0, k2 = 50.0

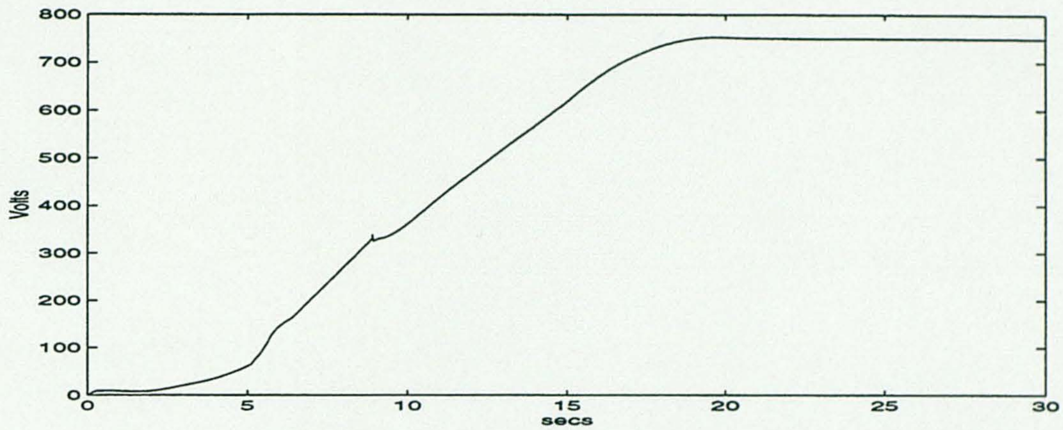

Figure 3.12: Plot of the combined PID control law for k0 = 7.0, k1 = 16.0, k2 = 50.0

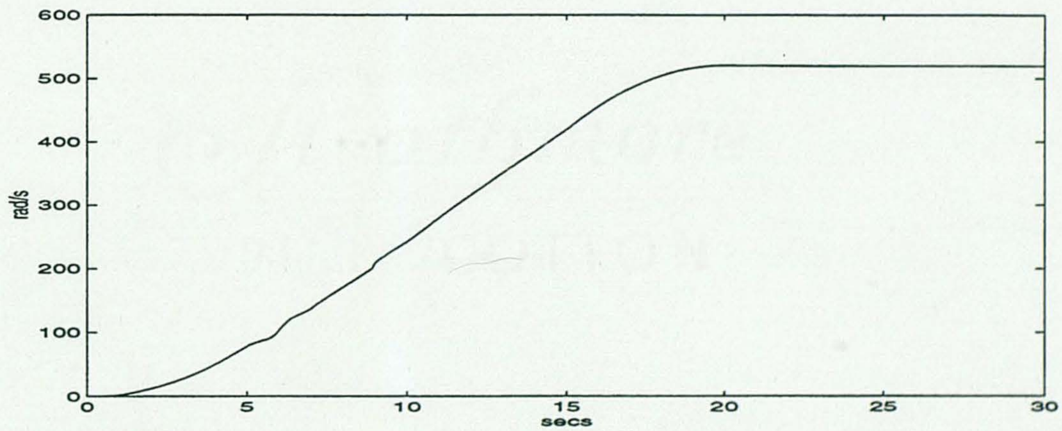

Figure 3.13: Plot of actual motor speed for k0 = 7.0, k1 = 16.0, k2 = 50.0

Figure 3.14: Plot of armature current for k0 = 7.0, k1 = 16.0, k2 = 50.0

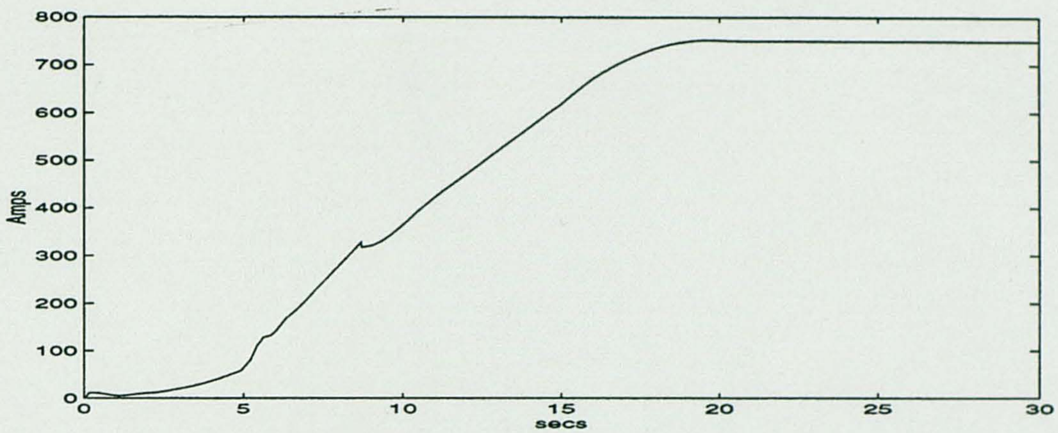

Figure 3.15: Plot of field current for k0 = 7.0, k1 = 16.0, k2 = 50.0

Robust control

In this final section in the chapter on designing control laws, we address the situation more commonly encountered in real life, that is, the system under study contains significant but bounded uncertainties. The background information on this approach includes a discussion of the generalized matching conditions and their importance in developing a robust control law. The theory is then applied to the system equations for the two cases of motor speed. Finally, simulation results are presented which help to show the validity of this approach.

Background

In order to apply the robust control method to this problem, we must show that the system meets the so-called Generalized Matching Conditions (GMC's) [22]. One important requirement for a system to meet the GMC's is that the only type of interconnections between the subsystems may be that of feedback. That is, the system must be written in the following form:

$$
\begin{aligned}
\dot{x_1} &= f_1(x_1,t) + \Delta f_1(x_1,v_1,t) + g_1(x_1,x_2,v_1,t) \\
\dot{x_2} &= f_2(x_1,x_2,t) + \Delta f_2(x_1,x_2,v_2,t) + g_2(x_1,x_2,u,x_2,t).
\end{aligned}
$$

In the equations above, $f_i(\cdot)$ denote local dynamics of subsystems including feedbacks from subsystems j, where $j < i$. $g_i(\cdot)$ denote the cascaded structure within the system. v_i, $i = 1, 2$ represent the uncertainties in the subsystems.

The GMC's include as special cases many physical systems which are series connections of nonlinear subsystems. The GMC's were originally introduced for linear uncertain systems but were later extended to nonlinear uncertain systems. The GMC's are important in the robust control design because, as shown in the GMC proof, those systems which contain unmatched uncertainties satisfying the GMC's can be fully compensated by a properly designed robust control.

The GMC's include five major conditions:

1. Controllability condition.
2. Condition to avoid singularity problem.
3. Condition to reduce the effort of finding a robust control.
4. Condition for simplicity of mathematical development.
5. Condition requiring uncertainties to be bounded.

As previously mentioned, a cascaded system is a special case of the generalized matching conditions. For our problem of the series DC motor, it has been shown that the system may be transformed into a cascaded system. We may then apply the recursive design approach in developing a robust control law for the system with time-varying uncertainties with the assurance that, for a properly designed robust control, the uncertainties may be fully compensated.

Application of Robust Control to the Problem

As before, we first consider the situation when the motor is operating above base speed and $R_p < \infty$. However, we need to modify the system equations for this case. Recall that we previously chose to introduce the variable $x_2 = \lambda = \phi_f(i_f)L_a i_a$. Since we now assume that L_a is not known exactly, we must redefine x_2 as $\phi_f(i_f)i_a$.

The original system equations are

$$\begin{aligned}
\frac{di_a}{dt} &= \frac{1}{L_a}V - \frac{R_a}{L_a}i_a - \frac{R_p}{L_a}(i_a - i_f) - \frac{K_m}{L_a}\phi_f(i_f)\omega \\
\frac{d\phi_f}{dt} &= R_f i_f + R_p(i_a - i_f)
\end{aligned}$$

$$\frac{d\omega}{dt} = \frac{K_m}{J}\phi_f(i_f)i_a - \frac{B}{J}\omega - \frac{\tau_L}{J}.$$

Using the new definition of x_2 along with the original definition of x_1, namely, $x_1 = \omega$, we may write

$$\begin{aligned}
\frac{d\lambda}{dt} &= \dot{\phi}_f(i_f)i_a + \frac{di_a}{dt}\phi_f(i_f) \\
&= -R_f i_f i_a + R_p(i_a^2 - i_a i_f) + \frac{1}{L_a}V\phi_f(i_f) - \frac{R_a}{L_a}\phi_f(i_f)i_a \\
&\quad -\frac{R_p}{L_a}(i_a - i_f)\phi_f(i_f) - \frac{K_m}{L_a}\phi_f^2(i_f)\omega.
\end{aligned}$$

Then the system equations in terms of x_1 and x_2 with $u = V$ are:

$$\dot{x}_1 = \frac{K_m}{J}x_2 - \frac{B}{J}x_1 - \frac{\tau_L}{J} \tag{3.34}$$

$$\begin{aligned}
\dot{x}_2 &= -R_f i_a i_f + R_p(i_a^2 - i_a i_f) - \frac{K_m}{L_a}\phi_f^2(i_f)x_1 \\
&\quad + \frac{R_p}{L_a}\phi_f(i_f)i_f - \frac{R_a + R_p}{L_a}x_2 + \frac{\phi_f(i_f)}{L_a}u.
\end{aligned} \tag{3.35}$$

Let $z_1 = x_1 - \omega_0$ and rewrite the first equation in terms of z_1 for the desired speed ω_0 to yield

$$\dot{z}_1 = -\frac{B}{J}z_1 - \frac{B}{J}\omega_0 + \frac{K_m}{J}x_2 - \frac{\tau_L}{J}.$$

We now wish to select the appropriate value for x_2^d such that the following terms go to zero:

$$-\frac{B}{J}\omega_0 + \frac{K_m}{J}x_2^d - \frac{\tau_L}{J} = 0.$$

However, in this case, unlike the previous cases, we do not know the load torque τ_L or the inductance L_a exactly. Instead, we must use expected nominal values in proceeding with the nominal control design and then include a robust control term to handle the unknown quantities.

The nominal values are

$$L_a \in [L_{a_0} - \delta_1, L_{a_0} + \delta_1] \qquad \tau_L \in [\tau_{L_0} - \delta_2, \tau_{L_0} + \delta_2]$$

where $\delta_1 = \kappa_1 L_{a_0}$ and $\delta_2 = \kappa_2 \tau_{L_0}$, $0.0 \le \kappa_1, \kappa_2 \le \bar{\kappa}$, with $\bar{\kappa} < 1.0$. Typical values for the variations, κ, include $\kappa_1 = 0.1$ and $\kappa_2 = 0.1$. That is, we may reasonably

expect up to a 10% variation in the nominal value of armature inductance and a 10% variation in load-torque. Therefore we choose

$$x_2^d = \frac{1}{K_m}(B\omega_0 + \tau_{L_0}) + u_{R_{11}}. \tag{3.36}$$

We choose τ_{L_0} since it is the middle value of the uncertainty range. This is done to make the robust control term as small as possible. By choosing the middle value, the most deviation that can occur is ϵ. If we were to choose the upper bound and the term was actually closer to the lower bound, then we would have a deviation of nearly 2ϵ. As a further explanation, consider the system represented by $\dot{x} = a + u$. Inserting a robust control term changes the system to $a + u = (a + u^d) + (u - u^d)$ where $u_R = (u - u^d)$. If we make $|u + u^d|$ as small as possible, then u_R is small.

The term $u_{R_{11}}$ is the robust control term, designated by the subscript R. The subscript 11 indicates that the control is for the first equation for the first case. Later in this case will we introduce the term $u_{R_{12}}$. Then, when we consider the motor below base speed, which has been designated as case 2, we will introduce the terms $u_{R_{21}}$ and $u_{R_{22}}$, where it will be seen that $u_{R_{21}}$ is simply equal to $u_{R_{11}}$.

Replacing x_2 with $z_2 + x_2^d$ in the the first system equation yields:

$$\dot{z}_1 = -\frac{B}{J}z_1 - \frac{B}{J}\omega_0 + \frac{K_m}{J}x_2^d + \frac{K_m}{J}z_2 - \frac{\tau_L}{J}. \tag{3.37}$$

To design the robust control $u_{R_{11}}$, we first chose the following Lyapunov function:

$$V(z) = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2.$$

However, the resulting computer simulations, while verifying stability, revealed that the control law based upon this particular Lyapunov function has a very poor time response. The transient error is quite large and the time to reach stability is significant. Thus, a variation of the previous Lyapunov functions is chosen. By comparing the initial results of the robust design to the control law developed for the related case when perfect knowledge was assumed, we can select the following Lyapunov function:

$$V(z) = \frac{1}{2}z_1^2 + \frac{L_a^2}{2}z_2^2. \tag{3.38}$$

Then,

$$\dot{V}(z) = z_1\dot{z}_1 + L_a^2 z_2\dot{z}_2.$$

We begin by examining the first half of the Lyapunov equation, $z_1\dot{z}_1$. This product yields,

$$\begin{aligned} z_1\dot{z}_1 &= -\frac{B}{J}z_1^2 + \frac{K_m}{J}\left[\frac{1}{K_m}(\tau_{L_0} + B\omega_0) + u_{R_{11}}\right] z_1 \\ &\quad -\frac{B}{J}\omega_0 z_1 - \frac{\tau_L}{J}z_1 + \frac{K_m}{J}z_1 z_2 \end{aligned}$$

where we have substituted for the value of x_2^d. For now, we drop the term $\frac{K_m}{J}z_1z_2$ from our analysis. This term is compensated for by the second robust control term, $u_{R_{12}}$.

Thus we are left with:

$$\begin{aligned} z_1\dot{z}_1 - \frac{K_m}{J}z_1z_2 &= -\frac{B}{J}z_1^2 + \left[\frac{\tau_{L_0}}{J} + \frac{B}{J}\omega_0 \right. \\ &\quad \left. + \frac{K_m}{J}u_{R_{11}} - \frac{B}{J}\omega_0 - \frac{\tau_L}{J}\right] z_1. \end{aligned} \tag{3.39}$$

Substituting for τ_L with nominal value and combining terms yields

$$\text{LHS of (3.39)} = -\frac{B}{J}z_1^2 + \frac{1}{J}[\tau_{L_0}(1-(1\pm\kappa_2)) + K_m u_{R_{11}}]z_1.$$

The coefficients which contain uncertainties are:

$$\frac{\tau_{L_0}}{J}[1-(1\pm\kappa_2)].$$

Select the bounding function, ρ_1, to be equal to the worst case, i.e. the largest possible uncertainty (largest possible numerators and smallest possible denominators),

$$\rho_1 = \frac{\tau_{L_0}}{J}[1-(1-\kappa_2)] = \frac{\tau_{L_0}}{J}\kappa_2.$$

With the bound so chosen, we use the robust control term $u_{R_{11}}$ to compensate for the uncertainties. Since ρ_1 represents the maximum value, we must change the equation to an inequality and write

$$\text{LHS of (3.39)} \leq -\frac{B}{J}z_1^2 + \frac{1}{J}\left(\rho_1|z_1| + K_m u_{R_{11}} z_1\right).$$

One control law which might appear to be appealing at first is

$$u_{R_{11}} = -\frac{1}{K_m} sgn(z_1)\rho_1.$$

Such a choice would yield

$$\begin{aligned}\text{LHS of (3.39)} &\leq -\frac{B}{J}z_1^2 + \frac{1}{J}\left(\rho_1|z_1| - sgn(z_1)\rho_1 z_1\right)\\ &\leq -\frac{B}{J}z_1^2.\end{aligned}$$

Unfortunately, this control is not differentiable. It is also very difficult to physically implement. Instead, we attempt to design a control law which behaves similarly, but which is differentiable and capable of implementation.

We choose the robust control law

$$u_{R_{11}} = -\frac{1}{K_m}(\frac{1}{\epsilon_1}\rho_1^2)z_1 \tag{3.40}$$

where ϵ_1 is an indication of the accuracy of the control. Typical values include 1 and 0.1. Making these changes and taking the absolute value of z_1 for the term with the coefficient ρ,

$$\text{LHS of (3.39)} \leq -\frac{B}{J}z_1^2 + \frac{1}{J}\left(\rho_1|z_1| - K_m\frac{1}{K_m}(\frac{1}{\epsilon_1}\rho_1^2)z_1^2\right).$$

This reduces to,

$$\begin{aligned}\text{LHS of (3.39)} &\leq -\frac{B}{J}z_1^2 + \frac{1}{J}\left(\rho_1|z_1| - \frac{1}{\epsilon_1}\rho_1^2 z_1^2\right)\\ &\leq -\frac{B}{J}z_1^2 + \frac{\epsilon_1}{4J}\end{aligned}$$

where we have made use of the triangular inequality,

$$a^2 + b^2 \geq 2ab$$

with

$$a = \frac{1}{\sqrt{\epsilon_1}}\rho_1|z_1|, \qquad b = \frac{\sqrt{\epsilon_1}}{2}.$$

We must wait until we complete the entire control design for this case before we may discuss the stability implications of the expression above.

Although there are better choices of robust control laws than the one chosen, this particular law produces a control which is both adequate to compensate the desired terms and easy to differentiate. The second property is a major consideration due to the fact that the derivative of this first control must be included in the design of the second control.

The derivative of the first robust control term is

$$\dot{u}_{R_{11}} = -\frac{1}{K_m}\left(\frac{1}{\epsilon_1}\rho_1^2\right)\dot{z}_1.$$

Since this term will be needed later, we rewrite it in a more complete form:

$$\dot{u}_{R_{11}} = -\frac{1}{K_m}\left(\frac{1}{\epsilon_1}\rho_1^2\right)\left(-\frac{B}{J}x_1 + \frac{K_m}{J}x_2 - \frac{\tau_L}{J}\right).$$

With a part of the system stabilized, we turn our attention to designing a control to handle the remaining terms.

