Computational Study of Traveling Wave Solutions and Global Stability of Predator-Prey Models

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COMPUTATIONAL STUDY OF TRAVELING WAVE SOLUTIONS AND GLOBAL STABILITY OF PREDATOR-PREY MODELS

by

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A dissertation submitted in partial fulfilment of the requirements
for the degree of Doctor of Philosophy
in the Department of Mathematics
in the College of Sciences
at the University of Central Florida
Orlando, Florida

Fall Term
2016

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ABSTRACT

In this thesis, we study two types of reaction-diffusion systems which have direct applications in understanding wide range of phenomena in chemical reaction, biological pattern formation and theoretical ecology.

The first part of this thesis is on propagating traveling waves in a class of reaction-diffusion systems which model isothermal autocatalytic chemical reactions as well as microbial growth and competition in a flow reactor. In the context of isothermal autocatalytic systems, two different cases will be studied. The first is autocatalytic chemical reaction of order m without decay. The second is chemical reaction of order m with a decay of order l, where m and l are positive integers and $m > l \geq 1$. A typical system is $A + 2B \to 3B$ and $B \to C$ involving three chemical species, a reactant A and an autocatalyst B and C an inert chemical species.

We use numerical computation to give more accurate estimates on minimum speed of traveling waves for autocatalytic reaction without decay, providing useful insight in the study of stability of traveling waves. For autocatalytic reaction of order $m = 2$ with linear decay $l = 1$, which has a particular important role in biological pattern formation, it is shown numerically that there exist multiple traveling waves with 1, 2 and 3 peaks with certain choices of parameters.

The second part of this thesis is on the global stability of diffusive predator-prey system of Leslie Type and Holling-Tanner Type in a bounded domain $\Omega \subset R^N$ with no-flux boundary condition. By using a new approach, we establish much improved global asymptotic stability of a unique positive equilibrium solution. We also show the result can be extended to more general type of systems with heterogeneous environment and/or other kind of kinetic terms.

I dedicate this dissertation to my husband, Xi Chen. I met him five years ago at the University of Central Florida. Xi is such a smart and kind person who brings lots and lots of happiness into my life. His unconditional love and support helped me to go through this long journey and overcome several obstacles This dissertation is not only the result of my effort but also the fruit of his love and care.

ACKNOWLEDGMENTS

I want to express my deepest appreciation to my advisor, Dr. Yuanwei Qi, for his teaching and support throughout my graduate study. Without his kind and encouraging guidance and persistent help, my doctoral study would not have been possible. I am very grateful to him for his brilliant thoughts and patiently spending time to teach me. In addition, I am sincerely thankful to my committee members, Dr. David Rollins, Dr. Zhisheng Shuai and Dr. Lei Zhai for help through teaching, conversations, comments and suggestions.

I wish to acknowledge the financial support by the Mathematics Department of University of Central Florida through my doctoral study. Finally, I want to thank my parents, my mother Yun Li and my father Guoqiang Zhu for supporting me to pursue my doctors degree in US and their unconditional love.

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CHAPTER 1: INTRODUCTION

In this thesis, we study two types of reaction-diffusion systems which have direct applications in understanding wide range of phenomena in chemical reaction, biological pattern formation and theoretical ecology.

In the first part, we study two reaction-diffusion systems. The first is isothermal autocatalytic chemical reaction between two chemical species A and B taking the form:

$$A + mB \longrightarrow (m+1)B \qquad \text{with rate } k[A][B]^m \tag{1.1}$$

where $m \geq 1$ is an integer and $k > 0$ is a rate constant. We say (1.1) is an isothemal autocatalytic chemical reaction of (m+1)-order. The autocatalytic step has been used successfully in models of real-world chemical reactions [46] and [60]. For example in [60], the cubic autocatalytic step is an useful model for the iodate-arsenous acid reaction. The resulting reaction-diffusion system is

$$(I) \begin{cases} \frac{\partial u}{\partial t} = D_A \frac{\partial^2 u}{\partial x^2} - kuv^m, \\ \frac{\partial v}{\partial t} = D_B \frac{\partial^2 v}{\partial x^2} + kuv^m. \end{cases} \tag{1.2}$$

It is well known that (I) has propagating traveling waves, see [31] and [64]. One important issue is to find minimum traveling wave speed since the corresponding traveling wave is most stable and plays a significant role in determining the global dynamics of general solutions, see [14]. The theoretical results show there exist bounds for the traveling wave speed and the existence and non-existence of the traveling wave under certain conditions. To fill in the gap of the theoretical study, we use computational methods to give more accurate estimates on minimum speed of traveling waves for autocatalytic reaction.

The second system is chemical reaction of order m with a decay of order l, where m and l are positive integers and $m > l \geq 1$. A typical system in autocatalysis is $A + 2B \to 3B$ and $B \to C$ involving three chemical species, a reactant A and an auto-catalyst B and C an inert chemical species. The corresponding system is

$$(II) \begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - uv^m, \\ \frac{\partial v}{\partial t} = D\frac{\partial^2 v}{\partial x^2} + uv^m - kv^l. \end{cases} \tag{1.3}$$

The case of autocatalytic reaction of order $m = 2$ with linear decay $l = 1$, is the famous Gray-Scott model and has a particular important role in chemical waves and pattern formation. We use numerical methods to show that there exist multiple traveling waves with 1, 2 and 3 peaks with certain choices of parameters. The existence of multiple traveling waves which have distinctive number of local maxima is also proved. Our computational result shows a new and very distinctive feature of Gray-Scott type of models in generating rich and structurally different traveling pulses than related models in literature. Our numerical result on (I) is published in a journal paper and that on (II) is under revision, see [55, 58]. We present our results on (I) in chapter two and those of (II) in chapter three.

In the second part, we study global stability of two predator-prey models

$$(III) \begin{cases} u_t = d_1 \triangle u + u(\lambda - \alpha u - \beta v), \quad (x,t) \in \Omega \times (0,\infty) \\ v_t = d_2 \triangle v + \mu v \left(1 - \frac{v}{u}\right), \quad (x,t) \in \Omega \times (0,\infty) \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad (x,t) \in \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) > 0, \quad v(x,0) = v_0(x) \geq 0 (\not\equiv 0), \quad x \in \bar{\Omega}. \end{cases}$$

and

$$
(IV) \begin{cases} u_t = d_1 \triangle u + au - u^2 - \frac{uv}{m+u}, & (x,t) \in \Omega \times (0,\infty) \\ v_t = d_2 \triangle v + bv - \frac{v^2}{\gamma u} & (x,t) \in \Omega \times (0,\infty) \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) > 0,\ v(x,0) = v_0(x) \geq 0 (\not\equiv 0) & x \in \bar{\Omega}. \end{cases}
$$

Here the quantities u and v represent prey and predator populations respectively. The two diffusion coefficients d_1 and d_2 are positive constants.

In chapter four and chapter five, we study the global stability of diffusive predator-prey systems of Leslie type and Holling-Tanner type in a bounded domain $\Omega \subset R^N$ with no-flux boundary condition. These systems are used in modeling a wide range of ecological phenomena. And the question of global stability in predator-prey systems is a very important problem in ecology. Several well-known methods have been used to prove global stability of the unique positive equilibrium of a predator-prey system. For example, in [29], the author constructed a Liapunov function for the predator-prey system to prove the global stability. In [35], the author used a combination of Dulac criterion, the method of comparison and the Poincare-Bendixson theorem to prove the global stability. Also in [9] and [10], the authors proved the global stability of the positive equilibrium by using the comparison method and iteration. In this thesis, we study the predator-prey systems by using a new approach. We establish much improved global asymptotic stability of a unique positive equilibrium solution. We also show the result can be extended to more general type of systems with heterogeneous environment and/or other kind of kinetic terms. Our new results on (III) and (IV) have been published in two journal papers and can be found in Qi and Zhu [56, 57].

1.1 Traveling Wave of Auto-catalytic Systems

In this section, we give a brief description of our result on system

$$(I)\begin{cases} \frac{\partial u}{\partial t} = D_A \frac{\partial^2 u}{\partial x^2} - kuv^m, \\ \frac{\partial v}{\partial t} = D_B \frac{\partial^2 v}{\partial x^2} + kuv^m. \end{cases} \tag{1.4}$$

We study the system in R^1 with the initial conditions:

$$u(x,0) = a_0, \;\; v(x,0) = g(x), \;\; \forall\, x \in R,$$

where a_0 represents the uniform distribution for the reactant and $g(x)$ is a nonnegative function with compact support.

By standard non-dimensional transformation, we have

$$\begin{cases} u_t = u_{xx} - uv^m, \;\; x \in R, \; t > 0 \\ v_t = dv_{xx} + uv^m, \;\; x \in R, \; t > 0 \\ u(x,0) = 1, \;\; v(x,0) = g(x), \quad \forall x \in R, \; t = 0. \end{cases} \tag{1.5}$$

The normalized traveling wave problem is as follows. Given $C > 0$, let $(u(x,t), v(x,t)) = (\alpha(z), \beta(z))$, where $z = x - Ct$. The traveling wave problem to (1.5) is to solve

$$\begin{cases} \alpha_{zz} + C\alpha_z = \alpha\beta^m, & \alpha \geqslant 0 \qquad \forall\, z \in \mathbb{R}, \\ d\beta_{zz} + C\beta_z = -\alpha\beta^m, & \beta \geqslant 0 \qquad \forall\, z \in \mathbb{R}, \\ \lim_{z\to\infty}(\ \alpha(z), \beta(z)\ ) = (1,0), \\ \lim_{z\to-\infty}(\ \alpha(z), \beta(z)\ ) = (0,1). \end{cases} \tag{1.6}$$

Here $C > 0$ is the constant traveling speed. The detailed analysis of this ODE system (1.6) will be shown in chapter two. The primary concern of the computational studies in the thesis is based on the following results of Chen and Qi [11]:

Theorem 1. *Let the boundary condition of traveling wave at $-\infty$ be fixed as $(0,1)$.*

(i) Suppose $d < 1$ and $m \geqslant 2$. A unique (up to translation) traveling wave solution exists for (1.6) *if $C \geq C_1(d) \equiv 4d/\sqrt{1+4d}$. On the other hand, there exists no solution for if $C \leq C_2(d) \equiv d/\sqrt{K(m)}$, where $K(m)$ is a constant, which increases with m. In particular, $K(1) = 1/4$, $K(2) = 2$.*

(ii) Suppose $d \geqslant 1$ and $m \geqslant 1$. There exists a positive constant $C_{\min}$ such that (1.6) *admits a traveling wave if and only if $C \geqslant C_{\min}$. In addition, $C_{\min}$ is bounded by*

$$\sqrt{\frac{d}{K(m)}} \leqslant C_{\min} \leqslant \sqrt{\frac{d}{K(m)}}\, \frac{1}{\sqrt{1 - (1-\frac{1}{d})\frac{\sqrt{4K(m)+1}-1}{\sqrt{4K(m)+1}+1}}}.$$

Apparently, there is a gap between the two bounds $C_1(d)$ and $C_2(d)$. So we ask:

What is the minimum speed for the system (1.5)?

The importance of this question is that for mono-stable type of problems, the experience of a single equation tells us it is the minimum speed traveling wave which is most relevant for the study of stability. In this thesis, we use numerical analysis to fill in the gap for C_{min} for different cases of d using Matlab and verified the results by Mathematica. We also did Regression Analysis to catch the dependence of C_{min} on d in an analytic formulation. The detailed computational results will be shown in Chapter 2.

1.2 Traveling Wave of Auto-catalytic Systems with Decay

The traveling wave problem with decay is far more complex. For example, the system

$$\begin{cases} u_t = u_{xx} - uv^m \\ v_t = dv_{xx} + uv^m - kv^m, \end{cases} \tag{1.7}$$

where $m \geq 1$ and $k > 0$ is a rate constant, was first studied in [64] for a special case and later in [31] for a general case when $m = 1$. Other related results appeared in [26], [27]. Whereas $m > 1$ case has been studied in [25] and [66]. The results again are in the classical mono-stable category for existence of traveling waves. But, for the system

$$\begin{cases} u_t = u_{xx} - uv^m \\ v_t = dv_{xx} + uv^m - kv, \end{cases} \tag{1.8}$$

where $m > 1$ and $k > 0$, the situation is totally different.

The main result on traveling wave of system (1.8) that we use in numerical analysis is proved by Chen et. al. [14] as following:

Theorem 2. *Let $m > 1$ and $D > 0$ be given constants.*

1. *There exist positive constants M_1, M_2, and M_3 that depend only on m and D such that for each $\varepsilon > 0$,* (3.6) *admits no solution if $c \geqslant \max\{\sqrt{M_1/\varepsilon}, M_2\}$ or if $c \leqslant \gamma - M_3\varepsilon$.*

2. *For each sufficiently small positive ε and each integer L satisfying $1 \leqslant L \leqslant \varepsilon^{-1/4}$, there exists a constant $c_L = L\gamma[1 + O(\varepsilon + [L-1]^2\varepsilon|\ln\varepsilon|)]$ such that when $c = c_L$, the system* (3.6) *admits a solution, unique up to a translation. The solution is an L-**hump solution** in the sense that $w := [1+\varepsilon u]v^{m-1}$ admits exactly L local maxima and $L-1$ interior local minima. In addition, if denote the interior points of local minima of w by $\{a_i\}_{i=2}^{L-1}$ and points of local maxima by $\{b_i\}_{i=1}^{L}$ with $-\infty = a_1 < b_1 < a_2 < b_2 < \cdots < b_L < a_{L+1} = \infty$, then*

$$w(b_i) = M + O(i[L+1-i]\varepsilon), \ \ G(w(a_{i+1})) = i(L-i)\sigma\gamma\varepsilon + O(i^2L^2\varepsilon^2|\ln\varepsilon|) \ \ \forall\, i = 1, \cdots, L.$$

Furthermore, $\|w'^2 - G(w)\|_{L^\infty(\mathbb{R})} = O(L^2\varepsilon)$ and

$$\lim_{\varepsilon\searrow 0} w(b_i + z) = \lim_{\varepsilon\searrow 0} v(b_i + z) = W(z)$$

uniformly in $i = 1, \cdots, L$ and locally uniformly in $z \in \mathbb{R}$, where W is the unique solution of

$$W'' = W - W^m \quad \textit{in } \mathbb{R}, \qquad W(0) = M, \qquad W'(0) = 0, \tag{1.9}$$

where

$$G(s) = s^2 - \frac{2s_+^{m+1}}{m+1}, \ \ \alpha = \frac{1}{m-1}, \ \ M = \Big(\frac{m+1}{2}\Big)^{\alpha}, \ \ \sigma = 4\int_0^M \sqrt{G(s)}ds, \ \ \gamma = \frac{2\alpha}{D}\int_0^M \frac{s^m ds}{\sqrt{G(s)}}, \tag{1.10}$$

$s_+ = \max\{s, 0\}$.

In spite of deep theoretical results obtained so far, intuitively, we would like to ask:

What is the traveling wave solution structure for the system (1.8)?

The main purpose of the computational study for the system with decay is to study the dependence of traveling wave solution structure for (1.7) on the boundary value at $-\infty$. And the numerical computation for (1.7) catches the corresponding traveling wave with one, two and three peaks of w, respectively. The result not only verifies the mathematical proof, but also provides more detailed information about the solutions and help us to gain insight into the complex interaction of diffusion and nonlinear reaction terms. But, the difficulty is that we need to integrate the solutions with high order nonlinearities over an extended interval. We use numerical analysis to verify theoretical results for various cases of c_L using Matlab, which implements explicit fourth-order Runge-Kutta method for the computation. To make sure the computation is accurate, we check the results by using the double precision build-in solver NDsolve from Mathematica. Our numerical results show very interesting features of the system and can be found in [55] and [58].

1.3 Leslie-type and Holling-Tanner type Predator-prey Models

In chapter four, we study the diffusive Leslie-type predator-prey model

$$(III)\begin{cases} u_t = d_1\triangle u + u(\lambda - \alpha u - \beta v), \quad (x,t)\in\Omega\times(0,\infty) \\ v_t = d_2\triangle v + \mu v\left(1-\frac{v}{u}\right), \quad (x,t)\in\Omega\times(0,\infty) \\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0, \quad (x,t)\in\partial\Omega\times(0,\infty) \\ u(x,0)=u_0(x)>0, \quad v(x,0)=v_0(x)\geq 0(\not\equiv 0), \quad x\in\bar{\Omega}. \end{cases}$$

Here $u(x,t)$ and $v(x,t)$ are the density of prey and predator, respectively, Ω is a bounded domain with smooth boundary $\partial\Omega$, λ, μ, α, and β are positive constants. We assume throughout this thesis that the two diffusion coefficients d_1 and d_2 are positive and equal. The no-flux boundary condition is imposed to guarantee that the ecosystem is not disturbed by exterior factors which may influence population densities through flow cross the boundary.

The system is a well established population model and is widely studied in literature, see [19, 29]. In particular, the following result was proved in [19] by construction of Lyapunov function.

Theorem. Suppose $d_1,\ d_2$ are positive constants, and $\alpha > \beta$ or $\alpha/\beta > s_0 \in (1/5, 1/4)$, then the unique positive equilibrium point

$$(u^*, v^*) = \left(\frac{\lambda}{\alpha+\beta}, \frac{\lambda}{\alpha+\beta}\right)$$

is globally asymptotically stable in the sense that every solution to (III) satisfies

$$\lim_{t\to\infty} (u, v) = (u^*, v^*) \qquad \text{uniformly in } \Omega.$$

It is interesting to ask whether (u^*, v^*) is a global attractor under all combination of the parameters $(\alpha, \beta, \lambda, \mu)$. As a matter of fact, the following open question was proposed in [29]:

Open Question: Is (u^*, v^*) asymptotically stable for all combination of α and β?

But, it seems to us that the above result and the open problem should not ignore the role of μ in their statements, which definitely plays an important part in the dynamics of solutions.

In this thesis, we prove a new global stability result for the positive equilibrium by using a novel comparison argument, which is different from the one used in literature such as [9]. Our main result [57] is:

Theorem 3. *Suppose the two diffusion coefficients are constants and* $d_1 = d_2 > 0$*, and* $(\alpha, \beta, \lambda, \mu)$ *are positive constants. Then,* (u^*, v^*) *is globally asymptotically stable if* $\mu > \beta\lambda/\alpha$*.*

In chapter five, the work is concerned with the study of the diffusive Holling-Tanner-type predator-prey system (IV) in a bounded domain $\Omega \subset R^N$ with no-flux boundary condition.

$$(IV)\begin{cases} u_t = d_1 \triangle u + au - u^2 - \frac{uv}{m+u}, & (x,t) \in \Omega \times (0,\infty) \\ v_t = d_2 \triangle v + bv - \frac{v^2}{\gamma u} & (x,t) \in \Omega \times (0,\infty) \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) > 0,\ v(x,0) = v_0(x) \geq 0 (\not\equiv 0) & x \in \bar{\Omega}. \end{cases}$$

Here $u(x,t)$ and $v(x,t)$ are the density of prey and predator, respectively, Ω is a bounded domain with smooth boundary $\partial\Omega$, $\lambda,\ \mu$, α, and β are positive constants. We assume throughout this thesis that the two diffusion coefficients d_1 and d_2 are positive and equal, but not necessarily constants. From now on, we shall use d to represent the common value. They may depend on both spatial and time variable but strictly positive in $\bar{\Omega}$. The no-flux boundary condition is imposed to guarantee that the ecosystem is not disturbed by exterior factors which may influence population flow cross the boundary, and therefore internal forces are the sole reason to generate dynamical behavior of the system.

