Entanglement and Coherence in Classical and Quantum Optics

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ABSTRACT

We explore the concepts of coherence and entanglement as they apply to both the classical and quantum natures of light. In the classical domain, we take inspiration from the tools and concepts developed in foundational quantum mechanics and quantum information science to gain a better understanding of classical coherence theory of light with multiple degrees of freedom (DoFs). First, we use polarization and spatial parity DoFs to demonstrate the notion of classical entanglement, and show that Bell’s measure can serve as a useful tool in distinguishing between classical optical coherence theory. Second, we establish a methodical yet versatile approach called ‘optical coherency matrix tomography’ for reconstructing the coherency matrix of an electromagnetic beam with multiple DoFs. This technique exploits the analogy between this problem in classical optics and that of tomographically reconstructing the density matrix associated with multipartite quantum states in quantum information science. Third, we report the first experimental measurements of the $4 \times 4$ coherency matrix associated with an electromagnetic beam in which polarization and a spatial DoF are relevant, ranging from the traditional two-point Young’s double slit to spatial parity and orbital angular momentum modes.

In the quantum domain, we use the modal structure of classical fields to develop qubits and structure Hilbert spaces for use in quantum information processing. Advancing to three-qubit logic gates is an important step towards the success of optical schemes for quantum computing. We experimentally implement a variety of two- and three- qubit, linear and deterministic, single-photon, controlled, quantum logic gates using polarization and spatial parity qubits. Lastly, we demonstrate the implementation of two-qubit single-photon logic using polarization and orbital angular momentum qubits.
I dedicate this work to my grandmother Jamila Karkhanawala, who passed away during the course of my PhD studies.
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LIST OF ABBREVIATIONS

1D one-dimensional
2D two-dimensional
4D four-dimensional
APD avalanche photodiode
BBO $\beta$-barium-borate
BB84 Bennett-Brassard 1984
BC beam combiner/beam coupler
BS beam splitter
CHSH Clauser-Horne-Shimony-Holt
CNOT controlled-NOT
D detector/diagonal/diffuser
DoF(s) degree(s) of freedom
FC fiber coupler
GHZ Greenberger-Horne-Zeilinger
GT Glan-Thomson
H horizontal
HWP half-wave plate
IF interference filter
KLM Knill-Laflamme-Milburn
MMF multi-mode fiber
MZI Mach-Zehnder interferometer
NLC nonlinear crystal
OAM orbital angular momentum
<table>
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<tr>
<td>OCM T</td>
<td>optical coherency matrix tomography</td>
</tr>
<tr>
<td>PA</td>
<td>parity analyzer</td>
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<tr>
<td>PBS</td>
<td>polarizing beam splitter</td>
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<tr>
<td>PD</td>
<td>phase delay</td>
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<tr>
<td>PF</td>
<td>parity flipper</td>
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<tr>
<td>PR</td>
<td>parity rotator</td>
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<tr>
<td>PS</td>
<td>phase shifter/polarization scrambler</td>
</tr>
<tr>
<td>PS-MZI</td>
<td>parity-sensitive Mach-Zehnder interferometer</td>
</tr>
<tr>
<td>PS-SLM</td>
<td>polarization-sensitive spatial light modulator</td>
</tr>
<tr>
<td>QIP</td>
<td>quantum information processing</td>
</tr>
<tr>
<td>QKD</td>
<td>quantum key distribution</td>
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<tr>
<td>QND</td>
<td>quantum non-demolition</td>
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<td>QWP</td>
<td>quarter-wave plate</td>
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<tr>
<td>R/RHC</td>
<td>right-hand circular</td>
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<tr>
<td>SF</td>
<td>spatial filter/spatial flipper</td>
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<td>SLM</td>
<td>spatial light modulator</td>
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<tr>
<td>SMF</td>
<td>single-mode fiber</td>
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<tr>
<td>SPDC</td>
<td>spontaneous parametric down-conversion</td>
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<td>SPQL</td>
<td>single-photon quantum logic</td>
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<tr>
<td>V</td>
<td>vertical</td>
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<td>WP</td>
<td>wave plate</td>
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LIST OF SYMBOLS

\( \hat{\rho} \)  density matrix
\( B \)  Bell’s measure
\( D_{\text{pol}} \)  degree of polarization
\( D_{\text{par}} \)  degree of parity
\( \mathbf{G} \)  coherency matrix
\( \mathbf{G}_p \)  polarization coherency matrix
\( \mathbf{G}_s \)  spatial-DoF coherency matrix
\( \otimes \)  direct product
\( |H\rangle \)  horizontal polarization state
\( |V\rangle \)  vertical polarization state
\( |e\rangle \)  even parity state
\( |o\rangle \)  odd parity state

In order of occurrence.
CHAPTER 1: INTRODUCTION

In this work, using various optical modes of light, namely, polarization, spatial position, spatial parity and orbital angular momentum, I present novel ideas regarding coherence and entanglement in both classical and quantum optics. In the classical domain, the polarization and spatial parity degrees of freedom (DoFs) are used to demonstrate the notion of classical entanglement, and show that how the Bell’s measure from foundational quantum mechanics can serve as a useful tool in classical optical coherence theory. We expound the use of the coherency matrix as a complete descriptor of the second-order coherence properties of an electromagnetic beam with multiple DoFs, and present the first experimental measurements of the coherency matrices associated with classical beams with two binary DoFs. In the quantum domain, we implement a variety of two- and three- qubit single-photon controlled quantum logic gates using polarization, spatial parity, and orbital angular momentum qubits. The relationship between the themes presented in this thesis is depicted in Figure 1.1. The degrees of freedom, optical modes, and qubits used in the experiments are illustrated in Figure 1.2.

Figure 1.1: Relationship between the themes presented in this thesis.
1.1 Quantum tools for classical coherence

The statistical description of optical fields in classical coherence theory is the foundation for many applications in metrology, microscopy, lithography, and astronomy. The coherence properties of an electromagnetic field are generally characterized by statistical fluctuations in its spatial, spectral and polarization degrees of freedom [1–5]. The traditional coherence measures for individual DoFs, such as the visibility of interference for the spatial and temporal DoFs, and the degree of polarization for the polarization DoF, are adequate only when the DoFs remain separable or uncorrelated as the beam propagates through a medium or an optical system.

Partial coherence is commonly attributed to underlying statistical fluctuations originating at the source or arising upon passage of a coherent beam through a random medium. However, in certain situations, such as when the beam scatters from a rough surface, or undergoes a polarization dependent transformation, one or more of these DoFs might become correlated or classically entangled. In such cases, the coherence measures pertaining to a single DoF are insufficient and measures spanning multiple DoFs are required to extract the information contained in their correlations. This less acknowledged source of uncertainty (partial coherence) stems from the act of ignoring a DoF of a beam when observing another DoF coupled to (or classically entangled with) it. If two DoFs of a coherent beam are correlated, then measurements of either DoF that are insensitive to
the other reveal apparent partial coherence.

The notion of classical entanglement has been the subject of recent investigations [6–10], and arises as a consequence of the mathematical similarity between the Hilbert space structure of multipartite states in quantum mechanics and the ordinary multi-DoF state of a classical beam of light. Once this correspondence has been established, all the tools and measures developed in quantum information science become readily available for application to classical coherence theory.

In Chapter 2, from recent advances in quantum optics, I adapt the use of Bell’s inequality and address the issue coherence, for a model optical beam with two binary DoFs: polarization and spatial parity. I demonstrate that Bell’s measure (BM), which is commonly used in tests of quantum non-locality, may be employed as a quantitative tool in classical optical coherence to delineate native incoherence associated with statistical fluctuations from correlation- (or, entanglement-) based incoherence. The description of a classical beam having multiple DoFs may be cast into the formalism of tensor-product Hilbert spaces and is thus isomorphic to the representation of multi-partite states in quantum mechanics. Studying classical optical coherence from this novel quantum-inspired perspective, we show that correlation-based partial coherence may migrate from one DoF to another through transformations spanning both DoFs. By introducing a new quantity into coherence theory that we designate the degree of intrinsic coherence of a DoF, we provide a unique ordering of physically admissible partially coherent beams. We perform a series of experiments to test the use of the Bell’s measure in three different scenarios: when the beam is coherent with coupled polarization and parity, when it is partially coherent with coupled polarization and parity, and when the parity-polarization coupling is random. This approach challenges the traditional view on spatial-coherence for partially polarized light, and provides a new measure of intrinsic coherence that is impervious to polarization effects. Our results demonstrate the applicability of the concepts recently developed in quantum information science to classical optical coherence theory and optical signal processing.
The two-point coherence of an electromagnetic field is represented completely by a $4 \times 4$ coherency matrix $G$ that encodes the joint polarization–spatial-field correlations. In Chapter 3, I describe a systematic sequence of cascaded spatial and polarization projective measurements that are sufficient to tomographically reconstruct $G$ – a task that, to the best of our knowledge, has not yet been realized. Our approach benefits from the correspondence between this reconstruction problem in classical optics and that of quantum state tomography for two-photon states in quantum optics. Identifying $G$ uniquely determines all the measurable correlation characteristics of the field and, thus, lifts ambiguities that arise from reliance on traditional scalar descriptors, especially when the field’s DoFs are correlated or classically entangled.

Situations where two binary DoFs of an optical field are considered, which may be termed as quaternary optics, the second-order coherence properties are expressed completely by a $4 \times 4$ Hermitian coherency matrix $G$. In Chapter 4, I present a methodical yet versatile approach – optical coherency matrix tomography (OCmT) – to measuring the complex elements of $4 \times 4$ coherency matrices $G$ by appropriating the quantum-state-tomography strategy. To demonstrate the universality of our approach, we implement it with coherent and partially coherent fields having coupled or uncoupled DoFs in three distinct settings involving pairs of points [9, 11, 12], spatial-parity modes [13–18], and orbital angular momentum (OAM) modes [19] – each together with polarization [Figure 1.2]. We identify the minimal set of linearly independent, joint spatial-polarization projective measurements that enable a unique reconstruction of $G$. Since $G$ is a complete representation of the field, its reconstruction obviates the need to measure directly any coherency descriptors (all of which are scalar functions of the complex elements of $G$) and, moreover, allows for unambiguous identification of classical entanglement.
1.2 Single-photon quantum logic with polarization and spatial qubits

The theory of quantum computation is based on the idea of using quantum mechanics to perform computations, instead of classical physics. A quantum computer, as the name suggests, is a device which makes use of purely quantum mechanical phenomena, such as superposition and entanglement, to carry out data computations. The realization of such a device remains one of today’s primary motivations for advancements in quantum optics, and by and large in quantum mechanics.

The Church-Turing thesis states that ‘Any algorithmic process can be simulated efficiently using a Turing machine’ [20]. A classical computer is able to perform any computation, and simulate any system, including a quantum computer. But there exist certain noteworthy problems that a classical computer is unable to solve efficiently. One such problem is the factoring of large numbers. A classical computer using the best existing factoring algorithm will require a stupendous $10^{10}$ years for a number that is 400 digits long, requiring time that is a polynomial function of the size of the problem [21]. However, an algorithm formulated by Peter Shor in 1994 [22] suggests that the problem can be solved in less than three years on a quantum computer. Hence a quantum computer with even modest resources would be profoundly more powerful, with regard to certain tasks, than any classical computer running the best known algorithms. Another motivation comes from the problem posed by the eventual failure of Moore’s law due to size limitations on conventional fabrication technology. One way out of this problem would be to shift to a different computing paradigm, such as quantum computing.

Any physical system that could be used as the basis for realizing a quantum computer must meet two important criteria. If it is to be feasible for arbitrary quantum computation, it should be scalable and efficient. Scalability for a quantum computer means that adding a large enough number of qubits and gates to the computer is not forbidden by any physical law. Without that, a quantum computer will always be of limited use and will never show an advantage over classical computa-
tion. An efficient scheme would allow a scaled-up version of a prototype computer using resources, such as space, time, the number of input sources and components, etc., that only follows polynomial increase relative to the size of the computation. In that case, an inefficient quantum computer requires exponentially increasing resources and the theoretically suggested exponential increase in computing power is undermined by the additional resources, since even a classical computer can show exponential gain with exponential resources available.

As mentioned above, quantum computing relies on purely quantum mechanical phenomena such as superposition of states and entanglement. The superposition of states allows for quantum parallelism, in which the device is able to act not just on the eigenstates of the input, but also on their superpositions, and hence extract global information [21]. Quantum parallelism is one of the hallmarks of quantum computation. Entanglement on the other hand is considered a necessary resource, but not by itself sufficient for speed-up of quantum algorithms over classical ones. In fact, it has even been suggested that a quantum computer relying on mixed, separable states may still surpass its classical counterpart [23]. It is probable that it is not entanglement that is essential, but simply a set of states that cannot be expressed by a small set of parameters [24, 25]. Nevertheless, it is still considered beneficial to bring entanglement into play, but its role in quantum computing still remains a subject of intense investigation.

The quantum circuit model provides the starting point for comprehending the operation of a quantum computer. Despite the existence of other models, the reason that makes it simplistic is because it is primarily the model for a classical circuit generalized to allow for quantum effects. A quantum circuit is most easily represented by a diagram, in which a qubit is represented by horizontal lines, and the direction of the computation is from left to right. The next simplest component, represented by a square containing a letter, is the single-qubit gate. These gates act to rotate the state of a single qubit, and the letter represents the operation performed. Multi-qubit gates follow the same convention, but acting on more than one qubit, and commonly perform a controlled opera-
tion, in which the outcome is dependent upon the state of one of the input qubits. The use of these components enables us to construct quantum circuits corresponding to algorithms.

There are many implementations of quantum computing but not one that is widely accepted as the building basis for the first quantum computer that will be able to give significant performance advantages over classical computers. Optical implementations seem to have attracted considerable interest for two main reasons. First, an optical scheme for quantum computation integrates naturally with ideas for quantum communication and fast transfer of quantum information. Secondly, optical quantum computation is largely unaffected by the most deleterious effect that plagues other schemes, decoherence. Implementations such as the ones based on ion traps and nuclear magnetic resonance suffer from the fact that the physical systems used to register the qubits interact strongly with their surroundings. That results in short lifetimes for the qubits and associated qubit entanglement. This makes it almost unpractical to perform a computation, particularly one of appreciable size, before the information carried by the qubits is lost to interaction mechanisms that couple the system with the environment. Single photons, on the other hand, are potentially free of decoherence; as a consequence entanglement between photons is also much more robust. In addition, manipulating photons with the means of polarization or the spatial mode of a photon can be more reliable since very mature and precise technology can be utilized. The primary advantage of optical schemes, however, is related to its largest disadvantage. Photons do not interact with each other, meaning it is at the moment impossible to directly entangle them in vacuum. The most widely used mean to entangle them is during their production through nonlinear media. Even with this limitation, optical quantum computation is a quickly advancing field.

As mentioned above, the circuit representation of quantum computation requires qubits, single- and two-qubit gates. In an optical scheme, the first two elements can nowadays be readily accessible. The third requirement is of particular difficulty since in photonic schemes, the carriers of information do not interact in a linear medium. It was assumed for some time that two-photon gates require
photon interaction through nonlinear means. No known material or process is characterized by a nonlinearity strong enough to lead to two photon interaction with a success rate high enough to allow scalable quantum computations. The strength of these nonlinearities fall short by several orders of magnitude, thus extended advancements in the fields of nonlinear materials or laser power will be necessary to make that scheme feasible. An alternative was to induce effective interactions using projective measurements that could be nonlinear in nature. In essence, this approach could allow for nondeterministic two-photon gates that would be failing in the vast majority of measurements but in a way that will still make possible the registration of a successful one. The problem with this approach was that it appeared that the gates failed at very high rates that scalable quantum computing could not be achieved. In 2001 Knill, Laflamme and Milburn (KLM) challenged this assertion and developed a protocol that allows for efficient, scalable quantum computation using only single photon sources, linear optical elements and projective measurements [26, 27].

The information-carrying capacity of a single photon may be harvested by encoding information in its multiple degrees of freedom (DoF): spatial, temporal, and polarization. Yet, quantum operations utilizing more than two qubits in more than one DoF of the photon have not been previously observed. In Chapter 5, using polarization and spatial parity-symmetry of the transverse field in two orthogonal directions, I report the first experimental demonstration of single-photon three-qubit, linear, deterministic, controlled quantum gates implemented by using a polarization-sensitive spatial light modulator (SLM). The SLM is shown to be a robust, versatile, non-interferometric device capable of implementing a wide range of controlled unitary operations without the need for active stabilization. We also generate and tomographically measure single-photon maximally entangled three-qubit states, namely the GHZ and W states, in the joint polarization and spatial-parity symmetry space. Our technique provides access to a wide range of three qubit states that do not rely on probabilities and post-selection for use in few-qubit quantum information processing algorithms.

And finally in Chapter 6, I demonstrate an implementation of a single-photon linear and deter-
ministic quantum logic gate using polarization and orbital angular momentum (OAM) qubits. The concept is very similar to the gates described in the previous chapter, however this version obviates the need for an interferometer in the analysis stage - a huge improvement from the experimental point of view.
CHAPTER 2: BELL’S MEASURE IN CLASSICAL OPTICAL COHERENCE

2.1 Introduction

Optical interference is one of the most fundamental pathways to gaining insight into the nature of light [28]. Indeed, optical coherence is generally assessed by the ability of light to interfere [3,5,29,30]. In typical interference experiments, one degree of freedom (DoF) is singled out, be it space, time, or polarization. The need for re-appraising the foundations of coherence theory has recently been appreciated in cases where multiple DoFs are relevant. For example, the inadequacy of traditional coherence measures has been realized in the case of describing spatial and polarization DoFs when a vector field illuminates Young’s double slits [4]. These developments highlight the need for a general formulation of multi-DoF coherence.

When the DoFs of an optical beam are uncoupled, the outcomes of coherence measurements pertaining to each DoF are independent. However, spatial, spectral, and polarization DoFs are often coupled or correlated in real beams. Strong coupling between the DoFs of a coherent beam prevents the observation of interference in any single DoF when the detection scheme is not capable of resolving the other DoFs. For example, when a coherent beam illuminates two slits with orthogonal polarizers placed on each slit – thereby coupling space to polarization – double-slit interference is not observed. Alternatively, a coherent beam with spatially varying polarization may appear to be partially polarized, or even unpolarized, when these variations are at a scale finer than the spatial resolution of the detector. Such deterministic effects are further complicated in the case of partially coherent beams. Therefore, the apparent degree of coherence of any DoF obtained from an inter-

ference experiment – one that targets this specific DoF and is insensitive to the others – inevitably incorporates two contributions: one from partial coherence associated with statistical fluctuations, and another arising from deterministic coupling, or entanglement, with other unresolved DoFs. Although the study of coherence has occupied optics for most of the past century [3, 5, 29, 30], the delineation of these two sources of apparent partial coherence remains an open question to be resolved.

In order to address this fundamental question, we present here a general approach to intra-DoF interferometry that allows for observing interference across multiple DoFs. We draw our inspiration from an analogous problem that arises in quantum optics in the context of two-photon states. When two photons are strongly correlated, or entangled, neither photon manifests first-order coherence (as in the case of a classical beam with two coupled DoFs). Nevertheless, high-visibility ‘two-photon interference’ (second-order coherence) is exhibited in suitable arrangements [31–33]. We show here that such concepts developed for multipartite quantum states [20,34], although rooted in a different physical theory, enable precise characterization of the optical coherence of a multi-DoF classical beam.

We restrict our investigation to optical beams with two binary DoFs in order to focus on the salient new concepts. The description of such a beam when the DoFs are coupled is mathematically isomorphic to quantum entanglement [35, 36] in two-qubit systems [37] such as two polarized photons. In light of this correspondence, we propose the use of Bell’s measure [38], which is typically used to quantify quantum entanglement (correlations) between the polarization of two photons, to instead quantify the entanglement (correlations) between the DoFs of a classical beam through joint measurements. Recent theoretical studies on the analogy between one- and two-photon states of light, on the one hand, and classical optical beams, on the other, have helped establish a common framework to study discretized DoFs in both the classical and quantum domains [8, 16, 39–44].
It may initially appear confusing that Bell’s measure, which is typically associated with quantum-mechanical features of two-photon states, bears relevance to classical coherence, which stems intrinsically from one-photon effects. Nevertheless, Bell’s measure is not inherently related to quantum phenomena. In fact, in Bell’s derivation of his inequality [38], quantum mechanics plays no role – only the principles of probability theory are invoked. There is nothing, therefore, that prevents the use of Bell’s measure in classical optics as a measure of classical entanglement (or, correlation), a point that has been obfuscated heretofore by its use exclusively in quantum optics. The physical interpretation of Bell’s measure is, of course, altogether different when applied to fourth-order correlations of two-photon states and second-order correlations of a classical beam having two DoFs. Bell’s measure evaluated for two photons with space-like separation is used to ascertain the validity of the premise of local realism. Violating Bell’s inequality here rules out local hidden-variable theories as models for explaining the observed correlations demonstrated by an entangled-photon pair. The physical interpretation of Bell’s measure is altogether different when applied to a classical beam with two DoFs. In this context, Bell’s measure quantifies the entanglement (correlation) ‘resource’ needed to construct a beam with the observed coherence properties, and violating Bell’s inequality indicates the impossibility of constructing such a beam by mixing any number of elementary beams, each with uncoupled DoFs.

By combining Bell’s measure with the traditional coherence measure for each DoF, we introduce a new quantity – the ‘degree of accessible coherence’. This measure quantifies the total degree of coherence potentially ‘available’ to each DoF. For example, a low degree of coherence for the spatial DoF associated with a high degree of accessible coherence indicates the existence of ‘hidden’ coherence that stems from coupling with another DoF. This hidden coherence may be ‘unveiled’, at the expense of the coherence of the other DoF, via deterministic unitary transformations spanning both DoFs, so that the apparent coherence of each DoF changes, without altering the overall beam coherence quantified by the field entropy. This type of swapping of coherence between DoFs
suggests that coherence of a multi-DoF optical beam should be viewed as a ‘resource’ that may be exchanged and shared among the DoFs, much like quantum entanglement is considered a resource in a multi-partite quantum system [45].

Our formulation helps clarify some conceptual issues in optical coherence theory and suggests potential applications exploiting the coupling among polarization, spatial, and spectral modes of the optical field to develop new protocols for optical communication through random channels, and to improve methods of metrology and imaging under conditions of scattering from random media and reflection from rough surfaces.

2.2 Polarization and spatial parity

We start by outlining a formalism to describe the coherence of a classical optical beam with two binary DoFs. There are many realizations of such a beam. In the optical beam model we consider here, the two binary DoFs are polarization and spatial parity along one direction [13–15] of a quasi-monochromatic source. This model may be put in one-to-one correspondence with other configurations such as the traditional two-point vector fields, whereupon one binary DoF is position and the other is polarization. In Appendix A we set forth this correspondence in detail.