Introduce the new variable z_2 and consider the second half of the Lyapunov function. With $z_2 = x_2 - x_2^d$, then, $\dot{z}_2 = \dot{x}_2 - \dot{u}_{R_{11}}$ and $x_2 = z_2 + x_2^d$. The second system equation in the new variable is then:

$$\begin{aligned}
\dot{z}_2 \;=\; & -\frac{K_m}{L_a}\phi_f^2(i_f)z_1 - \frac{K_m}{L_a}\phi_f^2(i_f)\omega_0 - \frac{R_a+R_p}{L_a}z_2 \\
& -\frac{R_a+R_p}{L_a}x_2^d - R_f i_a i_f + R_p(i_a^2 - i_a i_f) \\
& +\frac{R_p}{L_a}\phi_f(i_f)i_f + \frac{\phi_f(i_f)}{L_a}u - \dot{u}_{R_{11}} \qquad (3.41)
\end{aligned}$$

and the second half of the Lyapunov function plus the term dropped before is

$$\begin{aligned}
L_a^2 z_2\dot{z}_2 + \frac{K_m}{J}z_1 z_2 \;=\; & -K_m L_a\phi_f^2(i_f)z_1 z_2 - K_m L_a\phi_f^2(i_f)\omega_0 z_2 - (R_a+R_p)L_a z_2^2 \\
& -(R_a+R_p)L_a x_2^d z_2 - R_f L_a^2 i_a i_f z_2 + R_p L_a^2(i_a^2 - i_a i_f)z_2 \\
& +R_p L_a\phi_f(i_f)i_f z_2 + \phi_f(i_f)L_a u z_2 - L_a^2\dot{u}_{R_{11}}z_2 \\
& +\frac{K_m}{J}z_1 z_2. \qquad (3.42)
\end{aligned}$$

Factoring out L_a and rearranging the terms yields:

$$\begin{aligned}
\text{LHS of (3.42)} \;=\; & -(R_a+R_p)L_a z_2^2 + z_2 L_a[-K_m\phi_f^2(i_f)z_1 - K_m\phi_f^2(i_f)\omega_0 \\
& -(R_a+R_p)x_2^d - R_f L_a i_a i_f + R_p L_a(i_a^2 - i_a i_f) \\
& +R_p\phi_f(i_f)i_f + \phi_f(i_f)u - L_a\dot{u}_{R_{11}} + \frac{K_m}{JL_a}z_1].
\end{aligned}$$

Replace x_2^d, $\dot{u}_{R_{11}}$, and $u_{R_{11}}$ in the equation with their actual values to rewrite (3.42) as:

$$\begin{aligned}
\text{LHS of (3.42)} \;=\; & -(R_a+R_p)L_a z_2^2 + L_a z_2[-K_m\phi_f^2(i_f)x_1 + \frac{R_a+R_p}{K_m}\left(\frac{1}{\epsilon_1}\right)\rho_1^2 z_1 \\
& -R_f L_a i_a i_f + R_p L_a(i_a^2 - i_a i_f) + R_p\phi_f(i_f)i_f + \frac{K_m}{JL_a}z_1 \\
& -\frac{BL_a}{JK_m}\left(\frac{1}{\epsilon_1}\right)\rho_1^2 x_1 + \frac{L_a}{J}\left(\frac{1}{\epsilon_1}\right)\rho_1^2 x_2 \\
& -\frac{L_a\tau_L}{JK_m}\left(\frac{1}{\epsilon}\right)\rho_1^2 - \frac{R_a+R_p}{K_m}(B\omega_0 + \tau_{L_0}) + \phi_f(i_f)u].
\end{aligned}$$

Collect the terms which contain uncertainties:

$$\begin{aligned}
& -L_a R_f i_a i_f + R_p L_a(i_a^2 - i_a i_f) + \frac{K_m}{JL_a}z_1 - \frac{BL_a}{JK_m}\left(\frac{1}{\epsilon_1}\right)\rho_1^2 x_1 \\
& +\frac{L_a}{J}\left(\frac{1}{\epsilon_1}\right)\rho_1^2 x_2 - \frac{L_a\tau_L}{JK_m}\left(\frac{1}{\epsilon_1}\right)\rho_1^2.
\end{aligned}$$

First consider the term with the uncertainty in the denominator, that is $\frac{K_m}{JL_a}z_1$. The uncertain coefficient, $\frac{1}{L_a}$, may be rewritten and bounded as follows:

$$\begin{aligned}
\frac{1}{L_a} &= \frac{1}{L_{a_0}} + \left(\frac{1}{L_a} - \frac{1}{L_{a_0}}\right) \\
&\le \frac{1}{L_{a_0}} + \left(\frac{\kappa_1}{L_{a_0}(1-\kappa_1)}\right).
\end{aligned}$$

Thus the term $\frac{K_m}{JL_a}z_1$ may be rewritten as the sum of two terms, a known term and an uncertain term:

$$\frac{K_m}{JL_a}z_1 \le \frac{K_m}{JL_{a_0}}z_1 + \frac{\kappa_1 K_m}{JL_{a_0}(1-\kappa_1)}z_1.$$

Similarly, the other terms with uncertainties may also be split into known and unknown terms.

Bound the uncertainty above and the remaining uncertainties with the function ρ_2 by setting the uncertain terms equal to their maximum values and by taking the absolute value of sign varying terms:

$$\begin{aligned}
\rho_2 \;=\; & L_{a_0}\kappa_1\Big[R_f i_a i_f + R_p(i_a^2 - i_a i_f) + \frac{B}{JK_m}\left(\frac{1}{\epsilon_1}\right)\rho_1^2|x_1| \\
& +\frac{1}{\epsilon_1}\rho_1^2|z_2|\Big] + L_{a_0}(1+\kappa_1)\frac{\tau_{L_0}(1+\kappa_2)}{JK_m}\left(\frac{1}{\epsilon_1}\right)\rho_1^2 + \frac{K_m\kappa_1}{JL_{a_0}(1-\kappa_1)}|z_1|. \quad (3.43)
\end{aligned}$$

Note that it is not necessary to take the absolute value of the term $(i_a - i_f)$. From Chiasson [6] we know that above base speed $i_a > i_f$. Substituting ρ_2 into the second half of the Lyapunov equation yields

$$\begin{aligned}
\text{LHS of (3.42)} \leq\ & -(R_a+R_p)L_a z_2^2 + z_2 L_a \Big[-K_m\phi_f^2(i_f)x_1 - \frac{R_a+R_p}{K_m}(B\omega_0+\tau_{L_0}) \\
& + \frac{(R_a+R_p)}{K_m}\left(\frac{1}{\epsilon_1}\right)\rho_1^2 z_1 + R_p\phi_f(i_f)i_f \\
& - L_{a_0}R_f i_a i_f + R_p L_{a_0}(i_a^2 - i_a i_f) + \frac{K_m}{JL_{a_0}}z_1 - \frac{BL_{a_0}}{JK_m}\left(\frac{1}{\epsilon_1}\right)\rho_1^2 x_1 \\
& + \frac{L_{a_0}}{J}\frac{1}{\epsilon_1}\rho_1^2 x_2 + \phi_f(i_f)u\Big] + L_a\rho_2|z_2|.
\end{aligned}$$

Choose a control, u, to cancel terms, recalling that only nominal values may be used for L_a and τ_L:

$$\begin{aligned}
u =\ & \frac{1}{\phi_f(i_f)}[K_m\phi_f^2(i_f)x_1 + \frac{R_a+R_p}{K_m}(B\omega_0+\tau_{L_0}) \\
& - \frac{(R_a+R_p)}{K_m}\left(\frac{1}{\epsilon_1}\right)\rho_1^2 z_1 - R_p\phi_f(i_f)i_f \\
& + R_f L_{a_0} i_a i_f - \frac{K_m}{JL_{a_0}}z_1 + \frac{BL_{a_0}}{JK_m}\left(\frac{1}{\epsilon_1}\right)\rho_1^2 x_1 \\
& - \frac{L_{a_0}}{J}\left(\frac{1}{\epsilon_1}\right)\rho_1^2 x_2 - R_p L_{a_0}(i_a^2 - i_a i_f) + u_{R_{12}}]
\end{aligned}$$

and replace x_1, z_1, and x_2,

$$\begin{aligned}
u =\ & \frac{1}{\phi_f(i_f)}[K_m\phi_f^2(i_f)\omega + \frac{R_a+R_p}{K_m}(B\omega_0+\tau_{L_0}) \\
& - \frac{(R_a+R_p)}{K_m}\left(\frac{1}{\epsilon_1}\right)\rho_1^2(\omega-\omega_0) - R_p\phi_f(i_f)i_f \\
& + R_f L_{a_0} i_a i_f - \frac{K_m}{JL_{a_0}}(\omega-\omega_0) + \frac{BL_{a_0}}{JK_m}\left(\frac{1}{\epsilon_1}\right)\rho_1^2\omega \\
& - \frac{L_{a_0}}{J}\left(\frac{1}{\epsilon_1}\right)\rho_1^2\phi_f(i_f)i_a - R_p L_{a_0}(i_a^2 - i_a i_f) + u_{R_{12}}]. \qquad (3.44)
\end{aligned}$$

Then

$$\text{LHS of (3.42)} \leq -(R_a+R_p)L_a z_2^2 + L_a[\rho_2|z_2| + u_{R_{12}}z_2].$$

The following is a commonly used robust control term which is basically an approximation of the signum function previously discussed (i.e. $-sgn(\mu)\rho_2$):

$$u_{R_{12}} = -\frac{\mu^2 + \epsilon_2^2 e^{-2\beta t}}{|\mu|^3 + \epsilon_2^3 e^{-3\beta t}}\mu\rho_2$$

where $\mu = \rho_2 z_2$. This function is only differentiable if β is chosen to be zero. Thus the control law becomes

$$u_{R_{12}} = -\frac{\mu^2+\epsilon_2^2}{|\mu|^3+\epsilon_2^3}\mu\rho_2. \tag{3.45}$$

Substituting this robust control back into the equation yields,

$$\text{LHS of (3.42)} \le -(R_a+R_p)L_a z_2^2 + L_a\left[\rho_2|z_2| - \frac{\mu^2+\epsilon_2^2}{|\mu|^3+\epsilon_2^3}\mu\rho_2 z_2\right].$$

We need to determine the result of introducing the robust control law. We may proceed with our analysis by making use of Holder's inequality [18]:

$$\frac{1}{p}a^p + \frac{1}{q}b^q \ge ab \Rightarrow 1 \ge \frac{ab}{\frac{1}{p}a^p + \frac{1}{q}b^q}$$

where $1 < p < \infty$ and $\frac{1}{p}+\frac{1}{q} = 1$.

Find the common denominator of the term:

$$\begin{aligned} &\rho_2|z_2| - \frac{\mu^2+\epsilon_2^2}{|\mu|^3+\epsilon_2^3}\mu\rho_2 z_2 \\ = &\ \frac{\epsilon_2^2|\mu| - \epsilon_2|\mu|^2}{|\mu|^3+\epsilon_2^3}\epsilon_2. \end{aligned}$$

Split the fraction into two halves and examine each one separately. First consider the term $\frac{\epsilon_2^2|\mu|}{|\mu|^3+\epsilon_2^3}$. Write the fraction as

$$\frac{ab}{\frac{a^p}{p}+\frac{b^q}{q}} = C_1\frac{\epsilon_2^2|\mu|}{|\mu|^3+\epsilon_2^3}$$

and solve for $a, b, p, q,$ and C_1.

Choose $a = |\mu|$ and $p = 3$. Then $q = 3/2$ and

$$\frac{ab}{\frac{a^p}{p}+\frac{b^q}{q}} = \frac{|\mu|b}{\frac{1}{3}|\mu|^3 + \frac{2}{3}b^{\frac{3}{2}}} = C_1\frac{\epsilon_2^2|\mu|}{|\mu|^3+\epsilon_2^3}.$$

Choose $b = \left(\frac{1}{2}\right)^{\frac{2}{3}}\epsilon_2^2$. Then

$$\begin{aligned} \frac{|\mu|(\frac{1}{2})^{\frac{2}{3}}\epsilon_2^2}{\frac{1}{3}[|\mu|^3 + 2((\frac{1}{2})^{\frac{2}{3}}\epsilon_2^2)^{\frac{3}{2}}]} &= \frac{(\frac{1}{2})^{\frac{2}{3}}|\mu|\epsilon_2^2}{\frac{1}{3}[|\mu|^3 + 2\cdot\frac{1}{2}\epsilon_2^3]} \\ \frac{(\frac{1}{2})^{\frac{2}{3}}|\mu|^2}{\frac{1}{3}[|\mu|^3+\epsilon_2^3]} &= C_1\frac{|\mu|\epsilon_2^2}{|\mu|^3+\epsilon_2^3} \end{aligned}$$

$$C_1 = 3\left(\frac{1}{2}\right)^{\frac{2}{3}}.$$

Application of the triangular inequality shows

$$\frac{ab}{\frac{a^p}{p}+\frac{b^q}{q}} = C_1\frac{\epsilon_2^2|\mu|}{|\mu|^3+\epsilon_2^3} \leq 1 \Rightarrow \frac{\epsilon_2^2|\mu|}{|\mu|^3+\epsilon_2^3} \leq \frac{1}{C_1}.$$

The second fraction,

$$-\frac{\epsilon_2\mu^2}{|\mu|^3+\epsilon_2^3},$$

may be treated in a similar fashion. In order to use Holder's inequality, however, we must examine the fraction in its positive form, namely, $\frac{\epsilon_2\mu^2}{|\mu|^3+\epsilon_2^3}$. Once again we choose $p = 3$ which implies $q = \frac{3}{2}$. We then choose $a = \epsilon_2$ and $b = (\frac{1}{2})^{\frac{2}{3}}|\mu|^2$. Solving for C_2 in the equation

$$\frac{ab}{\frac{a^p}{p}+\frac{b^q}{q}} = C_2\frac{\epsilon_2|\mu|^2}{|\mu|^3+\epsilon_2^3},$$

yields $C_2 = C_1 = 3(\frac{1}{2})^{\frac{2}{3}}$.

Rewrite the original expression and incorporate these results to yield:

$$\frac{\epsilon_2^2|\mu|-\epsilon_2|\mu|^2}{|\mu|^3+\epsilon_2^3}\epsilon_2 \leq \left(\frac{1}{C_1}+\frac{1}{C_2}\right)\epsilon_2 = \left(\frac{2}{3(\frac{1}{2}^{\frac{2}{3}})}\right)\epsilon_2.$$

Combine these two results for the two halves of the Lyapunov equation:

$$\begin{aligned}
\dot{V}(z) &= z_1\dot{z_1} - \frac{K_m}{J}z_1z_2 + z_2\dot{z_2} + \frac{K_m}{J}z_1z_2 \\
&\leq -\frac{B}{J}z_1^2 - (R_a+R_p)L_a z_2^2 + L_a\left[\frac{\epsilon_2}{4} + \frac{2}{3}(\frac{1}{2})^{-\frac{2}{3}}\epsilon_2\right] \\
\dot{V}(z) &= z_1\dot{z_1} + z_2\dot{z_2} \leq -\frac{B}{J}z_1^2 - (R_a+R_p)L_a z_2^2 + (1.308267)L_a\epsilon_2.
\end{aligned}$$

We introduce an additional theorem to interpret the stability result of this expression. First noting that, in general, $(R_a+R_p)L_a \ll \frac{B}{J}$, we rewrite the stability equation as

$$\begin{aligned}
\dot{V}(z) &= z_1\dot{z_1} + z_2\dot{z_2} \\
&\leq -\frac{B}{J}z_1^2 - (R_a+R_p)L_a z_2^2 + (1.308267)L_a\epsilon_2 \\
&\leq -(R_a+R_p)L_a(z_1^2+z_2^2) + (1.308267)L_a\epsilon_2 \\
&= -2(R_a+R_p)L_a V(z) + (1.308267)L_a\epsilon_2.
\end{aligned}$$

Theorem 5:

Suppose V_e is the Lyapunov function defined as

$$V_e = k_m x_e^T P x_e$$

where P is a symmetric positive definite matrix, and that its time derivative may be expressed as

$$\dot{V}_e \leq -\lambda_e V_e - \epsilon e^{-\beta t}.$$

Then V_e converges to zero exponentially and so does the state x_e. □

In our case, $\beta = 0$. Therefore, the theorem proves asymptotic stability.

We now consider the case when the motor operates below base speed. For this case, $R_p \to \infty$ and $i_a = i_f = i$. As previously mentioned, the system equations in terms of the variables x_1 and x_2 need to be modified for the case when there are uncertain terms. x_1 may be defined as before, namely $x_1 = \omega$. For the second variable, x_2, we use the transformation $x_2 = \lambda = \phi_f(i)i$.