It is easy to verify that system (IV) has a unique positive equilibrium (u^*, v^*), where

$$u^* = \frac{1}{2}(a - m - b\gamma + \sqrt{(a-m-b\gamma)^2 + 4am}), \qquad v^* = b\gamma u^*.$$

The model is a well established one to describe real ecological interactions of various populations such as lynx and hare, sparrow and sparrow hawk, see [43, 65, 67] and is widely studied in literature in recent years, see [9], [10], [19], [29], [39], [50] and [62] and the reference therein. In particular, the following result was proved by Peng and Wang in [50] by the construction of a Lyapunov function and linear analysis.

Theorem (PW). Assume that the parameters $m, a, b, \gamma, d_1, d_2$ are all positive. Then for system (IV):

1. The positive equilibrium (u^*, v^*) is locally asymptotically stable if

$$m^2 + 2(a+b)m + a^2 - 2ab\gamma > 0.$$

2. The positive equilibrium (u^*, v^*) is globally asymptotically stable if

$$m > b\gamma, \qquad \text{and} \quad (m+K)[b\gamma + 2(m + u^* + Ka)] > (a+m)b\gamma,$$

where

$$K = \frac{1}{2}\left(a - m + \sqrt{(a-m)^2 + 4a(m - b\gamma)}\right).$$

Our main result [56] for the Holling-Tanner type system is:

Theorem 4. *Suppose $d_1 = d_2 = d = d(x,t)$ is strictly positive, bounded and continuous in $\bar{\Omega} \times [0, \infty)$, a, b, γ and m are positive constants, $\gamma^{-1} > a/(m+a)$, then the positive equilibrium solution (u^*, v^*) is globally asymptotically stable in the sense that every solution to (IV) satisfies*

$$\lim_{t\to\infty}(u, v) = (u^*, v^*) \qquad \textit{uniformly in } \Omega.$$

The organization of the paper is that we introduce the autocatalytic system without decay in chapter two which contains the background, the theoretical results, numerical analysis and the detailed computational approach of the system (I). In chapter three, we will discuss the autocatalytic system with decay on (II). Chapter four and five cover the results we get for the global stability of Leslie type and Holling-Tanner type predator-prey models, respectively.

CHAPTER 2: THE AUTOCATALYTIC SYSTEMS WITHOUT DECAY

In this chapter, we study the system

$$(I)\begin{cases} u_t = u_{xx} - uv^m, & x \in R,\ t > 0 \\ v_t = dv_{xx} + uv^m, & x \in R,\ t > 0 \\ u(x,0) = 1, & v(x,0) = g(x) \quad \forall x \in R,\ t = 0. \end{cases} \tag{2.1}$$

We will start by reviewing theoretical results obtained by other authors, in particular my supervisor Dr. Qi and his collaborators on the existence and non-existence of a traveling wave solution. Then, we describe the formulation of the system in a format which is more convenient in performing computation. Finally, we present our computational results.

2.1 The Literature Review

For quadratic autocatalysis, i.e. when $m = 1$, a traveling wave solution exists if and only if $C \geq 2\sqrt{d}$, as was proved by Billingham and Needham [5]. In this case, the minimum speed is exactly $2\sqrt{d}$. For the case of $m > 1$, there was a great improvement of the bounds obtained in [5] by Chen and Qi [11]. The main results of [11] are:

Theorem 5. *(i) Suppose $d < 1$ and $m \geqslant 2$. A unique (up to translation) traveling wave solution exists for (I) if $C \geq 4d/\sqrt{1+4d}$. On the other hand, there exists no solution for if $C \leq d/\sqrt{K(m)}$, where $K(m)$ is a constant, which increases with m. In particular, $K(1) = 1/4$, $K(2) = 2$.*

(ii) Suppose $d \geqslant 1$ and $m \geqslant 1$. There exists a positive constant $C_{\min}$ such that (I) admits a traveling

wave if and only if $C \geqslant C_{\min}$. In addition, $C_{\min}$ is bounded by

$$\sqrt{\frac{d}{K(m)}} \leqslant C_{\min} \leqslant \sqrt{\frac{d}{K(m)}}\ \frac{1}{\sqrt{1-(1-\frac{1}{d})\frac{\sqrt{4K(m)+1}-1}{\sqrt{4K(m)+1}+1}}}.$$

Remark. The above result was obtained using comparison with $d = 1$ case for $d > 1$ and therefore it is a sharp bound when $0 < d - 1 \ll 1$. But, for $d < 1$, the result is shown by a direct proof and it deviates from the single equation case of $d = 1$ even for d close to 1.

2.2 Preliminary Analysis

We provide some details in the analysis of (I). Given $C > 0$, let $(u(x,t), v(x,t)) = (\alpha(z), \beta(z))$, where $z = x - Ct$. The traveling wave solution $(\alpha, \beta) \in [C^2(\mathbb{R})]^2$ satisfies

$$\begin{cases} \alpha_{zz} + C\alpha_z = \alpha\beta^m, & \alpha \geqslant 0 \qquad \forall\, z \in \mathbb{R}, \\ d\beta_{zz} + C\beta_z = -\alpha\beta^m, & \beta \geqslant 0 \qquad \forall\, z \in \mathbb{R}, \\ \lim_{z\to\infty}(\ \alpha(z), \beta(z)\ ) = (1,0), \\ \lim_{z\to-\infty}(\ \alpha(z), \beta(z)\ ) = (0,1). \end{cases} \tag{2.2}$$

First, by adding the two equations, we get

$$[\alpha_z + C\alpha + d\beta_z + C\beta]_z = 0.$$

Then, an integration yields a conservation law

$$\alpha_z + d\beta_z + C(\alpha + \beta) = 1,$$

which justifies the boundary condition at $x = \infty$ and enables us to reduce the order of traveling wave problem from four to three. Moreover, if $d = 1$, the conservation law above gives $\alpha = 1 - \beta$. The system is reduced to a classical mono-stable scalar equation

$$\beta_{zz} + C\beta_z + \beta^m(1-\beta) = 0.$$

Introducing $\omega = \beta_z$, the system (2.2) is equivalent to:

$$\begin{cases} \alpha_z = C(1-\alpha-\beta) - d\omega, \\ \beta_z = \omega, \\ \omega_z = -d^{-1}(\alpha\beta^m + C\omega), \\ \lim_{z\to\infty}(\ \alpha(z), \beta(z), \omega(z)\ ) = (1,0,0), \\ \lim_{z\to-\infty}(\ \alpha(z), \beta(z), \omega(z)\ ) = (0,1,0). \end{cases} \tag{2.3}$$

The following properties of traveling wave solutions are give by Chen and Qi [11]:

Proposition 1. *The systems (2.2) and (2.3) are equivalent. Any solution (α, β) to (2.2) or (α, β, ω) to (2.3) has the following properties:*

(1) $\alpha_z > 0 > \beta_z$*, on* $\mathbb{R}$

(2) (a) $\alpha + \beta < 1$ *on* $\mathbb{R}$ *if* $d < 1$*, (b)* $\alpha + \beta \equiv 1$ *if* $d = 1$*, and (c)* $\alpha + \beta > 1$ *if* $d > 1$

(3) $C = \int_{-\infty}^{\infty} \alpha(z)\beta^m(z)dz > 0$

(4) The equilibrium point $(0, 1, 0)$ *of (2.3) is a saddle with a two-dimensional stable manifold and a one-dimensional unstable manifold. The eigenvalues and associated eigenvectors are:*

$$\lambda_1 = -Cd^{-1}, \quad e_{\lambda_1} = (0, -1, -\lambda_1)^T$$

$$\lambda_2 = -\frac{1}{2}(\sqrt{C^2+4}+C), \quad e_{\lambda_2} = [\lambda_2(d\lambda_2+C), -1, -\lambda_2]^T$$

$$\lambda_3 = \frac{1}{2}(\sqrt{C^2+4}-C), \quad e_{\lambda_3} = [\lambda_3(d\lambda_3+C), -1, -\lambda_3]^T$$

(5) When $m > 1$, the equilibrium point $(1,0,0)$ of (2.3) is degenerate; it has a two-dimensional stable manifold and a one-dimensional center manifold. The eigenvalues and associated eigenvectors are:

$$\mu_1 = -C, \quad e_{\mu_1} = (1,0,0)^T$$

$$\mu_2 = -Cd^{-1}, \quad e_{\mu_2} = (0,1,-Cd^{-1})^T$$

$$\mu_3 = 0, \quad e_{\mu_3} = (1,-1,0)^T$$

By first making the following change of variables, we transform the third order system (2.3) to a second order system:

$$y = Cz/d, \quad A(y) = d\alpha(y)/C^2, \quad \kappa = d/C \quad and \;\; s = 1-\beta$$

with s as independent variable, we get: $\alpha = \frac{C^2}{d}A \Rightarrow \alpha_z = \frac{C^2}{d}A_z$. then

$$\omega = \beta_z = \frac{d\beta}{dz} = \frac{d\beta}{ds}\frac{ds}{dz} = \frac{d\beta}{dx}\frac{ds}{dy}\frac{dy}{dz} = -\frac{ds}{dy}\frac{C}{d} = -\frac{C}{d}s_y.$$

Using the change of variables for $\alpha,\ \beta,\ \alpha_z,\ and\ \omega$, we get the following:

$$\frac{C^3}{d^2}A_y = C[1-\frac{C^2}{d}A-(1-s)] - d(-\frac{C}{d}s_y)$$

implies

$$A_y = -dA + \frac{d^2}{C^2}(s+s_y) = \kappa(s+s_y) - dA.$$

Also since,

$$\omega_z = \beta_{zz} = \frac{d}{dz}(\frac{d\beta}{dz}) = \frac{d}{dz}(-\frac{C}{d}s_y) = -\frac{C}{d}(\frac{d}{dz}(\frac{ds}{dy})) = -\frac{C}{d}(\frac{d^2s}{dy^2}\frac{dy}{dz}) = -\frac{C^2}{d^2}s_{yy},$$

we will have

$$-\frac{C^2}{d^2}s_{yy} = -d^{-1}\left[\frac{C^2}{d}A(1-s)^m - \frac{C^2}{d}s_y\right],$$

which implies

$$s_{yy} + s_y = A(1-s)^m \;\; on \;\; \mathbb{R}.$$

Now, we use s as independent variable. Let $P(s) = s_y$, then

$$s_{yy} = \frac{dP}{ds}\frac{ds}{dy} = P'P \; and \; A_y = A(s)_y = \frac{dA}{ds}\frac{ds}{dy} = PA'.$$

Hence we get the system

$$\begin{cases} PP' = A[1-s]^m - P & \forall s \in [0,1], \\ PA' = \kappa^2[P+s] - dA & \forall s \in [0,1], \\ P(s) > 0, \; A(s) > 0 & \forall\, s \in (0,1), \\ P(0) = 0, \; A(0) = 0. \end{cases} \tag{2.4}$$

It is clear that (A, P) is a traveling wave of (2.4) if and only if $P(s) \searrow 0$ as $s \nearrow 1$.

The formulation in (2.4) of traveling wave problem as a second order ODE system in phase plane gives us an alternative way to design numerical schemes. Let

$$P(s) = \lambda s + a_1 s^2 + O(s^3), \quad A(s) = \lambda(1+\lambda)s + b_1 s^2 + O(s^3) \qquad \text{for } 0 < s \ll 1,$$

after simple computation, we can get that

$$\lambda = \frac{\sqrt{4\kappa^2 + d^2} - d}{2}, \quad a_1 = \frac{b_1(d + 2\lambda)}{\lambda(d-1)}, \quad b_1 = -\frac{m\lambda^2(1+\lambda)}{6\lambda^2 + 2\lambda d + 3\lambda + d}.$$

To derive $P(s) = \lambda s + a_1 s^2 + O(s^3)$ and $A(s) = \lambda(1+\lambda)s + b_1 s^2 + O(s^3)$, first let

$$P(s) \approx c_1 s + O(s^2) \quad and \quad A(s) \approx c_2 s + O(s^2),$$

such that

$$P'(s) \approx c_1 \quad and \quad A'(s) \approx c_2.$$

We substitute P, P' and A into the first equation of (2.4), we get

$$c_1^2 s \approx c_2 s(1-s)^m - c_1 s,$$

which implies

$$c_1^2 + c_1 - c_2 = 0.$$

This gives us that

$$c_1 = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4c_2} \;\; or \;\; c_2 = c_1^2 + c_1.$$

Again, we substitute P, P' and A into the second equation of (2.4), we get

$$c_1 c_2 s = \kappa^2(c_1 s + s) - d c_2 s$$

such that

$$c_2(c_1 + d) = \kappa^2(c_1 + 1).$$

Since $c_2 = c_1^2 + c_1$ from the previous step, we will have

$$(c_1^2 + c_1)(c_1 + d) = \kappa^2(c_1 + 1),$$

which implies that

$$c_1 = \frac{-d \pm \sqrt{d^2 + 4\kappa^2}}{2}.$$

We know that $P(s) > 0$ for all $s \in (0, 1)$ and $P(s) \approx c_1 s$, then we have

$$\lambda = c_1 = \frac{1}{2}(\sqrt{d^2 + 4\kappa^2} - d),$$

and

$$c_2 = \lambda(\lambda + 1) \quad for \quad A(s) = \lambda(\lambda + 1)s + O(s^2).$$

No we have

$$P(s) = \lambda s + a_1 s^2, \quad A(s) = \lambda(1 + \lambda)s + b_1 s^2 \qquad \text{for } 0 < s \ll 1, \ \lambda = \frac{\sqrt{4\kappa^2 + d^2} - d}{2},$$

we can use the similar process to calculate a_1 and b_1, such that we get

$$a_1 = \frac{b_1(d + 2\lambda)}{\lambda(d - 1)}, \quad b_1 = -\frac{m\lambda^2(1 + \lambda)}{6\lambda^2 + 2\lambda d + 3\lambda + d}.$$

The primary concern for our numerical analysis is: what is the minimum speed for the system? In next section, we will use numerical analysis to find the specific minimum speed for different cases of systems. The importance of the minimum speed is that for mono-stable type of problems, the minimum speed of traveling wave which is most relevant for the study of stability.

2.2 Numerical Analysis

The theoretical results show there exist bounds for the traveling wave speed and the existence and non-existence of the traveling wave under certain conditions. In this section, we use numerical analysis to find the minimum speeds for different cases of d and m for the system without decay. To fill in the gap of the theoretical study, we use computational methods to gives more accurate estimates on minimum speed of traveling waves for autocatalytic reaction without decay, also providing useful insight in the study of stability of traveling waves. We have done extensive computation on two cases, $m = 2$ and $m = 2.5$ to determine C_{min} for various values of d. The method works well for wide range of d value from d fairly small such as $d = 0.1$ to rather big value of $d = 5$. Then we also use regression analysis try to catch the dependence of C_{min} on d analytically. By using a unique traveling wave solution (P, A) in (2.4), from the previous section we have that

$$P(s) = \lambda s + a_1 s^2, \quad A(s) = \lambda(1+\lambda)s + b_1 s^2 \qquad \text{for } 0 < s \ll 1,$$

with

$$\lambda = \frac{\sqrt{4\kappa^2 + d^2} - d}{2}, \quad a_1 = \frac{b_1(d + 2\lambda)}{\lambda(d-1)}, \quad b_1 = -\frac{m\lambda^2(1+\lambda)}{6\lambda^2 + 2\lambda d + 3\lambda + d}.$$

Our numerical scheme has the following key ingredients:

(i) It uses the above asymptotic expansion at $s = 0$ as initial input for $0 < s \ll 1$ and then use Matlab to compute the solution up to $s = 1$.

(ii) The algorithm in Matlab is the explicit fourth-order Runge-Kutta method. The criterion to judge whether the resulting solution is a traveling wave is to check whether $P(s) > 0$ on $(0, 1)$ and $|P(1)|$ is less than a preset upper bound of the order 10^{-6}.

(iii) All results were checked and confirmed by using double-precision Mathematica.

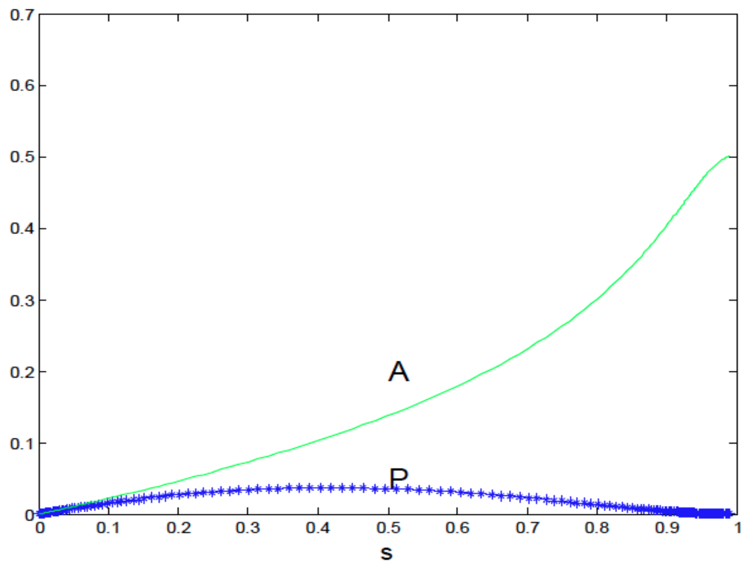

Figure 2.1: The results for traveling wave solutions (A, P) with $C_{min} = 0.0989$, $m = 2$ and $d = 0.1$.

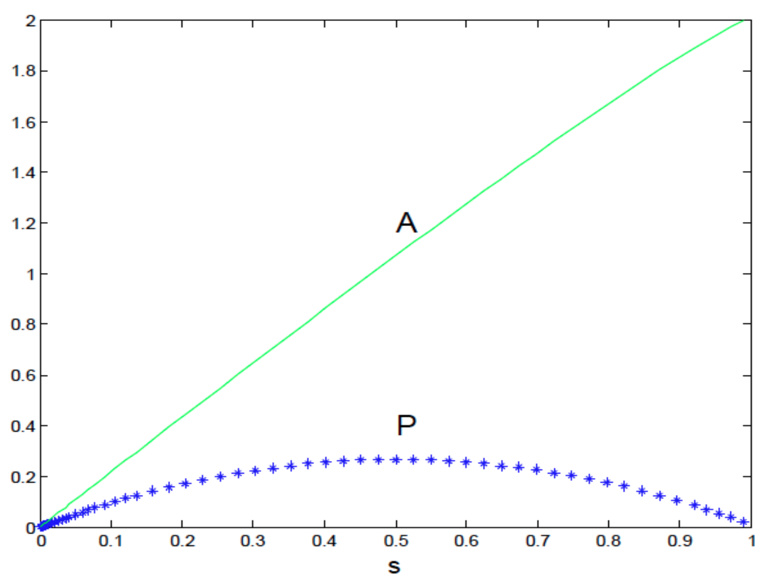

Figure 2.2: The results for traveling wave solutions (A, P) with $C_{min} = 0.7749$, $m = 2$ and $d = 1.2$.