We consider the spatial parity of one-dimensional scalar beams of the form:

\[ E(x) = E_e \psi_e(x) + E_o \psi_o(x), \]  

(2.1)

where \( \psi_e(x) \) is an even function and \( \psi_o(x) = \text{sgn}(x) \psi_e(x) \) is odd (\( \text{sgn}(x) \) is the sign function and \( \int dx |\psi_e(x)|^2 = 1 \); ref. [13]). Such a beam may be represented by a vector:

\[ \mathbf{J}_{\text{par}} = \begin{bmatrix} E_e & E_o \end{bmatrix}^T, \]  

(2.2)
(‘T’ refers to the vector conjugate transpose) in terms of its even $E_e$ and odd $E_o$ components, much like a Jones vector is used to describe polarization, which is the second binary DoF [46]. Consequently, beams in parity space may be represented geometrically by points on a Poincaré sphere [14]. Spatial parity may be manipulated with a phase plate or a spatial light modulator (SLM) that implements a phase difference $\varphi$ between the two halves of the plane, i.e., a phase function $\frac{\varphi}{2} \text{sgn}(x)$ [13]. This device ‘rotates’ the parity around a major circle on the parity Poincaré sphere in analogy to a half-wave plate (HWP) rotating polarization [13–15]; for example, it transforms an even beam:

$$J_{\text{par}} = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$$  \hspace{1cm} (2.3)

to a superposition of even and odd modes:

$$J_{\text{par}} = \begin{bmatrix} \cos \frac{\varphi}{2} & i \sin \frac{\varphi}{2} \end{bmatrix}^T. \hspace{1cm} (2.4)$$

The second binary DoF we consider is polarization whose representation using a vector is well-known. Together, the polarization and spatial parity of a coherent paraxial vector beam $E(x)$ are represented by a four-dimensional (4D) polarization-parity Jones vector:

$$J = \begin{bmatrix} E_{He} & E_{Ho} & E_{Ve} & E_{Vo} \end{bmatrix}^T,$$  \hspace{1cm} (2.5)

where ‘H’ and ‘V’ refer to the horizontal and vertical polarization components, respectively.

The polarization-parity coherence properties are described by a $4 \times 4$ coherency matrix:

$$G = \langle J \ast J^T \rangle$$  \hspace{1cm} (2.6)

(ref. [11]), where $\langle \ldots \rangle$ denotes the expected value. This beam is thus described in a linear vector space that is the direct product of the polarization and spatial-parity subspaces. Such a represen-
tation is in fact isomorphic to the quantum-mechanical description of two-qubit states [47, 48], where the two-qubit Hilbert space is the direct product of the Hilbert spaces of the individual qubits. A $2 \times 2$ parity-insensitive polarization coherency matrix $G_{\text{pol}}$ is then obtained by tracing over the parity DoF in $G$, and is measured with a detector that integrates over space. Likewise, a $2 \times 2$ polarization-insensitive spatial-parity coherency matrix $G_{\text{par}}$ is defined by tracing over the polarization DoF in $G$. We use the reduced $G_{\text{pol}}$ ($G_{\text{par}}$) to define the degree of polarization (parity) coherence $D_{\text{pol}}$ ($D_{\text{par}}$); see Methods. $D_{\text{pol}}$ and $D_{\text{par}}$ quantify the observed or apparent coherence of each DoF when the other DoF is unresolved, ranging from 1 for complete coherence to 0 for complete incoherence. We also define a degree of overall beam coherence $S$ using the beam’s linear entropy [40,49] $S_L$:

$$S = 1 - S_L = \frac{4}{3} \left( \text{Tr} \{ G^2 \} - \frac{1}{4} \right) , \quad (2.7)$$

where $S = 0$ ($S = 1$) corresponds to an incoherent (coherent) beam and ‘Tr’ refers to the matrix trace. This scalar measure quantifies the intrinsic randomness or statistical fluctuations in the beam across the DoFs and is independent of any deterministic coupling, or entanglement, between the DoFs. It corresponds in quantum mechanics to the degree of purity of the quantum state and is estimated from the density matrix. We have dropped the frequency-dependence of $S$ since we consider quasi-monochromatic sources.

We are interested here in beams for which polarization and spatial parity are coupled or entangled. For example, a coherent linearly polarized beam (at 45°) with even parity:

$$J = \frac{1}{\sqrt{2}} \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]^T , \quad (2.8)$$
has uncoupled DoFs. \( J \) may be factored into a direct product:

\[
J = J_{\text{pol}} \otimes J_{\text{par}},
\]

with

\[
J_{\text{pol}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix}^T,
\]

(2.10)

and

\[
J_{\text{par}} = \begin{bmatrix} 1 & 0 \end{bmatrix}^T.
\]

(2.11)

The coherency matrix for each DoF here corresponds to complete overall beam coherence: \( S = 1 \) and \( D_{\text{pol}} = D_{\text{par}} = 1 \). These DoFs may be readily coupled using a polarization-sensitive SLM (PS-SLM), which introduces a phase \( \frac{\varphi}{2} \text{sgn}(x) \) into the H component, but not the V component [18] (Figure 2.1(a)). The DoFs are now coupled via this unitary deterministic transformation, and

\[
J = \frac{1}{\sqrt{2}} \begin{bmatrix} \cos \frac{\varphi}{2} & i \sin \frac{\varphi}{2} & 1 & 0 \end{bmatrix}^T.
\]

(2.12)

can no longer be factored into a direct product of \( J_{\text{pol}} \) and \( J_{\text{par}} \). The coupling strength is determined by \( \varphi \): when \( \varphi = 0 \), there is no coupling; when \( \varphi = \pi \), the coupling is strongest. In the latter case (\( \varphi = \pi \)), we observe a surprising result: both \( G_{\text{pol}} \) and \( G_{\text{par}} \) exhibit complete incoherence (\( D_{\text{pol}} = D_{\text{par}} = 0 \)) despite the absence of any randomness in the beam (\( S = 1 \)). Hence, a coherence ‘deficit’ is apparent for a coherent beam with coupled DoFs when we rely on the traditional measures of optical coherence that are related to each DoF separately. We intuitively expect that any useful coherence measure should remain invariant under the action of unitary deterministic transformations that do not introduce randomness, such as that implemented by the PS-SLM. We proceed to demonstrate that adopting Bell’s measure to quantify the entanglement between the DoFs accounts precisely for this apparent coherence deficit.
Figure 2.1: Experimental setup. a-c, Setup for beam synthesis (a) and analysis (b,c). The source is an H-polarized, even-mode beam, 808 nm laser diode. HWP: half-wave plate; PBS: polarizing beam splitter; SLM: spatial light modulator (Hamamatsu X10468-02); PS-SLM: polarization sensitive SLM; PS-MZI: parity-sensitive Mach-Zehnder interferometer; $\theta$ and $\theta_a$ ($\varphi$ and $\varphi_a$) are the polarization (parity) rotation angles in the synthesis and analysis stages, respectively. A quarter-wave plate is added to evaluate the polarization Stokes parameters and obtain $D_{\text{pol}}$. Similar modification is made to the parity analysis system to obtain parity Stokes parameters and $D_{\text{par}}$. The system in (c) is a cascade of the two systems that are arranged in parallel in (b).
2.3 Polarization and spatial parity analysis

We next describe the procedure for analyzing these optical beams in terms of polarization (with parity ignored), parity (with polarization ignored), and combined polarization and parity. We analyze polarization alone in a linear basis rotated by $\theta_a$ while integrating over space (Figure 2.1(b)). We use the output signals to define normalized probabilities $P_H$ and $P_V$ ($P_H + P_V = 1$). The analogous system to analyze parity alone separates the even and odd components [13–15] and projects the beam onto a linear parity basis rotated by $\varphi_a$ (Figure 2.1(b)). The outcomes are used to define the probabilities $P_e$ and $P_o$ ($P_e + P_o = 1$). Combined polarization-parity measurements were carried out by concatenating polarization and parity analysis stages (Figure 2.1(c)), resulting in probabilities $P_{He}$, $P_{Ho}$, $P_{Ve}$ and $P_{Vo}$ for the four outcomes ($P_{He} + P_{Ho} + P_{Ve} + P_{Vo} = 1$). If polarization and parity are uncoupled, then $P_{He} = P_H P_e$, $P_{Ho} = P_H P_o$, etc.

We now proceed to quantifying the entanglement between polarization and parity using these measurements. We define a real, normalized correlation function $C(\theta_a, \varphi_a)$ ($|C| \leq 1$), defined as an expected value over the joint-measurement probabilities after assigning the symmetric values $c_{\text{pol}} = \pm 1$ ($c_{\text{par}} = \pm 1$) to the dichotomic H (e) and V (o) polarization (parity) outcomes:

$$C(\theta_a, \varphi_a) = \sum c_{\text{pol}} c_{\text{par}} P_{\text{pol,par}}(\theta_a, \varphi_a) = P_{He} - P_{Ho} - P_{Ve} + P_{Vo}. \quad (2.13)$$

Only if polarization and parity are uncoupled does $C$ separate into a product:

$$C(\theta_a, \varphi_a) = C_{\text{pol}}(\theta_a) C_{\text{par}}(\varphi_a), \quad (2.14)$$

where $C_{\text{pol}}(\theta_a) = P_H - P_V$ and $C_{\text{par}}(\varphi_a) = P_e - P_o$. Since $C$ is a functional of $G$, it is useful to extract a single scalar quantity that provides a measure of the entanglement between the DoFs. We propose the use of the Clauser-Horne-Shimony-Holt (CHSH) formulation [50] of Bell’s measure,
which has enabled the experimental confirmation of quantum non-locality when applied to entangled bipartite states [51, 52]. The CHSH measure is a linear combination of four points chosen from $C(\theta_a, \varphi_a)$:

$$B = |C(\theta_a, \varphi_a) + C(\theta_a, \varphi'_a) + C(\theta'_a, \varphi_a) - C(\theta'_a, \varphi'_a)|. \quad (2.15)$$

$B$ is thus a function of four pairs of angular settings involving four angles, two for polarization ($\theta_a$ and $\theta'_a$) and two for parity ($\varphi_a$ and $\varphi'_a$). By maximizing $B$ over all angular settings, we obtain a single parameter $B_{\text{max}}$ that we refer to hereafter as Bell’s measure. It can be shown that $B_{\text{max}} \leq 2\sqrt{2}$ [53]. While this so-called Tsirelson’s bound was originally derived in a quantum mechanical setting, the isomorphism between the classical description of an optical beam with two binary DoFs and the quantum-mechanical description of a two-qubit system ensures that this bound applies to both.

Several relations that are not immediately obvious may be shown to hold. First, $D_{\text{pol}} = D_{\text{par}}$ for all coherent beams, even beams with no symmetry between the two DoFs (see Methods). This condition does not necessarily hold for partially coherent beams. Second, for coherent beams, strict complementarities hold between $D_{\text{pol}}$ and $B_{\text{max}}$, and naturally between $D_{\text{par}}$ and $B_{\text{max}}$:

$$4D_{\text{pol}}^2 + B_{\text{max}}^2 = 4D_{\text{par}}^2 + B_{\text{max}}^2 = 8. \quad (2.16)$$

This complementarity is relaxed for partially coherent beams and becomes instead two independent inequalities:

$$4D_{\text{pol}}^2 + B_{\text{max}}^2 \leq 8 \quad (2.17)$$

and

$$4D_{\text{par}}^2 + B_{\text{max}}^2 \leq 8. \quad (2.18)$$
These observations motivate introducing two quantities: the degree of accessible coherence $S_{\text{pol}}$ of the polarization DoF, and the degree of accessible coherence $S_{\text{par}}$ of the parity DoF, defined as:

$$S_{\text{pol}} = \frac{D_{\text{pol}}^2}{2} + \left(\frac{B_{\text{max}}}{2\sqrt{2}}\right)^2, \quad S_{\text{par}} = \frac{D_{\text{par}}^2}{2} + \left(\frac{B_{\text{max}}}{2\sqrt{2}}\right)^2.$$

(2.19)

Several salutary features advocate the utility of $S_{\text{pol}}$ and $S_{\text{par}}$ as measures of coherence:

1. they are invariant under unitary deterministic transformations that affect only one DoF;
2. for coherent beams, $S_{\text{pol}}$ and $S_{\text{par}}$ are also invariant under arbitrary unitary deterministic transformations that couple the two DoFs;
3. $S_{\text{pol}} = S_{\text{par}} = 1$ only for coherent beams ($S = 1$), regardless of the coupling between the two DoFs;
4. $S_{\text{pol}} = S_{\text{par}} = 0$ only for an incoherent beam ($S = 0$); and
5. $S_{\text{pol}} = 0$ ($S_{\text{par}} = 0$) only if the polarization (parity) DoF lacks coherence and is uncoupled from the other DoF.

$S_{\text{pol}}$ and $S_{\text{par}}$ represent the coherence resource available to the DoFs, whether manifest in measurements of each DoF ($D_{\text{pol}}$ and $D_{\text{par}}$) or hidden in their coupling (quantified by $B_{\text{max}}$). A high value of $S_{\text{pol}}$ and a low value of $D_{\text{pol}}$, for example, implies that the initially low $D_{\text{pol}}$ may be increased by undoing the coupling between polarization and parity via a unitary deterministic transformation spanning both DoFs that does not alter the beam entropy. The hidden coherence thus ‘migrates’ from the entanglement between the DoFs and becomes manifest in measurements of the DoF itself. We demonstrate experimentally the usefulness of these two new quantities, $S_{\text{pol}}$ and $S_{\text{par}}$, in three distinct configurations that delineate the role of entanglement between two DoFs and that of statistical fluctuations in determining the apparent coherence of each DoF. We emphasize that the discussion above and the experiments described below apply in their entirety to other pairs of binary DoFs, and in particular to two-point vector correlations. In Appendix A we present a parallel treatment for the case of a vector field in a Young’s double-slit arrangement.
2.4 Experiment A: Coherent beam with coupled polarization and parity

We start in all three experiments with a 45° linearly polarized even-mode coherent beam:

\[
J_{\text{in}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}^T.
\]  

(2.20)

The PS-SLM set to phase \( \frac{\varphi}{2} \) \( \text{sgn}(x) \) couples polarization and parity [18] (Figure 2.2). The field consequently exhibits partial coherence for each DoF, \( D_{\text{pol}} = D_{\text{par}} = |\cos \frac{\varphi}{2}| \), even though no randomness has been introduced. At \( \varphi = \pi \), polarization and parity are maximally entangled, \( D_{\text{pol}} = D_{\text{par}} = 0 \), and \( B_{\text{max}} \) attains its peak value. Although \( D_{\text{pol}} \) and \( D_{\text{par}} \) reveal apparent partial coherence, we find that \( S_{\text{pol}} = S_{\text{par}} = 1 \) for all \( \varphi \). This reveals that the overall beam is in fact coherent (\( S = 1 \)) and the apparent partial coherence is instead due entirely to entanglement between the two DoFs and not to statistical fluctuations. Furthermore, the hidden coherence may be unveiled via a unitary deterministic transformation spanning both DoFs that nullifies their entanglement, in this case using the PS-SLM phase \( -\frac{\varphi}{2} \text{sgn}(x) \), whereupon the hidden coherence is made manifest in each DoF directly (\( D_{\text{pol}} \) and \( D_{\text{par}} = 1 \)).
Figure 2.2: Experiment A: coherent beam with coupled polarization and parity. (a) Measured $D_{\text{pol}}$ and $D_{\text{par}}$. Solid curves are the theoretical values $D_{\text{pol}} = D_{\text{par}} = |\cos \frac{\varphi}{2}|$. Inset: beam preparation set-up. (b) Measured correlation functions $C(\theta_a, \varphi_a)$ for three source configurations: $\varphi = 0$, yielding theoretically $C(\theta_a, \varphi_a) = \sin \theta_a \cos \varphi_a$; $\varphi = \frac{\pi}{2}$, yielding $C(\theta_a, \varphi_a) = \sin \theta_a \sin \left( \varphi_a \frac{\pi}{4} \right) - \frac{1}{\sqrt{2}} \cos \theta_a \cos \left( \varphi_a \frac{\pi}{4} \right)$; and $\varphi = \pi$, yielding $C(\theta_a, \varphi_a) = -\cos \left( \theta_a - \varphi_a \right)$. The quality of the fit between data and theory for these values of $\varphi$, in the root-mean-square sense, is given by 0.936, 0.944 and 0.903, respectively. (c) Measured $B_{\text{max}}$ while varying $\varphi$. Solid curve: theoretical values, $B_{\text{max}} = 2 \sqrt{1 + \sin^2 \frac{\varphi}{2}}$. (d) $S_{\text{pol}}$ and $S_{\text{par}}$ calculated from (a) and (c). Solid curve: theoretical expectation, $S_{\text{pol}} = S_{\text{par}} = 1$. 
2.5 Experiment B: Partially coherent beam with coupled polarization and parity

Starting with $J_{in}$, we scramble the polarization ($D_{pol} = 0$) using a variable polarization rotator without affecting the spatial DoF ($D_{par} = 1$). The two DoFs of this partially coherent beam are then coupled via a PS-SLM (Figure 2.3). The beam remains unpolarized for all $\varphi$ (Figure 2.3(a)), while parity coherence drops from $D_{par} = 1$ (when the DoFs are uncoupled, $\varphi = 0$) to $D_{par} = 0$ (when the coupling is maximal, $\varphi = \pi$), and $B_{max}$ concomitantly attains a maximum (Figure 2.3(c)). Here $S_{par} = 1$ is fixed at 0.5 (Figure 2.3(d)), while $S_{pol}$ rises from 0 (at $\varphi = 0$) to 0.5 ($\varphi = \pi$). We interpret these observations as a migration of coherence from the initially coherent parity DoF into the initially incoherent polarization DoF via their entanglement. At $\varphi = \pi$, the initially asymmetric beam ($D_{pol} = 0$, $D_{par} = 1$, $S_{pol} = 0$, $S_{par} = 0.5$) becomes symmetric with respect to the two entangled DoFs ($D_{pol} = D_{par} = 0$, $S_{pol} = S_{par} = 0.5$). Using a PS-SLM, one may undo this entanglement, leading to coherence migration back to the parity DoF, thereby returning to a spatially coherent but unpolarized beam.
Figure 2.3: Experiment B: partially coherent beam with coupled polarization and parity. (a) Measured $D_{\text{pol}}$ and $D_{\text{par}}$. Solid curves are the theoretical values $D_{\text{pol}} = 0$ and $D_{\text{par}} = |\cos\frac{\varphi}{2}|$. Inset: beam preparation set-up. The polarization rotator is a liquid-crystal variable polarization retarder (Meadowlark Optics, LVR-200) followed by a quarter-wave plate. (b) Measured correlation functions $C(\theta_a, \varphi_a)$ for three source configurations: $\varphi = 0$, yielding theoretically $C(\theta_a, \varphi_a) = 0$; $\varphi = \frac{\pi}{2}$, yielding $C(\theta_a, \varphi_a) = -\frac{1}{\sqrt{2}} \cos \theta_a \sin (\varphi_a + \frac{\pi}{4})$; and $\varphi = \pi$, yielding $C(\theta_a, \varphi_a) = -\cos \theta_a \cos \varphi_a$. The quality of the fit between data and theory for these values of $\varphi$, in the root-mean-square sense, is given by 0.867, 0.867 and 0.981, respectively. (c) Measured $B_{\text{max}}$ while varying $\varphi$. Solid curve: theoretical values, $B_{\text{max}} = 2|\sin\frac{\varphi}{2}|$. (d) $S_{\text{pol}}$ and $S_{\text{par}}$ calculated from (a) and (c). Solid curves: theoretical expectation, $S_{\text{pol}} = \frac{1}{2} \sin^2 \frac{\varphi}{2}$ and $S_{\text{par}} = \frac{1}{2}$. 24
2.6 Experiment C: Beam with random parity-polarization coupling

Finally, we examine the case where the two initially uncoupled DoFs of $J_{\text{in}}$ are coupled through an interaction that involves an element of randomness, such as occurs in a turbulent medium. Specifically, we consider an interaction that alternates between two states, one that couples polarization and parity (with probability $P$) and another that does not (probability $1 - P$). We implement this stochastic interaction by randomly toggling between two phase distributions on the PS-SLM, the first is $\frac{\pi}{2} \text{sgn}(x)$ which couples the DoFs, and the second is zero phase which does not, and $P$ is the fraction of the detection time that the PS-SLM displays the first phase pattern (Figure 2.4). Therefore, $P = 0$ results in the beam remaining coherent with uncoupled DoFs, $P = 1$ maximally couples the polarization and parity of the coherent beam, and intermediate values correspond to partially coherent beams whose DoFs are coupled. Unlike the previous two experiments, $S$ does indeed change. $D_{\text{pol}}$ and $D_{\text{par}}$ decrease monotonically from full coherence to complete incoherence (Figure 2.4(a)), while $S$ does not change monotonically because the beam is coherent at both extrema points $P = 0$ and $P = 1$. This trend is clearly not captured by $D_{\text{pol}}$ and $D_{\text{par}}$ as the entanglement between the DoFs and the randomness introduced by the system both contribute to a decrease in $D_{\text{pol}}$ and $D_{\text{par}}$. Examining $B_{\text{max}}$ (Figure 2.4(c)), however, reveals that the contribution of statistical fluctuations to $D_{\text{pol}}$ and $D_{\text{par}}$ is high ($B_{\text{max}} < 2$), but the contribution of entanglement subsequently dominates with an increase in $P$ ($B_{\text{max}} > 2$). Evaluating $S_{\text{pol}}$ and $S_{\text{par}}$ (Figure 2.4(d)) further confirms the initial decrease and subsequent increase in the accessible coherence of both DoFs with $P$. 

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Figure 2.4: Experiment C: beam with random polarization-parity coupling. (a) Measured $D_{\text{pol}}$ and $D_{\text{par}}$. Solid curves are the theoretical values $D_{\text{pol}} = D_{\text{par}} = 1 - P$. Inset: beam preparation set-up. (b) Measured correlation functions $C(\theta_a, \varphi_a)$ for three source configurations: $P = 0.25$, yielding theoretically $C(\theta_a, \varphi_a) = \frac{3}{4} \sin \theta_a \cos \varphi_a - \frac{1}{4} \cos (\theta_a - \varphi_a)$; $P = 0.5$, yielding $C(\theta_a, \varphi_a) = \frac{1}{2} \sin \theta_a \cos \varphi_a - \frac{1}{2} \cos (\theta_a - \varphi_a)$; and $P = 0.75$, yielding $C(\theta_a, \varphi_a) = \frac{1}{4} \sin \theta_a \cos \varphi_a - \frac{3}{4} \cos (\theta_a - \varphi_a)$. The case $P = 0$ corresponds to Figure 2.2(b) with $\varphi = 0$, and $P = 1$ corresponds to Figure 2.2(b) with $\varphi = \pi$. (c) Measured $B_{\text{max}}$ while varying $P$. Solid curve: theoretical expectation, $B_{\text{max}} = 2\sqrt{1 - 2P + 3P^2}$. The quality of the fit between data and theory for these values of $P$, in the root-mean-square sense, is given by 0.988, 0.961 and 0.933, respectively. (d) $S_{\text{pol}}$ and $S_{\text{par}}$ calculated from (a) and (c). Solid curves: theoretical expectation, $S_{\text{pol}} = S_{\text{par}} = S = P^2 + (1 - P)^2$. 

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2.7 Reduced representation of two binary DoFs

From these experiments, we clearly see that $S_{\text{pol}}$ and $S_{\text{par}}$, which account for the correlation between the DoFs, give a more complete description of the beam’s state of coherence than that afforded by the traditional measures $D_{\text{pol}}$ and $D_{\text{par}}$. Finding $D_{\text{pol}} = 0$, for example, signifies that either the beam is randomly polarized or that the beam is coherent but polarization is deterministically coupled to another DoF. $S_{\text{pol}}$ delineates these two cases: $S_{\text{pol}} = 1$ points to strong correlations to another DoF while $S_{\text{pol}} = 0$ indicates that the beam is unpolarized due to statistical fluctuations.

