2017

Coloring Graphs with Forbidden Minors

Martin Rolek
University of Central Florida

STARS Citation

Rolek, Martin, "Coloring Graphs with Forbidden Minors" (2017). *Electronic Theses and Dissertations*. 5400.
https://stars.library.ucf.edu/etd/5400

COLORING GRAPHS WITH FORBIDDEN MINORS

by

MARTIN ROLEK
MS University of Central Florida, 2014
BS University of Central Florida, 2012
AA Seminole State College, 2010

A dissertation submitted in partial fulfilment of the requirements
for the degree of Doctor of Philosophy
in the Department of Mathematics
in the College of Sciences
at the University of Central Florida
Orlando, Florida

Spring Term
2017

Major Professor: Zi-Xia Song

ABSTRACT

A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. My research is motivated by the famous Hadwiger's Conjecture from 1943 which states that every graph with no K_t-minor is $(t-1)$-colorable. This conjecture has been proved true for $t \leq 6$, but remains open for all $t \geq 7$. For $t = 7$, it is not even yet known if a graph with no K_7-minor is 7-colorable. We begin by showing that every graph with no K_t-minor is $(2t-6)$-colorable for $t = 7, 8, 9$, in the process giving a shorter and computer-free proof of the known results for $t = 7, 8$. We also show that this result extends to larger values of t if Mader's bound for the extremal function for K_t-minors is true. Additionally, we show that any graph with no K_8^--minor is 9-colorable, and any graph with no $K_8^=$-minor is 8-colorable. The Kempe-chain method developed for our proofs of the above results may be of independent interest. We also use Mader's H-Wege theorem to establish some sufficient conditions for a graph to contain a K_8-minor.

Another motivation for my research is a well-known conjecture of Erdős and Lovász from 1968, the Double-Critical Graph Conjecture. A connected graph G is *double-critical* if for all edges $xy \in E(G)$, $\chi(G-x-y) = \chi(G)-2$. Erdős and Lovász conjectured that the only double-critical t-chromatic graph is the complete graph K_t. This conjecture has been show to be true for $t \leq 5$ and remains open for $t \geq 6$. It has further been shown that any non-complete, double-critical, t-chromatic graph contains K_t as a minor for $t \leq 8$. We give a shorter proof of this result for $t = 7$, a computer-free proof for $t = 8$, and extend the result to show that G contains a K_9-minor for all $t \geq 9$. Finally, we show that the Double-Critical Graph Conjecture is true for double-critical graphs with chromatic number $t \leq 8$ if such graphs are claw-free.

To my fiancée Victoria for being the light in my life, and to my parents for their constant support.

ACKNOWLEDGMENTS

I want to thank my advisor, Zi-Xia Song, for the knowledge and guidance she has given me over my academic career. It was her undergraduate Graph Theory course six years ago which led me down this path, and I thank her for it. I would also like to thank Dr. Joseph Brennan, Dr. Ronald DeMara, Dr. Michael Reid, and Dr. Yue Zhao, for serving on my committee.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1: INTRODUCTION

We begin this dissertation with an overview of basic concepts and definitions in graph theory. We then proceed with a discussion of historical results which have motivated our work and include our own results in these areas.

1.1 Basic Definitions

A graph G consists of a vertex set $V(G)$ and an edge set $E(G)$ such that each edge is associated with two vertices, called its *ends*. If an edge $e \in E(G)$ has ends $x, y \in V(G)$, we may write $e = xy$ and say that e *joins* x and y in G, and that x and y are *adjacent* or *neighbors* in G. A *loop* is an edge whose ends are both the same vertex. *Multiple edges* are distinct edges which share the same two ends. A graph is *simple* if it contains no loops or multiple edges. All graphs considered in this dissertation are simple graphs.

The *complement* of a graph G, denoted $\overline{G}$, is the graph with vertex set $V(G)$ and edge set $\{xy : x, y \in V(G) \text{ and } xy \notin E(G)\}$. Two graphs G_1 and G_2 are *isomorphic* if there exists a bijection $f : V(G_1) \to V(G_2)$ such that $xy \in E(G_1)$ if and only if $f(x)f(y) \in E(G_2)$. A graph H is a *subgraph* of a graph G, denoted $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $S \subseteq V(G)$, the subgraph of G *induced* by S, denoted $G[S]$, is the subgraph of G with vertex set S and edge set $\{xy \in E(G) : x, y \in S\}$. Given graphs G and H, we say that G is H*-free* if G does not contain an induced subgraph isomorphic to H.

A *path* P in a graph G is a subgraph of G with $V(P) = \{v_1, v_2, \ldots, v_k\}$ and $E(P) = \{v_iv_{i+1} : 1 \leq i \leq k-1\}$, and we may write $P = v_1, v_2, \ldots, v_k$, where the vertices are said to be written in *path order* or simply *order*. The vertices v_1 and v_k are the *ends* of P, and $v_2, \ldots, v_{k-1}$ are *internal* vertices of P. If a path P has ends v_1 and v_k, we say P is a v_1, v_k*-path* or a *path from* v_1 *to* v_k. Two paths P_1 and P_2 are *disjoint* if they share no common vertices; and *internally disjoint* if they share no common internal vertices. We define the *length* of a path to be its number of edges. We may denote a path of length k by P_k when its specific vertex and edge sets are unimportant. A *cycle* is a path whose first and last vertices are joined by an edge, and we may write $C = v_1, v_2, \dots, v_k$, where the vertices are said to be written in *cyclic order*. The *length* of a cycle is also its number of edges, and we may similarly denote a cycle of length k by C_k when details are unimportant.

We define *vertex deletion* and *edge deletion* as follows. For a vertex set $S \subseteq V(G)$, the graph obtained from G by deleting S, denoted $G - S$, is the subgraph $G[V(G) \setminus S]$ of G induced by $V(G) \setminus S$. If $S = \{x\}$, we simply write $G - x$ instead of $G - S$. For an edge set $E \subseteq E(G)$, the graph obtained from G by deleting E, denoted $G - E$, is the subgraph of G with vertex set $V(G)$ and edge set $E(G) \setminus E$. If $E = \{e\}$, we simply write $G - e$ instead of $G - E$. We define *edge addition* as follows. Given two vertices $x, y \in V(G)$ such that $xy \notin E(G)$, we define $G + xy$ to be the graph with vertex set $V(G)$ and edge set $E(G) \cup \{xy\}$.

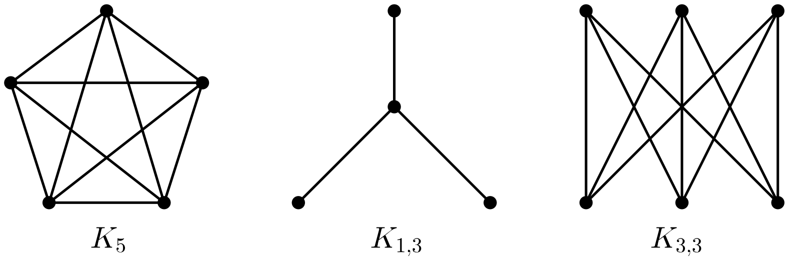

Figure 1.1: The complete graph K_5, the claw $K_{1,3}$, and the complete bipartite graph $K_{3,3}$.

The *complete graph* K_t is a graph with t vertices such that every pair of vertices are adjacent. We denote the graphs obtained from K_t by deleting one or two edges by K_t^- and $K_t^=$, respectively. A *complete* k*-partite graph* G has a partition of its vertex set into k independent sets $A_1, \dots, A_k$ such that A_i is complete to A_j for all $i \neq j$, and if $|A_i| = t_i$ for all i we denote this graph by $K_{t_1,\dots,t_k}$.

When $k = 2$, G is a *complete bipartite graph*, and in this case we say that G has a *bipartition* (A_1, A_2). A graph is *bipartite* if it is a subgraph of a complete bipartite graph. The complete bipartite graph $K_{1,t}$ is called a *star*, where $t \geq 1$ is an integer. When $t = 3$, $K_{1,3}$ is a *claw*.

The *neighborhood* in G of a vertex set $S \subseteq V(G)$, denoted $N_G(S)$, is the set of its neighbors in $V(G) \setminus S$, that is, $N_G(S) = \{v \in V(G) \setminus S : xv \in E(G) \text{ for some } x \in S\}$. The *closed neighborhood* in G of a vertex set $S \subseteq V(G)$, denoted $N_G[S]$, is $N_G[S] = N_G(S) \cup S$. If $S = \{x\}$, we will simply write $N_G(x)$ and $N_G[x]$. The *degree* of a vertex $x \in V(G)$, denoted $d_G(x)$, is defined to be $|N_G(x)|$. Let $\delta(G) := \min\{d_G(x) : x \in V(G)\}$ and $\Delta(G) := \max\{d_G(x) : x \in V(G)\}$. Then $\delta(G)$ and $\Delta(G)$ are the *minimum degree* and *maximum degree* G, respectively.

Given two disjoint vertex sets $A, B \subseteq V(G)$, we say that A is *complete* (resp. *anticomplete*) to B if for every $a \in A$ and $b \in B$ we have $ab \in E(G)$ (resp. $ab \notin E(G)$), and A is *mixed* on B if A is neither complete nor anticomplete to B. We use $e_G(A, B)$ to denote the number of edges in G with one end in A and the other end in B. If $A = \{x\}$, we simply write $e_G(x, B)$ and say that x is complete to, anticomplete to, or mixed on B.

A *clique* in a graph G is a subgraph of G isomorphic to a complete graph. An *independent set* in a graph G is a vertex set $S \subseteq V(G)$ such that if $x, y \in S$, then $xy \notin E(G)$. Let $\omega(G) := \max\{|V(H)| : H \subseteq G \text{ and } H \text{ is a clique}\}$ and $\alpha(G) := \max\{|S| : S \subseteq V(G) \text{ and } S \text{ is an independent set}\}$. Then $\omega(G)$ and $\alpha(G)$ are the *clique number* and *independence number* of G, respectively.

A graph G is *k-connected* if for any set $S \subseteq V(G)$ with $|S| < k$ and any $x, y \in V(G) \setminus S$, there exists an x, y-path in $G - S$. Equivalently, by Menger's Theorem [47], a graph G is k-connected if for any $x, y \in V(G)$ there exist k internally disjoint x, y-paths in G. If G is 1-connected, we simply say G is *connected.* If G is not connected, we say G is *disconnected.* Given a connected graph G, if $G - S$ is disconnected, we say S is a *separating set* of G, and if $G - x$ is disconnected for some

$x \in V(G)$, then x is a *cut-vertex*. A vertex set $S \subseteq V(G)$ is *connected* if $G[S]$ is connected.

A graph G is *t-colorable* if there exists a function $c : V(G) \to \{1, \ldots, t\}$ such that for any $xy \in E(G)$, $c(x) \neq c(y)$. For each $i \in \{1, 2, \ldots, t\}$, the set of vertices $V_i := \{v \in V(G) : c(v) = i\}$ is called a *color class* (of c). The *chromatic number* of a graph G, denoted $\chi(G)$, is the minimum integer t such that G is t-colorable. If $\chi(G) = t$, we also say that G is *t-chromatic*.

A graph G contains a *subdivision* of a graph H if a subgraph of G can be obtained from H by replacing the edges of H with pairwise internally disjoint paths such that none of these paths has an internal vertex in $V(H)$. In this case, we also say that G contains an H-subdivision. We define *edge contraction* as follows. Given an edge $xy \in E(G)$, the graph obtained from G by contracting xy, denoted G/xy, is the graph with vertex set $V(G - \{x, y\}) \cup \{z\}$ and edge set $E(G - \{x, y\}) \cup \{zv : v \in N_G(\{x, y\})\}$. In other words, G/xy is the graph obtained from G by deleting x and y and adding a new vertex z joined to all vertices in $N_G(x) \cup N_G(y) \setminus \{x, y\}$ (see Figure 1.2). If $S \subseteq V(G)$ is connected, then by G/S we mean the graph with vertex set $V(G - S) \cup \{z\}$ and edge set $E(G - S) \cup \{zv : v \in N_G(S)\}$. In this case, we may also call G/S the graph obtained from G by *contracting S to a single vertex*. A graph H is a *minor* of a graph G, denoted $G > H$, if H can be obtained from a subgraph of G by some sequence (possibly empty) of edge contractions. In this case, we may also say that G contains an H-minor. If G does not contain an H-minor, then G is *H-minor-free*.

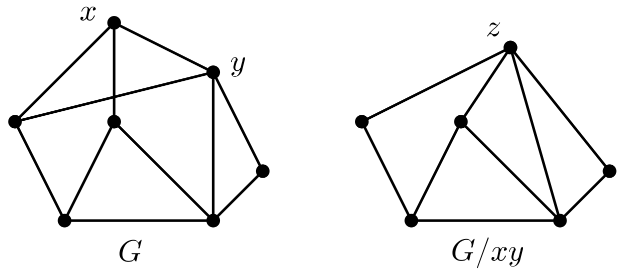

Figure 1.2: An example of edge contraction.

1.2 The Double-Critical Graph Conjecture

One of the motivations for this dissertation is the following well-known conjecture of Erdős and Lovász [19] from 1966.

Conjecture 1.2.1 Erdős-Lovász Tihany Conjecture. (Erdős and Lovász [19]) *For any integers* $s, t \geq 2$ *and any graph* G *with* $\omega(G) < \chi(G) = s + t - 1$*, there exist disjoint subgraphs* G_1 *and* G_2 *of* G *such that* $\chi(G_1) \geq s$ *and* $\chi(G_2) \geq t$*.*

The Erdős-Lovász Tihany Conjecture is hard. To date, the Erdős-Lovász Tihany Conjecture has been shown to be true only for values of $(s, t) \in \{(2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (3, 5)\}$. The case $(2, 2)$ is trivial. The case $(2, 3)$ was shown by Brown and Jung in 1969 [5]. Mozhan [48] and Stiebitz [68] each independently showed the case $(2, 4)$ in 1987. The cases $(3, 3)$, $(3, 4)$, and $(3, 5)$ were also settled by Stiebitz in 1987 [69].

Recent work on the Erdős-Lovász Tihany Conjecture has focused on proving the conjecture for certain classes of graphs. Kostochka and Stiebitz [41] showed the conjecture holds for line graphs. Balogh, Kostochka, Prince, and Stiebitz [4] then showed that the conjecture holds for all quasi-line graphs and all graphs G with $\alpha(G) = 2$. More recently, Chudnovsky, Fradkin, and Plumettaz [10] proved the following slight weaking of the Erdős-Lovász Tihany Conjecture for claw-free graphs, the proof of which is long and relies heavily on the structure theorem for claw-free graphs developed by Chudnovsky and Seymour in [12].

Theorem 1.2.2 (Chudnovsky, Fradkin, and Plumettaz [10]) *Let* G *be a claw-free graph with* $\chi(G) > \omega(G)$*. Then there exists a clique* K *with* $|V(K)| \leq 5$ *such that* $\chi(G - K) > \chi(G) - |V(K)|$*.*

The most recent result related to the Erdős-Lovász Tihany Conjecture is another slight weakening due to Stiebitz [70], who showed that for integers $s, t \geq 2$, any graph G with $\omega(G) < \chi(G) = s + t - 1$ contains disjoint subgraphs G_1 and G_2 of G with either $\chi(G_1) \geq s$ and $\text{col}(G_2) \geq t$, or $\text{col}(G_1) \geq s$ and $\chi(G_2) \geq t$. Here $\text{col}(H)$ denotes the *coloring number* of a graph H, i.e. the smallest positive integer k such that there is an ordering of the vertices of G in which each vertex is preceded by fewer than k of its neighbors.

If we restrict $s = 2$ in the statement of the Erdős-Lovász Tihany Conjecture, then the conjecture states that for any graph G with $\chi(G) > \omega(G) \geq 2$, there exists an edge $xy \in E(G)$ such that $\chi(G - x - y) \geq \chi(G) - 1$. If no such edge exists, then $\chi(G - x - y) = \chi(G) - 2$ for every edge $xy \in E(G)$. This motivates the following definition of double-critical graphs. A connected graph G is *double-critical* if for every edge $xy \in E(G)$, $\chi(G - x - y) = \chi(G) - 2$. The only known example of a double-critical, t-chromatic graph for any integer $t \geq 2$ is the complete graph K_t. Any double-critical, t-chromatic graph $G \neq K_t$ would be a counterexample to the Erdős-Lovász Tihany Conjecture, and this motivates the following special case of the conjecture.

Conjecture 1.2.3 Double-Critical Graph Conjecture. (Erdős and Lovász [19]) *For any integer* $t \geq 2$*, the only double-critical,* t*-chromatic graph is* K_t*.*

Since the Double-Critical Graph Conjecture is equivalent to the Erdős-Lovász Tihany Conjecture with $s = 2$, we see from the discussion above that the Double-Critical Graph Conjecture has been settled in the affirmative for $t \leq 5$ [48, 68], for line graphs [41], and for quasi-line graphs and graphs with independence number two [4]. The Double-Critical Graph Conjecture remains open for all $t \geq 6$.

Theorem 1.2.4 (Brown and Jung [5]; Mozhan [48]; Stiebitz [68]) *For* $t \in \{1, \ldots, 5\}$*, the only double-critical,* t*-chromatic graph is the complete graph* K_t*.*

The Double-Critical Graph Conjecture is also hard, and some weakenings of it have been studied. In 2010, Kawarabayashi, Pedersen, and Toft [37] proposed the following, which we call Hadwiger's Conjecture for Double-Critical Graphs.

Conjecture 1.2.5 Hadwiger's Conjecture for Double-Critical Graphs. (Kawarabayashi, Pedersen, and Toft [37]) *For any integer* $t \geq 2$*, any double-critical,* t*-chromatic graph contains a* K_t*-minor.*

Hadwiger's Conjecture for Double-Critical Graphs clearly holds for $t \leq 5$ by the above results on the Double-Critical Graph Conjecture. In the same paper as the proposal of the conjecture, Kawarabayashi, Pedersen, and Toft [37] also proved Hadwiger's Conjecture for Double-Critical Graphs for the cases $t = 6$ and $t = 7$. As a further weakening of the Double-Critical Graph Conjecture, Pedersen [50] showed that any double-critical, 8-chromatic graph contains K_8^- as a minor. Albar and Gonçalves [1] later proved that any double-critical, 8-chromatic graph contains K_8 as a minor. We summarize these results as follows.

Theorem 1.2.6 (Kawarabayashi, Pedersen, and Toft [37], Albar and Gonçalves [1]) *For any integer* $t \in \{6, 7, 8\}$*, every double-critical,* t*-chromatic graph contains* K_t *as a minor.*

We note here that for Theorem 1.2.6, the proof by Kawarabayashi, Pedersen, and Toft [37] of the case $t = 7$ is long, and the proof by Albar and Gonçalves [1] of the case $t = 8$ is computer-assisted. In Chapter 2, we provide a significantly shorter and computer-free proof of Theorem 1.2.6 for the cases $t = 7, 8$, and we also extend the theorem to the case $t = 9$. We note here that while the methods we use in proving the case $t = 9$ do not utilize a computer, we do make use of Theorem 1.4.3, whose proof in [67] was computer-assisted. Hence our proof for the case $t = 9$ is not strictly computer-free. We will actually prove the following slightly stronger result in Section 2.3.

Theorem 1.2.7 [59] *For integers* k, t *with* $k \in \{1, \ldots, 9\}$ *and* $t \geq k$*, every double-critical,* t*-*

chromatic graph contains K_k as a minor.

As a different weakening of the Double-Critical Graph Conjecture, the conjecture has been studied for claw-free graphs. We note that Theorem 1.2.2 does not completely settle the Double-Critical Graph Conjecture for claw-free graphs. Recently, Huang and Yu [31] proved that the only double-critical, 6-chromatic, claw-free graph is K_6. We give an alternative, shorter proof of their result, and further prove the following in Section 2.4.

Theorem 1.2.8 [57] *If G is a claw-free, double-critical, t-chromatic graph with $t \in \{6, 7, 8\}$, then G is isomorphic to K_t.*

Kawarabayashi, Pedersen, and Toft [37] proved that no two vertices of degree $t+1$ can be adjacent in a double-critical, t-chromatic graph for $t \geq 6$. One of the main results of Chapter 2 is the following improvement of their result, and it is crucial in the proof of Theorem 1.2.8.

Theorem 1.2.9 [57] *If G is a double-critical, t-chromatic graph with $t \geq 6$, then no vertex of degree $t+1$ is adjacent to a vertex of degree $t+1, t+2$, or $t+3$.*

We prove Theorem 1.2.9 in Section 2.2. Since Theorem 1.2.9 applies to all double-critical graphs, not just claw-free graphs, it will be useful in any future work on double-critical graphs.

1.3 Hadwiger's Conjecture

In this section, we introduce the main motivation for my research.

A graph is *planar* if it can be drawn in the plane in such a way that any two distinct edges do not intersect except possibly at a common end. If no such drawing exists, the graph is *nonplanar*. One of the earliest motivating questions in the field of graph theory was the 4-color conjecture, namely, are planar graphs 4-colorable? In 1890, Heawood [29] gave a short proof showing that five colors suffices. The question proved very difficult, and it was only with the aid of computers that it was able to be answered in the affirmative. In 1977, Appel and Haken [2, 3] first proved the Four Color theorem. A much shorter, but still computer assisted, proof was given by Robertson, Sanders, Seymour, and Thomas in 1997 [54].

Theorem 1.3.1 Four Color Theorem. (Appel and Haken [2, 3]) *Every planar graph is* 4*-colorable.*

It is well-known that the complete graph K_5 and complete bipartite graph $K_{3,3}$ are nonplanar. In the 1930s it was shown that, in a sense, these two examples characterize all nonplanar graphs. Kuratowski showed in 1930 [43] that a graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$, and Wagner showed in 1937 [73] that excluding minors of K_5 and $K_{3,3}$ suffices, summarized as follows.

Theorem 1.3.2 Kuratowski's Theorem. (Kuratowski [43]) *A graph is planar if and only if it does not contain* K_5 *or* $K_{3,3}$ *as a subdivision.*

Theorem 1.3.3 Wagner's Theorem. (Wagner [73]) *A graph is planar if and only if it does not contain* K_5 *or* $K_{3,3}$ *as a minor.*

We note now that the graph $K_{3,3}$, and indeed any bipartite graph, is 2-colorable. Inspired by Wagner's Theorem , in 1943, Hadwiger [25] conjectured the following.

Conjecture 1.3.4 Hadwiger's Conjecture. (Hadwiger [25]) *For any integer* $t \geq 1$*, every* t*-chromatic graph contains a* K_t*-minor.*

An equivalent formulation of Hadwiger's Conjecture states that any graph with no K_t-minor is $(t-1)$-colorable. Hadwiger's Conjecture is trivially true for $t \leq 3$. It is easy for $t = 4$ and was shown by both Hadwiger [25] and Dirac [17]. An alternative and quite short proof for $t = 4$ has also been given by Woodall [74]. Wagner [73] proved that the case $t = 5$ of Hadwiger's Conjecture is, in fact, equivalent to the Four Color Theorem. The same was shown for the case $t = 6$ by Robertson, Seymour, and Thomas [55] in their proof of the following.

Theorem 1.3.5 (Robertson, Seymour, and Thomas [55]) *If G does not contain K_6 as a minor, then G is 5-colorable.*

Hadwiger's Conjecture has also been verified for certain classes of graphs. In 2004, Reed and Seymour [53] proved that Hadwiger's Conjecture holds for line graphs, where such graphs are permitted to have multiple edges. In 2008, Chudnovsky and Fradkin [9] proved that Hadwiger's Conjecture holds for quasi-line graphs. Plummer, Stiebitz, and Toft [51] proved in 2003 that Hadwiger's Conjecture holds for every H-free graph G with $\alpha(G) = 2$, where H is any graph with $|V(H)| = 4$ and $\alpha(H) = 2$. This was strengthened by Kriesell [42] in 2010, who showed that Hadwiger's Conjecture holds for every H-free graph G with $\alpha(G) = 2$, where H is any graph with $|V(H)| = 5$ and $\alpha(H) = 2$. More recently, Song and Thomas [66] showed that Hadwiger's conjecture holds for graphs G with $\alpha(G) \geq 3$ and no induced cycle of length between 4 and $2\alpha(G) - 1$.

Hadwiger's Conjecture remains open for $t \geq 7$, and so the first open case of Hadwiger's Conjecture is proving that graphs with no K_7-minor are 6-colorable. However, it is not yet known if graphs with no K_7-minor are even 7-colorable. In fact, it has only been recently shown by Albar and Gonca̧lves [1] that K_7-minor-free graphs are 8-colorable. There have been several other partial results towards the case $t = 7$ of Hadwiger's Conjecture. Kawarabayashi and Toft [39] proved that every graph with neither K_7 nor $K_{4,4}$ as a minor is 6-colorable. Jakobsen [33, 34] proved that every graph with no K_7^--minor is 7-colorable and every graph with no $K_7^=$-minor is 6-colorable. We note that the result of Kawarabayashi and Toft and the latter result of Jakobsen are best possible since the complete graph K_6 is 6-chromatic and does not contain any of K_7, $K_7^{=}$, or $K_{4,4}$ as a minor.

In addition to showing that K_7-minor-free graphs are 8-colorable, Albar and Gonca̧lves [1] showed K_8-minor-free graphs are 10-colorable, which we summarize as follows.

Theorem 1.3.6 (Albar and Gonca̧lves [1]) *If a graph is K_7-minor-free, then it is 8-colorable. If a graph is K_8-minor-free, then it is 10-colorable.*

The proof of Theorem 1.3.6 given by Albar and Gonçalves in [1] is long and computer-assisted. In Chapter 3 we provide a much shorter and computer-free proof of Theorem 1.3.6 by using our powerful Lemma 1.5.3. We additionally use our Lemma 1.5.3 to extend Theorem 1.3.6 to show that K_9-minor-free graphs are 12-colorable, although we note that this result does rely on the extremal function for K_9-minors (Theorem 1.4.3) which was proved with computer assistance in [67]. The main result of Chapter 3 is summarized as follows.

Theorem 1.3.7 [58] *For all $t \in \{7, 8, 9\}$, any graph with no K_t-minor is $(2t - 6)$ colorable.*

Since Theorem 1.3.7 does utilize the extremal function for K_t-minors introduced in Section 1.4, extending it to values of $t \geq 10$ is hampered by the fact that the extremal function for K_t-minors has not yet been proved for $t \geq 10$. In Section 1.4, we see that the bound on the extremal function from Theorem 1.4.1 with $t \leq 7$ extends to $t \in \{8, 9\}$ except for a small number of counterexamples. In Section 3.5 we introduce Conjecture 3.5.2, which claims that this same bound extends to all $t \geq 10$, except for some counterexamples which are all $(t - 1)$-colorable. If we assume that Conjecture 3.5.2 is true, then we are able to show with Theorem 3.5.3 that for any integer $t \geq 5$, a graph with no K_t-minor is $(2t - 6)$-colorable. The method which we use to prove Theorem 3.5.3 is different from that used in the proof of Theorem 1.3.7, and so this provides an alternate proof of Theorem 1.3.7. Theorem 3.5.3 represents the first result on coloring K_t-minor-free graphs for general values of t.

The Kempe Chain method developed in Lemma 1.5.3, which is crucial in the proof of Theorem 1.3.7 presented in Section 3.2, is then used to prove the following two new results.

Theorem 1.3.8 [58] *Every graph with no* K_8^-*-minor is* 9*-colorable.*

Theorem 1.3.9 [58] *Every graph with no* $K_8^=$*-minor is* 8*-colorable.*

Our proofs of Theorem 1.3.8 and Theorem 1.3.9 are short and will also appear in Chapter 3.

More information on Hadwiger's Conjecture can be found in an earlier survey by Toft [71] and a very recent informative survey by Seymour [62].

1.4 The Extremal Functions for K_t-minors

The extremal function for a graph H, first introduced by Turán in 1941 [72], gives the maximum number of edges in a graph G which does not contain H as a subgraph. Turán studied the extremal function for complete graphs K_t, and completely characterized the K_t-free graphs with the maximum number of edges, namely, complete $(t-1)$-partite graphs now known as Turán graphs. The extremal function is naturally extended to graph minors as follows. Given a graph H and an integer $n \geq |V(H)|$, the *extremal function for* H*-minors* is the minimum integer $p = p(H)$ such that any graph with n vertices and at least p edges contains H as a minor.

The extremal function for K_t-minors was first shown in 1964 for $t \leq 5$ by Dirac [15] and then in 1968 for $t \leq 7$ by Mader [44]. The case $t = 6$ was also proved by Györi [24] in 1982, independent of Mader.

Theorem 1.4.1 (Dirac [15], Mader [44]) *For* $t \leq 7$*, any graph on* $n \geq t$ *vertices with at least* $(t-2)n - \binom{t-1}{2} + 1$ *edges has a* K_t*-minor.*

While this bound holds for $t \leq 7$, for larger values of t, counterexamples to this bound have been found. To describe some of these counterexamples, we must define an (H_1, H_2, k)-*cockade*, which we do recursively. For graphs H_1, H_2 and an integer k, we define any graph isomorphic to either of H_1 or H_2 to be an (H_1, H_2, k)-cockade. Now, given two (H_1, H_2, k)-cockades G_1 and G_2, any graph G obtained from the disjoint union of G_1 and G_2 by identifying a clique of size k in each of G_1 and G_2 is an (H_1, H_2, k)-cockade. Every (H_1, H_2, k)-cockade can be constructed in this fashion. If $H_1 = H_2 = H$, then we simply write (H, k)-cockade.

The extremal function for K_t-minors for $t = 8$ was shown in 1994 by Jørgensen [35] and for $t = 9$ in 2006 by Song and Thomas [67].

Theorem 1.4.2 (Jørgensen [35]) *Every graph on* $n \geq 8$ *vertices with at least* $6n - 20$ *edges either contains* K_8 *as a minor or is isomorphic to a* $(K_{2,2,2,2,2}, 5)$*-cockade.*

Theorem 1.4.3 (Song and Thomas [67]) *Every graph on* $n \geq 9$ *vertices with at least* $7n-27$ *edges either contains* K_9 *as a minor, or is isomorphic to* $K_{2,2,2,3,3}$*, or is isomorphic to a* $(K_{1,2,2,2,2,2,2}, 6)$*-cockade.*

The problem remains open for $t \geq 10$, though some partial results for $t = 10$ and $t = 11$ have been given by Song in [65].

Note that it follows immediately from Theorem 1.4.1, Theorem 1.4.2, Theorem 1.4.3, and Proposition 1.5.1(i) that for any integer $0 \leq t \leq 9$, any graph with no K_t-minor is $(2t - 5)$-colorable.

The extremal function for K_t^--minors and $K_t^=$-minors has also been studied.

The extremal function for K_t^--minors was found for $t \in \{5, 6\}$ in 1964 by Dirac [15]. It was then shown for $t = 7$ in 1983 by Jakobsen [33, 34]. Most recently, it was shown in 2005 for $t = 8$ by Song [64]. The extremal function problem for K_t^--minors remains open for $t \geq 9$.

Theorem 1.4.4 (Dirac [15]) *For* $t = 5, 6$*, if* G *is a graph with* $n \geq t$ *vertices and at least* $\left(t - \frac{5}{2}\right) n - \frac{1}{2}(t-3)(t-1)$ *edges, then* G *contains* K_t^- *as a minor, or* G *is a* $(K_{t-1}, t-3)$*-cockade.*

Theorem 1.4.5 (Jakobsen [33, 34]) *If* G *is a graph with* $n \geq 7$ *vertices and at least* $\frac{9}{2}n - 12$ *edges, then* G *contains* K_7^- *as a minor, or* G *is a* $(K_{2,2,2,2},\ K_6,\ 4)$*-cockade.*

Theorem 1.4.6 (Song [64]) *If* G *is a graph with* $n \geq 8$ *vertices and at least* $\frac{1}{2}(11n - 35)$ *edges, then* G *contains* K_8^- *as a minor, or* G *is a* $(K_{1,2,2,2,2},\ K_7,\ 5)$*-cockade.*

The extremal function for $K_t^=$-minors was found for $t \in \{5, 6\}$ in 1964 by Dirac [15]. It was found for $t \in \{7, 8\}$ in 1971 and 1972, respectively, by Jakobsen [32, 33]. The extremal function problem for $K_t^=$-minors remains open for $t \geq 9$.