Again, the original equations are

$$\begin{aligned}
\frac{di}{dt} &= \frac{1}{L_a}V - \frac{R_a}{L_a}i - \frac{K_m}{L_a}\phi_f(i)\omega \\
\frac{d\phi_f}{dt} &= -R_f i \\
\frac{d\omega}{dt} &= \frac{K_m}{J}\phi_f(i)i - \frac{B}{J}\omega - \frac{\tau_L}{J}.
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{d\lambda}{dt} &= \frac{\partial}{\partial i}[\phi_f(i)i]\frac{di}{dt} \\
&= F\left[-\frac{R_a}{L_a}i - \frac{K_m}{L_a}\phi_f(i)\omega + \frac{V}{L_a}\right]
\end{aligned}$$

where

$$F(i, \phi_f(i), \partial\phi_f(i)/\partial i) = \frac{\partial \phi_f(i)}{\partial i}i + \phi_f(i).$$

In terms of x_1 and x_2 the system equations are then

$$\dot{x}_1 = \frac{K_m}{J}x_2 - \frac{B}{J}x_1 - \frac{\tau_L}{J} \tag{3.46}$$

$$\dot{x_2} = F[-\frac{R_a}{L_a}i - \frac{K_m}{L_a}\phi_f(i)x_1 + \frac{1}{L_a}u]. \tag{3.47}$$

Rewriting the first equation in terms of z_1 for the desired speed ω_0 yields

$$\dot{z_1} = -\frac{B}{J}z_1 - \frac{B}{J}\omega_0 + \frac{K_m}{J}x_2 - \frac{\tau_L}{J}. \tag{3.48}$$

We now wish to select the appropriate value for x_2^d such that the following terms go to zero:

$$-\frac{B}{J}\omega_0 + \frac{K_m}{J}x_2^d - \frac{\tau_L}{J} = 0.$$

As before, we need to replace the unknown values L_a and τ_L with nominal values:

$$x_2^d = \frac{1}{K_m}(B\omega_0 + \tau_{L_0}) + u_{R_{21}}.$$

Several Lyapunov function candidates were considered in the attempt to design a control law for this case. Based upon the results of the case when the system was assumed to be perfectly known and through simple trial and error, the following Lyapunov function was chosen:

$$V(z) = \frac{1}{2}z_1^2 + \frac{L_a^3}{2}z_2^2. \tag{3.49}$$

Then,

$$\dot{V}(z) = z_1\dot{z_1} + L_a^3 z_2\dot{z_2}.$$

We can approach the control design by examining the Lyapunov equation in parts. Examination of the first part of the Lyapunov equation, namely, $z_1\dot{z_1}$ has already been completed for the previous case when the motor operates above base speed. Therefore, the analysis does not need to be repeated. Instead, we simply restate the main results. The first robust control term is

$$u_{R_{21}} = u_{R_{11}} = -\frac{1}{K_m}(\frac{1}{\epsilon_1}\rho_1^2)z_1.$$

With this stated, we now need only to examine the second part of the equation. Introduce the new variable $z_2 = x_2 - x_2^d$. Thus, $\dot{z_2} = \dot{x_2} - \dot{u}_{R_{21}}$ and $x_2 = z_2 + x_2^d$. The second system equation in the new variable is then:

$$\dot{z_2} = F[-\frac{R_a}{L_a}i - \frac{K_m}{L_a}\phi_f(i)z_1 - \frac{K_m}{L_a}\phi_f(i)\omega_0 + \frac{1}{L_a}u] - \dot{u}_{R_{21}}. \tag{3.50}$$

The second part of the Lyapunov equation including the term discarded from the first part is

$$
\begin{aligned}
L_a^3 z_2 \dot{z}_2 + \frac{K_m}{J} z_1 z_2 &= F[-R_a L_a^2 i - K_m L_a^2 \phi_f(i) x_1 \\
&\quad + L_a^2 u - \frac{L_a^3}{F} \dot{u}_{R_{21}}] z_2 + \frac{K_m}{J} z_1 z_2.
\end{aligned}
$$

Factor out the term L_a^2 and bring the term $\frac{K_m}{J} z_1 z_2$ inside the brackets:

$$
\begin{aligned}
L_a^3 z_2 \dot{z}_2 + \frac{K_m}{J} z_1 z_2 &= L_a^2 F[-R_a i - K_m \phi_f(i) x_1 \\
&\quad + u - \frac{L_a}{F} \dot{u}_{R_{21}} + \frac{K_m}{F J L_a^2} z_1] z_2. \qquad (3.51)
\end{aligned}
$$

Replace $\dot{u}_{R_{21}}$ with its actual value:

$$
\begin{aligned}
-\frac{L_a}{F} \dot{u}_{R_{21}} &= -\frac{L_a}{F} \left[-\frac{1}{K_m} \left(\frac{1}{\epsilon_1} \rho_1^2 \right) \right] \dot{z}_1 \\
&= -\frac{L_a}{F} \left[-\frac{1}{K_m} \left(\frac{1}{\epsilon_1} \rho_1^2 \right) \right] \left(-\frac{B}{J} x_1 + \frac{K_m}{J} x_2 - \frac{\tau_L}{J} \right) \\
&= -\frac{B L_a}{F J K_m} \left(\frac{1}{\epsilon_1} \right) \rho_1^2 x_1 + \frac{L_a}{F J} \left(\frac{1}{\epsilon_1} \right) \rho_1^2 x_2 \\
&\quad -\frac{L_a}{F J K_m} \left(\frac{1}{\epsilon_1} \right) \rho_1^2 \tau_L
\end{aligned}
$$

and rewrite the equation as

$$
\begin{aligned}
\text{LHS of (3.51)} &= F L_a^2 [-R_a i - K_m \phi_f(i_f) x_1 - \frac{L_a}{F J K_m} \left(\frac{1}{\epsilon_1} \right) \rho_1^2 \tau_L \\
&\quad -\frac{B L_a}{F J K_m} \left(\frac{1}{\epsilon_1} \right) \rho_1^2 x_1 + \frac{L_a}{F J} \left(\frac{1}{\epsilon_1} \right) \rho_1^2 x_2 + \frac{K_m}{F J L_a^2} z_1 + u] z_2.
\end{aligned}
$$

Gather the terms which contain uncertainties:

$$
\begin{aligned}
&-\frac{B L_a}{F J K_m} \left(\frac{1}{\epsilon_1} \right) \rho_1^2 x_1 + \frac{L_a}{F J} \left(\frac{1}{\epsilon_1} \right) \rho_1^2 x_2 \\
&-\frac{L_a \tau_L}{F J K_m} \left(\frac{1}{\epsilon_1} \right) \rho_1^2 + \frac{K_m}{F J L_a^2} z_1.
\end{aligned}
$$

First consider the term with an uncertainty in the denominator, $\frac{F K_m}{J L_a^2} z_1$. The uncertainty may be rewritten and bounded as follows:

$$
\begin{aligned}
\frac{1}{L_a^2} &= \frac{1}{L_{a_0}^2} + \left(\frac{1}{L_a^2} - \frac{1}{L_{a_0}^2} \right) \\
&\leq \frac{1}{L_{a_0}} + \left(\frac{2\kappa_1 + \kappa_1^2}{L_{a_0}^2 (1 - \kappa_1)^2} \right).
\end{aligned}
$$

Thus the term $\frac{K_m}{FJL_a^2} z_1$ may be rewritten as the sum of two terms, a known term and an uncertain term:

$$\frac{K_m}{FJL_a^2} z_1 \leq \frac{K_m}{FJL_{a_0}^2} z_1 + \frac{(2\kappa_1 + \kappa_1^2) K_m}{(1-\kappa_1)^2 FJL_{a_0}^2} z_1.$$

Bound the uncertain term above and the remaining uncertainties with the function ρ_2 by taking the maximum value of the uncertain terms and taking the absolute value of sign varying terms:

$$\begin{aligned} \rho_2 &= \frac{BL_{a_0}\kappa_1}{FJK_m}\left(\frac{1}{\epsilon_1}\right)\rho_1^2|x_1| + \frac{L_{a_0}\kappa_1}{FJ}\left(\frac{1}{\epsilon_1}\right)\rho_1^2|x_2| \\ &+ \frac{K_m(2\kappa_1+\kappa_1^2)}{FJL_{a_0}^2(1-\kappa_1)^2}|z_1| + L_{a_0}(1+\kappa_1)\frac{1}{FJK_m}\tau_{L_0}(1+\kappa_2)\left(\frac{1}{\epsilon_1}\right)\rho_1^2. \quad (3.52) \end{aligned}$$

Then,

$$\begin{aligned} \text{LHS of (3.51)} \leq\ & L_a^2 F[-R_a i - K_m\phi_f(i)x_1 \\ & -\frac{BL_{a_0}}{FJK_m}\left(\frac{1}{\epsilon_1}\right)\rho_1^2 x_1 + \frac{L_{a_0}}{FJ}\left(\frac{1}{\epsilon_1}\right)\rho_1^2 x_2 \\ & + \frac{K_m}{FJL_{a_0}^2} z_1 + u] z_2 + FL_a^2 \rho_2 |z_2|. \end{aligned}$$

Choose the following control term

$$\begin{aligned} u &= R_a i + K_m\phi_f(i_f)x_1 + \frac{BL_{a_0}}{FJK_m}\left(\frac{1}{\epsilon_1}\right)\rho_1^2 x_1 \\ & -\frac{L_{a_0}}{FJ}\left(\frac{1}{\epsilon_1}\right)\rho_1^2 x_2 - \frac{K_m}{FJL_{a_0}^2} z_1 + u_{R_{22}} - \frac{G}{F} z_2, \end{aligned}$$

where once again we include a term of the form $-Gz_2$ in order to generate a negative definite term in the second variable. Replace x_1, x_2, and z_1 to yield,

$$\begin{aligned} u &= R_a i + K_m\phi_f(i_f)\omega \\ & + \frac{BL_{a_0}}{FJK_m}\left(\frac{1}{\epsilon_1}\right)\rho_1^2\omega - \frac{L_{a_0}}{FJ}\left(\frac{1}{\epsilon_1}\right)\rho_1^2(\phi_f(i)i) \\ & - \frac{K_m}{FJL_{a_0}^2}(\omega - \omega_0) + u_{R_{22}} - \frac{G}{F} z_2. \end{aligned} \quad (3.53)$$

Then,

$$z_2\dot{z}_2 \leq FL_a^2[\rho_2|z_2| + u_{R_{22}} z_2 - \frac{G}{F} z_2^2].$$

As before choose the robust control law

$$u_{R_{22}} = -\frac{\mu^2 + \epsilon_2^2}{|\mu|^3 + \epsilon_2^3}\mu\rho_2$$

which yields

$$\text{LHS of(3.51)} \leq FL_a^2 \frac{2}{3(\frac{1}{2})^{\frac{2}{3}}}\epsilon_2 - GL_a^2 z_2^2.$$

And the complete Lyapunov result is

$$\dot{V}(z) \leq -\frac{B}{J}z_1^2 - GL_a^2 z_2^2 + L_a^2 F\left[\left(\frac{1}{4} + \frac{2}{3(\frac{1}{2})^{\frac{2}{3}}}\right)\epsilon_2\right].$$

Note that if G is chosen to be less than 10^5 then $GL_a^2 \ll \frac{B}{J}$ and the equation may be written as

$$\dot{V}(z) \leq -2GL_a^2 V(z) + L_a^2 F\left[\left(\frac{1}{4} + \frac{2}{3(\frac{1}{2})^{\frac{2}{3}}}\right)\epsilon_2\right]$$

for which asymptotic stability may be proved through the use of the theorem as before.

Simulation

Due to the fact that the control for this case contains several gain parameters, the simulation is more complicated than in previous cases. In general, ϵ_1 should be chosen greater than ϵ_2 and the value of G should be chosen to be within a reasonable range. The simulation must also be altered to test the robustness of the control. After several simulations, the following values for gain were chosen:

$$\epsilon_{11} = 25.0 \quad \epsilon_{12} = 0.1 \quad \epsilon_{21} = 50.0 \quad \epsilon_{22} = 0.3 \quad G = 20.0$$

In the simulations, the values of L_a and τ_L were varied within the specified limits of 10%. The first few figures show the results using the robust control law when no perturbations exist. The last two figures present results when minor perturbations occur.