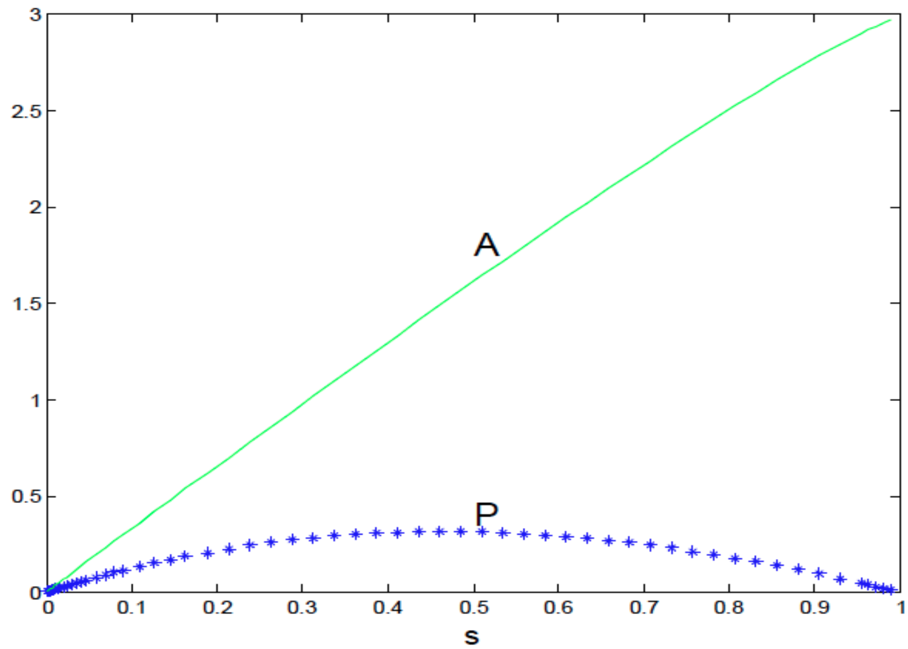

Figure 2.3: The results for traveling wave solutions (A, P) with $C_{min} = 0.6345$, $m = 2.5$ and $d = 1.2$.

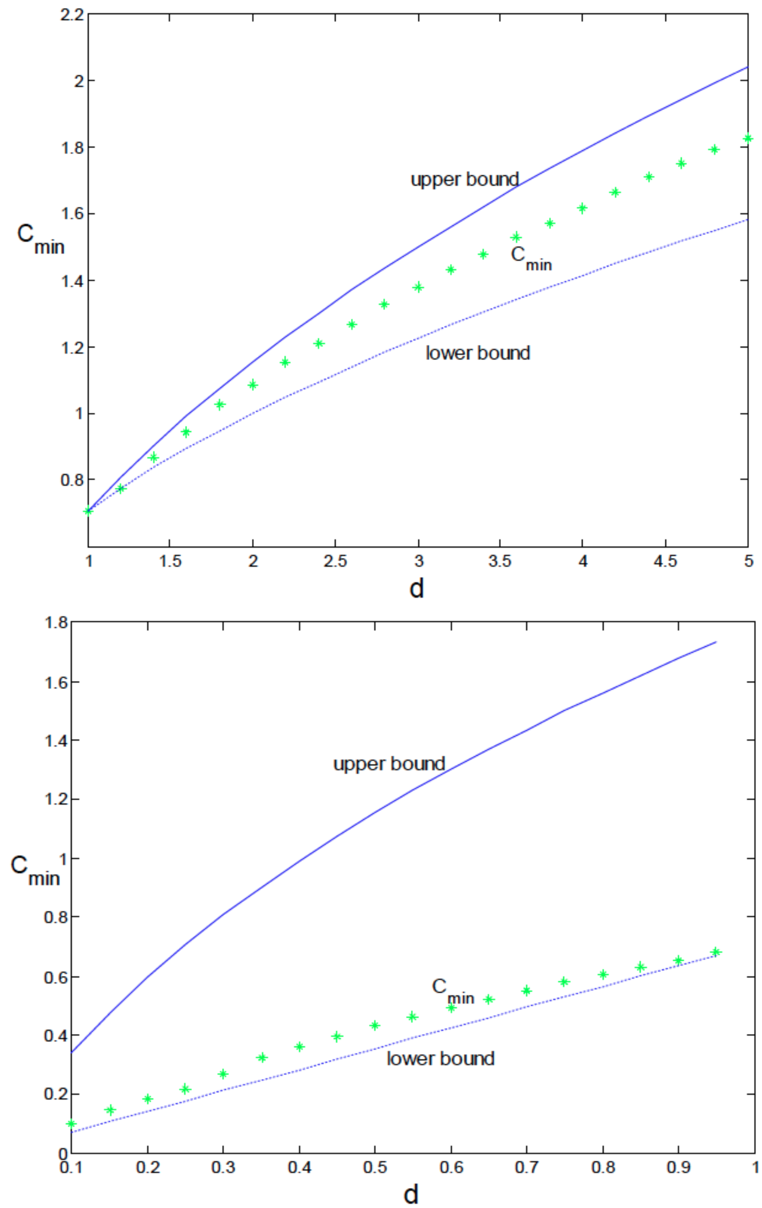

Figure 2.4: We compare the numerically computed C_{min} with the lower bound, shown as blue line and upper bound, shown as blue "–" from Theorem 1 for $d > 1$ and $d < 1$, respectively. It is clear that for $d > 1$, it demonstrates a good match of theoretical result with numerical computation. But, for $0 < d < 1$, the numerical results shows the minimum speed is far below that of theoretical result pointing out further refinement is needed to improve the estimate theoretically.

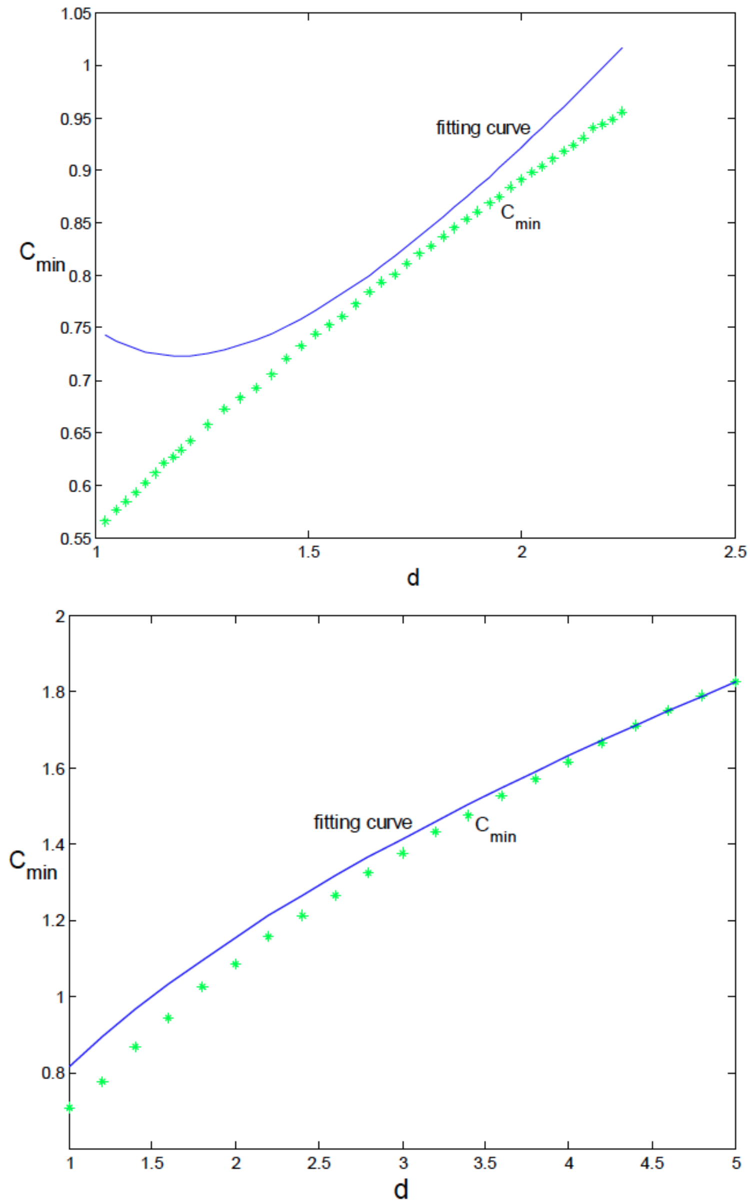

Figure 2.5: We also did Regression Analysis trying to catch the dependence of C_{min} on d analytically. By using a curve of the form $C_1\sqrt{d} + C_2 + \frac{C_3}{d} + \frac{C_4}{d^2}$. The best fitting curve when $m = 2.5$ is $\frac{1}{2}\sqrt{d} - \frac{1}{5} + \frac{1}{2d} - \frac{1}{20d^2}$. The best fitting curve when $m = 2$ is $\frac{2.5}{3}\sqrt{d}$.

Also, we will present some computational results on (2.1) with two special cases $m = 1$ and $m = 2$. The purpose is two folds. On one hand, computation can verify and confirm analytical results, in particular, whether the spreading of local disturbance of v is of order $O(\sqrt{d})$ when $d > 1$. On the other hand, it can help us to gain insight into the complex interaction of diffusion and nonlinear reaction terms and how their interaction determines the behavior of solutions.

In all examples of computation, the initial conditions are $u(x, 0) = 1$ and $v(x, 0)$ has a compact support. We take the spatial domain to be a large interval centered at zero and use periodic boundary conditions.

Figure 2.6 is the result of computation of $m = 1$, $d = 2$ with initial condition of v to be $v(x, 0) = 1$ in $[-1, 1]$, and zero otherwise. The spatial domain is $[-40, 40]$. The reaction starts from the central region and spreads out with the speed C, which is approximately $2\sqrt{d}$, the minimum speed, before v becomes very flat, approaching 1. This is in agreement with the theoretical result of [12].

Figure 2.7 is the result of computation of $m = 2$ with the other conditions same as the above case. The reaction again starts from the central region and spread out with the estimated speed of $2.5\sqrt{d}/3$, before v becomes very flat, approaching 1 as time $t \gg 1$.

Figure 2.8 is the result of computation of $m = 2$ and $d = 4$ with other conditions same as the above case. The reaction again starts from the central region and spread out with the estimated speed of $2.5\sqrt{d}/3$, before v becomes very flat, approaching 1 as time $t \gg 1$.

Figure 2.9 is the result of computation of $m = 1$ and $d = 3$ with initial condition of u to be

$$u(x, 0) = \frac{\pi}{2} \sin\left(\frac{\pi}{100}(50 + x)\right), \quad -50 < x < 50$$

and the spatial domain is $[-50, 50]$. The reaction again starts from the central region and spread out with the estimated speed of in the range of $7\sqrt{d}/12 < C < 3\sqrt{d}/4$, (with other values of $d > 1$ also computed to confirm the range). But instead of converging to 1, v converges to a fixed bell-shaped profile as time $t \gg 1$. In addition, u becomes two-hump from the initial one-hump and keep the same profile with diminished height as t increases before eventually tending to zero.

Figure 2.10 is the result of computation of $m = 2$ and $d = 3$ with other conditions same as the above case. The solutions demonstrate the same kind of qualitative behavior as the above case except the speed range is in $7\sqrt{d}/12 < C < 5\sqrt{d}/6$.