Theorem 1.4.7 (Dirac [15], Jakobsen [32, 33]) *For any integer t with* $5 \leq t \leq 8$*, every graph with* $n \geq t$ *vertices and at least* $(t-3)n - \frac{1}{2}(t-1)(t-4)$ *edges either contains a* $K_t^=$*-minor or is a* $(K_{t-1}, t-4)$*-cockade.*

For general graphs H, some extremal function results are known. The extremal function for $K_{3,3}$-minors has been shown by Hall [26]; for integers $t \geq 2$, Chudnovsky, Reed, and Seymour [11] gave the extremal function for $K_{2,t}$-minors; Ding [14] has proved the extremal function for $K_{2,2,2}$-minors; and the extremal function for P-minors has been shown by Hendry and Wood [30], where P is the Petersen graph.

1.5 Contraction-Critical Graphs

A graph G is *t-color-critical*, or simply *t-critical*, if G is t-chromatic and $\chi(H) < \chi(G)$ for any proper subgraph H of G. A graph G is *t-contraction-critical* if G is t-chromatic and $\chi(H) < \chi(G)$ for any proper minor H of G.

The motivation behind studying contraction-critical graphs is that if a minimum counterexample to Hadwiger's Conjecture exists, then it can be taken to be contraction-critical. This is clear, since if a graph G does not contain K_t as a minor, then no minor H of G can contain K_t as a minor either. The only known examples of contraction-critical graphs are the complete graphs K_t. Contraction-critical graphs were first studied by Dirac [18, 16]. The following basic properties of contraction-critical graphs are a result of this initial work.

Proposition 1.5.1 (Dirac [18, 16]) *If G is a non-complete k-contraction-critical graph, then the following hold:*

(i) $\delta(G) \geq k$,

(ii) *for any* $x \in V(G)$, $\alpha(G[N_G(x)]) \leq d(x) - k + 2$,

(iii) *no minimal separating set S of G can be partitioned into a clique and an independent set, and*

(iv) *for* $k \geq 5$, *G is 5-connected.*

As an improvement of Proposition 1.5.1(iv), in 1968 Mader [46] proved the following deep and long-standing result on the connectivity of contraction-critical graphs.

Theorem 1.5.2 (Mader [46]) *If G is a k-contraction-critical graph with $k \geq 6$, then*

(i) *G is 6-connected for* $k = 6$, *and*

(ii) *G is 7-connected for* $k \geq 7$.

It seems very difficult to improve Theorem 1.5.2 for small values of k. For larger values of k, some better results can be found. Kawarabayashi [36] has shown that any minimal non-complete k-contraction-critical graph with no K_k-minor is $\lceil \frac{2k}{27} \rceil$-connected, while Kawarabayashi and Yu [40] have improved that by showing that any minimal such graph is $\lceil \frac{k}{9} \rceil$-connected. Chen, Hu, and

Song [8] recently improved the bound further by showing that any minimal such graph is $\lceil \frac{k}{6} \rceil$-connected.

The following Lemma 1.5.3 represents the linchpin of our arguments proving Theorem 1.3.7, Theorem 1.3.8, and Theorem 1.3.9. Lemma 1.5.3 turns out to be incredibly powerful, as it provides a way to circumvent the connectivity restriction of Theorem 1.5.2 for small values of k. A path consisting of vertices of only two (alternating) colors is a *Kempe chain*. Given a graph G, any $e \notin E(G)$ is a *missing edge* of G. Lemma 1.5.3 turns out to be very powerfule, as it provides us with many paths, specifically Kempe chains, connecting ends of missing edges without requiring the graph G to have high connectivity. We prove Lemma 1.5.3 in Chapter 3.

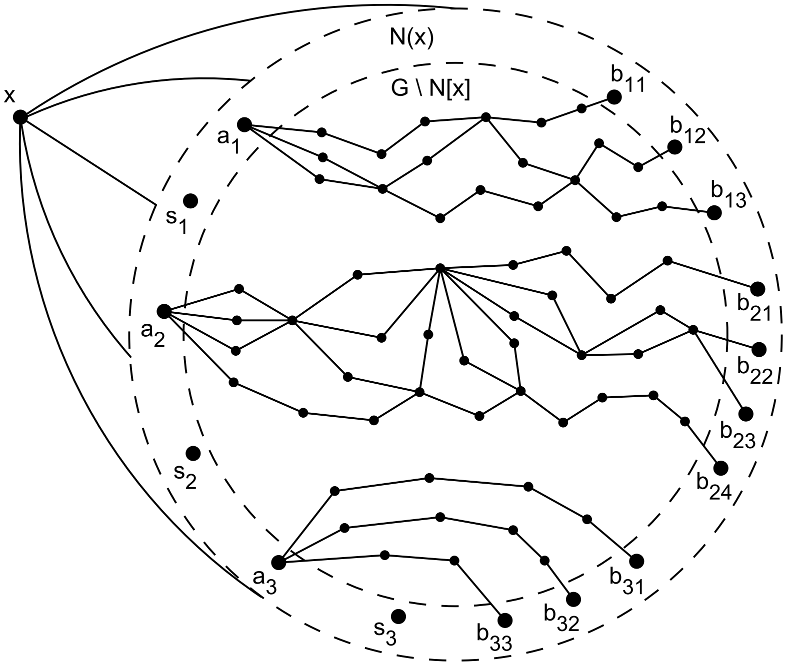

Figure 1.3: Kempe chains given by Lemma 1.5.3 with ends in $N_G(x)$. Here, $S = \{s_1, s_2, s_3\}$ and $M = \{\{a_1b_{11}, a_1b_{12}, a_1b_{13}\}, \{a_2b_{21}, a_2b_{22}\}, \{a_3b_{31}\}\}$.

Lemma 1.5.3 [58] *Let G be any k-contraction-critical graph. Let $x \in V(G)$ be a vertex of degree $k + s$ with $\alpha(G[N_G(x)]) = s + 2$ and let $S \subset N_G(x)$ with $|S| = s + 2$ be any independent* *set, where $k \geq 4$ and $s \geq 0$ are integers. If $N_G(x) \setminus S$ is not a clique, then for any $M = \{\{a_1b_{11}, \dots, a_1b_{1r_1}\}, \{a_2b_{21}, \dots, a_2b_{2r_2}\}, \dots, \{a_mb_{m1}, \dots, a_mb_{mr_m}\}\}$, where $m, r_i \geq 1$, $r_1 + r_2 + \cdots + r_m + m \leq k - 2$, the vertices $a_1, \dots, a_m, b_{11}, \dots, b_{mr_m} \in N_G(x) \setminus S$ are all distinct, and for any $i \in \{1, 2, \dots, m\}$, the set $\{a_ib_{i1}, \dots, a_ib_{ir_i}\}$ consists of r_i missing edges of $G[N_G(x) \setminus S]$ with a_i as a common end, then for each $i \in \{1, 2, \dots, m\}$ there exist paths $P_{i1}, \dots, P_{ir_i}$ in G such that each P_{ij} has ends a_i, b_{ij} and all its internal vertices in $V(G) \setminus N_G[x]$ for all $j = 1, 2, \dots, r_i$. Moreover, for any $1 \leq i < \ell \leq m$, the paths $P_{i1}, \dots, P_{ir_i}$ are vertex-disjoint from the paths $P_{\ell 1}, \dots, P_{\ell r_\ell}$.*

1.6 Mader's H-Wege Theorem

In addition to Lemma 1.5.3, another important tool that can be used to help find minors in graphs of low connectivity is a deep result of Mader [45], referred to as Mader's H-Wege Theorem, which states the following.

Theorem 1.6.1 Mader's H-Wege Theorem. (Mader [45]) *Let G be a graph, let $S \subseteq V(G)$ be an independent set, and let $k \geq 0$ be an integer. Then exactly one of the following holds:*

- (i) *There are k paths of G, each with distinct ends both in S, such that each $v \in V(G) \setminus S$ is in at most one of the paths.*
- (ii) *There exist $W \subseteq V(G) \setminus S$ and a partition $Y_1, \dots, Y_n$ of $V(G) \setminus (S \cup W)$, and for $1 \leq i \leq n$ a subset $X_i \subseteq Y_i$, such that*
  - (a) $|W| + \sum_{i=1}^{n} \left\lfloor \frac{1}{2}|X_i| \right\rfloor < k$,
  - (b) *no vertex in $Y_i \setminus X_i$ has a neighbor in $V(G) \setminus (W \cup Y_i)$, and*
  - (c) *every path of $G \setminus W$ with distinct ends both in S has an edge with both ends in Y_i for some i.*

Theorem 1.6.1 is often referred to in the literature as Mader's S-Paths Theorem. An alternative and much shorter proof of Theorem 1.6.1 has been given by Schrijver [63]. Given a graph G, let $H_1, \ldots, H_t$ be subsets of $V(G)$. We say a path in G with ends u, v is *good* if there exist distinct $i, j \in \{1, \ldots, n\}$ such that $u \in H_i$ and $v \in H_j$. Note that any vertex in $H_i \cap H_j$ with $i \neq j$ is considered to be a good path consisting of only a single vertex. In 1993, Robertson, Seymour, and Thomas [55] gave a slight modification of Mader's H-Wege Theorem (1.6.1) which allowed for slightly easier application (see Figure 1.4).

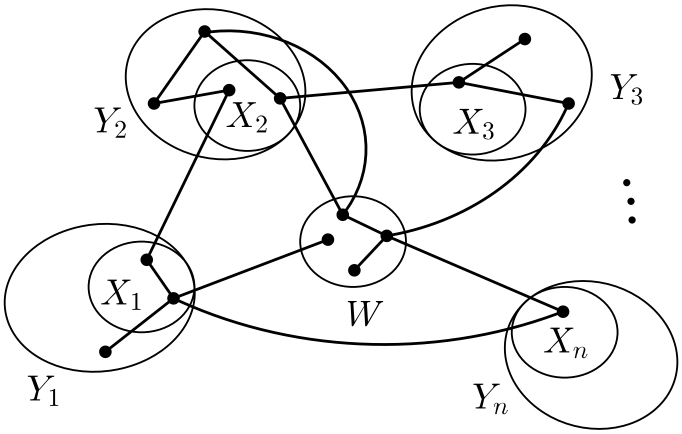

Figure 1.4: The sets $W, X_1, Y_1, \ldots, X_n, Y_n$ and some edges allowed by Theorem 1.6.2(ii).

Theorem 1.6.2 (Robertson, Seymour, and Thomas [55]) *Let G be a graph, let $H_1, \ldots, H_t$ be subsets of $V(G)$, and let $k \geq 0$ be an integer. Then exactly one of the following holds:*

(i) *There are k good paths of G, mutually vertex-disjoint.*

(ii) *There exists $W \subseteq V(G)$ and a partition $Y_1, \ldots, Y_n$ of $V(G) \setminus W$, and for $1 \leq i \leq n$ a subset $X_i \subseteq Y_i$, such that*

- (a) $|W| + \sum_{i=1}^{n} \left\lfloor \frac{1}{2}|X_i| \right\rfloor < k$,
- (b) *for $1 \leq i \leq n$, no vertex in $Y_i \setminus X_i$ has a neighbor in $V(G) \setminus (W \cup Y_i)$, and $Y_i \cap H_j \subseteq X_i$ for $1 \leq j \leq t$, and*

(c) *every good path P in G with $V(P) \cap W = \emptyset$ has an edge with both ends in Y_i for some $1 \leq i \leq n$.*

Using Theorem 1.6.2, Robertson, Seymour, and Thomas [55] were able to prove the following theorem utilized in their paper on the case $t = 6$ of Hadwiger's Conjecture.

Theorem 1.6.3 (Robertson, Seymour, and Thomas [55]) *Let G be a 6-connected graph such that $G - x$ is nonplanar for all $x \in V(G)$. If G contains three different subgraphs isomorphic to K_4, say L_1, L_2, and L_3, such that $|L_i \cap L_j| \leq 2$ for $1 \leq i < j \leq 3$, then G contains a K_6-minor.*

The application of Theorem 1.6.2 in [55] is complex and long. Kawarabayashi and Toft [39] used Theorem 1.6.2 in the same manner to prove a result extending Theorem 1.6.3 in their proof that every graph with no K_7-minor or $K_{4,4}$-minor is 6-colorable.

Theorem 1.6.4 (Kawarabayashi and Toft [39]) *Let G be a 7-connected graph with at least 19 vertices. If G contains three different subgraphs isomorphic to K_5, say L_1, L_2, and L_3, such that $|L_1 \cup L_2 \cup L_3| \geq 12$, then G contains a K_7-minor.*

Kawarabayashi, Luo, Niu, and Zhang [38] then extended Theorem 1.6.3 and Theorem 1.6.4 to find a K_t-minor for $t \geq 5$, again using Theorem 1.6.2.

Theorem 1.6.5 (Kawarabayashi, Luo, Niu, and Zhang [38]) *Let G be a $(t + 2)$-connected graph, where $t \geq 5$. If G contains three different subgraphs isomorphic to K_t, say L_1, L_2, and L_3, such that $|L_1 \cup L_2 \cup L_3| \geq 3t - 3$, then G contains a K_{t+2}-minor.*

Both Theorem 1.6.3 and Theorem 1.6.4 have had application in results related to Hadwiger's Conjecture. The first new case given by Theorem 1.6.5 is that when $t = 6$. In this case, we require a graph to be 8-connected to be able to find a K_8-minor. As discussed in Section 1.5, t-contraction-critical graphs have not yet been shown to be 8-connected for any small values of t. The best result for small values of t is given by Theorem 1.5.2, namely that t-contraction-critical graphs are 7-connected for $t \geq 7$. In Section 4.1, we prove Theorem 1.6.6 which can be used to find a K_8-minor in a 7-connected graph, albeit with additional restrictions.

Theorem 1.6.6 [56] *Let G be a 7-connected graph with $\delta(G) \geq 8$, and let $H_1, H_2, H_3 \subseteq V(G)$ be such that $G[H_1]$, $G[H_2]$, and $G[H_3]$ are three different subgraphs of G isomorphic to K_6. Then G contains a K_8-minor if all of the following conditions are satisfied:*

- (A) *for any minimum separating set S of G, $G-S$ has at most two components, $\Delta(G[S]) \leq 4$, and S cannot be partitioned into two sets such that one induces a clique in G and the other induces an independent set in G,*
- (B) $|H_1 \cap H_2| = 1$*, and the vertex in $H_1 \cap H_2$ has at most 11 neighbors in G, and*
- (C) $H_1 \cap H_2 \cap H_3 = \emptyset$*, and* $|(H_1 \cap H_2) \cup (H_2 \cap H_3) \cup (H_3 \cap H_1)| \leq 4$.

While the conditions required for Theorem 1.6.6 may initially seem restrictive, most of them follow from the properties of contraction-critical graphs. If G is k-contraction-critical for $k \geq 8$, then G is 7-connected by Theorem 1.5.2(ii) and has $\delta(G) \geq k$ by Proposition 1.5.1. It is also possible in k-contraction-critical graphs that vertices of minimum degree can have two disjoint cliques in their neighborhood. Suppose S is a minimum separating set of a k-contraction-critical graph G. If we can select two vertices x and y of minimum degree from distinct components of $G-S$ such that $G[N_G(x)]$ and $G[N_G(y)]$ each contain two disjoint cliques, then from this, (B) and (C) should follow. Only condition (A) may be difficult to verify in general.

CHAPTER 2: DOUBLE-CRITICAL GRAPHS

2.1 Graphs on a Small Number of Vertices

We begin this chapter by introducing several Lemmas that will be necessary in Section 2.3. The first Lemma is a result of Jørgensen [35].

Lemma 2.1.1 (Jørgensen [35]) *Let G be a graph with $n \leq 11$ vertices and $\delta(G) \geq 6$ such that for every vertex x in G, $G - x$ does not contain K_6 as a minor. Then G is one of the graphs $K_{2,2,2,2}$, $K_{3,3,3}$ or the complement of the Petersen graph.*

Given two graphs G_1 and G_2, the *union* $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. From Lemma 2.1.1, we deduce the following lemma which we will apply in a manner similar to that of Lemma 3.1.2.

Lemma 2.1.2 [59] *For $t \in \{1, 2, \ldots, 5\}$, let G be a graph with $n \leq 2t$ vertices and $\delta(G) \geq t$. Then $G > K_t \cup K_1$.*

Proof. Let G and t be given as in the statement. Consider the graph G' obtained from G by adding $6-t$ vertices, each adjacent to all other vertices in the graph. Then G' has at most $2t+6-t = t+6 \leq 11$ vertices and has $\delta(G') \geq t + 6 - t = 6$. Since none of $K_{2,2,2,2}$, $K_{3,3,3}$, or the complement of the Petersen graph has a vertex adjacent to all other vertices in the graph, it follows from Lemma 2.1.1 that G' contains some vertex x such that $G' - x > K_6$. If $x \in V(G)$, then $G - x > K_t$, and if $x \in V(G')$, then $G > K_{t+1}$, so in either case it follows that $G > K_t \cup K_1$. ■

We will also need the following technical Lemma of Song and Thomas [67]. We note here that the proof of Lemma 2.1.3 is computer-assisted.

Lemma 2.1.3 (Song and Thomas [67]) *Let* $n \in \{9, 10, \ldots, 13\}$ *and let* G *be a graph on* n *vertices with* $\delta(G) \geq 7$*. Then either* $G > K_7 \cup K_1$*, or* G *satisfies the following two properties.*

- (A) *Either* G *is isomorphic to* $K_{1,2,2,2,2}$*, or* G *has four distinct vertices* a_1, b_1, a_2, b_2 *such that* $a_1a_2, b_1b_2 \notin E(G)$ *and for* $i \in \{1, 2\}$*, the vertex* a_i *is adjacent to* b_i*, the vertices* a_i *and* b_i *have at most four common neighbors, and* $G + a_1a_2 + b_1b_2 > K_8$*.*
- (B) *For any two sets* $A, B \subseteq V(G)$ *of cardinality at least five such that neither is complete and* $A \cup B$ *includes all vertices of* G *of degree at most* $|G| - 2$*, either*
  - $(B1)$ *there exist* $a \in A$ *and* $b \in B$ *such that* $G' > K_8$*, where* G' *is obtained from* G *by adding all edges* aa' *and* bb' *for* $a' \in A - \{a\}$ *and* $b' \in B - \{b\}$*, or*
  - $(B2)$ *there exist* $a \in A - B$ *and* $b \in B - A$ *such that* $ab \in E(G)$ *and the vertices* a *and* b *have at most five common neighbors in G, or*
  - $(B3)$ *one of* A *and* B *contains the other and* $G + ab > K_7 \cup K_1$ *for all distinct nonadjacent vertices* $a, b \in A \cap B$*.*

2.2 Properties of Noncomplete, Double Critical Graphs

In this section we introduce several known properties of noncomplete, double-critical t-chromatic graphs. We also prove some new results, including our main result of this chapter, Theorem 1.2.9 which follows as an immediate corollary of Theorem 2.2.6. We begin with some properties first developed by Kawarabayashi, Pedersen, and Toft in [37], the first paper to explore these graphs. Note that if G is a noncomplete, double-critical, t-chromatic graph, then it follows from Theorem 1.2.4 that $t \geq 6$.

Proposition 2.2.1 (Kawarabayashi, Pedersen, and Toft [37]) *If* G *is a non-complete, double-critical,* t*-chromatic graph, then all of the following are true:*

(i) *G does not contain K_{t-1} as a subgraph.*

(ii) *for all edges xy, every proper coloring $c : V(G)\backslash\{x, y\} \rightarrow \{1, 2, \ldots, t-2\}$ of $G-\{x, y\}$, and any non-empty sequence $j_1, j_2, \ldots, j_i$ of i different colors from $\{1, 2, \ldots, t-2\}$, there is a path of order $i + 2$ with vertices $x, v_1, v_2, \ldots, v_i, y$ in order such that v_k is colored j_k for all $k \in \{1, 2, \ldots, i\}$.*

(iii) *for any edge $xy \in E(G)$, x and y have at least one common neighbor in every color class of any $(t-2)$-coloring of $G - \{x, y\}$, in particular, every edge $xy \in E(G)$ belongs to at least $t-2$ triangles.*

(iv) *there exists at least one edge $xy \in E(G)$ such that x and y share a common non-neighbor in G.*

(v) *for any edge $xy \in E(G)$, the subgraph of G induced by $N_G(x) \setminus N_G[y]$ contains no isolated vertices. In particular, no vertex can have degree one in $\overline{G[N_G(x)]}$.*

(vi) $\delta(G) \geq t + 1$.

(vii) *for any vertex $x \in V(G)$, $\alpha(G[N_G(x)]) \leq d_G(x) - t + 1$. Furthermore, for any vertex y in a maximum independent set $A \subseteq N_G(x)$, we have $|N_G(x) \cap N_G(y)| \leq d_G(x) - \alpha(N_G(x)) - 1$.*

$(viii)$ *for any vertex x with at least one non-neighbor in G, $\chi(G[N_G(x)]) \leq t - 3$.*

(ix) *for any $x \in V(G)$ with $d_G(x) = t + 1$, $\overline{G[N_G(x)]}$ consists of isolated vertices and cycles of length at least five.*

(x) *no two vertices of degree $t + 1$ are adjacent in G.*

(xi) *G is 6-connected and no minimal separating set of G can be partitioned into two sets A and B such that A is an independent set and $G[B]$ is complete.*

We will first give a slight improvement of Proposition 2.2.1(xi). It seems hard to use the main idea in the proof of Proposition 2.2.1(xi) to prove that any non-complete, double-critical, t-chromatic graph is 7-connected. Instead, we can say a bit more about minimal separating sets of size 6 in such graphs. We say two proper vertex-colorings c_1 and c_2 of a graph G are *equivalent* if, for all $x, y \in V(G)$, $c_1(x) = c_1(y)$ if and only if $c_2(x) = c_2(y)$. For any $A \subseteq V(G)$, we say that two vertex-colorings c_1 and c_2 of G are *equivalent on* A if the restrictions $c_1|_A$ and $c_2|_A$ to A are equivalent on the subgraph $G[A]$. Let S be a separating set of G, and let G_1, G_2 be connected subgraphs of G such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = G[S]$. If c_1 is a t-coloring of G_1 and c_2 is a t-coloring of G_2 such that c_1 and c_2 are equivalent on S, then it is clear that c_1 and c_2 can be combined to give a t-coloring of G by a suitable permutation of the color classes of, say c_2.

Lemma 2.2.2 [59] *Suppose G is a non-complete, double-critical, t-chromatic graph. If S is a minimal separating set of G with $|S| = 6$, then either $G[S] \subseteq K_{3,3}$ or $G[S] \subseteq K_{2,2,2}$.*

Proof. Suppose G is a non-complete, double-critical, t-chromatic graph. By Proposition 2.2.1(xi), G is 6-connected. Let $S = \{v_1, \dots, v_6\} \subset V(G)$ be a minimal separating set of G such that neither $G[S] \subseteq K_{3,3}$ nor $G[S] \subseteq K_{2,2,2}$. Let H be a component of $G - S$, and let $G_1 = G[V(H) \cup S]$ and $G_2 = G - V(H)$. Then $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = S$. Since $t \geq 6$ by Theorem 1.2.4, we have $\delta(G) \geq 7$ by Proposition 2.2.1(vi). In particular, since $|S| = 6$, there must exist at least one edge in each of $G_1 - S$ and $G_2 - S$. It follows then that both G_1 and G_2 are $(t-2)$-colorable. Let c_1 and c_2 be $(t-2)$-colorings of G_1 and G_2, respectively. For any set $A \subseteq V(G)$ and $i \in \{1, 2\}$, define $|c_i(A)|$ to be the number of distinct colors assigned to the vertices of A by c_i. Utilizing a new color, say α, we will redefine the colorings c_1 and c_2 so that c_1 and c_2 are $(t-1)$-colorings of G_1 and G_2, respectively, and are equivalent on S. This yields a contradiction, as c_1 and c_2, after a suitable permutation of the colors of c_2, can be combined to give a $(t-1)$-coloring of G.

By Proposition 2.2.1(xi), $\alpha(G[S]) \leq 4$ and so neither c_1 nor c_2 applies the same color to more than four vertices of S. Suppose that one of the colorings c_1 and c_2, say c_1, assigns the same color to four vertices of S, say $c_1(v_3) = c_1(v_4) = c_1(v_5) = c_1(v_6)$. Then $\{v_3, v_4, v_5, v_6\}$ is an independent set in G. Since $G[S] \not\subseteq K_{2,2,2}$, we have $c_2(v_1) \neq c_2(v_2)$. Now redefining $c_2(v_3) = c_2(v_4) = c_2(v_5) = c_2(v_6) = \alpha$ and $c_1(v_1) = \alpha$ will, after a suitable permutation of the colors of c_2, make c_1 and c_2 equivalent on S using $t-1$ colors. Hence neither c_1 nor c_2 assigns the same color to four distinct vertices of S.

Next suppose that one of the colorings c_1 and c_2, say c_1, assigns the same color to three vertices of S, say $c_1(v_4) = c_1(v_5) = c_1(v_6)$. Then $\{v_4, v_5, v_6\}$ is an independent set in G. Since $G[S] \not\subseteq K_{3,3}$, we have $|c_2(\{v_1, v_2, v_3\})| \geq 2$. If $|c_2(\{v_1, v_2, v_3\})| = 2$, we may assume that $c_2(v_2) = c_2(v_3)$. Then $\{v_2, v_3\}$ is an independent set. Then redefining $c_2(v_4) = c_2(v_5) = c_2(v_6) = \alpha$ and $c_1(v_2) = c_1(v_3) = \alpha$ will, after a suitable permutation of the colors of c_2, make c_1 and c_2 equivalent on S using $t-1$ colors, a contradiction. Thus $|c_2(\{v_1, v_2, v_3\})| = 3$ and so c_2 assigns distinct colors to each of v_1, v_2, and v_3. We redefine $c_2(v_4) = c_2(v_5) = c_2(v_6) = \alpha$. Clearly c_1 and c_2 are equivalent on S if c_1 assigns distinct colors to each of v_1, v_2, v_3. Thus $|c_1(\{v_1, v_2, v_3\})| \leq 2$. Since $G[S] \not\subseteq K_{3,3}$, we have $|c_1(\{v_1, v_2, v_3\})| = 2$. We may assume that $c_1(v_2) = c_1(v_3)$. Now redefining $c_1(v_3) = \alpha$ yields, after a suitable permutation of the colors of c_2, that c_1 and c_2 are equivalent on S. This proves that neither c_1 nor c_2 assigns the same color to three distinct vertices of S. Thus $|c_i(S)| \geq 3$ for $i \in \{1, 2\}$. Since $G[S] \not\subseteq K_{2,2,2}$ and neither c_1 nor c_2 assigns the same color to more than two vertices of S, we have $|c_i(S)| \geq 4$ for $i \in \{1, 2\}$.

Now by symmetry we may assume that $|c_1(S)| \geq |c_2(S)|$. Clearly c_1 and c_2 are not equivalent on S, for otherwise c_1 and c_2, after a suitable permutation of the colors of c_2, can be combined to give a $(t-2)$-coloring of G, a contradiction. Thus $|c_2(S)| \leq 5$. Suppose that $|c_2(S)| = 5$. Then either $|c_1(S)| = 5$ or $|c_1(S)| = 6$. Then we can make c_1 and c_2 equivalent on S by assigning the color α to one of the two vertices of S that are colored the same color by c_2 and, if $|c_1(S)| = 5$, similarly assigning the color α to one of the two vertices of S that are colored the same color by c_1. Thus $|c_2(S)| = 4$. Since neither c_1 nor c_2 assigns the same color to more than two distinct vertices of S, we may assume that $c_2(v_3) = c_2(v_4)$ and $c_2(v_5) = c_2(v_6)$. Then $v_3v_4, v_5v_6 \notin E(G)$. Since

$G[S] \not\subseteq K_{2,2,2}$, we have $v_1v_2 \in E(G)$. Thus $c_1(v_1) \neq c_1(v_2)$. We may assume that $c_1(v_3) \neq c_1(v_4)$ as c_1 and c_2 are not equivalent on S. If $|c_1(S)| = 6$, then redefining $c_1(v_5) = c_1(v_6) = \alpha$ and $c_2(v_3) = \alpha$ will, after a suitable permutation of the colors of c_2, make c_1 and c_2 equivalent on S using $t-1$ colors, a contradiction. Suppose now $|c_1(S)| = 5$ and that, say, v_5 is one of the two vertices of S assigned the same color by S. If $c_1(v_5) = c_1(v_6)$, then we redefine $c_2(v_3) = \alpha$; if $c_1(v_5) = c_1(v_3)$, say, then we redefine $c_2(v_3) = c_2(v_5) = \alpha$; and if $c_1(v_5) = c_1(v_1)$, say, then we redefine $c_1(v_3) = c_1(v_4) = \alpha$ and $c_2(v_2) = c_2(v_5) = \alpha$. In each case we see, after a suitable permutation of the colors of c_2, that c_1 and c_2 are equivalent on S using $t-1$ colors, a contradiction. Hence we may assume that, say, v_1 and v_3 are the two vertices of S assigned the same color by c_1. Now redfining $c_1(v_5) = c_1(v_6) = \alpha$ and $c_2(v_1) = c_2(v_3) = \alpha$ yields that c_1 and c_2 are $(t-1)$-colorings equivalent on S, a contradiction. Thus $|c_1(S)| = 4$. Suppose $c_1(v_5) = c_1(v_6)$. Since $v_1v_2 \in E(G)$, we may assume that $c_1(v_1) = c_1(v_3)$. Now redefining $c_1(v_3) = c_1(v_4) = \alpha$ will make c_1 and c_2 equivalent on S. Thus $c_1(v_5) \neq c_1(v_6)$. Let A and B be the two color classes of c_1 on S with $|A| = |B| = 2$. Suppose $v_1 \in A$ and $v_2 \in B$. Since $G[S] \not\subseteq K_{2,2,2}$, we cannot have either $\{v_3, v_4\} \subseteq A \cup B$ or $\{v_5, v_6\} \subseteq A \cup B$. Hence we may assume that, say, $v_3 \in A$ and $v_5 \in B$. Redefining $c_1(v_5) = c_1(v_6) = \alpha$ and $c_2(v_1) = c_2(v_3) = \alpha$ will make c_1 and c_2 equivalent on S. Thus $\{v_1, v_2\} \not\subseteq A \cup B$. Suppose $v_1 \in A$. By symmetry, we may assume $B = \{v_3, v_5\}$. If $v_4 \in A$, then we redefine $c_1(v_5) = c_1(v_6) = \alpha$ and $c_2(v_1) = c_2(v_4) = \alpha$; and if $v_6 \in A$, then we redefine $c_1(v_5) = c_1(v_6) = \alpha$ and $c_2(v_3) = \alpha$. In either case we see, after a suitable permutation of the colors of c_2, that c_1 and c_2 are equivalent on S using $t-1$ colors, a contradiction. Hence we may assume by symmetry that $A = \{v_4, v_6\}$ and $B = \{v_3, v_5\}$. Now redefining $c_1(v_5) = c_1(v_6) = \alpha$ and $c_2(v_3) = \alpha$ will make c_1 and c_2 equivalent on S. This completes the proof of Lemma 2.2.2. ■

Lemma 2.2.3 [57] *Let G be a double-critical, t-chromatic graph and let $x \in V(G)$. If $d_G(x) = |V(G)| - 1$, then $G - x$ is a double-critical, $(t-1)$-chromatic graph.*

Proof. Let uv be any edge of $G - x$. Clearly, $\chi(G - x) = t - 1$. Since G is double-critical, $\chi(G - \{u, v\}) = t - 2$ and so $\chi(G - \{u, v, x\}) = t - 3$ because x is adjacent to all the other vertices in $G - \{u, v\}$. Hence $G - x$ is double-critical and $(t - 1)$-chromatic. ■

Lemma 2.2.4 [57] *If G is a non-complete, double-critical, t-chromatic graph, then for any $x \in V(G)$ with at least one non-neighbor in G, $\chi(G - N_G[x]) \geq 3$. In particular, $G - N_G[x]$ must contain an odd cycle, and so $d_G(x) \leq |V(G)| - 4$.*

Proof. Let x be any vertex in G with $d_G(x) < |V(G)| - 1$ and let $H = G - N_G[x]$. Suppose that $\chi(H) \leq 2$. Since $d_G(x) < |V(G)| - 1$, H contains at least one vertex. Let $y \in V(H)$ be adjacent to a vertex $z \in N_G(x)$. This is possible because G is connected. If H has no edge, then $G - (V(H) \cup \{z\})$ has a $(t - 2)$-coloring c, which can be extended to a $(t - 1)$-coloring of G by assigning all vertices in $V(H)$ the color $c(x)$ and assigning a new color to the vertex z, a contradiction. Thus H must contain at least one edge, and so $\chi(H) = 2$. Let (A, B) be a bipartition of H. Now $G - H$ has a $(t - 2)$-coloring c', which again can be extended to a $(t - 1)$-coloring of G by assigning all vertices in A the color $c'(x)$ and all vertices in B the same new color, a contradiction. This proves that $\chi(H) \geq 3$, and so H must contain an odd cycle. Therefore $d_G(x) \leq |V(G)| - 4$. ■

Lemma 2.2.5 [57] *Let G be a double-critical, t-chromatic graph. For any edge $xy \in E(G)$, let c be any $(t - 2)$-coloring of $G - \{x, y\}$ with color classes $V_1, V_2, \ldots, V_{t-2}$. Then the following two statements are true.*

- (i) *For any $i, j \in \{1, 2, \ldots, t - 2\}$ with $i \neq j$, if $N_G(x) \cap N_G(y) \cap V_i$ is anti-complete to $N_G(x) \cap V_j$, then there exists at least one edge between $(N_G(y) \setminus N_G(x)) \cap V_i$ and $N_G(x) \cap V_j$ in G. In particular, $(N_G(y) \setminus N_G(x)) \cap V_i \neq \emptyset$.*
- (ii) *Assume that $d_G(x) = t + 1$ and y belongs to a cycle of length $k \geq 5$ in $\overline{G[N_G(x)]}$.*
    - (a) *If $k \geq 7$, then $d_G(y) \geq t + e(\overline{G[N_G(x)]}) - 4$;*

(b) *If* $k = 6$*, then* $d_G(y) \geq \max\{t+2, t+e(\overline{G[N_G(x)]})-5\}$*; and*

(c) *If* $k = 5$*, then* $d_G(y) \geq \max\{t+2, t+e(\overline{G[N_G(x)]})-6\}$.

