A Multiple Case Study Exploring the Relationship Between Engagement in Model-Eliciting Activities and Pre-Service Secondary Mathematics Teachers' Mathematical Knowledge for Teaching Algebra

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A MULTIPLE CASE STUDY EXPLORING THE RELATIONSHIP BETWEEN ENGAGEMENT IN MODEL-ELICITING ACTIVITIES AND PRE-SERVICE SECONDARY MATHEMATICS TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING ALGEBRA

by

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ABSTRACT

The goal of this research study was to explore the nature of the relationship between engagement in model-eliciting activities (MEAs) and pre-service secondary mathematics teachers' (PSMTs') mathematical knowledge for teaching (MKT) algebra. The data collection took place in an undergraduate mathematics education content course for secondary mathematics education majors. In this multiple case study, PSMTs were given a Learning Mathematics for Teaching (LMT) pre-assessment designed to measure their MKT algebra, and based on those results, three participants were selected with varying levels of knowledge. This was done to ensure varied cases were represented in order to be able to examine and describe multiple perspectives. The three examined cases were Oriana, a PSMT with high MKT, Bianca, a PSMT with medium MKT, and Helaine, a PSMT with low MKT. Over the course of five weeks, the three PSMTs were recorded exploring three MEAs, participated in two interviews, and submitted written reflections. The extensive amount of data collected in this study allowed the researcher to deeply explore the PSMTs' MKT algebra in relation to the given MEAs, with a focus on three specific constructs—bridging, trimming, and decompressing— based on the Knowledge of Algebra for Teaching (KAT) framework. The results of this study suggest that engaging in MEAs could elicit PSMTs' MKT algebra, and in some cases such tasks were beneficial to their trimming, bridging, and decompressing abilities. Exploring MEAs immersed the PSMTs in generating descriptions, explanations, and constructions, that helped reveal how they interpreted mathematical situations that they encountered. The tasks served as useful tools for PSMTs to have deep discussions and productive discourse on various algebra topics, and make many different mathematical connections in the process.

To my daughter, Olivia—

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LIST OF ACRONYMS

CCSSM	Common Core State Standards for Mathematics
KAT	Knowledge of Algebra for Teaching
LMT	Learning Mathematics for Teaching
MEA	Model Eliciting Activity
MKT	Mathematical Knowledge for Teaching
MMP	Models and Modeling Perspective
PSMT	Pre-service Secondary Mathematics Teacher(s)

CHAPTER ONE: INTRODUCTION

One of the standard explanations of why mathematics tends to get so much time and attention in formal educational settings has been the idea that mathematics is a useful tool to better understand everyday life, prepare for future employment, engage in intelligent citizenship, and make sense of other subjects that students are expected to learn (Pollak, 2016). In fact, Niss (1996) explained that mathematics is necessary because it is an important part of culture and society, and it should be seen as a powerful tool for better understanding and mastering present or future real-world situations. A natural question is: what might mathematics educators do to prepare all students to use mathematics for everyday life, future employment, and informed citizenship? Since the 1960s, much work has been done in the field of mathematics education in order to understand how students engage in the application of mathematics to real-world contexts. Generally, this process of using mathematics to solve real-world problems has come to be called mathematical modeling (Niss, Blum, & Galbraith, 2007; Pollak, 2016).

Beyond this general description, the process of teaching and learning mathematical modeling consists of a wide range of goals and backgrounds. For instance, Julie and Mudaly (2007) describe such differences by characterizing *modeling as a vehicle* — where modeling is used to develop mathematical understanding, or *modeling as content* — where modeling is used to develop the skills of modeling. The various perspectives on mathematical modeling can be attributed to the goals that are evident in what constitutes a mathematical modeling task, how the mathematical modeling process is described, and the purpose of the research or modeling itself (Plunkett, 2016). Researchers (Blomhøj, 2009; Kaiser & Sriraman, 2006) have looked at the various definitions and goals of mathematical modeling and developed broad classifications for the different perspectives of mathematical modeling. These perspectives include: (1) realistic modeling, (2) educational modeling, (3) models and modeling perspective (sometimes called contextual modeling), (4) socio-critical modeling, and (5) epistemological modeling. These perspectives will be discussed in detail in Chapter two. It is important to note the different views of mathematical modeling because there is not a single-agreed upon definition; rather, researchers have put forth different definitions, descriptions, and assumptions of what mathematical modeling entails (Cirillo, Pelesko, Felton-Koestler, & Rubel, 2016). Thus, the following paragraph is included to broadly describe the goals of mathematical modeling within each perspective, and clarify how mathematical modeling is viewed in this study.

In realistic modeling, mathematical modeling is viewed as *content*, and the goal is to develop skills to model and understand authentic, real-world scenarios. On the other hand, educational modeling views modeling as both *content* and *a vehicle* as there is an equal emphasis on both developing the skills to model a real-world scenario and understanding mathematics. Similarly, in the models and modeling perspective (MMP), modeling is viewed as *content* and a *vehicle* as well, however, there is a stronger emphasis on the learning of mathematics; the goal in this perspective is to develop a deep understanding of mathematics through a modeling context. Likewise, the goal in epistemological modeling is to develop formal mathematical reasoning, and thus, modeling is viewed as a *vehicle*, but not as *content* within this perspective. Lastly, in the socio-critical perspective, the aim is to develop mathematical modeling skills in order to make decisions in society; thus, modeling is viewed as both *content* and a *vehicle*.