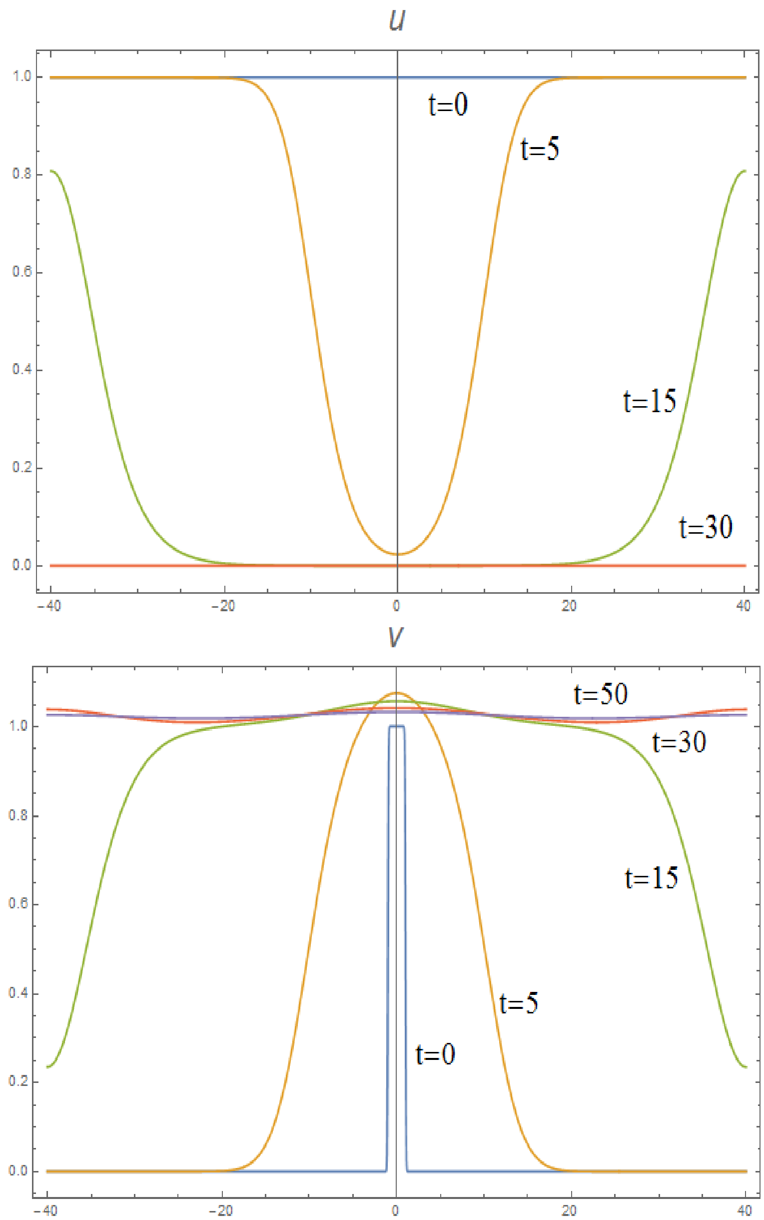

Figure 2.6: System (2.1) with $m = 1$ and $d = 2$

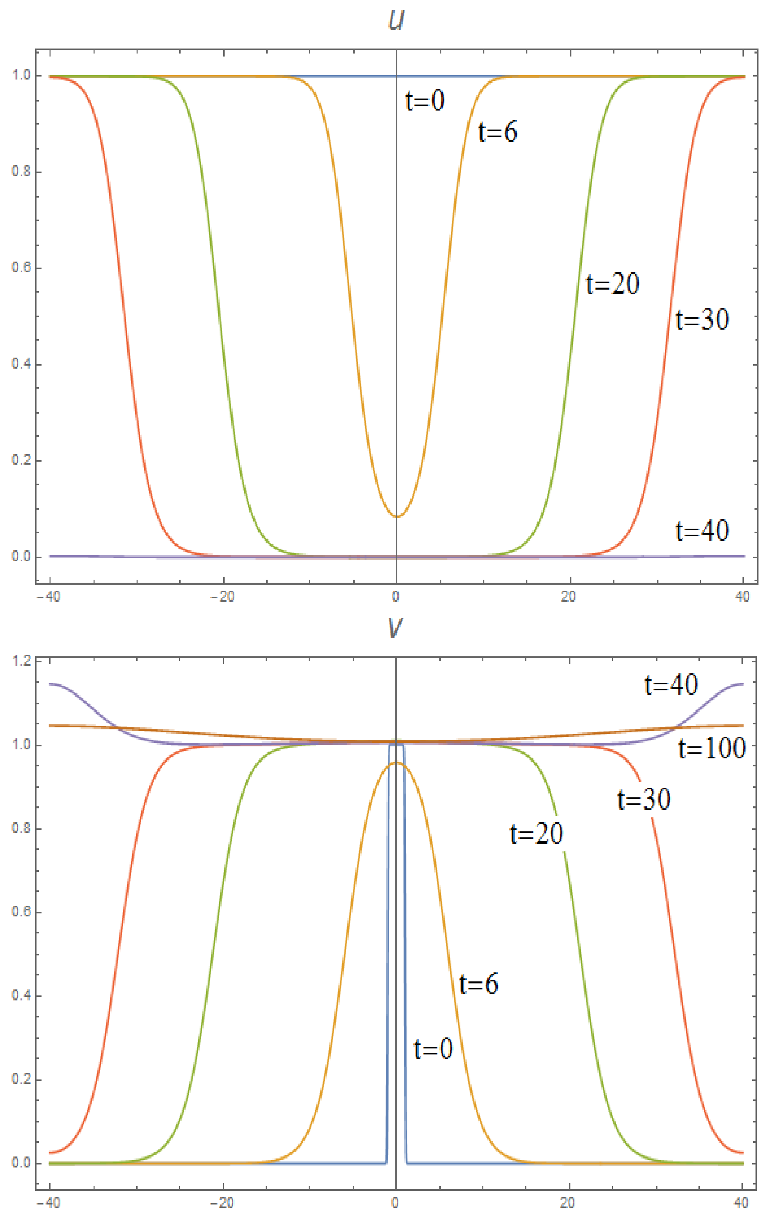

Figure 2.7: System (2.1) with $m = 2$ and $d = 2$

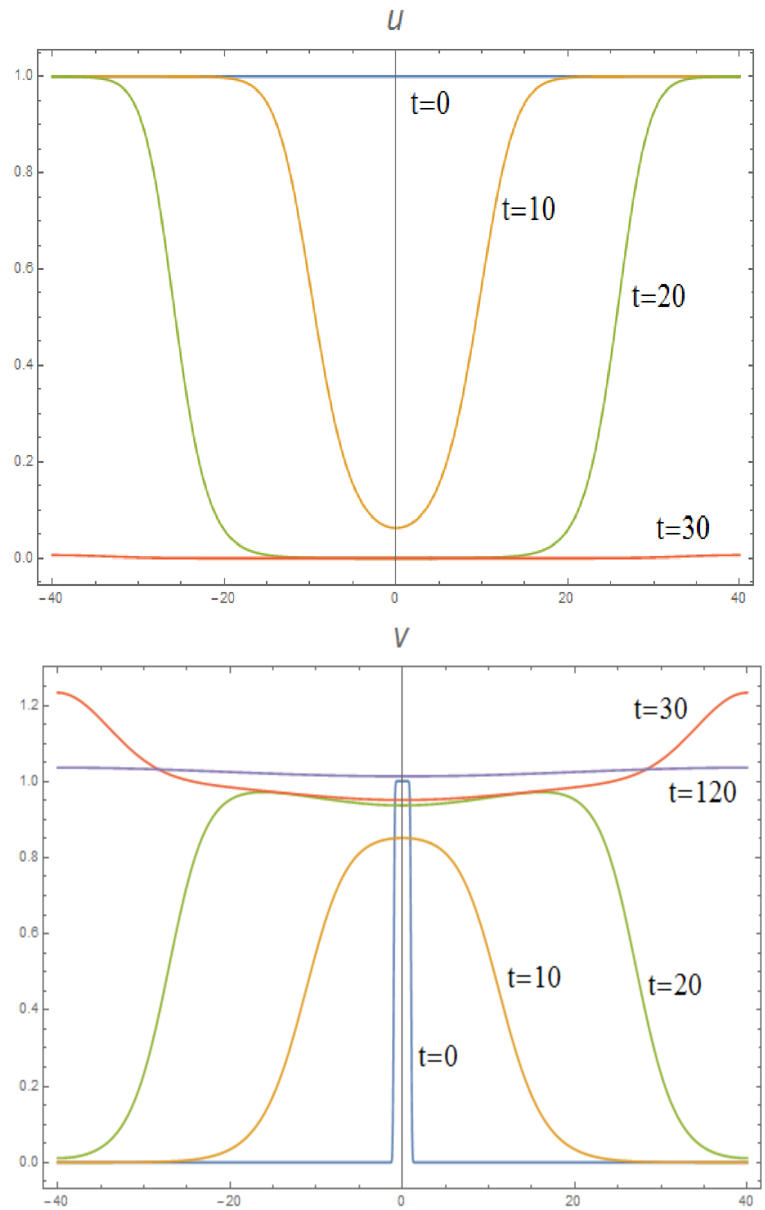

Figure 2.8: System (2.1) with $m = 2$ and $d = 4$

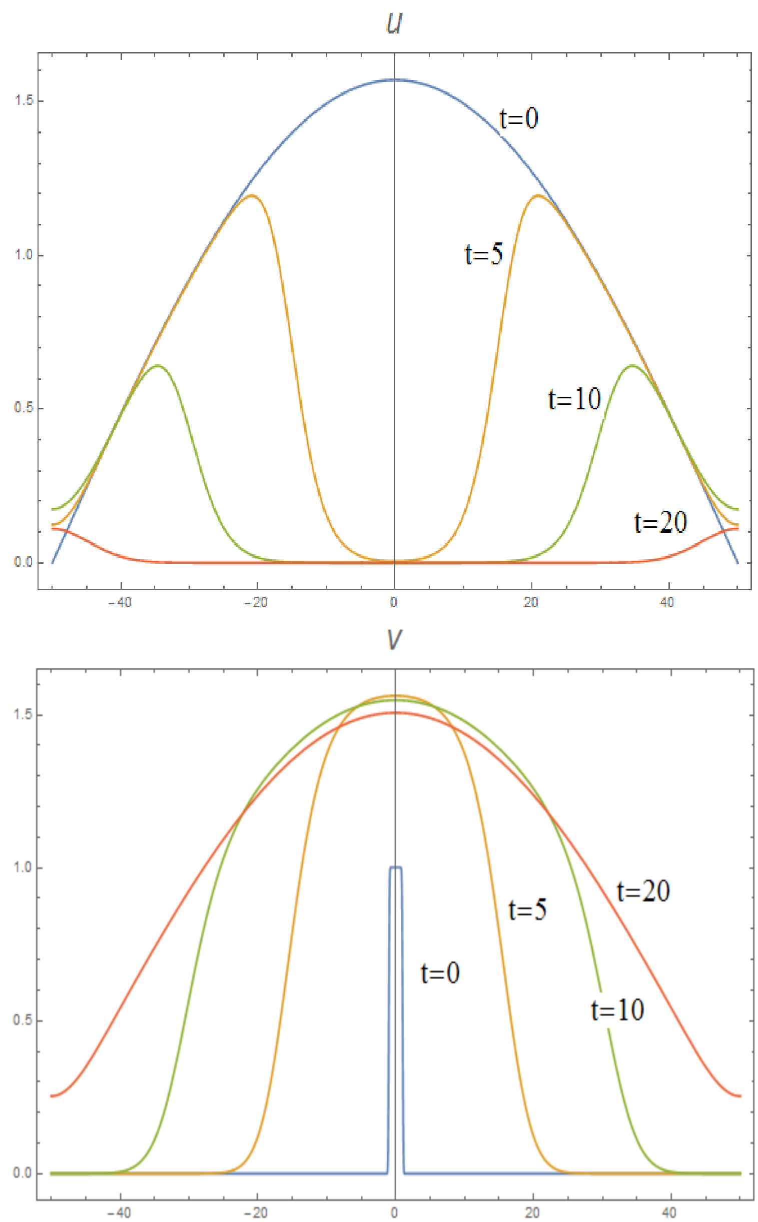

Figure 2.9: System (2.1) with $m = 1$ and $d = 3$

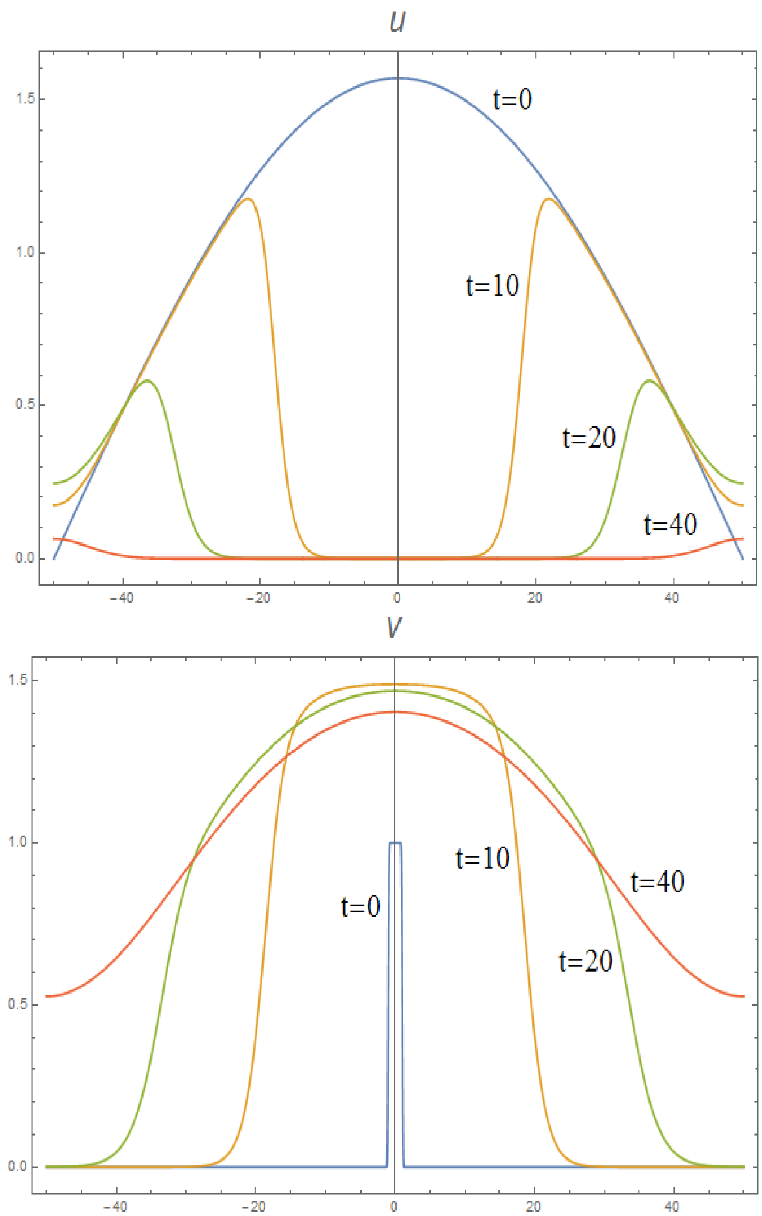

Figure 2.10: System (2.1) with $m = 2$ and $d = 3$

2.3 Summary and Algorithms

Numerical computation is a powerful tool in understanding complex solution structure of traveling wave problems. The result is giving good insight into the problem which provides good lead in our further analysis. It can help us to get some idea about the complex interaction of diffusion and nonlinear reaction terms and how their interaction determines the behavior of solutions. There are many methods to solve an ordinary differential equations system, such as the forward Euler method toward higher-order methods, linear multistep methods (MS) and Runge-Kutta methods (RK). The method we applied here is the explicit fourth-order Runge-Kutta Method and the reason is that RK methods maintain the structure of one-step methods, and increase their accuracy at the price of an increase of functional evaluations at each time level. A consequence is that RK methods are more suitable than MS methods at adapting the stepsize, provided that an efficient estimator of the local error is available.

CHAPTER 3: THE AUTOCATALYTIC SYSTEMS WITH DECAY

In this chapter, we consider

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} - uv^m, \\ \\ \dfrac{\partial v}{\partial t} = D\dfrac{\partial^2 v}{\partial x^2} + uv^m - kv^l. \end{cases} \tag{3.1}$$

It models chemical reaction of the form

$$A + mB \to (m+1)B \qquad \text{with rate } uv^m \text{ and}$$

$$B \to C \qquad \text{with rate } kv^l$$

with C an inert chemical species. Here, D, a positive constant, is the ratio of diffusion coefficients of chemical species B to that of A, $m \geq 1$ is a positive constant not necessarily an integer, k and $l \geq 1$ are both positive constants. We assume throughout that $1 \leq l < m$.

First, we will review the previous theoretical results obtained by other authors and discuss the existence and non-existence of a traveling wave solution. Then, we will show how to use computational method to reveal the complex solution structure of traveling wave solution to system (3.1).

3.1 The Literature Review

Many models in mathematical biology take the form of system (3.1), see [23, 49, 63]. In particular, $m = 2$ and $l = 1$ is the famous Gray-Scott model in biological pattern formation, one of the popular models proposed for replicating experimental results in early 1990's, see [42, 49]. The most exciting feature of the diffusive Gray-Scott system with feeding is self-replicating traveling pulse (traveling wave). It has been extensively studied, see [16, 17, 34], but the phenomenon and underlying mechanism is not completely understood.

Our main concern is using computational method to show the existence and stability of traveling wave to (3.1). A traveling wave solution for (3.1) links one equilibrium point to another. Since any equilibrium point of the system is of the form $(a, 0)$, $a \in \mathbb{R}$. Hence, any traveling wave $u(x,t) = u(x - Ct)$, $v(x,t) = v(x - Ct)$, with $C > 0$ as the speed, must link one equilibrium point $(a_0, 0)$ at $x = -\infty$ to another one $(a_1, 0)$ at $x = \infty$ with $a_1 > a_0 > 0$. Thus, we consider traveling wave problem

$$\begin{cases} u'' + Cu' = uv^m, & u' > 0 \quad \text{in } \mathbb{R}, \\ Dv'' + Cv' = kv^l - uv^m, & v > 0 \quad \text{in } \mathbb{R}, \\ u(-\infty) = u_0, \quad v(-\infty) = 0, & v(\infty) = 0, \quad u(\infty) < \infty. \end{cases} \tag{3.2}$$

The important implications of such a setting are (i) v has no monotonicity which is a strong contrast to the auto-catalytic chemical reaction without decay, (ii) any equilibrium point $(a, 0)$ with $a > 0$ is a saddle point, and (iii) there is no corresponding single equation to compare with. Indeed, it is not too hard to show that v is increasing coming out of $x = -\infty$ and reaches its 1st local maximum value and then it starts to decrease and may oscillate a few times. Whereas $u' > 0$ in $\mathbb{R}$ for a traveling wave. In addition, define

$$x_1(C) := \sup\{z \in \mathbb{R} \mid v' > 0 \text{ in } (-\infty, z)\}, \qquad x_2(C) := \sup\{z \in \mathbb{R} \mid v > 0 \text{ in } (-\infty, z)\}. \tag{3.3}$$

For each constant $C \geqslant 0$, we consider the initial value problem, for $(u, v) = (u(x, C), v(x, C))$,

$$\begin{cases} u'' + Cu' = uv_+^m & \text{in } \mathbb{R}, \\ Dv'' + Cv' - v = -uv_+^m & \text{in } \mathbb{R}, \\ [u, v] = [h, e^{\lambda x}] + O(1)e^{m\lambda x} & \text{as } x \to -\infty, \end{cases} \tag{3.4}$$

where λ is the positive root of $\lambda^2 d + C\lambda = 1$ and $v_+ := \max\{v, 0\}$. We denote

$$\begin{aligned} \mathcal{A} &:= \{C \geqslant 0 \mid x_2(C) < \infty\}, \\ \mathcal{B} &:= \{C \geqslant 0 \mid x_2(C) = \infty, \lim_{x\to\infty} u(x, C) = \infty\}, \\ \mathcal{C} &:= \{C \geqslant 0 \mid x_2(C) = \infty, \lim_{x\to\infty} u(x, C) < \infty\}. \end{aligned} \tag{3.5}$$

The approach in [14] is to use speed C as a shooting parameter. In fact, the following is a summary of key technical results in [14].

Lemma 1. *Suppose $d, h > 0$ are fixed. For each $c \geqslant 0$, problem* (3.4)*, with $\lambda = (\sqrt{C^2 + 4d} - C)/2d$ and $v_+ := \max\{v, 0\}$, admits a unique solution. The solution depends on C continuously and satisfies $u' > 0$ in $\mathbb{R}$. In addition, $x_1(C) < \infty, v'(x_1(C), C) = 0 > v''(x_1(C), C)$, and one and only one of the following holds:*

(1) $x_2(C) < \infty$ *and* $v(x_2(C), C) = 0$;

(2) $x_2(C) = \infty$ *and* $\lim_{x\to\infty} u(x, C) = \infty$*;*

(3) $x_2(C) = \infty$ *and* $\lim_{x\to\infty} u(x, C) < \infty$*. In this case,* $\lim_{x\to\infty} v(x, C) = 0$*, so* (C, u, v) *solves* (3.2).

Moreover, $\mathcal{A}$ and $\mathcal{B}$ are open, $0 \in \mathcal{A}$, and $[M, \infty) \subset \mathcal{B}$ for some $M \gg 1$. Thus, $\mathcal{C}$ is non-empty and problem (3.2) *admits a solution for some $C > 0$.*

The work in [15] is on the case of $h \gg 1$. By make the following change of scale and variables:

$$\varepsilon = h^{-\frac{m}{m-1}}, \quad u = [1+\varepsilon a]h, \quad v = h^{-\frac{1}{m-1}} b, \quad C = c\varepsilon,$$

(3.4) is transformed and the existence of traveling wave is equivalent to finding $(a, b, c) \in C^2(\mathbb{R}) \times C^2(\mathbb{R}) \times (0, \infty)$ which satisfy

$$\begin{cases} a'' + c\varepsilon a' = [1+\varepsilon a]b^m, & a' > 0 \qquad\qquad \text{in } \mathbb{R}, \\ db'' + c\varepsilon b' = b - [1+\varepsilon a]b^m, & b > 0 \qquad\qquad \text{in } \mathbb{R}, \\ a(-\infty) = 0, \quad b(-\infty) = 0, & a(\infty) = 0, \quad b(\infty) < \infty. \end{cases} \tag{3.6}$$

Our computational results are based on the following Theorem proved by Chen, Qi and Zhang [14] for the system (3.1).

Theorem 6. *Let $m > 1$ and $D > 0$ be given constants.*

1. *There exist positive constants M_1, M_2, and M_3 that depend only on m and D such that for each $\varepsilon > 0$,* (3.6) *admits no solution if $c \geqslant \max\{\sqrt{M_1/\varepsilon}, M_2\}$ or if $c \leqslant \gamma - M_3\varepsilon$.*

2. *For each sufficiently small positive ε and each integer L satisfying $1 \leqslant L \leqslant \varepsilon^{-1/4}$, there exists a constant $c_L = L\gamma[1 + O(\varepsilon + [L-1]^2\varepsilon|\ln\varepsilon|)]$ such that when $c = c_L$, the system* (3.6) *admits a solution, unique up to a translation. The solution is an L*-**hump solution** *in the sense that $w := [1+\varepsilon u]v^{m-1}$ admits exactly L local maxima and $L-1$ interior local minima. In addition, if denote the interior points of local minima of w by $\{a_i\}_{i=2}^{L-1}$ and points of local maxima by $\{b_i\}_{i=1}^{L}$ with $-\infty = a_1 < b_1 < a_2 < b_2 < \cdots < b_L < a_{L+1} = \infty$, then*

$$w(b_i) = M + O(i[L+1-i]\varepsilon), \;\; G(w(a_{i+1})) = i(L-i)\sigma\gamma\varepsilon + O(i^2L^2\varepsilon^2|\ln\varepsilon|) \;\; \forall\, i = 1, \cdots, L.$$

Furthermore, $\|w'^2 - G(w)\|_{L^\infty(\mathbb{R})} = O(L^2\varepsilon)$ *and*

$$\lim_{\varepsilon\searrow 0} w(b_i+z) = \lim_{\varepsilon\searrow 0} v(b_i+z) = W(z)$$

uniformly in $i = 1, \cdots, L$ *and locally uniformly in* $z \in \mathbb{R}$*, where* W *is the unique solution of*

$$W'' = W - W^m \quad \textit{in } \mathbb{R}, \qquad W(0) = M, \qquad W'(0) = 0, \tag{3.7}$$

where

$$G(s) = s^2 - \frac{2s_+^{m+1}}{m+1}, \;\; \alpha = \frac{1}{m-1}, \; M = \Big(\frac{m+1}{2}\Big)^{\alpha}, \;\; \sigma = 4\int_0^M \sqrt{G(s)}ds, \;\; \gamma = \frac{2\alpha}{D}\int_0^M \frac{s^m ds}{\sqrt{G(s)}}, \tag{3.8}$$

$s_+ = \max\{s, 0\}$.

3.2 The Computational Approach

The numerical computation for (3.2) is to catch the corresponding traveling wave with one, two and three peaks of w, respectively. We also did some computation for (3.1). In all Figures 3.6-3.8, the initial conditions are $u(x, 0) = 1$ and $v(x, 0)$ has a compact support with v to be $v(x, 0) = 1$ in $[-1, 1]$ and zero otherwise. We take the spatial domain to be a large interval centered at zero and use periodic boundary conditions. The result not only verifies the mathematical proof, but also provides more detailed information about the solutions. But, the difficulty is that we need to integrate the solutions with high order nonlinearities over an extended interval. We use numerical analysis to verify theoretical results for various cases of c_L using Matlab, which implements explicit fourth-order Runge-Kutta method for the computation. To make sure the computation is accurate, we check the results by using the double precision build-in solver NDsolve from Mathematica.

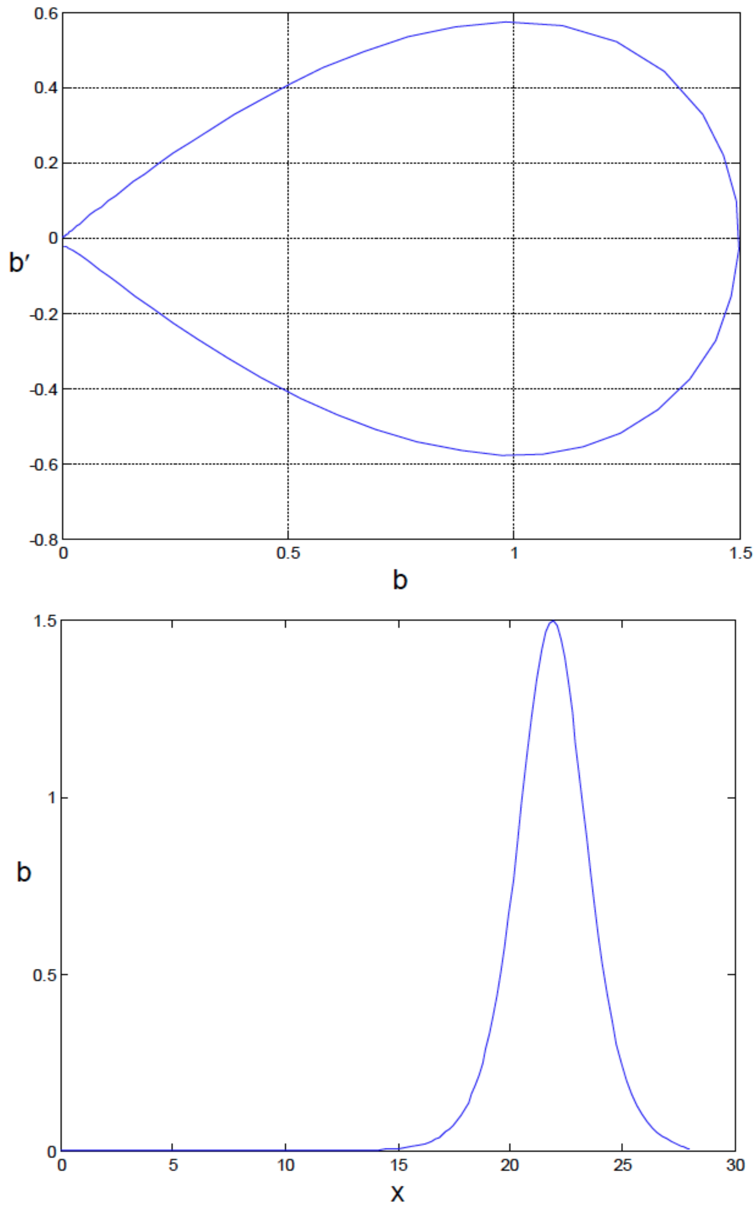

Figure 3.1: This represents the numerical result for (3.6) for 1-peak solution $w = (1 + \varepsilon a)b$ where $\varepsilon = 0.00025$, $c = c_1 = \gamma = 6$ and the integration interval is $[0, 28]$. As shown in the figure, the solutions only have one peak on the $b - b'$ phase plane.