If we represent an arbitrary partially coherent beam with two binary DoFs by the three invariants $S_{\text{pol}}, S_{\text{par}},$ and $S$, then the locus of all physically admissible beams corresponds to the points constituting the volume displayed in Figure 2.5(a) (see Methods). Each point in the volume represents the set of all partially coherent beams that may be transformed into each other via unitary deterministic transformations that do not couple the two DoFs (and hence do not change $S_{\text{pol}}, S_{\text{par}}$, or $S$). Moving a point representing a beam downwards, thus reducing $S$, requires introducing more randomness into the beam (increasing its entropy). Moving upwards, on the other hand, requires reducing the beam disorder, thereby increasing $S$, and is achieved using filtering devices. Moving in a horizontal constant-$S$ plane is achieved using unitary deterministic devices that couple the two DoFs. For example, a PS-SLM couples the DoFs in Experiment B and the generated beams lie along the red line in Figure 2.5(a). The lines A-1 and A-3 in Figure 2.5(a) represent beams having uncoupled DoFs. The degree of overall coherence $S = \frac{1}{3}$ holds special significance: any partial coherent beam with $S \leq \frac{1}{3}$ and uncoupled DoFs must have one of the DoFs lacking any coherence ($S_{\text{pol}} = 0$ or $S_{\text{par}} = 0$).

Just as the Poincaré sphere enables the visualization of the properties of binary DoFs (such as polarization), the canoe-shaped volume in Figure 2.5(a) enables one to visualize the properties of beams having two binary DoFs, in lieu of a Poincaré hypersphere that cannot be embedded.
in three dimensions, and hence cannot be visualized. This new representation offers many new insights into the global properties of partially coherent beams. We describe two ways in which this volume may be utilized. First, beams sharing the same value of $S$ cannot take on arbitrary values of $S_{\text{pol}}$ and $S_{\text{par}}$. Instead, $S$ uniquely defines the vertices of a concave-sided triangular area (Figure 2.5(b)) that results from the intersection of a horizontal constant-$S$ plane with the volume in Figure 2.5(a). This area represents the physically admissible pairs of values of $S_{\text{pol}}$ and $S_{\text{par}}$ of partially coherent beams that are consistent with the value of $S$. Such a domain may be spanned, and the corresponding beams transformed into each other, via unitary deterministic devices that couple the two DoFs. Alternatively, starting with a beam of given $S_{\text{pol}}$ and $S_{\text{par}}$, we may predict the corresponding $S$ within a small uncertainty window. By launching a vertical line from the point $(S_{\text{pol}}, S_{\text{par}}, 0)$, the length of the short segment resulting from its intersection with the volume corresponds to the uncertainty in the value of $S$ compatible with such a beam (on average, the uncertainty in determining $S$ is $\sim 6\%$). $S_{\text{pol}}$ and $S_{\text{par}}$ are thus endowed with predictive powers with respect to estimating the degree of overall beam coherence.

A surprising result that becomes clear by adopting our new perspective is highlighted in Figure 2.5(b). Experiment B starts at point 1 with a beam that is spatially coherent but unpolarized ($D_{\text{pol}} = 0$ and $D_{\text{par}} = 1$). Using a polarization-sensitive parity rotator (implemented by a PS-SLM) we couple the two DoFs and reach point 2 where the beam is symmetric ($D_{\text{pol}} = D_{\text{par}} = 0$, $S_{\text{pol}} = S_{\text{par}} = 0.5$). Now, a device in which the roles of polarization and parity in the PS-SLM are reversed, i.e., a parity-sensitive polarization rotator, would allow us to reach point 3 which represents a beam that is polarized but spatially incoherent ($D_{\text{pol}} = 1$ and $D_{\text{par}} = 0$). In effect, the initial coherence in the parity DoF is ‘swapped’ into the polarization DoF using unitary deterministic devices without changing the degree of overall beam coherence or reducing the beam energy. This example highlights the surprising results that become relatively obvious by examining classical optical coherence from our proposed perspective. Adopting this quantum-inspired ordering
of partially coherent beams reinforces the usefulness of the concept of coherence as a quantifiable resource shared among a beam’s multiple DoFs.

Figure 2.5: Parametrizing partially coherent beams by $S_{\text{pol}}$, $S_{\text{par}}$ and $S$. (a) The locus of all physically admissible partially coherent beams. Loci of experiments A, B and C are highlighted. The brightness-coded plot in the $S = 0$ plane represents the thickness of the volume above it. (b) Three sections through the volume in (a) at $S = \frac{1}{6}$, $\frac{1}{3}$ and $\frac{2}{3}$ projected onto a horizontal plane. Point 1 corresponds to an unpolarized spatially coherent beam ($D_{\text{pol}} = 0, D_{\text{par}} = 1$), point 3 corresponds to a spatially incoherent polarized beam ($D_{\text{pol}} = 1, D_{\text{par}} = 0$), both with uncoupled DoFs, and point 2 corresponds to maximally coupling the DoFs of the beams at 1 or 3.
2.8 Discussion and conclusions

When two DoFs of a coherent beam are coupled via a deterministic device, each DoF by itself (with the other DoF ignored) becomes partially coherent or even incoherent, although no randomness has been introduced. This acquired uncertainty results from limiting the space to two-dimensional (2D) when it is actually 4D (assuming each DoF is binary). Because of the mathematical similarity between the description of this classical beam and the quantum-mechanical representation of an entangled two-qubit system, it is natural to examine the relevance of Bell’s measure to the classical beam.

Of course, testing the non-locality exhibited by bipartite quantum systems, which was the original motivation for developing Bell’s measure, is not an issue in the classical paradigm. Nevertheless, there is special significance associated with $B_{\text{max}} > 2$, which in the quantum paradigm implies a violation of local-realism. Exceeding this bound in classical coherence signifies that the correlation function $C$ associated with the beam does not admit a modal decomposition of the type:

$$C(\theta_a, \varphi_a) = \int d\xi p(\xi) C_{\text{pol}}(\theta_a; \xi) C_{\text{par}}(\varphi_a; \xi),$$

where $p(\xi)$ is a probability distribution ($p(\xi)$ is real, $p(\xi) \geq 0$, and $\int d\xi p(\xi) = 1$) over any set of random variables, discrete or continuous, denoted by $\xi$ [50]. Since $C$ factors only when the two DoFs are uncoupled, this statement implies that beams having $B_{\text{max}} > 2$ cannot be produced by random mixing of any number of optical beams, if each beam is constrained to have uncoupled DoFs but may otherwise each have an arbitrary state of coherence (see Methods for examples).

We have used Bell’s measure here to quantify the correlation between the two DoFs. Other potential correlation measures have been studied in the quantum information theory literature, such as concurrence [47]. Adopting any of these measures will result in a volume representation shaped
differently than that in Figure 2.5(a). The optimal measure of correlation between the two DoFs in the context of classical coherence is one that reduces the volume in Figure 2.5(a) to a surface. In that case \( S_{\text{pol}} \) and \( S_{\text{par}} \) uniquely determine \( S \), and a fixed \( S \) establishes an exact complementarity between \( S_{\text{pol}} \) and \( S_{\text{par}} \). It remains an open question whether such a measure exists, and if not, how closely this limit may be approached.

We have experimentally confirmed this paradigm using polarization and spatial parity as representative DoFs. We have shown that Bell’s measure identifies the apparent uncertainty acquired in each DoF as a result of their entanglement, independently of the native uncertainty associated with statistical fluctuations. The polarization-parity results reported here are applicable to any pair of binary DoFs and may be generalized to multi-modal classical beams or even continuous DoFs. In this case, a higher-order Bell’s measure appropriate for the higher-dimensional DoFs can be used [54, 55].

Our results demonstrate, more generally, that the mathematical machinery developed over the past two decades in quantum information theory is of direct relevance to the much older discipline of classical optical coherence theory. In order to obtain a one-to-one correspondence, one needs only to interpret the levels of a quantum system as the modes of a DoF of the beam and the multiple quantum systems as independent DoFs. We predict that the appreciation of this correspondence will help stimulate the introduction of new optical metrology schemes, while also enriching classical optical coherence theory and information optics [56, 57] in general.
2.9 Methods

2.9.1 Definitions of \(G_{\text{pol}}\), \(G_{\text{par}}\), \(D_{\text{pol}}\) and \(D_{\text{par}}\)

In general, the \(4 \times 4\) polarization-parity coherency matrix is:

\[
G = \begin{pmatrix}
G_{\text{He,He}} & G_{\text{He,Ho}} & G_{\text{He,Ve}} & G_{\text{He,Vo}} \\
G_{\text{Ho,He}} & G_{\text{Ho,Ho}} & G_{\text{Ho,Ve}} & G_{\text{Ho,Vo}} \\
G_{\text{Ve,He}} & G_{\text{Ve,Ho}} & G_{\text{Ve,Ve}} & G_{\text{Ve,Vo}} \\
G_{\text{Vo,He}} & G_{\text{Vo,Ho}} & G_{\text{Vo,Ve}} & G_{\text{Vo,Vo}} \\
\end{pmatrix},
\]  

(2.22)

where \(G^\dagger = G\). Each element in \(G\) results from averaging the appropriate products of field components. For example, \(G_{\text{Ve,Ho}} = \langle E_{\text{Ve}}^* E_{\text{Ho}} \rangle\), and so on. We normalize \(G\) such that \(\text{Tr}\{G\} = 1\), where ‘Tr’ refers to the matrix trace operation. This matrix may be written in the block form:

\[
G = \begin{pmatrix}
G_{\text{HH}} & G_{\text{HV}} \\
G_{\text{VH}} & G_{\text{VV}} \\
\end{pmatrix}
\]  

(2.23)

with each of the \(2 \times 2\) submatrices having the appropriate parity elements. Conversely, \(G\) may be arranged into the form:

\[
G = \begin{pmatrix}
G_{\text{ee}} & G_{\text{eo}} \\
G_{\text{oe}} & G_{\text{oo}} \\
\end{pmatrix}
\]  

(2.24)

with the \(2 \times 2\) submatrices having the appropriate polarization elements. From this general form we obtain the reduced \(2 \times 2\) polarization and parity coherency matrices:

\[
G_{\text{pol}} = \begin{pmatrix}
G_{\text{He,He}} + G_{\text{Ho,Ho}} & G_{\text{He,Ve}} + G_{\text{Ho,Vo}} \\
G_{\text{Ve,He}} + G_{\text{Vo,Ho}} & G_{\text{Ve,Ve}} + G_{\text{Vo,Vo}} \\
\end{pmatrix},
\]  

(2.25)
\[ \mathbf{G}_{\text{pol}} = \begin{pmatrix} \mathbf{G}_{\text{He},\text{He}} + \mathbf{G}_{\text{Ve},\text{Ve}} & \mathbf{G}_{\text{He},\text{Ho}} + \mathbf{G}_{\text{Ve},\text{Vo}} \\ \mathbf{G}_{\text{Ho},\text{He}} + \mathbf{G}_{\text{Vo},\text{Ve}} & \mathbf{G}_{\text{Ho},\text{Ho}} + \mathbf{G}_{\text{Vo},\text{Vo}} \end{pmatrix}, \quad (2.26) \]

respectively. We define the degree of polarization coherence as \( D_{\text{pol}} = |\lambda_1 - \lambda_2| \), where \( \lambda_1 \) and \( \lambda_2 \) are the eigenvalues of \( \mathbf{G}_{\text{pol}} \). Similarly, \( D_{\text{par}} \) is the absolute value of the difference between the eigenvalues of \( \mathbf{G}_{\text{par}} \). Using these definitions, we obtain the theoretical expressions for \( D_{\text{pol}} \) and \( D_{\text{par}} \) corresponding to the three experiments reported in Figures 2.2, 2.3, and 2.4.

### 2.9.2 Proof that \( D_{\text{pol}} = D_{\text{par}} \) for any coherent beam

Consider a coherent beam where \( \mathbf{G} = \mathbf{J} \ast \mathbf{J}^T \) with no need for averaging. One may always write \( \mathbf{J} \) in the Schmidt form:

\[ \mathbf{J} = \begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & \mu_2 \end{bmatrix}^T, \quad (2.27) \]

with no loss of generality [58]. The vector \( \mathbf{J} \) may be reduced to the Schmidt form through unitary deterministic transformations that affect each DoF separately and without introducing coupling between them [48]. Using the definitions given above, it is straightforward to show that both reduced coherency matrices in this basis are equal:

\[ \mathbf{G}_{\text{pol}} = \mathbf{G}_{\text{par}} = \begin{pmatrix} \mu_1^2 & 0 \\ 0 & \mu_2^2 \end{pmatrix}, \quad (2.28) \]

thus leading necessarily to \( D_{\text{pol}} = D_{\text{par}} = |\mu_1^2 - \mu_2^2| \).
2.9.3 Experiment A

The field is coherent with components:

\[ \mathbf{J} = \frac{1}{\sqrt{2}} \begin{bmatrix} \cos \frac{\varphi}{2} & i \sin \frac{\varphi}{2} & 1 & 0 \end{bmatrix}^T, \]  

so that \( \mathbf{G} \) and the reduced coherency matrices are:

\[ \mathbf{G} = \frac{1}{2} \begin{pmatrix} \cos^2 \frac{\varphi}{2} & -i \cos \frac{\varphi}{2} \sin \frac{\varphi}{2} & \cos \frac{\varphi}{2} & 0 \\ i \cos \frac{\varphi}{2} \sin \frac{\varphi}{2} & \sin^2 \frac{\varphi}{2} & i \sin \frac{\varphi}{2} & 0 \\ \cos \frac{\varphi}{2} & -i \sin \frac{\varphi}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]  

\[ \mathbf{G}_{\text{pol}} = \frac{1}{2} \begin{pmatrix} 1 & \cos \frac{\varphi}{2} \\ \cos \frac{\varphi}{2} & 1 \end{pmatrix}, \]  

\[ \mathbf{G}_{\text{par}} = \frac{1}{2} \begin{pmatrix} 1 + \cos^2 \frac{\varphi}{2} & -i \cos \frac{\varphi}{2} \sin \frac{\varphi}{2} \\ i \cos \frac{\varphi}{2} \sin \frac{\varphi}{2} & \sin^2 \frac{\varphi}{2} \end{pmatrix}, \]

respectively. Here \( D_{\text{pol}} = D_{\text{par}} = |\cos \frac{\varphi}{2}| \), \( S = 1 \), and the polarization-parity correlation function \( C \) is:

\[ C(\theta_a, \varphi_a) = \sin \theta_a \sin \left( \varphi_a + \frac{\varphi}{2} \right) - \cos \theta_a \cos \left( \varphi_a + \frac{\varphi}{2} \right) \sin \frac{\varphi}{2}. \]  

It can be shown that:

\[ B_{\text{max}} = 2 \sqrt{1 + \sin^2 \frac{\varphi}{2}}, \]  

which is isomorphic to the case of a pure two-qubit state [48]. Consequently,

\[ S_{\text{pol}} = S_{\text{par}} = 1. \]
2.9.4 Experiment B

The initial field is separable in polarization and parity, and the coherency matrix $G$ and the reduced coherency matrices after the action of the PS-SLM are given by:

$$G = \frac{1}{2} \begin{pmatrix}
\cos^2 \frac{\varphi}{2} & -i \cos \frac{\varphi}{2} \sin \frac{\varphi}{2} & 0 & 0 \\
 i \cos \frac{\varphi}{2} \sin \frac{\varphi}{2} & \sin^2 \frac{\varphi}{2} & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0
\end{pmatrix}, \quad (2.36)$$

$$G_{\text{pol}} = \frac{1}{2} \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}, \quad (2.37)$$

$$G_{\text{par}} = \frac{1}{2} \begin{pmatrix}
1 + \cos^2 \frac{\varphi}{2} & -i \cos \frac{\varphi}{2} \sin \frac{\varphi}{2} \\
 i \cos \frac{\varphi}{2} \sin \frac{\varphi}{2} & \sin^2 \frac{\varphi}{2}
\end{pmatrix}, \quad (2.38)$$

respectively. Here, $D_{\text{pol}} = 0$, $D_{\text{par}} = |\cos \frac{\varphi}{2}|$, $S = \frac{1}{3}$ and,

$$C(\theta_a, \varphi_a) = \frac{1}{2} \cos \theta_a \{\cos (\varphi_a + \varphi) - \cos \varphi_a\}, \quad (2.39)$$

resulting in:

$$B_{\text{max}} = 2|\sin \frac{\varphi}{2}|. \quad (2.40)$$

Consequently:

$$S_{\text{pol}} = \frac{1}{2} \sin^2 \frac{\varphi}{2}, \quad (2.41)$$

and

$$S_{\text{par}} = \frac{1}{2}. \quad (2.42)$$
2.9.5  Experiment C

The two fields between which the configuration toggles are \( \frac{1}{\sqrt{2}} [ 0 \ i \ 1 \ 0 ]^T \), with probability \( P \) and \( \frac{1}{\sqrt{2}} [ 1 \ 0 \ 1 \ 0 ]^T \) with probability \( 1 - P \). The coherency matrix \( G \) and the reduced coherency matrices are:

\[
G = \frac{1}{2} \begin{pmatrix}
1 - P & 0 & 1 - P & 0 \\
0 & P & iP & 0 \\
1 - P & -iP & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

(2.43)

\[
G_{\text{pol}} = \frac{1}{2} \begin{pmatrix}
1 & 1 - P \\
1 - P & 1
\end{pmatrix},
\]

(2.44)

\[
G_{\text{par}} = \frac{1}{2} \begin{pmatrix}
2 - P & 0 \\
0 & P
\end{pmatrix},
\]

(2.45)

respectively. Here, \( D_{\text{pol}} = D_{\text{par}} = 1 - P \), and

\[
C(\theta_a, \varphi_a) = (1 - P) \sin \theta_a \cos \varphi_a - P \cos (\theta_a - \varphi_a),
\]

(2.46)

resulting in,

\[
B_{\text{max}} = 2 \sqrt{1 - 2P + 3P^2},
\]

(2.47)

and

\[
S_{\text{pol}} = S_{\text{par}} = S = P^2 + (1 - P^2).
\]

(2.48)
2.9.6 Determining the volume in Figure 2.5(a)

Numerical simulations of random coherency matrices were performed to produce the volume shown in Figure 2.5(a), which represents the locus of all physically admissible partially coherent beams characterized by two binary DoFs (polarization and spatial parity here). Each point in the volume corresponds to the family of optical beams that share the same values of the three parameters $S$, $S_{\text{pol}}$, and $S_{\text{par}}$. The points were produced by randomly generating diagonal $4 \times 4$ coherency matrices $G$ with the following constraints: the eigenvalues of $G$ are positive, real, and their sum is 1. The diagonal $4 \times 4$ coherency matrices were generated with ranks (number of non-zero eigenvalues) 1, 2, 3, and 4, and different degeneracy (number of equal eigenvalues). Next, the diagonal coherency matrices were transformed using randomly generated $4 \times 4$ unitary transformations. The unitary transformations were generated randomly with probability distribution given by the Haar Measure on $U(N)$ by diagonalizing a random Hermitian matrix [59]. We then obtain $S$, $S_{\text{pol}}$, and $S_{\text{par}}$ for each generated matrix $G$ and plot the volume in Figure 2.5(a).

2.9.7 Optical beams with $B_{\text{max}} = 2$

(a) Consider Experiment A when $\varphi = 0$ and $B_{\text{max}} = 2$. The beam may be factored in terms of its DoFs:

$$G = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = G_{\text{pol}} \otimes G_{\text{par}} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$  (2.49)

The beam is coherent, the polarization is linear at $45^\circ$, and the parity is even.

(b) Consider Experiment B when $\varphi = \pi$ and $B_{\text{max}} = 2$. The beam here is partially coherent, and
may be written as an equal-weighted incoherent mixture of two beams having coherency matrices $G_1$ and $G_2$:

$$G = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = p_1 G_1 + p_2 G_2$$

$$= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

(2.50)

where $p_1 = p_2 = \frac{1}{2}$. Both $G_1$ and $G_2$ correspond to coherent beams that have uncoupled polarization and parity having the Jones vectors:

$$J_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \end{bmatrix}^T \otimes \begin{bmatrix} 0 & 1 \end{bmatrix}^T,$$

(2.51)

and

$$J_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 \end{bmatrix}^T \otimes \begin{bmatrix} 1 & 0 \end{bmatrix}^T.$$

(2.52)

The first beam has H polarization and odd parity, while the second has V polarization and even parity.
Consider Experiment B when $\varphi = \frac{\pi}{3}$ and $B_{\text{max}} = 1$. The beam here is partially coherent and may be written as an equal-weighted incoherent mixture of two beams with coherency matrices $G_1$ and $G_2$:

$$
\mathbf{G} = \frac{1}{2} \begin{pmatrix}
\frac{3}{4} & i\frac{\sqrt{3}}{4} & 0 & 0 \\
-\frac{i\sqrt{3}}{4} & \frac{1}{4} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} = p_1 \mathbf{G}_1 + p_2 \mathbf{G}_2
$$

where $p_1 = p_2 = \frac{1}{2}$. Both $\mathbf{G}_1$ and $\mathbf{G}_2$ correspond to coherent beams that have uncoupled polarization and parity having the Jones vectors:

$$
\mathbf{J}_1 = \begin{bmatrix}
\frac{\sqrt{3}}{2} & i\frac{1}{2} & 0 & 0
\end{bmatrix}^T = \begin{bmatrix}
1 & 0
\end{bmatrix}^T \otimes \begin{bmatrix}
\frac{\sqrt{3}}{2} & i\frac{1}{2}
\end{bmatrix}^T,
$$

and

$$
\mathbf{J}_2 = \begin{bmatrix}
0 & 0 & 1 & 0
\end{bmatrix}^T = \begin{bmatrix}
0 & 1
\end{bmatrix}^T \otimes \begin{bmatrix}
1 & 0
\end{bmatrix}^T.
$$

The first beam has H polarization and a superposition of even and odd parity, while the second has V polarization and even parity.
CHAPTER 3: OPTICAL TWO-POINT VECTOR FIELD CORRELATIONS

3.1 Introduction

The concepts of partial polarization at a point in an optical field and partial spatial coherence in a scalar field are both well understood [2]. Partially coherent electromagnetic fields, in which both of these aspects are inextricably linked, have received considerable attention over the past decade or so [4,9,60,61]. It is well-established that the coherence of an electromagnetic field [62] (quantified by two-point field correlations) is represented by a $4 \times 4$ coherency matrix $G$ [11]. This matrix is a complete representation of second-order field correlations (first-order coherence) for any two points in the field. Therefore, all proposed measures of vector-field coherence are in fact scalar functions of the elements of $G$ [4,61]. Furthermore, the importance of $G$ stems from its predictive power: it determines the coherence properties after any subsequent linear manipulation of the field at these two points.