The purpose of this study is to explore the nature of the relationship between engagement in mathematical modeling and pre-service secondary mathematics teachers' (PSMT) mathematical knowledge for teaching (MKT). Ball, Thames and Phelps (2008) define MKT as a deep understanding of mathematics that is needed for teachers to effectively teach mathematics.

Accordingly, mathematical modeling is being explored in relation to mathematical understanding, and thus, modeling is viewed as both a *vehicle* and *content* in this study, with a stronger emphasis on the understanding of mathematics (the *vehicle* outlook). Furthermore, because this study aims to examine PSMT' deep understanding of mathematics through a modeling context, the models and modeling perspective will be used as a guiding framework. A more detailed explanation will be presented in Chapter two.

Throughout the years, many researchers have described how mathematical modeling can be used as a tool to develop mathematical competencies that help students solve problems arising in everyday life, society, and the workplace (Blum & Ferri 2009; Lesh et al., 2000; Maiorca & Stohlmann, 2016; Niss, 1996; Zbiek, 1998;), and how mathematical modeling provides a venue through which learners can engage in mathematics in robust ways (Zbiek & Conner, 2006). However, the Program for International Student Assessment (PISA), which assesses education systems by testing the skills and knowledge of students worldwide, has indicated that students struggle in solving mathematical modeling tasks (OECD, 2012). Consequently, mathematical modeling has been brought to the forefront of teaching and learning mathematics through imperative works both internationally and in the United States.

Internationally, many countries such as Germany, Singapore, and China have explored numerous aspects of mathematical modeling—how students learn the modeling process, how teachers can incorporate mathematical modeling effectively in their classrooms, how mathematical modeling can be incorporated in curricula, etc. (Stillman, Kaiser, Blum & Brown, 2013). Furthermore, countries like Germany, Singapore, Sweden, and South Africa have reformed their mathematics standards to increase the emphasis on mathematical modeling (Ferri, 2013). As a result, mathematical modeling research in the K-16 setting has expanded with numerous books being published encompassing mathematical modeling in mathematics education (Kaiser, Blum, Ferri, & Stillman, 2011; Lesh, Galbraith, Haines, & Hurford, 2010; Stillman, Kaiser, Blum, & Brown 2013; Stillman, Blum, & Biembengut, 2015).

Similarly, in the United States, works such as the *Common Core State Standards for Mathematics* (CCSSM) (The National Governors Association [NGA] Center for Best Practices & The Council of Chief State School Officers [CCSSO], 2010), *Next Generation Science Standards* (NGSS) (National Science Teachers Association, 2013), *The Annual Perspectives in Mathematics Education 2016* (APME) (Hirsch & McDuffie, 2016) and the *Guidelines for Assessment and Instruction in Mathematical Modeling Education* (GAIMME) (Garfunkel et al., 2016) report have made mathematical modeling prominent in the field of mathematics education. Although mathematical modeling has received amplified attention in the past ten years, its significance in K-12 mathematics classrooms was discussed as early as the 1966 *Report of the Modeling Committee* by the School Mathematics Study Group (SMSG), in which the report advised that mathematical models and modeling be incorporated in mathematics classrooms early in grades seven to nine, and should be a thread worked in throughout the sequence of topics (SMSG, 1966). However, similar recommendations, like those found in the CCSSM (NGA Center for Best Practices & CCSSO, 2010) have only recently been implemented in the mathematics K-12 curriculum.

The CCSSM (or a closely aligned or adapted version of them), which are being used in most states and territories across the U.S., have outlined a description and perspective of mathematical modeling within the standards. Mathematical modeling is described as a practice, where mathematically proficient students apply mathematics to solve real-world problems that might arise in everyday life and in the workplace (NGA & CCSSO, 2010). Furthermore, CCSSM defines the modeling process as "the process of choosing and using appropriate mathematics and statistics to analyze empirical situations to understand them better, and to improve decisions" (NGA & CCSSO, 2010, p. 72). This cyclic modeling process is illustrated in Figure 1.

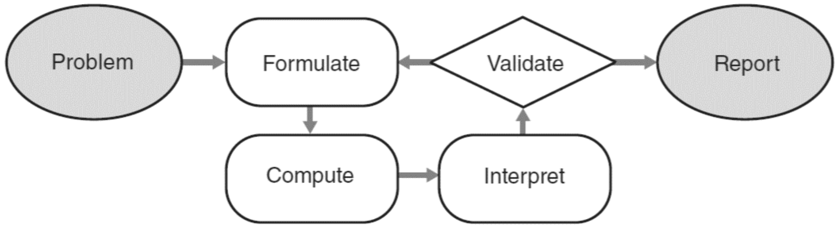


Figure 1. The Modeling Process in CCSSM (NGA & CCSSO, 2010, p. 72).