Proof. Let $G, x, y,$ and c be as given in the statement. For any $i, j \in \{1, 2, \ldots, t-2\}$ with $i \neq j$, if $N_G(x) \cap N_G(y) \cap V_i$ is anti-complete to $N_G(x) \cap V_j$, then G is non-complete. By Proposition 2.2.1(ii), there must exist a path x, u_j, u_i, y in G such that $c(u_j) = j$ and $c(u_i) = i$. Clearly, $u_j u_i \in E(G)$ and $u_j \in N_G(x) \cap V_j$. Since $N_G(x) \cap N_G(y) \cap V_i$ is anti-complete to $N_G(x) \cap V_j$, we see that $u_i \in (N_G(y) \setminus N_G(x)) \cap V_i$. This proves Lemma 2.2.5(i).

To prove Lemma 2.2.5(ii), let $H = \overline{G[N_G(x)]}$. Assume that $d_G(x) = t+1$ and that y belongs to a cycle, say C_k, of H, where $k \geq 5$. By Proposition 2.2.1(x), $d_G(y) \geq t+2$, and by Proposition 2.2.1(ix), H is the union of isolated vertices and cycles of length at least five. Clearly, $|N_G(x) \cap N_G(y)| = t-2$. By Proposition 2.2.1(iii), we may assume that $V_i \cap (N_G(x) \cap N_G(y)) = \{v_i\}$ for all $i \in \{1, \ldots, t-2\}$. Then $N_G(x) \cap N_G(y) = \{v_1, \ldots, v_{t-2}\}$. Let $\{a, b\} = N_G(x) \setminus N_G[y]$. Since a and b are neighbors of y in a cycle of length at least 5 in H, $ab \in E(G)$. We may further assume that $a \in V_1$ and $b \in V_2$. Then v_1, a, y, b, v_2 forms a path on five vertices of C_k, since $v_1, a \in V_1$ and $v_2, b \in V_2$. If $k \geq 6$, then $v_1 v_2 \in E(G)$ and both v_1 and v_2 have precisely one non-neighbor in $\{v_3, v_4, \ldots, v_{t-2}\}$. We may assume that $v_1 v_3 \notin E(G)$ and $v_2 v_\ell \notin E(G)$, where $\ell = 3$ if $k = 6$, and $\ell = 4$ if $k \geq 7$. For any $i, j \in \{3, 4, \ldots, t-2\}$ with $i \neq j$, if $v_i v_j \notin E(G)$, then by Lemma 2.2.5(i), there exists $v'_j \in V_j \setminus \{v_j\}$ such that $v'_j y \in E(G)$. By symmetry, there exists $v'_i \in V_i \setminus \{v_i\}$ such that $v'_i y \in E(G)$. Therefore, if C is any cycle in $H - V(C_k)$ and $V_m \cap V(C) \neq \emptyset$ for some $m \in \{3, 4, \ldots, t-2\}$, then y is adjacent to a vertex from $V_m \setminus \{v_m\}$.

Assume that $k = 5$. Then $v_1 v_2 \notin E(G)$, and so $d_G(y) \geq |N_G(x) \cap N_G(y)| + |\{x\}| + |E(H - V(C_k))| = (t-2) + 1 + (e(H) - 5) = t + e(H) - 6$. Next assume that $k = 6$. Then $v_\ell = v_3$. Since both $N_G(x) \cap N_G(y) \cap V_1$ and $N_G(x) \cap N_G(y) \cap V_2$ are anti-complete to $N_G(x) \cap V_3$, by Lemma 2.2.5(i), $N_G(y) \cap (V_1 \setminus \{a, v_1\}) \neq \emptyset$ and $N_G(y) \cap (V_2 \setminus \{b, v_2\}) \neq \emptyset$. Then $d_G(y) \geq$

$|N_G(x) \cap N_G(y)| + |\{x\}| + |N_G(y) \cap (V_1 \setminus \{a, v_1\})| + |N_G(y) \cap (V_2 \setminus \{b, v_2\})| + |E(H - V(C_k))| \geq (t-2) + 1 + 1 + 1 + (|E(H)| - 6) = t + |E(H)| - 5$. Finally assume that $k \geq 7$. Then $v_\ell = v_4$. Since $N_G(x) \cap N_G(y) \cap V_1$ is anti-complete to $N_G(x) \cap V_3$ and $N_G(x) \cap N_G(y) \cap V_2$ is anti-complete to $N_G(x) \cap V_4$, by Lemma 2.2.5(i), $N_G(y) \cap (V_1 \setminus \{a, v_1\}) \neq \emptyset$ and $N_G(y) \cap (V_2 \setminus \{b, v_2\}) \neq \emptyset$. As observed earlier, for any $i, j \in \{3, 4, \ldots, t-2\}$ with $i \neq j$ and $v_i v_j \notin E(G)$, y has at least one neighbor in each of $V_i \setminus \{v_i\}$ and $V_j \setminus \{v_j\}$ in G. Hence $d_G(y) \geq |N_G(x) \cap N_G(y)| + |\{x\}| + |N_G(y) \cap (V_1 \setminus \{a, v_1\})| + |N_G(y) \cap (V_2 \setminus \{b, v_2\})| + |V(C_k) \setminus \{a, b, v_1, v_2, y\}| + |E(H - V(C_k))| \geq (t-2) + 1 + 1 + 1 + (k-5) + (|E(H)| - k) = t + e(H) - 4$. Note that since $k \geq 7$, $|E(H)| \geq 7$, and so $d_G(y) \geq t + |E(H)| - 4 > t + 2$. This completes the proof of Lemma 2.2.5(ii). ■

We now conclude this section with the following result. It is clear that Theorem 2.2.6 immediately implies Theorem 1.2.9.

Theorem 2.2.6 [57] *If G is a non-complete, double-critical, t-chromatic graph with $t \geq 6$, then for any vertex $x \in V(G)$ with $d_G(x) = t+1$, the following hold:*

- (i) $e(\overline{G[N_G(x)]}) \geq 8$*; and*
- (ii) *for any vertex $y \in N_G(x)$, $d_G(y) \geq t+4$. Furthermore, if $d_G(y) = t+4$ then $|N_G(x) \cap N_G(y)| = t-2$ and $\overline{G[N_G(x)]}$ contains either only one cycle, which is isomorphic to C_8, or exactly two cycles, each of which is isomorphic to C_5.*

Proof. Let G and x be as given in the statement. Let $H = \overline{G[N_G(x)]}$. Then $|V(H)| = t+1$. Note that if $d_G(x) = |V(G)| - 1$, then it follows from Proposition 2.2.1(vi) that G is isomorphic to K_{t+1}, a contradiction. Thus $d_G(x) < |V(G)| - 1$. Now by Proposition 2.2.1(vii) and Proposition 2.2.1(viii) applied to the vertex x, $\alpha(\overline{H}) \leq 2$ and $\chi(\overline{H}) \leq t-3$. Let c' be any $(t-3)$-coloring of $\overline{H}$. Then each color class of c' contains at most two vertices. Since $|V(H)| = t+1$, we see that at least four color classes of c' must each contain two vertices. By Proposition 2.2.1(v), $\overline{H}$ has at least eight vertices of degree two in H and so $e(H) \geq 8$. This proves Theorem 2.2.6(i).

To prove Theorem 2.2.6(ii), let $y \in N_G(x)$. Since $d_G(x) = t+1$, by Proposition 2.2.1(ix), either $|N_G(x) \cap N_G(y)| = t$ or $|N_G(x) \cap N_G(y)| = t-2$. Assume that $|N_G(x) \cap N_G(y)| = t-2$. Then y belongs to a cycle of length $k \geq 5$ in H because H is a disjoint union of isolated vertices and cycles by Proposition 2.2.1(ix). By Theorem 2.2.6(i), $e(H) \geq 8$. Note that if $5 \leq k \leq 7$, then by Proposition 2.2.1(ix), H has at least two cycles of length at least 5, and so $e(H) \geq k+5 \geq 10$. Thus by Lemma 2.2.5(ii), $d_G(y) \geq t+4$. If $d_G(y) = t+4$, then it follows from Lemma 2.2.5(ii) that either $k = 8$ and H is isomorphic to $C_8 \cup \overline{K_{t-7}}$ or $k = 5$ and H is isomorphic to $C_5 \cup C_5 \cup \overline{K_{t-9}}$. So we may assume that $|N_G(x) \cap N_G(y)| = t$. Let c be any $(t-2)$-coloring of $G - \{x, y\}$ with color classes $V_1, V_2, \ldots, V_{t-2}$. Since $\alpha(\overline{H}) \leq 2$, we may further assume that $N_G(x) \cap V_1 = \{v_1, v_1'\}$, $N_G(x) \cap V_2 = \{v_2, v_2'\}$ and $N_G(x) \cap V_i = \{v_i\}$ for all $i \in \{3, 4, \ldots, t-2\}$. Then $v_1v_1', v_2v_2' \in E(H)$. By Proposition 2.2.1(i) applied to the vertex x, $e_H(\{v_1, v_1', v_2, v_2'\}, \{v_3, v_4, \ldots, v_{t-2}\}) \leq 4$. By Theorem 2.2.6(i), $e(H) \geq 8$. Thus there must exist at least four vertices in $\{v_3, v_4, \ldots, v_{t-2}\}$, say v_3, v_4, v_5, v_6, such that $d_H(v_j) = 2$ and y is adjacent to at least one vertex of $V_j \setminus \{v_j\}$ in G for all $j \in \{3, 4, 5, 6\}$. Therefore $|N_G(y) \setminus N_G[x]| \geq 4$ and so $d_G(y) = |N_G[x] \cap N_G(y)| + |N_G(y) \setminus N_G[x]| \geq (t+1) + 4 = t+5$. This completes the proof of Theorem 2.2.6. ■

2.3 Minors in Double-Critical Graphs

In this section, we will prove Theorem 1.2.7. To accomplish this, we will actually prove the following much stronger result, from which Theorem 1.2.7 follows.

Theorem 2.3.1 [59] *For $t \in \{6, 7, 8, 9\}$, let G be a $(t-3)$-connected graph with $t+1 \leq \delta(G)$. If every edge of G is contained in at least $t-2$ triangles and for any minimal separating set S of G and any $x \in S$, $G[S \setminus \{x\}]$ is not a clique, then $G > K_t$.*

Proof. Let G be a graph as in the statement with n vertices. By assumption, we have

(1) $t+1 \leq \delta(G)$ *and* $\delta(N_G(x)) \geq t-2$ *for any* x *in* G*; and*

(2) G *is* $(t-3)$*-connected and for any minimal separating set* S *of* G *and any* $x \in S$, $G[S \backslash \{x\}]$ *is not complete.*

By Theorem 1.4.1, Theorem 1.4.2, Theorem 1.4.3, (1), and (2), it follows that

(3) $t+1 \leq \delta(G) \leq 2t-5$.

We first show that the statement is true for $t = 6$. Assume $t = 6$. Then G is 3-connected with $\delta(G) = 7$. The statement is trivially true if G is complete, so we may assume G is not complete. Let $x \in V(G)$ be a vertex with $d_G(x) = 7$. By (1), $\delta(N_G(x)) \geq 4$, and so $|E(G[N_G(x)])| \geq 14$. If $|E(G[N_G(x)])| \geq 16$, then by Theorem 1.4.1, $N_G(x) > K_5$, and so $G > G[N_G[x]] > K_6$. If $|E(G[N_G(x)])| = 15$, then let K be a component of $G - N_G[x]$. By (2), $|N_G(K)| \geq 3$ and $G[N_G(K)]$ is not complete. Let $x, y \in N_G(K)$ be non-adjacent in $G[N_G(x)]$ and let P be an x,y-path with interior vertices in K. We see that $G > K_6$ by contracting all but one of the edges of P. So we may assume that $|E(G[N_G(x)])| = 14$, and so $G[N_G(x)]$ is 4-regular and $\overline{G[N_G(x)]}$ is 2-regular. Thus $\overline{G[N_G(x)]}$ is then either isomorphic to C_7 or to $C_4 \cup C_3$, and in both cases it is easy to see that $G[N_G(x)] > K_5$, and thus $G > K_6$, as desired. Hence we may assume $t \in \{7, 8, 9\}$.

Suppose for a contradiction that G does not contain K_t as a minor. We next prove the following.

(4) *Let* $x \in V(G)$ *be such that* $t+1 \leq d_G(x) \leq 2t-5$. *Then there is no component* K *of* $G - N_G[x]$ *such that* $N_G(K') \cap M \subseteq N_G(K)$ *for every component* K' *of* $G - N_G[x]$, *where* M *is the set of vertices of* $N_G(x)$ *not adjacent to all other vertices of* $N_G(x)$.

Suppose such a component K exists. Among all vertices x with $t+1 \leq d_G(x) \leq 2t-5$ for which such a component exists, choose x to be of minimal degree. We first prove that $M \subseteq N_G(K)$.

Suppose for a contradiction that $M \setminus N_G(K) \neq \emptyset$, and let $y \in M \setminus N_G(K)$ be such that $d_G(y)$ is minimum. Clearly, $d_G(y) < d_G(x)$ since y has no neighbor outside $N_G[x]$. Let J be the component of $G - N_G[y]$ containing K. We claim that $N_G(x) \setminus N_G[y] \nsubseteq V(J)$. Suppose to the contrary that $N_G(x) \setminus N_G[y] \subseteq V(J)$. Let K' be any other component of $G - N_G[x]$, and let J' be the component of $G - N_G[y]$ containing K'. If $G - N_G[y]$ contains only one component, then J is a component which trivially satisfies that $N_G(J') \cap M_y \subseteq N_G(J)$ for every component J' of $G - N_G[y]$, where M_y is the set of vertices of $N_G(y)$ not adjacent to all other vertices of $N_G(y)$, contradicting the choice of x since $d_G(y) < d_G(x)$. Hence we may assume $J' \neq J$. Then $J' \cap (N_G(x) \setminus N_G[y]) = \emptyset$, and so it follows that $N_G(J') = N_G(K') \subseteq N_G(y)$. Since $(N_G(x) \setminus N_G[y]) \subseteq M$, we see $(N_G(K') \cap M) \subseteq (N_G(K) \cap N_G(y))$. Thus $N_G(J') \cap M_y \subseteq N_G(J)$, where M_y is the set of vertices of $N_G(y)$ not adjacent to all other vertices of $N_G(y)$, again contradicting the choice of x. Our claim that $N_G(x) \setminus N_G[y] \nsubseteq V(J)$ follows.

Now let $H = G[N_G(x) \setminus (N_G[y] \cup N_G(K))]$. Clearly, $V(H) \subseteq M$. We have $d_G(z) \geq d_G(y)$ for all $z \in V(H)$ by the choice of y. Let $k = |V(H)|$. If $k = 1$, then $V(H)$ is complete to $N_G(y)$, since no vertex $z \in V(H)$ has a neighbor outside $N_G[x]$ by the choice of x and K. But then the vertex y and component H contradict the choice of x, and so $k \geq 2$. On the other hand $k \leq d_G(x) - d_G(y) \leq (2t - 5) - (t + 1) = t - 6 \leq 3$ and so $t \geq 8$. Notice that $k = 2$ when $t = 8$. From (1) applied to y, we deduce that $N_G(x) \cap N_G(y)$ has minimum degree at least $t - 3$. Let $L = G[(N_G(x) \cap N_G[y]) \cup V(H)]$. Then $E(L)$ consists of edges of $N_G(x) \cap N_G(y)$, edges incident to y, and edges incident to vertices in $V(H)$. Clearly, $e_G(L - H, H) = \sum_{z \in V(H)}(d_G(z) -$

$1) - 2|E(H)| \geq k(d_G(y) - 1) - 2|E(H)|$. Thus

$$\begin{aligned} |E(L)| &\geq |E(G[N_G(x) \cap N_G(y)])| + d_G(y) - 1 + e_G(L - H, H) + |E(H)| \\ &\geq \frac{(t-3)(d_G(y)-1)}{2} + d_G(y) - 1 + k(d_G(y) - 1) - |E(H)| \\ &\geq \frac{(t-3)(d_G(y)-1)}{2} + d_G(y) - 1 + k(d_G(y) - 1) - \frac{1}{2}k(k-1) \\ &\geq \begin{cases} 5(d_G(y) + 2) + \frac{d_G(y)}{2} - \frac{33}{2}, & \text{if} \quad t = 8 \\ 6(d_G(y) + k) + (k-2)d_G(y) - 4 - 7k - \frac{1}{2}k(k-1), & \text{if} \quad t = 9 \end{cases} \\ &\geq (t-3)|V(L)| - \binom{t-2}{2} + 1, \end{aligned}$$

because $d_G(y) \geq t + 1$ and $2 \leq k \leq t - 6$. If $t = 9$, since $12 \leq |V(L)| \leq 13$ the graph L is not a $(K_{2,2,2,2,2}, 5)$-cockade. By Theorem 1.4.1 and Theorem 1.4.2, $G[N_G(x)] > L > K_{t-1}$. Thus $G > G[N_G[x]] > K_t$, a contradiction. This proves that $M \subseteq N_G(K)$.

If $N_G(x) > K_{t-2} \cup K_1$, then $N_G(x)$ has a vertex y such that $G[N_G(x) \backslash \{y\}] > K_{t-2}$. If $y \notin M$, then $G[N_G(x)] > K_{t-1}$. Otherwise, by contracting the connected set $V(K) \cup \{y\}$ we can contract K_{t-1} onto $N_G(x)$ since $M \subseteq N_G(K)$. Thus in either case $G > K_t$, a contradiction. Thus $G[N_G(x)] \not> K_{t-2} \cup K_1$. If $t \leq 8$, then by Lemma 2.1.1 and Lemma 2.1.2, we have $t = 8$ and $G[N_G(x)]$ is either isomorphic to $K_{3,3,3}$ or $\overline{P}$, where $\overline{P}$ is the complement of the Petersen graph. It can be easily checked that $\overline{P} + xy > K_7$ for any $xy \in E(P)$. By (2), $|N_G(K)| \geq 5$ and $N_G(K)$ is not complete. Let $x, y \in N_G(K)$ be non-adjacent vertices in $N_G(x)$ and let Q be an x, y-path with interior vertices in K. We see that $G > K_8$ by contracting all but one of the edges of Q, a contradiction. Thus $G[N_G(x)]$ is isomorphic to $K_{3,3,3}$, and so $M = N_G(x)$. Let $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ be the vertex sets of two disjoint triangles of $\overline{G[N_G(x)]}$. Suppose $G - N_G[x]$ is either 2-connected or has at most two vertices. Clearly, the vertices a_i and b_i have at least two common neighbors in $G - N_G[x]$ for $i \in \{1, 2\}$, since every edge of G belongs to at least $t - 2$ triangles. Let u and u' (resp. w and w') be two distinct common neighbors of a_1 and b_1 (resp. a_2 and b_2) in $G - N_G[x]$.

By Menger's Theorem, $G - N_G[x]$ contains two disjoint paths from $\{u, u'\}$ to $\{w, w'\}$ and so $G > G[N_G[x]] + a_1a_2 + b_1b_2 > K_8$, a contradiction. Thus $G - N_G[x]$ has at least three vertices and is not 2-connected. If $G - N_G[x]$ is disconnected, let $H_1 = K$, and let H_2 be another connected component of $G - N_G[x]$. If $G - N_G[x]$ has a cut-vertex, say w, let H_1 be a connected component of $G - N_G[x] - w$, and let $H_2 = G - N_G[x] - V(H_1)$. In either case, H_1 and H_2 are disjoint connected subgraphs of $G - N_G[x]$ such that $M \subseteq N_G(H_1) \cup N_G(H_2)$, since $M \subseteq N_G(K)$. By (2), $G[N_G(H_i) \cap N_G(x)]$ is not complete and $|N_G(H_i) \cap N(x)| \geq 4$. By the pigeonhole principle, we see that each of $N_G(H_1)$ and $N_G(H_2)$ must contain a missing edge of $G[N_G(x)]$. If, say, H_2 only contains one missing edge e of $G[N_G(x)]$, then $|N_G(H_2) \cap N_G(x)| = 4$ and $|N_G(H_1) \cap N_G(x)| \geq 5$, and so $G[N_G(H_1)]$ contains at least two missing edges of $G[N_G(x)]$, at least one of which is disjoint from e. Hence we may assume that $a_1a_2 \in E(\overline{G[N_G(H_1)]})$ and $b_1b_2 \in E(\overline{G[N_G(H_2)]})$. By contracting H_1 onto a_1 and H_2 onto b_1 we see that $G > G[N_G[x]] + a_1a_2 + b_1b_2 > K_8$, a contradiction. This proves that $t = 9$, and so by Lemma 2.1.3, we may assume that $G[N_G(x)]$ satisfies properties (A) and (B).

Since $d_G(x) \geq 10$, $G[N_G(x)]$ is not isomorphic to $K_{1,2,2,2,2}$. If $G - N_G[x]$ is 2-connected or has at most two vertices, then by property (A) and (2), the set $N_G(x)$ has four distinct vertices $a_1, b_1, a_2,$ and b_2 such that $a_1a_2, b_1b_2 \notin E(G)$, $G[N_G(x)] + a_1a_2 + b_1b_2 > K_8$, and for $i \in \{1, 2\}$, the vertex a_i is adjacent to b_i, and the vertices a_i and b_i have at least two common neighbors in $G - N_G[x]$. Let u, u' (resp. w, w') be two distinct common neighbors of a_1 and b_1 (resp. a_2 and b_2) in $G - N_G[x]$. By Menger's Theorem, $G - N_G[x]$ contains two disjoint paths from $\{u, u'\}$ to $\{w, w'\}$, and so $G > G[N_G[x]] + a_1a_2 + b_1b_2 > K_9$, a contradiction.

Thus $G - N_G[x]$ has at least three vertices and is not 2-connected. If $G - N_G[x]$ is disconnected, let $H_1 = K$, and let H_2 be another connected component of $G - N_G[x]$. If $G - N_G[x]$ has a cut-vertex, say w, let H_1 be a connected component of $G - N_G[x] - w$, and let $H_2 = G - N_G[x] - V(H_1)$. In either case, H_1 and H_2 are disjoint connected subgraphs of $G - N_G[x]$ such that $M \subseteq N_G(H_1) \cup$

$N_G(H_2)$ since $M \subseteq N_G(K)$. For $i \in \{1,2\}$, let $A_i = N_G(H_i) \cap N_G(x)$. By (2), $G[A_i]$ is not complete and $|A_i| \geq 5$ for $i \in \{1,2\}$. By property (B), A_1 and A_2 satisfy at least one of the properties (B1), (B2), or (B3).

Suppose first that A_1 and A_2 satisfy property (B1). Then there exist $a_i \in A_i$ such that $G[N_G(x)] + \{a_1a : a \in A_1 \setminus \{a_1\}\} + \{a_2a : a \in A_2 \setminus \{a_2\}\} > K_8$. By contracting the connected sets $V(H_1) \cup \{a_1\}$ and $V(H_2) \cup \{a_2\}$ to single vertices, we see that $G > K_9$, a contradiction. Suppose next that A_1 and A_2 satisfy property (B2). Then there exist $a_1 \in A_1 \setminus A_2$ and $a_2 \in A_2 \setminus A_1$ such that $a_1a_2 \in E(G)$ and the vertices a_1 and a_2 have at most five common neighbors in $N_G(x)$. Thus $a_1, a_2 \in M$ by (1), and, by another application of (1), there exists a common neighbor $u \in V(G) \setminus N_G[x]$ of a_1 and a_2. But $a_1 \notin A_2$ and $a_2 \notin A_1$, and hence $u \notin V(H_1) \cup V(H_2)$. Thus $G - N_G[x]$ is disconnected and $H_1 = K$. But then $a_2 \in M \subseteq N_G(K) = N_G(H_1)$, a contradiction. Thus we may assume that A_1 and A_2 satisfy (B3), and hence $A_i \subseteq A_{3-i}$ for some $i \in \{1,2\}$. As $M \subseteq A_1 \cup A_2$, we have $M \subseteq N_G(H_{3-i})$. Since A_i is not complete, let $a, b \in A_i$ be distinct and nonadjacent. By property (B3), $G[N_G(x)] + ab > K_7 \cup K_1$. Let P be an a, b-path with interior in $V_G(H_i)$. By contracting all but one of the edges of the path P, and by contracting H_{3-i} similarly as above, we see that $G > K_9$, a contradiction. This completes the proof of (4).

(5) $G - N_G[x]$ *is disconnected for every vertex* $x \in V(G)$ *of degree at most* $2t-5$.

If $G - N_G[x]$ is not null, then it is disconnected by (4). Thus we may assume that x is adjacent to all other vertices of G. Let $H = G - x$. Then $|V(H)| = d_G(x)$ and $\delta(H) \geq t$. Thus $|E(H)| \geq \frac{td_G(x)}{2} > (t-3)d_G(x) - \binom{t-2}{2} + 1$ because $d_G(x) \leq 2t-5$. By Theorem 1.4.1 and Theorem 1.4.2, $G - x$ has a K_{t-1} minor, and so the graph G has a K_t minor, a contradiction. This proves (5).

(6) *Let* $x \in V(G)$ *be such that* $t+1 \leq d_G(x) \leq 2t-5$. *Then there is no component* K *of* $G - N_G[x]$ *such that* $d_G(y) \geq 2t-4$ *for every vertex* $y \in V(K)$.

Assume that such a component K exists. Let $G_1 = G - K$ and $G_2 = G[K \cup N_G(K)]$. Let d_1 be the maximum number of edges that can be added to G_2 by contracting edges of G with at least one end in G_1. More precisely, let d_1 be the largest integer so that G_1 contains disjoint sets of vertices $V_1, V_2, \ldots, V_p$ so that $G_1[V_j]$ is connected, $|N_G(K) \cap V_j| = 1$ for $1 \leq j \leq p = |N_G(K)|$, and so that the graph obtained from G_1 by contracting $V_1, V_2, \ldots, V_p$ and deleting $V(G_1) - (\bigcup_j V_j)$ has $|E(G[N_G(K)])| + d_1$ edges. Let G_2' be the graph with $V(G_2') = V(G_2)$ and $|E(G_2')| = |E(G_2)| + d_1$ obtained from G by contracting each set $V_1, V_2, \ldots, V_p$ to a single vertex and deleting $V(G_1) - (\bigcup_j V_j)$. By (1), $|V(G_2')| \geq t + 2$. If $|E(G_2')| \geq (t-2)|V(G_2')| - \binom{t-1}{2} + 2$, then by Theorem 1.4.1 and Theorem 1.4.2, $G > G_2' > K_t$, a contradiction. Thus

$$\begin{aligned}
|E(G_2)| &= |E(G_2')| - d_1 \\
&\leq (t-2)|V(G_2)| - \binom{t-1}{2} + 1 - d_1 \\
&= (t-2)|N_G(K)| + (t-2)|V(K)| - \binom{t-1}{2} + 1 - d_1.
\end{aligned}$$

By contracting the edge xz, where $z \in N_G(K)$ has minimum degree δ in $G[N_G(K)]$, we see that $d_1 \geq |N_G(K)| - \delta - 1$, and hence

$$|E(G_2)| \leq (t-3)|N_G(K)| + (t-2)|V(K)| - \binom{t-1}{2} + 2 + \delta. \tag{a}$$

Let $k = e_G(N_G(K), K)$. We have $|E(G_2)| = |E(K)| + k + |E(N_G(K))|$ and

$$2|E(K)| \geq (2t-4)|V(K)| - k, \tag{b}$$

and hence

$$|E(G_2)| \geq (t-2)|V(K)| + \frac{k}{2} + \frac{\delta|N_G(K)|}{2}. \tag{c}$$

Since $G[N_G(x)]$ has minimum degree at least $t-2$, it follows that the subgraph $G[N_G(K)]$ of $G[N_G(x)]$ has minimum degree at least $(t-2)-(d_G(x)-|N_G(K)|)$. Thus $\delta \geq (t-2)-(d_G(x)-|N_G(K)|) \geq |N_G(K)|-t+3$. From (a) and (c) we get

$$\frac{k}{2} \leq (t-3)|N_G(K)| - \frac{\delta(|N_G(K)|-2)}{2} - \binom{t-1}{2} + 2 \leq \begin{cases} \frac{15}{2} & \text{if} \quad t=7 \\ 12 & \text{if} \quad t=8 \\ 18 & \text{if} \quad t=9 \end{cases} . \tag{d}$$

Since G does not contain K_t as a minor, it follows that K does not contain K_t as a minor. Hence from (b), Theorem 1.4.1, Theorem 1.4.2, and Theorem 1.4.3, we get

$$\frac{k}{2} \geq \binom{t-1}{2} - 1 = \begin{cases} 14 & \text{if} \quad t=7 \\ 20 & \text{if} \quad t=8 \\ 27 & \text{if} \quad t=9 \end{cases} ,$$

contradicting (d). This proves (6).

By (3) there is a vertex $x \in V(G)$ with $t+1 \leq d_G(x) \leq 2t-5$. Choose such a vertex x so that $G-N_G[x]$ has a component K with $|V(K)|$ minimum. Then choose a vertex $y \in V(K)$ of least degree in G. Thus $t+1 \leq d_G(y) \leq 2t-5$ by (1) and (6). Let L be the component of $G-N_G[y]$ containing x. We claim that $N_G(L)$ contains all vertices of $N_G(y)$ that are not adjacent to all other vertices of $N_G(y)$. Indeed, let $z \in N_G(y)$ be not adjacent to some vertex of $N_G(y) \setminus \{z\}$. We may assume that $z \notin N_G(x)$, for otherwise $z \in N_G(L)$. Thus $z \in V(K)$, and hence $d_G(z) \geq d_G(y)$ by the choice of y. Thus z has a neighbor $z' \in N_G[x] \cup V(K-N_G[y])$. Then $z' \in V(L)$, for otherwise the component of $G-N_G[y]$ containing z' would be a proper subgraph of K, contradicting our choice of x and K. Thus $z \in N_G(L)$. This proves our claim that $N_G(L)$ contains all vertices z not adjacent to all other vertices of $N_G(y)$, contrary to (4). This contradiction completes the proof of Theorem 2.3.1. ■

We are now ready to prove Theorem 1.2.7.