3.2 Methodology

Despite its fundamental importance, the elements of $G$ have not been directly measured in their entirety, heretofore, and proposed approaches to achieving this goal are lacking. Here, we present a systematic measurement methodology to reconstruct $G$ for an electromagnetic field. Moreover, this approach is applicable to any two (or more) discrete degrees of freedom (DoFs) of an optical field. Underpinning this strategy is a finite set of optical measurements (following appropriate field

transformations and projections) that may be inverted to yield the complex elements of \( G \). The choice of the particular measurements to be implemented is elucidated by highlighting the correspondence between the problem of identifying the elements of \( G \) in classical optics and identifying the complex elements of the density matrix associated with two-photon states in quantum optics – a process typically known as quantum state tomography [63, 64]. In light of this correspondence, we call the approach described here optical coherency matrix tomography, applied in the current context to the particular case of two-point polarization correlations.

### 3.2.1 Formalism for polarization DoF

We first consider the polarization DoF at a single point in a quasi-monochromatic beam, which is represented by a \( 2 \times 2 \) Hermitian polarization coherency matrix:

\[
\mathbf{G}_p = \begin{pmatrix} G_{HH} & G_{HV} \\ G_{VH} & G_{VV} \end{pmatrix} = \frac{1}{2} \sum_{l=0}^{3} S^p_l \hat{\sigma}_l = \frac{1}{2} \begin{pmatrix} S^p_0 + S^p_1 & S^p_2 - i S^p_3 \\ S^p_2 + i S^p_3 & S^p_0 - S^p_1 \end{pmatrix},
\]

(3.1)

where \( G_{jj'} = \langle E_j(\vec{r}) E^*_j(\vec{r}) \rangle \), \( G_{jj'} = G^*_{j'j} \), \( E_j(\vec{r}) \) is the horizontal (H) or vertical (V) field component at a point \( \vec{r} \), \{\( S^p_l \)\} are the Stokes parameters, and \( \hat{\sigma}_l \) \( (l = 0, \ldots, 3) \) are the Pauli matrices defined as:

\[
\begin{align*}
\hat{\sigma}_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \hat{\sigma}_1 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & \hat{\sigma}_2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \hat{\sigma}_3 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
\end{align*}
\]

(3.2)

The degree of polarization \( D_p \) is defined as:

\[
D_p = \frac{1}{S^p_0} \sqrt{(S^p_1)^2 + (S^p_2)^2 + (S^p_3)^2}.
\]

(3.3)
This representation contains four real parameters that may be identified experimentally via the four polarization projections shown in Figure 3.1(a): the total power $I_0$ and that of the H, 45°, and right-hand circular (RHC) polarization components, corresponding to $I_1$, $I_2$, and $I_3$, respectively. These measurements yield the Stokes parameters since $S_j^p = 2I_j - I_0$, $j = 0, \ldots, 3$. Note that normalizing the measurements with respect to $I_0$ yields a unity-trace $G_p$, which is now in one-to-one correspondence with the density operator used in quantum optics to describe one-photon polarization states [65,66], or, more generally, any two-level quantum system.
3.2.2 Formalism for two-point spatial DoF

Such a formalism may also be used to capture the two-point spatial correlations in a scalar field. The corresponding representation of spatial coherence using two-point correlations at \( \vec{r}_a \) and \( \vec{r}_b \) is a spatial coherency matrix:

\[
G_s = \begin{pmatrix}
G_{aa} & G_{ab} \\
G_{ba} & G_{bb}
\end{pmatrix} = \frac{1}{2} \sum_{m=0}^{3} S_s^m \hat{\sigma}_m = \frac{1}{2} \begin{pmatrix}
S_0^s + S_1^s & S_2^s - i S_3^s \\
S_2^s + i S_3^s & S_0^s - S_1^s
\end{pmatrix}, \quad (3.4)
\]

where \( G_{kk'} = \langle E(\vec{r}_k) E^*(\vec{r}_{k'}) \rangle \), \( G_{kk'} = G_{k'k}^* \), \( k, k' = a, b \), and \( \{ S_s^m \} \) are the two-point spatial Stokes parameters, and \( \hat{\sigma}_m (m = 0, \ldots, 3) \) are the Pauli matrices defined in Equation (3.2). The elements of \( G_s \) may be obtained experimentally using the spatial projective measurements shown in Figure 3.1(b) without recourse to recording spatial interference patterns: the total power \( I_0 \); the power \( I_1 \) at one position, e.g., \( \vec{r}_a \) (corresponding to the H-polarization measurement above); the power \( I_2 \) after symmetric mixing of the field at \( \vec{r}_a \) and \( \vec{r}_b \) (corresponding to the 45°-polarization measurement); and \( I_3 \) obtained similarly to \( I_2 \) except for a \( \frac{\pi}{2} \) phase shift placed at \( \vec{r}_b \) (corresponding to the RHC-polarization measurement). As with the case of polarization above, \( G_s \) is normalized such that \( S_0^s = 1 \) puts \( G_s \) in one-to-one correspondence with density operators that describe two-level quantum systems. The visibility of Young’s interferogram produced by the field at these two points is then simply:

\[
V = 2 |\Re \{ G_{ab} \} | = |S_2^s|, \quad (3.5)
\]

which is clearly not sufficient to retrieve all the parameters of \( G_s \). In analogy to \( D_p \), we define a spatial counterpart of the degree of polarization as the degree of spatial coherence:

\[
D_s = \frac{1}{S_0^s} \sqrt{(S_1^s)^2 + (S_2^s)^2 + (S_3^s)^2}. \quad (3.6)
\]
This quantity has a clear interpretation: it is the maximum observable visibility from the field at \( \vec{r}_a \) and \( \vec{r}_b \) after an arbitrary unitary transformation is implemented. That is, after inserting relative phases and/or mixing the field from \( \vec{r}_a \) and \( \vec{r}_b \).

### 3.2.3 Joint formalism for polarization and spatial DoFs

We now consider an electromagnetic field, in which case both the spatial coherence and polarization DoFs must be considered simultaneously. Coherence at two points \( \vec{r}_a \) and \( \vec{r}_b \) is then captured by a \( 4 \times 4 \) Hermitian two-point vector coherency matrix [9, 11]:

\[
G = \begin{pmatrix}
G_{Ha,Ha} & G_{Ha,Hb} & G_{Ha,Va} & G_{Ha,Vb} \\
G_{Hb,Ha} & G_{Hb,Hb} & G_{Hb,Va} & G_{Hb,Vb} \\
G_{Va,Ha} & G_{Va,Hb} & G_{Va,Va} & G_{Va,Vb} \\
G_{Vb,Ha} & G_{Vb,Hb} & G_{Vb,Va} & G_{Vb,Vb}
\end{pmatrix} 
\]  \hspace{1cm} (3.7)

Here \( G_{jk,j'k'} = \langle E_j(\vec{r}_k)E_{j'}^*(\vec{r}_{k'}) \rangle \), \( G_{j,k,j',k'} = G_{j',k',j,k} \), \( j, j' = H, V \), and \( k, k' = a, b \). The Hermiticity of \( G \) reduces the number of real parameters necessary to uniquely specify it to 16. It is critical to note that it is not possible to reconstruct \( G \) from independent polarization and spatial measurements. To observe this fact, first ignore the polarization DoF. The spatial coherence is then characterized by a reduced spatial coherency matrix (obtained by carrying out a partial trace over the polarization DoF in \( G \)):

\[
G_s^{(r)} = \begin{pmatrix}
G_{Ha,Ha} + G_{Va,Va} & G_{Ha,Hb} + G_{Va,Vb} \\
G_{Hb,Ha} + G_{Vb,Va} & G_{Hb,Hb} + G_{Vb,Vb}
\end{pmatrix} 
\]  \hspace{1cm} (3.8)

The polarization-independent spatial-coherence measures \( V \) and \( D_s \) may be determined from \( G_s^{(r)} \). Similarly, by ignoring the spatial DoF we obtain a reduced polarization coherency matrix for the
total field at both $\vec{r}_a$ and $\vec{r}_b$ (i.e., without spatially resolving the two points):

$$G_p^{(r)} = \begin{pmatrix}
G_{Ha,Ha} + G_{Hb,Hb} & G_{Ha,Va} + G_{Hb,Vb} \\
G_{Va,Ha} + G_{Vb,Hb} & G_{Va,Va} + G_{Vb,Vb}
\end{pmatrix},$$

(3.9)

which may be used to determine $D_p$. By inspection, it is clear that $G_p^{(r)}$ and $G_s^{(r)}$ are insufficient to reconstruct $G$. The elements of $G$ that are missing from $G_p^{(r)}$ and $G_s^{(r)}$ are those that account for correlations between polarization and spatial DoFs. Determining these elements requires joint polarization-spatial measurements.

### 3.2.4 Reconstruction of coherency matrix $G$

In addressing the task of completely reconstructing $G$, we take inspiration from an analogous problem in quantum optics, where the polarization of two-photon states of light is encoded by a $4 \times 4$ density matrix $\hat{\rho}$ in the Hilbert space formed of the direct product of the Hilbert spaces that characterize the polarization of each photon [65]. An isomorphism between $\hat{\rho}$ (for the quantum state) and $G$ (for the classical field) is established by identifying the vector spaces for the state of each photon in the former with the vector spaces representing the DoFs of the classical beam. That is, we identify mathematically the polarization of the first photon, for example, in the two-photon state with the polarization of the classical field. Then we identify the polarization of the second photon with the spatial DoF of the classical field.

The measurements necessary to reconstruct $\hat{\rho}$ for composite quantum systems were identified by Wootters [67] (see also Refs. [68, 69]). For a quantum system comprising two subsystems, the necessary measurements to reconstruct $\hat{\rho}$ are the correlation of pairs of projective measurements, one for each subsystem, chosen from the sets of measurements that are sufficient to reconstruct the state of each subsystem. In other words, the measurements required to completely specify the
subsystems are, surprisingly, sufficient to specify the complete system – so long as they are carried out jointly. Therefore, in the case of a two-photon polarization state, each photon is directed to the four polarization analyzers used in Figure 3.1(a) [63, 64]. In the two-photon measurement scheme (Figure 3.2) 16 measurements are obtained by pairing polarization measurements implemented in the paths of photons traveling to the right (P₁) and left (P₂).

The isomorphism between \( \hat{\rho} \) and \( G \) guarantees that an analogous measurement strategy is sufficient to uniquely reconstruct \( G \) for the classical beam. The required measurements to reconstruct \( G \) correspond to cascading pairs of \( 4 \times 4 = 16 \) projections, one for each DoF of the classical beam. These (real) tomographic measurements that span both DoFs may then be inverted to obtain the complex elements of \( G \). An arrangement where spatial measurements follow polarization projections is shown in Figure 3.1(c). The order of the projective measurements may also be reversed where polarization measurements follow spatial projections. To the best of our knowledge and despite the fundamental importance of \( G \), the experimental scheme we have described here for tomographically reconstructing \( G \) has not been realized to date.

We relate the real measurements \( I_{jk} \) to the complex elements of \( G \) by first defining a new set of Stokes parameters:

\[
\mathbf{G} = \frac{1}{4} \sum_{l,m=0}^{3} S_{lm} \hat{\sigma}_l^p \otimes \hat{\sigma}_m^s, \tag{3.10}
\]

(\( \otimes \) is the direct product). The elements of \( G \) are then given explicitly as:

\[
\mathbf{G} = \frac{1}{4} \begin{pmatrix}
S_{00} + S_{01} + S_{10} + S_{11} & S_{02} + S_{12} - i(S_{03} + S_{13}) & S_{20} + S_{21} - i(S_{30} + S_{31}) & S_{22} - S_{33} - i(S_{23} + S_{32}) \\
S_{02} + S_{12} + i(S_{03} + S_{13}) & S_{00} - S_{01} + S_{10} - S_{11} & S_{22} + S_{33} + i(S_{23} - S_{32}) & S_{20} - S_{21} - i(S_{30} - S_{31}) \\
S_{20} + S_{21} + i(S_{30} + S_{31}) & S_{22} + S_{33} - i(S_{23} - S_{32}) & S_{00} + S_{01} - S_{10} - S_{11} & S_{02} - S_{12} - i(S_{03} - S_{13}) \\
S_{22} - S_{33} - i(S_{23} + S_{32}) & S_{20} - S_{21} + i(S_{30} - S_{31}) & S_{02} - S_{12} + i(S_{03} - S_{13}) & S_{00} - S_{01} - S_{10} + S_{11}
\end{pmatrix}, \tag{3.11}
\]
In this formulation the (real) multi-DoF Stokes parameters $S_{lm}$ are determined by the measurements following the relationship $S_{lm} = 4I_{lm} - 2I_{l0} - 2I_{0m} + I_{00}$, $l, m = 0, \ldots, 3$. Therefore, once the multi-DoF Stokes parameters $S_{lm}$ are obtained, they may be substituted into Equation (3.11), thereby completing the inversion of the measurements and tomographic reconstruction of $G$. Thus, there is no need to carry out separate polarization measurements at $\vec{r}_a$ or $\vec{r}_b$. Note that if $G$ is normalized to unit trace ($S_{00} = 1$), then complete coherence is achieved when $\text{Tr}\{G^2\} = 1$, and complete spatial and polarization incoherence occur when $\text{Tr}\{G^2\} = \frac{1}{4}$.

Figure 3.2: Setup for measuring the two-photon polarization density matrix $\hat{\rho}$ through projective measurements on photons $P_1$ and $P_2$ emitted from a two-photon source $S$. $C_{jk}$ is the probability of coincidence detection after polarization projections $j$ and $k$ (shown here is the particular measurement $C_{12}$ out of 16 potential measurements). See Figure 3.1 for a definition of the components.
3.3 Results

Reconstructing $G$ uniquely defines the coherence state of the field at any two points, thereby lifting ambiguities that arise from the use of only a few scalar parameters, especially when the DoFs are correlated. To highlight this crucial feature of $G$, we describe below six examples of fields where we compare $V$, $D_s$, or $D_p$ – extracted from traditional measurements – to the information extracted from the reconstructed $G$. We present a pictorial depiction of the coherency matrices in Figure 3.3 that appropriates the methodology common in quantum state tomography.

3.3.1 $G_1$: spatially coherent, horizontally polarized field

$G_1$ corresponds to a spatially coherent, horizontally polarized field with equal amplitudes at $\vec{r}_a$ and $\vec{r}_b$, in which case $V = D_s = D_p = 1$ [Figure 3.3(a)]. Here, it is clear that $G_1$ separates into a direct product of the coherency matrices for the uncoupled DoFs:

$$G_1 = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right)^p \otimes \frac{1}{2} \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right)^s.$$

(3.12)

This indicates that the two DoFs are independent [as is also clear in Figure 3.3(a)] and, since $\text{Tr}\{G_1^2\} = 1$, each is fully coherent.

3.3.2 $G_2$: classically entangled field

The importance of reconstructing $G$ becomes apparent when examining fields in which the spatial and polarization DoFs are correlated, or classically entangled. Consider $G_2$ that corresponds to a coherent field with orthogonal polarizations at $\vec{r}_a$ and $\vec{r}_b$ [Figure 3.3(b)]. Although there is no randomness, the usual indicators of coherence applied to each DoF reveal an apparent complete
lack of coherence: \( V = D_s = D_p = 0 \). This is in contradistinction to the fact that \( \text{Tr}\{G_2^2\} = 1 \), which confirms the global coherence of the field. Such an ambiguity is resolved by noting from Figure 3.3(b) that \( G_2 \) cannot be factorized into a direct product of polarization and spatial coherency matrices, as is the case for \( G_1 \) above. Therefore, the apparent lack of coherence is due to the classical entanglement between the two DoFs and not due to intrinsic randomness or fluctuations [9].

### 3.3.3 \( G_3 \): spatially coherent, randomly polarized field

\( G_3 \) shown in Figure 3.3(c) corresponds to a spatially coherent field with randomized polarization: \( V = D_s = 1 \) and \( D_p = 0 \). Here,

\[
G_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^p \otimes \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^s; \tag{3.13}
\]

i.e., the two DoFs are independent and the former lacks coherence. Both features are observed clearly in Figure 3.3(c).

### 3.3.4 \( G_4 \): spatially incoherent, horizontally polarized field

Consider the previous case of \( G_3 \), with the role of the two DoFs reversed. In this scenario, \( G_4 \) corresponds to a horizontally polarized field that is spatially incoherent: \( V = D_s = 0 \) and \( D_p = 1 \) [Fig. 3(d)]. Here,

\[
G_4 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^p \otimes \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^s; \tag{3.14}
\]

i.e., the two DoFs are independent and the latter lacks coherence. Both features are clear in Figure 3.3(d), albeit with the role of polarization and space reversed with respect to Figure 3.3(c).
3.3.5 $G_5$: spatially incoherent, unpolarized field

$G_5$ corresponds to an unpolarized, spatially incoherent field – a maximum entropy state of the electromagnetic field [Figure 3.3(e)]. In this case the field globally lacks any coherence $\text{Tr}\{G_5^2\} = \frac{1}{3}$, and, similarly, the reduced coherency matrices each reveal a lack of coherence ($V = D_s = D_p = 0$). By examining $G_5$ as shown in Figure 3.3(e), it is clear that $G_5 = \frac{1}{4}(\hat{\sigma}_0^p \otimes \hat{\sigma}_0^s)$, corresponding to intrinsic randomness in all DoFs of the field.

3.3.6 $G_6$: partially coherent, partially polarized field

$G_6$ corresponds to a partially coherent, partially polarized field with correlated DoFs [Figure 3.3(f)]. This state arises if the polarization is randomized at one of the two points from an initially coherent polarized field. The coherence measures obtained from the reduced coherency matrices are $V = D_s = D_p = \frac{1}{2}$. The apparent lack of coherence here has two sources. First, incoherence stemming from intrinsic randomness, which is revealed by noting that $\text{Tr}\{G_6^2\} = (\frac{5}{8}) < 1$. Second, since $G_6$ cannot be factorized into a direct product of reduced matrices, a fraction of the apparent incoherence is due to the correlation, or classical entanglement, between the DoFs.
Figure 3.3: (a)-(f) Pictorial depictions of the real part of the coherency matrices $G$ for the fields described in the text; the imaginary parts of the elements of $G$ in all these cases are zero. The labels correspond to the indices of the elements of $G$ in Equation (3.7).
3.4 Discussion

Reconstruction of the $4 \times 4$ coherency matrix $G$ therefore enables the identification of the origin of lack of coherence as determined from $D_s$ and $D_p$. The lack of spatial coherence in $G_4$, polarization coherence in $G_3$, and both spatial and polarization coherence in $G_5$, are all due to intrinsic randomness in the DoFs. On the other hand, in $G_2$ the apparent absence of polarization and spatial coherence is due to coupling between the DoFs. Relying solely on separate polarization and spatial measures leads to ambiguities, which are lifted after reconstructing $G$, since $G$ provides a complete description of the field at two points. Furthermore, other parameters (such as the recently proposed Bell’s measure [9]) may also be calculated from $G$ and used as descriptors. Moreover, reconstructing $G$ for a beam before and after transmission through a system that couples different DoFs, as in a photonic crystal or anisotropic scatterers, will help elucidate the system’s characteristics.

In conclusion, we have demonstrated that the necessary measurements for characterizing the joint polarization-spatial coherence properties of an electromagnetic field are cascades of the projective measurements needed for each DoF separately. The principle of this method may be extended to other DoFs (such as optical orbital angular momentum or other spatial mode), by using appropriate projective measurements [70, 71]. This result hinges on the fact that the vector space describing the properties of an electromagnetic field having multiple DoFs [6, 9] is the direct product of the vector spaces corresponding to each DoF. Consequently, multi-DoF classical beams of light may be placed in one-to-one correspondence with states of multipartite quantum systems, and quantum state tomographic strategies may thus be employed in the classical setting.
CHAPTER 4: OPTICAL COHERENCY MATRIX TOMOGRAPHY

4.1 Introduction

The statistical fluctuations of light in space and time may be characterized by a hierarchy of correlation functions for electromagnetic field components [2,3]. These functions, not the optical fields themselves, provide a description of light in terms of observable quantities [72]. The theory of optical coherence investigates the properties of these correlation functions pertaining to the temporal, spatial, and polarization degrees of freedom (DoFs). When these DoFs are uncoupled (or uncorrelated), simple measures of coherence for each DoF suffice, such as coherence time, coherence area, and degree of polarization [62]. However, when the DoFs are coupled, such measures lose their utility and more sophisticated approaches are required, such as the mutual coherence function [73], the beam coherence-polarization matrix [4,5,60], or the $4 \times 4$ field correlation matrix for a pair of points in an electromagnetic field [9,11,12].

While the importance of coupling between DoFs was recognized decades ago, as in Mandel’s seminal work on optical cross-spectral purity (the absence of spatial-spectral coupling) [74], recent advances have led to a host of scenarios wherein such coupling is critical. For example, vector beams correlate polarization with spatial position [75], scattering from complex photonic structures and devices may couple the relevant field DoFs [76,77], and reliance on multimode optical fibers for spatial multiplexing is reviving interest in joint polarization-spatial-mode characterization [78]. In exploring these settings, it has recently proven fruitful to adopt the Hilbert-space formulation used in quantum mechanics to the needs of classical coherence theory [9,12] – an approach that has early prescient antecedents [65,79]. In the context of coupling between multiple DoFs, such a treatment

necessitates introducing the notion of ‘classical entanglement’ [6–10, 80–82]. In quantum mechanics, states associated with bipartite systems that do not separate into products of states belonging to the Hilbert space of each particle are said to be entangled [25]. As a consequence of the mathematical similarity between the Hilbert spaces of multi-partite quantum states and multi-DoF classical optical fields, a corresponding concept of classical entanglement indicates the non-separability of the beam into uncoupled DoFs. After the initial suggestion by Spreeuw [6], a substantial body of work has accumulated in the past five years in which classical entanglement is exploited in solving long-standing problems in polarization optics [42, 43, 83], delineating the contributions of non-separability and intrinsic randomness to the coherence of an optical beam [9, 84], introducing new metrology schemes [85], and implementing classical analogs of quantum information processing protocols, such as teleportation [86, 87], and super-dense coding [88], etc.

A fundamental capability that has remained elusive in classical optics is the complete identification of the coherence function for a beam with coupled DoFs. In quantum mechanics, the task of measuring all the elements of a density matrix is known as ‘quantum state tomography’ [63, 64]. The corresponding procedure for multi-DoF beams in classical optics has been studied theoretically [12], but has not been demonstrated experimentally heretofore. Even in the simplest case of two binary DoFs [60] (e.g., polarization, a bimodal waveguide [89, 90], two coupled single-mode waveguides [91, 92], spatial-parity modes [13–15, 17, 18], etc.), the associated $4 \times 4$ coherency matrix $G$, which is a complete representation of second-order coherence [9, 12], has not been measured in its entirety to date.

### 4.2 Methodology

We present a methodical approach – optical coherency matrix tomography (OCmT) – for measuring the complex elements of $4 \times 4$ coherency matrices $G$ by appropriating the quantum-state-
tomography strategy. To demonstrate the universality of our approach, we implement it with coherent and partially coherent fields having coupled or uncoupled DoFs in three distinct settings involving pairs of points [9, 11, 12], spatial-parity modes [13–15, 17, 18], and orbital angular momentum (OAM) modes [19] – each together with polarization. We identify the minimal set of linearly independent, joint spatial-polarization projective measurements that enable a unique reconstruction of \( \mathbf{G} \). Since \( \mathbf{G} \) is a complete representation of the field, its reconstruction obviates the need to measure directly any coherence descriptors (all of which are scalar functions of the complex elements of \( \mathbf{G} \)) and, moreover, allows for unambiguous identification of classical entanglement.