Congruent to the international research discussed previously, mathematical modeling research is also being conducted in the U.S. Lesh, Doerr, and colleagues, renowned in the field of mathematics education for conducting mathematical modeling research, provided the foundation for using mathematical modeling as a context to develop mathematical understanding (Brady, Lesh, & Sevis, 2015; Doerr & O'Neil, 2011; Lesh & Doerr, 2003; Lesh, Middleton, Caylor, & Gupta, 2008). Lesh, Hoover, Hole, Kelly and Post (2000) described six principles to help researchers, teachers, curriculum designers, and other stakeholders create thought-revealing activities they refer to as model-eliciting activities (MEAs). The authors describe model-eliciting activities as:

> activities that focus on the development of constructs (models or conceptual systems that are embedded in a variety of representational systems) that provide the conceptual foundations for deeper and higher order understandings of many of the most powerful ideas in mathematics and science curricula. The descriptions, explanations, and constructions that students generate while working on them directly reveal how they are interpreting the mathematical situations that they encounter by disclosing how these situations are being mathematized (e.g., quantified, organized, coordinatized, dimensionalized) or interpreted (p. 592).

Notably, these studies demonstrated that such tasks can help develop conceptual mathematical understanding in students (Carlson et al., 2003; Doerr & O'Neil, 2011 Lesh et al., 2008). They can similarly be useful for investigating (or promoting) the development of teachers' pedagogical knowledge (Lesh & Clarke, 2000). In fact, Doerr and Lesh (2003) indicated that in the same manner that MEAs can provide a foundation for the development of powerful mathematical reasoning in learners, they also can effectively guide teacher educators in helping pre-service teachers develop both their subject matter knowledge, as well as their pedagogical content knowledge.

The rising awareness of the value of mathematical modeling — as an essential part of the mathematical experiences for all students — requires focused attention on what is needed in terms of teacher preparation. The emphasis on mathematical modeling in the K-16 mathematics curriculum is relatively new, which indicates that most teachers (both pre-service and in-service) who are expected to teach mathematical modeling, or teach mathematics through modeling, have not themselves explored the various educational aspects involved in solving modeling tasks in a systematic way (Phillips, 2016). This serves as an important premise for this study, because one of the major motivations for conducting this research is to engage PSMT in mathematical modeling tasks, and the mathematical modeling process, in order to gain some pedagogical and subject matter insights of mathematical modeling.

Not only has mathematical modeling been valued in the mathematical experiences of K-12 students, but Doerr and Lesh (2003) have also revealed that implementing "a models and modeling perspective on teachers' development brings to the foreground a focus on the ways teachers interpret their practice" (p. 126). They also explained model-eliciting activities have the capability of helping teachers: (a) "reveal their current ways of thinking"; (b) "test, revise, and refine those ways of thinking"; (c) "share with colleagues for replication"; and (d) "reuse their ways of thinking in multiple contexts" (p. 133). This is important because it reveals that MEAs can be used by mathematics teacher educators to help PSMT develop their subject-matter and pedagogical content knowledge. However, according to the ICMI *Study on Applications and Modeling in Mathematics Education*, mathematics teacher education programs rarely include a perspective on mathematical modeling or the use of modeling in pre-service teachers' courses (Blum, Galbraith, Henn, & Niss, 2007). This study aims to address this gap by incorporating mathematical modeling tasks in a pre-service secondary mathematics teachers' course, and examining the aforementioned capabilities of MEAs.

Additionally, the literature suggests that for student learning to take place with understanding, teachers need to attain deep and broad content knowledge for teaching mathematics (Ball, Thames, & Phelps, 2008). There is an abundance of research on the significance of teachers' mathematical knowledge (Ball, 1988, 1990; Leinhardt & Smith, 1985; Ma, 1999; Shulman, 1986), Many studies have associated teachers' mathematical knowledge for teaching (MKT) with the effectiveness of mathematics instruction, (Baumert et al., 2010; Eisenhart et al., 1993; Hill et al., 2008) as well as achievement gains in students (Hill, Rowan, & Ball, 2005). Research also indicates that many teachers in the United States still lack the deep mathematical knowledge necessary for effective teaching (Hill & Ball, 2004; Ma, 1999; Tatto et al., 2008). Nevertheless, it is now more critical than ever to focus on developing teachers' mathematical knowledge for teaching in order to support students in meeting challenging learning goals and practices, like mathematical modeling, set out by rigorous standards such as the CCSSM (Selling, Garcia, & Ball, 2016). Because pre-service teachers tend to attain much of their content knowledge in their own education and mathematics education courses (Sahin, 2015), it is imperative that pre-service teachers engage in MEAs in order to develop deeper content knowledge for teaching. Thus, the need for continued research to uphold programmatic revision and development in the preparation of teachers is essential.

Although considerable progress has been achieved in the mathematical knowledge for teaching elementary grades, less work has been done when it comes to mathematical knowledge for teaching algebra in secondary grades (McCrory et al., 2012). The emphasis on requiring algebra for all students and offering it at earlier grades (Allensworth & Nomi, 2009; Loveless, 2008; National Mathematics Advisory Panel [NMAP], 2008) is resulting in more and more students enrolling in algebra courses (Ferrini-Mundy et al., 2012). As more students take algebra, more teachers are teaching algebra, yet the preparation of algebra teachers, in terms of their content knowledge for teaching, has not been widely researched (Stein et al., 2011). This means little research exists to inform decisions about mathematical knowledge for teaching algebra in the secondary setting (McCrory et al., 2012). This study aims to address this gap by exploring pre-service teachers' knowledge of introductory algebra topics such as: proportional reasoning, linear reasoning, and quadratic relationships. Introductory algebra content is particularly important to investigate with pre-service secondary mathematics teachers because the development of algebraic thinking is considered foundational to learning mathematics (Greenes et al., 2001). In fact, it has been revealed that resistance to algebra in the upper and post-secondary years could be lessened if students were to develop their algebraic thinking and reasoning at earlier levels of schooling (Cai & Moyer, 2008; Carraher et al., 2006). Therefore, it is important for pre-service secondary mathematics teachers to have a solid foundation, and a coherent view of early algebra content in order to develop their overall MKT. This study aims to help PSMT deepen their understanding of algebra topics by engaging in MEAs.