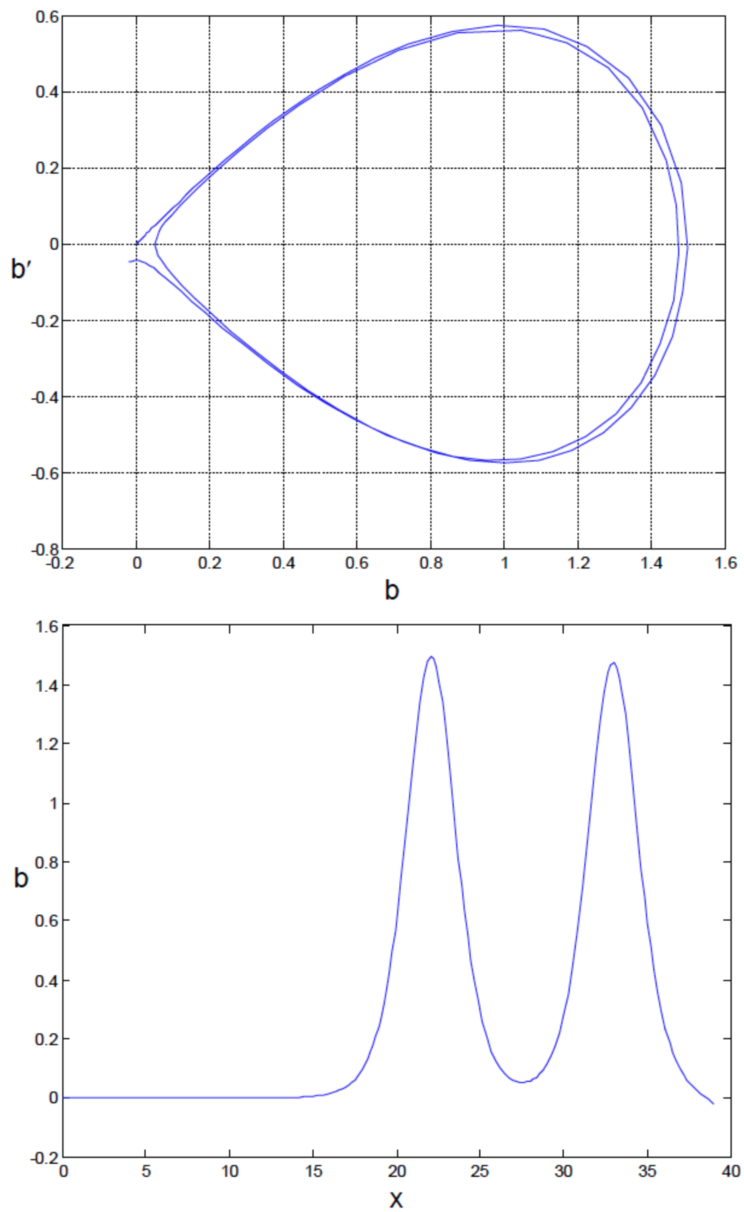

Figure 3.2: This is the results for (3.6) for 2-peak solution $w = (1 + \varepsilon a)b$, where $\varepsilon = 0.00025$, $c = c_2 = 2\gamma = 12$ and the integration interval is $[0, 39]$

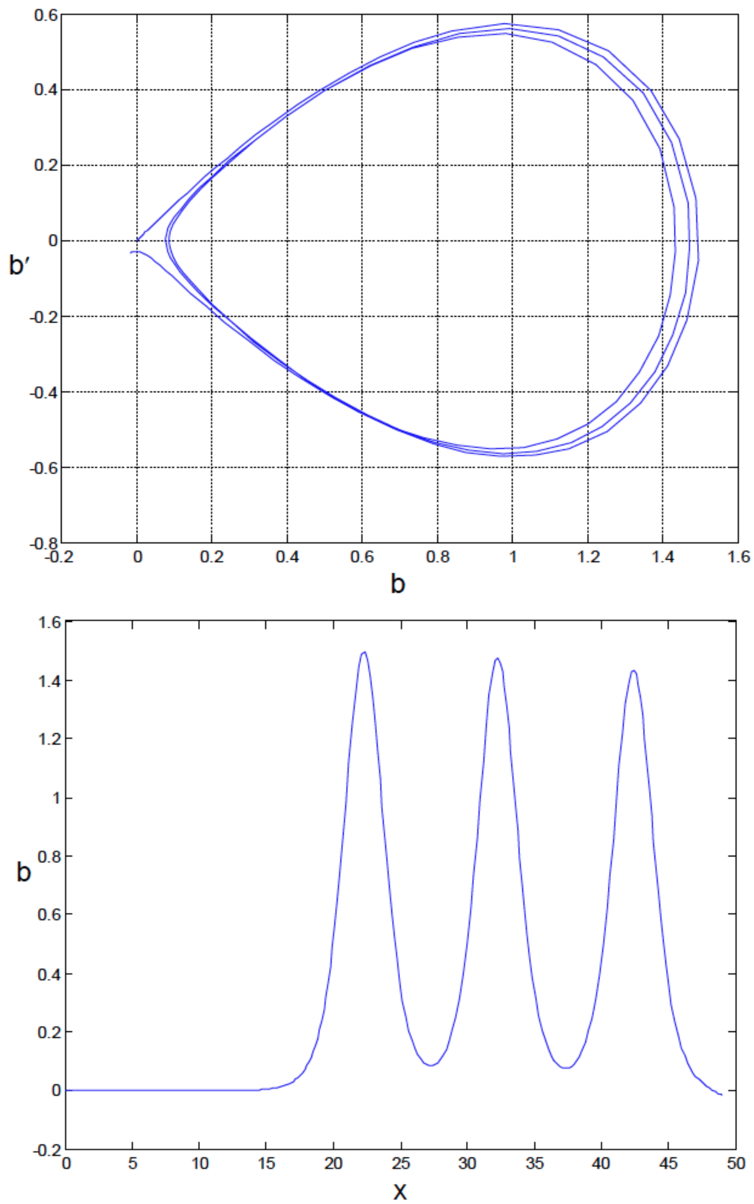

Figure 3.3: This is the results for (3.6) for 3-peak solution $w = (1 + \varepsilon a)b$, where $\varepsilon = 0.00025$, $c = c_3 = 3\gamma = 18$ and the integration interval is $[0, 50]$. The three-peak solutions are as expected on the $b - b'$ phase plane.

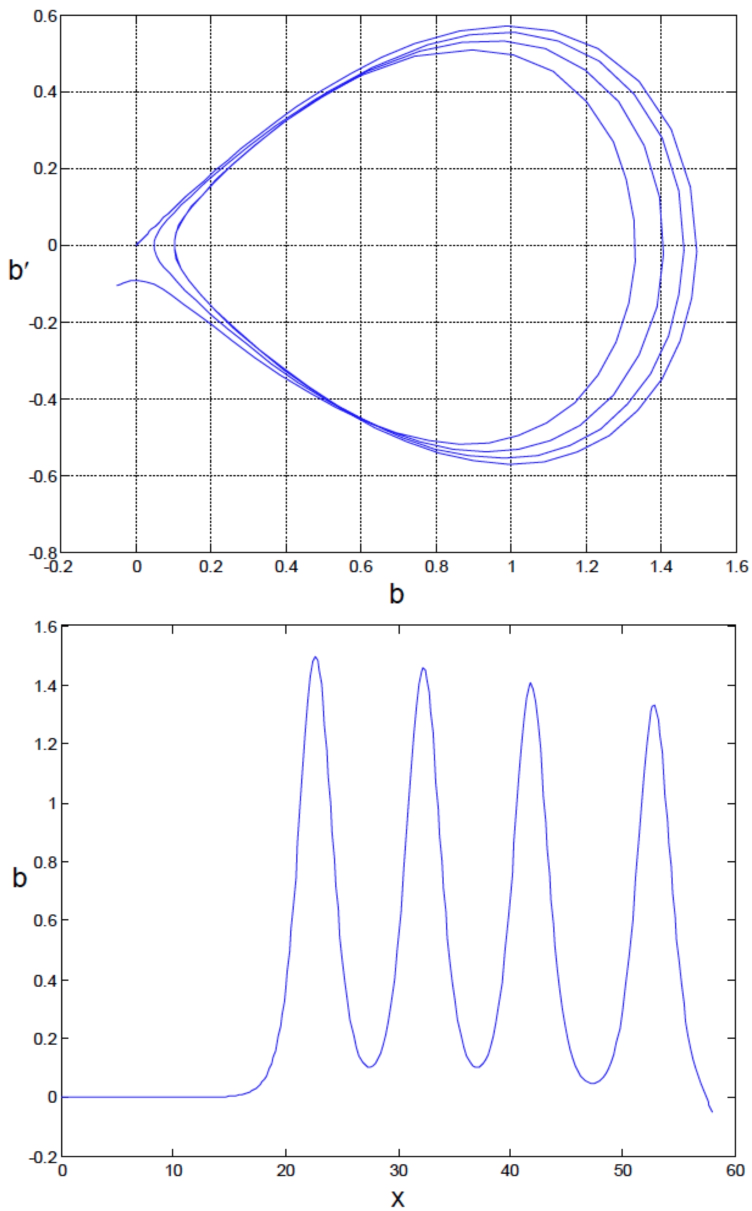

Figure 3.4: This shows the computation with $\varepsilon = 0.00038$, $c = c_3 = 3\gamma = 18$ on the finite interval $[0, 58]$.

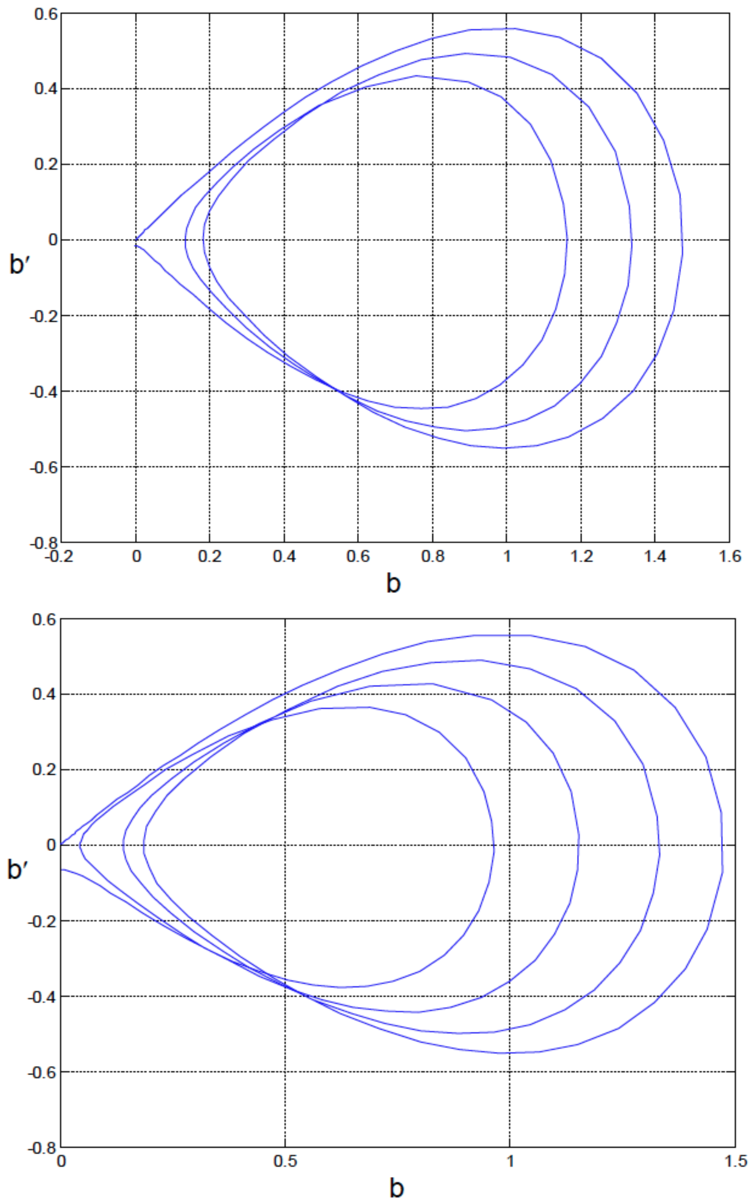

Figure 3.5: $\varepsilon = 0.0023$, $c = 12$ on the interval $[0, 55]$. A small change of ε from the setting of figure 3.1 results in 4-peak solution with the same speed. Moreover, this shows in $b - b'$ phase plane that small change in ε with same speed C gives different types of solutions.

In Figures 3.6 to 3.8, we present some computational results on (3.1) with $m = 2$, $l = 1$, $D = 4$ and the same initial conditions where $u(x, 0) = 1$ and $v(x, 0)$ has a compact support with v to be $v(x, 0) = 1$ in $[-1, 1]$ and zero otherwise. They show that when $k = 1$ and $k = 0.2$, the decay is very strong and v tends to zero very fast before any pattern to form effectively. But, for $k = 0.05$, v undergoes some very interesting evolution before decaying to zero eventually.

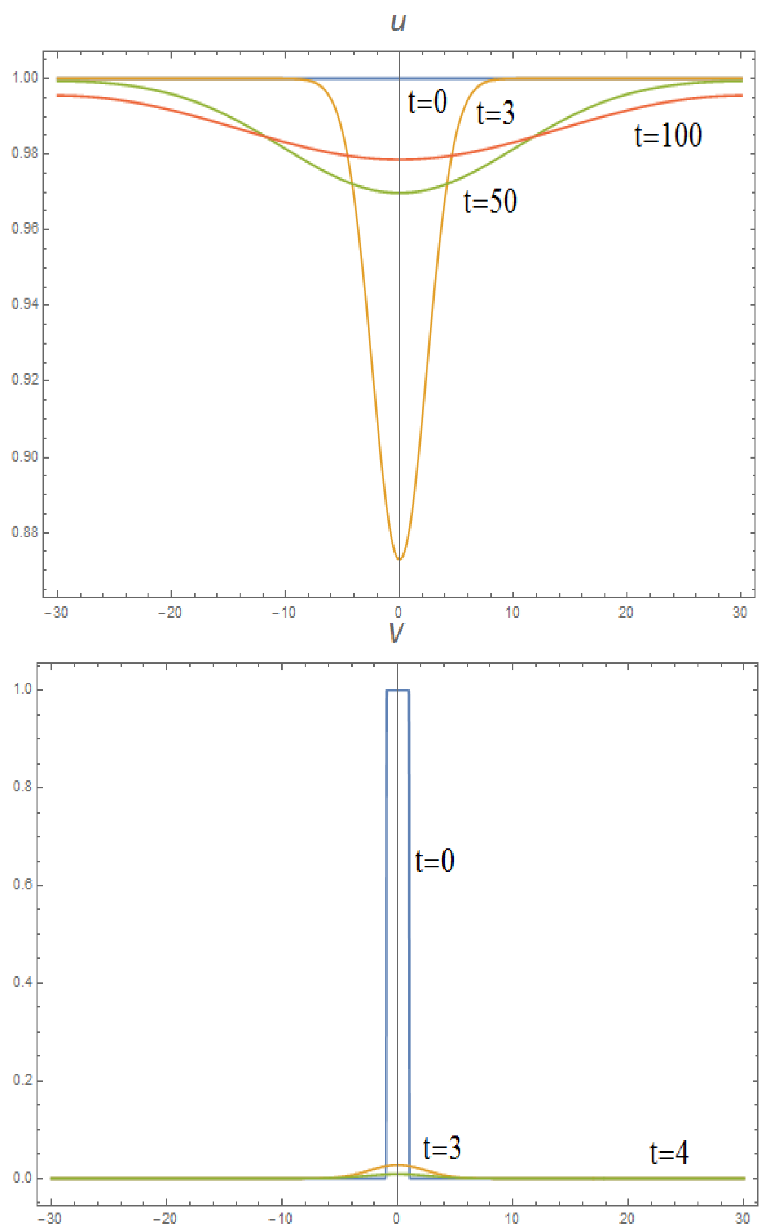

Figure 3.6: System (3.1) with $m = 2$, $l = 1$, $D = 4$ and $k = 1$

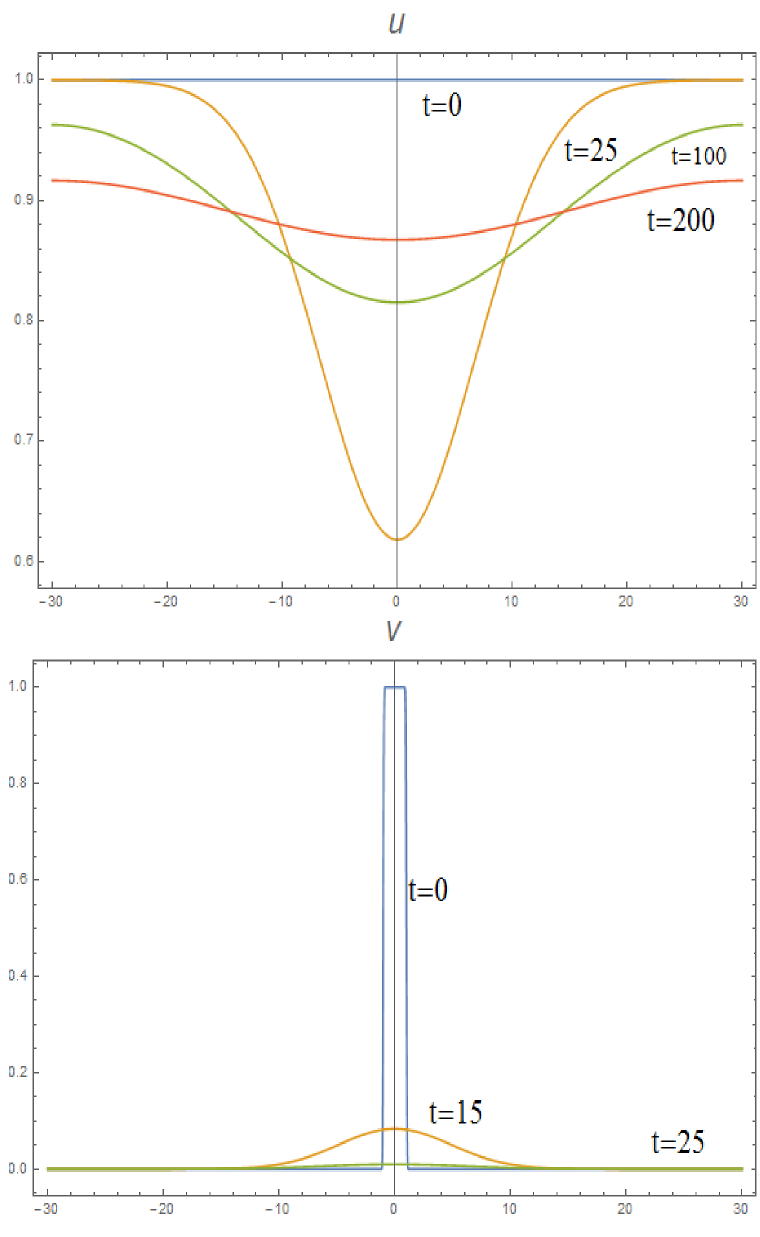

Figure 3.7: System (3.1) with $m = 2$, $l = 1$, $D = 4$ and $k = 0.2$

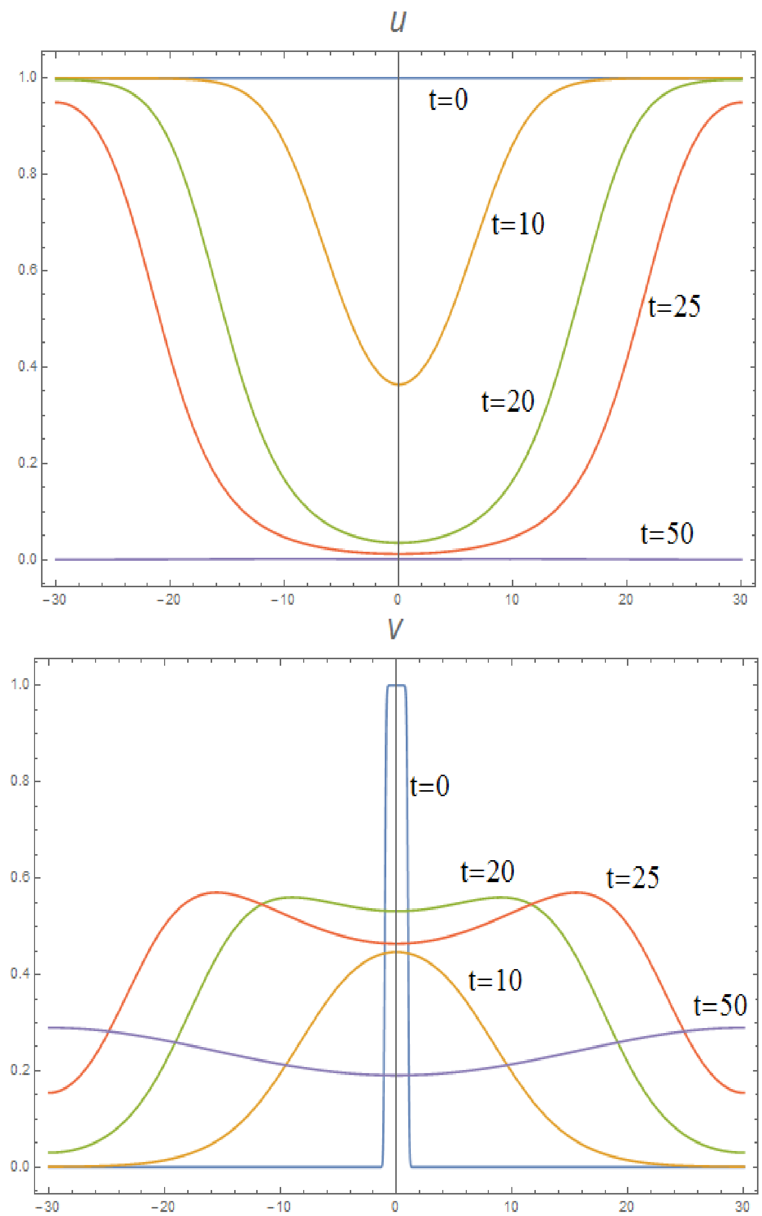

Figure 3.8: System (3.1) with $m = 2$, $l = 1$, $D = 4$ and $k = 0.05$

CHAPTER 4: GLOBAL STABILITY OF LESLIE-TYPE PREDATOR-PREY MODEL

The global stability of equilibrium states is an important and interesting problem to be studied in predator-prey model. The spatially homogeneous Leslie-type predator-prey model is the the ODE system