The coherence of an optical beam having a single binary DoF is represented by a \( 2 \times 2 \) Hermitian coherency matrix:

\[
\mathbf{G} = \begin{pmatrix}
G_{11} & G_{12} \\
G_{21} & G_{22}
\end{pmatrix},
\]

(4.1)

where \( G_{jj'} = \langle E_j E_{j'}^* \rangle, j, j' = 1, 2 \), and \( E_j \) is the field corresponding to one level of the DoF. For example, polarization is represented by the coherency matrix [62]:

\[
\mathbf{G}_p = \begin{pmatrix}
G_{HH} & G_{HV} \\
G_{VH} & G_{VV}
\end{pmatrix} = \frac{1}{2} \sum_{l=0}^{3} S^p_l \hat{\sigma}_l = \frac{1}{2} \begin{pmatrix}
S^p_0 + S^p_1 & S^p_2 - i S^p_3 \\
S^p_2 + i S^p_3 & S^p_0 - S^p_1
\end{pmatrix},
\]

(4.2)

where \( G_{jj'} = \langle E_j(\vec{r}) E_{j'}^*(\vec{r}) \rangle \), \( E_j(\vec{r}) \) is the horizontal (H) or vertical (V) field component at a point \( \vec{r} \), \( \{S^p_l\} \) are the Stokes parameters, and \( \{\hat{\sigma}_l\} \) are the Pauli matrices:

\[
\hat{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \hat{\sigma}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \hat{\sigma}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \hat{\sigma}_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
\]

(4.3)
Figure 4.1: Measurement scheme for optical coherency matrix tomography (OCmT). (a) Four projections $I^p_l, l = 0, \ldots, 3$ to obtain the Stokes parameters $\{S_l\}$ and reconstruct $G_p$ for the polarization DoF. (b) Similarly, four projections $I^s_m, m = 0, \ldots, 3$ to obtain the Stokes parameters $\{S_m\}$ and reconstruct $G_s$ for a binary spatial DoF. (c) OCmT enables the reconstruction of $G$ for the two binary DoFs in (a) and (b) considered simultaneously via 16 joint polarization-spatial measurements, each of which consists of a cascade of two projections – one from (a) and the other from (b).
Polarization coherence is quantified by the degree of polarization:

\[ D_p = \frac{1}{S_p^0} \sqrt{(S_p^1)^2 + (S_p^2)^2 + (S_p^3)^2} \]  

(4.4)

with \( D_p = 1 \) and \( D_p = 0 \) corresponding to purely polarized and completely unpolarized light, respectively. The Stokes parameters are evaluated via four projective polarization measurements [Figure 4.1(a)]: \( I_p^1, I_p^2, \) and \( I_p^3 \) correspond to the H, diagonal (D), and right-hand-circular (R) polarization components, respectively, in addition to the total power \( I_p^0 \) [93]; in which case \( S_p^0 = 2I_p^1 - I_p^0 \). The same formalism may be applied to other binary DoFs [Figure 4.1(b)]: (i) the scalar field at two points \( \vec{r}_a \) and \( \vec{r}_b \), \( E_a \) and \( E_b \), (ii) the spatial-parity even ‘e’ and odd ‘o’ modes of a scalar field \( E_e \) and \( E_o \) [13–15, 17, 18]; or (iii) a pair of OAM modes, e.g., \( E_0 \) and \( E_1 \) corresponding to OAM \( \ell = 0 \) and 1, respectively [19, 94].

When two binary DoFs of the field are relevant, e.g., the first is polarization ‘p’ and the second is a spatial ‘s’ DoF with modes identified as ‘a’ and ‘b’ [Figure 4.1(c)] – the corresponding coherency matrix \( G \) is now \( 4 \times 4 \) [9, 12]:

\[
G = \begin{pmatrix}
G_{Ha,Ha} & G_{Ha,Hb} & G_{Ha,Va} & G_{Ha,Vb} \\
G_{Hb,Ha} & G_{Hb,Hb} & G_{Hb,Va} & G_{Hb,Vb} \\
G_{Va,Ha} & G_{Va,Hb} & G_{Va,Va} & G_{Va,Vb} \\
G_{Vb,Ha} & G_{Vb,Hb} & G_{Vb,Va} & G_{Vb,Vb}
\end{pmatrix} = \frac{1}{4} \sum_{l,m=0}^{3} S_{lm} \hat{\sigma}_l^p \otimes \hat{\sigma}_m^s, 
\]  

(4.5)

where \( G_{jk,j'k'} = \langle E_{jk} E_{j'k'}^* \rangle \), \( G_{jk,j'k'} = G_{j'k',jk}^* \), \( E_{jk} \) is a field component with \( j = H, V \) and \( k = a, b \), \{\( S_{lm} \)\} are the two-DoF Stokes parameters, and \{\( \hat{\sigma}_l^p \)\} and \{\( \hat{\sigma}_m^s \)\} are the Pauli matrices on the polarization- and spatial-DoF Hilbert subspaces, respectively [12]. In determining the coherence descriptors of each DoF independently of the other, one first traces over the other DoF to obtain a reduced coherency matrix [9]. The reduced polarization coherency matrix \( G_p \), obtained by tracing
over the *spatial* DoF in $G$, is given by:

$$G_p = \begin{pmatrix} G_{Ha,Ha} + G_{Hb,Hb} & G_{Ha,Va} + G_{Hb,Vb} \\ G_{Va,Ha} + G_{Vb,Hb} & G_{Va,Va} + G_{Vb,Vb} \end{pmatrix}, \quad (4.6)$$

while the reduced coherency matrix for the *spatial* DoF $G_s$, obtained after tracing over *polarization* in $G$ (see Ref. [12]), is given by:

$$G_s = \begin{pmatrix} G_{Ha,Ha} + G_{Va,Va} & G_{Ha,Hb} + G_{Va,Vb} \\ G_{Hb,Ha} + G_{Vb,Va} & G_{Hb,Hb} + G_{Vb,Vb} \end{pmatrix}. \quad (4.7)$$

The elements of the reduced coherency matrices are measured by a system sensitive to one DoF, but not to the other. When the two DoFs are uncoupled, then $G = G_p \otimes G_s$, otherwise the elements of $G_p$ and $G_s$ lack information about the correlations between the two DoFs that is contained in $G$. Such correlations are only measurable by a system that is sensitive to both DoFs via joint polarization-spatial measurements.

We pose the following question: what are the necessary and sufficient measurements to reconstruct an arbitrary $G$ for two binary DoFs? This question was solved by Wootters [67] in the context of reconstructing the density matrix $\hat{\rho}$ for a bipartite quantum system. He showed that the measurements carried out on each subsystem to reconstruct its reduced density matrix are sufficient to reconstruct $\hat{\rho}$ when carried out *jointly* – a methodology known as quantum state tomography [63,64]. In our context of a classical optical beam having two binary DoFs, the analogy with the quantum setting allows us to exploit the same strategy. Regardless of the specific form of $G$, the necessary measurements to carry out OCmT and reconstruct $G$ [Figure 4.1(c)] are those used to reconstruct the reduced coherency matrices [Figure 4.1(a),(b)] carried out in cascades of pairs of projections – one for polarization and the other for the spatial DoF. Each measurement yields a real number $I_{lm}$
(projection $l$ for polarization and $m$ for the spatial DoF) corresponding to the projection of a tomo-
graphic slice through $G$. The 16 combinations of polarization-spatial measurements are inverted
to obtain the two-DoF Stokes parameters, $S_{lm} = 4I_{lm} - 2I_{l0} - 2I_{0m} + I_{00}$, and hence reconstruct
$G$ (see Ref. [12] for details).

4.3 Experiments

We have performed a series of experiments implementing the OC$m$T scheme described above
using quasi-monochromatic beams having two binary DoFs: polarization and a spatial DoF. We
have measured the $4 \times 4$ coherency matrix $G$ for six different beams corresponding to distinct states
of light having the following properties:

$G_1$: the polarization and spatial DoFs are separable and both are coherent.
$G_2$: the polarization DoF is coherent while the spatial DoF lacks coherence.
$G_3$: both the polarization and spatial DoFs lack coherence.
$G_4$: the polarization and spatial DoFs are classically entangled.
$G_5$: the polarization and spatial DoFs are classically correlated.
$G_6$: this beam is a mixture of the separable-coherent beam $G_1$ and the classically entangled beam
$G_4$.

We use the sequence of polarization projections described earlier and present below the spatial
projections following the H projection (similar spatial projections are carried out following the V,
D and R polarization projections).
4.3.1 Polarization with spatial position

The first realization of the spatial DoF is the traditional two points, as in the Young’s double slit experiment. The polarization and position-coupled beam is prepared in one of six states \( \mathbf{G}_1 \) through \( \mathbf{G}_6 \); Figure 4.2(a) (see Appendix B for details). OCmT for a such a beam comprises of the polarization analysis followed by the spatial analysis; Figure 4.2(b). The spatial analysis may alternatively be carried out by extracting specific intensity points from the far-field intensity patterns for only two values of displacement \( x \) on a screen or an array of detectors; Figure 4.2(c). The four spatial projections are obtained by measuring the following: (1) the total power from both points \( I_{10} = I_H \) at \( x = 0 \); (2) the power from point ‘a’ \( I_{11} = I_{Ha} \) at \( x = 0 \); (3) the power in the far-field interference pattern \( I_{12} = I_{H(a+b)} \) at \( x = 0 \); and (4) the power at the value of \( x \) corresponding to a \( \frac{\pi}{2} \) phase shift \( I_{13} = I_{H(a+ib)} \); see Figure 4.2(d). It is important to note that the visibility of fringes is not the parameter sought here to characterize the spatial coherence at ‘a’ and ‘b’; instead the four points identified in Figure 4.2(d), together with the set of points obtained for the V, D, and R polarization projections, reveal the complete picture even when polarization and the spatial DoFs are classically entangled.
Figure 4.2: Polarization considered with spatial location in a Young’s double-slit configuration.
(a) Experimental setups for preparing polarization and position-coupled beams $G_1$ through $G_6$; D: diffuser that randomizes the beam spatially; PS: polarization scrambler that randomizes the beam polarization; HWP: half-wave plate. (b) Experimental setup delineating the stages of polarization analysis followed by spatial analysis. HWP: half-wave plate; QWP: quarter-wave plate; PBS: polarizing beam splitter; BC: 50:50 beam coupler; PD: phase delay element that introduces a phase shift $\varphi$. (c) Illustration depicting the equivalence between the phase shift $\varphi$ introduced by the phase element PD, and the lateral displacement $x$ on a screen upon which the far-field intensity pattern is projected. (d) Spatial profile measurements obtained by a CCD camera illustrating the spatial projective measurements for the H polarization projection; similarly for the V, D, and R projections.
4.3.2 Polarization with spatial parity

The second spatial-DoF realization makes use of one-dimensional even ‘e’ and odd ‘o’ spatial-parity modes with respect to \( x = 0 \). The polarization and spatial parity-coupled beam is prepared in one of six states \( G_1 \) through \( G_6 \); Figure 4.3(a) (see Appendix B for details). \( OCmT \) for a such a beam comprises of the polarization analysis followed by the spatial-parity analysis; Figure 4.3(b). The four spatial projections are obtained by measuring the power (integrated over the shaded areas in Figure 4.3(c)) in the following settings: (1) the total power \( I_{10} = I_H \) of the beam; (2) the power of the even component \( I_{11} = I_{He} \) obtained from a modified Mach-Zehnder interferometer that separates the beam into the different spatial-parity components [14]; (3) the power \( I_{12} = I_{H(e+o)} \) after blocking the half-plane \( x < 0 \), corresponding to a projection onto the \( e + o \) component; and (4) a projection onto the \( e + io \) component obtained from the power \( I_{13} = I_{H(e+io)} \) of the even component measured after first introducing a phase-step \( \frac{\pi}{2} \) between the two plane halves \( x < 0 \) and \( x \geq 0 \) implemented by a spatial light modulator (SLM); see Figure 4.3(c). This phase modulation was shown in Ref. [13–15, 17, 18] to produce a rotation on a major circle on a Poincaré sphere having the e and o modes as antipodes.
Figure 4.3: Polarization considered with spatial-parity modes. (a) Experimental setups for preparing polarization and spatial-parity-coupled beams $G_1$ through $G_6$: HWP: half-wave plate; SLM: spatial light modulator; PS: polarization scrambler that randomizes the beam polarization. (b) Experimental setup delineating the stages of polarization analysis followed by spatial-parity analysis. HWP: half-wave plate; QWP: quarter-wave plate; PBS: polarizing beam splitter; SLM: spatial light modulator; BS: beam splitter; SF: spatial flipper that flips the sign of the odd ‘o’ spatial-parity mode, leaving the even ‘e’ mode intact. (c) Corresponding spatial profile measurements obtained by a CCD camera illustrating the spatial projective measurements for the H polarization projection; similarly for the V, D, and R projections.
4.3.3 **Polarization with OAM**

The third realization exploits two low-order OAM modes $\ell = 0 (\ell_0)$ and $\ell = 1 (\ell_1)$. The polarization and OAM-coupled beam is prepared in one of six states $G_1$ through $G_6$; Figure 4.4(a) (see Appendix B for details). $OCmT$ for a such a beam comprises of the polarization analysis followed by the OAM mode analysis; Figure 4.4(b). The four spatial projections are obtained by measuring the following: (1) the total power $I_{10} = I_H$; (2) the power of the $\ell_0$ component $I_{10} = I_{H\ell_0}$ obtained using a spatial filter (a lens focusing to a single-mode fiber); (3) the power of the $\ell_0 + \ell_1$ component using a phase vortex with the dislocation displaced laterally with respect to the beam implemented using an SLM $I_{12} = I_{H(\ell_0 + \ell_1)}$ [95]; and (4) the power of the $\ell_0 + i\ell_1$ component using the same phase vortex but with the dislocation displaced vertically $I_{13} = I_{H(\ell_0 + i\ell_1)}$; see Figure 4.4(c).
Figure 4.4: Polarization considered with OAM modes. (a) Experimental setups for preparing polarization and OAM-coupled beams $G_1$ through $G_6$; HWP: half-wave plate; SLM: spatial light modulator; PS: polarization scrambler that randomizes the beam polarization. (b) Experimental setup delineating the stages of polarization analysis followed by OAM-mode analysis. HWP: half-wave plate; QWP: quarter-wave plate; PBS: polarizing beam splitter; SLM: spatial light modulator; FC: fiber coupler; SMF: single-mode fiber. (c) Corresponding spatial profile measurements obtained by a power-meter illustrating the spatial projective measurements for the H polarization projection; similarly for the V, D, and R projections. Spatial measurements are obtained by dislocating the phase singularity ($e^{i\phi}$, $0 \leq \phi < 2\pi$) relative to the beam $\Delta x$ along $x$ and $\Delta y$ along $y$. $I_H$ is obtained by adding the intensities $I_{\ell_0}$ at $\Delta x_{\text{max}} \gg \sigma$ ($\sigma$ is the beam width) and $I_{\ell_1}$ at $\Delta x = 0$, $I_{\ell_0+\ell_1}$ is obtained at $\Delta x_{\text{mid}}$, and $I_{\ell_0+2\ell_1}$ is obtained at $\Delta y_{\text{mid}}$ ($\Delta x_{\text{mid}}$ and $\Delta y_{\text{mid}}$ are calibrated using a Gaussian beam).
4.4 Results

We have measured the complex elements of $G$ for six different classes of beams comprising those with separable DoFs (both coherent, both incoherent, or in a hybrid coherent/incoherent configuration), non-separable DoFs (classically entangled or classically correlated), and mixtures of beams from the two separable and non-separable classes (see Appendix B for the complete results). In each experiment, the prepared beam passes first through polarization then spatial-DoF analysis stages (the order may be reversed without changing the outcome). In each of these realizations, permutations of the four polarization projection settings combined with the four spatial projection settings yield 16 measurements for $\text{OCmT}$, which are used to reconstruct $G$. We make use of a maximal-likelihood algorithm that exploits the constraints set by the trace, hermiticity, and semi-positive-definiteness of $G$ [96]. We portray the real and imaginary components of $G$ using the standard visualization from quantum state tomography. In each plot we provide the coherence descriptor for the polarization $D_p$ and spatial DoF $D_s$ obtained from their reduced coherency matrices, in addition to the linear entropy:

$$D_G = \frac{4}{3} \left\{ \text{Tr}(G^2) - \frac{1}{4} \right\},$$  

(4.8)

which serves as a measure of the overall beam coherence, where $D_G$ ranges from 0 (complete incoherence in all DoFs) to 1 (a coherent beam with no statistical fluctuations) [97]. Finally, we provide the fidelity:

$$F = \left[ \text{Tr}\left\{ \sqrt{\Gamma G \sqrt{\Gamma}} \right\} \right]^2$$

(4.9)

as a measure of the robustness of the reconstruction process via $\text{OCmT}$ [98], where $\Gamma$ is the theoretical matrix and $G$ is the measured matrix. A deviation between $\Gamma$ and $G$ yields a fidelity less than unity. The experimental sources of error that affect the fidelity include the introduction of aberra-
tions in the beam by various optical components, imperfections in the SLMs which cause a less than optimal coupling between the DoFs, imperfections in alignment which produce a less than optimal visibility in cases where interference is measured, and fluctuations in laser beam intensity, etc.

4.4.1 Beams with separable DoFs

We present in Figure 4.5 three examples of beams having separable DoFs, \( G = G_p \otimes G_s \). First, both polarization and spatial DoFs are coherent,

\[
G_1 = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) ^p \otimes \frac{1}{2} \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) ^s
\]  
(4.10)

[Figure 4.5(a)]. Second, a hybrid beam in which polarization is pure (along D) but the beam is spatially incoherent,

\[
G_2 = \frac{1}{2} \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) ^p \otimes \frac{1}{2} \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) ^s
\]  
(4.11)

[Figure 4.5(b)]. Third, a completely incoherent beam

\[
G_3 = \frac{1}{2} \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) ^p \otimes \frac{1}{2} \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) ^s
\]  
(4.12)

[Figure 4.5(c)]. In all three cases, the separability of the two DoFs is readily detected by visual inspection of \( G \) and confirmed by taking the direct product of the reduced coherency matrices.
Figure 4.5: Measurements of $\mathbf{G}$ for beams having separable DoFs. (a) $\mathbf{G}_1$: both DoFs are coherent; (b) $\mathbf{G}_2$: polarization is coherent but the spatial DoF is incoherent; and (c) $\mathbf{G}_3$: both DoFs are incoherent. In each case, the reduced coherency matrices $\mathbf{G}_p$ and $\mathbf{G}_s$ are also depicted. The imaginary components are negligible, and are not shown.
4.4.2 Beams with non-separable DoFs

We next present two fundamentally distinct classes of beams with non-separable $G$ in Figure 4.6(a),(b). First, $O_CmT$ of a classically entangled beam $G_4$ is shown in Figure 4.6(a), wherein the beam is fully coherent, and yet the measures extracted from reduced coherency matrices indicate complete incoherence. In such a beam, the polarization and spatial modes occur in pairs – e.g., $H$ with ‘a’ and $V$ with ‘b’ (but never $H$ with ‘b’ or $V$ with ‘a’). In the traditional view of the double slit experiment, such coupling will produce no interference fringes, and the lack of visibility may be interpreted as the absence of spatial coherence, despite the beam being perfectly spatially coherent. This coupling between the DoFs is in fact encoded in the non-zero off-diagonal elements of $G$ revealed once it is reconstructed through $O_CmT$, but cannot be obtained from $G_p$ or $G_s$.

Second, a classically correlated beam $G_5$ is shown in Figure 4.6(b) in which the same coupling between polarization and spatial modes occurs as in the previous example, except the different combinations are incoherently mixed and not linearly superposed. The partial global coherence – despite the complete lack of coherence for each DoF – is clear from the fact that not all the diagonal elements of $G_5$ are equal as is the case in $G_3$.

4.4.3 Mixture of beams with separable and non-separable DoFs

Finally, in Figure 4.6(c) we depict $G_6$ corresponding to a beam formed by statistically mixing the separable-coherent beam $G_1$ and the classically entangled beam $G_4$. The measurement of $G_6$ indicates that part of the apparent incoherence in this beam stems from the intrinsic randomness in the individual DoFs, and part of it from the correlation, or classical entanglement, between the two DoFs.
Figure 4.6: Measurements of $\mathbf{G}$ for beams having non-separable DoFs. (a) $\mathbf{G}_4$: classically entangled beam; and (b) $\mathbf{G}_5$: classically correlated beam. The imaginary components in both cases are negligible, and are not shown. (c) Measurement of the real and imaginary parts of $\mathbf{G}_6$: mixture of beams $\mathbf{G}_1$ and $\mathbf{G}_4$. In each case, the reduced coherency matrices $\mathbf{G}_p$ and $\mathbf{G}_s$ are also depicted.
4.5 Discussion

The reconstruction of $G$ allows for the unambiguous and complete mathematical expression of fields that are coherent, partially coherent, or incoherent, in either, or both, DoFs of an optical beam with two binary DoFs. The usefulness of this technique becomes specially apparent in cases where the DoFs are coupled or non-separable, and the traditional scalar measures of coherence provide a conflicting and fallacious account of beam coherence. The apparent absence of coherence in any DoF may be the result of intrinsic randomness due to statistical fluctuations, or due to the coupling or non-separability with another DoF. In the latter case, the measurement of $G$ also provides the way for implementing unitary transformations required to undo such coupling, and restore coherence in the DoFs. The application of our work can be easily seen in the myriad applications of coherence under conditions of coupled DoFs, particularly those involving localized vector beams, sub-diffraction imaging, nanophotonics, and propagation through disordered media. Measurement of $G$, before and after transmission though a system that couples various DoFs, will help determine the characteristics of the system. This technique may hence find important applications in crystallography, atmospheric optics, and systems involving photonic crystals or anisotropic scatters, etc.

In summary, we have experimentally demonstrated for the first time a methodical, yet versatile, approach to reconstructing the $4 \times 4$ coherency matrix $G$ of an optical beam having two binary DoFs, which we call optical coherency matrix tomography. We have explored three different physical realizations in which we combine polarization with spatial position, spatial parity, or orbital angular momentum modes. By exploiting the mathematical similarity with quantum state tomography of two photon states, we determine the minimal set of measurements required to reconstruct $G$. Although we have conducted the experiments for a beam with two binary DoFs, this methodology is equally applicable for a higher number of DoFs with $m$-ary levels each.
CHAPTER 5: SINGLE-PHOTON QUANTUM LOGIC WITH
POLARIZATION AND SPATIAL PARITY QUBITS

In England the famous scientist Professor Foulbody invented a machine which would
tell you at once, without opening the wrapper of a candy bar, whether or not there was
a golden ticket hidden underneath. The machine had a mechanical arm that shot out
with tremendous force and grabbed hold of anything that had the slightest bit of gold
inside it and, for a moment, it looked like the answer to everything.