Likewise, although teachers are expected to play a leading role in their students' classroom modeling experiences, modeling has not been thoroughly integrated in teacher education programs nor has it been comprehensively studied in the mathematics teacher education field (Blum, Galbraith, Henn, & Niss, 2007; Zbiek, 2016). After exploring International Community of Teachers of Mathematical Modelling and Applications publications from 2007 onward, examining the contents of Beyond Constructivism (Lesh & Doerr, 2003), and thorough searches of academic databases and search engines using key words such as "mathematical modeling" and "pre-service teachers", it was evident that the number of research studies that focus on mathematical modeling and pre-service teachers is limited. Such studies will be discussed in greater detail in Chapter two, the paragraph below is included to demonstrate the relevance for the inclusion of mathematical modeling in teacher education programs.

Zbiek (1998) explored 13 pre-service teachers' solutions of mathematical modeling tasks through the use of computing tools. She found that engaging in the mathematical modeling process can help pre-service teachers enhance their understanding of mathematics by seeking and learning new mathematical concepts and procedures while refining their mathematical models. Similarly, Zbiek and Conner (2006) found that mathematical modeling tasks can influence pre-service teachers' learning through effects on motivation and changes in understanding. Specifically, they revealed that mathematical modeling tasks helped pre-service teachers connect known mathematics to new contexts through the process of moving back and forth between "the real-world phenomenon" and "mathematical world" (p. 103). Gould (2013) examined teachers' (both pre-service and in-service) understanding of mathematical modeling, and found that teachers held many misconceptions regarding mathematical modeling, such as what the mathematical modeling cycle looks like, and the role of the real-world within the cycle.

While these studies begin to illuminate how learners (K-12 students, pre-service and practicing teachers) engage in mathematical modeling, there are still many unanswered questions about the teaching and learning of mathematical modeling in the U.S., specifically concerning pre-service teachers (Cai et al., 2014) and their mathematical knowledge for teaching (Phillips, 2016).

Statement of the Problem

Through the years, mathematical modeling has been shown to support the development of mathematical competencies that help students solve problems arising in everyday life, society, and the workplace (Blum & Ferri 2009; Lesh et al., 2000; Maiorca & Stohlmann, 2016; Niss, 1996; Zbiek, 1998), and the modeling process has also been shown to provide an avenue through which students can learn mathematics in profound ways (Zbiek, 1998; Zbiek & Conner, 2006). With the implementation of the CCSSM (or a variation of them) in the majority of the states in the U.S., mathematical modeling has come to the forefront of mathematics classrooms across the nation. As a result, teachers are expected to support their students' through mathematical modeling experiences. However, teachers have not experienced modeling thoroughly themselves in their preparation, because mathematics teacher education programs seldom integrate mathematical modeling (Blum, Galbraith, Henn, & Niss, 2007).

Moreover, mathematical modeling provides alternative ways of thinking about mathematics teaching and learning, ways in which teachers can produce useful conceptual tools that have powerful implications in the context of decision-making issues (Lesh & Doerr, 2003). This means that unique challenges may arise for teachers who do not have the necessary experience in the practice of modeling (Phillips, 2016). Although some of the research studies discussed above begin to reveal how learners engage in mathematical modeling and how mathematics is supported by engaging in modeling tasks, there is a gap between how mathematical modeling can help support pre-service teachers and pre-service teachers' mathematical knowledge for teaching (MKT).

It is important to explore modeling in relation to MKT because many studies have linked teachers' MKT to the effectiveness of mathematics instruction, (Baumert et al., 2010; Eisenhart et al., 1993; Hill et al., 2008) as well as achievement gains for students (Hill, Rowan, and Ball, 2005). Many researchers have argued that since MKT is a type of knowledge that can be applied and assessed, it can also be learned through increased experiences with tasks that focus on the application of mathematical knowledge to the work of teaching (Bass, 2005; Stylianides & Stylianides, 2014). It has also been suggested that particular mathematical tasks for teaching have the potential of eliciting various types of knowledge by teachers (Hill, Dean, & Goffney, 2007). Interestingly, Doerr and Lesh (2003) described MEAs as mathematical tasks that require the application of mathematical knowledge to a context of interpretations and analyzing, resembling the work of a teacher. Similar to mathematical modeling in mathematics education programs, there also exists a limited amount of research to help inform decisions about mathematical knowledge for teaching algebra topics in the secondary setting (McCrory et al., 2012), even though algebra is of particular importance in K-16 because it serves as a foundation and a gatekeeper for later mathematics courses (Ferrini-Mundy et al., 2012).