$$\begin{cases} u_t = u(\lambda - \alpha u) - \beta uv, \\ v_t = \mu v(1 - \frac{v}{u}). \end{cases}$$

where u is the population of the prey and v is the population of the predator, t is time, λ, α, β and μ are positive parameters. The predator consumes the prey at a rate according to the functional response, which is of Holling type I, i.e., βu. The ODE Leslie-Gower model has the unique positive equilibrium and several well-known approaches have been used to prove global stability of the positive equilibrium, see [19, 29].

We will add diffusive terms to the ODE system which takes into account the random movement of the predator and prey, with non-negative diffusion coefficients d_1 and d_2. The system is:

$$(III) \begin{cases} u_t = d_1 \Delta u + u(\lambda - \alpha u - \beta v), \quad (x,t) \in \Omega \times (0,\infty) \\ v_t = d_2 \Delta v + \mu v \left(1 - \frac{v}{u}\right), \quad (x,t) \in \Omega \times (0,\infty) \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad (x,t) \in \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) > 0, \quad v(x,0) = v_0(x) \geq 0 (\not\equiv 0), \quad x \in \bar{\Omega}. \end{cases} \tag{4.1}$$

Here $u(x,t)$ and $v(x,t)$ are the density of prey and predator, respectively, Ω is a bounded domain with smooth boundary $\partial\Omega$, λ, μ, α and β are positive constants. We assume the two diffusion coefficients d_1 and d_2 are positive and equal, but not necessarily constants. The no-flux boundary condition is imposed to guarantee that the ecosystem is not disturbed by exterior factors which may influence population flow cross the boundary. The system is a well established population model and is widely studied in literatures such as in [19, 29] by the construction of a Lyapunov function.

Equilibria. Equilibria of the ODE system and PDE system (4.1) are given by the solutions of the following system:

$$u(\lambda - \alpha u - \beta v) = 0, \quad \mu v\left(1 - \frac{v}{u}\right) = 0$$

It is easy to see that the equilibrium points are $(\frac{\lambda}{\alpha+\beta}, \frac{\lambda}{\alpha+\beta})$ and $(\frac{\lambda}{\alpha}, 0)$. We are only interested in the the positive equilibrium point $(\frac{\lambda}{\alpha+\beta}, \frac{\lambda}{\alpha+\beta})$, because this describes the coexistence of the predator and prey.

In this chapter, we will study the global stability of diffusive predator-prey system of Leslie-type in a bounded domain $\Omega \subset R^N$ with no-flux boundary condition. The world's leading mathematicians Du and Hsu gave a conjecture about the combination of α and β, see [19]. By using our new approach, we, in a way answer the open question and establish much improved global asymptotic stability of the unique positive equilibrium point. We shall prove the main results in next section. Also, we shall discuss results on the more general setting where Laplace operator is replaced by a spatial differential operator which is a uniform elliptic operator with variable coefficients, natural generalization of the one given in (III).

4.1 Main Results

Theorem 7. *Suppose the two diffusion coefficients are constants and $d_1 = d_2 > 0$, and $(\alpha, \beta, \lambda, \mu)$ are positive constants. Then, (u^*, v^*) is globally asymptotically stable if $\mu > \beta\lambda/\alpha$.*

Remark. Given any α and β, (u^*, v^*) is globally asymptotically stable if μ and λ are chosen suitably.

Remark. The above result can be interpreted as saying that if μ is suitably large, in relation to (α, β, λ), then the evolution of v in time will be adjusted sufficiently fast to any change in u so that both converge to the same equilibrium value as $t \to \infty$. It reveals more intimate relation of the various parameters to determine the large time behavior of solution than previous works.

Remark. It will be clear from our proof that a simplified version of our approach can yield (u^*, v^*) is globally asymptotically stable if $\alpha > \beta$, without the restriction to $d_1 = d_2 > 0$, which recovers the previous result of [29].

Remark. The method we use here is more flexible than the Lyapunov function method and the results cover more general settings such as when the Laplace operator is replaced by a uniform elliptic operator. In particular, the diffusion coefficients can depend on x. It means that the environment is non-homogeneous.

Let

$$\mathcal{L}u = \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}$$

be a uniform elliptic operator in Ω with continuous coefficients $a_{ij}(x), \ \ i, j = 1, \cdots, N$. Then, we can show a result similar to Theorem 7 for the following initial-boundary value problem:

$$(V) \begin{cases} u_t = \mathcal{L}u + u(\lambda - \alpha u - \beta v), \quad (x,t) \in \Omega \times (0, \infty) \\ v_t = \mathcal{L}v + \mu v \left(1 - \frac{v}{u}\right) \quad (x,t) \in \Omega \times (0, \infty) \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad (x,t) \in \partial\Omega \times (0, \infty) \\ u(x,0) = u_0(x) > 0, \quad v(x,0) = v_0(x) \geq 0 (\not\equiv 0), \quad x \in \bar{\Omega}. \end{cases}$$

Theorem 8. *Suppose $(\alpha, \beta, \lambda, \mu)$ are positive constants. Then, the unique positive equilibrium (u^*, v^*) of (V) is globally asymptotically stable if $\mu > \beta\lambda/\alpha$.*

Then, we study the following system where a more general type of reaction-terms is considered.

$$(VI)\begin{cases} u_t = d_1\Delta u + u(\lambda - \alpha u^{\sigma} - \beta v), & (x,t)\in\Omega\times(0,\infty) \\ v_t = d_2\Delta v + \mu v\left(1 - \frac{v}{u^{\sigma}}\right) & (x,t)\in\Omega\times(0,\infty) \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & (x,t)\in\partial\Omega\times(0,\infty) \\ u(x,0) = u_0(x) > 0, \quad v(x,0) = v_0(x) \geq 0(\not\equiv 0), & x\in\bar{\Omega}. \end{cases}$$

Here $0 < \sigma < 1$ and d_1, d_2 and $(\alpha, \beta, \lambda, \mu)$ are as in Theorem 7.

Theorem 9. *Suppose $d_1 = d_2 > 0$, $0 < \sigma < 1$ and $(\alpha, \beta, \lambda, \mu)$ are positive constants. Then, the unique positive equilibrium*

$$(u_\sigma^*, v_\sigma^*) = \left(\left(\frac{\lambda}{\alpha+\beta}\right)^{1/\sigma}, \left(\frac{\lambda}{\alpha+\beta}\right)^{1/\sigma}\right)$$

of (VI) is globally asymptotically stable if $\mu > \beta\lambda\sigma/\alpha$.

4.1.1 Proof of Theorem 7

Let $w = \frac{v}{u}$. It's easy to compute

$$w_t = \frac{v_t}{u} - \frac{u_t v}{u^2}, \quad \nabla w = \frac{\nabla v}{u} - \frac{\nabla u}{u^2}v,$$

$$\Delta w = \frac{\Delta v}{u} - \frac{v\Delta u}{u^2} - \frac{2\nabla u\cdot\nabla v}{u^2} + \frac{2|\nabla u|^2}{u^3}v.$$

The equation satisfied by w is

$$\begin{aligned} w_t - d\Delta w &= \mu\frac{v}{u}(1 - \frac{v}{u}) - \frac{v}{u}(\lambda - \alpha u - \beta v) + \frac{2d}{u}\nabla u\cdot\nabla w \\ &= w(\mu - \lambda + \alpha u - w(\mu - \beta u)) + \frac{2d}{u}\nabla u\cdot\nabla w \end{aligned} \tag{4.2}$$

Lemma 2. *Suppose $\mu > \frac{\beta}{\alpha}\lambda$ and $\varepsilon_1 > 0$ small. There exists T sufficiently large such that when $t \geq T$,*

$$u \leq \bar{u}_2(\varepsilon_1) \equiv \frac{\lambda}{\alpha}\left[1 - \frac{\beta(\mu - \beta\bar{u}_1)}{(\alpha+\beta)\mu - \beta\lambda}\right] + O(\varepsilon_1), \ \textit{in}\ \Omega,$$

where $\bar{u}_1 \equiv \frac{\lambda}{\alpha}$.

Proof. Since $v \geq 0$, it's clear that u satisfies

$$u_t - d_1\Delta u \leq u(\lambda - \alpha u) \text{ in } \Omega \times (0,\ \infty).$$

It is a well established fact that any positive solution of

$$\begin{cases} u_t - d_1\Delta u = u(\lambda - \alpha u), \quad \text{in } \Omega \times (0, \infty) \\ \frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial\Omega \times (0, \infty) \end{cases}$$

converges uniformly to $\frac{\lambda}{\alpha}$ as $t \to \infty$. Therefore, $\exists\, t_1 > 0$ such that

$$u(x,t) < \bar{u}_1(\varepsilon_1) \equiv \frac{\lambda}{\alpha} + \frac{\varepsilon_1}{5} \text{ in } \Omega \times [t_1,\ \infty).$$

Then,

$$w_t - d\Delta w \leq w[(\mu - \lambda + \alpha\bar{u}_1(\varepsilon_1)) - w(\mu - \beta\bar{u}_1(\varepsilon_1))] + \frac{2d}{u}\nabla u \cdot \nabla w.$$

We assume ε_1 is sufficiently small so that $\mu > \beta\bar{u}_1(\varepsilon_1)$. Hence, $w(x, t + t_1) \leq W(t)$, where $W(t)$ is a solution of

$$\begin{cases} W_t = W[(\mu - \lambda + \alpha\bar{u}_1(\varepsilon_1)) - W(\mu - \beta\bar{u}_1(\varepsilon_1))] \\ W(0) = \max_{\bar{\Omega}} W(x, t_1). \end{cases}$$

It is clear that $\exists\, t_2 > t_1$ such that

$$w(x,t) \le \bar{w}_1(\varepsilon_1) \equiv \frac{\mu - \lambda + \alpha \bar{u}_1(\varepsilon_1)}{\mu - \beta \bar{u}_1(\varepsilon_1)} + \frac{\varepsilon_1}{5} \;\text{ in } \Omega \times [t_2,\ \infty).$$

Substitute the above inequality into the first equation in (III), we have

$$u_t - d\Delta u \ge u[\lambda - \alpha u - \beta \bar{w}_1(\varepsilon_1) u] \text{ in } \Omega \times [t_2,\ \infty).$$

This, in turn, implies there exists $t_3 > t_2$ such that

$$u \ge \underline{u_1}(\varepsilon_1) \equiv \frac{\lambda}{\alpha + \beta \bar{w}_1(\varepsilon_1)} - \frac{\varepsilon_1}{5} \;\text{ in } \Omega \times [t_3,\ \infty).$$

By using the above inequality in the second equation, one obtains, there exists $t_4 > t_3$ such that

$$v \ge \underline{v_1}(\varepsilon_1) = \frac{\lambda}{\alpha + \beta \bar{w}_1(\varepsilon_1)} - \frac{\varepsilon_1}{4} \;\text{ in } \Omega \times [t_4,\ \infty).$$

Subsequently, when the above lower bound of v is used in the first equation of (III), we obtain

$$u_t - d\Delta u \le u[\lambda - \alpha u - \beta \underline{v_1}(\varepsilon_1)] \;\text{ in } \Omega \times [t_4,\ \infty).$$

This yields there exists $t_5 > t_4$ such that

$$u \le \bar{u}_2(\varepsilon_1) \equiv \frac{\lambda - \beta \underline{v_1}(\varepsilon_1)}{\alpha} + \frac{\varepsilon_1}{5} \;\text{ in } \Omega \times [t_5,\ \infty).$$

Simple computation shows

$$
\begin{aligned}
\bar{u}_2(\varepsilon_1) &= \frac{\lambda - \beta \underline{v_1}(\varepsilon_1)}{\alpha} + \frac{\varepsilon_1}{5} \\
&= \frac{\lambda}{\alpha}\left[1 - \frac{\beta}{\alpha + \beta \bar{w}_1(\varepsilon)}\right] + O(\varepsilon_1) \\
&= \frac{\lambda}{\alpha}\left[1 - \frac{\beta}{\alpha + \beta\left[\frac{\mu - \lambda + \alpha \bar{u}_1}{\mu - \beta \bar{u}_1}\right]}\right] + O(\varepsilon_1) \\
&= \frac{\lambda}{\alpha}\left[1 - \frac{\beta(\mu - \beta \bar{u}_1)}{(\alpha + \beta)\mu - \beta\lambda}\right] + O(\varepsilon_1).
\end{aligned}
\tag{4.3}
$$

This proves the lemma. $\square$

By repeating the above procedure, for any positive integer n, there exists t sufficiently large such that

$$u \le \bar{u}_{n+1}(\varepsilon_1) \equiv \frac{\lambda - \beta \underline{u_n}(\varepsilon_1)}{\alpha} + O(\varepsilon_1),$$

$$u \ge \underline{u_n}(\varepsilon_1) \equiv \frac{\lambda(\mu - \beta \bar{u}_n)}{(\alpha + \beta)\mu - \lambda\beta} + O(\varepsilon_1),$$

uniformly in Ω. Let $\varepsilon_1 = 0$, we have

$$\bar{u}_{n+1} = \frac{\lambda - \beta \underline{u_n}}{\alpha}, \quad \underline{u_n} = \frac{\lambda[\mu - \beta \bar{u}_n]}{(\alpha + \beta)\mu - \lambda\beta}, \quad n = 1, 2, \ldots$$

with $\bar{u}_1 > \bar{u}_2 > u^*$, $\underline{u_1} < u^*$. It's easy to see that $\{\bar{u}_n\}$ is a decreasing sequence with $\bar{u}_n > u^*$, $\forall n \ge 1$ and $\{\underline{u_n}\}$ is an increasing sequence with with $\underline{u}_n < u^*$, $\forall n \ge 1$. Suppose

$$\lim_{n\to\infty} \bar{u}_n = \bar{u}^* \; and \; \lim_{n\to\infty} \underline{u_n} = \underline{u}^*,$$

then

$$\bar{u}^* = \frac{\lambda - \beta \underline{u}^*}{\alpha},$$

$$\underline{u}^* = \frac{\lambda[\mu - \beta\bar{u}^*]}{(\alpha+\beta)\mu - \lambda\beta} = \frac{\lambda}{\alpha+\beta} + \frac{\lambda\beta}{(\alpha+\beta)\mu - \lambda\beta}\left(\frac{\lambda}{\alpha+\beta} - \bar{u}^*\right).$$

The combination of the two yields

$$\underline{u}^* = \frac{\lambda}{\alpha+\beta} + \frac{\lambda\beta}{(\alpha+\beta)\mu - \lambda\beta}\left(\frac{\lambda}{\alpha+\beta} - \frac{\lambda}{\alpha} + \frac{\beta\underline{u}^*}{\alpha}\right),$$

which has a unique solution $\underline{u}^* = u^*$. Consequently, $\bar{u}^* = u^*$. This shows

$$\lim_{n\to\infty} \bar{u}_n = \lim_{n\to\infty} \underline{u_n} = u^*.$$

Simply,

$$\bar{v}_n(\varepsilon_1) = \bar{u}_n(\varepsilon_1)\bar{w}_n(\varepsilon_1) + O(\varepsilon_1)$$

$$\underline{v_n}(\varepsilon_1) = \underline{u_n}(\varepsilon_1) + O(\varepsilon_1)$$

with

$$\bar{w}_n(\varepsilon_1) = \frac{\mu - \lambda + \alpha\bar{u}_n(\varepsilon_1)}{\mu - \beta\bar{u}_n(\varepsilon_1)} + O(\varepsilon_1).$$

Setting $\varepsilon_1 = 0$, we have

$$\bar{w}_n = \frac{\mu - \lambda + \alpha\bar{u}_n}{\mu - \beta\bar{u}_n},\ \bar{v}_n = \bar{u}_n\bar{w}_n,\ \underline{v_n} = \underline{u_n}.$$

$$\lim_{n\to\infty} \bar{w}_n = 1\ and\ \lim_{n\to\infty} \underline{v_n} = v^*.$$

Now, we show $\lim_{t\to\infty}(u, v) = (u^*, v^*)$ uniformly in Ω.

Proof of Theorem 7: $\forall\varepsilon > 0$, there exists $n_0 > 1$ such that when $n \geq n_0$,

$$|\bar{u}_n - u^*| + |\underline{u}_n - u^*| < \frac{\varepsilon}{4}. \tag{4.4}$$

Choose $\varepsilon_1 > 0$ sufficiently small such that

$$|\bar{u}_{n_0}(\varepsilon_1) - \bar{u}_{n_0}| + |\underline{u}_{n_0}(\varepsilon_1) - \underline{u}_{n_0}| < \frac{\varepsilon}{4} \tag{4.5}$$

and the same to $\underline{v}_n(\varepsilon_1)$, $\underline{v}_n$, $\bar{v}_n(\varepsilon_1)$, $\bar{v}_n$ and v^*.