Proof of Theorem 1.2.7. Let G be a double-critical, t-chromatic graph with $t \geq k$ as in the statement of Theorem 1.2.7. The assertion is trivially true if G is complete, so suppose not. By Theorem 1.2.4, we may assume that $t \geq 6$. By Proposition 2.2.1(vi), $\delta(G) \geq t+1 \geq k+1$. By Proposition 2.2.1(iii), every edge of G is contained in at least $t-2 \geq k-2$ triangles. By Proposition 2.2.1(xi), G is 6-connected and no minimal separating set of G can be partitioned into a clique and an independent set. In particular, if S is a minimal separating set of G and $x \in S$, then $S \setminus \{x\}$ does not induce a clique in G. By Theorem 2.3.1, $G > K_k$, as desired. ■

2.4 Claw-Free, Double-Critical Graphs

In this section we focus specifically on claw-free, double-critical graphs. Recall that a graph is *claw-free* if it does not contain the claw, $K_{1,3}$, as an induced subgraph. We first prove two lemmas before proving Theorem 1.2.8.

Lemma 2.4.1 [57] *Let G be a double-critical, t-chromatic graph with $t \geq 6$. If G is claw-free, then for any $x \in V(G)$, $d_G(x) \leq 2t-4$. Furthermore, if $d_G(x) < |V(G)|-1$, then $d_G(x) \leq 2t-6$.*

Proof. Let $x \in V(G)$ be a vertex of maximum degree in G, and let uv be any edge of $G-x$. Let c be any $(t-2)$-coloring of $G-\{u,v\}$ with color classes $V_1, V_2, \ldots, V_{t-2}$. We may assume that $x \in V_{t-2}$. Since G is claw-free, x can have at most two neighbors in each of $V_1, \ldots, V_{t-3}$. Additionally, x may be adjacent to u and v in G. Therefore $d_G(x) \leq 2t-4$. If $d_G(x) < |V(G)|-1$, then $\chi(G[N_G(x)]) \leq t-3$ by Proposition 2.2.1(viii). Since G is claw-free, each color class in any $(t-3)$-coloring of $G[N_G(x)]$ can contain at most two vertices, and so $d_G(x) \leq 2t-6$. ■

Lemma 2.4.2 *Let G be a double-critical, t-chromatic graph with $t \geq 6$. If G is claw-free, then for* *any $x \in V(G)$, $G[N_G(x)]$ is $(2t-1-d_G(x))$-connected.*

Proof. Let $x \in V(G)$ be any vertex and let S be a minimal separating set of $G[N_G(x)]$. Since G is claw-free, $G[N_G(x)]-S$ has two components, say C_1 and C_2, both of which must be cliques. Since $\delta(G[N_G(x)]) \geq t-2$ by Proposition 2.2.1(iii), we see that $|V(C_i) \cup S| \geq t-1$ for each $i \in \{1,2\}$. Then $d_G(x) = |V(C_1)| + |V(C_2)| + |S| = |V(C_1) \cup S| + |V(C_2) \cup S| - |S| \geq 2t-2-|S|$, and so $|S| \geq 2t-2-d_G(x)$.

Suppose that $|S| = 2t-2-d_G(x)$. Then $|V(C_1) \cup S| = |V(C_2) \cup S| = t-1$. Since $\delta(G[N_G(x)]) \geq t-2$, any vertex $v \in V(C_i)$ is complete to $S \cup (V(C_i) \setminus \{v\})$ for each $i \in \{1,2\}$. Hence, S is complete to $V(C_1) \cup V(C_2)$. Let $y \in V(C_1)$, and let c be any $(t-2)$-coloring of $G - \{x,y\}$. Then $|N_G(x) \cap N_G(y)| = |S \cup (V(C_1) \setminus \{y\})| = t-2$. By Proposition 2.2.1(iii), every vertex of $S \cup (V(C_1) \setminus \{y\})$ must be assigned a distinct color by c. Since $V(C_2)$ is complete to S and C_2 is a clique, every vertex of $V(C_2) \cup S$ must then be assigned a distinct color by c as well. Thus $|V(C_2) \cup S| \leq t-2$, contrary to the fact that $|V(C_2) \cup S| = t-1$. ■

We are now ready to prove Theorem 1.2.8. It is an easy consequence of Proposition 2.2.1 and Lemma 2.4.1 that Theorem 1.2.8 is true for $t = 6, 7$.

Proof of Theorem 1.2.8. Let G and t be as given in the statement. Suppose that G is not isomorphic to K_t. By Proposition 2.2.1(iv), there exists an edge $xy \in E(G)$ such that x and y have a common non-neighbor. By Proposition 2.2.1(vi) and Lemma 2.4.1, $t+1 \leq d_G(x) \leq 2t-6$ and $t+1 \leq d_G(y) \leq 2t-6$. Thus $t \geq 7$. If $t = 7$, then $d_G(x) = d_G(y) = 8$, which contradicts Proposition 2.2.1(x). Hence we may assume that $t = 8$. We next show that

(1) *G is 10-regular.*

By Lemma 2.2.3 and Theorem 1.2.8 for the case $t = 7$ above, we have $\Delta(G) \leq |V(G)| - 2$. By Proposition 2.2.1(vi) and Lemma 2.4.1, we see that $9 \leq d_G(x) \leq 10$ for all vertices $x \in V(G)$. By Theorem 1.2.9, G is 10-regular. This proves (1).

(2) *For any* $x \in V(G)$*,* $2 \leq \delta(\overline{G[N_G(x)]}) \leq \Delta(\overline{G[N_G(x)]}) \leq 3$.

Let $x \in V(G)$. Then x has at least one non-neighbor in G, otherwise G is isomorphic to K_{11} by (1), a contradiction. By Proposition 2.2.1(viii), $\chi(G[N_G(x)]) \leq 5$. Since G is claw-free, we see that $\alpha(G[N_G(x)]) = 2$, and so $\chi(G[N_G(x)]) = 5$ since every color class can contain at most two vertices. Thus every vertex of $N_G(x)$ has at least one non-neighbor in $G[N_G(x)]$. By Proposition 2.2.1(v) and Proposition 2.2.1(iii), $2 \leq \delta(\overline{G[N_G(x)]}) \leq \Delta(\overline{G[N_G(x)]}) \leq 3$. This proves (2).

(3) *For any* $x \in V(G)$*,* $\Delta(\overline{G[N_G(x)]}) = 3$*. That is,* $\overline{G[N_G(x)]}$ *is not 2-regular.*

Suppose that there exists a vertex $x \in V(G)$ such that $\overline{G[N_G(x)]}$ is 2-regular. Let $y \in N_G(x)$ and let c be any 6-coloring of $G - \{x, y\}$ with color classes $V_1, V_2, \ldots, V_6$. Let $W = N_G(x) \cap N_G(y)$. Then $|W| = 7$ because $\overline{G[N_G(x)]}$ is 2-regular. By Proposition 2.2.1(iii), we may assume that $|V_1 \cap W| = 2$ and $|V_i \cap W| = 1$ for each $i \in \{2, 3, 4, 5, 6\}$. Let $V_1 \cap W = \{v_1, u_1\}$ and $V_i \cap W = \{v_i\}$ for each $i \in \{2, 3, 4, 5, 6\}$. Since G is claw-free, we may further assume that $N_G(x) \cap V_2 = \{v_2, u_2\}$ and $N_G(x) \cap V_3 = \{v_3, u_3\}$. Clearly, $yu_2, yu_3 \notin E(G)$ and thus $u_2u_3 \in E(G)$ because G is claw-free. Since $\overline{G[N_G(x)]}$ is 2-regular, we see that $G[\{v_4, v_5, v_6\}]$ is not a clique. We may assume that $v_4v_5 \notin E(G)$. By Lemma 2.2.5(i), $N_G(y) \cap (V_j \setminus \{v_j\}) \neq \emptyset$ for each $j \in \{4, 5\}$. Let $w_4 \in V_4 \setminus \{v_4\}$ and $w_5 \in V_5 \setminus \{v_5\}$ be two other neighbors of y in G. Then $N_G(y) \setminus N_G[x] = \{w_4, w_5\}$ since G is 10-regular by (1). By Lemma 2.2.5(i), v_6 must be complete to $\{v_2, v_3, v_4, v_5\}$ in G. Notice that v_6 is complete to $\{u_2, u_3\}$ in G since $\overline{G[N_G(x)]}$ is 2-regular. Thus v_6 must be anti-complete to $\{v_1, u_1\}$ in G and so $G[\{x, v_1, u_1, v_6\}]$ is isomorphic to $K_{1,3}$, a contradiction. This proves (3).

From now on, we fix an arbitrary vertex $x \in V(G)$. Let $H = \overline{G[N_G(x)]}$. By (3), let $y \in N_G(x)$ with $|N_G(x) \cap N_G(y)| = 6$. We choose such a vertex $y \in N_G(x)$ so that $N_G(x) \setminus N_G[y]$ contains as many vertices of degree two in H as possible. Let c be any 6-coloring of $G - \{x, y\}$ with color classes $V_1, V_2, \ldots, V_6$. We may assume that $V_i \cap N_G(x) \cap N_G(y) = \{v_i\}$ for all $i \in \{1, 2, 3, 4, 5, 6\}$. Since G is claw-free, we may further assume that $N_G(x) \cap V_j = \{v_j, u_j\}$ for all $j \in \{1, 2, 3\}$. Notice that y is anti-complete to $\{u_1, u_2, u_3\}$ in G and since G is claw-free, $G[\{u_1, u_2, u_3\}]$ is isomorphic to K_3. Let $A = \{u_1, u_2, u_3\}$, $B = \{v_1, v_2, v_3\}$, and $C = \{v_4, v_5, v_6\}$.

(4) *B is not complete to C in G.*

Suppose that B is complete to C in G. Then $e_H(C, A) = \sum_{v \in C} d_H(v) - 2|E(H[C])| \geq 6 - 2|E(H[C])|$. For each $i \in \{1, 2, 3\}$, $u_i v_i, u_i y \notin E(G)$ and $d_H(u_i) \leq 3$. Thus $e_H(A, C) \leq 3$ and so $|E(H[C])| \geq 2$. Since G is claw-free, we have $|E(H[C])| = 2$. We may assume that $v_4 v_6 \notin E(H)$. Then $v_4 v_6 \in E(G)$ and $v_4 v_5, v_5 v_6 \notin E(G)$. Since $d_H(v_4) \geq 2$, $d_H(v_6) \geq 2$, and B is complete to C in G, we may assume that $u_2 v_4, u_3 v_6 \notin E(G)$. Note that H is not 3-regular since $e_H(A, C) \leq 3$ and $e_H(B, C) = 0$. By the choice of y, $d_H(u_1) = 2$ and $d_H(v_j) = 2$ for all $j \in \{4, 5, 6\}$. Since $d_H(u_2) = d_H(u_3) = 3$, by the choice of y again, $d_H(v_2) = d_H(v_3) = 3$. Thus $G[B]$ is isomorphic to $\overline{K_3}$ and so $G[\{x\} \cup B]$ is isomorphic to $K_{1,3}$, a contradiction. This proves (4).

(5) *$G[C]$ is isomorphic to K_3.*

Suppose that $G[C]$ contains a missing edge, say $v_4 v_5 \notin E(G)$. By Lemma 2.2.5(i), there exist $w_4 \in V_4 \setminus \{v_4\}$ and $w_5 \in V_5 \setminus \{v_5\}$ such that $y w_4, y w_5 \in E(G)$. By (4), we may assume that $v_3 v_j \notin E(G)$ for some $j \in \{4, 5, 6\}$. By Lemma 2.2.5(i), y has another neighbor, say w_3, in $V_3 \setminus \{v_3\}$. Since G is 10-regular by (1), $\{w_3, w_4, w_5\} = N_G(y) \setminus N_G[x]$, so by Lemma 2.2.5(i), $v_4 v_5$ is the only missing edge in $G[C]$ and $\{v_1, v_2\}$ is complete to C in G. If $e_H(A, C) = 3$, then $d_H(u_i) = 3$ for all $i \in \{1, 2, 3\}$. By the choice of y, $d_H(v_3) = 3$, or else we could replace y with u_3. Notice that for all $i \in \{4, 5, 6\}$, $e_H(\{v_i\}, A \cup \{v_3\}) \geq 1$, and so by the choice of y, $d_H(v_i) = 3$, or else we could replace y with v_3. Thus $e_H(A,C) \geq 5$, which is impossible. Hence $e_H(A,C) \leq 2$. Notice that $e_H(A,C) = (d_H(v_4)-1) + (d_H(v_5)-1) + d_H(v_6) - e_H(v_3,C) \geq 2$. It follows that $e_H(A,C) = 2$, $e_H(v_3,C) = 2$ and $d_H(v_i) = 2$ for all $i \in \{4,5,6\}$. Then $N_G(x) \setminus N_G[y]$ has at most one vertex of degree two in H, but $N_G(x) \setminus N_G[v_3]$ has two vertices of degree two in H, contradicting the choice of y. This proves (5).

(6) v_1u_1, v_2u_2, *and* v_3u_3 *are the only edges in* $H[A \cup B]$.

Suppose that $H[A \cup B]$ has at least four edges. By (5) and (2), $e_H(A \cup B, C) \geq 6$. On the other hand, $e_H(A \cup B, C) = \sum_{v \in A \cup B} d_H(v) - 2|E(H[A \cup B])| - 3 \leq 15 - 2|E(H[A \cup B])|$. It follows that $|E(H[A \cup B])| = 4$ and $A \cup B$ contains at most one vertex of degree two in H. Thus $e_H(A \cup B, C) \leq 7$ and so at least two vertices of C, say v_4 and v_5, are of degree two in H. Since $e_H(A,C) \leq 3$ and $G[C]$ is isomorphic to K_3 by (5), we may assume that $v_4v_3 \notin E(G)$. If $d_H(v_3) = 3$, then since $d_H(v_4) = 2$ and at most one vertex of $A \cup B$ has degree two in H, by the choice of y, exactly one of u_1, u_2, u_3 has degree two in H. Then $e_H(A \cup B, C) = 6$. Thus $d_H(v_j) = 2$ for all $j \in \{4,5,6\}$ and by the choice of y, each vertex of B is adjacent to at most one vertex of C in H. Thus $e_H(A \cup B, C) \leq 5$, a contradiction. Hence $d_H(v_3) = 2$. Now $d_H(u_i) = 3$ for all $i \in \{1,2,3\}$ because at most one vertex of $A \cup B$ has degree two in H. We see that $N_G(x) \setminus N_G[y]$ has no vertex of degree two in H but $N_G(x) \setminus N_G[u_3]$ has at least one vertex of degree two in H, contrary to the choice of y. This proves (6).

By (6), we see that for any $i \in \{1,2,3\}$, $v_iv_j \notin E(G)$ for some $j \in \{4,5,6\}$. By Lemma 2.2.5(i), let $w_i \in V_i \setminus \{v_i\}$ be such that $yw_i \in E(G)$ for all $i \in \{1,2,3\}$. Let $D = \{w_1, w_2, w_3\}$. Then $N_G(y) \setminus N_G[x] = D$ and $G[D]$ is isomorphic to K_3 because G is claw-free. Clearly, D is not complete to C in G, otherwise $G[\{y\} \cup D \cup C]$ is isomorphic to K_7, contrary to Proposition 2.2.1(i). We may assume that $w_3v_4 \notin E(G)$. For each $i \in \{1,2\}$, $v_iv_3, v_iu_3 \in E(G)$ by (6). Thus $v_1w_3, v_2w_3 \notin E(G)$ because G is claw-free. Notice that $w_3, x, v_1, v_2, v_4 \in N_G(y)$ and w_3 is anticomplete to $\{x, v_1, v_2, v_4\}$ in G. Thus $\Delta(\overline{G[N_G(y)]}) \geq 4$, contrary to (2). This completes the proof of Theorem 1.2.8. ■

CHAPTER 3: COLORING GRAPHS WITH FORBIDDEN MINORS

3.1 Preliminary Lemmas

In this section, we will prove several lemmas used throughout Chapter 3. Of particular importance is Lemma 1.5.3, which we prove first.

Proof of Lemma 1.5.3. Let G, x, S, and M be as given in the statement. Let H be obtained from G by contracting $S \cup \{x\}$ into a single vertex, say w. Then H is $(k-1)$-colorable. Let $c : V(H) \to \{1, 2, \ldots, k-1\}$ be a $(k-1)$-coloring of H. We may assume that $c(w) = 1$. Then each of the colors $2, \ldots, k-1$ must appear in $G[N_G(x) \setminus S]$, or else we could assign x the missing color and assign all vertices in S the color 1 to obtain a proper $(k-1)$-coloring of G, a contradiction. Since $|N_G(x) \setminus S| = k-2$, we have $c(u) \neq c(v)$ for any two distinct vertices $u, v \in N_G(x) \setminus S$. We next claim that for each $i \in \{1, 2, \ldots, m\}$ and each $j \in \{1, 2, \ldots, r_i\}$ there must exist a path between a_i and b_{ij} with its internal vertices in $V(G) \setminus N_G[x]$. Suppose not. Let $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, r_i\}$ be such that there is no such path between a_i and b_{ij}. Let H' be the subgraph of H induced by all vertices colored either $c(a_i)$ or $c(b_{ij})$ by the coloring c. Then $V(H') \cap N_G(x) = \{a_i, b_{ij}\}$. Notice that a_i and b_{ij} must belong to different components of H' as there is no path between a_i and b_{ij} with its internal vertices in $V(G) \setminus N_G[x]$. By switching the colors on the component of H' containing a_i, we obtain a $(k-1)$-coloring of H with the color $c(a_i)$ missing on $G[N_G(x) \setminus S]$, a contradiction. This proves that there must exist a path P_{ij} in H' with ends a_i, b_{ij} and all its internal vertices in $V(H') \setminus N_G[x] \subseteq V(G) \setminus N_G[x]$ for each $i \in \{1, 2, \ldots, m\}$ and each $j \in \{1, 2, \ldots, r_i\}$. Clearly, for any $1 \leq i < \ell \leq m$, the paths $P_{i1}, \ldots, P_{ir_i}$ are vertex-disjoint from the paths $P_{\ell 1}, \ldots, P_{\ell r_\ell}$, because no two vertices of $a_1, \ldots, a_r, b_{11}, \ldots, b_{mr_m}$ are assigned the same color by c. ■

We note here that if $r_1 = r_2 = \cdots = r_m = 1$ in the statement of Lemma 1.5.3, then we simply write $M = \{a_1b_{11}, a_2b_{21}, \ldots, a_mb_{m1}\}$, and so M is a matching of missing edges of $G[N_G(x) \setminus S]$. In this case, the paths $P_{11}, P_{21}, \ldots, P_{m1}$ are pairwise vertex-disjoint if $m \geq 2$. Similarly, if $m = 1$ in the statement of Lemma 1.5.3, then we simply write $M = \{a_1b_{11}, \ldots, a_1b_{1r_1}\}$. In this case, the paths $P_{11}, \ldots, P_{1r_1}$ have a_1 as a common end and are not necessarily pairwise internally vertex-disjoint if $r_1 \geq 2$.

Furthermore, we also note that if we keep the same set $S \subseteq N_G(x)$, we may be able to usefully apply Lemma 1.5.3 to two different sets M_1 and M_2 if we choose the missing edges a_ib_{ij} in each set carefully. The paths given by applying Lemma 1.5.3 to M_1 may intersect the paths given by applying Lemma 1.5.3 to M_2. However, since the paths provided by Lemma 1.5.3 are Kempe chains, we are able to specifically control which paths may intersect by our choices of a_ib_{ij} in M_1 and M_2.

We will also need the following lemma in the proofs of Theorem 1.3.8 and Theorem 1.3.9.

Lemma 3.1.1 [58] *For any 7-connected graph G, if G contains two different subgraphs isomorphic to K_6, then $G > K_8^-$.*

Proof. Let H_1 and H_2 be two different subgraphs of G such that both are isomorphic to K_6 with $V(H_1) = \{v_1, \ldots, v_6\}$ and $V(H_2) = \{w_1, \ldots, w_6\}$. Let $t = |V(H_1) \cap V(H_2)|$. Then $0 \leq t \leq 5$. We may assume that $v_i = w_i$ for all $i \leq t$ if $t \neq 0$. Assume first that $t = 5$. Then $G[V(H_1) \cup V(H_2)]$ is a subgraph of G isomorphic to K_7^-. Since G is 7-connected, it is easy to see that $G > K_8^-$ by contracting a component of $G - (V(H_1) \cup V(H_2))$ into a single vertex. So we may now assume that $t \leq 4$. Then there exist $6 - t$ pairwise disjoint paths $P_{t+1}, \ldots, P_6$ between $\{v_{t+1}, \ldots, v_6\}$ and $\{w_{t+1}, \ldots, w_6\}$ in $G - (V(H_1) \cap V(H_2))$. We may assume that P_i has ends v_i, w_i for all $i \in \{t + 1, \ldots, 6\}$. Then $G - \{v_1, \ldots, v_5, w_6\}$ is connected since G is 7-connected, so there must exist a path Q with one end, say x, in $(V(P_{t+1}) \cup \cdots \cup V(P_5)) \setminus \{v_{t+1}, \ldots v_5\}$, the other end, say y, in $V(P_6) \setminus \{w_6\}$, and no internal vertices in $\{v_1, \cdots, v_t\} \cup V(P_{t+1}) \cup \cdots \cup V(P_6)$ (possibly $x \in \{w_{t+1}, \ldots, w_6\}$ or $y = v_6$). We may assume that x lies on the path $P_5 - v_5$. Let P_5' be the subpath of P_5 with ends x and w_5, and let P_6' be the subpath of P_6 with ends y and v_6. Now contracting P_5' onto w_5, $P_5 - P_5'$ onto v_5, P_6' and $Q - x$ onto v_6, $P_6 - P_6'$ onto w_6, and each of $P_{t+1}, \ldots, P_4$ to a single vertex if $t < 4$, together with $v_1, \ldots, v_t$ if $t > 0$, yields a K_8^- minor in G, as desired. ■

The following Lemma 3.1.2 will only be used in the proof of Theorem 1.3.7. It can be obtained from the (computer-assisted) proof of Lemma 3.7 in [67]. Here we give a computer-free proof of Lemma 3.1.2 so that the proof of Theorem 1.3.7 is also computer-free for the cases $t = 7, 8$.

Lemma 3.1.2 (Song and Thomas [67]) *For* $7 \leq t \leq 9$, *let* H *be a graph with* $2t - 5$ *vertices and* $\alpha(H) = 2$. *Then* $H > K_{t-2} \cup K_1$.

Proof. Suppose that H has no $K_{t-2} \cup K_1$-minor. Then $\omega(H) \leq t - 3$. We claim that

(1) $\omega(H) \leq t - 4$.

Suppose $\omega(H) = t - 3$. Let $K \subseteq H$ be isomorphic to K_{t-3}. Then $|V(H) \setminus V(K)| = t - 2 \geq 5$. If $H - K$ contains an induced path on three vertices, say P, with ends y and z, then every vertex in $V(K)$ is adjacent to either y or z because $\alpha(H) = 2$. By contracting the path P into a single vertex, we see that $H[V(K) \cup V(P)] > K_{t-2}$, and so $H > K_{t-2} \cup K_1$, a contradiction. Thus $H - K$ does not contain an induced path on three vertices. Since $\alpha(H) = 2$, it follows that $H - K$ is a disjoint union of two cliques, say A_1 and A_2. For $i \in \{1, 2\}$, let

$$K_i = \{v \in V(K) : v \text{ is not adjacent to some vertex in } V(A_{3-i})\}.$$

Since $\alpha(H) = 2$ and $V(A_1)$ is anticomplete to $V(A_2)$, K_i is complete to $V(A_i)$ for each $i \in \{1, 2\}$. Thus $H - (K_i \cup V(A_i))$ is a clique for each $i \in \{1, 2\}$ and so, since K_1 and K_2 are disjoint, either $H - (K_1 \cup V(A_1))$ or $H - (K_2 \cup V(A_2))$ is a clique of size at least $t - 2$, contrary to the fact that $\omega(H) \leq t - 3$. This proves (1).

(2) *for any $y \in V(H)$ and any $A \subseteq N_H(y)$ with $|A| \geq 6$, either $H[A \cup \{y\}]$ contains two vertex-disjoint, induced paths on three vertices or $H[A]$ is a disjoint union of two cliques.*

Suppose $H[A]$ is not a disjoint union of two cliques. Then $H[A]$ is connected because $\alpha(H) = 2$. We next show that $H[A \cup \{y\}]$ contains two vertex-disjoint, induced paths on three vertices. By (1), $H[A]$ is not a clique and thus contains an induced path on three vertices, say P, with ends a and c, and $V(P) = \{a, b, c\}$. Let $\{d_1, d_2, \ldots, d_s\} = A \backslash V(P)$, where $s = |A| - 3 \geq 3$. Clearly $H[A \cup \{y\}]$ contains two vertex-disjoint, induced paths on three vertices if $H[\{d_1, d_2, \ldots, d_s\}]$ is not a clique, since yd_i is an edge for all $i \in \{1, 2, \ldots, s\}$. So we may assume that $H[\{d_1, d_2, \ldots, d_s\}]$ is isomorphic to K_s. First suppose that a is complete to $\{d_1, d_2, \ldots, d_s\}$. By (1), b is not complete to $\{d_1, d_2, \ldots, d_s\}$. We may assume that $bd_1 \notin E(H)$. Clearly $H[\{a, y, c\}]$ and $H[\{d_1, b, d_i\}]$ are two vertex-disjoint, induced paths on three vertices if $bd_i \in E(H)$ for some $i \neq 1$. So we may assume that $bd_i \notin E(H)$ for all $i \in \{1, 2, \ldots, s\}$. Now either $H[\{b, a, d_1\}]$ and $H[\{c, y, d_2\}]$ (if $cd_2 \notin E(H)$) or $H[\{a, d_2, c\}]$ and $H[\{b, y, d_1\}]$ (if $cd_2 \in E(H)$) are two vertex-disjoint, induced paths on three vertices. Next suppose that a is not complete to $\{d_1, d_2, \ldots, d_s\}$. We may assume that $ad_1 \notin E(H)$. Then $cd_1 \in E(H)$ because $\alpha(H) = 2$. By symmetry, we may assume that $cd_2 \notin E(H)$. Then $ad_2 \in E(H)$. Now either $H[\{c, d_1, d_2\}]$ and $H[\{a, y, d_3\}]$ (if $ad_3 \notin E(H)$) or $H[\{a, d_3, d_1\}]$ and $H[\{c, y, d_2\}]$ (if $ad_3 \in E(H)$) are two vertex-disjoint, induced paths on three vertices, as desired. This proves (2).

Let $\delta := \delta(H)$ and let $y \in V(H)$ be a vertex with $d(y) = \delta$. Let $J = H - N_H[y]$. Since $\alpha(H) = 2$,

J is a clique of size $2t-\delta-6$. By (1), $|J| = 2t-\delta-6 \le t-4$ and so $\delta \ge t-2$.

(3) $\delta = t-2$.

Suppose $\delta \ge t-1$. By Theorem 1.4.1, $(t-4)(2t-6) - \binom{t-3}{2} \ge |E(H-y)| \ge \delta|V(H)|/2 - \delta = \delta(|V(H)|-2)/2 \ge (t-1)(2t-7)/2$, which yields that $t = 9$ and $\delta = t-1 = 8$. Then H is a graph with $|V(H)| = 2t-5 = 13$. Clearly, J is isomorphic to K_4. Let $z \in N_H(y)$ be such that $|N_H(z) \cap V(J)|$ is maximum. Since $e_H(V(J), N_H(y)) \ge 20$, we have $|N_H(z) \cap V(J)| \ge 3$. If $|N_H(z) \cap V(J)| = 4$, then $H[\{z\} \cup V(J)]$ is isomorphic to K_5 and $|N_H(y) \setminus \{z\}| = 7$. Clearly $H > K_7 \cup K_1$ if $H[N_H[y] \setminus \{z\}]$ has two vertex-disjoint, induced paths on three vertices. By (2), $H[N_H[y] \setminus \{z\}]$ is thus a disjoint union of two cliques, say A_1 and A_2. By (1), we may assume that A_1 is isomorphic to K_3 and A_2 is isomorphic to K_4. Let $a \in V(A_1)$. By (1) again, a is not complete to $\{z\} \cup V(J)$ and thus $d_H(a) \le 7$, contrary to the fact that $\delta = 8$. Thus $|N_H(z) \cap V(J)| = 3$. Let $z' \in V(J)$ be the non-neighbor of z. By the choice of z, every vertex in $N_H(y)$ has at least one non-neighbor in $V(J)$ and so $\delta(H[N_H(y)]) \ge 4$. In particular, since $d(z) \ge 8$, $|N_H(z) \cap N_H(y)| \ge 4$. By (1), $H[N_H(z) \cap N_H(y)]$ is not a clique and so z' is adjacent to at least one vertex, say w, in $N_H(z) \cap N_H(y)$, because $\alpha(H) = 2$. Now the edge zw is dominating J, that is, every vertex in J is adjacent to either z or w. Notice that $|N_H(y) \setminus \{z, w\}| = 6$. If $H[N_H[y] \setminus \{z, w\}]$ contains two vertex-disjoint, induced paths on three vertices, say P_1 and P_2, then $H > K_7 \cup K_1$ by contracting the edge zw and the two paths P_1 and P_2 each into a distinct vertex, respectively, a contradiction. Thus $H[N_H[y] \setminus \{z, w\}]$ does not contain two vertex-disjoint, induced paths on three vertices. By (2), $H[N_H(y) \setminus \{z, w\}]$ is thus a disjoint union of two cliques, say B_1 and B_2. Since $\delta(H[N_H(y)]) \ge 4$, we must have both B_1 and B_2 isomorphic to K_3. By (1), $H[V(B_1) \cup \{z, w, y\}]$ is not a clique. Let $w' \in V(B_1)$ be such that either $ww' \notin E(H)$ or $zw' \notin E(H)$. Since w' is adjacent to at most three vertices of $V(J)$, we see that $d_H(w') \le 7$, contrary to the fact that $\delta = 8$. This proves (3).

By (3), $\delta = t - 2$. If $t = 7$, then H is a graph on nine vertices with $\delta(H) = 5$. Thus there exists a vertex $z \in V(H)$ such that $d_H(z) \geq 6$ and so $N_H[z]$ contains a subgraph isomorphic to K_4 because $\alpha(H[N_H(z)]) = 2$, contrary to (1). Hence $t \geq 8$. Now $H - N_H[y]$ is a clique of size $t - 4$ and $|N_H(y)| = t - 2 \geq 6$. Clearly $H > K_{t-2} \cup K_1$ if $H - N_H(y)$ contains two vertex-disjoint, induced paths on three vertices, a contradiction. Thus by (2), $N_H(y)$ is a disjoint union of two cliques, say A_1 and A_2. For $i = 1, 2$, let

$$K_i = \{v \in V(H - N_H[y]) : v \text{ is not adjacent to some vertex in } A_{3-i}\}.$$

Since $\alpha(H) = 2$ and $V(A_1)$ is anticomplete to $V(A_2)$, K_i is complete to $V(A_i)$ for each $i \in \{1, 2\}$. Thus, since $H[K_i]$ is a clique, $H - (K_i \cup V(A_i) \cup \{y\})$ is a clique for each $i \in \{1, 2\}$. Therefore, since K_1 and K_2 are disjoint, at least one of either $H - (K_1 \cup V(A_1) \cup \{y\})$ or $H - (K_2 \cup V(A_2) \cup \{y\})$ is a clique of size at least $t - 3$, contrary to (1). This completes the proof of Lemma 3.1.2. ■

Given two graphs G_1 and G_2, the *join* $G_1 + G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{xy : x \in V(G_1), y \in V(G_2)\}$. For our proof of Theorem 1.3.8, we will need the following lemma from [64].

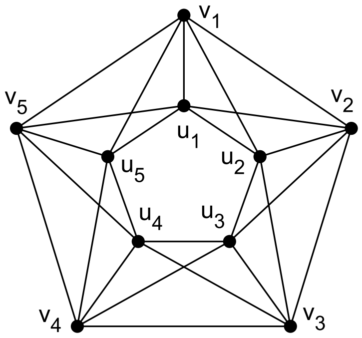

Figure 3.1: The graph J.