Accordingly, this study aims to address this gap by exploring the relationship between engagement in mathematical modeling and pre-service secondary mathematics teachers' MKT algebra. The purpose of this research is to answer the following question—*What is the nature of* *the relationship between engagement in model-eliciting activities (MEAs) and pre-service secondary mathematics teachers'(PSMT) mathematical knowledge for teaching (MKT) algebra?*

This introduction included a background of the relevance of this research study, in particular: the value of mathematical modeling tasks, such as model-eliciting activities, for attaining a deeper understanding of mathematics, the need for engaging pre-service teachers in mathematical modeling, and the importance of mathematical knowledge for teaching. Next, the literature related to topics in the current study will be discussed, including previous research on the various perspectives of mathematical modeling, previous research on mathematical modeling with pre-service teachers, and the literature on the needed mathematical knowledge for teaching. The design of the current study, instruments used, the sample of participants, and the mathematical modeling tasks will be described in Chapter three. Data analysis and results will be shared in Chapters four, five and six. Finally, the findings will be discussed in Chapter seven along with limitations, implications and directions for future research.

CHAPTER TWO: LITERATURE REVIEW

The purpose of this research study is to explore the nature of the relationship between pre-service secondary mathematics teachers' (PSMT) mathematical knowledge for teaching (MKT) algebra and engagement in Model-Eliciting Activities (MEAs). Accordingly, this chapter will provide a review of the literature on mathematical modeling and mathematical knowledge for teaching within the field of mathematics education. More specifically, this chapter will review research about the different mathematical modeling perspectives, mathematical modeling with pre-service teachers, teacher knowledge, and teachers' knowledge of algebra. At the end of the chapter, the study's conceptual framework, its relevance, and the manner in which the study aims to address a gap in the literature will be discussed.

Multiple Perspectives on Mathematical Modeling

Although many researchers agree that mathematical modeling should be integrated into mathematics classrooms, further conceptualization of how that can be done has not yet been developed (Kaiser & Sriraman, 2006; Lesh & Fennewald, 2010). The different outlooks on mathematical modeling are influenced by the perceived goal(s) of the modeling process along with the theoretical beliefs of the researchers (Ferri, 2013). Such differences often lead to various definitions, frameworks, research foci, and implications.

The differences in the goals behind selecting specific mathematical modeling tasks serve as defining characteristics in the different perspectives. Niss, Blum, and Galbraith (2007) along with Julie and Mudaly (2007) categorize and examine these differences in goals as, (1) facilitating the learning of mathematics and (2) exploring non-mathematical scenarios using mathematics as a tool. These classifications are not meant to represent a dichotomy, rather they should be seen as a duality, because it is possible to work towards different levels of each goal at the same time (Niss, Blum, & Galbraith, 2007).

The first category is described by Niss, Blum, and Galbraith (2007) as “modeling for the learning of mathematics” (p. 6) and by Julie and Mudaly (2007) describe it as “modeling as a vehicle” (p. 504). The end goal of this category is for the learner to gain a deeper understanding of mathematics through the use of contextual situations. However, the end goals for the second category are not explicitly learning the mathematics itself, rather the ability to go through the modeling process. Niss, Blum and Galbraith (2007) describe this learning outcome as the application of mathematics to solve problems outside of mathematics. Conversely, Julie and Mudaly (2007) call this category “modeling as content” (p. 504) and state that modeling itself is the learning goal without a specific focus on the mathematical content.

The following section is included to compare the different views of mathematical modeling, because there is not a single-agreed upon definition of mathematical modeling, or how it is done. Rather, researchers have put forth different definitions, descriptions, and assumptions of what mathematical modeling entails (Cirillo, Pelesko, Felton-Koestler, & Rubel, 2016). Accordingly, the subsequent section describes: (a) the goals of mathematical modeling, (b) the definition of a mathematical model, (c) the mathematical modeling cycle, and (d) the design of the modeling task in an effort to show the rich and diverse background of mathematical modeling, which provides a thorough context for this study.

Five Mathematical Modeling Perspectives

Along with the different purposes of mathematical modeling, the theoretical beliefs of the researcher play a key role in the varying perspectives on mathematical modeling (Ferri, 2013). Although many researchers agree that mathematical modeling has roots in the works of Piaget, Vygotsky, and pragmatists, other researchers have shown strong ethno-mathematics and socio-cultural influences in their works (Confrey & Maloney, 2006; Kaiser & Sriraman, 2006; Lesh & Doerr, 2003). Such influences create both subtle and foundational differences in the various frameworks and definitions of mathematical modeling.

Using the aforementioned categories of goals and varying theoretical backgrounds, Kaiser & Sriraman (2006) developed five broad classifications for the different perspectives within mathematical modeling research, they include:

1. Realistic modeling.
2. Educational modeling.
3. Socio-Critical modeling.
4. Epistemological modeling.
5. Models and modeling perspective, or contextual modeling.

It is important to note that these perspectives are not exclusive, meaning many researchers have worked and published in multiple perspectives (Kaiser, Sriraman, Blomhøj, & Garcia, 2007).

Realistic Modeling

When it comes to realistic modeling, the most defining aspect focuses on the goal of mathematical modeling, which is solving real-world problems. This perspective suggests that students must work with *realistic, authentic* and *messy* tasks. Kaiser, Schwarz, and Buchholtz (2011) describe such tasks as "problems that are only a little simplified and … recognized by people working in this field as being a problem they might meet in their daily work" (p. 592). These authentic tasks should require students to go through the mathematical modeling cycle because one of the goals of this perspective is to support students' ability to apply the modeling process. The main goal of this perspective is that the learner will develop the ability to solve real-life problems through engaging in these authentic tasks (Kaiser & Sriraman, 2006).