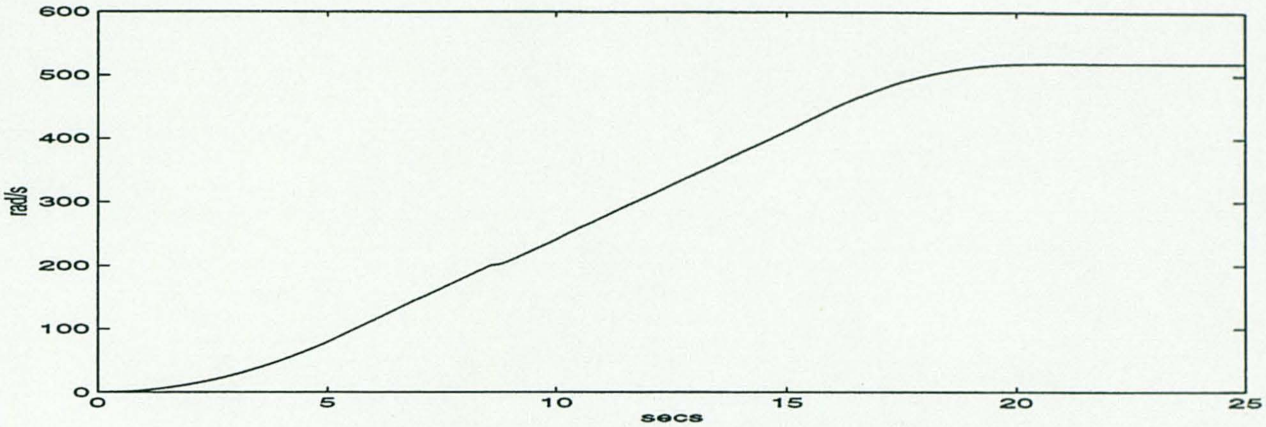

Figure 3.16: Plot of motor speed for nominal values

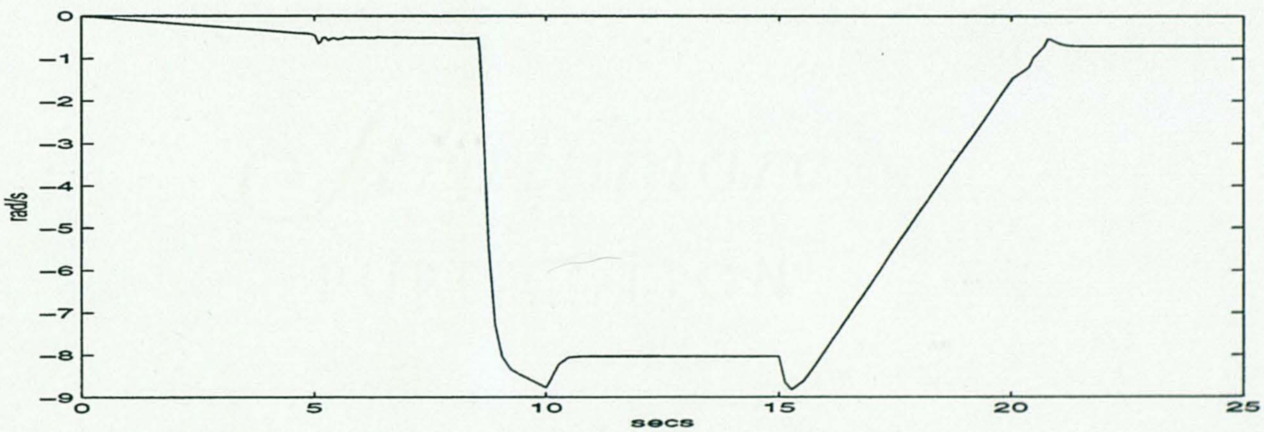

Figure 3.17: Plot of error for nominal values

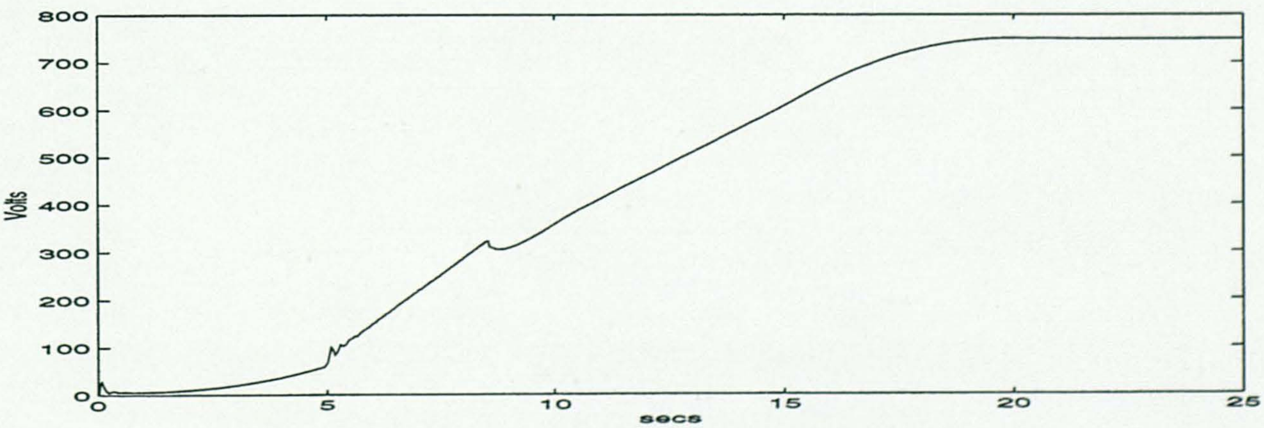

Figure 3.18: Plot of control law for nominal values

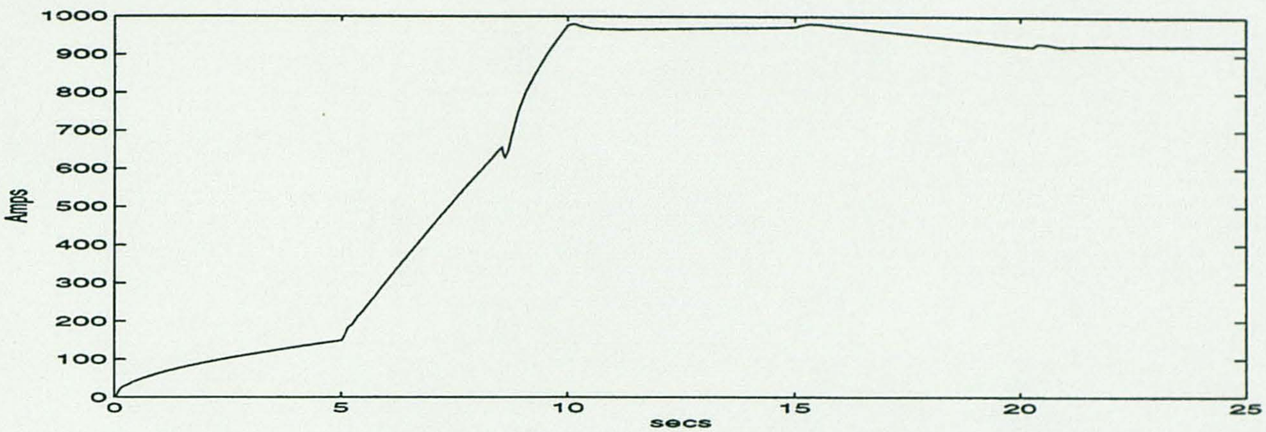

Figure 3.19: Plot of armature current for nominal values

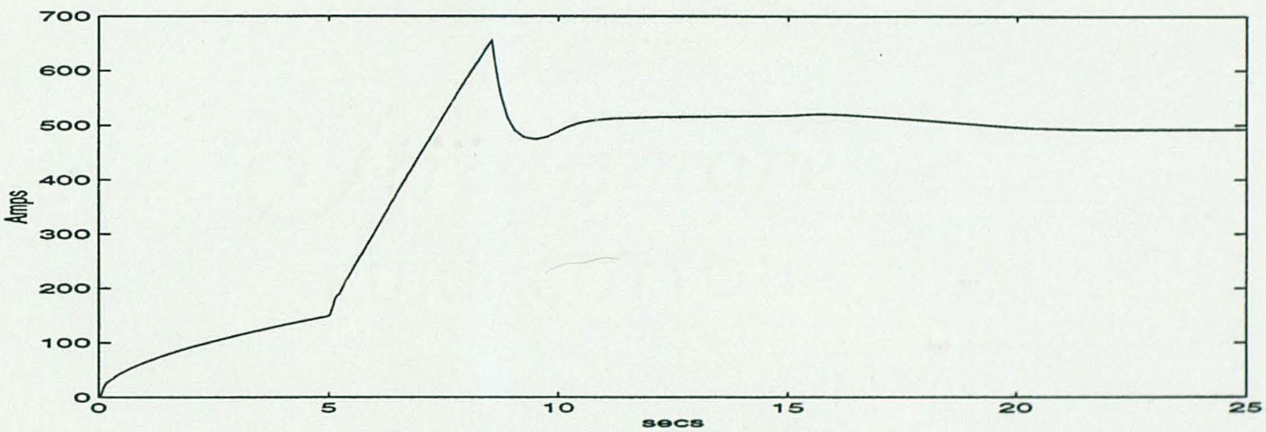

Figure 3.20: Plot of field current for nominal values

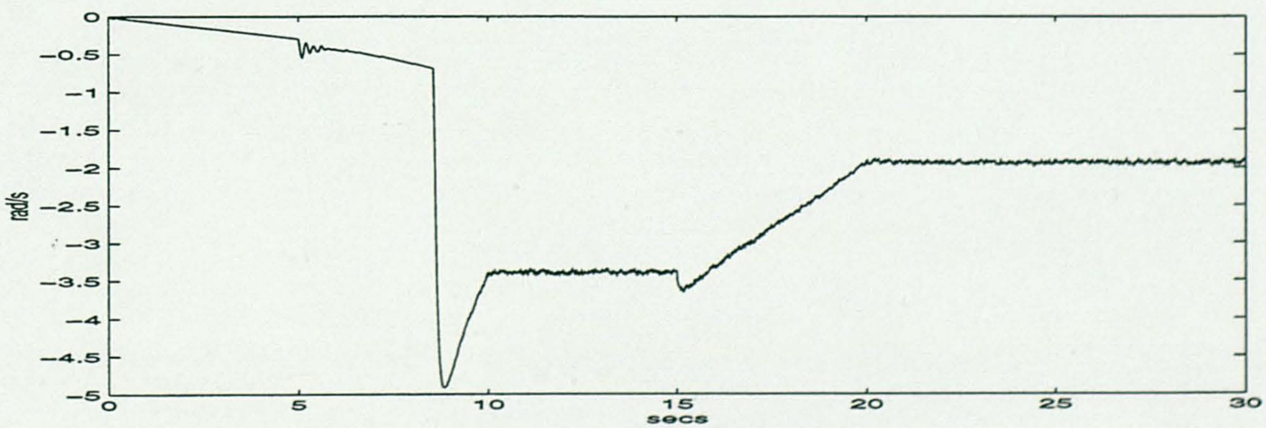

Figure 3.21: Plot of error for 10% increase in τ_L and L_a

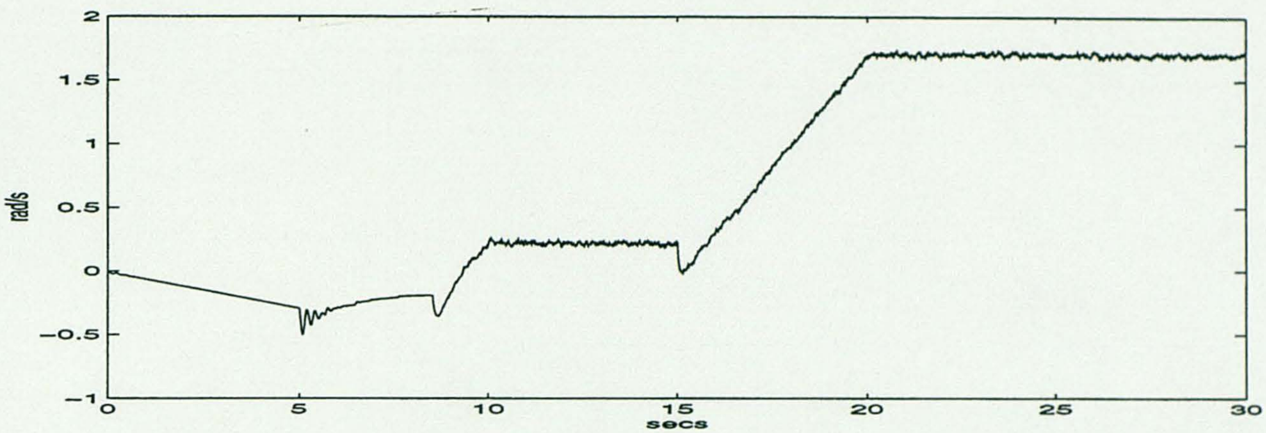

Figure 3.22: Plot of error for 10% decrease in τ_L and 10% increase in L_a

Comparison

In order to demonstrate the true power of the robust control law, simulations were performed which included perturbations from the nominal values of two system parameters. Many nonlinear systems are highly sensitive to changes in system parameters, as discussed in [28]. It is through the use of robust control, then, that we hope to compensate for this sensitivity.

First, the actual load torque was perturbed 10% from its nominal value using the same equation as before. The resulting error is shown in the first three figures for the three different cases considered. Then, a load torque with dynamic perturbation was chosen. The final three figures show the error for this load torque. In addition to perturbing the load torque, the value of the armature inductance (L_a) was perturbed by 10% as well. As can be seen, the robust control law performed very well.

It should be noted that the spikes in the control law for the robust case are artifacts of the algorithms used in SIMNON to simulate the system and reduce the error in calculations, and not an indication of an error in the equations of the control law.

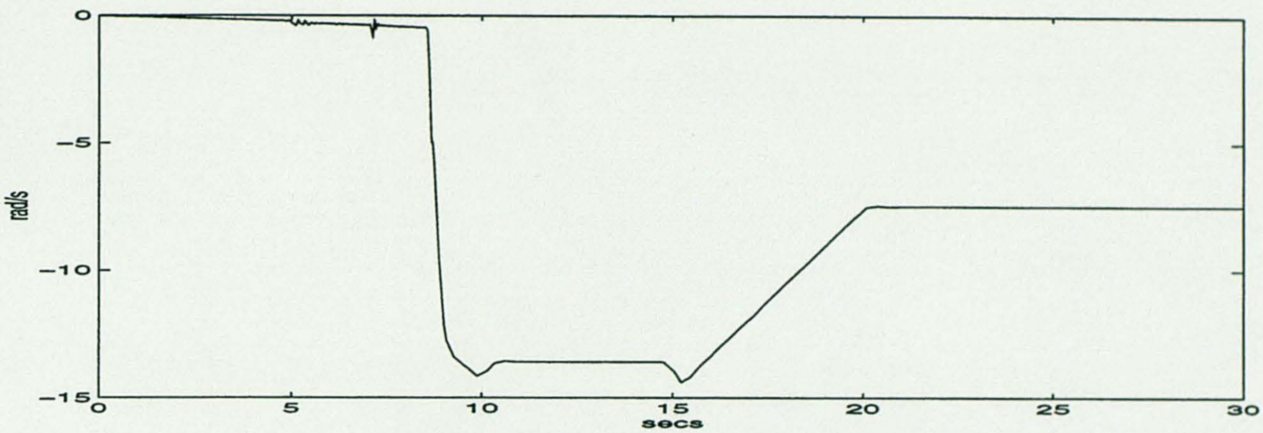

Figure 3.23: Plot of error for perfect knowledge case

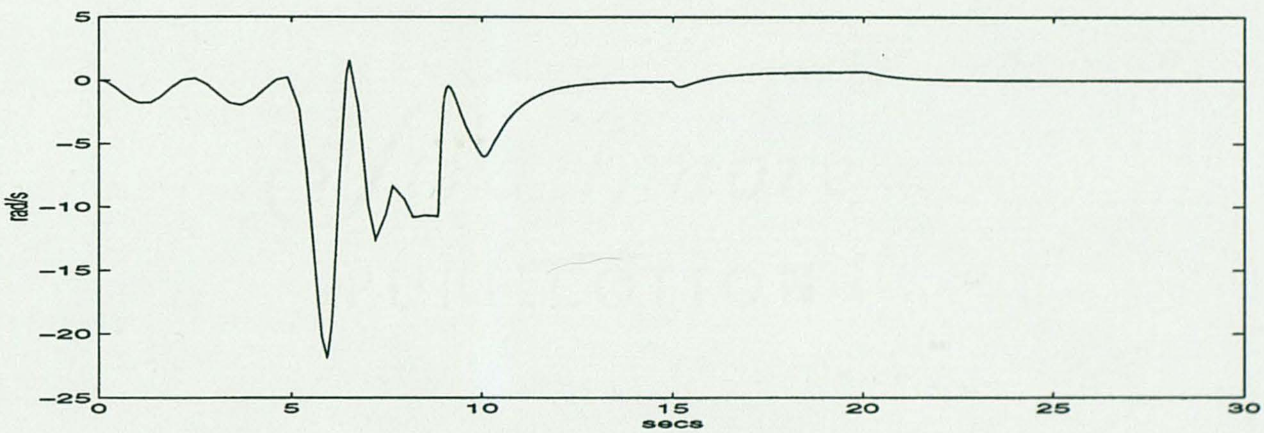

Figure 3.24: Plot of error for PID control case

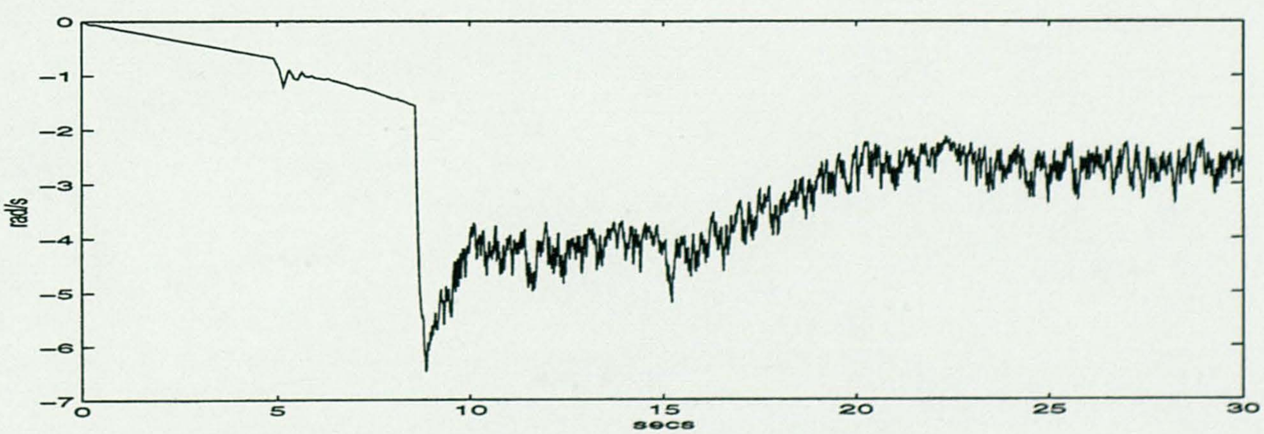

Figure 3.25: Plot of error for robust control case

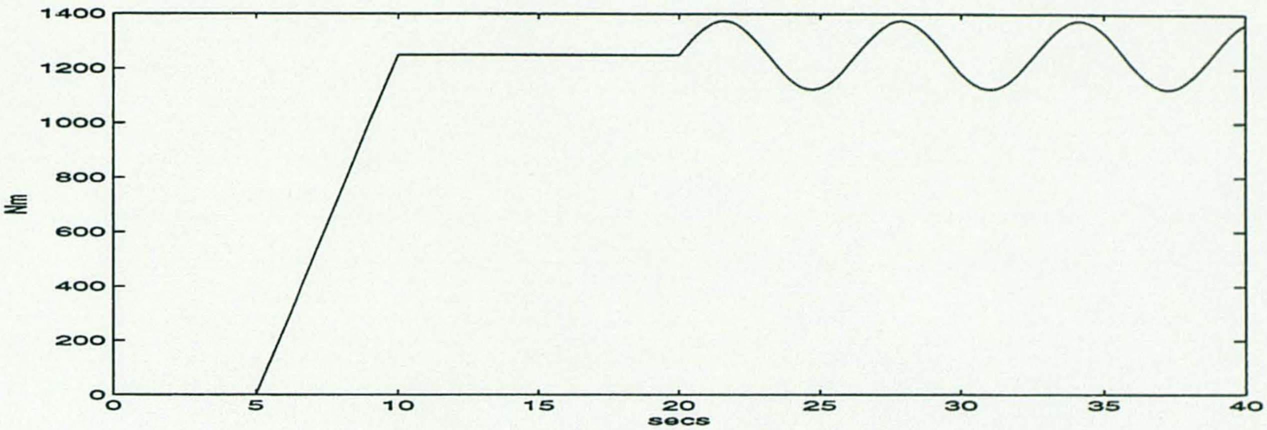

Figure 3.26: Plot of load torque with dynamic perturbation

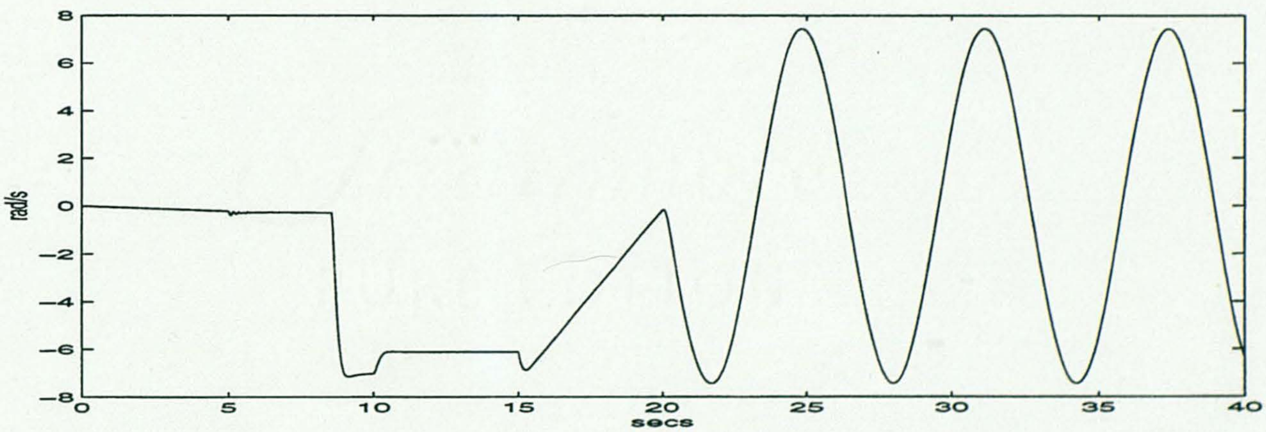

Figure 3.27: Plot of error for perfect knowledge case

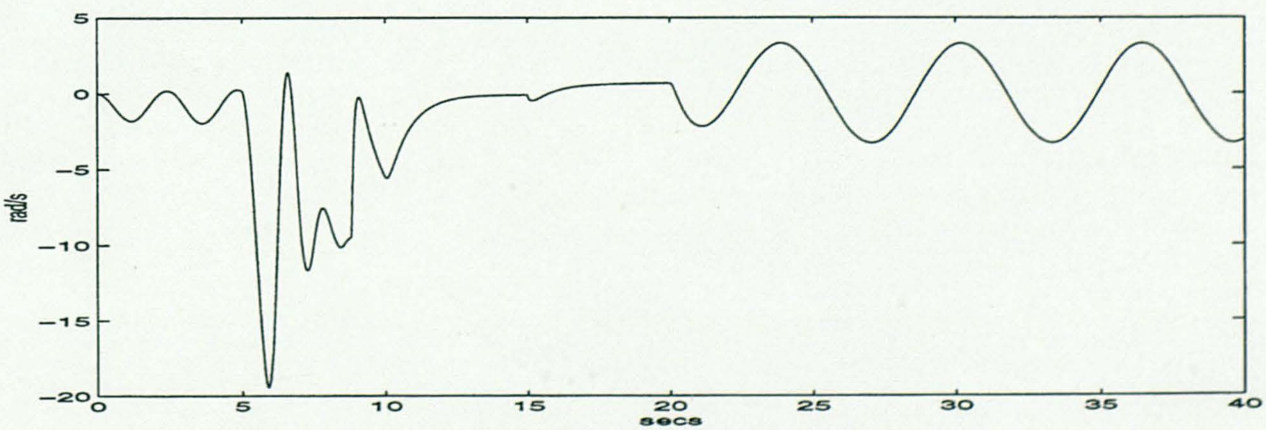

Figure 3.28: Plot of error for PID control case

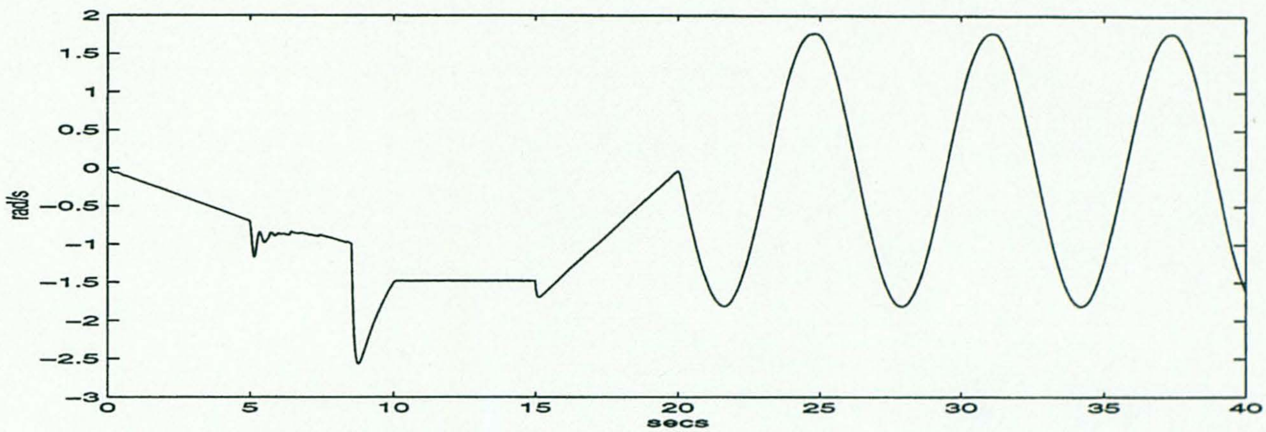

Figure 3.29: Plot of error for robust control case

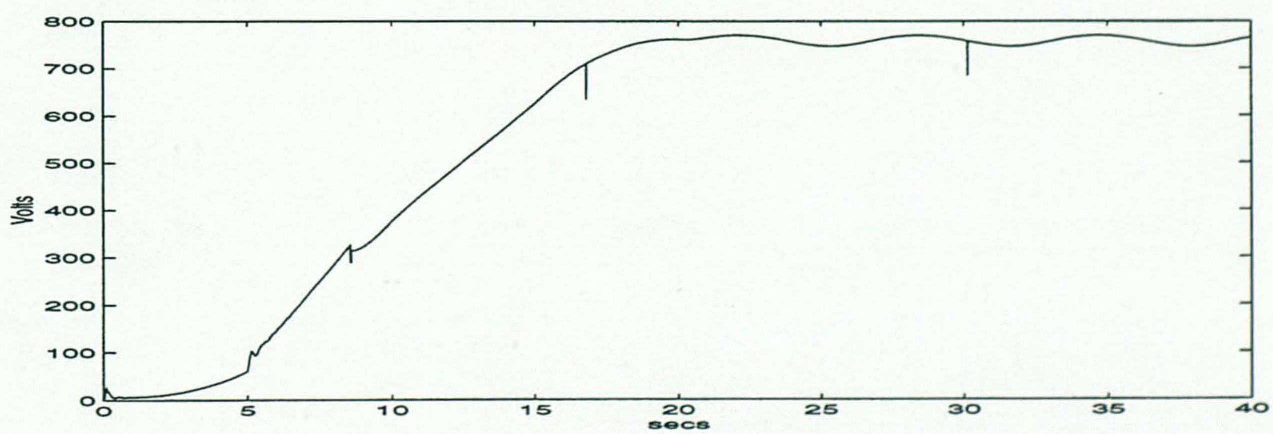

Figure 3.30: Plot of combined control law for robust control case

CHAPTER 4

CONCLUSIONS

It has been shown that the recursive design approach may be successfully applied to the problem of designing a robust control for the nonlinear model of a series DC motor.