Lemma 3.1.3 (Song [64]) *Let G be a graph with $8 \leq |V(G)| \leq 10$ and $\delta(G) \geq 5$. Then either $G > K_6^- \cup K_1$ or G is isomorphic to one of $\overline{C_8}$, $\overline{C_4} + \overline{C_4}$, $\overline{K_3} + C_5$, $\overline{K_2} + \overline{C_6}$, $K_{2,3,3}$, or J, where J is the graph depicted in Figure 3.1. In particular, all of these graphs are edge maximal (subject to not having a $K_6^- \cup K_1$-minor) with maximum degree at most $|V(G)| - 2$. Moreover, $\overline{C_8} > K_6$, $\overline{C_4} + \overline{C_4} > K_6$, and $J > K_6$.*

Notice that of the counterexamples listed in Lemma 3.1.3, only the graph J has ten vertices, and none has exactly nine vertices. We next prove the following lemma, which we will also need for the proof of Theorem 1.3.8.

Lemma 3.1.4 [58] *Let G be a graph with $|V(G)| = 10$ and $\alpha(G) = 2$. Then either $G > K_6^- \cup K_1$, or G contains a subgraph isomorphic to $K_5 \cup K_5$, or G is isomorphic to the graph J depicted in Figure 3.1.*

Proof. If $\delta(G) \geq 5$, then by Lemma 3.1.3, either $G > K_6^- \cup K_1$ or G is isomorphic to J. So we may assume that $\delta(G) \leq 4$. Let $x \in V(G)$ be such that $d_G(x) = \delta(G)$. Since $\alpha(G) = 2$, one can easily see that $G > K_6 \cup K_1$ if $d_G(x) \leq 3$. Hence we may further assume that $d_G(x) = 4$. Then $G - N_G[x]$ must be isomorphic to K_5 as $\alpha(G) = 2$. If $G[N_G[x]]$ is also isomorphic to K_5, then G contains a subgraph isomorphic to $K_5 \cup K_5$. Otherwise, some edge is missing from $G[N_G(x)]$, say $y, z \in N_G(x)$ with $yz \notin E(G)$. Then since $\alpha(G) = 2$, each vertex in $V(G) \setminus N_G[x]$ must be adjacent to either y or z. Thus by contracting $\{x, y, z\}$ to a single vertex, we see that $G > K_6 \cup K_1$, as desired. This completes the proof of Lemma 3.1.4. ■

3.2 Proof of Theorem 1.3.7

Suppose the assertion is false. Let G be a graph with no K_t-minor such that G is not $(2t-6)$-colorable. We may choose such a graph G so that it is $(2t-5)$-contraction-critical. Let $x \in V(G)$ be a vertex of minimum degree. Since $K_{2,2,2,3,3}$ and each $(K_{2,2,2,2,2}, 5)$-cockade are 5-colorable, and every $(K_{1,2,2,2,2,2}, 6)$-cockade is 6-colorable, it follows from Theorem 1.4.1, Theorem 1.4.2, and Theorem 1.4.3 that $d_G(x) \leq 2t-5$. On the other hand, since G is $(2t-5)$-contraction-critical, by Proposition 1.5.1(i), $d_G(x) \geq 2t-5$. Thus $d_G(x) = 2t-5 \geq t+2$. By Proposition 1.5.1(ii), we have $\alpha(G[N_G(x)]) = 2$. We next show that

(1) *G has no K_{t-1}-subgraph.*

Suppose G contains K_{t-1} as a subgraph. Let $H \subseteq G$ be isomorphic to K_{t-1}. Since $\delta(G) = d_G(x) \geq t+2$, every vertex in $v(H)$ is adjacent to at least one vertex in $V(G-H)$. Then $G-H$ is disconnected, since otherwise $G > K_t$ by contracting $G-H$ into a single vertex, a contradiction. Let G_1 be a component of $G-H$. Then $N_G(V(G_1)) \subseteq V(H)$ is a minimal separating set of G. In particular, $N_G(V(G_1))$ induces a clique in G, contrary to Proposition 1.5.1(iii). This proves (1).

(2) *For any $u \in N_G(x)$, $|N_G(x) \cap N_G(u)| \geq t-3$.*

Suppose that there exists a vertex $u \in N_G(x)$ such that $|N_G(x) \cap N_G(u)| \leq t-4$. Since $\alpha(G[N_G(x)]) = 2$, $N_G(x)$ contains a clique of size $|N_G(x) \setminus N_G[u]| \geq t-2$ and so $N_G[x]$ has a subgraph isomorphic to K_{t-1}, contrary to (1). This proves (2).

By Lemma 3.1.2, $G[N_G(x)] > K_{t-2} \cup K_1$. Let $y \in N_G(x)$ be such that $G[N_G(x) \setminus \{y\}] > K_{t-2}$. Clearly, y is not complete to $N_G(x) \setminus \{y\}$, for otherwise $G > N_G[x] > K_t$, a contradiction. Let $\{y_1, \ldots, y_p\} = N_G(x) \setminus N_G[y]$, where $p = 2t-5-|N_G(x) \cap N_G[y]|$. Then y is anticomplete to $\{y_1, y_2, \ldots, y_p\}$. By (1) and (2), $N_G[y] \cap N_G(x)$ is not a clique. Let uw be a missing edge in $G[N_G(y) \cap N_G(x)]$. By Lemma 1.5.3 applied to $N_G(x)$ with $k = 2t - 5$, $S = \{u, w\}$ and $M = \{yy_1, yy_2, \ldots, yy_p\}$, there exists a path P_i between y and y_i with its internal vertices in $V(G) \setminus N_G[x]$ for each $i \in \{1, 2, \ldots, p\}$. Note that the paths $P_1 - y_1, \ldots, P_p - y_p$ have y as a common end. By contracting all paths $P_i - y_i$ onto y, we see that $G > K_t$, a contradiction. ■

3.3 Proof of Theorem 1.3.8

Let G be a graph that does not contain K_8^- as a minor. Suppose for a contradiction that $\chi(G) \geq 10$. We may choose such a graph G so that it is 10-contraction-critical. Then by Proposition 1.5.1(i), $\delta(G) \geq 10$. On the other hand, since every $(K_{1,2,2,2,2}, K_7, 5)$-cockade is 7-colorable, by Theorem 1.4.6 we see that $\delta(G) \leq 10$. Thus $\delta(G) = 10$. Let $x \in V(G)$ be such that $d_G(x) = 10$. Since G has no K_8^--minor, by Proposition 1.5.1(ii) we have

(1) $\alpha(G[N_G(x)]) = 2$.

We next show that

(2) $G[N_G(x)]$ *is not isomorphic to the graph* J.

Suppose that $N_G(x)$ is isomorphic to the graph J. Let the vertices of J be labeled as depicted in Figure 3.1. By Lemma 1.5.3 applied to J with $S = \{v_2, v_5\}$ and $M = \{\{u_1u_3, u_1u_4, u_1v_3, u_1v_4\}, \{u_2u_5\}\}$ with $m = 2$, $r_1 = 4$, and $r_2 = 1$, there exist paths P_{11}, P_{12}, P_{13}, P_{14}, and P_{21} such that the paths P_{11}, P_{12}, P_{13}, P_{14}, and P_{21} have ends $\{u_1, u_3\}$, $\{u_1, u_4\}$, $\{u_1, v_3\}$, $\{u_1, v_4\}$, and $\{u_2, u_5\}$, respectively, and all their internal vertices in $V(G) \setminus N_G[x]$. Moreover, the paths P_{11}, P_{12}, P_{13}, and P_{14} are vertex-disjoint from the path P_{21}. By contracting $(V(P_{11}) \setminus \{u_3\}) \cup (V(P_{12}) \setminus \{u_4\}) \cup (V(P_{13}) \setminus \{v_3\}) \cup (V(P_{14}) \setminus \{v_4\})$ onto u_1, $V(P_{21}) \setminus \{u_2\}$ onto u_5, and $\{v_2, v_1, v_5\}$ into a single vertex, we see that $G > K_8$, a contradiction. This proves (2).

(3) $G[N_G(x)]$ *contains a subgraph isomorphic to* $K_5 \cup K_5$.

Suppose that $G[N_G(x)]$ does not contain a subgraph isomorphic to $K_5 \cup K_5$. Then by (1), (2), and Lemma 3.1.4, we see that $G[N_G(x)] > K_6^- \cup K_1$. Let $y \in N_G(x)$ be a vertex such that $G[N_G(x)\backslash\{y\}] > K_6^-$. Clearly, y is not complete to $N_G(x)\backslash\{y\}$, for otherwise $G > N_G[x] > K_8^-$, a contradiction. Let $\{y_1, \ldots, y_p\} = N_G(x) \setminus N_G[y]$, where $p = 10 - |N_G(x) \cap N_G[y]| \geq 1$. Then y is anticomplete to $\{y_1, y_2, \ldots, y_p\}$. Clearly, $G[N_G(x) \setminus \{y, y_i\}]$ is not a clique for all $i \in \{1, 2, \ldots, p\}$. By Lemma 1.5.3 applied p times to $G[N_G(x)]$ with $k = 10$, $s = 0$, and $m = 1$ (where for $i \in \{1, 2, \ldots, p\}$, we have $M = \{yy_i\}$ and $S = \{u_i, v_i\}$, where $u_i v_i$ is any missing edge in $G[N_G(x) \setminus \{y, y_i\}]$), there exists a path P_i between y and y_i with its internal vertices in $V(G) \setminus N_G[x]$ for each $i \in \{1, 2, \ldots, p\}$. Note that the paths $P_1, \ldots, P_p$ all have y as a common end. By contracting each set $V(P_i) \setminus \{y_i\}$ onto y, we see that $G > K_8^-$, a contradiction. This proves (3).

By (3), x belongs to two different subgraphs of G isomorphic to K_6. By Theorem 1.5.2(ii), G is 7-connected. By Lemma 3.1.1, $G > K_8^-$. This contradiction completes the proof of Theorem 1.3.8.

■

3.4 Proof of Theorem 1.3.9

Suppose the assertion is false. Let G be a graph with no $K_8^=$-minor such that $\chi(G) \geq 9$. We may choose such a graph G so that it is 9-contraction-critical. Let $x \in V(G)$ be a vertex of minimum degree. By Proposition 1.5.1(i), $d_G(x) \geq 9$. On the other hand, since each $(K_7, 4)$-cockade is 4-colorable, it follows from Theorem 1.4.7 for $p = 8$ that $d_G(x) \leq 9$. Thus $d_G(x) = 9$. It follows from Theorem 1.4.7 for $p = 8$ again that

(1) *G contains at least 28 vertices of degree 9.*

Since G has no $K_8^=$-minor, by Proposition 1.5.1(ii),

(2) $\alpha(G[N(x)]) = 2$.

We next show that

(3) *$N_G(x)$ contains a subgraph isomorphic to K_5.*

Suppose that $N_G(x)$ does not contain a subgraph isomorphic to K_5. Then $\omega(G[N_G(x)]) \leq 4$ and by (2), $\delta(G[N_G(x)]) \geq 4$. We claim that $\delta(G[N_G(x)]) = 4$. Suppose that $\delta(G[N_G(x)]) \geq 5$. By Lemma 3.1.3 applied to $N_G(x)$, we see that $N_G(x) > K_6^- \cup K_1$. Let $y \in N_G(x)$ be such that $G[N_G(x)] - y > K_6^-$. Clearly y has at least two non-neighbors in $N_G(x) - y$, otherwise $G[N_G[x]] > K_8^=$, a contradiction. Let $\{y_1, y_2, \ldots, y_p\} = N_G(x) \setminus N_G[y]$ be all non-neighbors of y in $N_G(x)$, where $p = |N_G(x) \setminus N_G[y]| \geq 2$. Since $\omega(G[N_G(x)]) \leq 4$, $G[N_G(x) \cap N_G(y)]$ must have a missing edge, say uv. By Lemma 1.5.3 applied to $G[N_G(x)]$ with $S = \{u, v\}$ and $M = \{yy_1, \ldots, yy_p\}$, there exist p paths $P_1, P_2, \ldots, P_p$ such that each path P_i has ends $\{y, y_i\}$ and all its internal vertices in $V(G) \setminus N_G[x]$. By contracting all the edges of each $P_i - y_i$ onto y for all $i \in \{1, 2, \ldots, p\}$, we see that $G > K_8^-$, a contradiction. This proves that $\delta(G[N_G(x)]) = 4$, as claimed.

Let $y \in N_G(x)$ be such that y has degree four in $G[N_G(x)]$ with the number of edges in $G[N_G(x) \cap N_G(y)]$ maximum. Let $Z = \{z_1, z_2, z_3, z_4\}$ be the set of all neighbors of y in $N_G(x)$. Since $\omega(G[N_G(x)]) \leq 4$, $G[N_G(x) \cap N_G[y]]$ is not complete. We may assume that $z_1z_2 \notin E(G)$. By (2), $G[N_G(x) \setminus N_G[y]]$ is isomorphic to K_4. Let $W = \{w_1, w_2, w_3, w_4\} = N_G(x) \setminus N_G[y]$. We next show that

(3.1) *each of* z_3, z_4 *has at most one neighbor in* W.

Suppose, say z_4, is adjacent to at least two vertices in W. Then the subgraph $G[W \cup \{z_4\}]$ has a $K_5^=$-minor and thus $G[N_G[x]] > K_8^=$ if z_3 is adjacent to all vertices in W (by contracting the path z_1yz_2 into a single vertex), a contradiction. Thus we may assume that z_3 is not adjacent to $w_1, \ldots, w_k$, where $1 \leq k \leq 4$. By Lemma 1.5.3 applied to $G[N_G(x)]$ with $S = \{z_1, z_2\}$ and $M = \{z_3w_1, \ldots, z_3w_k\}$, there exist k paths $P_1, P_2, \ldots, P_k$ such that for each $i = 1, 2, \ldots, k$, the path P_i has ends $\{z_3, w_i\}$ and all its internal vertices in $V(G) \setminus N_G[x]$. By contracting each $P_i - w_i$ onto z_3 and contracting $\{z_1, y, z_2\}$ into a single vertex, we see that $G > K_8^=$, a contradiction. This proves (3.1).

We next claim that $G[N_G[y] \cap N_G(x)]$ is isomorphic to K_5^-. Suppose $z_3z_4 \notin E(G)$. By symmetry, we may apply (3.1) to the missing edge z_3z_4 in $G[N_G(x)]$, and so we see that each of z_1 and z_2 has at most one neighbor in W. Hence $e_G(Z, W) \leq 4$. On the other hand, since $\alpha(G[N_G(x)]) = 2$, each w_i must be adjacent to at least one of the vertices in each of $\{z_1, z_2\}$ and $\{z_3, z_4\}$, for all $i \in \{1, 2, 3, 4\}$. Thus $e_G(W, Z) \geq 8$, a contradiction. This proves that $z_3z_4 \in E(G)$ and thus $G[N_G(x) \cap N_G[y]]$ does not have two independent missing edges. Next if $z_1z_3 \notin E(G)$, then $z_2z_3 \in E(G)$ because $\alpha(G[N_G(x)]) = 2$. Since $G[N_G(x) \cap N_G[y]]$ does not have two independent missing edges, we see that $z_2z_4 \in E(G)$. If $z_1z_4 \notin E(G)$, then since $\delta(G[N_G(x)]) \geq 4$ and $N_G(x)$ does not contain a subgraph isomorphic to K_5, we may assume z_1 is adjacent to w_2, w_3, w_4. Then since $\alpha(G[N_G(x)]) = 2$ by (2), we have w_1 complete to $\{z_2, z_3, z_4\}$. By symmetry, we may apply (3.1) to each of the missing edges z_1z_2, z_1z_3, and z_1z_4 to conclude that z_2, z_3, and z_4 have no other neighbors in W. But now w_2 has four neighbors in $N_G(x)$ and $G[N_G(x) \cap N_G[w_2]]$ has 5 edges, contrary to our choice of y. Hence $z_1z_4 \in E(G)$ and $G[N_G(x) \cap N_G[y]]$ is isomorphic to $K_5^=$. Since $\omega(G[N_G(x)]) \leq 4$, we may assume that $z_1w_1 \notin E(G)$. Then w_1 must be adjacent to both z_2 and z_3 by (2). By symmetry again, we may apply (3.1) to the missing edges z_1z_2 and z_1z_3, to see that $\{z_2, z_3\}$ is anticomplete to $\{w_2, w_3, w_4\}$ and that z_4 has at most one neighbor in W. By (2), z_1 is complete to $\{w_2, w_3, w_4\}$. Since z_4 has at most one neighbor in W, we may assume that $w_4z_4 \notin E(G)$. Now w_4 has degree four in $N_G(x)$ with $N_G(x) \cap N_G[w_4]$ is isomorphic to K_5^-, contrary to the choice of y. Thus $G[N_G(x) \cap N_G[y]]$ isomorphic to K_5^- with z_1z_2 the only missing edge, as claimed.

Since $\delta(G[N_G(x)]) = 4$, each of z_1 and z_2 has at least one neighbor in W. By (2), each of $w_1, \ldots, w_4$ is adjacent to at least one of z_1 and z_2, and so either z_1 or z_2 has at least two neighbors in W. By symmetry, we may assume that $|N_G(z_1) \cap W| \geq |N_G(z_2) \cap W|$. On the other hand, each vertex in Z has at least one non-neighbor in W as $\omega(G[N_G(x)]) \leq 4$. Thus, z_1 has either two or three neighbors in W. We consider the following two cases.

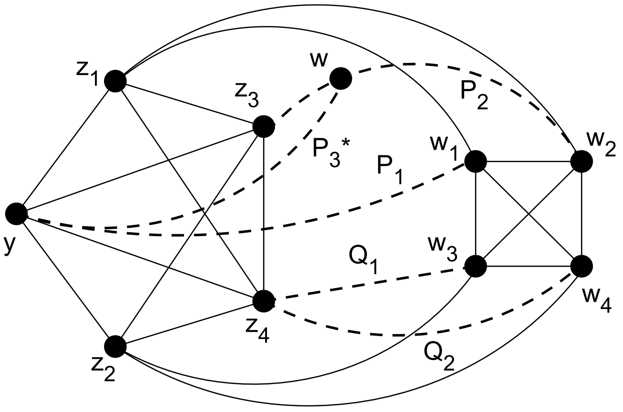

Figure 3.2: Finding a $K_8^=$-minor when z_1 has exactly two neighbors in W.

First, assume that z_1 has exactly two neighbors in W, say w_1, w_2. Then z_2 must have exactly two neighbors in W, namely w_3 and w_4. By (3.1), each of z_3 and z_4 has at most one neighbor in W. We may assume that z_4 is not adjacent to w_3 and w_4, and that z_3 is not adjacent to w_2. By Lemma 1.5.3

applied twice to $G[N_G(x)]$ with $S = \{z_1, z_2\}$ and $M \in \{\{yw_1, z_3w_2, z_4w_3\}, \{yw_2, z_4w_4\}\}$, there exist three vertex-disjoint paths P_1, P_2, Q_1 and two vertex-disjoint paths P_3, Q_2 such that the paths P_1, P_2, P_3, Q_1, and Q_2 have ends $\{y, w_1\}$, $\{z_3, w_2\}$, $\{y, w_2\}$, $\{z_4, w_3\}$ and $\{z_4, w_4\}$, respectively, and all their internal vertices in $V(G) \setminus N_G[x]$, as depicted in Figure 3.2. Notice that for all $i \in \{1, 2, 3\}$ and all $j \in \{1, 2\}$, the ends of each P_i are distinct from the ends of each Q_j, and so each P_i is chosen to be vertex-disjoint from Q_j when applying Lemma 1.5.3. Similarly, P_1 and P_2 are chosen to be vertex-disjoint, but P_3 is not necessarily internally disjoint from either of P_1 or P_2. If P_3 and P_2 have only w_2 in common, then contracting $P_1 - w_1$ and $P_3 - w_2$ onto y, contracting $P_2 - w_2$ onto z_3, contracting $Q_1 - w_3$ and $Q_2 - w_4$ onto z_4, and contracting each of w_1w_3 and w_2w_4 into distinct vertices yields a $K_8^=$-minor in G, a contradiction. Thus P_3 and P_2 must have an internal vertex in common. Let w be the first vertex on P_3 (when P_3 is read in order from y to w_2) that is also on P_2. Then $w \notin V(P_1) \cup \{z_3\}$. Let P_3' be the subpath of P_3 from y to w and P_2' be the subpath of P_2 from w to w_2. Notice that $P_3' - w$ is vertex-disjoint from P_2 but not necessarily internally disjoint from P_1. Now contracting $P_1 - w_1$ and $P_3' - w$ onto y, contracting P_2' onto w_2, contracting $P_2 - P_2'$ onto z_3, contracting $Q_1 - w_3$ and $Q_2 - w_4$ onto z_4, and contracting each of w_1w_3 and w_2w_4 into distinct vertices yields another $K_8^=$-minor in G, a contradiction.

It remains to consider the case when z_1 has exactly three neighbors, say w_1, w_2, w_3 in W. Then z_2 is adjacent to w_4. By (3.1), we may assume that w_1 is not adjacent to z_3 and z_4, and that w_3 is not adjacent to z_4. By Lemma 1.5.3 applied twice to $G[N_G(x)]$ with $S = \{z_1, z_2\}$ and $M \in \{\{yw_2, z_3w_1, z_4w_3\}, \{z_4w_1\}\}$, there exist vertex-disjoint paths P_1, Q_1, Q_2 and another path Q_3 such that the paths $P_1, Q_1, Q_2,$ and Q_3 have ends $\{y, w_2\}$, $\{z_3, w_1\}$, $\{z_4, w_3\}$, and $\{z_4, w_1\}$, respectively, and all their internal vertices in $V(G) \setminus N_G[x]$, as depicted in Figure 3.3. Notice that P_1 is vertex-disjoint from Q_j for all $j \in \{1, 2, 3\}$, but that Q_3 is not necessarily internally disjoint from either of Q_1 or Q_2. If Q_3 and Q_2 have only z_4 in common, then we obtain a $K_8^=$-minor by contracting each of P_1 and z_2w_4 into distinct vertices, contracting $Q_1 - z_3$ and $Q_3 - z_4$ onto w_1, and contracting $Q_2 - z_4$ onto w_3, a contradiction. Thus Q_3 and Q_2 must have an internal vertex in common. Let w be the first vertex on Q_3 (when Q_3 is read from w_1 to z_4) that is also on Q_2. Then $w \notin V(Q_1) \cup \{w_3\}$. Let Q_3' be the subpath of Q_3 from w_1 to w and Q_2' be the subpath of Q_2 from w to z_4. Notice that $Q_3' - w$ is vertex-disjoint from Q_2 but not necessarily internally disjoint from Q_1. Now we obtain another $K_8^{=}$-minor by contracting each of P_1 and z_2w_4 into distinct vertices, contracting $Q_1 - z_3$ and $Q_3' - w$ onto w_1, contracting Q_2' onto z_4, and contracting $Q_2 - Q_2'$ onto w_3, a contradiction. This completes the proof of (3).

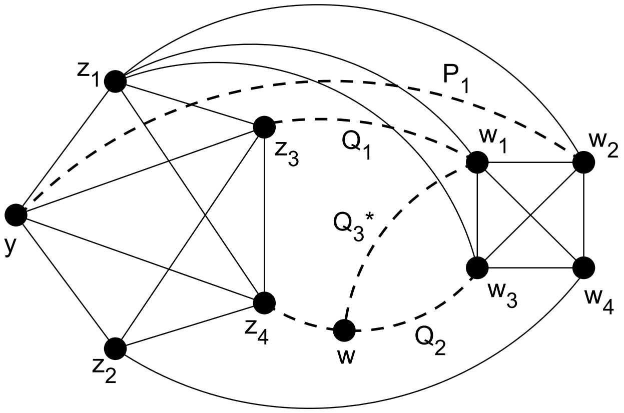

Figure 3.3: Finding a $K_8^{=}$-minor when z_1 has exactly three neighbors in W.

By (3), every vertex of degree 9 belongs to some subgraph of G isomorphic to K_6. By (1), G contains at least five different subgraphs isomorphic to K_6. By Theorem 1.5.2(ii), G is 7-connected and thus $G > K_8^-$ by Lemma 3.1.1. This contradiction completes the proof of Theorem 1.3.9. ■

3.5 K_t-minor free graphs for $t \geq 10$

Our Theorem 1.3.7, which we proved in Section 3.2, relies on the extremal function for K_t-minors for $t \in \{7, 8, 9\}$, namely Theorem 1.4.1, Theorem 1.4.2, and Theorem 1.4.3. As mentioned in Section 1.4, the extremal function for K_t-minors remains open for all $t \geq 10$. By noting that any (H_1, H_2, k)-cockade not isomorphic to either of H_1 or H_2 is at most k-connected and that any complete multipartite graph $K_{k_1,\dots,k_r}$ has a fixed number of vertices, Seymour and Thomas [67] proposed the following conjecture.

Conjecture 3.5.1 (*Seymour and Thomas [67]) For every* $t \geq 1$ *there exists a constant* $N = N(t)$ *such that every* $(t-2)$*-connected graph on* $n \geq N$ *vertices with at least* $(t-2)n - \binom{t-1}{2} + 1$ *edges has a* K_t*-minor.*

By the results mentioned in Section 1.4, Conjecture 3.5.1 is true for $t \leq 9$. However, as mentioned in Section 1.5, it seems very hard to prove that even 8-contraction-critical, non-complete graphs are 8-connected. Hence Conjecture 3.5.1 cannot be easily applied to prove results on the coloring of K_t-minor free graphs for $t \geq 10$.

By noting that any (H_1, H_2, k)-cockade is $\max\{\chi(H_1), \chi(H_2)\}$-colorable and any complete multipartite graph $K_{k_1,\dots,k_r}$ is r-colorable, we instead propose the following conjecture.

Conjecture 3.5.2 [58] *For every* $t \geq 1$*, every graph* G *on* n *vertices with at least* $(t-2)n - \binom{t-1}{2} + 1$ *edges either has a* K_t*-minor or is* $(t-1)$*-colorable.*

Again by the results mentioned in Section 1.4, Conjecture 3.5.2 is true for $t \leq 9$. To end this chapter, we apply our versatile Lemma 1.5.3 along with an idea different from that used in the proof of Theorem 1.3.7 in Section 3.2 (namely, considering the chromatic number of $G[N_G(x)]$ instead of showing that $N_G(x) > K_{t-2} \cup K_1$) to prove that the truth of Conjecture 3.5.2 implies that every graph with no K_t-minor is $(2t-6)$-colorable for all $t \geq 5$. Since Conjecture 3.5.2 is true for $t \leq 9$, we see that the following Theorem 3.5.3 implies Theorem 1.3.7.

Theorem 3.5.3 [58] *If Conjecture 3.5.2 is true, then every graph with no K_t-minor is $(2t-6)$-colorable for all $t \geq 5$.*

Proof. Suppose the assertion is false. By Wagner's Theorem and the Four Color Theorem for $t = 5$, Theorem 1.3.5 for $t = 6$, and Theorem 1.3.7 for $t \in \{7, 8, 9\}$, we see the conclusion holds for all $t \in \{5, \ldots, 9\}$. Hence $t \geq 10$. Among all minimum counterexamples, we choose G so that G has no K_t-minor and G is $(2t-5)$-contraction-critical. Let $x \in V(G)$ be such that $d_G(x) = \delta(G)$. By the assumed truth of Conjecture 3.5.2, we see that $d_G(x) \leq 2t-5$. On the other hand, by Proposition 1.5.1(i), we see that $d_G(x) \geq 2t-5$. Hence $d_G(x) = 2t-5$. By Proposition 1.5.1(ii), we have

(1) $\alpha(G[N_G(x)]) = 2$.

Our strategy now will be to examine the subgraph $G[N_G(x)]$ and its chromatic number. We first show that

(2) $\omega(G[N_G(x)]) \leq t-3$*, and so* $\delta(G[N_G(x)]) \geq t-3$.

Suppose $\omega(G[N_G(x)]) \geq t-2$. Let $H \subseteq G[N_G[x]]$ be isomorphic to K_{t-1}. Since $\delta(G) = 2t-5$, every vertex in $V(H)$ is adjacent to $t-3$ vertices in $V(G-H)$. Then $G-H$ is disconnected, for otherwise $G > K_t$ by contracting $G-H$ into a single vertex, a contradiction. Let G_1 be a component of $G-H$. Then $N_G(G_1)$ is a minimal separating set of G. In particular, $N_G(G_1)$ is a clique, contrary to Proposition 1.5.1(iii). This proves that $\omega(G[N_G(x)]) \leq t-3$. By (1), we see that $\delta(G[N_G(x)]) \geq t-3$. This proves (2).

(3) $\chi(G[N_G(x)]) = t-2$.

Suppose to the contrary that $\chi(G[N_G(x)]) \neq t-2$. By (1), it is clear that $\chi(G[N_G(x)]) \geq t-2$. Thus $\chi(G[N_G(x)]) = p$ for some $p \geq t-1$. Let $V_1, \ldots, V_p$ be the color classes of any p-coloring of $G[N_G(x)]$. We may assume that the color classes are ordered so that $V_i = \{a_i\}$ for $i \in \{1, 2, \ldots, 2p-2t+5\}$ and $V_i = \{a_i, b_i\}$ for $i \in \{2p-2t+6, \ldots, p\}$. Let $r = 2p-2t+6 \geq 4$. Since $\chi(G[N_G(x)]) = p$, we see that there is at least one edge between any pair of color classes $V_1, \ldots, V_p$ in G. Hence $\{a_1, a_2, \ldots, a_{r-1}\}$ induces a clique in $G[N_G(x)]$, and so $r \leq t-2$ by (2). Furthermore, a_i is adjacent to either a_j or b_j for each $i \in \{1, 2, \ldots, r-1\}$ and each $j \in \{r, \ldots, p\}$. Notice that if $p = t-1$, then $r = 4$. Suppose either that $p \geq t$ or that $p = t-1$ and a_1, a_2 and a_3 have a common neighbor in $N(x) \setminus \{a_1, a_2, a_3\}$, say a_4. By Lemma 1.5.3 applied to $G[N_G(x)]$ with $S = \{a_r, b_r\}$ and $M = \{a_{r+1}b_{r+1}, \ldots, a_pb_p\}$, there exist $p-r$ pairwise vertex-disjoint paths $P_{r+1}, \ldots, P_p$ such that each P_j has ends $\{a_j, b_j\}$ and all its internal vertices in $V(G) \setminus N_G[x]$. By contracting each P_j to a single vertex for all $j \in \{r+1, \ldots, p\}$, together with $x, a_1, \ldots, a_{r-1}$ (if $t \geq p$) or together with x, a_1, a_2, a_3, and a_4 (if $p = t-1$, where $a_4 = a_r$ is a common neighbor of a_1, a_2, and a_3), we obtain a clique minor with $(p-r)+r = p \geq t$ vertices in the former case and $(p-r)+r+1 = p+1 = t$ vertices in the latter case, a contradiction. Thus $p = t-1$ and a_1, a_2, and a_3 have no common neighbor in $N_G(x) \setminus \{a_1, a_2, a_3\}$.

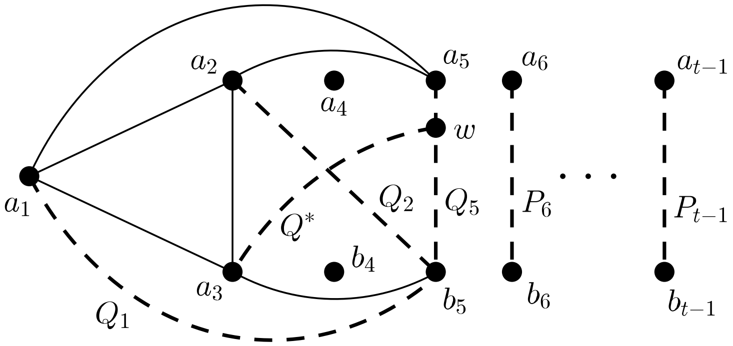

Figure 3.4: Finding a K_t-minor when $\chi(G[N_G(x)]) = t-1$.