This perspective emphasizes the application of the mathematical modeling process, but there is not a single agreed-upon cycle that researchers use. An example of a modeling cycle with roots in this perspective was developed by Blum (2011), which is a modified version of Blum & Leiss (2007). This mathematical modeling cycle has been adapted by various researchers within this perspective (Biccard & Wessels, 2011; Frejd & Arleback, 2011; Ludwig & Reit, 2013). In this modeling cycle, the modeler begins with the given *real situation and problem*. Subsequently, once the problem is understood, a *situation model* is constructed. Then, the modeler simplifies the problem by identifying key aspects or variables to structure a *real model*. This step is critical because the original modeling task is presented in a *real, authentic, and messy* context. Next, *mathematization* occurs in which the real model is transformed into a *mathematical model* (such as an equation, a graph, etc.). This mathematical model allows the modeler to work with the mathematics and produce some *mathematical results*. Once the mathematical results are produced, they must be interpreted back into the real situation, which yields *real results*. Once the real results are produced, validation of these results must occur in order to report or expose the results of the given task, or the modeler goes back and creates another situation model (Blum, 2011).

Figure 2 shows mathematical modeling steps and sub-processes—the steps result in products of the modeling cycle (such as a situation model, mathematical results, and real results), and the sub-processes explain the actions of the modeler during the modeling process (numbered

1-7 on the right side of the diagram). This allows the modeler's activities to be highlighted through each part of the modeling cycle (Blomhøj & Jensen, 2003).

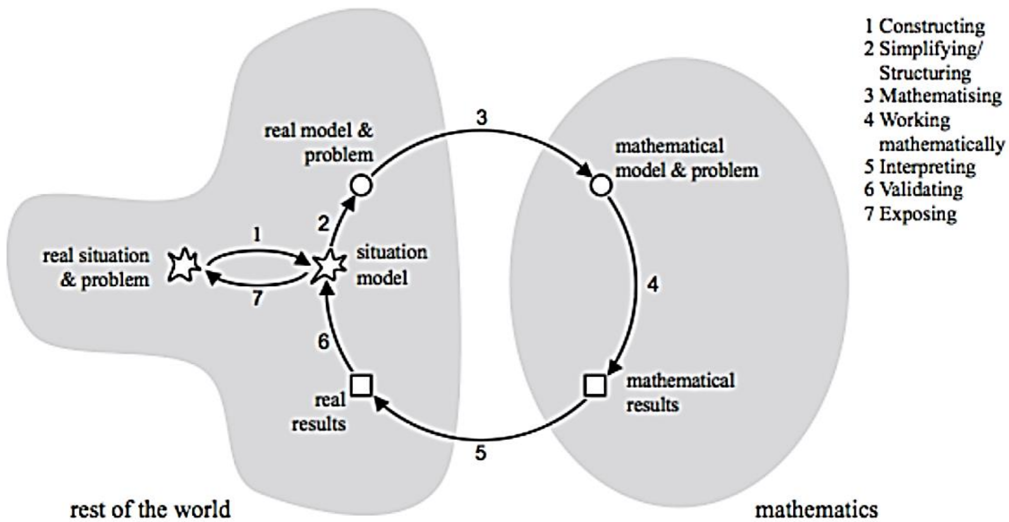


Figure 2. A Modeling Cycle from the Realistic Perspective (Blum, 2011, p. 18).

Educational Modeling

The goal of educational modeling is not only to develop mathematical modeling competencies (like realistic modeling), but also to learn mathematics (Blomhøj, 2009). Within the educational modeling perspective both "modeling as content" and "modeling as a vehicle" (Julie & Mudaly, 2007) are simultaneously emphasized. Since this perspective highlights the relationship between the real-world and mathematics as a vital element in the teaching and learning of mathematics, Kaiser and Sriraman (2006) described it as a continuation of Hans Freudenthal's late work (1973, 1981) in the field of mathematics education. The dual focus on mathematical modeling competencies and mathematics in educational modeling may result in the original tasks being more simplified than the messy authentic tasks described in the realistic modeling perspective. Furthermore, this view of modeling uses contexts to develop mathematical concepts, which leads to mathematical modeling. In other words, the goal of mathematical modeling tasks in this perspective is to motivate the need for learning mathematics (Kaiser, Sriraman, Blomhøj & Garcia, 2007).

Mathematical models are created in the educational perspective when the real problem is analyzed by the modelers relative to their interests, which leads to a real model that contains the necessary characteristics of the original real situation. Then, the real model is translated into mathematics, or mathematized. When the work is complete within the mathematics world, and conclusions are drawn, they must be "re-translated" back to the real-world in order to validate the model. This modeling process may require the modeler to go through the cycle several times (Blum & Niss, 1991). This process is very similar to the mathematical modeling process described in the realistic perspective, which ought to be expected since both share a common goal of advocating for the importance of modeling competencies. However, the two outlooks are separated by the notion that researchers within the educational perspective also examine mathematical learning not only mathematical modeling.