Initially, the system was examined under the assumption that all system parameters were perfectly known. After transforming the system into a cascaded structure, we were able to easily apply the recursive design approach. The resulting control law when simulated produced excellent results. The maximum error was seen to be approximately 7 rad/s and occurred when expected, namely during a change in both load torque and in the control law during the transition through base speed. The final speed of the motor almost exactly matched the reference speed for nearly zero steady state error.

In the second case, the system was examined under the assumption that no information was available concerning the load torque. Use of the proportional-integral technique enabled the development of a control law without the need to know the load torque. In addition, a proportional-derivative control term was included to reduce the destabilizing effect of the PI controller. Once again the results were quite promising. A maximum error of 8.5 rad/s occurred during the expected time when load torque was changing and the motor passed through base speed. The steady state error was once again nearly zero.

In the third case, the system was examined under the assumption that two of its parameters were unknown. However, it was assumed that the parameters varied within a certain percentage of expected nominal values. Several cases were simulated to test the robustness of the control law. The law performed well when the uncertainties fell within the designed range.

Finally, the three control laws were applied to the case when the load torque contained dynamic perturbation. It was in this application that the robust control

law proved its strength. The perfect control law was unable to compensate for the continually varying load torque, while the robust control law was able to minimize the error quite well.

Although we only considered the cases when load torque and armature inductance were unknown, the approach as presented could be easily extended to handle additional uncertainties. Further research could be conducted by including additional nonlinear terms in the system equation. Or one might choose to consider the possibility of the existence of uncertainties in other terms such as the moment of inertia or flux. In any case, the superior performance of the robust control law demonstrates its value in design theory and application.

As manufacturing standards continue to demand greater precision and performance from robots and other computer controlled mechanisms, the need for more precise, robust control laws becomes greater too. The more complete the model of a system, the greater the precision that can be achieved. Such modeling usually requires that the system be represented by nonlinear equations which may contain uncertain terms. This, then, provides our motivation for continuing to develop and refine techniques of nonlinear control and to apply these techniques to physical systems. As shown, the recursive design approach may be used to develop a robust control law for the series DC motor with generally acceptable results.

APPENDICES

APPENDIX A

COMPLETE DERIVATION STEPS

The System is Perfectly Known

The original motor equations are

$$\begin{aligned} L_a di_a/dt &= V - R_a i_a - R_p(i_a - i_f) - K_m \phi_f(i_f)\omega \\ d\phi_f/dt &= -R_f i_f + R_p(i_a - i_f) \\ J d\omega/dt &= K_m \phi_f(i_f) i_a - B\omega - \tau_L. \end{aligned}$$

The equations must be rewritten in cascaded form in order to apply recursive design. We introduce the following variable transformation:

$$\lambda = \phi_f(i_f) L_a i_a.$$

Taking the derivative yields:

$$d\lambda/dt = \dot{\lambda} = \dot{\phi}_f(i_f) L_a i_a + \frac{d}{dt}[L_a i_a]\phi_f(i_f)$$

$$\dot{\phi}_f(i_f) = \frac{d\phi_f}{dt} = -R_f i_f + R_p(i_a - i_f)$$

$$\frac{d[L_a i_a]}{dt} = V - R_a i_a - R_p(i_a - i_f) - K_m \phi_f(i_f)\omega$$

$$\frac{d\lambda}{dt} = L_a i_a[-R_f i_f + R_p(i_a - i_f)] + \phi_f(i_f)[V - R_a i_a - R_p(i_a - i_f) - K_m \phi_f(i_f)\omega]$$

$$\begin{aligned} &= -L_a R_f i_a i_f + R_p L_a(i_a^2 - i_f i_a) + V\phi_f(i_f) - R_a i_a \phi_f(i_f) \\ &\quad -R_p(i_a - i_f)\phi_f(i_f) - K_m \phi_f^2(i_f)\omega. \end{aligned}$$

The complete steps involved in writing the derivative of the Lyapunov equation are presented below.

$$\dot{V}(z) = z_1\dot{z_1} + z_2\dot{z_2}$$

$$\begin{aligned} \dot{V}(z) &= -\frac{B}{J}z_1^2 - \frac{B}{J}\omega_0 z_1 + \frac{K_m}{JL_a}z_1 z_2 + \frac{K_m}{JL_a}x_2^d z_1 - \frac{\tau_L}{J}z_1 \\ &\quad -K_m\phi_f^2(i_f)z_1 z_2 - K_m\phi_f^2(i_f)\omega_0 z_2 - \frac{R_a + R_p}{L_a}z_2^2 \\ &\quad -\frac{R_a + R_p}{L_a}x_2^d z_2 - L_a R_f i_a i_f z_2 + R_p L_a(i_a^2 - i_f i_a)z_2 \\ &\quad +R_p\phi_f(i_f)i_f z_2 + \phi_f(i_f)z_2 u. \end{aligned}$$

Grouping terms yields,

$$\begin{aligned}\dot{V}(z) &= -\frac{B}{J}z_1^2 - \frac{R_a+R_p}{L_a}z_2^2 - \frac{B}{J}\omega_0 z_1 - \frac{\tau_L}{J}z_1 + \frac{K_m}{JL_a}x_2^d z_1 \\ &\quad +z_2[-K_m\phi_f^2(i_f)\omega_0 - \frac{R_a+R_p}{L_a}x_2^d - L_aR_fi_ai_f \\ &\quad +R_pL_a(i_a^2 - i_fi_a) + R_p\phi_f(i_f)i_f + \frac{K_m}{JL_a}z_1 \\ &\quad -K_m\phi_f^2(i_f)z_1 + \phi_f(i_f)u].\end{aligned}$$

Rewrite equation by substituting the value for x_2^d:

$$\begin{aligned}\dot{V}(z) &= -\frac{B}{J}z_1^2 - \frac{R_a+R_p}{L_a}z_2^2 - \frac{B}{J}\omega_0 z_1 - \frac{\tau_L}{J}z_1 + \frac{B}{J}\omega_0 z_1 + \frac{\tau_L}{J}z_1 \\ &\quad +z_2[-K_m\phi_f^2(i_f)\omega_0 - \frac{R_a+R_p}{K_m}(\tau_L + B\omega_0) - L_aR_fi_ai_f \\ &\quad +R_pL_a(i_a^2 - i_fi_a) + R_p\phi_f(i_f)i_f + \frac{K_m}{JL_a}z_1 \\ &\quad -K_m\phi_f^2(i_f)z_1 + \phi_f(i_f)u] \\ \\ &= -\frac{B}{J}z_1^2 - \frac{R_a+R_p}{L_a}z_2^2 + z_2[-K_m\phi_f^2(i_f)\omega_0 \\ &\quad -\frac{R_a+R_p}{K_m}(\tau_L + B\omega_0) - L_aR_fi_ai_f + R_pL_a(i_a^2 - i_fi_a) \\ &\quad +R_p\phi_f(i_f)i_f + \frac{K_m}{JL_a}z_1 - K_m\phi_f^2(i_f)z_1 + \phi_f(i_f)u].\end{aligned}$$

The derivation of the control law involves the following steps:

$$\begin{aligned}u &= \frac{1}{\phi_f(i_f)}[K_m\phi_f^2(i_f)\omega_0 + \frac{R_a+R_p}{K_m}(\tau_L + B\omega_0) + L_aR_fi_ai_f \\ &\quad -R_pL_a(i_a^2 - i_fi_a) - R_p\phi_f(i_f)i_f - \frac{K_m}{JL_a}z_1 + K_m\phi_f^2(i_f)z_1].\end{aligned}$$

Rewriting u for $z_1 = x_1 - \omega_0$ yields:

$$\begin{aligned}u &= \frac{1}{\phi_f(i_f)}[K_m\phi_f^2(i_f)\omega_0 + \frac{R_a+R_p}{K_m}(\tau_L + B\omega_0) + L_aR_fi_ai_f \\ &\quad -R_pL_a(i_a^2 - i_fi_a) - R_p\phi_f(i_f)i_f - \frac{K_m}{JL_a}x_1 + \frac{K_m}{JL_a}\omega_0 \\ &\quad +K_m\phi_f^2(i_f)x_1 - K_m\phi_f^2(i_f)\omega_0].\end{aligned}$$

Finally, replacing x_1 with ω:

$$\begin{aligned} u &= \frac{1}{\phi_f(i_f)}\Big[\frac{R_a+R_p}{K_m}(\tau_L+B\omega_0)+L_aR_fi_ai_f \\ &\quad -R_pL_a(i_a^2-i_fi_a)-R_p\phi_f(i_f)i_f \\ &\quad -\frac{K_m}{JL_a}(\omega-\omega_0)+K_m\phi_f^2(i_f)\omega\Big]. \end{aligned}$$

For the second case, when the motor operates below base speed,

$$\begin{aligned} L_a di/dt &= V-R_ai-K_m\phi_f(i)\omega \\ d\phi_f/dt &= -R_fi \\ Jd\omega/dt &= K_m\phi_f(i)i-B\omega-\tau_L. \end{aligned}$$

Let

$$\lambda = \phi_f(i)L_ai.$$

Taking the derivative yields:

$$\begin{aligned} d\lambda/dt &= \frac{\partial}{\partial i}[\phi_f(i)i]L_a\frac{di}{dt} \\ &= \left[\frac{\partial\phi_f(i)}{\partial i}i+\frac{\partial i}{\partial i}\phi_f(i)\right]\cdot(V-R_ai-K_m\phi_f(i)\omega) \\ &= \left[\frac{\partial\phi_f(i)}{\partial i}i+\phi_f(i)\right]\cdot(V-R_ai-K_m\phi_f(i)\omega). \end{aligned}$$

The additional figures below reveal the effect on the system error for different values of G_1.