Since each of $a_1, a_2,$ and a_3 is adjacent to either a_5 or b_5, by symmetry, we may assume that a_5 is adjacent to a_1 and a_2, but not to a_3. Then b_5 is adjacent to a_3. We may assume that b_5 is not adjacent to a_1. For the worst case scenario, we may further assume that b_5 is not adjacent to a_2. By Lemma 1.5.3 applied twice to $G[N_G(x)]$ with $S = \{a_4, b_4\}$ and $M \in \{\{\{a_6b_6\}, \ldots, \{a_pb_p\}, \{b_5a_1, b_5a_2, b_5a_5\}\}, \{a_5a_3\}\}$, there exist pairwise vertex-disjoint paths $P_6, \ldots, P_p$ such that each P_j has ends $\{a_j, b_j\}$ and all its internal vertices in $V(G) \setminus N_G[x]$, and paths Q_1, Q_2, Q_5, Q with ends $\{b_5, a_1\}$, $\{b_5, a_2\}$, $\{b_5, a_5\}$, and $\{a_5, a_3\}$, respectively, and all their internal vertices in $V(G) \setminus N_G[x]$. Notice that each P_j is vertex-disjoint from $Q_1, Q_2, Q_5,$ and Q, and that Q is vertex-disjoint from Q_1 and Q_2 but not necessarily from Q_5. Let w be the first vertex on Q (when read from a_3 to a_5) that is also on Q_5. Note that w could be a_5. Let Q' be the subpath of Q between w and a_3, as depicted in Figure 3.4. By contracting each P_j to a single vertex for all $j \in \{6, \ldots, p\}$, contracting $Q_1 - a_1$ and $Q_2 - a_2$ onto b_5, contracting $Q' - w$ onto a_3, and contracting $Q_5 - b_5$ onto a_5, together with the vertices $x, a_1, a_2,$ and a_3, we obtain a K_t-minor, a contradiction. This proves (3).

(4) $\delta(G[N_G(x)]) \geq t - 2$.

By (1) and (2), we have $\delta(G[N_G(x)]) \geq t - 3$. Suppose there exists a vertex $y \in N_G(x)$ such that y has exactly $t - 3$ neighbors in $N_G(x)$. Then $N_G(x) \setminus N_G[y]$ induces a clique in $G[N_G(x)]$ with $t - 3$ vertices. Furthermore, by (2), $G[N_G(x) \cap N_G(y)]$ must have some missing edge, say uv. Then every vertex of $N_G(x) \setminus N_G[y]$ is adjacent to at least one of u and v. Thus, by contracting $\{u, y, v\}$ to a single vertex, we can see that $G[N_G(x)] > K_{t-2} \cup K_1$.

Now we can assume without loss of generality that $y \in N_G(x)$ is such that $G[N_G(x) \setminus \{y\}] > K_{t-2}$. Clearly y is not adjacent to every vertex in $N_G(x) \setminus \{y\}$, or else $G > G[N_G[x]] > K_t$, a contradiction. Let $\{y_1, \ldots, y_p\} = N_G(x) \setminus N_G[y]$, where $p \geq 1$. Again, by (2), $G[N_G(y)]$ must have some missing edge, say uv. By Lemma 1.5.3 applied to $G[N_G(x)]$ with $S = \{u, v\}$

and $M = \{yy_1, \ldots, yy_p\}$, there exist paths $P_1, \ldots, P_p$ such that each P_j has ends $\{y, y_j\}$ and all internal vertices in $V(G) \setminus N_G[x]$. Now by contracting each $P_j - y_j$ onto y, we see that $G > K_t$, a contradiction. This proves (3).

By (2) and (4), $N_G(x)$ does not contain K_{t-2} as a subgraph and $\delta(G[N_G(x)]) \geq t-2$. By (3), $\chi(G[N_G(x)]) = t-2$. Let $V_1, \ldots, V_{t-2}$ be the color classes of any $(t-2)$-coloring of $G[N_G(x)]$. By (1), we may assume that the color classes are ordered so that $V_1 = \{a_1\}$ and $V_j = \{a_j, b_j\}$ for $j \in \{2, \ldots, t-2\}$. Since $\chi(G[N_G(x)]) = t-2$, we see that there is at least one edge between any pair of color classes $V_1, \ldots, V_{t-2}$ in G. By (4), a_1 must be complete to some color class $V_i \in \{V_2, \ldots, V_{t-2}\}$, say V_2. By (4) again, a_2 and b_2 must have one common neighbor in some color class $V_i \in \{V_2, \ldots, V_{t-2}\}$, say V_3. We may assume that a_3 is adjacent to both a_2 and b_2. By symmetry, we may further assume that b_3 is adjacent to a_2. By Lemma 1.5.3 applied to $G[N_G(x)]$ with $S = \{a_2, b_2\}$ and $M = \{\{b_3a_1, b_3a_3\}, \{a_4b_4\}, \ldots, \{a_{t-2}b_{t-2}\}\}$, there exist paths P_1, P_2 and pairwise vertex-disjoint paths $Q_4, \ldots, Q_{t-2}$ such that P_1 and P_2 have ends $\{b_3, a_1\}$ and $\{b_3, a_3\}$, respectively, each Q_j has ends $\{a_j, b_j\}$, and all such paths have their internal vertices in $V(G) \setminus N_G[x]$. By contracting $P_1 - a_1$ and $P_2 - a_3$ onto b_3, contracting the edge b_2a_3 onto a_3, and contracting each Q_j into a single vertex for $4 \leq j \leq t-2$, we see that $G > K_t$, a contradiction. This contradiction completes the proof of Theorem 3.5.3. ■

CHAPTER 4: FINDING K_8-MINORS USING MADER'S H-WEGE THEOREM

4.1 Proof of Theorem 1.6.6

We will now proceed with the proof of Theorem 1.6.6.

Proof of Theorem 1.6.6. Let $G, H_1, H_2,$ and H_3 be given as in the statement of Theorem 1.6.6. Suppose for a contradiction that G does not contain a K_8-minor. Let $M := (H_1 \cap H_2) \cup (H_2 \cap H_3) \cup (H_3 \cap H_1)$. Then $1 \leq |M| \leq 4$ by (B) and (C). Note that any vertex $x \in M$ corresponds to a good path on a single vertex. We also note here that there is symmetry between H_1 and H_2, and in general there is no symmetry between H_3 and either H_1 or H_2.

(1) *G does not have eight disjoint good paths.*

For suppose $P_1, \dots, P_8$ are disjoint good paths. Then for any distinct $i, j \in \{1, 2, \dots, 8\}$, P_i has an end in two of the sets H_1, H_2, H_3, and similarly for P_j. Hence there exists $k \in \{1, 2, 3\}$ such that H_k contains an end of each of P_i, P_j. Therefore, it follows that contracting each of these good paths to a single vertex gives a K_8-minor in G, a contradiction. This proves (1).

From Theorem 1.6.2 and (1), we can immediately conclude the following (see Figure 1.4):

(2) *There exists a set $W \subseteq V(G)$ and a partition $Y_1, \dots, Y_n$ of $V(G) \setminus W$, and for $1 \leq i \leq n$ a subset $X_i \subseteq Y_i$, such that*

- (i) $|W| + \sum_{i=1}^{n} \left\lfloor \frac{1}{2}|X_i| \right\rfloor \leq 7$,
- (ii) *no vertex in $Y_i \setminus X_i$ has a neighbor in $V(G) \setminus (W \cup Y_i)$, and $Y_i \cap (H_1 \cup H_2 \cup H_3) \subseteq X_i$ for $1 \leq i \leq n$, and*

(iii) *every good path disjoint from* W *has an edge with both ends in* Y_i *for some* i.

Let us choose the sets $W, Y_1, \ldots, Y_n, X_1, \ldots, X_n$ as in (2) such that W is maximum. We may assume that $Y_i \neq \emptyset$ for all $i \in \{1, 2, \ldots, n\}$.

(3) $M \subseteq W$.

Since each $v \in M$ corresponds to a good path consisting of only a single vertex, (3) follows immediately from (2.iii).

(4) $n \geq 2$.

If $n = 1$, then $H_1 \cup H_2 \cup H_3 \subseteq W \cup X_1$ by (2.ii), but since $|H_1 \cup H_2 \cup H_3| = 18 - |M|$ by (C), and since $M \subseteq W$ by (3), we then have $|W| + \left\lfloor \frac{1}{2}|X_1| \right\rfloor \geq 9$, contradicting (2.i). This proves (4).

(5) $X_i \neq \emptyset$ *for any* $i \in \{1, 2, \ldots, n\}$.

Suppose for a contradiction that $X_1 = \emptyset$, say. Then $Y_1 \setminus X_1 = Y_1$, and by (2.ii) every vertex of Y_1 has neighbors only in $Y_1 \cup W$. Let $y_1 \in Y_1$. By (4), $Y_2 \neq \emptyset$, so let $y_2 \in Y_2$. Then every y_1, y_2-path in G must meet W. Since G is 7-connected, this implies $|W| \geq 7$. By (2.i), we conclude that $|W| = 7$ and that $|X_i| < 2$ for all $i \in \{2, 3, \ldots, n\}$. Since $H_1 \cup H_2 \cup H_3 \subseteq W \cup X_1 \cup \cdots \cup X_n$ by (2.ii), and since $|H_1 \cup H_2 \cup H_3| = 18 - |M|$ and $|M| \leq 4$ by (C), we deduce that at least 7 sets X_i must be non-empty. It follows that $n \geq 7$. Thus we have shown

(5.1) $|W| = 7$, $n \geq 7$, *and* $|X_i| \leq 1$ *for all* $i \in \{2, 3, \ldots, n\}$.

Now we show

(5.2) $X_i \neq \emptyset$ *for any* $i \in \{2, 3, \ldots, n\}$.

Suppose that $X_2 = \emptyset$, say. The sets Y_1 and Y_2 are non-empty, and, for $i \in \{1, 2\}$, any $y \in Y_i$ has neighbors only in $W \cup Y_i$ by (2.ii). Thus $G[Y_1]$ and $G[Y_2]$ each contain at least one component of $G - W$. Since $n \geq 7$, and $Y_i \cap Y_j = \emptyset$ when $i \neq j$, there is at least one more component of $G - W$ disjoint from $G[Y_1 \cup Y_2]$, contradicting (A). This proves (5.2).

It follows immediately from (5.2) and (A) applied to the minimum separating set W of G that

(5.3) $G - W - Y_1$ *is connected.*

Since $|W| = 7$ and by (C), there is at most one H_i such that $H_i \subseteq W$. Say, $H_1 \setminus W \neq \emptyset$ and $H_j \setminus W \neq \emptyset$ for some $j \in \{2, 3\}$. Let $u_1 \in H_1 \setminus W$ and $u_j \in H_j \setminus W$. By (2.ii) and since $X_1 = \emptyset$, $H_1 \cup H_j \subseteq W \cup \bigcup_{i=2}^{n} X_i$, and so $u_1, u_j \notin Y_1$. By (5.3), there exists a u_1, u_j-path P which avoids W. Then P is a good path, so by (2.iii) some edge of P has both ends in Y_k for some $k \in \{2, 3, \ldots, n\}$. Say $k = 2$ and $z_1 z_j$ is an edge of P with both ends in Y_2, where u_1, z_1, z_j, u_j appear on P in order (note that u_i and z_i are not necessarily distinct for $i \in \{1, j\}$). If $z_1 \in Y_2 \setminus X_2$, then by (2.ii) and since P avoids W and $u_1 \notin Y_2 \setminus X_2$, some vertex of P between u_1 and z_1 must belong to X_2. Otherwise $z_1 \in X_2$. Similarly, either $z_j \in X_2$ or some vertex of P between u_j and z_j belongs to X_2. Hence $|X_2| \geq 2$, contradicting (5.1). This completes the proof of (5).

(6) $|X_i|$ *is odd for all* $i \in \{1, 2, \ldots, n\}$.

Suppose that $|X_1|$ is even, say. By (5), $|X_1| \geq 2$. Let $x \in X_1$. Define $W' = W \cup \{x\}$, $X_1' = X_1 \setminus \{x\}$, $Y_1' = Y_1 \setminus \{x\}$, and $X_i' = X_i, Y_i' = Y_i$ for $i \in \{2, 3, \ldots, n\}$. Then $W, X_1', \ldots, X_n', Y_1', \ldots, Y_n'$ satisfy (2), contradicting our choice of W as maximum. This proves (6).

Now for $j \in \{1, 2, 3\}$ let us define the set Z_j to be the union of the vertex sets of all paths P meeting H_j and avoiding W such that P has no edge with both ends in Y_i for any $i \in \{1, 2, \ldots, n\}$. In particular, note that any such path P is not a good path. The following is clear from (2.ii), (2.iii), and our definition of Z_j:

(7) *For* $j \in \{1, 2, 3\}$,

(i) $H_j \setminus W \subseteq Z_j \subseteq V(G) \setminus W$,

(ii) $Z_j \subseteq X_1 \cup \cdots \cup X_n$, *and*

(iii) *the sets* Z_1, Z_2, Z_3 *are mutually disjoint.*

(8) *For any* $j, k \in \{1, 2, 3\}$ *with* $j \neq k$, *every* Z_j, Z_k*-path avoiding* W *has at least 2 vertices in* X_i *for some* $i \in \{1, 2, \ldots, n\}$.

For suppose that G has a path Q avoiding W with ends $u \in Z_1$, say, and $v \in Z_j$ for some $j \in \{2, 3\}$. By the definition of Z_1 and Z_j, there exists both a path P of $G - W$ from some $u' \in H_1$ to u and a path R of $G - W$ from v to some $v' \in H_j$, such that both P and R have no edge with both ends in Y_i for any $i \in \{1, 2, \ldots, n\}$ (possibly $u = u'$ or $v = v'$, and the corresponding path P or R consists of only a single vertex). Let S be a subpath of $P \cup Q \cup R$ with ends u' and v'. Then S is a good path avoiding W and so has an edge e with both ends in Y_i for some $i \in \{1, 2, \ldots, n\}$ by (2.iii). Since $V(P) \subseteq Z_1$ and $V(R) \subseteq Z_j$ by our choice of P and R, this edge e must belong to Q by the definition of Z_1 and Z_j. By (7.ii) and the definition of Z_1 and Z_j, u and v belong to $X_1 \cup \cdots \cup X_n$, so by (2.ii) the part of Q from u to the first end of e must contain a vertex from X_i and the part of Q from the second end of e to v must similarly contain a vertex from X_i, as required, where Q is read from u' to v'. This proves (8).

(9) $|W| \leq 6$.

Suppose $|W| > 6$. By (2.i), $|W| = 7$. First suppose $G - W$ has some component which contains vertices from at least two of the sets Z_i. Say at least one vertex of each of Z_1 and Z_j for some $j \in \{2, 3\}$ belong to the same component of $G - W$. Thus there must exist some Z_1, Z_j-path in G which avoids W. By (8), at least two vertices of this path belong to the same X_k for some

$k \in \{1, 2, \ldots, n\}$. But then $|W| + \sum_{i=1}^{n} \left\lfloor \frac{1}{2}|X_i| \right\rfloor \geq 8$, contradicting (2.i).

Thus we may suppose that no component of $G - W$ contains vertices from more than one Z_i. Since $G - W$ contains at most two components by (A), this means there exists $j \in \{1, 2, 3\}$ such that $Z_j = \emptyset$. Thus, $H_j \subseteq W$ by the definition of Z_j. Further, since $|W| = 7$, no other $H_i \subseteq W$ for $i \in \{1, 2, 3\}$, $i \neq j$. Thus $G - W$ is disconnected, with $Z_i, Z_{i'}$ belonging to separate components of $G - W$, where $\{i, i', j\} = \{1, 2, 3\}$. But then W is a separating set of G with $|W| = 7$ and $H_j \subseteq W$, contradicting (A). This proves (9).

(10) $|W| \leq 5$.

Suppose $|W| > 5$. By (9), $|W| = 6$. We first show the following:

(10.1) *There do not exist vertices $x_1, x_2, x_3 \in X_i$ for some $i \in \{1, 2, \ldots, n\}$ such that for $j \in \{1, 2, 3\}$, $x_j \in Z_j$, and such that there exist vertices $y_1, y_2, y_3 \in Y_i$ (not necessarily distinct from the x_j) and internally disjoint paths $P_1, P_2, P_3, Q_{12}, Q_{13}, Q_{23} \subseteq Y_i$ where for $j, k \in \{1, 2, 3\}$ with $j \neq k$, P_j has ends x_j, y_j, and Q_{jk} has ends y_j, y_k. (See Figure 4.1.)*

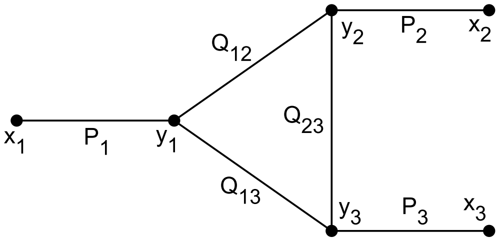

Figure 4.1: The arrangement of paths and vertices forbidden by (10.1).

By (8) and (2.i), we may assume that $\{x_1, x_2, x_3\} = X_1$, say. Since $|W| = 6$ and $X_i \cap (W) = \emptyset$, it not hard to see that $H_k \subseteq W \cup X_1$ for at most one $k \in \{1, 2, 3\}$, and so without loss of generality we may assume that Z_1 and Z_ℓ, say, each have a vertex in $G - W - X_1$, where $\ell \in \{2, 3\}$. As the proof is identical for both values of ℓ, we will assume that $\ell = 2$ for the sake of notational clarity. Say $z_1 \in Z_1 \setminus \{x_1\}$ and $z_2 \in Z_2 \setminus \{x_2\}$ are vertices of $G - W - X_1$. Then Z_1 and Z_2 both belong to the same component C of $G - W - x_3$ since $P_1 \cup Q_{12} \cup P_2 \subseteq G - W - x_3$ and the sets Z_1, Z_2 are each connected by definition. If there exists a z_1, z_2-path P in C which avoids at least one of x_1 and x_2, then by (8) and the fact that $\{x_1, x_2, x_3\} = X_1$, there exists $i \in \{2, 3, \dots, n\}$ such that X_i contains two vertices of P, and so $|W| + \sum_{i=1}^{n} \left\lfloor \frac{1}{2}|X_i| \right\rfloor \geq 8$, contradicting (2.i). Thus we may assume that every z_1, z_2-path in C includes both vertices x_1 and x_2. In particular, C is not 2-connected and $C - x_i$ is disconnected for each $i \in \{1, 2\}$. Let C_i be the component of $C - \{x_1, x_2\}$ which contains z_i for each $i \in \{1, 2\}$. It is clear that for $i \in \{1, 2\}$, no vertex of C_i can be adjacent to any vertex of $P_1 \cup P_2 \cup P_3 \cup Q_{12} \cup Q_{13} \cup Q_{23} \setminus \{x_1, x_2, x_3\}$, since then a z_1, z_2-path in C avoiding one of x_1 or x_2 can be found. We further claim that no vertex of C_i is adjacent to x_3 for $i \in \{1, 2\}$, that is $x_3 \notin V(C_i)$. Indeed, if $x_3 \in V(C_1)$, say, then there is a path P from z_1 to x_3 contained in C_1. In particular, P is a Z_1, Z_3-path avoiding $W \cup \{x_1, x_2\}$, so by (8), we will have $|W| + \sum_{i=1}^{n} \left\lfloor \frac{1}{2}|X_i| \right\rfloor \geq 8$, contradicting (2.i).

Since G is 7-connected, there exist 7 internally disjoint z_1, z_2-paths in G. We have shown that at most one of these paths is contained in C, so at least 6 of these paths must meet $W \cup \{x_3\}$. As $N_G(C_i) \subseteq W \cup \{x_i\}$ for $i \in \{1, 2\}$ and $|W| = 6$, none of these paths meets x_3, and so every vertex of W belongs to one such path. If $H_3 \subseteq W \cup \{x_3\}$, then by contracting each of $C_i \cup \{x_i\}$, P_i, and $Q_{13} \cup P_3 \cup Q_{23} \setminus \{y_1, y_2\}$ to a single vertex for $i \in \{1, 2\}$ and contracting Q_{12} to a single edge, we obtain a K_8-minor of G, a contradiction.

Thus $H_3 \not\subseteq W \cup \{x_3\}$, and in particular that some vertex $z_3 \in Z_3$ is in $G - W - X_1$. Let C_3 be the component of $G - W - X_1$ which contains Z_3. By the same argument as above, no vertex of C_3 is adjacent to any vertex in $P_1 \cup P_2 \cup P_3 \cup Q_{12} \cup Q_{23} \cup Q_{31} \cup \{x_1, x_2\}$. In particular, the graph $G - W - x_3$ is disconnected and C_3 is one of its components. Thus $N_G(C_3) \subseteq W \cup \{x_3\}$. Now, $H_3 \subseteq C_3 \cup W \cup \{x_3\}$. Since G is 7-connected, there exist 6 disjoint paths $R_1, \dots, R_6$ of G with one end in H_3 and the other end in $W \cup \{x_3\}$. Additionally, we may select these paths $R_1, \ldots, R_6$ such that all internal vertices of these paths belong to C_3. Since $|W \cup \{x_3\}| = 7$, by (A), $G - W - x_3$ has two components, one of which contains both Z_1 and Z_2. Thus, the above arguments apply, and so the component C_i of $G - W - X_1$ containing z_i satisfies $N_G(C_i) = W \cup \{x_i\}$ for $i \in \{1, 2\}$. Therefore, contracting each of $R_1, \ldots, R_6, P_1 \cup C_1, P_2 \cup C_2$, and P_3 to a single vertex and each of Q_{12}, Q_{13}, Q_{23} to a single edge gives a K_8-minor in G, a contradiction. This proves (10.1).

As G is 7-connected and $|W| = 6$, $G - W$ is connected. Also since $|W| = 6$, it is easy to see that there are at least two sets Z_j such that $|Z_j| \geq 2$, say Z_1 and Z_ℓ for some $\ell \in \{2, 3\}$. Again the proof is identical for both values of ℓ, so for notational convenience we will assume $\ell = 2$. If there exist two disjoint Z_1, Z_2-paths in $G - W$, then by (8), $|W| + \sum_{i=1}^{n} \left\lfloor \frac{1}{2}|X_i| \right\rfloor \geq 8$, contradicting (2.i). Thus there is at most one disjoint Z_1, Z_2-path P in $G - W$. $G - W$ is connected, so we may choose P as short as possible with ends $z_j \in Z_j$ for $j \in \{1, 2\}$. By (8), at least two vertices of P belong to the same set X_i for some $i \in \{1, 2, \ldots, n\}$. Let P' be the longest subpath of P such that P' has distinct ends in the same set X_i, say $x_1, x_2 \in X_1$, and such that z_1, x_1, x_2, z_2 appear on P in order (note that z_i and x_i are not necessarily distinct for $i \in \{1, 2\}$). By an application of Menger's Theorem [47], there exists a vertex $p \in V(P)$ such that $G - W - p$ is disconnected. Since $W \cup \{p\}$ is a separating set in G with $|W \cup \{p\}| = 7$, we have $H_3 \not\subseteq W \cup \{p\}$ by (A), and so $Z_3 \neq \emptyset$ and there exists a shortest Z_3, P-path Q in $G - W$. Let t be the unique vertex in $V(P) \cap V(Q)$, and let $z_3 \in Z_3$ be the other end of Q. We next prove the following.

(10.2) $t \notin \{x_1, x_2\}$.

If $t = x_1$, say, then we first claim $x_1 \notin Z_3$. For if $x_1 \in Z_3$, then the subpath of P from x_1 to z_1 is a Z_3, Z_1-path avoiding W, and so must contain two vertices from some X_i by (8). By the maximality of P', these two vertices cannot belong to X_1, and therefore we will have $|W| + \sum_{i=1}^{n} \left\lfloor \frac{1}{2}|X_i| \right\rfloor \geq 8$, contradicting (2.i). So $x_1 \notin Z_3$, and thus Q contains at least two vertices. The subpath R of $P \cup Q$

with ends z_1, z_3 is a Z_1, Z_3-path avoiding W. By (8), two vertices of R belong to the same X_i. Let Q' be the longest subpath of R such that the ends of Q' belong to the same set X_i. By (2.i) and the maximality of P', the ends of Q' must be x_1 and, say x_3, where $X_1 = \{x_1, x_2, x_3\}$ and z_1, x_1, x_3, z_3 appear on Q in order (again, z_3 and x_3 are not necessarily distinct). Hence $Q' \subseteq Q$.

We now claim that $x_j \in Z_j$ for $j \in \{1, 2, 3\}$. The proof is similar for all j, so we will suppose $x_1 \notin Z_1$. Since the subpath P^* of P from z_1 to x_1 does not meet W, by the definition of the set Z_1, P^* must contain some edge with both ends in Y_i for some $i \in \{1, 2, \ldots, n\}$. By (2.ii), (7.ii), and the maximality of P', we conclude that $i \neq 1$. Say $e \in E(P^*)$ has both ends in Y_2. Then the subpath of P^* from z_1 to the first end of e must contain some vertex from X_2, and the subpath of P^* from the second end of e to x_1 must contain another vertex of X_2 by (2.ii) and the fact that P^* avoids W. But then $|W| + \sum_{i=1}^{n} \left\lfloor \frac{1}{2}|X_i| \right\rfloor \geq 8$, contradicting (2.i). This proves the claim that $x_j \in Z_j$ for $j \in \{1, 2, 3\}$. In particular, we have $z_j = x_j$ for $j \in \{1, 2, 3\}$ by the minimality of P and Q.

We next claim that $V(P' \cup Q') \subseteq Y_1$. Assume to the contrary that P', say, has a vertex outside Y_1. No interior vertex of P' can belong to X_1 since the minimality of Q implies $x_3 \notin V(P')$ and since $X_1 = \{x_1, x_2, x_3\}$. By (2.i) no two interior vertices of P' can belong to the same set X_i, and so we must have $P' \subseteq X_1 \cup \cdots \cup X_n$ by (2.ii). But then since $x_1 \in Z_1$ and $x_2 \in Z_2$, and by the definition of Z_j, we have that $P' \subseteq Z_1$ and $P' \subseteq Z_2$, contradicting (7.iii). The proof in the case that Q' has a vertex outside Y_1 is similar, and so our claim that $P' \cup Q' \subseteq Y_1$ follows.

Now since $x_2 \notin H_3$, $|Z_j| \geq 2$ for $j \in \{1, 2\}$, and $|Z_3| \geq 1$, and by (A), there exist at least two sets Z_j belonging to the same component C of $G - W - x_2$. By (8) and (2.i), for $j, k \in \{1, 2, 3\}$ with $j \neq k$, any Z_j, Z_k-path in C must contain both x_1 and x_3, as otherwise $|W| + \sum_{i=1}^{n} \left\lfloor \frac{1}{2}|X_i| \right\rfloor \geq 8$, a contradiction. Thus we may assume $j = 1$ and $k = 3$, since $x_1 \in Z_1$ and $x_3 \in Z_3$. Let R' be the subpath of any Z_1, Z_3-path in C such that the ends of R' are x_1 and x_3. By the same argument applied to $P' \cup Q'$ above, it can be shown that $R' \subseteq Y_1$. Since R' avoids x_2, some subpath of R' is a P', Q'-path containing at least one edge. Therefore, for $i \in \{1, 2, 3\}$, we have vertices $x_1, x_2, x_3 \in X_1$ such that $x_i \in Z_i$, and within $P' \cup Q' \cup R'$ we can find vertices $y_1, y_2, y_3 \in Y_1$ and internally disjoint paths $P_1, P_2, P_3, Q_{12}, Q_{23}, Q_{31}$ where P_i has ends x_i, y_i and Q_{ij} has ends y_i, y_j (here $\{x_2\} = \{y_2\} = V(P_2)$). This contradicts (10.1), and so (10.2) follows.

If $t \in P - P'$, then we may assume z_1, t, x_1 appear on P in order (possibly $t = z_1$, but $t \neq x_1$ by (10.2)). Then the subpath of $P \cup Q$ with ends z_1, z_3 is a Z_1, Z_3-path avoiding W. As neither x_1 nor x_2 belong to this subpath, by (8), we will have $|W| + \sum_{i=1}^{n} \left\lfloor \frac{1}{2}|X_i| \right\rfloor \geq 8$, contradicting (2.i). Hence by this argument and (10.2), we may assume that $t \in P' - \{x_1, x_2\}$. Let R_j be the subpath of $P \cup Q$ with ends z_j, z_3 for $j \in \{1, 2\}$. Then for $j \in \{1, 2\}$, R_j is a Z_j, Z_3-path avoiding W, and so by (8) contains two vertices in some set X_i. By (2.i), R_j contains x_j and, say, x_3 for $j \in \{1, 2\}$. Let Q' be the subpath of Q with ends x_3, t.

By the same argument as in the proof of (10.2), it can be shown that $x_j \in Z_j$, and thus that $x_j = z_j$ for $j \in \{1, 2, 3\}$. Also by the same argument as in the proof of (10.2), we can show that $P' \cup Q' \subseteq Y_1$.

Suppose $t \neq z_3$ and consider the graph $G' := G - W - t$. Then $z_j \in V(G')$ for $j \in \{1, 2, 3\}$. Since G' has at most two components by (A), at least two of the vertices $z_j = x_j$ must belong to the same component of G', say x_1 and x_2 belong to the component C of G'. Thus there is a Z_1, Z_2-path R in C. In particular, R is a path in G' with ends x_1, x_2 which avoids t. Using the same argument as above, we can show that $R \subseteq Y_1$. Therefore it is easy to see that $P' \cup Q' \cup R$ contains some set of paths which contradicts (10.1).

Thus we may assume that $t = z_3 = x_3$, and so Q' consists of only the single vertex t. Again consider the graph $G' := G - W - t$. If Z_1 and Z_2 belong to the same component C of G', a similar argument to the above allows us to find an x_1, x_2-path R avoiding t such that $R \subseteq Y_1$, and therefore such that $P' \cup Q' \cup R$ contains some set of paths contradicting (10.1). Thus Z_1 and Z_2 must belong to distinct components of G'. But then $H_3 \not\subseteq W \cup \{t\}$ by (A). Hence there is a vertex $z_3' \in H_3$ in G' such that $z_3' \in Z_3$. As G' has at most two components by (A), z_3' must belong to the same component as one of Z_1 or Z_2, say Z_1. Then there is a Z_1, Z_3-path in G' which avoids both x_3 and x_2, and thus by (8) we will have $|W| + \sum_{i=1}^{n} \left\lfloor \frac{1}{2}|X_i| \right\rfloor \geq 8$, contradicting (2.i). This completes the proof of (10).

The following two statements are immediate consequences of (10).

(11) *For* $i \in \{1, 2, \ldots, n\}$, *if* $|X_i| = 1$, *then* $Y_i = X_i$.

Suppose, say, $|X_1| = 1$ and $Y_1 \setminus X_1 \neq \emptyset$. By (2.ii), $W \cup X_1$ separates $Y_1 \setminus X_1$ from $V(G) \setminus X_1 \cup W$. But $|W \cup X_1| \leq 6$ by (10), contradicting that G is 7-connected.

(12) $Z_j \neq \emptyset$ *for any* $j \in \{1, 2, 3\}$.

Since $|W| \leq 5$ by (10), we have $|H_j \setminus W| \geq 6 - |W| \geq 1$ for all $j \in \{1, 2, 3\}$. Thus, for any $j \in \{1, 2, 3\}$, by (7.i), $Z_j \supseteq H_j \setminus W \neq \emptyset$.

(13) *For all* $j \in \{1, 2, 3\}$ *and* $i \in \{1, 2, \ldots, n\}$, *if* $z \in Z_j \cap X_i$ *has a neighbor in* $G - W - Z_j$, *then* $|X_i| \geq 3$.

Suppose $z \in Z_1 \cap X_1$, say, has a neighbor y in $G - W - Z_1$ be a neighbor of z. By the definition of the set Z_1 and by (2.ii), $y \in Y_1$, and so $|Y_1| \geq 2$. By (11), $|X_1| \neq 1$, and so by (6), $|X_1| \geq 3$.

(14) *There are at least two sets* Z_j *such that every* $z \in Z_j$ *has a neighbor in* $G - W - Z_j$ *for* $j \in \{1, 2, 3\}$.