Seminal work in educational modeling includes research conducted by Zbiek and Conner (2006). In their study, the two researchers developed a "diagrammatic model of mathematical modeling as a process that allows for mathematical understandings to be identified as learners are engaged in modeling tasks." (Zbiek & Conner, 2006, p. 89). Their diagrammatic (Figure 3 below) demonstrates how the modeler transitions between the two worlds— the real-world and the mathematical one. The boxes in the diagram represent steps within the modeling cycle and the two-direction arrows represent the sub-processes. The mathematical modeling process begins with the real-world, where students explore the real-world situation they are given. In the real-world, modelers *specify*—or identify the conditions and assumptions (C&A) of the given scenario and *observe mathematically*—or examine what happens while exploring with mathematics. This includes identifying variables and constraints in the given situation and using mathematical concepts to describe the problem. Then, students identify mathematical ideas—properties and parameters (P&P), they can relate to the mathematical world, or entity. This process is called *mathematizing* and it results in moving from the real-world to the mathematical entity. Once the modeler is in the mathematical world, *combining* takes places, in which properties and parameters are explained and justified to match the mathematical entity. At this point, modelers can also *analyze*— or mathematically manipulate, and *associate*—connect to the real-world. The last step is for the modeler to move back to the real-world by going through three activities: (1) *highlighting*—clearly describe any unidentified properties and parameters on the mathematical entity; (2) *interpreting*—translate mathematical results into the real-world context; and (3) *examining*— validate conclusions with the goal of the task. Throughout the entire process, the modelers are expected to apply the two sub processes of *aligning*—constantly compare the current state to the former ones and *communicating*—put forth ideas about the task between modelers (Zbiek & Conner, 2006).

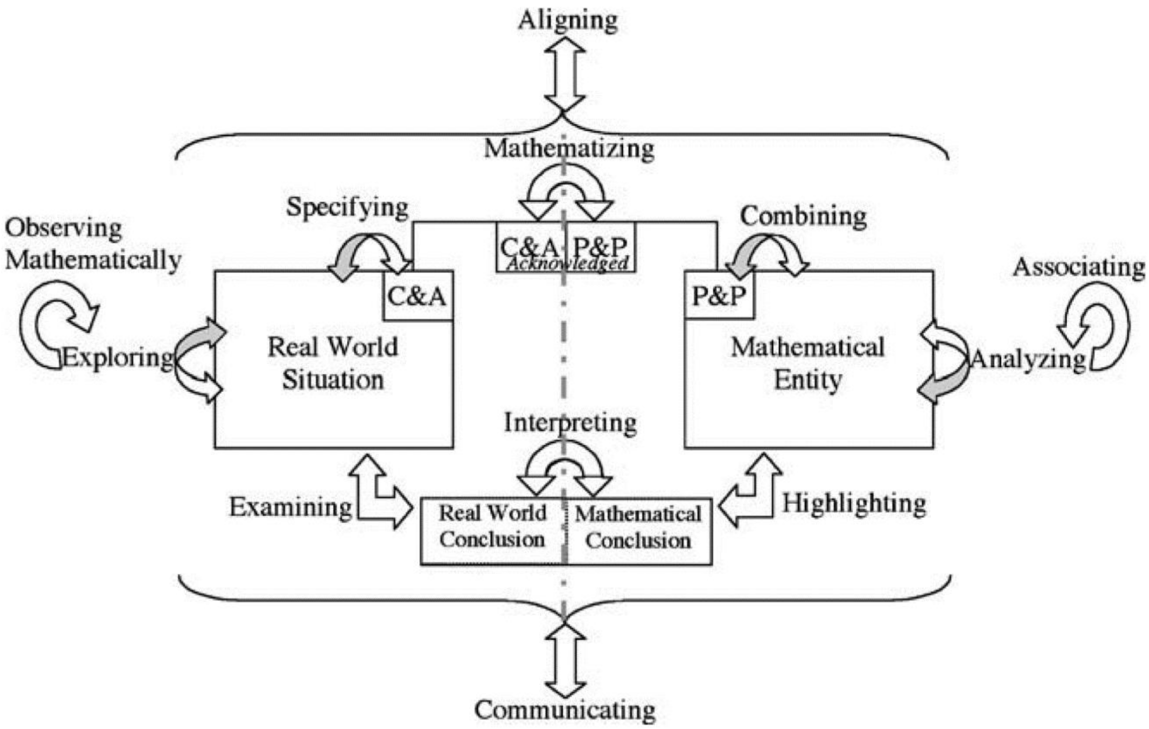


Figure 3. A Modeling Cycle from the Educational Perspective (Zbiek & Conner, 2006, p. 98).

Comparing the two mathematical cycles described above (Blum & Niss, 1991; Zbiek & Conner, 2006), one can see that both begin in the real-world, mathematizing takes place to create a mathematical model, a mathematical solution is found and translated back into the real-world and then validated. This perspective is also similar to the realistic perspective discussed above in that the mathematical modeling task should be authentic and realistic.

Socio-Critical Modeling

The socio-critical modeling perspective focuses on the societal role that mathematics can play. This can be described as *modeling as critic* (Barbosa, 2006) with the goal of using mathematical concepts to make decisions in society. The goals of this perspective lie inside and outside of mathematics. The main goal within mathematics is to demonstrate the power of mathematics in making decisions, and how it can be used as a tool to make decisions. However, the end goal of this perspective is to make decisions about society and to critically understand the mathematical modeling process (Kaiser et al., 2007).