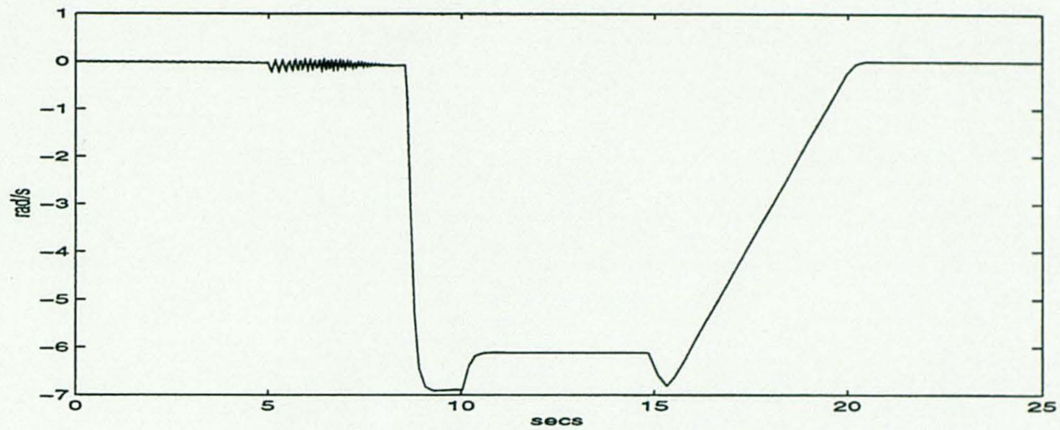

Figure A.1: Plot of error for $G_1 = 1.0$

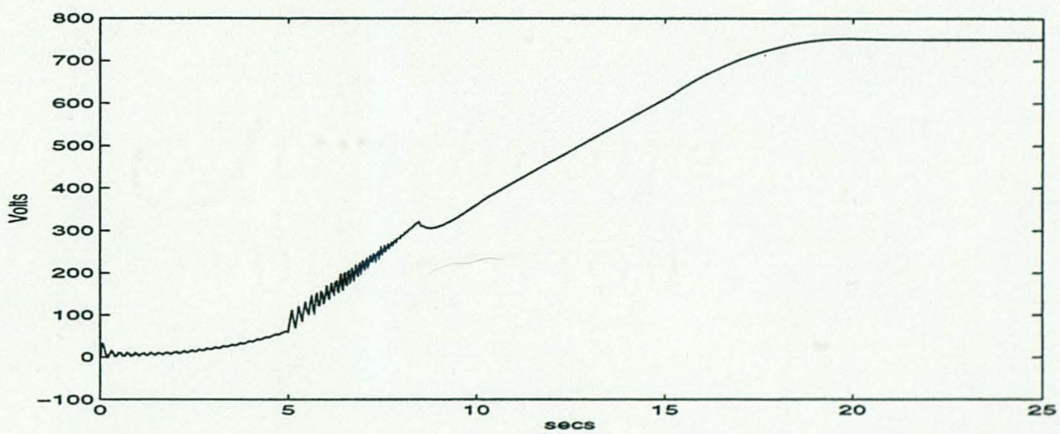

Figure A.2: Plot of the combined control law for $G_1 = 1.0$

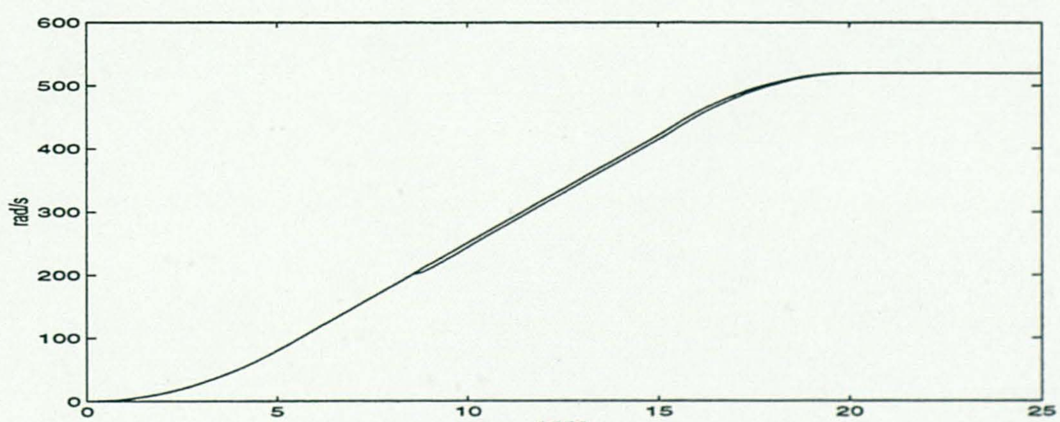

Figure A.3: Plot of both actual motor speed and reference speed for $G_1 = 1.0$

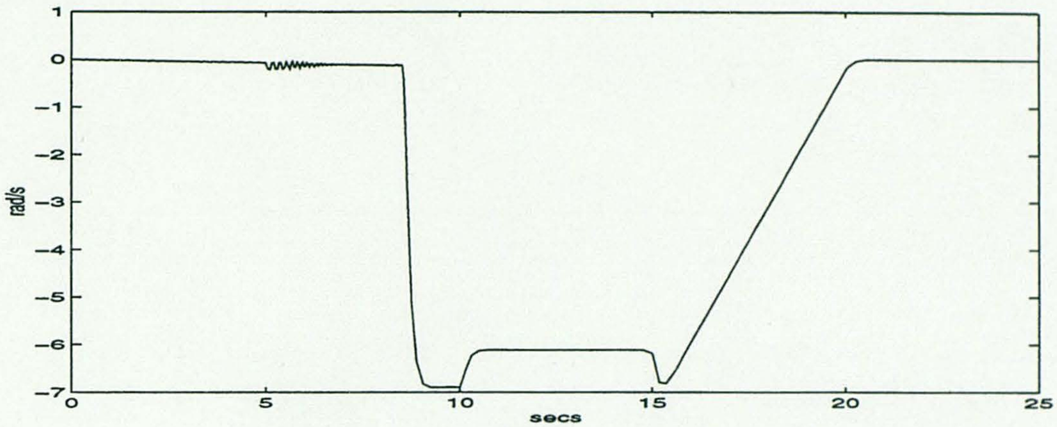

Figure A.4: Plot of error for $G_1 = 5.0$

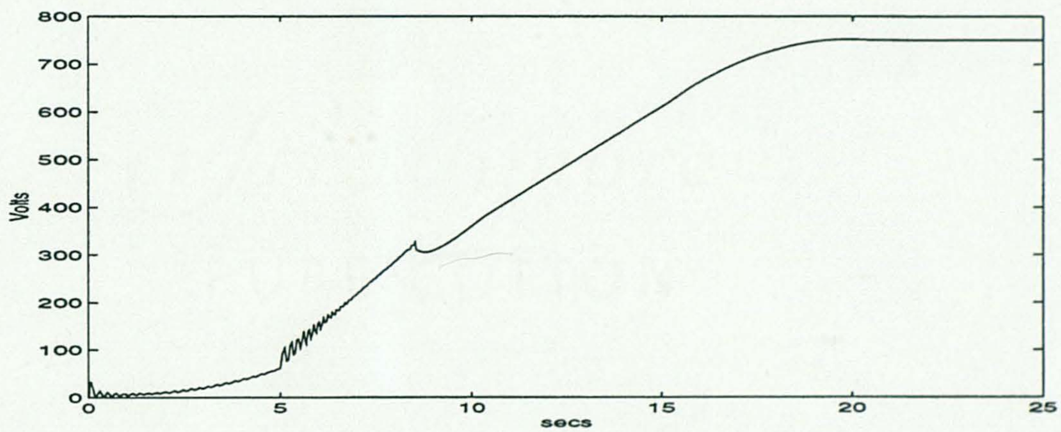

Figure A.5: Plot of the combined control law for $G_1 = 5.0$

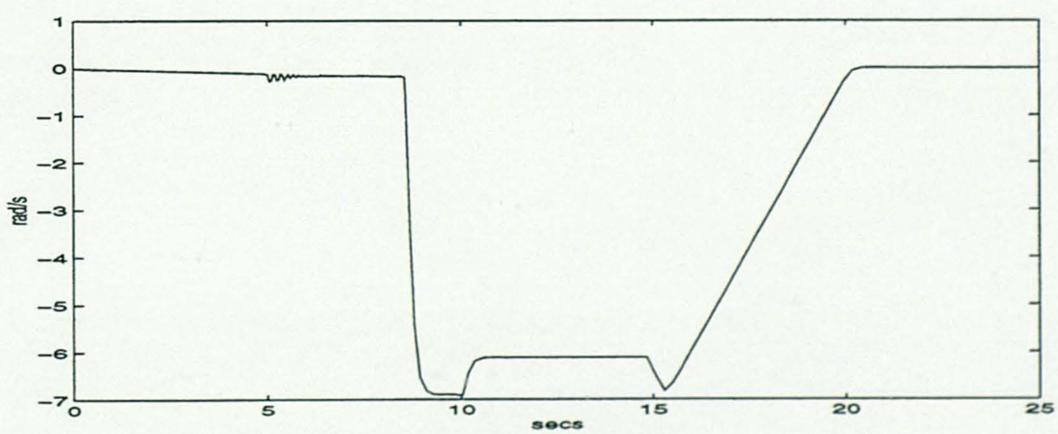

Figure A.6: Plot of error for $G_1 = 10.0$

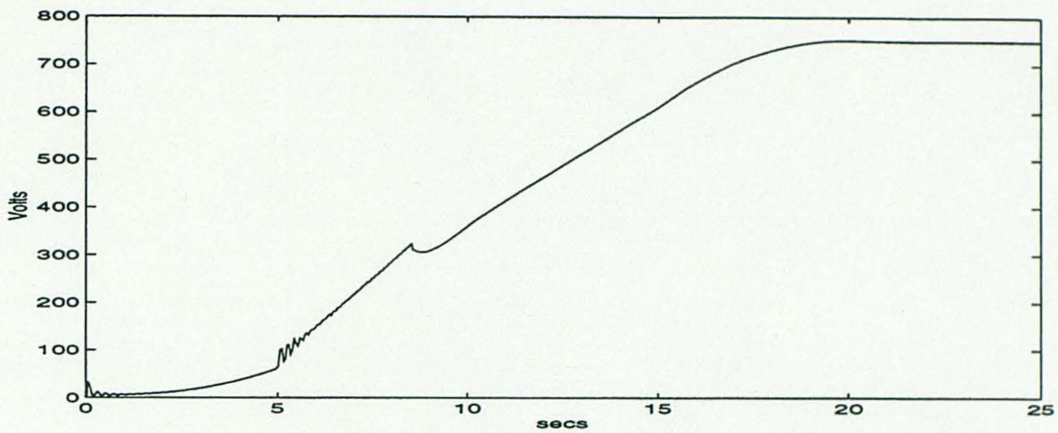

Figure A.7: Plot of the combined control law for $G_1 = 10.0$

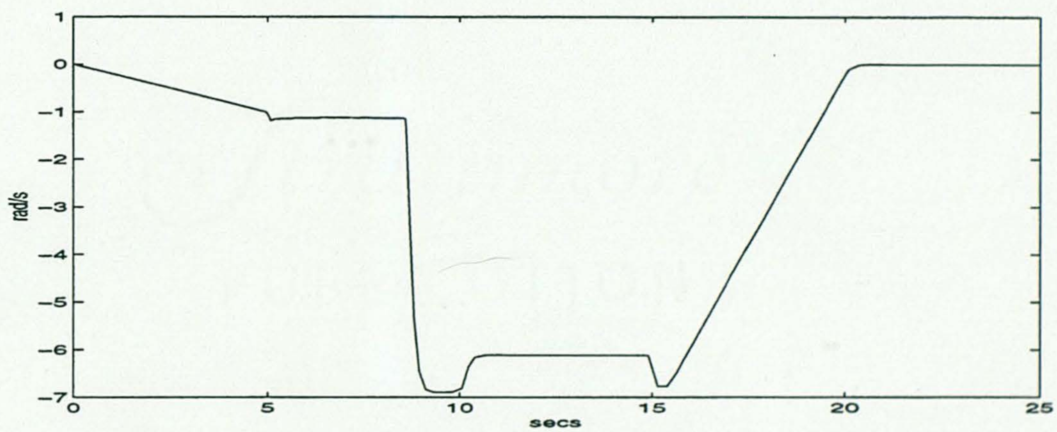

Figure A.8: Plot of error for $G_1 = 100.0$

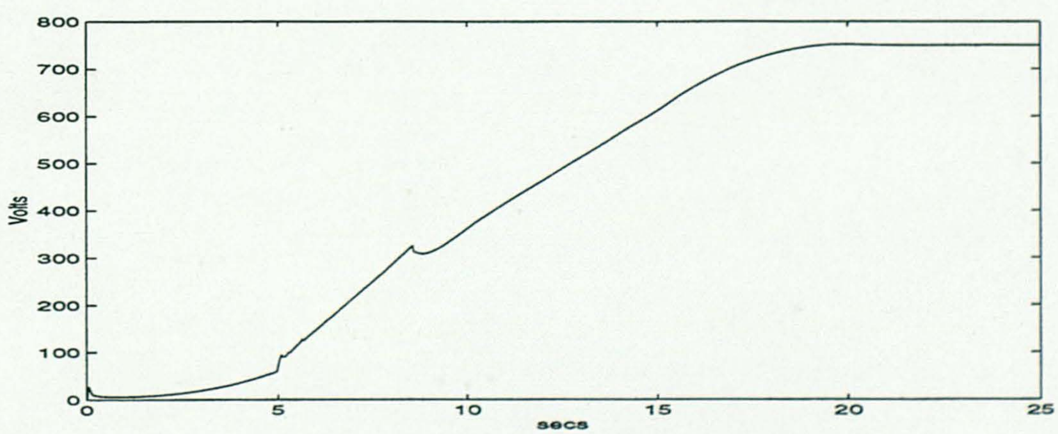

Figure A.9: Plot of the combined control law for $G_1 = 100.0$

Load Torque is Unknown; PI Control is Used

Solve $PA + A^T P = -Q$:

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -k_0 & -k_1 \end{bmatrix} + \begin{bmatrix} 0 & -k_0 \\ 1 & -k_1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -k_0 p_{12} & p_{11} - k_1 p_{12} \\ -k_0 p_{22} & p_{21} - k_1 p_{22} \end{bmatrix} + \begin{bmatrix} -k_0 p_{21} & -k_0 p_{22} \\ p_{11} - k_1 p_{21} & p_{12} - k_1 p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The following four equations result:

$$\begin{aligned} -k_0 p_{12} - k_0 p_{21} &= -1 \\ p_{11} - k_1 p_{21} - k_0 p_{22} &= 0 \\ p_{11} - k_1 p_{12} - k_0 p_{22} &= 0 \\ p_{12} + p_{21} - 2k_1 p_{22} &= -1. \end{aligned}$$

From this we obtain

$$\begin{aligned} p_{11} &= \frac{k_0^2 + k_0 + k_1^2}{2k_0 k_1} \\ p_{12} &= \frac{1}{2k_0} \\ p_{21} &= \frac{1}{2k_0} \\ p_{22} &= \frac{k_0 + 1}{2k_0 k_1}. \end{aligned}$$

So

$$P = \begin{bmatrix} \frac{k_0^2 + k_0 + k_1^2}{2k_0 k_1} & \frac{1}{2k_0} \\ \frac{1}{2k_0} & \frac{k_0 + 1}{2k_0 k_1} \end{bmatrix}.$$

Rewriting $\dot{x_1}$ and $\dot{x_2}$ in terms of z yields

$$\begin{aligned} \dot{x_1} &= -k_0 z_0 - k_1 x_1 + \frac{K_m}{JL_a} z_2 - \frac{B}{J}\omega_0 - \dot{\omega_0} \\ &= -k_0 z_0 - k_1 z_1 + \frac{K_m}{JL_a} z_2 - \frac{B}{J}\omega_0 - \dot{\omega_0} \end{aligned}$$

$$
\begin{aligned}
\dot{x_2} &= -L_a R_f i_a i_f + R_p L_a (i_a^2 - i_f i_a) + R_p \phi_f(i_f) i_f \\
&\quad -K_m \phi_f^2(i_f) x_1 - K_m \phi_f^2(i_f)\omega_0 - \frac{R_a + R_p}{L_a} x_2 + \phi_f(i_f) u \\
&= -L_a R_f i_a i_f + R_p L_a (i_a^2 - i_f i_a) + R_p \phi_f(i_f) i_f + \phi_f(i_f) u \\
&\quad -K_m \phi_f^2(i_f) z_1 - K_m \phi_f^2(i_f)\omega_0 - \frac{R_a + R_p}{L_a} x_2.
\end{aligned}
$$

The steps to rewrite x_2 are

$$
\begin{aligned}
x_2 &= z_2 - \frac{JL_a}{K_m} k_0 x_0 - \frac{JL_a}{K_m}(k_1 - B/J) z_1 \\
&= z_2 - \frac{JL_a}{K_m} k_0 x_0 - \frac{JL_a}{K_m}(k_1 z_1 - (B/J) z_1) \\
&= z_2 - \frac{JL_a}{K_m} k_0 x_0 - \frac{JL_a}{K_m} k_1 z_1 + \frac{BL_a}{K_m} z_1.
\end{aligned}
$$

So

$$
\begin{aligned}
-\frac{R_a + R_p}{L_a} x_2 &= -\frac{R_a + R_p}{L_a} z_2 + \frac{J}{K_m}(R_a + R_p) k_0 x_0 \\
&\quad + \frac{J}{K_m}(R_a + R_p) k_1 z_1 - \frac{B}{K_m}(R_a + R_p) z_1.
\end{aligned}
$$

Calculating the control law involves the following steps.

Rewrite u in terms of x,

$$
\begin{aligned}
u &= \frac{1}{\phi_f(i_f)} \Big[-(k_1 + k_2) x_2 - k_0 (k_1 + k_2) \frac{JL_a}{K_m} x_0 \\
&\quad -(k_1 + k_2) \frac{JL_a}{K_m}(k_1 - B/J) x_1 + \Big(K_m \phi_f^2(i_f) + \frac{B}{K_m}(R_a + R_p) \\
&\quad + \frac{JL_a}{K_m} k_1^2 - k_1 \frac{J}{K_m}(R_a + R_p) - k_0 \frac{JL_a}{K_m} - k_1 \frac{BL_a}{K_m} \Big) x_1 \\
&\quad + \left(\frac{BL_a}{K_m} k_1 + K_m \phi_f^2(i_f) - \frac{B^2 L_a}{JK_m} \right) \omega_0 \\
&\quad + \left(\frac{JL_a}{K_m} k_1 - \frac{BL_a}{K_m} \right) \dot{\omega_0} \\
&\quad -k_0 \frac{J}{K_m}(R_a + R_p) x_0 + L_a R_f i_a i_f - R_p L_a (i_a^2 - i_f i_a) - R_p \phi_f(i_f) i_f \Big],
\end{aligned}
$$

and then rewrite in terms of the original variables,

$$
= \frac{1}{\phi_f(i_f)} \left[-(k_1 + k_2) L_a i_a \phi_f(i_f) + L_a R_f i_a i_f - R_p L_a (i_a^2 - i_f i_a) \right.
$$

$$
\begin{aligned}
&-R_p\phi_f(i_f)i_f - k_0(k_1+k_2)\frac{JL_a}{K_m}x_0 - k_0\frac{J}{K_m}(R_a+R_p)x_0 \\
&+\Big(-k_1(k_1+k_2)\frac{JL_a}{K_m} + (k_1+k_2)\frac{BL_a}{K_m} + K_m\phi_f^2(i_f) \\
&+\frac{B}{K_m}(R_a+R_p) + k_1^2\frac{JL_a}{K_m} - k_1\frac{J}{K_m}(R_a+R_p) \\
&-k_0\frac{JL_a}{K_m} - k_1\frac{BL_a}{K_m}\Big)(\omega-\omega_0) + \left(\frac{JL_a}{K_m}k_1 - \frac{BL_a}{K_m}\right)\dot{\omega_0} \\
&+\left(\frac{BL_a}{K_m}k_1 + K_m\phi_f^2(i_f) - \frac{B^2L_a}{JK_m}\right)\omega_0\Big].
\end{aligned}
$$

Finally,

$$
\begin{aligned}
u \;=\; & \frac{1}{\phi_f(i_f)}\Big[-(k_1+k_2)L_a i_a\phi_f(i_f) + L_aR_f i_a i_f - R_pL_a(i_a^2 - i_f i_a) \\
&-R_p\phi_f(i_f)i_f - k_0\frac{J}{K_m}(R_a+R_p)x_0 - k_0(k_1+k_2)\frac{JL_a}{K_m}x_0 \\
&+\Big(k_2\frac{BL_a}{K_m} + \frac{B}{K_m}(R_a+R_p) - k_1k_2\frac{JL_a}{K_m} \\
&-k_1\frac{J}{K_m}(R_a+R_p) - k_0\frac{JL_a}{K_m}\Big)(\omega-\omega_0) \\
&+\left(\frac{JL_a}{K_m}k_1 - \frac{BL_a}{K_m}\right)\dot{\omega_0} \\
&+K_m\phi_f^2(i_f)\omega + \left(k_1\frac{BL_a}{K_m} - \frac{B^2L_a}{JK_m}\right)\omega_0\Big]
\end{aligned}
$$

For the case below base speed, the steps are similar.

The additional figures presented below provide an indication of the effect of varying the P, I, and D gains in the control law on the steady state error.

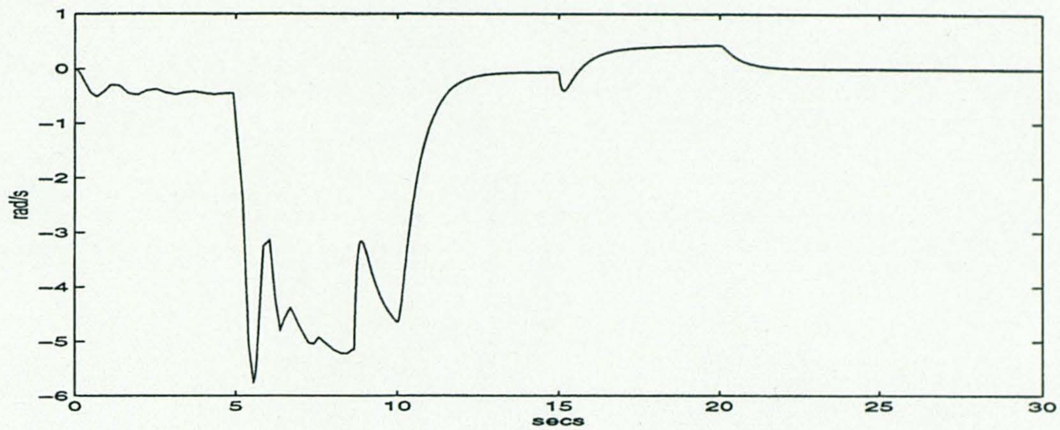

Figure A.10: Error plot for k0 = 17.0, k1 = 12.0, k2 = 190.0

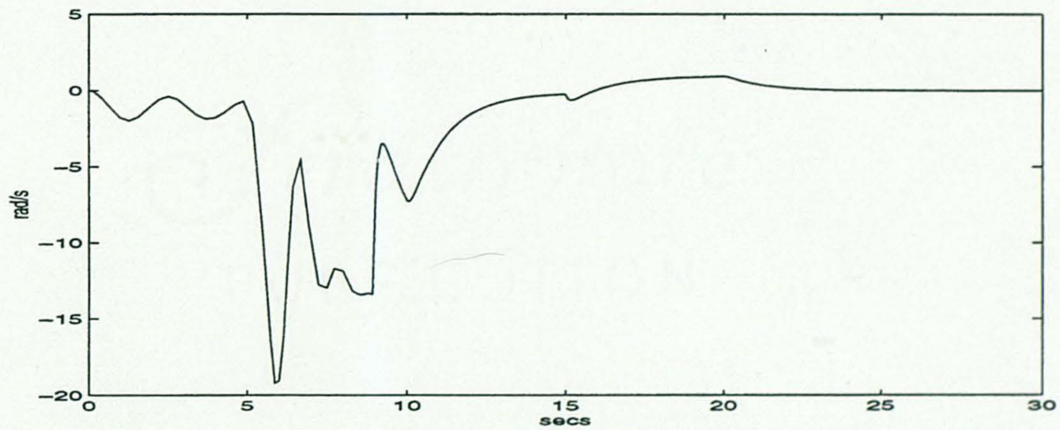

Figure A.11: Error plot for k0 = 7.0, k1 = 10.0, k2 = 50.0

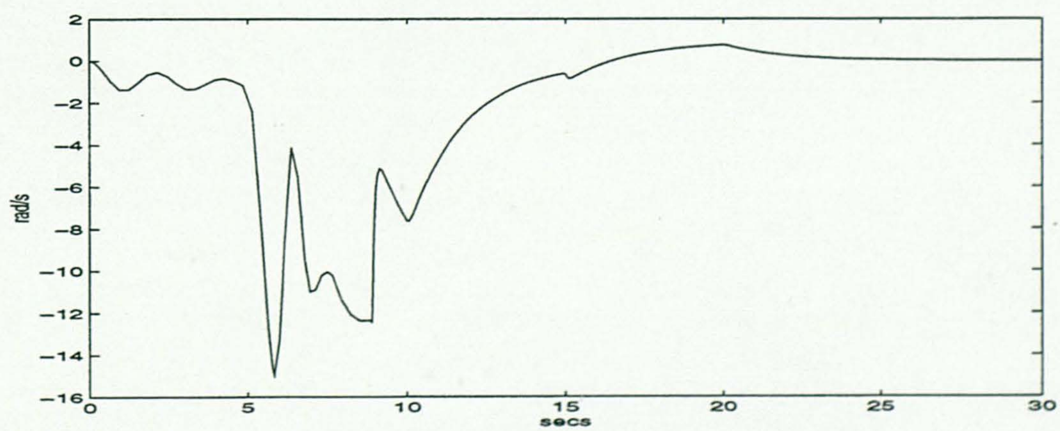

Figure A.12: Error plot for k0 = 7.0, k1 = 16.0, k2 = 45.0

Uncertainties Exist

Apply the triangular inequality to the following expression:

$$z_1\dot{z_1} \leq -\frac{B}{J}z_1^2 + \rho_1|z_1| - \frac{1}{\epsilon}\rho_1^2 z_1^2.$$