– Roald Dahl, Charlie and the Chocolate Factory

5.1 Introduction

In a span of three decades, quantum information science has become a rapidly developing inter-
disciplinary area encompassing quantum mechanics, computer science, information theory and
cryptology [20]. It is well known that a quantum computer with even modest resources would be
profoundly more powerful, with regard to certain tasks, than any classical computer running the
best known algorithms. Even though the theoretical branch of the field has prospered, any physical
implementation of a quantum device is fundamentally governed by the laws of physics. Hence for
quantum computation to be of any meaningful utility would involve improvements at the device
level. The community that is working over the challenges related to quantum computing hardware
have each devised a way they think is the best: laser-cooled ion traps [99, 100], cavity quantum-
electrodynamics [101], nuclear magnetic resonance [102], semiconductor devices [103, 104], su-
perconducting devices [105, 106], and quantum computing with photons [27, 107], etc.

Photonic implementations of quantum gates [108–111] have attracted considerable interest for three main reasons [27, 112, 113]. First, an optical scheme integrates naturally with existing technologies for secure quantum communication and fast transfer of quantum information [114]. Second, photonic gates offer fast switching as compared to any other physical system. Third and most importantly, optical implementations are largely unaffected by the most deleterious effect that plagues other schemes, namely decoherence. Single photons are potentially free of decoherence, and as a consequence, entanglement between photons is also much more robust. This primary advantage, however, is related to its largest disadvantage – photons do not interact with each other, meaning it is impossible to directly entangle them in vacuum. Moreover, it was assumed for some time that two-photon gates require photon interaction through nonlinear means [115, 116]. However, no known material, structure [117], or process [118] is characterized by a nonlinearity strong enough to lead to two-photon interaction with a success rate high enough to allow scalable quantum computing [119]. In 2001, Knill, Laflamme and Milburn (KLM) challenged this assertion and developed a protocol that allows for efficient, scalable quantum computing using only single photon sources, linear optical elements such as mirrors, phase shifters and beam splitters, projective measurements and single photon detectors [26]. However, despite their simplicity, such implementations are probabilistic and rely on the postselection of measurements. The computation only becomes deterministic asymptotically in the limit of infinite resources in terms of ancilla photons and optical components, which renders such schemes impractical for large scale experimental implementation.

Photons have different degrees of freedom (DoFs): polarization, spatial path, transverse spatial mode, time of arrival, spectral frequency and orbital angular momentum. The information-carrying capacity of a single photon may be garnered by encoding information in all modes of its spatial, temporal, and polarization degrees DoFs. This is of course hampered by the difficulty of generating arbitrary entangled states in a Hilbert space of high dimension, and the challenge of implement-
ing transformations spanning the modes of different DoFs in order to implement various logic operations [120, 121]. In contrast to the KLM proposal, which is scalable but indeterministic, single-photon quantum logic (SPQL) enables deterministic computation at the expense of requiring exponential resources as the qubits are scaled up. Nevertheless, few-qubit SPQL continues to evoke a strong interest, and research on this front has seen steady progress in both the theoretical and experimental domains.

Any discrete finite-dimensional unitary operator can be implemented in the laboratory as a multiport system comprising of optical devices such as beam splitters and phase shifters [122]. One of the earliest proposals propounding SPQL for the simulation of small-scale quantum circuits relies on the representation of \( n \) qubits by a single photon in a network of interferometers with \( 2^n \) paths [123]. To curtail the exponential rise in complexity and apparatus size with an increase in \( n \), a hybrid approach to quantum computing was proposed, in which \( L \) DoFs of single photons are entangled via quantum non-demolition measurements (QND), and the computation is carried out in smaller spatially separated sub-systems [124]. This technique is hampered by the inability to achieve a cross-phase-modulation of \( \pi \) at the single-photon level during the QND measurement. Other theoretical proposals have been reported for implementing two-qubit SPQL using polarization and orbital angular momentum (OAM) [125], and polarization and spatial modes [126]. Experimental demonstrations of single-photon quantum logic have been limited to two qubits, using polarization and multi-path [127–129], and polarization and spatial modes [130]. It includes the experimental demonstration of two-qubit controlled-NOT (CNOT) and SWAP gates using polarization and momentum qubits [127, 128], implementation of the Deutsch algorithm with polarization and spatial transverse modes [131], quantum key distribution (QKD) without the need of a shared reference frame [132], mounting an intense attack on Bennett-Brassard 1984 (BB84) QKD [133], and hyperentanglement-assisted Bell state analysis using polarization and momentum qubits [134], etc. A theoretical proposal for two- and three-qubit SPQL based on photon
polarization in conjunction with spatial-parity symmetry of the photon field transverse spatial distribution [135] in two orthogonal directions has recently been reported [18]. It was argued that a simple polarization-sensitive spatial light modulator (PS-SLM) may be readily used to couple polarization to spatial parity, and thereby implement two- and three-qubit controlled unitary gates.

In this chapter, we report experimental demonstration of linear and deterministic, two- and three-qubit SPQL with polarization and spatial parity qubits of single photons. The centerpiece of our experiment is a PS-SLM, which is used to implement various deterministic quantum unitary operations in the joint Hilbert space of polarization and spatial parity qubits. The gates utilize the photon polarization as the control qubit, and the photon spatial-parity symmetry of the transverse field along the x and y directions as the target qubits, and their roles may be reversed with a minor change to the optical setup. Furthermore, we demonstrate its versatility by using it in the following different configurations: CNOT action on the x parity qubit alone with the y parity qubit left intact (CNOT\textsubscript{x}), CNOT action on the y parity qubit alone with the x parity qubit left intact (CNOT\textsubscript{y}), CNOT action on both the x and y parity qubits simultaneously (CNOT\textsubscript{xy}), separable controlled rotation $\hat{R}_x(\pi)$ on the x parity qubit and $\hat{R}_y(\pi/2)$ on the y parity qubit, and joint controlled rotation $\hat{R}_{xy}(\pi/2)$ on both the x and y parity qubits. The phase distribution imparted to the SLM selects the desired configuration. We also use the SLM to generate and tomographically measure maximally entangled states, namely, a two-qubit Bell state, and three-qubit Greenberger-Horne-Zeilinger (GHZ) and W states [136], in polarization and spatial-parity symmetry DoFs of a single photon.

Our technique is shown to be a robust and viable means for photonic implementation of few-qubit quantum information processing (QIP) applications. Multiple SLMs may be cascaded together for realizing few-qubit QIP algorithms, and for implementing error correction codes [137]. The wide array of three-qubit states accessible via this technique may be used to improve the violations in experimental tests of local realistic theories [138], and for enhancing the sensitivity in various
methods for quantum metrology [139]. The second photon gives access to three extra qubits, thereby allowing for the creation of six-qubit cluster states [140], and tests of computation in that paradigm [141–143].

5.2 Polarization and spatial parity

The polarization of a single photon, as the traditional embodiment of an optical qubit, has found widespread applications, from fundamental tests of quantum mechanics, as in the first violation of Bell’s inequality [51, 144], to quantum information science, as in the first experimental demonstration of quantum teleportation [145], etc. This popularity stems from the ease of generating, manipulating and detecting states of polarization by means of simple linear optical elements such as retarders and polarizing beamsplitters. This qubit may be expressed in various bases, such as $\{ |H\rangle, |V\rangle \}$, $\{ |D^+\rangle = \frac{1}{\sqrt{2}} (|H\rangle + |V\rangle), |D^-\rangle = \frac{1}{\sqrt{2}} (|H\rangle - |V\rangle) \}$, and $\{ |R^+\rangle = \frac{1}{\sqrt{2}} (|H\rangle + i|V\rangle), |R^-\rangle = \frac{1}{\sqrt{2}} (|H\rangle - i|V\rangle) \}$, where $|H\rangle$ and $|V\rangle$ correspond to the horizontal and vertical linear polarization components, respectively. The Pauli operator $\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for this qubit, which flips the polarization from $|H\rangle$ to $|V\rangle$, and vice versa, is implemented by a half-wave plate (HWP) with the fast axis oriented at 45° with respect to the horizontal. The Pauli operator $\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which introduces a sign flip in $|V\rangle$, but not in $|H\rangle$, is realized by using a HWP with the fast axis along the horizontal. A rotation is achieved by a stack of quarter-wave plate (QWP), HWP and QWP at angles 0°, $\theta$ and 90° respectively. And finally a polarizing beam splitter (PBS) projects this qubit onto the $\{ |H\rangle, |V\rangle \}$ basis.

A corresponding Hilbert space is constructed for spatial parity modes along one transverse coordinate [9,13–15], say $x$. This qubit may be expressed in various bases, such as $\{ |e\rangle, |o\rangle \}$, where $|e\rangle$ and $|o\rangle$ correspond to even and odd functions of the photon field distribution along $x$, respectively,
and are depicted as antipodal points on the parity Poincare sphere. It may also be expressed in other bases such as $\{ |d^+\rangle = \frac{1}{\sqrt{2}} (|e\rangle + |o\rangle), |d^-\rangle = \frac{1}{\sqrt{2}} (|e\rangle - |o\rangle) \}$, and $\{ |r^+\rangle = \frac{1}{\sqrt{2}} (|e\rangle + i|o\rangle), |r^-\rangle = \frac{1}{\sqrt{2}} (|e\rangle - i|o\rangle) \}$. The utility of spatial parity lies in the ease of its manipulation using simple linear optical components. For example, a parity rotator, a device that implements a rotation operator:

$$\hat{R}_x(\theta) = \begin{pmatrix} \cos \frac{\theta}{2} & i \sin \frac{\theta}{2} \\ i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix},$$

for the parity qubit, is realized by introducing a phase difference $\theta$ between the two halves of the transverse plane. For $\theta = \pi$, the Pauli operator $\hat{\sigma}_x$ is implemented, which flips the parity from $|e\rangle$ to $|o\rangle$, and vice versa; an action that is isomorphic to that introduced by a HWP at 45° with respect to $|H\rangle$ for the polarization qubit. The Pauli operator $\hat{\sigma}_z$ is realized by a spatial flipper that implements the transformation $|e\rangle \rightarrow |e\rangle$ and $|o\rangle \rightarrow -|o\rangle$ with the help of a mirror or parity prism; an action analogous to that introduced by a HWP with fast axis aligned to the horizontal for the polarization qubit. A parity analyzer, comprising of a Mach-Zehnder interferometer (MZI) with a spatial flipper in one arm, projects onto the parity basis $\{ |e\rangle, |o\rangle \}$, in much the same way a PBS projects onto the polarization basis $\{ |H\rangle, |V\rangle \}$. Note that all of these operations may be also used for spatial parity along the $y$ transverse coordinate by appropriate rotations in physical space. Furthermore, as we show below, $x$ and $y$ spatial parity operators may be combined together [17] – along with polarization [18]. The polarization, x-parity, and y-parity qubits, together with their operators are depicted in Figure 5.1.

Polarization and spatial parity are both physical degrees of freedom of the single photon with each representing a logical qubit. Together, they form a joint 4-dimensional (two-qubit) Hilbert space spanned by the hybrid polarization-parity basis $\{ |He\rangle, |Ho\rangle, |Ve\rangle, |Vo\rangle \}$. A unique advantage of this physical realization is that two-qubit operations spanning the joint Hilbert space are readily performed with available optical technology.
Figure 5.1: Toolbox for three-qubit quantum logic with polarization and spatial parity qubits. The polarization qubit ($|\text{H}\rangle$: horizontal and $|\text{V}\rangle$: vertical, and their superpositions) is depicted on the surface of a Poincaré sphere (shaded in yellow). The Pauli $X$ operator ($\hat{\sigma}_x$) is implemented with the help of a half-wave plate (HWP) with the fast axis oriented at 45°. The Pauli $Z$ operator ($\hat{\sigma}_z$) is implemented with the help of a HWP with the fast axis oriented at 0°. A rotation operator $\hat{R}(\theta)$ is implemented with a stack of quarter-wave plate (QWP), a HWP, and a second QWP at angles 0°, $\theta$, and 90°, respectively. A polarizing beamsplitter (PBS) projects on $|\text{H}\rangle$ and $|\text{V}\rangle$ basis. The parity qubit is represented on the surface of the Poincaré sphere (shaded in pink). A parity flipper (PF) is implemented with a phase step of $\pi$ in the transverse $x$ plane. A spatial flipper (SF) is implemented with a Dove prism on its side. A parity rotator (PR) is a device that rotates the parity by an angle $\theta$. A parity analyzer (PA) projects on the basis $|\text{e}\rangle$ and $|\text{o}\rangle$. Similar operators are implemented for the $y$-parity qubit.
5.3 SLM as a controlled-unitary gate

The central example that we exploit here is a polarization-sensitive SLM [146] (or simply SLM hereafter). Since available SLMs are based on liquid-crystal technology, they spatially modulate the transverse phase for one polarization component (considered $|H\rangle$ throughout), while leaving the orthogonal polarization component ($|V\rangle$) unscathed. In this manner, a coupling between the polarization and spatial degrees of freedoms is introduced, a feature of the SLM that has so far received little attention. This feature may be exploited to couple polarization and spatial parity, thereby entangling the corresponding logical qubits. A salient advantage of the SLM is that it does not require interferometric stability for operation, and is hence less prone to decoherence and noise effects.

If we consider x-parity qubit along with polarization, then implementing a phase-step $\theta$ rotates the parity associated with the $|H\rangle$ polarization, while leaving the parity of the $|V\rangle$ polarization unchanged. If we put the basis $\{|Ve\rangle, |Vo\rangle, |He\rangle, |Ho\rangle\}$ in correspondence with the logical basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, then this unitary operation is described by the matrix:

$$
\hat{U}_x(\theta) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \cos \frac{\theta}{2} & i \sin \frac{\theta}{2} \\
0 & 0 & i \sin \frac{\theta}{2} & \cos \frac{\theta}{2}
\end{pmatrix}.
$$

Note that polarization and parity here play the roles of the control and target qubits, respectively. This is a particularly attractive feature since the phase $\theta$ may be dynamically varied in real time electronically, thus enabling access to a whole family of two-qubit gates in a single device. In the special case of $\theta = \pi$, for example, we obtain a controlled-NOT (CNOT) gate. The correspondence between the ideal truth table of a logical CNOT gate and the measured truth table of a polarization-parity-CNOT gate implemented by an SLM is shown in Figure 5.2(b). The impact of the phase step
$\theta = \pi$ on the four basis states is illustrated in Figure 2b: when the polarization is $|H\rangle$, the parity state is flipped; when the polarization is $|V\rangle$, the parity state remains invariant. Another crucial feature of the SLM is that it implements the phase modulation along both transverse coordinates $x$ and $y$. Therefore, controlled unitary gates (such as the CNOT) using the $y$-parity as a target qubit are realized by simply rotating the orientation of the phase step, as shown in Figure 5.2(c). Moreover, since operations on $x$- and $y$-parity commute, they may be implemented simultaneously on the same SLM, thus paving the way for constructing three-qubit gates, as we demonstrate below.

With recourse to three qubits, a larger number of operations can be performed. As before, when the control qubit is $|V\rangle(|0\rangle)$, the parity qubits are unchanged. When the control qubit is $|H\rangle(|1\rangle)$, then the parity qubit(s) undergo a rotation dependent on the phase distribution imparted on the SLM. For example, a phase step of $\pi$ along the $x$ direction implements a CNOT gate for the $x$-parity qubit, leaving the $y$-parity qubit unchanged (CNOT$_X$). Similarly, a phase step of $\pi$ in the $y$ direction on the SLM implements a CNOT gate for the $y$-parity qubit, with the $x$-parity qubit unmodified (CNOT$_Y$). By displaying a phase distribution in which there is a phase step of $\pi$ between adjacent quadrants, a CNOT gate for both $x$-parity and $y$-parity qubits (CNOT$_{XY}$) is implemented. By using a phase step other than $\pi$, a general controlled rotation is implemented, which may be separable $\hat{R}_x(\phi_x)\hat{R}_y(\phi_y)$, or joint $\hat{R}_{xy}(\phi_{xy})$. Implementations of these generalized rotations for three-qubits with as SLM is shown in Figure 5.3.
Figure 5.2: Implementation of a two-qubit CNOT gate with a spatial light modulator (SLM) with polarization as the control qubit and spatial parity as the target qubit. (a) Ideal operator for a CNOT gate, circuit representation, implementation with SLM, and measurement of operator for two-qubit CNOT gate. The measurement basis $\{|Ve\rangle, |Vo\rangle, |He\rangle, |Ho\rangle\}$ corresponds to the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, respectively. (b) Operation of an SLM as a two-qubit CNOT gate with polarization as the control qubit and spatial-parity $x$ as the target qubit. (c) Operation of an SLM as a two-qubit CNOT gate with polarization as the control qubit and spatial-parity $y$ as the target qubit. The parity qubit in (b) and (c) only flips when the polarization qubit is $|H\rangle$ i.e., the control qubit is $|1\rangle$. 
Figure 5.3: Operation of an SLM as a three-qubit quantum gate and corresponding circuit schematics. The rotations implemented are: (a) $R_X(\theta)$, (b) $R_Y(\theta)$, (c) $R_X(\theta)R_Y(\theta)$, (d) $R_X(\theta_1)R_Y(\theta_2)$, and (e) $R_{XY}(\theta)$. 

- Figure a: $\theta/2 \quad \theta/2$
- Figure b: $\theta/2 \quad \theta/2$
- Figure c: $0 \quad -\theta$
- Figure d: $\theta/2 \quad \theta/2 \quad \theta/2 \quad \theta/2$
- Figure e: $-\theta/2 \quad \theta/2$
5.4 Experimental setup

The arrangement used in characterizing the two- and three-qubit polarization-parity gates realized using an SLM is depicted schematically in Figure 5.4. The single-photon-source is a heralded photon from a photon pair produced by type-I collinear spontaneous parametric down-conversion (SPDC). A vertically polarized monochromatic pump laser with an even spatial profile from a semiconductor diode laser (Coherent cube 405-50, 405 nm, 50 mW) impinges on a 1.5-mm-thick $\beta$-barium-borate (BBO) crystal. The pump is subsequently removed using a Glan-Thompson polarizer, and an interference filter centered at 810 nm with 10-nm bandwidth removes residual pump photons and sets the bandwidth of the photon pairs. One of the two horizontally polarized photons heralds the arrival of the other by coupling through a single-mode fiber (SMF) to a single-photon-sensitive avalanche photodiode, APD (SPCM-AQR). Detection of this photon heralds the arrival of its twin at the experimental setup. The heralded photons are collected after the setup through a multimode fiber to another APD.

The setup is divided into three stages for state preparation, control, and analysis of the heralded photon. Coupling the trigger photon into a SMF projects the heralded photon onto a single, even-parity spatial mode, such that its state may be written as $|\psi\rangle = |\text{He}\rangle$. This state may be further modified using a sequence of a SLM (SLM$_1$ to prepare the parity state) and a HWP (to rotate the polarization). The state control is implemented using the controlled-unitary quantum gate realized with a polarization sensitive SLM (SLM$_2$). In the case of x-parity, we use a step phase pattern ($\theta$) on SLM$_2$ along x; similarly for y-parity. State analysis cascades a polarization projection (a HWP and a PBS) followed by a parity projection (SLM$_3$ and a MZI).
Figure 5.4: Experimental setup showing state preparation, state control and state measurement stages for implementing quantum logic with polarization and spatial-parity qubits. SF: Spatial filter; NLC: Nonlinear crystal; GT: Glan-Thomson polarizer; BS: Beamsplitter; SLM: Spatial light modulator; HWP: Half-wave plate; WP: wave plate (half- or quarter- depending upon the measurement); PBS: Polarizing beamsplitter; SF_x: Spatial flipper in x; MZI: Mach-Zehnder interferometer; IF: Interference filter; FC: Fiber coupler; MMF: Multi-mode fiber; SMF: Single-mode fiber; D: Detector. All measurements are made in coincidence between D_1 and D_2.
5.5 Results

5.5.1 Two-qubit SPQL

By varying the phase step $\theta$, we have implemented and characterized two-qubit three single-photon quantum logic (SPQL) gates in which polarization is the control qubit and x-parity is the target qubit. The three settings are:

1. $\theta = 0$ corresponding to the identity gate,
2. $\theta = \frac{\pi}{2}$ corresponding to a $\sqrt{\text{CNOT}}$ gate, and
3. $\theta = \pi$ corresponding to a CNOT gate.

In each configuration, we start with $|V\rangle$ polarization – logical qubit $|0\rangle$ – and subject it either to a Pauli Z operator (which leaves $|V\rangle$ invariant), a Hadamard gate (transforming $|V\rangle$ to $|D^+\rangle$), or a Pauli X operator (transforming $|V\rangle$ to $|H\rangle$). We thus probe two aspects of the SPQL gate: the strength of the coupling between the polarization and parity qubits and also the kind of rotation in parity space affected by polarization. The correspondence between the quantum logic circuits and the physical realizations highlight the simplicity of varying the implemented gate.

The measurement results are presented in Figure 5-4 for the nine different combinations of coupling strength and parity rotation. In each setting, we carry out two-qubit quantum state tomography (in polarization and parity) to reconstruct the density operator $\hat{\rho}$, and plot the real and imaginary parts. When $\theta = 0$, the gate does not change the input state. The measured density matrices therefore correspond to $|Ve\rangle\langle Ve|$, $|D^+e\rangle\langle D^+e|$, and $|He\rangle\langle He|$, respectively. When $\theta = \pi$, for $|H\rangle$, the gate corresponds to a Pauli X operator on spatial parity, while for $|V\rangle$, it corresponds to the identity. We note that for $|D\rangle$ polarization, the factorized two-qubit state becomes maximally entangled.
Figure 5.5: Density matrices for two-qubit states measured via quantum state tomography for three different rotation angles: $\theta = 0$, $\frac{\pi}{2}$, and $\pi$. (a) For $|\psi_i\rangle = |V\rangle$, for all three settings $\theta = 0$, $\frac{\pi}{2}$, and $\pi$, $\hat{\rho} = |V\rangle \langle V|$, as the SLM remains impervious to the vertical polarization qubit. (b) For $|\psi_i\rangle = |D\rangle$, if $\theta = 0$, $\hat{\rho} = \frac{1}{2} (|H\rangle \langle H| + |V\rangle \langle V| + |H\rangle \langle V| + |V\rangle \langle H|)$, which indicates that the state remains separable; if $\theta = \frac{\pi}{2}$, indicates partial entanglement; and if $\theta = \pi$, $\hat{\rho} = \frac{1}{2} (|H\rangle \langle H| + |V\rangle \langle V| + i |H\rangle \langle V| - |V\rangle \langle H|)$, which shows maximum entanglement. (c) For $|\psi_i\rangle = |H\rangle$, if $\theta = 0$, $\hat{\rho} = |H\rangle \langle H|$, and as expected, no rotation occurs in the parity qubit; if $\theta = \frac{\pi}{2}$, the parity qubit undergoes partial rotation with $\hat{\rho} = \frac{1}{2} (|H\rangle \langle H| + |H\rangle \langle H| + i |H\rangle \langle H| - |H\rangle \langle H|)$; and if $\theta = \pi$, the parity qubit undergoes a complete flip, and $\hat{\rho}$ for the resulting state is $|H\rangle \langle H|$. 
5.5.2 Three-qubit SPQL

In this section, results for three-qubit SPQL are presented, where polarization is the control qubit and x-parity and y-parity are the target qubits. The polarization and spatial parity of the heralded photon are prepared in any of the eight basis states \( \{ |H\rangle, |V\rangle \} \otimes \{ |e_x\rangle, |o_x\rangle \} \otimes \{ |e_y\rangle, |o_y\rangle \} \) by using the appropriate phase distribution on SLM\(_1\) and angle of rotation of HWP\(_1\). SLM\(_2\) implements the controlled gate operation. When the control qubit is \( |V\rangle(0)\), the parity qubits are unchanged. When the control qubit is \( |H\rangle(1)\), then the parity qubit(s) undergo a rotation dependent on the phase distribution imparted on the SLM. For example, a phase step of \( \pi \) along the \( x \) direction implements a CNOT gate for the x-parity qubit, leaving the y-parity qubit unchanged (CNOT\(_X\)). Similarly, a phase step of \( \pi \) in the \( y \) direction on the SLM implements a CNOT gate for the y-parity qubit, with the x-parity qubit unmodified (CNOT\(_Y\)). By displaying a phase distribution in which there is a phase step of \( \pi \) between adjacent quadrants, a CNOT gate for both x-parity and y-parity qubits is implemented (CNOT\(_{XY}\)). By using a phase step other than \( \pi \), a general controlled rotation is implemented, which may be separable \( \hat{R}_x(\phi_x)\hat{R}_y(\phi_y) \), or joint \( \hat{R}_{xy}(\phi_{xy}) \). In Figure 5.6, we show the measurements of operators, or truth tables, for five different operations: CNOT\(_X\), CNOT\(_Y\), CNOT\(_{XY}\), \( \hat{R}_{xy}(\phi_{xy}) \) and \( \hat{R}_x(\phi_x)\hat{R}_y(\phi_y) \). This is achieved by generating inputs in all eight basis states and taking projections of the output states along all eight basis states.