Furthermore, there exists $t_M \gg 1$ such that when $t \gg t_M$,

$$\underline{u}_{n_0}(\varepsilon_1) \leq u(x,t) \leq \bar{u}_{n_0}(\varepsilon_1) \text{ in } \Omega.$$

Hence, by (4.4) and (4.5), when $t \gg t_M$,

$$|u(x,t) - u^*| < \varepsilon \text{ in } \Omega.$$

This proves $\lim_{t\to\infty} u(x,t) = u^*$ uniformly in Ω. Similarly, $\lim_{t\to\infty} v(x,t) = v^*$ uniformly in Ω.

□

Remark. It is clear from the proof of Lemma 2 that if $\beta < \alpha$, using $u \leq \widetilde{u}_1 = \lambda/\alpha$ in the second equation of (III) we get, ignoring ε_1, $v \leq \widetilde{v}_1 = \lambda/\alpha$, which in turn, when used in the first equation gives

$$u \geq \underset{\sim}{u_1} = \frac{\lambda}{\alpha} - \frac{\beta}{a}\widetilde{u}_1.$$

This will enable us to obtain, from the second equation, $v \geq \underset{\sim}{u_1}$ and subsequently, from the first equation,

$$u \leq \widetilde{u}_2 = \frac{\lambda}{\alpha} - \frac{\beta}{\alpha}\underset{\sim}{u_1}.$$

An iteration of the above procedure resulted in two sequences

$$\widetilde{u}_{n+1} = \frac{\lambda}{\alpha} - \frac{\beta}{a}\underset{\sim}{u}_n, \quad \underset{\sim}{u}_n = \frac{\lambda}{\alpha} - \frac{\beta}{\alpha}\widetilde{u}_n$$

with $\widetilde{u}_n > u^*$, $\underset{\sim}{u}_n < u^*$ and both converge to u^*. This recovers the result of [19] when $\alpha > \beta$, but without the restriction that d_1, $d_2 > 0$ must be constants.

4.2 More General Settings

Let $w = \frac{v}{u^\sigma}$. It's easy to compute

$$w_t = \frac{v_t}{u^\sigma} - \frac{\sigma u_t v}{u^{\sigma+1}}, \quad \nabla w = \frac{\nabla v}{u^\sigma} - \frac{\sigma \nabla u}{u^{\sigma+1}} v,$$

$$\Delta w = \frac{\Delta v}{u^\sigma} - \frac{\sigma v \Delta u}{u^{\sigma+1}} - \frac{2\sigma \nabla u \cdot \nabla v}{u^{\sigma+1}} + \frac{\sigma(\sigma+1)|\nabla u|^2}{u^{\sigma+2}} v.$$

The equation satisfied by w is

$$w_t - d\Delta w = w[(\mu - \sigma\lambda + \sigma\alpha u^\sigma) - w(\mu - \beta u^\sigma)] + \frac{2d\sigma \nabla u \cdot \nabla w}{u} + \frac{d\sigma(\sigma-1)|\nabla u|^2 w}{u^2}$$

Lemma 3. *Suppose $\mu > \frac{\beta\sigma\lambda}{\alpha}$ where $0 < \sigma < 1$ and $\varepsilon_1 > 0$ small. There exists T sufficiently large such that when $t \geq T$,*

$$u^\sigma \leq \bar{u}_2^\sigma(\varepsilon_1) \equiv \frac{\lambda}{\alpha}\left(1 - \frac{\beta(\mu - \beta\sigma\bar{u}_1^\sigma)}{\alpha\mu + \beta\mu - \beta\sigma\lambda}\right) + O(\varepsilon_1), \quad \textit{in } \Omega,$$

where $\bar{u}_1^\sigma \equiv \frac{\lambda}{\alpha}$.

Proof. Since $v \geq 0$, it's clear that u satisfies

$$u_t - d_1 \Delta u \leq u(\lambda - \alpha u^\sigma) \ \text{ in } \Omega \times (0,\ \infty),$$

from which we derive that $\exists\, t_1 > 0$ such that

$$u^\sigma(x,t) < \bar{u}_1^\sigma(\varepsilon_1) \equiv \frac{\lambda}{\alpha} + \frac{\varepsilon_1}{5} \ \text{ in } \Omega \times [t_1,\ \infty).$$

It is a well established fact that any positive solution of

$$\begin{cases} u_t - d_1 \Delta u = u(\lambda - \alpha u^\sigma), & \text{in } \Omega \times (0, \infty) \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega \times (0, \infty) \end{cases}$$

converges uniformly to $(\frac{\lambda}{\alpha})^{1/\sigma}$ as $t \to \infty$. Then,

$$w_t - d\Delta w \leq w[(\mu - \lambda\sigma + \alpha\sigma \bar{u}_1^\sigma(\varepsilon_1)) - w(\mu - \beta\sigma \bar{u}_1^\sigma(\varepsilon_1))] + \frac{2d\sigma \nabla u \cdot \nabla w}{u}.$$

We assume ε_1 is sufficiently small so that $\mu > \beta \bar{u}_1^\sigma(\varepsilon_1)$. Hence, $w(x, t + t_1) \leq W(t)$, where $W(t)$ is a solution of

$$\begin{cases} W_t = W[(\mu - \lambda\sigma + \alpha\sigma \bar{u}_1^\sigma(\varepsilon_1)) - W(\mu - \beta\sigma \bar{u}_1^\sigma(\varepsilon_1))] \\ W(0) = \max_{\bar{\Omega}} W(x, t_1). \end{cases}$$

It is clear that $\exists\, t_2 > t_1$ such that

$$w(x,t) \leq \bar{w}_1(\varepsilon_1) \equiv \frac{\mu - \sigma\lambda + \alpha\sigma \bar{u}_1^\sigma(\varepsilon_1)}{\mu - \beta\sigma \bar{u}_1^\sigma(\varepsilon_1)} + \frac{\varepsilon_1}{5} \ \text{ in } \Omega \times [t_2,\ \infty).$$

Substitute the above inequality into the first equation in (VI), we have

$$u_t - d\Delta u \geq u[\lambda - \alpha u^\sigma - \beta \bar{w}_1(\varepsilon_1) u^\sigma] \text{ in } \Omega \times [t_2, \infty).$$

This, in turn, implies there exists $t_3 > t_2$ such that

$$u^\sigma \geq \underline{u_1^\sigma}(\varepsilon_1) \equiv \frac{\lambda}{\alpha + \beta \bar{w}_1(\varepsilon_1)} - \frac{\varepsilon_1}{5} \text{ in } \Omega \times [t_3, \infty).$$

By using the above inequality in the second equation, one obtains, there exists $t_4 > t_3$ such that

$$v \geq \underline{v_1}(\varepsilon_1) = \frac{\lambda}{\alpha + \beta \underline{w_1}(\varepsilon_1)} - \frac{\varepsilon_1}{4} \text{ in } \Omega \times [t_4, \infty).$$

Subsequently, when the above lower bound of v is used in the first equation of (VI), we obtain

$$u_t - d\Delta u \leq u[\lambda - \alpha u^\sigma - \beta \underline{v_1}(\varepsilon_1)] \text{ in } \Omega \times [t_4, \infty).$$

This yields there exists $t_5 > t_4$ such that

$$u^\sigma \leq \bar{u}_2^\sigma(\varepsilon_1) \equiv \frac{\lambda - \beta \underline{v_1}(\varepsilon_1)}{\alpha} + \frac{\varepsilon_1}{5} \text{ in } \Omega \times [t_5, \infty).$$

Simple computation shows

$$\begin{aligned} \bar{u}_2^\sigma(\varepsilon_1) &= \frac{\lambda - \beta \underline{v_1}(\varepsilon_1)}{\alpha} + \frac{\varepsilon_1}{5} \\ &= \frac{\lambda}{\alpha}\left(1 - \frac{\beta}{\alpha + \beta \bar{w}_1(\varepsilon_1)}\right) + O(\varepsilon_1) \\ &= \frac{\lambda}{\alpha}\left(1 - \frac{\beta(\mu - \beta\sigma \bar{u}_1^\sigma)}{\alpha\mu + \beta\mu - \beta\sigma\lambda}\right) + O(\varepsilon_1). \end{aligned} \tag{4.6}$$

□

By repeating the above procedure, for any positive integer n, there exists t sufficiently large such that

$$u^\sigma \leq \bar{u}^\sigma_{n+1}(\varepsilon_1) \equiv \frac{\lambda - \beta \underline{u_n^\sigma}(\varepsilon_1)}{\alpha} + O(\varepsilon_1),$$

$$u^\sigma \geq \underline{u_n^\sigma}(\varepsilon_1) \equiv \frac{\lambda(\mu - \beta\sigma\bar{u}_n^\sigma)}{(\alpha+\beta)\mu - \lambda\beta\sigma} + O(\varepsilon_1),$$

uniformly in Ω. Let $\varepsilon_1 = 0$, we have

$$\bar{u}^\sigma_{n+1} = \frac{\lambda - \beta\sigma\underline{u_n^\sigma}}{\alpha}, \;\; \underline{u_n^\sigma} = \frac{\lambda[\mu - \beta\sigma\bar{u}_n^\sigma]}{(\alpha+\beta)\mu - \lambda\beta\sigma}, \;\; n = 1, 2, \ldots$$

with $\bar{u}_1^\sigma > \bar{u}_2^\sigma > (u^*)^\sigma$, $\underline{u_1^\sigma} < (u^*)^\sigma$. It's easy to see that $\{\bar{u}_n^\sigma\}$ is a decreasing sequence and $\{\underline{u_n^\sigma}\}$ is an increasing sequence with

$$\lim_{n\to\infty} \bar{u}_n^\sigma = \lim_{n\to\infty} \underline{u_n^\sigma} = (u_\sigma^*)^\sigma.$$

Simply,

$$\bar{v}_n(\varepsilon_1) = \bar{u}_n^\sigma(\varepsilon_1)\bar{w}_n(\varepsilon_1) + O(\varepsilon_1)$$

$$\underline{v_n}(\varepsilon_1) = \underline{u_n^\sigma}(\varepsilon_1) + O(\varepsilon_1)$$

with

$$\bar{w}_n(\varepsilon_1) = \frac{\mu - \sigma\lambda + \alpha\sigma\bar{u}_n^\sigma(\varepsilon_1)}{\mu - \beta\sigma\bar{u}_n^\sigma(\varepsilon_1)} + O(\varepsilon_1).$$

Setting $\varepsilon_1 = 0$, we have

$$\bar{w}_n = \frac{\mu - \lambda\sigma + \alpha\sigma\bar{u}_n^\sigma}{\mu - \beta\sigma\bar{u}_n^\sigma}, \; \bar{v}_n = \bar{u}_n^\sigma\bar{w}_n, \; \underline{v_n} = \underline{u_n^\sigma}.$$

$$\lim_{n\to\infty} \bar{w}_n = 1 \text{ and } \lim_{n\to\infty} \underline{v_n} = v_\sigma^*.$$

Now, we show $\lim_{t\to\infty}(u, v) = (u_\sigma^*, v_\sigma^*)$ uniformly in Ω. The proof of Theorem 8 and 9 are the same as in the proof of Theorem 7.

It is clear that we can combine the features of (V) and (VI) to get a more complex model and still the same result as in Theorem 7. The model we develop here is new and can be applied to many interesting reaction-diffusion type models where the stability of a unique positive equilibrium solution is a key issue to be studied, such as the Holling-Tanner predator-prey model which will be discussed in Chapter 5.

CHAPTER 5: GLOBAL STABILITY OF A DIFFUSIVE HOLLING-TANNER PREDATOR-PREY MODEL

We will study the global stability in a diffusive version of the Holling-Tanner predator-prey model, which takes into account the random movement of the predator and prey. First, let's review the spatially homogeneous Holling-Tanner predator-prey model of the the ODE system

$$\begin{cases} u_t = u(a-u) - \frac{uv}{m+u}, \\ v_t = v(b - \frac{v}{\gamma u}). \end{cases}$$

where u is the population of the prey and v is the population of the predator, t is time, a, b, m and γ are positive parameters. The predator consumes the prey according to the functional response, which is of Holling type II, i.e., $\frac{u}{m+u}$. The ODE Holling-Tanner model has a unique positive equilibrium and several well-known approaches have been used to prove global stability of the positive equilibrium, see [50].

We will consider the random movement of the predator and prey by adding diffusive term to the ODE system, with non-negative diffusion coefficients d_1 and d_2:

$$(IV)\begin{cases} u_t = d_1\Delta u + u(a-u) - \frac{uv}{m+u}, \quad (x,t) \in \Omega \times (0,\infty) \\ u_t = d_2\Delta v + v(b - \frac{v}{\gamma u}) \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad (x,t) \in \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) > 0, \quad v(x,0) = v_0(x) \geq 0 (\not\equiv 0), \quad x \in \bar{\Omega}. \end{cases} \tag{5.1}$$

Equilibria. Equilibria of the ODE system and PDE system (5.1) are given by the solutions of the following system:

$$u(a-u) - \frac{uv}{m+u} = 0, \quad v(b - \frac{v}{\gamma u}) = 0$$

It is easy to see that the equilibrium points are $(a, 0)$ and

$$u^* = \frac{1}{2}(a - m - b\gamma + \sqrt{(a - m - b\gamma)^2 + 4am}), \qquad v^* = b\gamma u^*.$$

We are only interested in (u^*, v^*), because that is the states when the predator and prey are coexistence. In this chapter we study the global stability of equilibrium states of diffusive predator-prey system of Holling-Tanner type in a bounded domain $\Omega \subset R^N$ with no-flux boundary condition. By using a novel approach, we establish much improved global asymptotic stability of the unique positive equilibrium solution than works in the literature. We also show that the result can be extended to more general type of systems with heterogeneous environment and/or other kind of kinetic terms.

5.1 Main Results

Theorem 10. *Suppose $d_1 = d_2 = d = d(x,t)$ is strictly positive, bounded and continuous in $\bar{\Omega} \times [0, \infty)$, a, b, γ and m are positive constants, $\gamma^{-1} > a/(m + a)$, then the positive equilibrium solution (u^*, v^*) is globally asymptotically stable in the sense that every solution to (IV) satisfies*

$$\lim_{t\to\infty}(u, v) = (u^*, v^*) \qquad \textit{uniformly in } \Omega.$$

Remark. The above result covers more ground than the result of [50] or [9]. In particular, if $\gamma \leq 1$, for all choices of a, b, m we have global asymptotic stability, or when $b \geq (\gamma - 1)a/\gamma$, our assumption is weaker than $m > b\gamma$.

Remark. The method we use here is more flexible than the Lyapunov function method and more powerful than those one used in [9], and the results covers more general settings such as when the Laplace operator is replaced by a uniform elliptic operator. It means that we can cover cases with heterogeneous environment.

Let

$$\mathcal{L}u = \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}$$

be a uniform elliptic operator in Ω with continuous coefficients $a_{ij}(x),\ \ i,j = 1,\cdots,N$. Then, we can show a result similar to Theorem 10 for the following initial-boundary value problem:

$$(VII)\begin{cases} u_t = \mathcal{L}u + au - u^2 - \frac{uv}{m+u}, & (x,t) \in \Omega \times (0,\infty) \\ v_t = \mathcal{L}v + bv - \frac{v^2}{\gamma u} & (x,t) \in \Omega \times (0,\infty) \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) > 0,\ v(x,0) = v_0(x) \geq 0 (\not\equiv 0) & x \in \bar{\Omega}. \end{cases}$$

Theorem 11. *Suppose a, b, γ, m are positive constants satisfying the assumption in Theorem 10 and $\mathcal{L}$ a uniform elliptic operator in Ω with continuous coefficients . Then, the unique positive equilibrium (u^*, v^*) of (VII) is globally asymptotically stable.*

5.1.1 Proof of Theorem 10

Proposition 2. *Suppose $d_1 = d_2 > 0$ are constants.*

(1) $u^* > a - b\gamma$.

(2) (u^*, v^*) *is locally asymptotically stable if* $\gamma^{-1} > a/(m + a)$.

(3) $\gamma^{-1} > a/(m + a)$ *implies* $u^* > a - b$.

Proof. It is easy to verify that

$$(a-m-b\gamma)^2+4am=(a+m-b\gamma)^2+4mb\gamma>(a+m-b\gamma)^2$$

and therefore,

$$u^*>\frac{a-m-b\gamma+a+m-b\gamma}{2}=a-b\gamma.$$

This proves the first statement.

For (2), from the proof of Theorem 2.1 in [50] it follows, when $d_1=d_2$, that (u^*,v^*) is locally asymptotically stable if and only if

$$2(u^*)-(a-m-b\gamma)>0 \tag{5.2}$$

and

$$2(u^*)^2-(a-m-b\gamma)u^*+bm>0 \tag{5.3}$$

If $\gamma\le 1$, both are trivially true by (1) and then expression of u^*.

If $\gamma>1$, we only need to show that u^* is larger than the largest positive root of the quadratic function in (5.3) under the condition that $a>m+b$, which is only possible if $\gamma<2$ by the condition $\gamma^{-1}>a/(m+a)$. Using the assumption that $\gamma^{-1}>a/(m+a)$, it is easy to verify that

$$u^*>a-b\gamma\ge\frac{a-m-b}{2}$$

if $\gamma a\ge(2\gamma-1)b$. The only case we need to consider is $\gamma a<(2\gamma-1)b$. But, it contradicts $a>m+b>(\gamma-1)a+b$. Hence, (2) holds.