The steps involved are:

$$a^2 + b^2 \geq 2ab$$

with

$$a = \frac{1}{\sqrt{\epsilon}}\rho_1|z_1|, \qquad b = \frac{\sqrt{\epsilon}}{2}$$

$$a^2 = \frac{1}{\epsilon}\rho_1^2|z_1|^2 \qquad b^2 = \frac{\epsilon}{4} \qquad 2ab = \left(\frac{1}{\sqrt{\epsilon}}\rho_1|z_1|\right)\frac{\sqrt{\epsilon}}{2} = \rho_1|z_1|.$$

Therefore,

$$\begin{aligned} a^2 + b^2 \geq 2ab \Rightarrow 2ab - a^2 \leq b^2 \\ \rho_1|z_1| - \frac{1}{\epsilon}\rho_1^2|z_1|^2 \leq \frac{\epsilon}{4}, \end{aligned}$$

and

$$z_1\dot{z_1} \leq -\frac{B}{J}z_1^2 + \frac{\epsilon}{4}.$$

The second half of the Lyapunov function plus the term dropped before is

$$\begin{aligned} L_a^2 z_2\dot{z_2} + \frac{K_m}{J}z_1 z_2 &= -K_m L_a \phi_f^2(i_f) z_1 z_2 - K_m L_a \phi_f^2(i_f)\omega_0 z_2 - (R_a + R_p)L_a z_2^2 \\ &\quad -(R_a + R_p)L_a x_2^d z_2 - R_f L_a^2 i_a i_f z_2 + R_p L_a^2(i_a^2 - i_a i_f) z_2 \\ &\quad +R_p L_a \phi_f(i_f) i_f z_2 + \phi_f(i_f) L_a u z_2 - L_a^2 \dot{u}_{R_{11}} z_2 \\ &\quad +\frac{K_m}{J} z_1 z_2. \end{aligned}$$

Factoring out L_a and rearranging the terms:

$$\begin{aligned} \text{LHS of (3.42)} &= -(R_a + R_p)L_a z_2^2 + z_2 L_a[-K_m \phi_f^2(i_f) z_1 - K_m \phi_f^2(i_f)\omega_0 \\ &\quad -(R_a + R_p)x_2^d - R_f L_a i_a i_f + R_p L_a(i_a^2 - i_a i_f) \\ &\quad +R_p \phi_f(i_f) i_f + \phi_f(i_f) u - L_a \dot{u}_{R_{11}} + \frac{K_m}{J L_a} z_1]. \end{aligned}$$

Replacing x_2^d in the equation with its actual value yields:

$$\begin{aligned}
\text{LHS of (3.42)} &= -(R_a+R_p)L_a z_2^2 + z_2 L_a[-K_m\phi_f^2(i_f)z_1 \\
&\quad -K_m\phi_f^2(i_f)\omega_0 - \frac{(R_a+R_p)}{K_m}(B\omega_0+\tau_{L_0}) - (R_a+R_p)u_{R_{11}} \\
&\quad -R_f L_a i_a i_f + R_p L_a(i_a^2 - i_a i_f) + R_p\phi_f(i_f)i_f + \phi_f(i_f)u \\
&\quad -L_a\dot{u}_{R_{11}} + \frac{K_m}{JL_a}z_1].
\end{aligned}$$

Replace $\dot{u}_{R_{11}}$ with its actual value:

$$\begin{aligned}
-L_a\dot{u}_{R_{11}} &= -L_a\left[-\frac{1}{K_m}\left(\frac{1}{\epsilon_1}\right)\rho_1^2\left(-\frac{B}{J}x_1+\frac{K_m}{J}x_2-\frac{\tau_L}{J}\right)\right] \\
&= -\frac{BL_a}{JK_m}\left(\frac{1}{\epsilon_1}\right)\rho_1^2 x_1 + \frac{L_a}{J}\frac{1}{\epsilon_1}\rho_1^2 x_2 - \frac{L_a\tau_L}{JK_m}\left(\frac{1}{\epsilon_1}\right)\rho_1^2.
\end{aligned}$$

Replace $u_{R_{11}}$ with its actual value:

$$\begin{aligned}
-(R_a+R_p)u_{R_{11}} &= -(R_a+R_p)\left[-\frac{1}{K_m}\left(\frac{1}{\epsilon_1}\right)\rho_1^2 z_1\right] \\
&= \frac{(R_a+R_p)}{K_m}\left(\frac{1}{\epsilon_1}\right)\rho_1^2 z_1.
\end{aligned}$$

Combining all of these substitutions we may rewrite (3.42) as:

$$\begin{aligned}
\text{LHS of (3.42)} &= -(R_a+R_p)L_a z_2^2 + L_a z_2[-K_m\phi_f^2(i_f)x_1 + \frac{R_a+R_p}{K_m}\left(\frac{1}{\epsilon_1}\right)\rho_1^2 z_1 \\
&\quad -R_f L_a i_a i_f + R_p L_a(i_a^2 - i_a i_f) + R_p\phi_f(i_f)i_f + \frac{K_m}{JL_a}z_1 \\
&\quad -\frac{BL_a}{JK_m}\left(\frac{1}{\epsilon_1}\right)\rho_1^2 x_1 + \frac{L_a}{J}\left(\frac{1}{\epsilon_1}\right)\rho_1^2 x_2 \\
&\quad -\frac{L_a\tau_L}{JK_m}\left(\frac{1}{\epsilon}\right)\rho_1^2 - \frac{R_a+R_p}{K_m}(B\omega_0+\tau_{L_0}) + \phi_f(i_f)u].
\end{aligned}$$

First consider the term with the uncertainty in the denominator, that is $\frac{K_m}{JL_a}z_1$. The uncertain coefficient, $\frac{1}{L_a}$, may be rewritten and bounded as follows:

$$\begin{aligned}
\frac{1}{L_a} &= \frac{1}{L_{a_0}} + \left(\frac{1}{L_a} - \frac{1}{L_{a_0}}\right) \\
&= \frac{1}{L_{a_0}} + \left(\frac{L_{a_0}-L_a}{L_a L_{a_0}}\right) \\
&\le \frac{1}{L_{a_0}} + \left(\frac{L_{a_0}-L_{a_0}(1-\kappa_1)}{L_{a_0}L_{a_0}(1-\kappa_1)}\right) \\
&\le \frac{1}{L_{a_0}} + \left(\frac{\kappa_1}{L_{a_0}(1-\kappa_1)}\right).
\end{aligned}$$

Thus the term $\frac{K_m}{JL_a}z_1$ may be rewritten as the sum of two terms, a known term and an uncertain term:

$$\frac{K_m}{JL_a}z_1 \leq \frac{K_m}{JL_{a_0}}z_1 + \frac{\kappa_1 K_m}{JL_{a_0}(1-\kappa_1)}z_1.$$

All of the steps in finding the common denominator are

$$\begin{aligned}
& \rho_2|z_2| - \frac{\mu^2+\epsilon^2}{|\mu|^3+\epsilon^3}\mu\rho_2 z_2 \\
= \; & |\mu| - \frac{\mu^2+\epsilon^2}{|\mu|^3+\epsilon^3}\mu^2 \\
= \; & \frac{|\mu|^4+\epsilon^3|\mu| - |\mu|^4 - \epsilon^2\mu^2}{|\mu|^3+\epsilon^3} \\
= \; & \frac{\epsilon^3|\mu| - \epsilon^2|\mu|^2}{|\mu|^3+\epsilon^3} \\
= \; & \frac{\epsilon^2|\mu| - \epsilon|\mu|^2}{|\mu|^3+\epsilon^3}\epsilon.
\end{aligned}$$

Below base speed, consider the term with an uncertainty in the denominator, $\frac{FK_m}{JL_a^2}z_1$. The uncertainty may be rewritten and bounded as follows:

$$\begin{aligned}
\frac{1}{L_a^2} &= \frac{1}{L_{a_0}^2} + \left(\frac{1}{L_a^2} - \frac{1}{L_{a_0}^2}\right) \\
&= \frac{1}{L_{a_0}^2} + \left(\frac{L_{a_0}^2 - L_a^2}{L_a^2 L_{a_0}^2}\right) \\
&\leq \frac{1}{L_{a_0}^2} + \left(\frac{L_{a_0}^2 - L_{a_0}^2(1+\kappa_1)^2}{L_{a_0}(1-\kappa_1)^2 L_{a_0}^2}\right) \\
&= \frac{1}{L_{a_0}^2} + \left(\frac{1-(1+\kappa_1)^2}{L_{a_0}^2(1-\kappa_1)^2}\right) \\
&\leq \frac{1}{L_{a_0}} + \left(\frac{2\kappa_1+\kappa_1^2}{L_{a_0}^2(1-\kappa_1)^2}\right).
\end{aligned}$$

Thus the term $\frac{K_m}{FJL_a^2}z_1$ may be rewritten as the sum of two terms, a known term and an uncertain term:

$$\frac{K_m}{FJL_a^2}z_1 \leq \frac{K_m}{FJL_{a_0}^2}z_1 + \frac{(2\kappa_1+\kappa_1^2)K_m}{(1-\kappa_1)^2 FJL_{a_0}^2}z_1.$$

APPENDIX B

SIMNON FILES

The System is Perfectly Known

The following SIMNON file was used to simulate the case when it assumed that the system is perfectly known.

```
CONTINUOUS SYSTEM dcperf

"ia1 = armature current for case 1
"ia2 = armature current for case 2
"if = field current; if < ia above base speed
"                    if = ia below base speed
"w = speed
"phi = flux
"iphi = current associated with flux
"u = control


STATE ia1 ia2 if w
DER dia1 dia2 dif dw
TIME t
OUTPUT u e phi iphi iia wref

"Electrical dynamics

"Case 1 – over base speed

dia1 = IF ABS(w) > wbase THEN ia1h ELSE ia1l
dif = IF ABS(w) > wbase THEN ifh ELSE ifl

"Two different equations are required to handle the armature current
"for case1. The h subscript indicates above base speed, the l
"subscript indicates below base speed.
ia1h = (u - Ra*ia1 - Rp*(ia1 - if) - Km*phi*w)/La
ia1l = (u - Rf*ia1 - Ra*ia1 - Km*phi*w)/(dphidi + La)

"Two different equations are required to handle the field current
"for case1
ifh = (-Rf*if + Rp*(ia1 - if))/dphidi
ifl = (u - Rf*if - Ra*if - Km*phi*w)/(dphidi + La)

"Case 2 – under base speed

"For this case, armature current equals field current (ia = if)
dia2 = (u - Rf*ia2 - Ra*ia2 - Km*phi*w)/(dphidi + La)

"The equation for armature current changes at base speed
iia = IF ABS(w) < wbase THEN ia2 ELSE ia1

"The current associated with the flux depends upon whether the motor
"is above or below base speed.
```

```
iphi = IF ABS(w) < wbase THEN ia2 ELSE if

"Mechanical motion – the same for both cases
dw = (Km*phi*iia - B*w - tauL)/J

"Control

"Input voltage (control) must be bounded above by 1000V
uu = IF w < wbase THEN u2 ELSE u1
u = IF ABS(uu) > 1000 THEN SIGN(uu)*1000 ELSE uu

"Case 1
u1 = (term1 + term2 + term3)/phii
term1 = (Ra + Rp)*(tauL0 + B*wref)/Km
term2 = La0*Rf*ia1*if - Rp*La0*ia1*(ia1 - if) - Rp*phii*if
term3 = Km*(wref - w)/(J*La0) + Km*phii*phii*w

phii = IF phi < 1.0 THEN 1.0 ELSE phi

"Case 2
u2 = -Km*(w - wref)/(J*La0*La0*F) + Km*phi*w + Ra*ia2 - u21
u21 = (B*G)*(phii*La0*ia2 - La0*(tauL0 + B*wref)/Km)/(F*J*La0)
F = dphidi*ia2 + phii

"Error
e = w - wref

"Simulated load torque (nominal value)
tauL0 = IF t < 5 THEN 0.0 ELSE (IF t > 10 THEN 1250.0 ELSE 1250.0*(t-5.0)/5.0)

"Simulated load torque with 10% perturbation
tauL = IF t < 5 THEN 0.0 ELSE (IF t > 10 THEN 1125.0 ELSE 1125.0*(t-5.0)/5.0)

"Derivative of flux with respect to current
dphidi = IF ABS(iphi) < 169.2 THEN .108 ELSE d2
d2 = IF ABS(iphi) < 200 THEN .09066 ELSE d3
d3 = IF ABS(iphi) < 246.2 THEN .076 ELSE d4
d4 = IF ABS(iphi) < 284.7 THEN .0493 ELSE d5
d5 = IF ABS(iphi) < 307.6 THEN .04266 ELSE d6
d6 = IF ABS(iphi) < 353.8 THEN .03866 ELSE d7
d7 = IF ABS(iphi) < 407.7 THEN .0266 ELSE d8
d8 = IF ABS(iphi) < 500.1 THEN .02266 ELSE d9
d9 = IF ABS(iphi) < 631 THEN .01733 ELSE d10
d10 = 0.01466

"Integration of flux with respect to current
phi = IF ABS(iphi) < 169.2 THEN .108*iphi ELSE dd2
dd2 = IF ABS(iphi) < 200 THEN .09066*(iphi - 169.2) + 18.2736 ELSE dd3
```

```
dd3 = IF ABS(iphi) < 246.2 THEN .07600*(iphi - 200) + 21.0660 ELSE dd4
dd4 = IF ABS(iphi) < 284.7 THEN .04930*(iphi - 246.7) + 24.5770 ELSE dd5
dd5 = IF ABS(iphi) < 307.6 THEN .04266*(iphi - 284.7) + 26.4750 ELSE dd6
dd6 = IF ABS(iphi) < 353.8 THEN .03866*(iphi - 307.6) + 27.4520 ELSE dd7
dd7 = IF ABS(iphi) < 407.7 THEN .02660*(iphi - 353.8) + 29.2380 ELSE dd8
dd8 = IF ABS(iphi) < 500.1 THEN .02266*(iphi - 407.7) + 30.6720 ELSE dd9
dd9 = IF ABS(iphi) < 631 THEN .01733*(iphi - 500.1) + 32.7660 ELSE dd10
dd10 = IF ABS(iphi) < 1000 THEN .01466*(iphi - 631) + 35.0340 ELSE dd11
dd11 = 40.4335

"Constants
B : 0.1 "N*m/rad/s
J : 3.0 "Kg*m*m
Km : 0.04329 "N*m/(Wb*A)
La0 : 0.0014 "H (nominal value)
La : 0.00154 "H (10% perturbation)
Ra : 0.00989 "ohm
Rf : 0.01485 "ohm
Rp : 0.01696 "ohm

"Control parameter
G : 20.0

"Base speed
wbase : 200.0 "rad/s

"Parabolic equation to simulate reference speed
wref = IF t < 5 THEN 3.2*t*t ELSE IF t < 15 THEN 34.0*(t-5) + 80.0 ELSE wwref
wwref = IF t < 20 THEN -4.0*(t-20.0)*(t-20.0)+520.0 ELSE 520

END
```

The following is an example of the "go" file used to run the simulation of the perfect control case.

```
macro goperf
syst dcperf
init ia1:0.0
init ia2:0.0
init if:0.0
init w:0.0
error 1e-4
algor dopri45r
store w e u ia1 ia2 iia if phi iphi wref u1 u2 F
simu 0 30 0.001/0.01
end
```