Suppose to the contrary that there exist two sets Z_j such that some $z_j \in Z_j$ has no neighbor in $G - W - Z_j$. The proof is identical no matter which two sets Z_j satisfy the above, so we will assume that for $j \in \{2,3\}$, there exists a vertex $z_j \in Z_j$ such that z_j has no neighbor in $G-W-Z_j$. For each $j \in \{1,2,3\}$, define $G_j := G - (W \cup Z_j)$ and define N_j to be the set of vertices in Z_j with a neighbor in G_j. Let $r := 7 - |W|$. Note that by (10), $r \geq 2$. Since $G - W$ is r-connected and $|Z_j| \geq 6 - |W| = r - 1$, either each vertex in Z_j has a neighbor in G_j, or at least r vertices of Z_j have a neighbor in G_j for $j \in \{1,2,3\}$. Thus, for $j \in \{2,3\}$, since z_j has no neighbor in G_j, $|N_j| \geq r$. By (13), there exists at least one $i \in \{1,2,\ldots,n\}$ such that $|X_i| \geq 3$. Let us reorder the sets Y_i, X_i so that $|X_1| \geq |X_2| \geq \cdots \geq |X_n|$, and let s be the integer such that $|X_i| \geq 3$ for all $i \in \{1,2,\ldots,s\}$ and $|X_i| = 1$ for all $i \in \{s+1, s+2, \ldots, n\}$. Again by (13), we see that $N_j \subseteq X_1 \cup \cdots \cup X_s$ for $j \in \{1,2,3\}$, and that

$$\left|\bigcup_{i=1}^{s} X_i\right| \geq \left|\bigcup_{j=1}^{3} N_j\right| \geq (r-1) + r + r = 3r - 1 = 20 - 3|W|.$$

By (2.i), it is easy to see that either we have $|X_1| = 5, |X_i| = 3$ for $i \in \{2,3,\ldots,s\}$, and $s = r-1$, or we have $|X_i| = 3$ for $i \in \{1,2,\ldots,s\}$ and $s = r$. By (2.i) and (13), the values of $|N_j|$ are restricted as follows. We may have $|N_1| = r$ or $r-1$. If $|N_1| = r-1$, then we may assume $|N_2| = r$ and either $|N_3| = r$ with $|X_1| = 5$ and $|X_i| = 3$ for all $i \in \{2,3,\ldots,s\}$ or $|N_3| = r+1$ with $|X_i| = 3$ for all $i \in \{1,2,\ldots,s\}$. If $|N_1| = r$ then $|N_2| = |N_3| = r$ and $|X_i| = 3$ for all $i \in \{1,2,\ldots,s\}$.

(14.1) *For any* $i \in \{1,2,\ldots,s\}$*, if* $Z_1 \cap X_i \neq \emptyset$*, then* $|Z_1 \cap X_i| = 1$*. If, in addition,* $|X_i| = 3$*, then* $|Z_j \cap X_i| = 1$ *for* $j \in \{1,2,3\}$.

For suppose that $|Z_1 \cap X_1| \geq 2$, say. By the above, $|N_1| \leq r$. Since $G-W$ is r-connected, we may select $|N_1|$ disjoint N_1, Z_2-paths with no internal vertices in $Z_1 \cup Z_2$. Then two of these paths must have an end in X_1. If $|X_1| = 3$, then since the paths have no internal vertices in $Z_1 \cup Z_2$, both paths with an end in X_1 must meet the third vertex of X_1, contradicting that the paths are disjoint. If $|X_1| = 5$, then $s = r-1$, and $|N_j| = r$ for $j \in \{2,3\}$. It is easy to see that since $|N_2| = |N_3| = r$, there must exist $i \in \{2,3,\ldots,s\}$ and $j \in \{2,3\}$, such that $|X_i \cap Z_j| = 2$. Say $|X_2 \cap Z_2| = 2$. Now $|X_2| = 3$, and so by applying the same argument to a set of r disjoint N_2, Z_3-paths, we obtain a contradiction. This same argument can again be used to prove the final part of the statement, and so (14.1) follows.

(14.2) *For $j \in \{2,3\}$, we can select a set of $|N_1|$ disjoint Z_1, Z_j-paths in $G - W$ such that each path P will have both ends in the same set X_i, and $P \subseteq Y_i$. Additionally, if $|X_1| = 5$, then a Z_1, Z_2-path Q_1, a Z_1, Z_3-path Q_2 and a Z_2, Z_3-path Q_3 with $Q_1 \cup Q_2 \cup Q_3 \subseteq Y_1$ can be chosen with Q_3 disjoint from $Q_1 \cup Q_2$.*

There are two primary cases to consider based on $|N_1|$. If $|N_1| = r$, then $|N_2| = |N_3| = r$, and for $i \in \{1,2,\ldots,s\}$ and $j \in \{1,2,3\}$, $|X_i| = 3$ and so by (14.1), $|X_i \cap Z_j| = 1$. For $j \in \{2,3\}$, we claim that if P is a Z_1, Z_j-path in $G - W$ with ends $z_1 \in Z_1$, $z_j \in Z_j$ say, such that $z_1, z_j \in X_i$ for some $i \in \{1,2,\ldots,s\}$ and no interior vertex of P is in $Z_1 \cup Z_j$, then $P \subseteq Y_i$. For if not, some interior vertex of P must belong to $Y_{i'}$ for some $i' \neq i$. It is clear from the definition of the sets Z_1 and Z_j that the neighbors on P of z_1 and z_j must belong to Y_i. Thus by (2.ii), the part of P from z_1 to $Y_{i'}$ must contain two vertices of X_i, and likewise the part of P from $Y_{i'}$ to z_j. But then $|X_i| \geq 4$, a contradiction, thus proving our claim. Now since $G - W$ is r-connected, for $j \in \{2,3\}$, we can select $r = |N_1|$ disjoint Z_1, Z_j-paths in $G - W$ with no interior vertex in $Z_1 \cup Z_j$. These paths must each have their two ends in the same set X_i by our claim, and so (14.2) follows in this case.

Now consider the case $|N_1| = r - 1$. The same argument as above can be applied to any set X_i with $|X_i \cap Z_1| = 1$ and $|X_i| = 3$ to find a path $P \subseteq Y_i$. So let us assume that $|X_1| = 5$, and so $|N_2| = |N_3| = r$. It is clear from (14.1) that we must have $|X_1 \cap Z_j| = 2$ for $j \in \{2,3\}$. Now for $j \in \{2,3\}$, by deleting a single vertex from $X_1 \cap Z_{5-j}$ if necessary, we may find a set of $r-1$ disjoint N_1, Z_j-paths with no interior vertex in $(\cup_{j=1}^{3} Z_j) \setminus (\cup_{i=1}^{n} X_i)$. Say $j = 2$, and let P be the N_1, Z_2-path with an end in $N_1 \cap X_1$. The subpath P' of P from $N_1 \subseteq Z_1$ to its first vertex in

Z_2, say x_2, is then clearly contained in Y_1. Now, consider a set of r disjoint N_2, N_3-paths with no interior vertex in $Z_2 \cup Z_3$. Let Q and Q' be the N_2, N_3-paths with one end in $Z_2 \cap X_1$. By extending the argument above, it is clear that the second end of each of Q and Q' belongs to $Z_3 \cap X_1$, and so $Q, Q' \subseteq Y_1$, since otherwise we would have $|X_1| \geq 6$, a contradiction. Since x_2 must be an end of either Q or Q', we see $P' \cap (Q \cup Q') \neq \emptyset$. So we may assume that the first vertex of $Q \cup Q'$ on P belongs to Q, where P is read from its end in $N_1 \cap X_1$ to x_2. Then for $j \in \{2, 3\}$, $P' \cup Q$ contains a subpath R_j with ends in $Z_1 \cap X_1$ and $Z_j \cap X_1$ such that $R_j \subseteq Y_1$, and R_j is disjoint from Q' (See Figure 4.2). This proves (14.2).

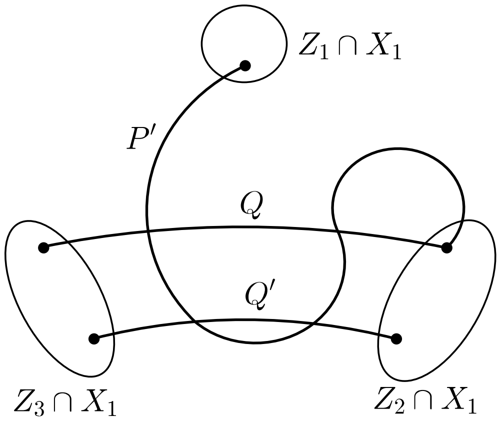

Figure 4.2: Z_i, Z_j-paths in Y_1 when $|X_1| = 5$.

By (14.2), let $\mathcal{P}'$ be a collection of $|N_1|$ Z_1, Z_2-paths and $|N_1|$ Z_1, Z_3-paths in $G - W$ where for each $P \in \mathcal{P}'$ there is some some $i \in \{1, 2, \ldots, s\}$ such that P has both ends in the same set X_i and $P \subseteq Y_i$. By (14.2), we may further select these paths $\mathcal{P}'$ so that if $|X_1| = 5$ there exists a Z_2, Z_3-path $Q \subseteq Y_1$ disjoint from $\bigcup_{P \in \mathcal{P}'} V(P)$. Note that the paths $P \in \mathcal{P}'$ are not necessarily pairwise internally disjoint. Now, since $H_1 \subseteq Z_1 \cup W$ by (7.i) and G is 7-connected, there exist 6 disjoint paths $P_1, \ldots, P_6$ in G with one end in H_1 the other end in $W \cup N_1$, and no internal vertices in $H_1 \cup W \cup N_1$. Let $\mathcal{P} \subseteq \mathcal{P}'$ be the subset consisting of paths which share an end with one of $P_1, \ldots, P_6$. From here, we consider two cases.

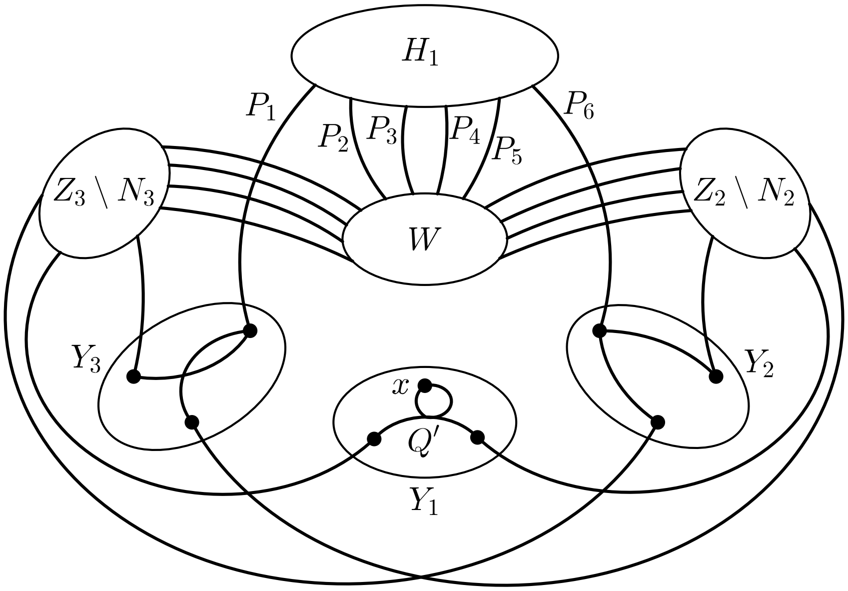

Figure 4.3: A K_8-minor in G when $N_1 \setminus H_1 \neq \emptyset$ and shown here with $s = 3$.

If there exists $x \in N_1$ such that $x \notin H_1$, then $|W \cup N_1| = |W| + r = 7$ because either $H_1 \subseteq W \cup N_1 \setminus \{x\}$ or $W \cup N_1$ is a separating set in G. We select the 6 paths $P_1, \ldots, P_6$ in $G - x$. Since $N_G(Z_1 \setminus N_1) \subseteq W \cup N_1$ and $N_1 \subseteq Z_1$, it is clear that $P_1, \ldots, P_6$ may be selected such that $\cup_{k=1}^{6} V(P_k) \subseteq W \cup Z_1$. Since $x \in N_1$, we must have $x \in X_i$ for some $i \in \{1, 2, \ldots, s\}$, say $x \in X_1$. As there exists a Z_1, Z_2-path $P \in \mathcal{P}'$ and a Z_1, Z_3-path $P' \in \mathcal{P}'$ such that $P, P' \subseteq Y_1$, we see that some subpath Q' of $P \cup P'$ is a Z_2, Z_3-path. Since x is not an end of any of $P_1, \ldots, P_6$, we see that Q' is disjoint from $\mathcal{P}$. Note that since $|N_1| = r$ in this case, $|X_i| = 3$ for $i \in \{1, 2, \ldots, s\}$. We also have $|N_j| = r$, and so $N_G(Z_j \setminus N_j) = N_j \cup W$ for $j \in \{2, 3\}$. Now by contracting each of $P_1, \ldots P_6, Z_2$, and Z_3 to a single vertex, and contracting Q' and each $P \in \mathcal{P}$ to a single edge, we obtain a K_8-minor in G, a contradiction (see Figure 4.3).

On the other hand, if $x \in H_1$ for every $x \in N_1$, then either $|N_1| = r - 1$ and $|W \cup N_1| = 6$, or $|N_1| = r$ and there exists a single vertex $w \in W \setminus H_1$ such that w is not an end of any path

$P_1, \ldots, P_6$ (by picking these paths in $G-w$, if necessary). We consider these two cases, separately.

First, suppose $|N_1| = r-1$ and $|W \cup N_1| = 6$. Then we must have $H_1 = W \cup N_1$. If $|X_1| = 5$, then let Q' be the Z_2, Z_3-path Q disjoint from $\bigcup\limits_{P \in \mathcal{P}'} V(P)$ given by (14.2). If $|X_1| = 3$, then since there are r sets X_i with $|X_i| = 3$ and $|N_1| = r-1$, we may assume that $N_1 \cap X_1 = \emptyset$. By (14.1) and since $|N_2| = r$, $|N_3| = r+1$, we have $N_2 \cap X_1 \neq \emptyset$ and $N_3 \cap X_1 \neq \emptyset$. As there are r disjoint Z_2, Z_3-paths in $G-W$, it is clear that a Z_2, Z_3-path Q' can be found such that $Q' \subseteq Y_1$. Now in either case, by contracting each of $P_1, \ldots P_6, Z_2$, and Z_3 to a single vertex, and contracting Q' and each $P \in \mathcal{P}$ to a single edge, as before, we obtain a K_8-minor in G, a contradiction.

Lastly, suppose $|N_1| = r$ and there exists a single vertex $w \in W \setminus H_1$ such that w is not an end of any path $P_1, \ldots, P_6$. Then $|X_i| = 3$, and by (14.1) $|Z_j \cap X_i| = 1$ for all for all $i \in \{1, 2, \ldots, s\}$ and $j \in \{1, 2, 3\}$. Furthermore, we have $|N_2| = |N_3| = r$, and so $|N_2 \cup W| = |N_3 \cup W| = 7$. Since G is 7-connected, in any set of 7 disjoint z_2, z_3-paths, one path Q', say, must meet w. It is clear that Q' is disjoint from $P_1, \ldots, P_6$ and every $P \in \mathcal{P}$ since $Q' \cap (Z_1 \cup Y_1 \cup \cdots \cup Y_s) = \emptyset$. Thus we may once again obtain a K_8-minor in G by contracting each of $P_1, \ldots P_6, Z_2$, and Z_3 to a single vertex, and contracting Q' and each $P \in \mathcal{P}$ to a single edge, a contradiction. This completes the proof of (14).

(15) $\max\{|Z_1|, |Z_2|, |Z_3|\} \geq 7 - |W|$.

Suppose to the contrary that $|Z_j| \leq 6 - |W|$ for all $j \in \{1, 2, 3\}$. Since $H_j \subseteq Z_j \cup W$, we have $|Z_j| \geq 6 - |W|$, and so $|Z_j| = 6 - |W|$ for $j \in \{1, 2, 3\}$. Since Z_j and W are mutually disjoint, it follows that $|Z_j \cup W| = |Z_j| + |W| = 6$, and so $Z_j \cup W$ induces the K_6-subgraph $G[H_j]$ for all $j \in \{1, 2, 3\}$. Therefore $W \subseteq H_1 \cap H_2 \cap H_3$. But $|W| \geq 1$ by (B) and (3), and $H_1 \cap H_2 \cap H_3 = \emptyset$ by (C), a contradiction. (15) follows.

(16) *At most one of* Z_1, Z_2, Z_3 *has at most* $6 - |W|$ *vertices.*

Suppose not. By (15), exactly two sets Z_j satisfy $|Z_j| \leq 6 - |W|$, say Z_1 and Z_k for some $k \in \{2,3\}$. By the same argument as in the proof of (15), we have that $Z_1 \cup W$ and $Z_k \cup W$ induce the K_6-subgraphs $G[H_1]$ and $G[H_k]$, respectively. Thus $W \subseteq H_1 \cap H_k$, and so we must have $k = 2$, $W = M = H_1 \cap H_2$, and $|W| = 1$ by (B) and (3). Further, since $H_j \subseteq Z_j \cup W$ by (7.i), we have $|Z_j| = 5$ for $j \in \{1,2\}$. Then $H_3 \cap W = \emptyset$ by (C), so $|Z_3| \geq 6$.

We first claim for any $i \in \{1,2,\dots,s\}$ that if $Z_j \cap X_i \neq \emptyset$ for some $j \in \{1,2\}$, then $|X_i| \geq 5$. Suppose $z \in Z_1 \cap X_1$ with $|X_1| < 5$. Then $|X_1| = 1$ or 3 by (6). If there exists $y \in Y_1 \setminus X_1$, then by (2.ii), $X_1 \cup W$ is a separating set in G with $|X_1 \cup W| \leq 4$, contradicting that G is 7-connected. Therefore $Y_1 = X_1$. Now by (2.iii) and the definition of the set Z_1, z can only have neighbors in $Z_1 \cup Y_1 \cup W \setminus \{z\}$. But $|Z_1 \cup Y_1 \cup W \setminus \{z\}| \leq |Z_1 \setminus \{z\}| + |Y_1 \setminus \{z\}| + |W| = 4 + 2 + 1 = 7$, contradicting that $\delta(G) \geq 8$. This proves our claim.

By (2.i), there are at most three sets X_i such that $|X_i| \geq 5$. Thus, by our above claim, we may assume that $Z_1 \cup Z_2 \subseteq X_1 \cup X_2 \cup X_3$ say. Then $|X_1 \cup X_2 \cup X_3| \leq 15$ by (2.i). Let $z \in Z_j \cap X_i$ for some $j \in \{1,2\}$ and some $i \in \{1,2,3\}$, say $z \in Z_1 \cap X_1$. Then z cannot be adjacent to a vertex $y \in X_k \setminus H_1$ unless $k = 1$, since by the definition of Z_1 any such vertex y would belong to Z_1 and then $|Z_1| \geq 6$, a contradiction. Thus it follows that $W \cup X_1 \cup X_2 \cup X_3 \setminus (Z_1 \cup Z_2)$ is a separating set in G with at most 6 vertices, contradicting that G is 7-connected. This completes the proof of (16).

We will utilize the following definition from the proof of (14) throughout the remainder of the proof of Theorem 1.6.6. For $j \in \{1,2,3\}$, define N_j to be the set of vertices of Z_j with a neighbor in $G - (Z_j \cup W)$.

(17) $|Z_j| \geq 7 - |W|$ *for* $j \in \{1,2,3\}$.

Suppose to the contrary. By (16), we have only one set Z_j with $|Z_j| \leq 6 - |W|$. Since $|Z_j| \geq$

$6 - |W|$ by (7.i), we have $|Z_j| = 6 - |W|$ for some $j \in \{1, 2, 3\}$. Hence $W \subseteq H_j$, and so $j \in \{1, 2\}$, say $|Z_1| = 6 - |W|$. Then $N_1 = Z_1$ since G is 7-connected. By (2.i), (13), and that G is 7-connected, each of $|N_2|$ and $|N_3|$ must be equal to one of $7 - |W|$ or $8 - |W|$, with at most one equal to the latter. We now prove the following.

(17.1) $|X_i| \leq 3$ *for all* $i \in \{1, 2, \ldots, n\}$.

For suppose $|X_1| > 3$. By (6), $|X_1| \geq 5$. By (2.i) and (13), we must have $|N_2| = |N_3| = 7 - |W|$, and additionally $|X_1| = 5$ and $|X_i| \leq 3$ for all $i \in \{2, 3, \ldots, n\}$. If $|X_1 \cap Z_1| \geq 3$, then $(Z_1 \setminus X_1) \cup (X_1 \setminus Z_1) \cup W$ is a separating set of G with cardinality

$$|(Z_1 \setminus X_1) \cup (X_1 \setminus Z_1) \cup W| = |Z_1 \setminus X_1| + |X_1 \setminus Z_1| + |W| \leq (3 - |W|) + 2 + |W| = 5,$$

contradicting that G is 7-connected. If $|X_1 \cap Z_1| = 2$ or 0, then it is easy to see that some other set X_i, say X_2, must have $|X_2 \cap N_j| = 2$ for some $j \in \{1, 2, 3\}$. But then $(N_j \setminus X_2) \cup (X_2 \setminus N_j) \cup W$ is a separating set of G with cardinality

$$|(N_j \setminus X_2) \cup (X_2 \setminus N_j) \cup W| = |N_j \setminus X_2| + |X_2 \setminus N_j| + |W| \leq (5 - |W|) + 1 + |W| = 6,$$

again contradicting that G is 7-connected. Thus we must have $|X_1 \cap Z_1| = 1$. If $|X_1 \cap N_j| \geq 3$ for $j \in \{2, 3\}$, then $(N_j \setminus X_1) \cup (X_1 \setminus N_j) \cup W$ is a separating set of G of cardinality at most 6, a contradiction. Thus we have $|X_1 \cap N_2| = |X_1 \cap N_3| = 2$. By a similar argument again, we must have $|X_i \cap N_j| = 1$ for all $i \in \{2, 3, \ldots, n\}$ and $j \in \{1, 2, 3\}$ such that $|X_i| = 3$.

Now by (B), (C), (7.i), and since $W \subseteq H_1$, we must have $|Z_2| \geq |H_2 \setminus W| \geq 5$ and $|Z_3| \geq |H_3 \setminus W| \geq 3$. By (14), either $|Z_2| = |N_2| = 7 - |W|$ or $|Z_3| = |N_3| = 7 - |W|$, and thus $|W| \leq 4$. Therefore $|Z_1| \geq 2$, and so there is a set, say X_2, with $|X_2 \cap Z_1| = 1$ and $|X_2| = 3$. Let $x \in X_2 \cap Z_1$. Then the neighbors of x in G must be in $Z_1 \cup W \cup Y_2$. As $|Z_1 \cup W \setminus \{x\}| = 5$, x

must have at least 3 neighbors in Y_2 since $\delta(G) \geq 8$. Thus $Y_2 \neq X_2$, and so $X_2 \cup W$ is a separating set in G of size $3 + |W|$. Since G is 7-connected, we conclude $|W| \geq 4$. Therefore $|W| = 4$. Since $|Z_2| \geq 5$ and now $|N_2| = 3$, we have $Z_2 \neq N_2$. Thus $Z_3 = N_3$ by (14). Since $|Z_3| = 3$, we see that $|H_1 \cap H_3| = 3$. But now $S := H_1 \cup X_2 \setminus (Z_1 \cap X_2)$ is a separating set in G with $|S| = 7$ and $\Delta(G[S]) \geq 5$, contradicting (A). This proves (17.1).

It is clear from (13) and (17.1) that $|N_1| + |N_2| + |N_3|$ must be divisible by 3, and so $\{|N_2|, |N_3|\} = \{7 - |W|, 8 - |W|\}$. Now let $x \in Z_1$. Since $Z_1 = N_1$, x belongs to some set, say X_1, such that $|X_1| = 3$. Since $Z_1 \cup X_1 \cup W \setminus (Z_1 \cap X_1)$ is a separating set of G, it is clear that $|Z_1 \cap X_1| = 1$. Now x can have neighbors in G only in $Y_1 \cup Z_1 \cup W$. As $W \cup Z_1 \subseteq H_1$, we see that x must have at least three neighbors in Y_1 since $\delta(G) \geq 8$, and so $Y_1 \neq X_1$. Thus, by (2.ii), $X_1 \cup W$ is a separating set of G with $|X_1 \cup W| = 3 + |W|$. Since G is 7-connected, we conclude $|W| \geq 4$. Since $|Z_2| \geq 5$, we see $|Z_2| \geq 9 - |W|$, and so $Z_2 \neq N_2$. By (14), $Z_3 = N_3$. Since $|H_3 \cap H_1| \leq 3$, we see $|Z_3| \geq 3$.

We claim that $X_1 \cap Z_3 \neq \emptyset$. Indeed, if $X_1 \cap Z_3 = \emptyset$, then $|X_1 \cap N_2| = 2$ and $N_2 \cup X_1 \cup W \setminus (X_1 \cap N_2)$ is a separating set of G with

$$|N_2 \cup X_1 \cup W \setminus (X_1 \cap N_2)| \leq |N_2| + |X_1| + |W| - |X_1 \cap N_2| = |N_2| + |W| - 1.$$

Since G is 7-connected, this gives $|N_2| \geq 8 - |W|$, and so $|N_2| = 8 - |W|$. Thus $|Z_3| = 7 - |W| \leq 3$ since $|W| \geq 4$, and since $|Z_3| \geq 3$ we must have $|W| = 4$ and $|Z_3| = 3$. Now it is not hard to see that there must exist some set X_i, say X_2, such that $|Z_3 \cap X_2| \geq 2$. But then $Z_3 \cup X_2 \cup W \setminus (Z_3 \cap X_2)$ is a separating set with $|Z_3 \cup X_2 \cup W \setminus (Z_3 \cap X_2)| \leq 6$, contradicting that G is 7-connected. This proves the claim. But now $S := H_1 \cup X_1 \setminus (H_1 \cap X_1)$ is a separating set in G with $|S| = 7$ and

$\Delta(G[S]) \geq 5$, contradicting (A). This proves (17).

(18) $|N_j| = 7 - |W|$ *for* $j \in \{1, 2, 3\}$.

Since G is 7-connected and $|Z_j| \geq 7 - |W|$ by (17), it follows that $|N_j| \geq 7 - |W|$ for all $j \in \{1, 2, 3\}$. By (13) and (2.i), it is easy to see that (18) follows.

(19) $|X_i| \leq 3$ *for all* $i \in \{1, 2, \ldots, n\}$.

By (18), $|N_1| + |N_2| + |N_3| = 21 - 3|W|$, and so it is clear by (13) and (2.i) that there must exist exactly $7 - |W|$ sets X_i with $|X_i| = 3$, and all other sets X_i have $|X_i| = 1$.

(20) $|X_i \cap N_j| = 1$ *for all* $j \in \{1, 2, 3\}$ *and* $i \in \{1, 2, \ldots, n\}$ *such that* $|X_i| = 3$.

For if $|X_i \cap N_j| \geq 2$, then $X_i \cup N_j \cup W \setminus (X_i \cap N_j)$ is a separating set in G with $|X_i \cup N_j \cup W \setminus (X_i \cap N_j)| \leq 6$, a contradiction.

(21) *If* $|W| \leq 3$*, then* $X_i = Y_i$ *for all* $i \in \{1, 2, \ldots, n\}$.

By (11), this is true if $|X_i| = 1$. So assume by (19) that, say, $|X_1| = 3$ and $X_1 \neq Y_1$. Then $X_1 \cup W$ is a separating set in G with $|X_1 \cup W| = 3 + |W|$. Since G is 7-connected, we must have $|W| \geq 4$, and (21) follows.

(22) *For* $i \in \{1, 2, \ldots, n\}$*, if* $|X_i| = 3$ *and* $X_i = Y_i$*, then* X_i *induces a* K_3*-subgraph of* G.

By (14) and the symmetry between Z_1 and Z_2, we may assume that $Z_1 = N_1$. Let $x \in Z_1$, and suppose by (13) and (19) that $x \in X_1$, say, where $|X_1| = 3$ and $X_1 = Y_1$. Then x can have neighbors in G only in $Z_1 \cup Y_1 \cup W$. Since $|Z_1 \cup W \setminus \{x\}| = 6$ by (18), and since $\delta(G) \geq 8$, x must have at least two neighbors in Y_1, namely the two vertices of $X_1 \setminus Z_1$. By (14), there exists $j \in \{2, 3\}$ such that $N_j = Z_j$. By (20), there exists $x' \in Z_j \cap X_1$. Then by the same argument, x' also must have at least two neighbors in Y_1. Since $|X_1| = 3$, it is clear that X_1 must then induce a

K_3-subgraph in G. This proves (22).

(23) $|W| \geq 2$.

Suppose to the contrary that $|W| \leq 1$. By (B) and (3), $|W| \geq 1$. Thus $|W| = 1$. Then $|N_3| = 6$ by (18). Since G is 7-connected, $G - W$ is 6-connected. So let $P_1, \ldots, P_6$ be disjoint paths in $G - W$ such that one end of P_i belongs to H_3 and the other end belongs to N_3 for each $i \in \{1, 2, \ldots, 6\}$. Note that it is possible some paths P_i may consist of only a single vertex. For each $i \in \{1, 2, \ldots, 6\}$, let x_i be the end of P_i in N_3, and by (13), (19), and (20), we may assume that $x_i \in X_i$, where $|X_i| = 3$. By (20) and (22), each x_i has a neighbor in each of Z_1 and Z_2 for $i \in \{1, 2, \ldots, 6\}$. But then contracting each of $P_1, \ldots, P_6, Z_1$, and $Z_2 \cup W$ to a single vertex gives a K_8-minor in G, a contradiction. This proves (23).

(24) $|W| \geq 3$.

Suppose to the contrary that $|W| \leq 2$. By (23), $|W| \geq 2$. Thus $|W| = 2$. First, consider the case that $W \subseteq H_1$, say. Then $W \cap H_3 = H_1 \cap H_3$. If $H_1 \cap H_3 \neq \emptyset$, then $|H_2 \backslash W| = |H_3 \backslash W| = 5$. By (14) and (18), either $Z_2 = H_2 \setminus W$ or $Z_3 = H_2 \setminus W$. If $H_1 \cap H_3 = \emptyset$, then $|Z_3| \geq |H_3| > 7 - |W|$, and so $Z_3 \neq N_3$ by (18). Hence $Z_2 = N_2 = H_2 \setminus W$ by (14) and (18). In either case, Z_j induces a K_5-subgraph of G for some $j \in \{2, 3\}$. Suppose $Z_2 = N_2 = H_2 \backslash W$, and let $x \in N_2 \cap X_1$, say. Then x can have neighbors only in $Z_2 \cup Y_1 \cup W$ by (2.ii) and the definition of Z_2. Since $|Z_2 \cup Y_1 \setminus \{x\}| = 6$ by (21), x must be adjacent to both vertices of W since $\delta(G) \geq 8$. It follows that Z_2 is complete to W. But then $Z_2 \cup W$ induces a K_7-subgraph of G, and so a K_8-minor can be easily found since G is 7-connected, a contradiction. A similar argument holds if $Z_3 = N_3 = H_3 \setminus W$.

Thus $W \not\subseteq H_j$ for any $j \in \{1, 2, 3\}$. If $|M| = 2$, then $W = M \subseteq H_j$ for some $j \in \{1, 2, 3\}$, and so we must have $|M| = 1$. Then if $W \cap H_3 = \emptyset$, then $W \subseteq H_j$ for some $j \in \{1, 2\}$, and so $|W \cap H_3| = 1$. By symmetry and (14), we may assume that $Z_1 = N_1$. Then by (14), either

$Z_2 = N_2$ or $Z_3 = N_3$. If $Z_2 = N_2$, then $Z_2 = H_2 \setminus W$ by (18). Let $x \in Z_2$ and suppose $x \in X_1$, say. Since x can have neighbors only in $Z_2 \cup Y_1 \cup W$, and since $|Z_2 \cup Y_1 \setminus \{x\}| = 6$ by (21), we see that x must be adjacent to both vertices of W since $\delta(G) \geq 8$. Thus it follows that Z_2 is complete to W, and by symmetry Z_1 is complete to W as well. Let $P_1, \ldots, P_5$ be disjoint paths in Z_3 with one end in $H_3 \setminus W$ and the other end in N_3, where possibly the paths P_k consist of only a single vertex. Then contracting each of $P_1, \ldots, P_5, Z_1$, and H_2 to a single vertex gives a K_8-minor in G (along with the vertex in $H_3 \cap W$) by (21) and (22), a contradiction. If instead $Z_3 = N_3$, a similar argument shows that Z_3 and Z_1 are complete to W, and suitable paths from H_2 to N_2 can be contracted along with Z_1, a Z_1, W-edge and Z_3 to give a K_8-minor. This proves (24).

(25) $|W| \geq 4$.

Suppose to the contrary that $|W| < 4$. By (24), $|W| = 3$. We may assume that $|W \cap H_1| \geq |W \cap H_2|$ by symmetry.

(25.1) $|M| \geq 2$.