The last statement follows from simple computation which shows that the inequality, under the assumption $a > b$, and $\gamma > 1$ is equivalent to $m > (\gamma - 1)(a - b)$. This completes the proof of the proposition. $\square$

Let $w = \frac{v}{u}$. It's easy to compute

$$w_t = \frac{v_t}{u} - \frac{u_t v}{u^2}, \quad \nabla w = \frac{\nabla v}{u} - \frac{\nabla u}{u^2} v,$$

$$\Delta w = \frac{\Delta v}{u} - \frac{v \Delta u}{u^2} - \frac{2\nabla u \cdot \nabla v}{u^2} + \frac{2|\nabla u|^2}{u^3} v.$$

The equation satisfied by w is

$$\begin{aligned} w_t - d\Delta w &= \mu \frac{v}{u} \left(b - \frac{v}{\gamma u}) - \frac{v}{u}(a - u - \frac{v}{m + u} \right) + \frac{2d}{u} \nabla u \cdot \nabla w \\ &= w \left(b - a + u + w \left(-\gamma^{-1} + \frac{u}{m + u} \right) \right) + \frac{2d}{u} \nabla u \cdot \nabla w \end{aligned} \tag{5.4}$$

Lemma 4. *Suppose $\gamma^{-1} > a/(m + a)$ and $\varepsilon_1 > 0$ small. There exists T sufficiently large such that when $t \geq T$,*

$$u \leq \bar{u}_2(\varepsilon_1) \equiv \frac{a - m + \sqrt{(a + m)^2 - 4b\gamma \underline{u}_1}}{2} + O(\varepsilon_1), \ \textit{in } \Omega,$$

where

$$\underline{u}_1 \equiv \frac{a - m - \bar{w}_1 + \sqrt{(a - m - \bar{w}_1)^2 + 4am}}{2}, \quad \bar{w}_1 = \frac{(b - a + \bar{u}_1)(m + \bar{u}_1)}{\gamma^{-1}(m + \bar{u}_1) - \bar{u}_1},$$

$\bar{u}_1 \equiv a.$

Proof. Since $v \geq 0$, it's clear that u satisfies

$$u_t - d\Delta u \leq u(a - u) \text{ in } \Omega \times (0, \infty).$$

By a simple comparison argument and the well established fact that any positive solution of

$$\begin{cases} u_t - d\Delta u = u(a-u), & \text{in } \Omega \times (0,\infty) \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega \times (0,\infty) \end{cases}$$

converges to a uniformly as $t \to \infty$, we obtain that $\forall \varepsilon_1 > 0$, $\exists\, t_1 > 0$ such that if $t \geq t_1$

$$u(x,t) < \bar{u}_1(\varepsilon_1) \equiv a + \frac{\varepsilon_1}{5} \text{ in } \Omega. \tag{5.5}$$

Then, for $t \geq t_1$,

$$w_t - d_1\Delta w \leq w\left(b - a + \bar{u}_1(\varepsilon_1) + w\left(-\gamma^{-1} + \frac{\bar{u}_1(\varepsilon_1)}{m + \bar{u}_1(\varepsilon_1)}\right)\right) + \frac{2d}{u}\nabla u \cdot \nabla w.$$

We assume ε_1 is sufficiently small so that $\gamma^{-1}(m + \bar{u}_1(\varepsilon_1)) > \bar{u}_1(\varepsilon_1)$. Since any positive solution $W(t)$ of the ODE

$$W_t = W\left(b - a + \bar{u}_1(\varepsilon_1) + W\left(-\gamma^{-1} + \frac{\bar{u}_1(\varepsilon_1)}{m + \bar{u}_1(\varepsilon_1)}\right)\right)$$

converges to the stable equilibrium point

$$W_0 = \frac{(b - a + \bar{u}_1(\varepsilon_1))(m + \bar{u}_1(\varepsilon_1))}{\gamma^{-1}(m + \bar{u}_1(\varepsilon_1)) - \bar{u}_1(\varepsilon_1)}, \tag{5.6}$$

a simple comparison argument yields that $\exists\, t_2 > t_1$ such that if $t \geq t_2$,

$$w(x,t) \leq \bar{w}_1(\varepsilon_1) \equiv W_0 + \frac{\varepsilon_1}{5} \text{ in } \Omega. \tag{5.7}$$

Consequently, $v \le \bar{w}_1(\varepsilon_1)u$ in Ω when $t \ge t_2$, and

$$\begin{aligned} u_t - d\Delta u &\ge u(a-u) - \frac{\bar{w}_1(\varepsilon_1)}{m+u}u^2 \\ &= \frac{u}{m+u}\left((a-u)(m+u) - \bar{w}_1(\varepsilon_1)u\right). \end{aligned}$$

The quadratic equation

$$(a-u)(m+u) - \bar{w}_1(\varepsilon_1)u = 0$$

has only one positive root,

$$R = \frac{a-m-\bar{w}_1(\varepsilon_1) + \sqrt{(a-m-\bar{w}_1(\varepsilon_1))^2 + 4am}}{2},$$

which is a stable equilibrium point of corresponding ODE

$$u_t = \frac{u}{m+u}\left((a-u)(m+u) - \bar{w}_1(\varepsilon_1)u\right)$$

and it attracts every positive solution. This, in turn, by comparison, implies there exists $t_3 > t_2$ such that if $t \ge t_3$,

$$u \ge \underline{u}_1(\varepsilon_1) \equiv \frac{a-m-\bar{w}_1(\varepsilon_1) + \sqrt{(a-m-\bar{w}_1(\varepsilon_1))^2 + 4am}}{2} - \frac{\varepsilon_1}{5} \text{ in } \Omega. \tag{5.8}$$

The above inequality, when used in the v equation, gives

$$v_t - d\Delta v \ge bv - \frac{v^2}{\gamma \underline{u}_1(\varepsilon_1)} \quad \text{in } \Omega \times [t_3, \infty).$$

Hence, there exists $t_4 > t_3$ such that if $t \ge t_4$,

$$v \ge \underline{v}_1(\varepsilon_1) = b\gamma \underline{u}_1(\varepsilon_1) - \frac{\varepsilon_1}{5} \text{ in } \Omega. \tag{5.9}$$

Substitute $v \geq \underline{v}_1(\varepsilon_1)$ into the u euqtion, we obtain

$$u_t - d\Delta u \leq au - u^2 - \frac{u\underline{v}_1(\varepsilon_1)}{m+u} \text{ in } \Omega \times [t_4,\ \infty).$$

A direct application of comparison principle then yields there exists $t_5 > t_4$ such that if $t \geq t_5$,

$$u \leq \bar{u}_2(\varepsilon_1) \equiv \frac{a - m + \sqrt{(a+m)^2 - 4\underline{v}_1(\varepsilon_1)}}{2} + \frac{\varepsilon_1}{5} \quad \text{in } \Omega. \tag{5.10}$$

Simple computation using (5.5)-(5.10) shows the experssion of $\bar{u}_2(\varepsilon_1)$ and that of $\underline{u}_1(\varepsilon_1)$ and $\bar{w}_1(\varepsilon_1)$ are valid. This proves the lemma. □

By repeating the above procedure, for any positive integer n, there exists T sufficiently large such that when $t \geq T$,

$$u \leq \bar{u}_{n+1}(\varepsilon_1) \equiv \frac{a - m + \sqrt{(a+m)^2 - 4\underline{v}_n(\varepsilon_1)}}{2} + \frac{\varepsilon_1}{5}, \text{ and}$$

$$u \geq \underline{u}_n(\varepsilon_1) \equiv \frac{a - m - \bar{w}_n + \sqrt{(a - m - \bar{w}_n)^2 + 4am}}{2} + \frac{\varepsilon_1}{5}$$

uniformly in Ω, where

$$\underline{v}_n(\varepsilon_1) = b\gamma\underline{u}_n - \frac{\varepsilon_1}{5}, \quad \bar{w}_n = \frac{(b - a + \bar{u}_n(\varepsilon_1))(m + \bar{u}_n(\varepsilon_1))}{\gamma^{-1}(m + \bar{u}_n(\varepsilon_1)) - \bar{u}_n(\varepsilon_1)} + \frac{\varepsilon_1}{5}.$$

It is clear that when $\varepsilon_1 = 0$, we have

$$\bar{u}_{n+1} = \frac{a - m + \sqrt{(a+m)^2 - 4b\gamma\underline{u}_n}}{2}, \quad \underline{u}_n = \frac{a - m - \bar{w}_n + \sqrt{(a - m - \bar{w}_n)^2 + 4am}}{2}$$

and

$$\underline{v}_n = b\gamma\underline{u}_n, \quad \bar{w}_n = \frac{(b - a + \bar{u}_n)(m + \bar{u}_n}{\gamma^{-1}(m + \bar{u}_n(\varepsilon_1)) - \bar{u}_n}, \quad \bar{v}_n = \bar{u}_n\bar{w}_n, \ \ n = 1, 2, \ldots.$$

with $\bar{u}_1 = a > u^*$, $\bar{w}_1 > b\gamma$ and $\underline{u}_1 < u^*$. It's easy to see that $u^* < \bar{u}_2 < \bar{u}_1$, $\bar{w}_n$ is an increasing function of $\bar{u}_n$ as long as $(m + \bar{u}_n)\gamma^{-1} - \bar{u}_n > 0$. This, together with $\bar{u}_2 < \bar{u}_1$, implies $\underline{u}_2 > \underline{u}_1$. A simple induction then shows, under the assumption of Theorem 10, $\{\bar{u}_n\}$ is a decreasing sequence and $\{\underline{u}_n\}$ is an increasing sequence with

$$\lim_{n\to\infty} \bar{u}_n = \lim_{n\to\infty} \underline{u}_n = u^*.$$

Consequently,

$$\lim_{n\to\infty} \bar{v}_n = \lim_{n\to\infty} \underline{v}_n = v^*$$

Now, we show $\lim_{t\to\infty}(u, v) = (u^*, v^*)$ uniformly in Ω.

Remark. The above procedure depends critically on two inequalities, (i) $\gamma^{-1} > a/(m + a)$ and (ii) $u^* > a - b$. It is shown the second condition follow from the first in Proposition 2.

Proof of Theorem 10: $\forall \varepsilon > 0$, there exists $n_0 > 1$ such that when $n \geq n_0$,

$$|\bar{u}_n - u^*| + |\underline{u}_n - u^*| < \frac{\varepsilon}{4}. \tag{5.11}$$

Choose $\varepsilon_1 > 0$ sufficiently small such that

$$|\bar{u}_{n_0}(\varepsilon_1) - \bar{u}_{n_0}| + |\underline{u}_{n_0}(\varepsilon_1) - \underline{u}_{n_0}| < \frac{\varepsilon}{4} \tag{5.12}$$

and the same to $\underline{v}_n(\varepsilon_1)$, $\underline{v}_n$, $\bar{v}_n(\varepsilon_1)$, $\bar{v}_n$ and v^*. Furthermore, there exists $t_M \gg 1$ such that when $t \geq t_M$,

$$\underline{u}_{n_0}(\varepsilon_1) \leq u(x,t) \leq \bar{u}_{n_0}(\varepsilon_1) \text{ in } \Omega.$$

Hence, by (5.11) and (5.12), when $t \geq t_M$,

$$|u(x,t) - u^*| < \varepsilon \ \text{ in } \Omega.$$

This proves $\lim_{t\to\infty} u(x,t) = u^*$ uniformly in Ω. Similarly, $\lim_{t\to\infty} v(x,t) = v^*$ uniformly in Ω.

$\square$

Remark. It is clear from the proof of Lemma 4 that if $\beta < \alpha$, using $u \leq \widetilde{u}_1 = \lambda/\alpha$ in the second equation of (IV) we get, ignoring ε_1, $v \leq \widetilde{v}_1 = \lambda/\alpha$, which in turn, when used in the first equation gives

$$u \geq \underset{\sim}{u}_1 = \frac{\lambda}{\alpha} - \frac{\beta}{a}\widetilde{u}_1.$$

This will enable us to obtain, from the second equation, $v \geq \underset{\sim}{u}_1$ and subsequently, from the first equation,

$$u \leq \widetilde{u}_2 = \frac{\lambda}{\alpha} - \frac{\beta}{\alpha}\underset{\sim}{u}_1.$$

An iteration of the above procedure resulted in two sequences

$$\widetilde{u}_{n+1} = \frac{\lambda}{\alpha} - \frac{\beta}{a}\underset{\sim}{u}_n, \qquad \underset{\sim}{u}_n = \frac{\lambda}{\alpha} - \frac{\beta}{\alpha}\widetilde{u}_n$$

with $\widetilde{u}_n > u^*$, $\underset{\sim}{u}_n < u^*$ and both converge to u^*. This recovers the result of [19] when $\alpha > \beta$ but without the restriction that $d_1,\ d_2 > 0$ must be constants.

5.2 More General Settings

It is easy to see that the proof of Theorem 11 follows exactly the same line of argument as in Theorem 10 and we omit the details. The method that we develop in this work is new and can be applied to many interesting reaction- diffusion type models where the stability of a unique positive equilibrium solution is a key issue to be studied. For example, the famous Geierer-Menhardt system is an interesting model worth of looking into.

CHAPTER 6: CONCLUSION

In this thesis, we first analyzed the behavior of the traveling wave solutions in auto-catalytic with/without decay. We determine the minimum speed for different cases of the traveling wave problems and verify the boundaries from the theoretical results and showed the existence of the traveling wave solutions under certain conditions. Even though the two systems are different by one linear decay term, both the theoretical results and numerical results show a big differences. From our study, we realized that the numerical computation is a powerful tool in understanding complex solution structure of traveling wave problem in systems shown in Chapter two and three. This is our first endeavor in this direction. The result is yielding good insight into the problem which provides good lead in our further analysis.

We shall do more computation in future and use more sophisticated algorithms to try to overcome a particular challenge which we did not elaborate in details, which is (1.8) has positive solutions (u, v) with v decaying to zero algebraically as $x^{-1/2}$ and u growing to ∞ also algebraically as $x^{1/2}$. They are not traveling waves. How to distinguish traveling wave from such solutions proves to be a challenge for us right now. Another direction we shall do computation is to study the bouyant instability when a fluid is involved in system (III) as in the famous iodate-arsenous-acid (IAA) reaction. When the fluid strength is increased, the original planar wave is destabilized, resulting cellular fingering. We shall use Hele-Shaw cell in two dimensional setting to approximate the full three dimensional Navies-Stokes equation.

In chapter four and chapter five, we developed a scheme to prove the global stability of the diffusive predator-prey system of Holling-Tanner type and Leslie type in a bounded domain $\Omega \in R^N$ with no-flux boundary condition. The method we develop in this thesis is new and can be applied to many interesting reaction-diffusion type models where the stability of a unique positive equilibrium solution is a key issue to be studied. For example, the famous Geierer-Menhardt system is an interesting model worth of looking into. It will be interesting to see how can we corporate other interesting features such as time delay into our scheme.

APPENDIX A: RUNGE-KUTTA (RK) METHODS

In this section, we use the results from [59]. Runge-Kutta methods maintain the structure of one-step methods, and increase their accuracy at the price of an increase of functional evaluations at each time level, thus scarifying linearity. RK methods is suitable at adapting the stepsize. In its most general form, an RK method can be written as

$$u_{n+1} = u_n + hF(t_u, u_n, h; f), \ \ n \geq 0,$$

where F is the increment function defined as follows:

$$F(t_n, u_n, h; f) = \sum_{i=1}^{s} b_i K_i,$$

$$K_i = f(t_n + c_i h, u_u + h \sum_{j=1}^{s} a_{ij} K_j), \ i = 1, 2, \ldots, s$$

and s denotes the number of stages of the method. The coefficients a_{ij}, c_i and b_i fully characterize and RK method and are usually collected in the so-called butcher array

$\mathbf{c}$	$\mathbf{A}$
	$\mathbf{b^T}$

where $\mathbf{A} = (\mathbf{a_{ij}}) \in \mathbb{R}^{\mathbf{s} \times \mathbf{s}}$, $\mathbf{b} = (\mathbf{b_1}, \ldots, \mathbf{b_s})^{\mathbf{T}} \in \mathbb{R}^{\mathbf{s}}$ and $\mathbf{c} = (\mathbf{c_1}, \ldots, \mathbf{c_s})^{\mathbf{T}} \in \mathbb{R}^{\mathbf{s}}$. We shall assume that the following condition holds

$$c_i = \sum_{j=1}^{s} a_{ij}, \ \ i = 1, \ldots, s. \tag{A.1}$$

If the coefficients a_{ij} in $\mathbf{A}$ are equal to zero for $j \geq i$, with $i = 1, 2, \ldots, s$, then each K_i can be explicitly computed in terms of the $i-1$ coefficients $K, \ldots, K_{i-1}$ that have already been determined. In such a case the RK method is explicit. Otherwise, it is implicit and solving a nonlinear system of size s is necessary for computing the coefficients K_i.

The way to derive an explicit RK method consists of enforcing that the highest number of terms in Taylor's expansion of the exact solution y_{n+1} about t_n coincide with those of the approximate solution u_{n+1}, assuming that we take one step of the RK method starting from the exact solution y_n. For example, if consider a 2-stage explicit RK method and assume to dispose at the n-th step of the exact solution y_n. Than

$$u_{n+1} = y_n + hF(t_n, y_n, h; f)y_n + h(b_1K_1 + b_2K_2),$$

$$K_1 = f(t_n, y_n), \quad K_2 = f(t_n + hc_2, y_n + hc_2K_1),$$

having assumed that (A.1) is satisfied. Expanding K_2 in a Taylor series in a neighborhood of t_n and truncating the expansion at the second order, we get

$$K_2 = f_n + hc_2(f_{n,t} + K_1 f_{n,y}) + \mathbf{O(h^2)}.$$

We have denoted by $f_{n,z}$ (for $z = t$ or $z = y$) the partial derivative of f with respect to z evaluated at (t_n, y_n). Then

$$u_{n+1} = y_n + hf_n(b_1 + b_2) + h^2c_2b_2(f_{n,t} + f_nf_{n,y}) + \mathbf{O(h^3)}.$$

If we perform the same expansion on the exact solution, we find

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2}y''_n + +\mathbf{O(h^3)} = \mathbf{y_n + hf_n + \frac{h^2}{2}(f_{n,t} + f_nf_{n,y}) + +O(h^3)}.$$

Forcing the coefficients in the two expansions above to agree, we obtain that the coefficients of the RK method must satisfy $b_1 + b_2 = 1$, $c_2b_2 = \frac{1}{2}$. Thus, there are infinitely many 2-stage explicit RK methods with second-order accuracy.

APPENDIX B: SOME BACKGROUND

In this section, we will introduce Sobolev Spaces from [6] and the Maximum Principle from [24].

Let $\Omega \subset R^N$ be an open set and let $p \in R$ with $1 \leq p \leq \infty$.

Definition. The Sobolev space [6] $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \left\{ u \subset L^p(\Omega), \exists g_1, \ldots, g_N \subset L^p(\Omega), s.t. \int_\Omega u \frac{\partial \phi}{\partial x_i} = -\int_\Omega g_i \phi \, \forall \phi \subset C_c^\infty(\Omega), \ \forall i = 1, \ldots, N \right\}.$$

We set

$$H^1(\Omega) = W^{1,2}(\Omega).$$

For $u \subset W^{1,p}(\Omega)$ we define $\frac{\partial u}{\partial x_i} = g_i$, and we write

$$\nabla u = grad\, u = (\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}).$$

The space $W^{1,p}(\Omega)$ is equipped with the norm

$$||u||_{W^{1,p}(\Omega)} = ||u||_p + \sum_{i=1}^N ||\frac{\partial u}{\partial x_i}||_p$$

or sometimes with the equivalent norm $(||u||_p^p + \sum_{i=1}^N ||\frac{\partial u}{\partial x_i}||_p^p)^{1/p}$ if $1 \leq p < \infty$.

The space $H^1(\Omega)$ is equipped with the scalar product

$$(u, v)_{H^1} = (u, v)_{L^2} + \sum_{i=1}^N (\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i})_{L^2} = \int_\Omega uv + \sum_{i=1}^N \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}.$$

The associated norm

$$||u||_{H^1} = (||u||_2^2 + \sum_{i=1}^N ||\frac{\partial u}{\partial x_i}||_2^2)^{1/2}$$

is equivalent to the $W^{1,2}$ norm.

Theorem (The maximum principle) [24]

Let $a \geq 0$. Suppose that $u(x,t)$ is continuous on $0 \leq x \leq L$, $0 \leq t \leq T$ and satisfies

$$u_t - au_{xx} \leq 0, \qquad \text{for } 0 < x < L, \; 0 < t \leq T,$$

$$u(0,t) \leq 0, \; u(L,t) \leq 0 \qquad \text{on } 0 \leq t \leq T,$$

$$u(x,t) \leq 0 \qquad \text{on } 0 \leq x \leq L.$$

Then $u(x,t) \leq 0$ for $0 \leq x \leq L$ and $0 \leq t \leq T$.

Corollary 1 [24] Let $a \geq 0$. Suppose that $u(x,t)$ is continuous on $0 \leq x \leq L$, $0 \leq t \leq T$ and satisfies

$$u_t - au_{xx} \geq 0, \qquad \text{for } 0 < x < L, \; 0 < t \leq T,$$

$$u(0,t) \geq 0, \; u(L,t) \geq 0 \qquad \text{for } 0 \leq t \leq T,$$

$$u(x,t) \geq 0 \qquad \text{for } 0 \leq x \leq L.$$

Then $u(x,t) \geq 0$ for $0 \leq x \leq L$ and $0 \leq t \leq T$.

Corollary 2 [24] Let $a \geq 0$. Suppose that $u(x,t)$ is defined and continuous on $0 \leq x \leq L$, $0 \leq t \leq T$ and satisfies

$$u_t = au_{xx} \qquad \text{in } 0 < x < L, \; 0 < t \leq T,$$

Let

$$M = \max\{\max_{0 \leq x \leq L} u(x,0), \; \max_{0 \leq t \leq T} u(0,t), \; \max_{0 \leq t \leq T} u(L,t)\}$$

$$m = \min\{\min_{0 \leq x \leq L} u(x,0), \; \min_{0 \leq t \leq T} u(0,t), \; \min_{0 \leq t \leq T} u(L,t)\}$$

Then

$$m \leq u(x, t) \leq M$$

for all $0 \leq x \leq L$, $0 \leq t \leq T$.

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