Next, we show measurements for single-photon three-qubit maximally entangled states: the GHZ and W states. Beginning with the state \( |H_e_x e_y\rangle \), the GHZ state \( \frac{1}{\sqrt{2}}\{-|H_o_x o_y\rangle + |V e_x e_y\rangle\} \) is prepared by rotating the polarization by \( \frac{\pi}{4} \) and use of a PS-SLM with phases of \( \frac{\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, -\frac{\pi}{2} \) in the four quadrants. The W state \( \frac{1}{\sqrt{3}}\{i|H_e_x o_y\rangle - |H_o_x e_y\rangle + |V e_x e_y\rangle\} \) is generated by rotating the polarization by an angle \( \theta_W \) and using a PS-SLM with phases of \( \pi, -\frac{\pi}{2}, 0, \frac{\pi}{2} \). In each case, the outcome density matrix of the generated state was measured via quantum state tomography (Figure 5.7). The fidelities were found to be 0.821 for the GHZ state and 0.828 for the W state.
Figure 5.6: Measurement of operator and circuit representation for (a) unity gate, (b) CNOT gate with x-parity qubit as target, (c) CNOT gate with y-parity qubit as target, (d) CNOT gate with both x-parity and y-parity qubits as targets, (e) a joint rotation $\hat{R}_{xy}(\frac{\pi}{2})$ on both x-parity and y-parity qubits, and (f) a rotation $\hat{R}_x(\pi)$ on x-parity qubit and a rotation $\hat{R}_y(\frac{\pi}{2})$ on y-parity qubit.
Figure 5.7: Quantum circuit representation, state preparation, and measurements for real and imaginary parts of density matrices for (a) GHZ state, and (b) W state, obtained via quantum state tomography.
5.6 Discussion

In conclusion, we have experimentally demonstrated linear, deterministic, single-photon, two- and three-qubit, quantum logic gates using polarization and spatial parity qubits and implemented by a simple optical device, the SLM. The average fidelity for two-qubit SPQL is found to be 93%, whereas the average fidelity for three-qubit SPQL is noted to be 83%. The performance of these gates is limited essentially due to two reasons. The first is due to the precision of phase selection on the SLM, and the second is due to edge effects between constant phase domains imparted on the SLM. Both these drawbacks can be reduced with advancements in SLM technology. Another factor that leads to the reduction in the fidelities of the measured states is the less than perfect alignment of the interferometers, with a visibility of 94%, used in the analysis of spatial parity. Hence, this error cannot be assigned to the state itself, but to the imperfect projection operators in the measurement process. It can be removed by conducting the experiment in a temperature controlled environment and by implementing active control in the interferometer.

We have shown how three qubits can be encoded in the polarization and spatial parity DoFs of a single photon. Likewise, we can use the second photon, which is used for heralding in this experiment, to encode three extra qubits. The use of both photons opens up a host of interesting possibilities, such as the creation of six-qubit cluster states, production of exotic hyper-entangled states, and tests of quantum non-locality, etc.

Finally, this approach may also be applied to other DoFs, such as OAM, to realize quantum gates in which the polarization qubit acts as the control and the OAM qubit acts as the target. In contrast to OAM states, the appealing features of spatial parity include the non-necessity of truncation of Hilbert space via modal filtering of photons using slits or pinholes, and the simplicity of constructing operators in spatial-parity space. Multiple gates may be readily cascaded, thereby paving the way to convenient implementations of few-qubit quantum information processing algorithms.
CHAPTER 6: SINGLE-PHOTON QUANTUM LOGIC WITH POLARIZATION AND ORBITAL ANGULAR MOMENTUM QUBITS

6.1 Introduction

The KLM proposal has garnered widespread attention due to the idea that universal quantum computation can be carried out in a simple fashion by using linear optical components and single-photon detectors [26]. However, the success of their proposal relies on two crucial factors. The first is the unrestricted extension of the Hilbert space, dictated by the ability to encode more information in a single photon. The second is the development of robust and reliable controlled operations. Because any quantum algorithm can be implemented using single-qubit operations and controlled-NOT (CNOT) gates alone, efficient schemes for CNOT gates still remains an area of active research. We propose a new and simple scheme for a CNOT gate based on a spatial light modulator (SLM), in which the control qubit is the polarization whereas the target qubits are the photon orbital angular momentum (OAM) states [19, 147].

6.2 Polarization and OAM

The use of polarization of a photon as a qubit has found widespread applications since the earliest days of quantum optics. Tests of nonlocality, in the form of Bell- and GHZ-inequality violations, and demonstration of quantum teleportation have all relied on the polarization of a photon. It is well known that a beam of light with an azimuthal phase dependence of the form $e^{i\ell \phi}$ carries an OAM of $\ell \hbar$ per photon. The OAM of single photons presents an infinite dimensional discrete Hilbert

space for applications in quantum computation and communication. Experiments demonstrating entanglement between OAM states of biphotons [148], even values as large as \( \ell = \pm 300 \) [149], have yielded good results. A CNOT gate with OAM and polarization of single photons was first proposed by Deng et al. [125] and experimentally demonstrated by Fiorentino et al. [127]. Their realizations depend on either a Mach-Zehnder interferometer or a Sagnac-type interferometer for selective modification of a photon’s OAM depending on its polarization. In this chapter, we present a much simpler scheme in which these interferometers have been replaced by a polarization-sensitive SLM, hence eliminating the need for precise alignment.

For the polarization qubit, we associate the vertical polarization \( |V\rangle \) with control qubit \( |0\rangle \), and the horizontal polarization \( |H\rangle \) with control qubit \( |1\rangle \). For OAM, the transverse Gaussian (G) mode profile \( |\ell_0\rangle \), with \( \ell = 0 \), is associated with target qubit \( |0\rangle \), and the first-order Laguerre-Gaussian (LG) modes \( |\ell_{\pm 1}\rangle \), with \( \ell = \pm 1 \), is associated with target qubit \( |1\rangle \). The polarization-sensitive SLM leaves the vertically polarized photons intact, and only modifies the phase of the horizontally polarized photons. The polarization selectivity depends on the orientation of the liquid crystals in the SLM. When the control qubit is \( |V\rangle \) and the target qubit is either \( |\ell_0\rangle \) or \( |\ell_1\rangle \), the SLM leaves the state of the photon as is. When the control qubit is \( |H\rangle \), the SLM flips \( |\ell_0\rangle \) to \( |\ell_{+1}\rangle \), and vice versa. In this simple way, the SLM realizes the operation of a CNOT gate: \( |00\rangle \rightarrow |00\rangle \), \( |01\rangle \rightarrow |01\rangle \), \( |10\rangle \rightarrow |11\rangle \), and \( |11\rangle \rightarrow |10\rangle \). This operation is illustrated in Figure 6.1.
Figure 6.1: Implementation of a two-qubit CNOT gate with a spatial light modulator with polarization as the control qubit and OAM as the target qubit.
6.3 Experimental Setup

A schematic of the experimental setup is shown in Figure 6.2. A pump laser at 405 nm shines on a BBO crystal in type-I collinear configuration and pairs of photons are produced via spontaneous parametric down-conversion. A Glan-Thompson polarizer oriented orthogonal to the polarization of the pump beam blocks it from propagating further. A non-polarizing beamplitter directs one of the pair of downconverted photons to a single-mode fiber coupled trigger, whereas the second photon travels through the experimental setup. This single-photon heralding scheme effectively implements a single-photon source, and all measurements are carried out in coincidence between the trigger and the output detector. The state of the incoming photon is $|H\ell_0\rangle$. A uniform phase distribution on spatial light modulator SLM$_1$ keeps the OAM in state $|\ell_0\rangle$, whereas a clockwise azimuthal phase distribution of $0 \rightarrow 2\pi$ radians converts the OAM to state $|\ell_{+1}\rangle$. By using the appropriate phase distribution on SLM$_1$ and angle of half-wave plate HWP$_1$, the polarization and OAM of the photon can be prepared in any of the four basis states $\{ |V\rangle, |H\rangle \} \otimes \{ |\ell_0\rangle, |\ell_{\pm 1}\rangle \}$. Spatial light modulator SLM$_2$ is the CNOT gate on which an anticlockwise azimuthal phase distribution of $0 \rightarrow 2\pi$ radians is displayed. The OAM state of a vertically polarized photon remains impervious. For a horizontally polarized photon, the G mode $|\ell_0\rangle$ converts to the first order LG mode $|\ell_{-1}\rangle$, whereas the first order LG mode $|\ell_{+1}\rangle$ converts back to the G mode $|\ell_0\rangle$. The two-qubit state is analyzed first by a polarization analyzer, followed by a single-mode fiber (SMF), which acts as an OAM analyzer. Only the G mode $|\ell_0\rangle$ is coupled into the SLM, and the LG modes $|\ell_{\pm 1}\rangle$ remain uncoupled due to their larger spatial extension.
Figure 6.2: Experimental setup showing state preparation, state control and state measurement stages for implementing quantum logic with polarization and OAM qubits. SF: Spatial filter; NLC: Nonlinear crystal; GT: Glan-Thomson polarizer; BS: Beamsplitter; SPP: Spiral phase plate; HWP: Half-wave plate; SLM: Spatial light modulator; WP: wave plate (half- or quarter-depending upon the measurement); PBS: Polarizing beamsplitter; IF: Interference filter; FC: Fiber coupler; MMF: Multi-mode fiber; SMF: Single-mode fiber; D: Detector. All measurements are made in coincidence between $D_1$ and $D_2$. 
6.4 Results

The measurement of the operator for the two-qubit CNOT gate with polarization as the control qubit and OAM as the target qubit is shown in Figure 6.3. Input states in all four basis states are generated, and for each input state, projective measurements are made along all four basis states to obtain this measurement.

Next we verify the operation of this CNOT gate by performing state tomography on the states generated after CNOT action. We select the input states \( |\psi_{in}\rangle = |V\ell_0\rangle, |\psi_{in}\rangle = |V\ell_1\rangle, |\psi_{in}\rangle = |H\ell_0\rangle, \) and \( |\psi_{in}\rangle = |H\ell_1\rangle, \) to obtain the outcomes \( |\psi_{out}\rangle = |V\ell_0\rangle, |\psi_{out}\rangle = |V\ell_1\rangle, |\psi_{out}\rangle = |H\ell_1\rangle, \) and \( |\psi_{out}\rangle = |H\ell_0\rangle, \) respectively. These measurements are shown in Figure 6.4.

Figure 6.3: Measurement of operator for two-qubit CNOT gate with polarization as the control qubit and OAM as the target qubit.
Figure 6.4: Measurements of density matrices of the outcomes of the two-qubit CNOT gate obtained via state tomography. a) $|\psi_{\text{in}}\rangle = |V\ell_0\rangle$, $|\psi_{\text{out}}\rangle = |V\ell_0\rangle$, b) $|\psi_{\text{in}}\rangle = |V\ell_1\rangle$, $|\psi_{\text{out}}\rangle = |V\ell_1\rangle$, c) $|\psi_{\text{in}}\rangle = |H\ell_0\rangle$, $|\psi_{\text{out}}\rangle = |H\ell_1\rangle$, and d) $|\psi_{\text{in}}\rangle = |H\ell_1\rangle$, $|\psi_{\text{out}}\rangle = |H\ell_0\rangle$.

6.5 Discussion

In conclusion, we have proposed a linear and deterministic CNOT gate using polarization and OAM of single photons. The polarization qubit controls the photon orbital angular momentum qubits. The hallmark of this approach is that the CNOT gate is implemented by a simple optical device, the SLM. Using a similar approach, higher dimensions of OAM may be utilized to increase the information capacity of the single photon.
CHAPTER 7: CONCLUSION

In conclusion, using various optical modes of light, namely, polarization, spatial position, spatial parity and orbital angular momentum, we have explored novel ideas regarding coherence and entanglement in classical and quantum optics.

We have used Bell’s measure to quantify correlation between two binary DoFs. Adopting this quantum inspired ordering of partially coherent beams reinforces the usefulness of coherence as a quantifiable resource shared among a beam’s multiple DoFs. This concept can be explored further by including a third DoF, and applying three-qubit measures of entanglement, such as the 3-tangle, to classical optical coherence.

We have shown the strategy for measuring the coherency matrix for two-point vector field correlations. We then performed the first experimental measurement of the coherency matrix $G$ associated with an electromagnetic beam having two binary degrees of freedom via a technique called optical coherency matrix tomography. Additional DoFs may be included in future experiments by applying this cascaded measurement technique to multiple DoFs.

We have shown that a PS-SLM can be used as a linear, deterministic quantum gate with polarization and spatial parity qubits, or polarization and OAM qubits. Spatial-parity qubits provide the advantages of ease of manipulation and invariance to propagation. OAM qubits provide the advantages of unrestricted Hilbert space and non-interferometric projections. In future experiments, the second photon may be used to encode additional qubits, which would enable the creation of cluster states, and provide states for tests of quantum non-locality.
APPENDIX A: FORMULATION OF BELL’S MEASURE AND THE DEGREE OF ACCESSIBLE COHERENCE IN YOUNG’S DOUBLE-SLIT EXPERIMENT
A.1 Introduction

Although the foundations of optical coherence theory were established in the first half of the twentieth century [3, 5, 29], there has been a recent revival of interest in these foundations [60]. This revival is driven by recognition that the traditional description of coherence using correlation function and the visibility of interferograms do not provide adequate account of several important configurations. For example, while the original double-slit experiment performed by Young, which ushered in the wave-optics framework, used unpolarized light (the concept of optical polarization in fact was not yet even recognized), the traditional framework of describing this experiment predicts no interferogram [4]! Furthermore, it is now widely recognized that deterministic unitary devices (such as wave plates, for example) may alter the visibility of the double-slit interferogram, thereby rendering the visibility an inadequate measure of coherence [61, 150].

We have presented in Chapter 2 a formulation of the coherence of an optical beam having two binary DoFs that is inspired by the direct-product Hilbert space formalism commonly used in the quantum-mechanical description of multi-partite states. The model beam we investigated had polarization and spatial parity along 1D as the two DoFs of interest. This formalism may be used to describe any two binary DoFs of a classical optical beam. We demonstrate here the generality of our approach by applying it to the more traditional scenario of two-point spatial correlations for a vector electromagnetic field. We use the Young double-slit arrangement to apply our formalism.

A.2 Review of previous coherence measures

Wolf has suggested the use of the spectral degree of coherence $\mu(r_1, r_2; \omega)$, based on a $2 \times 2$ cross-spectral density matrix, as a measure of spatial coherence between two points $r_1$ and $r_2$ in the context of a Young’s double-pinhole interference experiment [4]. An earlier approach by Gori
introduced a $2 \times 2$ beam coherence-polarization matrix to describe both the spatial (two-point) coherence and (pointwise) polarization properties [60]. The degree of coherence in both formulations is closely related to the visibility of interference fringes produced by light after passing through the two pinholes. Such measures, however, were found to not be invariant under local unitary transformations [151], (i.e., implemented at each pinhole), and the visibility of interference was shown to change when such transformations are implemented [150], thereby casting doubt on the usefulness of these coherence measures. Réfrégier et al. have proposed an alternative $2 \times 2$ normalized mutual coherence matrix [61] with the aim of defining a measure of coherence that remains invariant under local unitary transformations. Their approach relies on evaluating two degrees of coherence, $\mu_S$ and $\mu_I$ instead of a single invariant quantitative measure.

A.3 The global coherency matrix and reduced coherency matrices

In the methodology introduced in Chapter 2, we start from a $4 \times 4$ global coherency matrix that provides a complete description of the field. From this global coherency matrix, we obtain all the coherence measures. In the context of a two-point electromagnetic field, the global coherency matrix defined at the plane of the two pinholes is given by,

$$G = \begin{pmatrix}
\langle E_x(\mathbf{r}_1)E_x^*(\mathbf{r}_1) \rangle & \langle E_x(\mathbf{r}_1)E_x^*(\mathbf{r}_2) \rangle & \langle E_x(\mathbf{r}_1)E_y^*(\mathbf{r}_1) \rangle & \langle E_x(\mathbf{r}_1)E_y^*(\mathbf{r}_2) \rangle \\
\langle E_x(\mathbf{r}_2)E_x^*(\mathbf{r}_1) \rangle & \langle E_x(\mathbf{r}_2)E_x^*(\mathbf{r}_2) \rangle & \langle E_x(\mathbf{r}_2)E_y^*(\mathbf{r}_1) \rangle & \langle E_x(\mathbf{r}_2)E_y^*(\mathbf{r}_2) \rangle \\
\langle E_y(\mathbf{r}_1)E_x^*(\mathbf{r}_1) \rangle & \langle E_y(\mathbf{r}_1)E_x^*(\mathbf{r}_2) \rangle & \langle E_y(\mathbf{r}_1)E_y^*(\mathbf{r}_1) \rangle & \langle E_y(\mathbf{r}_1)E_y^*(\mathbf{r}_2) \rangle \\
\langle E_y(\mathbf{r}_2)E_x^*(\mathbf{r}_1) \rangle & \langle E_y(\mathbf{r}_2)E_x^*(\mathbf{r}_2) \rangle & \langle E_y(\mathbf{r}_2)E_y^*(\mathbf{r}_1) \rangle & \langle E_y(\mathbf{r}_2)E_y^*(\mathbf{r}_2) \rangle 
\end{pmatrix} \quad (A.1)$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are positions of the two pinholes. This matrix representation is mathematically isomorphic to the density matrix description of a two-qubit system in quantum mechanics. This matrix has been previously formulated in the context of the double-slit experiment in Ref. [11].
By tracing over the appropriate elements, we obtain the two reduced $2 \times 2$ coherency matrices that characterize the spatial and polarization DoFs independently of the other:

$$G_{\text{pol}} = \begin{pmatrix} \langle E_x(r_1)E^*_x(r_1) \rangle + E_x(r_2)E^*_x(r_2) & \langle E_x(r_1)E^*_y(r_1) \rangle + E_x(r_2)E^*_y(r_2) \\ \langle E_y(r_1)E^*_x(r_1) \rangle + E_y(r_2)E^*_x(r_2) & \langle E_y(r_1)E^*_y(r_1) \rangle + E_y(r_2)E^*_y(r_2) \end{pmatrix}, \quad (A.2)$$

$$G_{\text{sp}} = \begin{pmatrix} \langle E_x(r_1)E^*_x(r_1) \rangle + E_y(r_1)E^*_y(r_1) & \langle E_x(r_1)E^*_y(r_1) \rangle + E_y(r_1)E^*_y(r_1) \\ \langle E_x(r_2)E^*_x(r_2) \rangle + E_y(r_2)E^*_x(r_2) & \langle E_x(r_2)E^*_y(r_2) \rangle + E_y(r_2)E^*_y(r_2) \end{pmatrix}. \quad (A.3)$$

It is important to understand clearly the physical meaning of these two reduced matrices. The coherency matrix $G_{\text{sp}}$ describes the correlation between the fields at the two points $r_1$ and $r_2$ while ignoring polarization. In other words, the correlations are registered by two detectors with no polarization discrimination (and thus trace over the polarization DoF). The visibility of the Young’s fringes commonly indicates this correlation, with low visibility taken to indicate low coherence. We have uncovered here an alternate explanation, that there may also be deterministic correlations with another DoF, polarization in our context.

The second reduced matrix $G_{\text{pol}}$ describes the polarization coherence of the field. There is a crucial difference between $G_{\text{pol}}$ here and the usual polarization coherence matrix commonly used, which typically describes a single spatial point in the field. Our $G_{\text{pol}}$, on the other hand, describes the full field after integrating over space and not a single point. Determining $G_{\text{pol}}$ requires using a detector and polarization components that cover both slits, i.e., a detector with no spatial discrimination.
A.4 Degree of polarization and degree of spatial coherence

From these two reduced matrices, we use the standard approach to find the degree of polarization and the degree of coherence:

\[ D_{\text{pol}} = \sqrt{1 - \frac{4 \text{Det} G_{\text{pol}}}{(\text{Tr} G_{\text{pol}})^2}} \]  
\[ (A.4) \]

\[ D_{\text{sp}} = \sqrt{1 - \frac{4 \text{Det} G_{\text{sp}}}{(\text{Tr} G_{\text{sp}})^2}} \]  
\[ (A.5) \]

where ‘Tr’ and ‘Det’ indicate the matrix trace and determinant, respectively.

A.5 Accessible degree of polarization and accessible degree of spatial coherence

The accessible degree of polarization and the accessible degree of spatial coherence are then given by:

\[ S_{\text{pol}} = \frac{D_{\text{pol}}^2}{2} + \left( \frac{B_{\text{max}}}{2\sqrt{2}} \right)^2, \]  
\[ (A.6) \]

\[ S_{\text{sp}} = \frac{D_{\text{sp}}^2}{2} + \left( \frac{B_{\text{max}}}{2\sqrt{2}} \right)^2 \]  
\[ (A.7) \]

where \( B_{\text{max}} \) is the Bell’s measure obtained from the methodology outlined in the main text implemented in the two-point vector field configuration.

One possible optical arrangement is depicted schematically in Figure A.1. Here, we do not rely on the traditional route of registering a spatial interferogram and measuring a (potentially polarization-dependent) interference visibility. Instead, we produce a 2D intra-DoF interferogram parametrized by two real numbers, one associated with each DoF. For the spatial DoF (the two locations of the slits), we use a beam combiner with varying combining strength \( \cos \delta \). Such a beam combiner is
implemented in Fig. A.1 using a Mach-Zehnder interferometer having a variable phase shifter $\delta$ in one arm. The output field is then subjected to a polarization projective measurement identical to that used in the main text (using a polarization analyzer at $\gamma$). By varying the two angles $\delta$ and $\gamma$, we obtain a 2D interferogram $C(\delta, \gamma)$ using Equation 2.13. From this intra-DoF interferogram, we obtain $B_{\text{max}}$ and evaluate $S_{\text{pol}}$ and $S_{\text{sp}}$.