Load Torque is Unknown; PI Control is Used

This is the SIMNON file used to simulate the case when it is assumed that the load torque is unknown. A control utilizing PI design is used.

```
CONTINUOUS SYSTEM dcpi

"ia1 = armature current for case 1
"ia2 = armature current for case 2
"if = field current; if < ia above base speed
"                    if = ia below base speed
"w = speed
"x0 = integral of speed (w) for use in PI control
"phi = flux
"iphi = current associated with flux
"u = control


STATE ia1 ia2 if w x0
DER dia1 dia2 dif dw dx0
TIME t
OUTPUT u e phi iphi iia wref dwref

"Electrical dynamics

"Case 1 - over base speed

dia1 = IF ABS(w) > wbase THEN ia1h ELSE ia1l
dif = IF ABS(w) > wbase THEN ifh ELSE ifl

"Two different equations are required to handle the armature current
"for case1. The h subscript indicates above base speed, the l
"subscript indicates below base speed.
ia1h = (u - Ra*ia1 - Rp*(ia1 - if) - Km*phi*w)/La
ia1l = (u - Rf*ia1 - Ra*ia1 - Km*phi*w)/(dphidi + La)

"Two different equations are required to handle the field current
"for case1
ifh = (-Rf*if + Rp*(ia1 - if))/dphidi
ifl = (u - Rf*if - Ra*if - Km*phi*w)/(dphidi + La)

"Case 2 - under base speed

"For this case, armature current equals field current (ia = if)
dia2 = (u - Rf*ia2 - Ra*ia2 - Km*phi*w)/(dphidi + La)

"The equation for armature current changes at base speed
iia = IF ABS(w) < wbase THEN ia2 ELSE ia1

"The current associated with the flux depends upon whether the motor
"is above or below base speed.
```

```
iphi = IF ABS(w) < wbase THEN ia2 ELSE if

"Mechanical motion - the same for both cases
dw = (Km*phi*iia - B*w - tauL)/J

"Integral of speed for PI control
"Note: dx0 = x1    x1 = w - wref
dx0 = w - wref

"Control

"Input voltage (control) must be bounded above by 1000V
uu = IF w < wbase THEN u2 ELSE u1
u = IF ABS(uu) > 1000 THEN SIGN(uu)*1000 ELSE uu

"Case 1
u1 = (u11 + u12 + u13 + u14 +u15 + u16)/phii
u11 = -(k1 + k2)*La0*ia1*phii + La0*Rf*ia1*if - Rp*La0*ia1*(ia1 - if)
u12 = -Rp*phii*if - k0*(J/Km)*(Ra + Rp)*x0 - k0*(k1 + k2)*(J*La0/Km)*x0
u13 = (k2*B*La0/Km + B*(Ra + Rp)/Km - k1*k2*J*La0/Km)*(w - wref)
u14 = (-k1*J*(Ra + Rp)/Km - k0*J*La0/Km - k1*B*La0/Km)*(w - wref)
u15 = (k1*J*La0/Km - B*La0/Km)*dwref
u16 = (k1*B*La0/Km + Km*phii*phii)*w - (B*B*La0)/(J*Km)*wref

phii = IF phi < 1.0 THEN 1.0 ELSE phi

"Case 2
u2 = (u21 + u22 + u23 + u24)/F
u21 = -(k1 + k2)*La0*phii*ia2 + Ra*F*ia2 - k0*(k1 + k2)*(J*La0/Km)*x0
u22 = (k2*B*La0 - k1*k2*J*La0 - k0*J*La0)*(w - wref)/Km
u23 = (k1*J*La0 - B*La0)*dwref/Km
u24 = F*Km*phii*w + (k1*B*La0 - (B*B*La0)/J)*wref/Km
F = dphidi*ia2 + phii

"Error
e = w - wref

"Simulated load torque
tauL = IF t < 5 THEN 0.0 ELSE (IF t > 10 THEN 1250.0 ELSE 1250.0*(t-5.0)/5.0)

"Derivative of flux with respect to current
dphidi = IF ABS(iphi) < 169.2 THEN .108 ELSE d2
d2 = IF ABS(iphi) < 200 THEN .09066 ELSE d3
d3 = IF ABS(iphi) < 246.2 THEN .076 ELSE d4
d4 = IF ABS(iphi) < 284.7 THEN .0493 ELSE d5
d5 = IF ABS(iphi) < 307.6 THEN .04266 ELSE d6
d6 = IF ABS(iphi) < 353.8 THEN .03866 ELSE d7
d7 = IF ABS(iphi) < 407.7 THEN .0266 ELSE d8
d8 = IF ABS(iphi) < 500.1 THEN .02266 ELSE d9
```

```
d9 = IF ABS(iphi) < 631 THEN .01733 ELSE d10
d10 = 0.01466

"Integration of flux with respect to current
phi = IF ABS(iphi) < 169.2 THEN .108*iphi ELSE dd2
dd2 = IF ABS(iphi) < 200 THEN .09066*(iphi - 169.2) + 18.2736 ELSE dd3
dd3 = IF ABS(iphi) < 246.2 THEN .07600*(iphi - 200) + 21.0660 ELSE dd4
dd4 = IF ABS(iphi) < 284.7 THEN .04930*(iphi - 246.7) + 24.5770 ELSE dd5
dd5 = IF ABS(iphi) < 307.6 THEN .04266*(iphi - 284.7) + 26.4750 ELSE dd6
dd6 = IF ABS(iphi) < 353.8 THEN .03866*(iphi - 307.6) + 27.4520 ELSE dd7
dd7 = IF ABS(iphi) < 407.7 THEN .02660*(iphi - 353.8) + 29.2380 ELSE dd8
dd8 = IF ABS(iphi) < 500.1 THEN .02266*(iphi - 407.7) + 30.6720 ELSE dd9
dd9 = IF ABS(iphi) < 631 THEN .01733*(iphi - 500.1) + 32.7660 ELSE dd10
dd10 = IF ABS(iphi) < 1000 THEN .01466*(iphi - 631) + 35.0340 ELSE dd11
dd11 = 40.4335

"Constants
B : 0.1 "N*m/rad/s
J : 3.0 "Kg*m*m
Km : 0.04329 "N*m/(Wb*A)
La : 0.0014 "H
Ra : 0.00989 "ohm
Rf : 0.01485 "ohm
Rp : 0.01696 "ohm

"PI control
k0 : 7.0
k1 : 16.0
k2 : 50.0

"Base speed
wbase : 200.0 "rad/s

"Parabolic equation to simulate reference speed
wref = IF t < 5 THEN 3.2*t*t ELSE IF t < 15 THEN 34.0*(t-5) + 80.0 ELSE wwref
wwref = IF t < 20 THEN -4.0*(t-20.0)*(t-20.0)+520.0 ELSE 520

"First time derivative of the equation for reference speed
dwref = IF t < 5 THEN 6.4*t ELSE IF t < 15 THEN 34.0 ELSE dwwref
dwwref = IF t < 20 THEN -8.0*t + 160 ELSE 0.0

END
```

The following is an example of the "go" file used to run the simulation of the PI control case.

```
macro godcpi
syst dcpi
init ia1:0.0
init ia2:0.0
init if:0.0
init w:0.0
error 1e-4
algor dopri45r
store w e u ia1 ia2 iia if phi iphi wref u1 u2 F
simu 0 30 0.001/0.01
end
```

The System Contains Bounded Uncertainties

The following file was used to simulate the case when it assumed that the system contains bounded uncertainties. In addition to the values shown in this file, other values of La, tauL, K1, and K2 were used to test the robustness of the control law. The control parameters, Ep11, Ep12, Ep21, and Ep22, were varied until the best result was achieved.

```
CONTINUOUS SYSTEM dcrob

"ia1 = armature current for case 1
"ia2 = armature current for case 2
"if = field current; if < ia above base speed
"                    if = ia below base speed
"w = speed
"phi = flux
"iphi = current associated with flux
"u = control


STATE ia1 ia2 if w
DER dia1 dia2 dif dw
TIME t
OUTPUT u e phi iphi iia wref

"Electrical dynamics

"Case 1 - over base speed

dia1 = IF ABS(w) > wbase THEN ia1h ELSE ia1l
dif = IF ABS(w) > wbase THEN ifh ELSE ifl

"Two different equations are required to handle the armature current
"for case1. The h subscript indicates above base speed, the l
"subscript indicates below base speed.
ia1h = (u - Ra*ia1 - Rp*(ia1 - if) - Km*phi*w)/La
ia1l = (u - Rf*ia1 - Ra*ia1 - Km*phi*w)/(dphidi + La)

"Two different equations are required to handle the field current
"for case1
ifh = (-Rf*if + Rp*(ia1 - if))/dphidi
ifl = (u - Rf*if - Ra*if - Km*phi*w)/(dphidi + La)

"Case 2 - under base speed

"For this case, armature current equals field current (ia = if)
dia2 = (u - Rf*ia2 - Ra*ia2 - Km*phi*w)/(dphidi + La)

"The equation for armature current changes at base speed
iia = IF ABS(w) < wbase THEN ia2 ELSE ia1
```

```
"The current associated with the flux depends upon whether the motor
"is above or below base speed.
iphi = IF ABS(w) < wbase THEN ia2 ELSE if

"Mechanical motion - the same for both cases
dw = (Km*phi*iia - B*w - tauL)/J

"Control

"Input voltage (control) must be bounded above by 1000V
uu = IF w < wbase THEN u2 ELSE u1
u = IF ABS(uu) > 1000 THEN SIGN(uu)*1000 ELSE uu

"Case 1
u1 = (u11 + u12 + u13 + u14 + u15 + u16 + uR12)/phii
u11 = Km*phii*phii*w + (Ra + Rp)*(B*wref + tauL0)/Km
u12 = -(Ra + Rp)*(1/Ep11)*rho11*rho11*(w - wref)/Km
u13 = -Rp*phii*if + Rf*La0*ia1*if - Km*(w - wref)/(J*La0)
u14 = (B*La0)*(1/Ep11)*rho11*rho11*w/(J*Km)
u15 = -La0*(1/Ep11)*rho11*rho11*phii*ia1/J
u16 = -Rp*La0*ia1*(ia1 - if)
uR12 = -(mu1*mu1 + Ep12*Ep12)*(mu1*rho12)/den1
den1 = ABS(mu1)*ABS(mu1)*ABS(mu1) + Ep12*Ep12*Ep12
mu1 = rho12*(phii*ia1 - x2d1)
x2d1 = (1/Km)*(B*wref + tauL0) - (1/Km)*(1/Ep11)*rho11*rho11*(w - wref)
rho11 = tauL0*K2/J
rho12 = La0*K1*(Rf*ia1*if + Rp*ia1*(ia1 - if) + r1 + r2) + r3 + r4
r1 = B*(1/Ep11)*rho11*rho11*ABS(w)/(J*Km)
r2 = (1/Ep11)*rho11*rho11*ABS(phii*ia1 - x2d1)
r3 = La0*(1 + K1)*tauL0*(1 + K2)*(1/Ep11)*rho11*rho11/(J*Km)
r4 = Km*(K1)*ABS(w - wref)/((J*La0)*(1 - K1))

phii = IF phi < 1.0 THEN 1.0 ELSE phi

"Case 2
u2 = u21 + u22 + u23 + u24 + uR22
u21 = Ra*ia2 + Km*phii*w
u22 = (B*La0)*(1/Ep21)*rho21*rho21*w/(F*J*Km)
u23 = -La0*(1/Ep21)*rho21*rho21*phii*ia2/(F*J)
u24 = -Km*(w - wref)/(F*J*La0*La0) - (B*G)*(phii*ia2 - x2d2)/(F*J)
uR22 = -((mu2*mu2 + Ep22*Ep22)/den2)*mu2*rho22
den2 = ABS(mu2)*ABS(mu2)*ABS(mu2) + Ep22*Ep22*Ep22
mu2 = rho22*(phii*ia2 - x2d2)
x2d2 = (1/Km)*(B*wref + tauL0) - (1/Km)*(1/Ep21)*rho21*rho21*(w - wref)
rho21 = tauL0*K2/J
rho22 = r6 + r7 + r8 + r9
r6 = (B*La0*K1)*(1/Ep21)*rho21*rho21*ABS(w)/(F*J*Km)
r7 = La0*K1*(1/Ep21)*rho21*rho21*ABS(phii*ia2)/(F*J)
r8 = Km*(2*K1 + K1*K1)*ABS(w - wref)/(F*J*La0*La0*(1 - K1)*(1 - K1))
```

```
r9 = La0*(1 + K1)*tauL0*(1 + K2)*(1/Ep21)*rho21*rho21/(F*J*Km)

F = dphidi*ia2 + phii
"Error
e = w - wref

"Simulated load torque with dynamic perturbation
tauL = IF t < 5 THEN 0.0 ELSE tauL1
tauL1 = IF t > 10 THEN tauL2 ELSE 1250.0*(t - 5.0)/5.0
tauL2 = IF t < 20 THEN 1250.0 ELSE 1250.0+125.0*sin(t - 20)

"Simulated load torque (nominal value)
tauL0 = IF t < 5 THEN 0.0 ELSE (IF t > 10 THEN 1250.0 ELSE 1250.0*(t-5.0)/5.0)

"Derivative of flux with respect to current
dphidi = IF ABS(iphi) < 169.2 THEN .108 ELSE d2
d2 = IF ABS(IPHI) < 200 THEN .09066 ELSE d3
d3 = IF ABS(IPHI) < 246.2 THEN .076 ELSE d4
d4 = IF ABS(IPHI) < 284.7 THEN .0493 ELSE d5
d5 = IF ABS(IPHI) < 307.6 THEN .04266 ELSE d6
d6 = IF ABS(IPHI) < 353.8 THEN .03866 ELSE d7
d7 = IF ABS(IPHI) < 407.7 THEN .0266 ELSE d8
d8 = IF ABS(IPHI) < 500.1 THEN .02266 ELSE d9
d9 = IF ABS(IPHI) < 631 THEN .01733 ELSE d10
d10 = 0.01466

"Integration of flux with respect to current
phi = IF ABS(iphi) < 169.2 THEN .108*iphi ELSE dd2
dd2 = IF ABS(iphi) < 200 THEN .09066*(iphi - 169.2) + 18.2736 ELSE dd3
dd3 = IF ABS(iphi) < 246.2 THEN .07600*(iphi - 200) + 21.0660 ELSE dd4
dd4 = IF ABS(iphi) < 284.7 THEN .04930*(iphi - 246.7) + 24.5770 ELSE dd5
dd5 = IF ABS(iphi) < 307.6 THEN .04266*(iphi - 284.7) + 26.4750 ELSE dd6
dd6 = IF ABS(iphi) < 353.8 THEN .03866*(iphi - 307.6) + 27.4520 ELSE dd7
dd7 = IF ABS(iphi) < 407.7 THEN .02660*(iphi - 353.8) + 29.2380 ELSE dd8
dd8 = IF ABS(iphi) < 500.1 THEN .02266*(iphi - 407.7) + 30.6720 ELSE dd9
dd9 = IF ABS(iphi) < 631 THEN .01733*(iphi - 500.1) + 32.7660 ELSE dd10
dd10 = IF ABS(iphi) < 1000 THEN .01466*(iphi - 631) + 35.0340 ELSE dd11
dd11 = 40.4335

"System parameters
B : 0.1 "N*m/rad/s
J : 3.0 "Kg*m*m
Km : 0.04329 "N*m/(Wb*A)
La : 0.00154 "H (perturbed value)
Ra : 0.00989 "ohm
Rf : 0.01485 "ohm
Rp : 0.01696 "ohm

"Control law parameters
La0 : 0.0014
```

```
K1 : 0.1
K2 : 0.1
Ep11 : 25.0
Ep12 : 0.1
Ep21 : 50.0
Ep22 : 0.3
G : 20.0

"Base speed
wbase : 200.0 "rad/s

"Parabolic equation to simulate reference speed
wref = IF t < 5 THEN 3.2*t*t ELSE IF t < 15 THEN 34.0*(t-5) + 80.0 ELSE wwref
wwref = IF t < 20 THEN -4.0*(t-20.0)*(t-20.0)+520.0 ELSE 520

END
```

The following is an example of the "go" file used to run the simulation of the robust control case.

```
macro gorob
syst dcrob
init ia1:0.0
init ia2:0.0
init if:0.0
init w:0.0
error 1e-4
algor dopri45r
store w e u ia1 ia2 iia if phi iphi wref u1 u2 F
simu 0 30 0.001/0.01
end
```

BIBLIOGRAPHY

[1] Karl Johan Aström and Tore Hägglund, *Automatic Tuning of PID Controllers.* Research Triangle Park, NC: Instrument Society of America, 1988.

[2] Ramachandran Balasubramanian and Kong Hung Wong, A Microcomputer-Based Self-Tuning IP Controller for DC Machines, *IEEE Transactions on Industry Applications,* vol. IA-22, no. 6, pp. 989-999, November/December 1986.

[3] Claudio Bonivento, Ashot Nersisian, Alberto Tonielli, and Roberto Zanasi, A Cascade Structure for Robust Control Design, *IEEE Transactions on Automatic Control,* vol. 39, no. 4, pp. 846-849, April 1994.

[4] Chi-Tsong Chen, *Linear System Theory and Design.* Fort Worth: Saunders College Publishing, 1984.

[5] John Chiasson and Marc Bodson, Nonlinear Control of a Shunt DC Motor, *IEEE Transactions on Automatic Control,* vol. 38, no. 11, pp. 1662-1666, November 1993.

[6] John Chiasson, Nonlinear Differential-Geometric Techniques for Control of a Series DC Motor, *IEEE Transactions on Control Systems Technology,* vol. 2, no. 1, pp. 35-42, March 1994.

[7] Yasuhiko Dote, Application of Modern Control Techniques to Motor Control, *Proceedings of IEEE,* vol. 76, pp. 438-454, April 1988.

[8] A.E. Fitzgerald, Charles Kingsley, Jr., and Stephen D. Umans, *Electric Machinery.* New York: McGraw-Hill Publishing Company, 1990.

[9] Michael J. Grimble, *Robust Industrial Control: Optimal Design Approach for Polynomial Systems.* New York: Prentice Hall, 1994.

[10] Hassan K. Khalil, *Nonlinear Systems.* New York: Macmillan Publishing Co., 1992.

[11] Paul C. Krause, *Analysis of Electric Machinery.* New York: McGraw-Hill Book Company, 1986.

[12] Paul C. Krause and Oleg Wasynczuk, *Electromechanical Motion Devices.* New York: McGraw-Hill Book Company, 1989.

[13] W. Leonhard, *Control of Electrical Devices.* Berlin: Springer-Verlag, 1985.

[14] D.C. McFarlane and K. Glover, *Lecture Notes in Control and Information Sciences: No. 138 Robust Controller Design Using Normalized Coprime Factor Plant Descriptions.* Berlin: Springer-Verlag, 1990.

[15] Ronald R. Mohler, *Nonlinear Systems: Volume I, Dynamics and Control.* Englewood Cliffs, NJ: Prentice Hall, 1991.

[16] Motors, *Machine Design,* vol. 60, pp. 57-97, May 19, 1988.

[17] Syed A. Nasar, *Electric Machines and Power Systems, Volume 1: Electric Machines.* New York: McGraw-Hill, Inc., 1995.

[18] Arch W. Naylor and George R. Sell, *Linear Operator Theory in Engineering and Science.* New York: Springer-Verlag, 1982.

[19] Philip D. Olivier, Feedback Linearization of DC Motors, *IEEE Transactions on Industrial Electronics,* vol. 38, no. 6, pp. 498-501, December 1991.

[20] Charles L. Phillips and Royce D. Harbor, *Feedback Control Systems.* Englewood Cliffs, NJ: Prentice Hall, 1991.

[21] Zhihua Qu, Class Notes, EEL 6621, Summer 1994.

[22] Zhihua Qu, Class Notes, EEL 6680, Spring 1994.

[23] Zhihua Qu, Robust Control of Nonlinear Uncertain Systems Under Generalized Matching Conditions, *Automatica,* vol. 29, no. 4, pp. 985-998, 1993.

[24] Zhihua Qu, Robust Control of Nonlinear Uncertain Systems Without Generalized Matching Conditions: A Recursive-Interlacing Design, *IEEE Transactions on Automatic Control,* August 8, 1994.

[25] Raymond A. Serway, *Physics for Scientists and Engineers.* Philadelphia: Saunders College Publishing, 1986.

[26] Jean-Jacques E. Slotine and Weiping Li, *Applied Nonlinear Control.* Englewood Cliffs, NJ: Prentice Hall, 1991.

[27] Ralph J. Smith, *Circuits, Devices, and Systems.* New York: John Wiley & Sons, 1984.

[28] Michael Tabor, *Chaos and Integrability in Nonlinear Dynamics: An Introduction.* New York: John Wiley & Sons, 1989.

[29] Hansheng Wu and R.A. Willgoss, Robust Stabilization for a Class of Uncertain Nonlinear Dynamical System, *1991 International Conference on Industrial Electronics, Control, and Instrumentation,* pp. 1752-1757, 1991.