If $|M| = 1$, it is not hard to see that this is only possible if $|W \cap H_j| = 2$ for $j \in \{1, 2\}$ by (14) and (18). Then $|Z_3| \geq |H_3| = 6 > 7 - |W|$, and so $Z_3 \neq N_3$ by (18). By (14), (18), and (7.i), $N_j = Z_j = H_j \setminus W$ for $j \in \{1, 2\}$. Let $x \in Z_2 \cap X_1$, say. Then x can have neighbors only in $Z_2 \cup Y_1 \cup W$ by (2.ii) and the definition of Z_2. Since $Y_1 = X_1$ by (21) and $|Z_2 \cup Y_1 \setminus \{x\}| = 5$, it is clear that x must be complete to W since $\delta(G) \geq 8$, and so every vertex in W has at least one neighbor in Z_2. Now let $y \in Z_3 \setminus N_3$. Since G is 7-connected, there exist 7 x, y-paths in G, disjoint except for their ends. As any vertex of $Z_3 \setminus N_3$ can have neighbors only in $Z_3 \cup W$ and $|N_3 \cup W| = 7$, it is clear that each vertex of $N_3 \cup W$ is met by one of these paths. In particular, each vertex in W has at least one neighbor in Z_3. Additionally note by (22) that each set X_i with $|X_i| = 3$ induces a triangle for any $i \in \{1, 2, \ldots, n\}$. Since every vertex of H_1 either belongs to W or to N_1, we therefore deduce that every vertex in H_1 has some neighbor in each of Z_2 and Z_3.

Then Z_2 and Z_3 are adjacent via any set X_i with $|X_i| = 3$. By contracting each of Z_2 and Z_3 to a single vertex, we obtain a K_8-subgraph in G, a contradiction. This proves (25.1).

(25.2) $W = M$.

Suppose to the contrary that $W \neq M$. By (3) and (25.1), $|M| = 2$. Since $|H_1 \cap H_2| = 1$ by (B), we may assume $|H_1 \cap H_3| = 1$, say. Let $w \in W \setminus M$. If $w \in H_1$, then $|Z_j| \geq 5 > 7 - |W|$ and $Z_j \neq N_j$ by (18) for $j \in \{2, 3\}$, contradicting (14). So by symmetry, we may assume $w \in H_3$, say. Then $|Z_2| \geq 5$, and so $Z_2 \neq N_2$. Then by (14), $Z_j = N_j$ for $j \in \{1, 3\}$. Let $x \in Z_3 \cap X_1$, say. Then x can have neighbors only in $Z_3 \cup Y_1 \cup W$, and since $|Z_3 \cup Y_1 \setminus \{x\}| = 5$ by (21), it is clear that x must be complete to W since $\delta(G) \geq 8$. It follows that Z_3 is complete to W, and in particular that $y \in H_1 \cap H_2$ is complete to Z_3. But this contradicts that $d(y) \leq 11$ by (B). This proves (25.2).

From (25.2), we have $|M| = 3$. Thus either $|H_3 \cap H_j| = 1$ for $j \in \{1, 2\}$ or $|H_3 \cap H_1| = 2$ and $|H_3 \cap H_2| = 0$. In the former case, by symmetry and (14), we may assume $Z_1 = N_1$. In the latter case, $|Z_2| \geq 5 > 7 - |W|$, and so $Z_2 \neq N_2$ by (18), and by (14), $Z_1 = N_1$. So in either case, we have $Z_1 = N_1$. Let $x \in Z_1 \cap X_1$, say. Then x can have neighbors only in $Z_1 \cup Y_1 \cup W$. By (21), $Y_1 = X_1$, and so $|Z_1 \cup Y_1 \cup W \setminus \{x\}| = 8$. Thus x is complete to $Z_1 \cup Y_1 \cup W \setminus \{x\}$ since $\delta(G) \geq 8$, and it follows both that Z_1 is complete to W and that Z_1 induces a clique in G (note if $|H_3 \cap H_1| = 2$, then $Z_1 \neq H_1 \setminus W$). As $W = M$ also induces a clique, we see that $Z_1 \cup W$ induces a K_7-subgraph of G. Since G is 7-connected, a K_8-minor can be easily obtained, which is a contradiction. This proves (25).

(26) $|W| = 5$.

Suppose to the contrary that $|W| \neq 5$. By (10) and (25), $|W| = 4$. By symmetry, we may assume that $|W \cap H_1| \geq |W \cap H_2|$. Then it is not too hard to see that there are only three possibilities which satisfy (14) and (18):

(i) $|H_1 \cap H_3| = 2$ and $|H_2 \cap H_3| = 1$;

(ii) $|M| = 3$, $|H_1 \cap H_3| = 2$, and $|W \cap H_3| = 1$; or

(iii) $|H_1 \cap H_3| = 3$ and $Z_1 \neq H_1 \setminus W$.

Note that in each of these cases, $|Z_2| \geq 4$, so $Z_2 \neq N_2$, and thus by (14), $Z_j = N_j$ for $j \in \{1, 3\}$. We claim that for any $i \in \{1, 2, \ldots, n\}$, if $|X_i \cap Z_j| = 1$ for all $j \in \{1, 2, 3\}$, then $X_i = Y_i$. Suppose to the contrary that, say, $|X_1 \cap Z_j| = 1$ for $j \in \{1, 2, 3\}$, and that $X_1 \neq Y_1$. Then $S := X_1 \cup W$ is a separating set in G and so $|S| = 7$ by (19). But $x \in H_1 \cap H_3 \subseteq W$ has five neighbors in S, contradicting that $\Delta(G[S]) \leq 4$ by (A). This proves the claim.

It now follows that each $x \in Z_3$ belongs to some X_i where $X_i = Y_i$ by (13) and (19), since $Z_3 = N_3$. Say $x \in Z_3 \cap X_1$. As such an x can have neighbors only in $Z_3 \cup Y_1 \cup W$ and $|Z_3 \cup Y_1 \setminus \{x\}| = 4$, it is clear that x is complete to W since $\delta(G) \geq 8$. It follows that Z_3 is complete to W, and in particular that $y \in H_1 \cap H_2$ is complete to Z_3. But this contradicts that $d(y) \leq 11$ by (B). This proves (26).

By (18) and (26), we have $|N_j| = 2$ for all $j \in \{1, 2, 3\}$. In order to satisfy this and (14), it is not hard to see that, by symmetry, we must have $|H_1 \cap H_3| = 3$, $|M| = 4$, and $H_3 \supseteq (W \setminus M) \neq \emptyset$. Then $|Z_2| \geq |H_3 \setminus W| > 5$, and so $Z_2 \neq N_2$. By (14), $H_j \setminus W = Z_j = N_j$ for $j \in \{1, 3\}$.

(27) *For* $i \in \{1, 2, \ldots, n\}$*, if* $Y_i \neq X_i$*, then every vertex in* $X_i \cup W$ *has a neighbor in every component of* $G[Y_i \setminus X_i]$.

Suppose $Y_1 \neq X_1$, say. Then $|X_1| = 3$ by (11) and (19), and $S := X_1 \cup W$ is a separating set of G with $|S| = 8$. Thus for any component C of $G[Y_1 \setminus X_1]$, at least seven vertices of S have a neighbor in $V(C)$ since G is 7-connected. If only seven vertices of S have a neighbor in some component C of $G[Y_1 \setminus X_1]$, say $x \in S$ is anticomplete to $V(C)$, then $S \setminus \{x\}$ is a separating set with $|S \setminus \{x\}| = 7$, but then any vertex in $H_1 \cap H_3 \setminus \{x\}$ has at least five neighbors in $S \setminus \{x\}$, contradicting (A). Thus (27) follows.

(28) *For $i \in \{1, 2, \ldots, n\}$, the subgraph of G induced by Y_i is connected. Additionally, if $|X_i| = 3$ and $x, y \in X_i$, then the subgraph of G induced by $Y_i \setminus \{x, y\}$ is connected.*

If $X_i = Y_i$ this follows from (11) and (22). So we may assume that $X_1 \neq Y_1$, say. From (27) we see that each vertex of X_1 has at least one neighbor in every component of $G[Y_1 \setminus X_1]$, and so (28) follows.

(29) *For $i \in \{1, 2, \ldots, n\}$, if $|X_i| = 3$ then $X_i \neq Y_i$.*

For suppose there exists $i \in \{1, 2, \ldots, n\}$ such that $|X_i| = 3$ and $X_i = Y_i$, say $i = 1$. By (22), X_1 induces a K_3-subgraph of G. Let $x \in N_3 \cap X_1$. Then x can have neighbors only in $Z_3 \cup Y_1 \cup W$, and since $|Z_3 \cup Y_1 \setminus \{x\}| = 3$, we can see that x must be complete to W since $\delta(G) \geq 8$, and in particular x is complete to M. Thus $H' := M \cup X_1 \setminus N_2$ induces a clique in G with $|H'| = 6$. There must be one more set X_i with $|X_i| = 3$, say X_2. Then by (28), $Y_2 \setminus (N_j \cup N_k)$ induces a connected subgraph of G for all $j, k \in \{1, 2, 3\}$ with $j \neq k$, and it follows that there is an N_1, N_3-path P avoiding N_2, and a P, N_2-path Q disjoint from P except for its end, such that both P and Q are in Y_2. It is clear that every vertex in H' is adjacent to at least one end of P. Lastly, $W \cup N_2$ is a minimum separating set in G, and so it follows that every vertex of W has a neighbor in $Z_2 \setminus N_2$. Since X_1 induces a K_3, we see that every vertex of H' thus has a neighbor in Z_2. Then contracting each of $V(P)$ and $Z_2 \cup V(Q) \setminus V(P)$ to a single vertex will give a K_8-minor in G, a contradiction. This proves (29).

Now, by (18) and (26), $|N_j| = 3$ for all $j \in \{1, 2, 3\}$. By (13) and (19), there are two sets X_i with $|X_i| = 3$, say X_1 and X_2. It follows from (29) and (27) that the vertex $y \in H_1 \cap H_2$ has at least one neighbor in each of $Y_1 \setminus X_1$ and $Y_2 \setminus X_2$. Since $(H_1 \cup H_2) \cap (Y_i \setminus X_i) = \emptyset$ for $i \in \{1, 2\}$, we see y has at least twelve neighbors in G, contradicting (B). This contradiction completes the proof of Theorem 1.6.6. ■

CHAPTER 5: FUTURE WORK

In this chapter, we discuss possible extensions of our work in this dissertation, as well as other topics of interest.

5.1 Double-Critical Graph Conjecture

Our ultimate goal in this area is to prove the next open case of the Double-Critical Graph Conjecture, namely that the only double-critical, 6-chromatic graph is the complete graph K_6. In [68], Stiebitz showed that any non-complete, 5-chromatic, double-critical graph must contain a K_3, and hence a K_4, thereby obtaining a contradiction when the vertices of any edge disjoint from the K_4 are deleted. The additional color in a non-complete, 6-chromatic, double-critical graph prevents Stiebitz's method from being applied directly. It is not even clear that any non-complete, 6-chromatic, double-critical graph must contain K_4 as a subgraph.

We also hope to prove the next case of the Double-Critical Graph Conjecture for claw-free graphs. That is, we want to show that any 9-chromatic, double-critical, claw-free graph must be K_9. Our methods presented in Section 2.4 applied to this case show that any such graph must contain vertices of only degree 11 or 12. We believe that it should be possible, although very tedious, to prove this case by analyzing the neighborhoods of vertices of degree 11 or 12 in a manner similar to our examination of the neighborhoods of vertices of degree 10 in the proof of Theorem 1.2.8. One approach to simplify this would be to improve Theorem 1.2.9. If we can show that no vertex of degree $t+2$ is adjacent to a vertex of degree $t+2$ or $t+3$, then any 9-chromatic, double-critical, claw-free graph must be 12-regular.

5.2 Hadwiger's Conjecture

We hope to apply our powerful Lemma 1.5.3 to improve Theorem 1.3.7. Our main goal is to show that any K_7-minor-free graph is 7-colorable. Hadwiger's Conjecture says that any K_7-minor-free graph is 6-colorable. We have been able to make some partial progress on this particular problem. If G is an 8-contraction-critical, K_7-minor-free graph, then we are able to show in [61] that G contains at most one vertex of degree 8. If we can similarly restrict the number of vertices of degree 9 in such graphs, then it will follow from Theorem 1.4.1 that G contains a K_7-minor, a contradiction. This task, however, seems very hard.

Other applications of Lemma 1.5.3 are possible. If Theorem 1.4.6 can be extended to give an edge bound for K_9^--minors, then we believe that Lemma 1.5.3 can be used to extend our Theorem 1.3.8 to show that any K_9^--minor free graph is 11-colorable. Similarly, if Theorem 1.4.7 can be extended to give an edge bound for $K_9^=$-minors, then we believe that Lemma 1.5.3 can be used to extend Theorem 1.3.9 to show that any $K_9^=$-minor free graph is 10-colorable. In general, if a suitable analogue to Conjecture 3.5.2 holds true for K_t^--minors and $K_t^=$-minors, then we believe it can be shown that any K_t^--minor-free graph is $(2t-7)$-colorable and any $K_t^=$-minor-free graph is $(2t-8)$-colorable.

5.3 On $\mathcal{R}_{\min}(K_3, \mathcal{T}_k)$-saturated Graphs

Given graphs $G, H_1, \ldots, H_t$, we write $G \to (H_1, \ldots, H_t)$ if every t-edge-coloring of G contains a monochromatic H_i in color i for some $i \in \{1, 2, \ldots, t\}$. The classical *Ramsey number* $r(H_1, \ldots, H_t)$ is the minimum positive integer n such that $K_n \to (H_1, \ldots, H_t)$. A graph G is $(H_1, \ldots, H_t)$*-Ramsey-minimal* if $G \to (H_1, \ldots, H_t)$, but for any proper subgraph G' of G, $G' \not\to (H_1, \ldots, H_t)$. We define $\mathcal{R}_{\min}(H_1, \ldots, H_t)$ to be the family of $(H_1, \ldots, H_t)$-Ramsey-minimal graphs. It is straightforward to prove by induction that a graph G satisfies $G \to (H_1, \ldots, H_t)$ if and only if there exists a subgraph G' of G such that G' is $(H_1, \ldots, H_t)$-Ramsey-minimal. Ramsey's theorem [52] implies that $\mathcal{R}_{\min}(H_1, \ldots, H_t) \neq \emptyset$ for all integers t and all finite graphs $H_1, \ldots, H_t$. As pointed out in a recent paper of Fox, Grinshpun, Liebenau, Person, and Szabó [21], "it is still widely open to classify the graphs in $\mathcal{R}_{\min}(H_1, \ldots, H_t)$, or even to prove that these graphs have certain properties". Some properties of $\mathcal{R}_{\min}(H_1, \ldots, H_t)$ have been studied, such as the minimum degree $s(H_1, \ldots, H_t) := \min\{\delta(G) : G \in \mathcal{R}_{\min}(H_1, \ldots, H_t)\}$, which was first introduced by Burr, Erdős, and Lovász [6]. Recent results on $s(H_1, \ldots, H_t)$ can be found in [22, 21]. For more information on Ramsey-related topics, the readers are referred to a very recent informative survey due to Conlon, Fox, and Sudakov [13].

A graph G is $\mathcal{R}_{\min}(H_1, \ldots, H_t)$-*saturated* if no element of $\mathcal{R}_{\min}(H_1, \ldots, H_t)$ is a subgraph of G, but for any edge e in $\overline{G}$, some element of $\mathcal{R}_{\min}(H_1, \ldots, H_t)$ is a subgraph of $G + e$. This notion was initiated by Nešetřil [49] in 1986 when he asked whether there are infinitely many $\mathcal{R}_{\min}(H_1, \ldots, H_t)$-saturated graphs. This was answered in the positive by Galluccio, Siminovits, and Simonyi [23]. We define $sat(n, \mathcal{R}_{\min}(H_1, \ldots, H_t))$ to be the minimum number of edges over all $\mathcal{R}_{\min}(H_1, \ldots, H_t)$-saturated graphs on n vertices. This notion was first discussed by Hanson and Toft [28] in 1987 when $H_1, \ldots, H_t$ are complete graphs. They proposed the following conjecture.

Conjecture 5.3.1 (Hanson and Toft [28]) *Let $r = r(K_{k_1}, \ldots, K_{k_t})$ be the classical Ramsey number for complete graphs. Then*

$$sat(n, \mathcal{R}_{\min}(K_{k_1}, \ldots, K_{k_t})) = \begin{cases} \binom{n}{2}, & n < r \\ (r-2)(n-r+2) + \binom{r-2}{2}, & n \geq r \end{cases}$$

Chen, Ferrara, Gould, Magnant, and Schmitt [7] proved that $sat(n, \mathcal{R}_{\min}(K_3, K_3)) = 4n - 10$ for

$n \geq 56$. This settles the first non-trivial case of Conjecture 5.3.1 for sufficiently large n, and is so far the only settled case. Ferrara, Kim, and Yeager [20] proved that $sat(n, \mathcal{R}_{\min}(m_1K_2, \ldots, m_tK_2)) = 3(m_1 + \cdots + m_t - t)$ for $m_1, \ldots, m_t \geq 1$ and $n > 3(m_1 + \cdots + m_t - t)$. The problem of finding $sat(n, \mathcal{R}_{\min}(K_3, T_k))$ was also explored in [7].

Proposition 5.3.2 (Chen, Ferrara, Gould, Magnant, and Schmitt [7]) *Let $k \geq 2$ and $t \geq 2$ be integers. Then*

$$\begin{aligned} sat(n, \mathcal{R}_{\min}(K_t, T_k)) \leq n(t-2)(k-1) - (t-2)^2(k-1)^2 - (t-2)\binom{k-1}{2} \\ + \binom{(t-2)(k-1)}{2} + \left\lfloor \frac{n}{k-1} \right\rfloor \binom{k-1}{2} + \binom{r}{2}, \end{aligned}$$

where $r = n \pmod{k-1}$.

It was conjectured in [7] that the upper bound in Proposition 5.3.2 is asymptotically correct. Note that there is only one tree on three vertices, namely, P_3. A slightly better result was obtained for $\mathcal{R}_{\min}(K_3, P_3)$-saturated graphs in [7].

Theorem 5.3.3 (Chen, Ferrara, Gould, Magnant, and Schmitt [7]) *For $n \geq 11$, $sat(n, \mathcal{R}_{\min}(K_3, P_3)) = \left\lfloor \frac{5n}{2} \right\rfloor - 5$.*

Motivated by Conjecture 5.3.1, we study the following problem in [60]. Let $\mathcal{T}_k$ be the family of all trees on k vertices. Instead of fixing a tree on k vertices as in Proposition 5.3.2, we will investigate $sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_k))$, where a graph G is $(K_3, \mathcal{T}_k)$-*Ramsey-minimal* if for any 2-coloring $c : E(G) \to \{\text{red, blue}\}$, G has either a red K_3 or a blue tree $T_k \in \mathcal{T}_k$, and we define $\mathcal{R}_{\min}(K_3, \mathcal{T}_k)$ to be the family of $(K_3, \mathcal{T}_k)$-Ramsey-minimal graphs. By Theorem 5.3.3, we see that $sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_3)) = \lfloor 5n/2 \rfloor - 5$ for $n \geq 11$. In [60], we prove the following two main results. We first establish the exact bound for $sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_4))$ for $n \geq 18$, and then obtain an asymptotic bound for $sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_k))$ for all $k \geq 5$ and $n \geq 2k + (\lceil k/2 \rceil + 1)\lceil k/2 \rceil - 2$.

Theorem 5.3.4 [60] *For* $n \geq 18$, $sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_4)) = \left\lfloor \frac{5n}{2} \right\rfloor$.

Theorem 5.3.5 [60] *For any integers* $k \geq 5$ *and* $n \geq 2k + (\lceil k/2 \rceil + 1)\lceil k/2 \rceil - 2$, *there exist constants* $c = \left(\frac{1}{2}\left\lceil \frac{k}{2} \right\rceil + \frac{3}{2}\right)k - 2$ *and* $C = 2k^2 - 6k + \frac{3}{2} - \left\lceil \frac{k}{2} \right\rceil \left(k - \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil - 1\right)$ *such that*

$$\left(\frac{3}{2} + \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil\right) n - c \leq sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_k)) \leq \left(\frac{3}{2} + \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil\right) n + C.$$

The constants c and C in Theorem 5.3.5 are both quadratic in k. We believe that the true value of $sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_k))$ is closer to the upper bound in Theorem 5.3.5.

For future work in this area, we plan to investigate $sat(n, \mathcal{R}_{\min}(K_3, T_k))$ for fixed trees T_k, rather than the family of trees $\mathcal{T}_k$. Further, we we believe the method developed in the proof of Theorem 5.3.5 can be used to find $sat(n, \mathcal{R}_{\min}(K_4, \mathcal{T}_k))$.

LIST OF REFERENCES

[1] B. Albar and D. Gonçalves, "On triangles in K_r-minor free graphs," *(submitted).* arXiv:1304.5468.

[2] K. Appel and W. Haken, "Every planar map is four colorable. Part I. Discharging," *Illinois J. Math.* **21** (1977), 429–490.

[3] K. Appel, W. Haken, and J. Koch, "Every planar map is four colorable. Part II. Reducibility," *Illinois J. Math.* **21** (1977), 491–567.

[4] J. Balogh, A. V. Kostochka, N. Prince, and M. Stiebitz, "The Erdős-Lovász Tihany conjecture for quasi-line graphs," *Discrete Math.* **309** (2009), 3985–3991.

[5] W. G. Brown and H. A. Jung, "On odd circuits in chromatic graphs." *Acta Math. Sci. Hung.* **20** (1969), 129–134.

[6] S. A. Burr, P. Erdős, and L. Lovász, "On graphs of Ramsey type," *Ars Combin.* **1** (1976), 167–190.

[7] G. Chen, M. Ferrara, R. J. Gould, C. Magnant, and J. Schmitt, "Saturation numbers for families of Ramsey-minimal graphs," *J. Combinatorics.* **2** (2011), 435–455.

[8] G. Chen, Z. Hu, and F. Song, "Linkage and Hadwiger's conjecture," *(submitted).*

[9] M. Chudnovsky and A. Fradkin, "Hadwiger's conjecture for quasi-line graphs," *J. Graph Theory.* **59** (2008), 17-33.

[10] M. Chudnovsky, A. Fradkin, and M. Plumettaz, "On the Erdős-Lovász Tihany Conjecture for Claw-Free Graphs," *(preprint).* arXiv:1309.1020v1.

[11] M. Chudnovsky, B. Reed, and P. Seymour, “The edge-density for $K_{2,t}$ minors,” *J. Combin. Theory, Ser. B.* **101** (2011), 18–46.

[12] M. Chudnovsky and P. Seymour, “Claw-free graphs. V. Global structure,” *J. Combin. Theory, Ser. B.* **98** (2008), 1373–1410.

[13] D. Conlon, J. Fox and B. Sudakov, “Recent developments in graph Ramsey theory,” *Surveys in Combinatorics.* **424** (2015), 49–118.

[14] G. Ding, “A characterisation of graphs with no octohedron minor,” *J. Graph Theory.* **74** (2013), 143–162.

[15] G. A. Dirac, “Homomorphism theorems for graphs,” *Math. Ann.* **153** (1964), 69–80.

[16] G. A. Dirac, “On the structure of 5- and 6-chromatic abstract graphs,” *J. Reine Angew. Math.* **214/215** (1964), 43–52.

[17] G. A. Dirac, “A property of 4-chromatic graphs and some remarks on critical graphs,” *J. London Math. Soc.* **27** (1952), 85–92.

[18] G. A. Dirac, “Trennende Knotenpunktmengen und Reduzibilität abstrakter Graphen mit Anwendung auf das Vierfarbenproblem,” *J. Reine Agew. Math.* **204** (1960), 116–131.

[19] P. Erdős, Problem 2. In: *Theory of Graphs (Proc. Colloq., Tihany, 1966).* Academic Press, New York, (1968), 361.

[20] M. Ferrara, J. Kim, and E. Yeager, “Ramsey-minimal saturation numbers for matchings,” *Discrete Math.* **322** (2014), 26–30.

[21] J. Fox, A. Grinshpun, A. Liebenau, Y. Person, and T. Szabó, “On the minimum degree of minimal Ramsey graphs for multiple colors,” *J. Combin. Theory, Ser. B.* **120** (2016), 64–82.

[22] J. Fox and K. Lin, "The minimum degree of Ramsey-minimal graphs," *J. Graph Theory.* **54** (2007), 167–177.

[23] A. Galluccio, M. Simonovits, and G. Simonyi, "On the structure of co-critical graphs," *Proc. of Graph Theory, Combinatorics, and Computing (Kalamazoo, MI)*, Wiley-Intersci. Publ., Wiley, New York, (1995), 1053–1071.

[24] E. Györi, "On the edge numbers of graphs with Hadwiger number 4 and 5," *Period. Math. Hung.* **13** (1982), 21–27.

[25] H. Hadwiger, "Über eine Klassifikation der Streckencomplexe," *Vierteljschr. Naturforsch. Ges. Zürich.* **88** (1943), 133–142.

[26] D. W. Hall, "A note on primitive skew curves," *Bull. Amer. Math. Soc.* **49** (1943), 935–936.

[27] P. Hall, "On Representatives of Subsets," *J. London Math. Soc.* **10** (1935), 26–30.

[28] D. Hanson and B. Toft, "Edge-colored saturated graphs," *J. Graph Theory.* **11** (1987), 191–196.

[29] P. J. Heawood, "Map-color theorem," *Q. J. Math.* **24** (1890), 332–339.

[30] K. Hendrey and D. R. Wood, "The extremal function for Petersen minors," *(submitted).* arXiv:1508.04541.

[31] H. Huang and A. Yu, "A note on the double-critical graph conjecture," *(submitted).*

[32] I. T. Jakobsen, "A homomorphism theorem with an application to the conjecture of Hadwiger," *Studia Sci. Math. Hungar.* **6** (1971), 151–160.

[33] I. T. Jakobsen, "On certain homomorphism properties of graphs I," *Math. Scand.* **31** (1972), 379–404.

[34] I. T. Jakobsen, "On certain homomorphism properties of graphs II," *Math. Scand.* **52** (1983), 229–261.

[35] L. K. Jørgensen, "Contractions to K_8," *J. Graph Theory.* **18** (1994), 431–448.

[36] K. Kawarabayashi, "On the connectivity of minimum and minimal counterexamples to Hadwiger's Conjecture," *J. Combin. Theory, Ser. B.* **97** (2007), 144–150.

[37] K. Kawarabayashi, A. S. Pedersen, and B. Toft, "Double-critical graphs and complete minors," *Electron. J. Combin.* **17(1)** (2010) Research Paper 87.

[38] K. Kawarabayashi, R. Luo, J. Niu, and C. Zhang, "On the structure of k-connected graphs without K_k-minor," *Euro. J. Combin.* **26** (2005), 293–308.

[39] K. Kawarabayashi and B. Toft, "Any 7-chromatic graph has K_7 or $K_{4,4}$ as a minor," *Combinatorica.* **25** (2005), 327–353.

[40] K. Kawarabayashi and G. Yu, "Connectivities for k-knitted graphs and for minimal counterexamples to Hadwiger's Conjecture," *J. Combin. Theory, Ser. B.* **103** (2013), 320–326.

[41] A. V. Kostochka and M. Stiebitz, "Partitions and edge colorings of multigraphs," *Electron. J. Combin.* **15** (2008), N25.

[42] M. Kriesell, "On Seymour's strengthening of Hadwiger's conjecture for graphs with certain forbidden subgraphs," *Discrete Math.* **310** (2010), 2714–2724.

[43] K. Kuratowski, "Sur le problème des courbes gauches en topologie," *Fund. Math.* **15** (1930), 271–283.

[44] W. Mader, "Homomorphiesätze für Graphen," *Math. Ann.* **178** (1968), 154–168.

[45] W. Mader, "Über die Maximalzahl kreuzungsfreier H-Wege," *Arch. Math. (Basel).* **31** (1978), 387–402.

[46] W. Mader, “Über trennende Eckenmengen in homomorphiekritischen Graphen,” *Math. Ann.* **175** (1968), 243–252.

[47] K. Menger, “Zur allgemeinen Kurventheorie,” *Fun. Math.* **10** (1927), 95–115.

[48] N. N. Mozhan, “On doubly critical graphs with the chromatic number five,” *Metody Diskretn. Anal.* **46** (1987), 50-59.

[49] J. Nešetřil, “Problem,” in: *Irregularities of Partitions.* (edited by G. Halász and V. T. Sós), Springer Verlag, Series Algorithms and Combinatorics, vol 8, (1989), 164. (Proc. Coll. held at Fertőd, Hungary 1986).

[50] A. S. Pedersen, “Complete and almost complete minors in double-critical 8-chromatic graphs,” *Electron. J. Combin.* **18(1)** (2011), Research Paper 80.

[51] M.D. Plummer, M. Stiebitz, and B. Toft, “On a special case of Hadwiger’s Conjecture,” *Discuss. Math. Graph Theory.* **23** (2003), 333–363.

[52] F. P. Ramsey, “On a problem of formal logic,” *Proc. London Math. Soc.* **30** (1930), 264–286.

[53] B. Reed and P. Seymour, “Hadwiger’s conjecture for line graphs,” *European J. Math.* **25** (2004), 873–876.

[54] N. Robertson, D. Sanders, P. Seymour, and R. Thomas, “The four-color theorem,” *J. Combin. Theory, Ser. B.* **70** (1997), 2–44.

[55] N. Robertson, P. Seymour, and R. Thomas, “Hadwiger’s conjecture for K_6-free graphs,” *Combinatorica.* **13** (1993), 279–361.

[56] M. Rolek, “Finding a K_8-minor using Mader’s H-Wege Theorem,” *(to be submitted).*

[57] M. Rolek and Z-X. Song, “Double-critical Graph Conjecture for Claw-free Graphs,” *to appear in Discrete Math.* arXiv:1610.00636

[58] M. Rolek and Z-X. Song, “Coloring Graphs with Forbidden Minors,” *(submitted).* arXiv:1606.05507

[59] M. Rolek and Z-X. Song, “Clique Minors in Double-critical Graphs,” *(submitted).* arXiv:1603.06964

[60] M. Rolek and Z-X. Song, “On $\mathcal{R}_{\min}(K_3, \mathcal{T}_k)$-saturated graphs,” *(submitted).*

[61] M. Rolek, Z-X. Song, and R. Thomas, “Every graph with no K_7-minor is 7-colorable,” *(in preparation).*

[62] P. Seymour, “Hadwiger’s conjecture,” in: *Open Problems in Mathematics.* (edited by J. Nash and M. Rassias), Springer, (2016), 417–437.

[63] A. Schrijver, “A Short Proof of Mader’s $\mathscr{S}$-Paths Theorem,” *J. Combin. Theory, Ser. B.* **82** (2001), 319–321.

[64] Z-X. Song, “The extremal function for K_8^- minors,” *J. Combin. Theory, Ser. B.* **95** (2005), 300–317.

[65] Z-X. Song, “Extremal functions for contractions of graphs,” Doctoral Dissertaton, Georgia Institute of Technology, (2004).

[66] Z-X. Song and B. Thomas, “Hadwiger’s conjecture for graphs with forbidden holes,” *(submitted).* arXiv:1607.06718

[67] Z.-X. Song and R. Thomas, “The extremal function for K_9 minors,” *J. Combin. Theory, Ser. B.* **96** (2006), 240–252.

[68] M. Stiebitz, “K_5 is the only double-critical 5-chromatic graph,” *Discrete Math.* **64** (1987), 91-93.

[69] M. Stiebitz, "On k-critical n-chromatic graphs," in: *Combinatorics (Eger, 1987). Colloq. Math. Soc. János Bolyai.* **52** (1987), 509–514.

[70] M. Stiebitz, "A relaxed version of the Erdős-Lovász Tihany conjecture," *to appear in J. Graph Theory.*

[71] B. Toft, "A survey of Hadwiger's Conjecture," in: *Surveys in Graph Theory* (edited by G. Chartrand and M. Jacobson), *Congr. Numer.* **115** (1996), 249–283.

[72] P. Turán, "On an extremal problem in graph theory," *Matematikai és Fizikai Lapok.* **48** (1941), 436–452.

[73] K. Wagner, "Über eine Eigenschaft der ebenen Komplexe," *Math. Ann.* **114** (1937) 570–590.

[74] D. R. Woodall, "A Short Proof of a Theorem of Dirac's About Hadwiger's Conjecture," *J. Graph Theory.* **16** (1992) 79–80.