A.6 Young’s double-slit experiments corresponding to experiments A, B, and C in Chapter 2

We now create a correspondence between the situation under consideration and the three experiments described in Chapter 2. One must keep in mind that the role of polarization here is isomorphic to spatial-parity, and that of spatial coherence is isomorphic to polarization, in the Chapter 2 experiments.

A.6.1 Experiment A

The source is spatially coherent and linearly polarized. Elements A and B are half-wave plates with rotation angles $\alpha/2$ and $\beta/2$ respectively. For simplicity, we take $\beta = 0$. Changing the polarization of one slit with respect to the other results in correlating polarization and spatial DoFs. As a result, we obtain $D_{\text{pol}} = D_{\text{sp}} = |\cos \alpha|$ and $S_{\text{pol}} = S_{\text{sp}} = 1$.

A.6.2 Experiment B

The source is spatially incoherent but linearly polarized. Elements A and B are half-wave plates with rotation angles $\alpha/2$ and $\beta/2$ respectively. For simplicity, we take $\beta = 0$. Changing the polarization of one slit with respect to the other results in correlating polarization and spatial DoFs. As a result,
we obtain $D_{\text{pol}} = |\cos \alpha|$, $D_{\text{sp}} = 0$, $S_{\text{pol}} = \frac{1}{2}$, and $S_{\text{sp}} = \frac{1}{2} \sin^2 \alpha$.

A.6.3 Experiment C

The source is spatially coherent and linearly polarized. Element A is a polarization rotator that switches between horizontal and vertical states many times within one measurement cycle, with $P$ being the probability that the state was horizontal. Element B is a half-wave plate with its axis aligned to the horizontal. We obtain $D_{\text{pol}} = D_{\text{sp}} = 1 - P$ and $S_{\text{pol}} = S_{\text{sp}} = P^2 + (1 - P^2)$.

Figure A.1: Experimental setup for evaluating $B_{\text{max}}$. Element A and Element B are half-wave plates with rotation angles $\alpha/2$ and $\beta/2$, respectively. The Mach-Zehnder interferometer with phase shifter $\delta$ implements a beam combiner of variable strength.
APPENDIX B: SUPPLEMENTARY INFORMATION FOR OPTICAL COHERENCY MATRIX TOMOGRAPHY
This document provides supplementary information for Chapter 4: Optical coherency matrix tomography. We provide experimental details for generating beams for which the polarization is coupled with a spatial DoF. We also provide the complete results of three sets of experiments, each of which probes a different spatial degree of freedom (DoF) of an optical beam in conjunction with polarization. In Chapter 4, we presented selections from these three experiments, which are given here in their entirety. The results for the two-point Young’s double slit case are given in Figure B.1, for the spatial parity modes in Figure B.2, and for the orbital angular momentum modes in Figure B.3.
B.1 Polarization and double slits

The polarization and position-coupled beams in states $G_1$ through $G_6$ are prepared as follows:

(a) $G_1$: the polarization and spatial DoFs are separable and both are coherent. This beam is produced by simply passing a horizontally polarized beam though the slits ‘a’ and ‘b’.

(b) $G_2$: the polarization DoF is coherent while the spatial DoF lacks coherence. This beam is produced by introducing spatial incoherence in a horizontally polarized beam, by using a diffuser, such as rotating ground glass, or a spatial light modulator (LCOS-SLM X10468-02, Hamamatsu Photonics K. K.).

(c) $G_3$: both the polarization and spatial DoFs lack coherence. This beam is produced by first introducing spatial incoherence in the beam, and then passing it through a polarization scrambler (Liquid Crystal Polarization Rotator, Meadowlark Optics).

(d) $G_4$: the polarization and spatial DoFs are classically entangled. This beam is produced by passing a vertically polarized beam through slit ‘a’, and a horizontally polarized beam through slit ‘b’. The vertically polarized beam is obtained by placing a half-wave plate (HWP) at 45° at slit ‘b’.

(e) $G_5$: the polarization and spatial DoFs are classically correlated. This beam is produced by passing a vertically polarized beam through slit ‘a’, and a horizontally polarized beam through slit ‘b’, and then introducing spatial incoherence in the beam.

(f) $G_6$: this beam is a mixture of the separable-coherent beam $G_1$ and the classically entangled beam $G_4$. It is produced by placing the polarization scrambler at slit ‘b’.
Figure B.1: Polarization and double slits. Measurements of $G$ for beams having separable DoFs. (a) $G_1$: both DoFs are coherent; (b) $G_2$: polarization is coherent but the spatial DoF is incoherent; and (c) $G_3$: both DoFs are incoherent. Measurements of $G$ for beams having non-separable DoFs. (a) $G_4$: classically entangled beam; (b) $G_5$: classically correlated beam; and (c) $G_6$: mixture of beams $G_1$ and $G_4$. The imaginary components in all cases are negligible, and are not shown.
B.2 Polarization and spatial parity

The polarization and spatial parity-coupled beams in states $G_1$ through $G_6$ are prepared as follows:

(a) $G_1$: the polarization and spatial DoFs are separable and both are coherent. This beam is produced by passing a diagonally polarized beam through the SLM having a zero phase step.

(b) $G_2$: the polarization DoF is coherent while the spatial DoF lacks coherence. This beam is produced by first passing a horizontally polarized beam through the SLM. The phase pattern on the SLM switches periodically between the zero phase step and the $\pi$ phase step. The beam then passes through a HWP at 22.5°.

(c) $G_3$: both the polarization and spatial DoFs lack coherence. This beam is produced by passing it through the SLM whose phase pattern switches periodically between the zero phase step and the $\pi$ phase step, and then scrambling the polarization.

(d) $G_4$: the polarization and spatial DoFs are classically entangled. The beam is prepared by passing a diagonally polarized beam through the SLM having a $\pi$ phase step.

(e) $G_5$: the polarization and spatial DoFs are classically correlated. This beam is produced by scrambling the polarization and passing it through the SLM having a $\pi$ phase step.

(f) $G_6$: this beam is a mixture of the separable-coherent beam $G_1$ and the classically entangled beam $G_4$. It is produced by passing a diagonally polarized beam through the SLM whose phase pattern switches periodically between the zero phase step and the $\pi$ phase step.
Figure B.2: Polarization and spatial-parity modes. Measurements of $G$ for beams having separable DoFs. (a) $G_1$: both DoFs are coherent; (b) $G_2$: polarization is coherent but the spatial DoF is incoherent; and (c) $G_3$: both DoFs are incoherent. The imaginary components in these cases are negligible, and are not shown. Measurements for the real and imaginary parts of $G$ for beams having non-separable DoFs. (a) $G_4$: classically entangled beam; (b) $G_5$: classically correlated beam; and (c) $G_6$: mixture of beams $G_1$ and $G_4$. 
B.3 Polarization and OAM modes

The polarization and OAM mode-coupled beams in states $G_1$ through $G_6$ are prepared as follows:

(a) $G_1$: the polarization and spatial DoFs are separable and both are coherent. This beam is produced by passing a diagonally polarized beam through the SLM having a zero phase distribution.

(b) $G_2$: the polarization DoF is coherent while the spatial DoF lacks coherence. This beam is produced by passing a horizontally polarized beam through the SLM. The phase pattern on the SLM switches periodically between the zero phase distribution and the phase vortex $e^{i\phi}$, $0 \leq \phi < 2\pi$. The beam then passes through a HWP at 22.5°.

(c) $G_3$: both the polarization and spatial DoFs lack coherence. This beam is produced by passing it through the SLM whose phase pattern switches periodically between the zero phase distribution and the phase vortex $e^{i\phi}$, $0 \leq \phi < 2\pi$, and then scrambling the polarization.

(d) $G_4$: the polarization and spatial DoFs are classically entangled. The beam is prepared by passing a diagonally polarized beam through the SLM having the phase vortex $e^{i\phi}$, $0 \leq \phi < 2\pi$.

(e) $G_5$: the polarization and spatial DoFs are classically correlated. This beam is produced by scrambling the polarization and passing it through the SLM having the phase vortex $e^{i\phi}$, $0 \leq \phi < 2\pi$.

(f) $G_6$: this beam is a mixture of the separable-coherent beam $G_1$ and the classically entangled beam $G_4$. It is produced by passing a diagonally polarized beam through the SLM whose phase pattern switches periodically between the zero phase distribution and the phase vortex $e^{i\phi}$, $0 \leq \phi < 2\pi$. 

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Figure B.3: Polarization and OAM modes. Measurements of $G$ for beams having separable DoFs. (a) $G_1$: both DoFs are coherent; (b) $G_2$: polarization is coherent but the spatial DoF is incoherent; and (c) $G_3$: both DoFs are incoherent. Measurements of $G$ for beams having non-separable DoFs. (a) $G_4$: classically entangled beam; (b) $G_5$: classically correlated beam; and (c) $G_6$: mixture of beams $G_1$ and $G_4$. The imaginary components in all cases are negligible, and are not shown.
APPENDIX C: OPTICAL COHERENCY MATRIX TOMOGRAPHY OF UNCONVENTIONAL BEAMS
Figure C.1: Examples of unconventional beams. Beams with coupled DoFs: a) polarization with spatial position, b) polarization with spatial parity modes, and c) polarization with orbital angular momentum modes.

We present the first experimental measurement of the coherency matrix $G$ associated with an electromagnetic beam having two binary degrees of freedom. We consider beams with polarization and a spatially varying phase: spatial parity modes and orbital angular momentum modes, respectively.

The statistical fluctuations of light are characterized by a hierarchy of classical or quantum correlations of components of the electromagnetic vector field at pairs of points in time and space. The theory of optical coherence, which describes the properties of these correlation functions or matrices, often focuses on only one or two of the underlying temporal/spectral, spatial, or polarization degrees of freedom (DoFs). For example, the temporal coherence function at a single position and for a single polarization component, and the polarization matrix at a single position and time, are based on a single DoF, and are used to define useful measures such as the coherence time, the coherence area, and the degree of polarization. In situations where multiple DoFs are involved, the correlation function/matrix defined for a single DoF, and measures based thereupon, are meaningful only if the DoF is uncoupled from the other DoFs. In the presence of such cou-
pling, disregarding one DoF can lead to misleading conclusions about the other. For example, it is possible for a deterministic field for which polarization is coupled to spatial distribution to exhibit an unpolarized character if polarization is measured by a spatially insensitive detector. Coupling between the polarization and the spatial DoFs occurs in beams with spatially dependent polarization, and is introduced when light undergoes polarization-dependent spatial transformations, such as scattering. A few examples of unconventional beams with coupled DoFs is given in Figure C.1.

We have recently investigated the coherence properties of optical beams with two DoFs, each of which is binary [9, 12]. In analogy with binary optics, we call this quaternary optics. The polarization DoF is of course inherently binary [93]. Examples of binary spatial DoF are propagation in a two-mode waveguide or two coupled single-mode waveguides, a beam in even-odd spatial parity-modes [17], or light in two orbital angular momentum (OAM) modes [19]. Since a quaternary optical beam may be described by a vector in a four-dimensional Hilbert space, it is mathematically similar to the quantum state of two particles, each in a binary quantum state. This correspondence has been exploited to use the tools and measures developed in quantum theory, including the notion of entangled states. In this spirit, coupling between two DoFs for classical quaternary light, which is described by a non-separable state, may be regarded as a form of classical entanglement. Bell’s measure has been proposed as a means of quantifying the coherence hidden in the coupled DoFs [9].

The second-order coherence properties of quaternary beams (in which the polarization is coupled to a spatially varying amplitude or phase) are expressed completely by a $4 \times 4$ coherency matrix $G$. The elements of $G$ embody the coherence of the individual DoFs as well as their mutual coherence. The elements of $G$ are obtained from a cascade of projective measurements of its individual DoFs via a technique we refer to as optical coherency matrix tomography (OcMT) [Figure C.2]. The Hermiticity of $G$ ensures that a set of 16 mutually independent projective measurements are sufficient to reconstruct the complex elements of $G$. The choice of measurements to be implemented
is provided by an equivalent problem in quantum optics, that of the reconstruction of the density matrix associated with two-photon quantum states.

We present the first experimental measurements of $G$ for beams in which the polarization is coupled to a spatial DoF: spatial parity modes and orbital angular momentum modes [Figure C.3]. We consider the following beams:

(a) $G_1$: A 45° polarized beam with OAM mode $\ell = 0$. The polarization and spatial DoFs are separable and both are coherent.

(b) $G_2$: A radially polarized beam with OAM mode $\ell = 1$. The polarization and spatial DoFs are classically entangled.

(c) $G_3$: A 45° polarized beam in a mixture of even and odd spatial parity modes. The polarization DoF is coherent while the spatial DoF lacks coherence.

(d) $G_4$: A beam with a 50:50 distribution of horizontal and vertical polarizations in an even spatial parity mode. The polarization and spatial DoFs are classically correlated.

Although we have conducted the experiments for a quaternary beam, the above methodology is equally valid for a higher number of DoFs with $m$-ary levels each.
Figure C.2: a) Methodical approach for optical coherency matrix tomography. A beam with two binary degrees of freedom (DoFs) is represented by a $4 \times 4$ coherency matrix $G$. The beam first undergoes analysis for the polarization DoF, followed in cascade by an analysis for the spatial DoF (spatial parity or OAM). $I_H$, $I_V$, $I_D$, and $I_R$ represent the intensities for horizontal, vertical, diagonal, and circular polarization measurements, respectively. $I_a$, $I_b$, $I_{a+b}$ and $I_{a+i}b$ are the equivalent for the spatial DoF (spatial parity or OAM). Schematic of the experimental setup showing beam preparation and analysis b) for beams with polarization and spatial parity, c) for beams with polarization and OAM. OAM: orbital angular momentum; HWP: half-wave plate; SLM: spatial light modulator; WP: wave plate (half or quarter); PBS: polarizing beam splitter; MZI: Mach-Zehnder interferometer; D: detector; FC: fiber coupler; SMF: single-mode fiber.
Figure C.3: Beam transverse intensity profiles with polarization distributions and measurement of the $4 \times 4$ elements of the associated coherency matrix $G$. $D_p$: degree of polarization; $D_s$: degree of coherence for the spatial DoF (spatial parity or OAM); $L$: Linear entropy as a measure of the overall beam coherence. Beams with polarization and orbital angular momentum modes: a) $G_1$: $45^\circ$ linearly polarized beam in an $\ell = 0$ OAM mode; the degrees of freedom (DoFs) are separable and coherent, b) $G_2$: radially polarized beam in an $\ell = 1$ OAM mode; the DoFs are classically entangled. Beams with polarization and spatial parity modes: c) $G_3$: $45^\circ$ linearly polarized beam in a mixture of even and odd modes, d) $G_4$: beam with a 50:50 distribution of horizontal and vertical polarizations in an even mode. Although the elements of $G$ are in general complex, the beams considered here have only significant real components. The imaginary components are negligible and are not shown.
The motivation for the formulation of a variable that is continuous in the Hilbert space comes directly from Einstein, Podolsky and Rosen. In their famous argument [35], they pointed out the inadequacies in the then prevailing notions of quantum mechanics. Their thought experiment was based on an ‘EPR state’ using continuous position-momentum variables. By doing so, they introduced two essential features of Quantum Mechanics: entanglement and non-locality. Due to challenges involved in identifying continuous quantum variables, experimentally accessing those variables, and implementation of operators for manipulating those variables, investigations into the EPR argument have mainly relied on Bohm’s version of the EPR states that inhabit a discretized Hilbert space [37]. Indeed, Bell himself erroneously concluded that EPR states with continuous variables will not exhibit quantum non-locality and will allow for a hidden-variables description of the system. Abouraddy et al. [13] have presented a proposal for a continuous variable based on the spatial-parity symmetry of the transverse field of photons, as summarized below.

D.1 Continuous spatial variables in parity space

We start by establishing a correspondence between the single-mode multiphoton electromagnetic-field spanned by the Fock-state basis \( \{ |n\rangle \} \) (pseudospin space) and the single-photon multimode electromagnetic-field spanned by a spatial-eigenmode basis \( \{ \phi_n(x) \} \) (spatial-parity space). Chen et al. [152] have introduced the following pseudospin operators for photons in analogy to the spin operator \( \sigma \) for spin-\( \frac{1}{2} \) systems:

\[
s_x = \sum_{n=0}^{\infty} (|2n+1\rangle \langle 2n| + |2n\rangle \langle n+1|),
\]

(D.1)

\[
s_y = i \sum_{n=0}^{\infty} (|2n+1\rangle \langle 2n| - |2n\rangle \langle n+1|),
\]

(D.2)
\[ s_z = \sum_{n=0}^{\infty} \left( |2n\rangle \langle 2n| - |2n+1\rangle \langle n+1| \right), \]  

and raising and lowering operators:

\[ s^+ = \sum_{n=0}^{\infty} |2n+1\rangle \langle 2n|, \]  

\[ s^- = \sum_{n=0}^{\infty} |2n\rangle \langle 2n+1|. \]

The construction of these operators in the pseudospin space has so far proven elusive. Abouraddy et al. [13] have devised a strategy to identify each Fock state with a spatial mode, \( |n\rangle \rightarrow \phi_n(x) \), which enables the construction of parity-space operators that are easily implementable in the spatial domain.

A pure single-photon multimode state in the 1D spatial domain, in a direction \( x \) orthogonal to the general direction of propagation of the photon, is given in general by:

\[ |\Psi\rangle = \int dx \psi(x) |1_x\rangle, \]  

where \( \int dx |\psi(x)|^2 = 1 \). The state function \( \psi \) may be decomposed in an orthonormal basis of square integrable functions, such as the set of Hermite-Gaussian functions,

\[ \psi(x) = \sum_{n=0}^{\infty} c_n \phi_n(x), \]  

where \( \sum_n |c_n|^2 = 1 \) and \( c_n = \int dx \phi_n^* \psi(x) \). Noting that \( \int \phi_n(x) |1_x\rangle = |n\rangle \), the state may be reformulated as:

\[ |\Psi\rangle = \sum_{n=0}^{\infty} c_n \int dx \phi_n(x) |1_x\rangle = \sum_{n=0}^{\infty} c_n |n\rangle. \]
The sequence of Hermite-Gaussian functions alternates between even and odd function, i.e., \( \phi_{2n}(-x) = \phi_{2n}(x) \) and \( \phi_{2n+1}(-x) = -\phi_{2n+1}(x), \forall n \). An association between alternation of even and odd integers \( n \) with the alternation of even and odd spatial-parity modes \( \phi_n(x) \) allows to express the state in terms of even (|\(e\rangle\)) and odd (|\(o\rangle\)) spatial-parity components:

\[
|\Psi\rangle = \sum_{n=0}^{\infty} c_{2n} |2n\rangle + \sum_{n=0}^{\infty} c_{2n+1} |2n + 1\rangle = \alpha |e\rangle + \beta |o\rangle,
\]

where \( \langle e|e\rangle = \langle o|o\rangle = 1 \), \( |\alpha|^2 = \sum_{2n} |c_{2n}|^2 \), \( |\beta|^2 = \sum_{2n+1} |c_{2n+1}|^2 \), and \( |\alpha|^2 + |\beta|^2 = 1 \).

The one-photon state in Equation D.6 can hence be given in three levels of description:

1. the spatial-parameter description of Equation D.7;
2. the discretized spatial eigenmode description given by Equation D.8;
3. and the spatial-parity-space description provided by Equation D.9.

Now we shall look at the experimental realizations of operators in spatial-parity space.

### D.2 Construction of operators in one-photon parity space

Simplicity is the hallmark of experimental realizations of operators in spatial-parity space, which does not involve any non-linearities or moving parts, in stark comparison with implementation in the Fock basis. We seek to implement spatial parity operators \( S_x, S_y \) and \( S_z \) isomorphic to the pseudospin operators \( s_x, s_y \) and \( s_z \) in the Fock basis. The operators obey the commutation relation

\[
[S_i, S_j] = i2\epsilon_{ijk}S_k, \quad \text{where } k = x, y, z \text{ and } \epsilon_{ijk} \text{ is the antisymmetric vector}.
\]
D.2.1 Parity flipper

The parity flipper is an optical system that transforms an even mode to an odd mode, and vice versa. It has the form \( h_{\text{PF}}(x, x') = e^{i\pi H(x)}\delta(x - x') \), where \( H(x) \) is the Heaviside step function, which equals unity for positive \( x \), and zero elsewhere. We choose to express the parity flipper as

\[
h(x) = -i R_{\pi}(x),
\]

where \( |R_{\pi}(x)| = 1 \), \( \forall x \), \( \angle R_{\pi}(x) = \frac{\pi}{2} \) for \( x \geq 0 \) and \( \angle R_{\pi}(x) = -\frac{\pi}{2} \) for \( x < 0 \). \( R_{\pi}(x) \) represents a glass plate that imparts a phase shift of \( \pi \) to one-half of the single-photon wavefront, leaving the other half intact along the \( x \) direction [Figure D.1(a)]. It can be shown that this operation corresponds to the \( S_x \) spatial-parity operation for spaces that are closed under it [13].

D.2.2 Spatial flipper

The spatial parity operator \( S_z \) corresponds to the transformation \( \psi(x) \rightarrow \psi(-x) \), and its effect is to flip the input optical field distribution along the \( x \) direction about the origin [Figure D.1(b)]. This transformation maybe mathematically expressed as [13]:

\[
|\Psi\rangle \rightarrow \int dx \psi(-x) |1_x\rangle = \sum_{n=0}^{\infty} c_{2n} |2n\rangle - \sum_{n=0}^{\infty} c_{2n+1} |2n + 1\rangle = \alpha |e\rangle - \beta |o\rangle = S_z |\Psi\rangle.
\]

The spatial flipper is easily implementable by mirrors, lenses or other simple optical components, e.g. a Dove prism. Using the commutation relation, the \( S_y \) operator can be implemented by using \( S_z \) and \( S_x \) operators.
D.2.3 Parity analyzer

Just like the polarizing beamsplitter separates the incoming state into the horizontal and vertical components, a parity analyzer is a device that separates the incoming one-photon state into its even and odd parity spatial components [13]. It can be implemented using a balanced Mach-Zehnder interferometer, one in which both the arms are perfectly the same length, and using a spatial flipper in one of the arms [Figure D.1(c)]. The spatial flipper may comprise of a mirror, and the extra mirror in one arm with respect to the other implements the flip.

The super-operator transformation $P$ between the input ports 0 and 1, and the output ports 2 and 3 may be expressed as [13]:

$$
\begin{pmatrix}
\phi_3(x) \\
\phi_2(x)
\end{pmatrix} = 
\begin{pmatrix}
P_o & iP_e \\
iP_e & -P_o
\end{pmatrix}
\begin{pmatrix}
\phi_0(x) \\
\phi_1(x)
\end{pmatrix}.
$$

The projection operators over the even and odd spatial-parity subspaces are defined as:

$$P_e = \frac{1}{2} \{ I + S_z \} = \sum_n |2n\rangle \langle 2n|,$$

$$P_o = \frac{1}{2} \{ I - S_z \} = \sum_n |2n+1\rangle \langle 2n+1|.$$
Figure D.1: Construction of operators in one-photon parity space: a) parity flipper, b) spatial flipper, and c) parity analyzer.
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