Socio-Critical modeling advocates for the learning of mathematics and mathematical modeling in a manner that satisfies both objectives (Barbosa, 2009). Mathematical models in this perspective are not neutral in nature, because they depend on their development process, and the development process depends on how the modelers see and understand the problem-situation, as well as how they adapt mathematical concepts into the situation. Therefore, this biased nature of mathematical modeling creates an authentic relationship between the mathematical model and the conditions applied to develop it (Barbosa, 2009). Additionally, this perspective brings up the idea of power behind mathematics, since mathematical modeling is used to make decisions in society, modelers who are able to both create, and critique mathematical models hold the power. Therefore, this perspective promotes mathematical modeling for students in order to develop the skills to become independent citizens (Blomhøj, 2009).

Furthermore, mathematical modeling in this perspective is defined as "a learning milieu" in which students investigate a real-world problem via mathematics (Barbosa, 2006, p. 294). The modeling process is defined as a school activity that is taken from everyday contexts (Barbosa, 2006), which is a much broader definition than how the other perspectives describe the modeling process. This perspective does not describe how the modeling cycle takes place, it simply describes the mathematical model as a mathematical representation of a presented real-world situation (Barbosa, 2007).

Epistemological Modeling

The epistemological perspective of mathematical modeling perceives mathematical modeling as a lens to establish general theories for the teaching and learning of mathematics

(Blomhøj, 2009). The only goal for this perspective is the development of mathematical understanding. In fact, unlike any of the other perspectives discussed above, this perspective does not require a real-world problem or situation to be modeled. Kaiser and Sriraman (2006) explain that in the epistemological perspective, every mathematical task can be described as a modeling task, and modeling is not limited to the mathematizing of non-mathematics problems. Importantly, modeling activities in this perspective often begin in the real-world but end in mathematics (Kaiser & Sriraman, 2006).

Emergent modeling is a subcategory of epistemological modeling, and it was developed by Koeno Gravemeijer and his colleagues through the Realistic Mathematics Education (RME) learning theory (Kaiser et al., 2007). The central goal for this type of modeling is focused on the learning process. The initial model that is developed by the learner is founded through an experience, and over time, the learner is able to recognize the model as it transitions from an informal experience to formal mathematical understanding (Gravemeijer, 2007). This outlook is rooted in the idea that mathematics should be connected to the learners' experiences, and embedded in human activity (Doorman & Gravemeijer, 2009). Moreover, Gravemeijer (2002) explains that the emergent modeling perspective was established as a result of the application of external representations in mathematics education that later develop intrinsic meaning.

Models and Modeling Perspective (MMP)/Contextual Modeling

The following mathematical modeling perspective is called *contextual perspective* in Kaiser and Sriraman (2006) and *models and modeling perspective* (MMP) by Lesh and Doerr (2003). It is also called model-eliciting perspective by Kaiser and Sriraman (2006). Nonetheless, this perspective describes a mathematical modeling framework that was developed by Lesh and colleagues; accordingly, this perspective will be referred to as the MMP from this point onwards.

The MMP adapts the goal of problem solving to contain the use of mathematical concepts and developing a deeper meaning to those concepts (Lesh & Doerr, 2003). This reveals an overlap to the goal of modeling in the educational perspectives because modeling in this perspective is also viewed as content as well as a vehicle. However, there is a stronger emphasis on the learning of mathematics (Kaiser & Sriraman, 2006). The fundamental difference between the models and modeling perspective and the educational perspective is the emphasis on model-eliciting activities (MEAs) and what constitutes a mathematical model. The MMP links mathematical modeling, and the learning that occurs through it, strongly to learning theories. Lesh and his colleagues (Lesh & Doerr, 2003; Lesh, Hoover, Hole, Kelly & Post, 2000) have been influenced by Piaget, and this influence is represented in their description of the mathematical modeling process and definition of models as:

> conceptual systems (consisting of elements, relations, operations, and rules governing interactions) that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviors of other system(s)—perhaps so that the other system can be manipulated or predicted intelligently… [and a mathematical model] focuses on structural characteristics of the relevant systems (Lesh & Doerr, 2003, p. 10).

This definition differs from the previously discussed perspectives in that both the realistic and the educational perspectives view mathematical models as mathematization of the real-world problem, while in this perspective, a mathematical model is described as a conceptual tool of a mathematical system that develops from a specific real-world situation (Lesh et al., 2003). Rather than viewing mathematical models as mathematical objects relating to a given real-world scenario like educational modeling, the MMP views them as systems, composed of different concepts, that map different characteristics of related systems.

Model-Eliciting Activities

As previously stated, in the MMP, there is a strong emphasis on MEAs. In order to construct mathematical models, as they are defined above, the real-world tasks in this perspective are specifically designed to promote mathematical development. These real-world tasks are called Model-Eliciting Activities (MEAs). MEAs are purposefully tailored around incorporating the mathematics process in the produced model. This serves as a major difference from mathematical modeling tasks within the realistic and education perspectives, because the goal of the design of an MEA is that through engaging in the task, the constructed model represents the mathematical process (Kaiser & Sriraman, 2006). As a result of the important implications of MEAs, Lesh and colleagues developed six guiding principles for designing them. These principles (which are summarized in Table 1 below) include: (1) the model construction principle; (2) the reality principle, (3) the self-assessment principle, (4) the model documentation principle, (5) the generalizability principle, and (6) the effective prototype principle (Lesh et al., 2000). According to the *model construction principle,* the task should require modelers to describe how they develop their mathematical model. The *reality principle* requires the task to be authentic and take place in the real-world. The authenticity of MEAs differ from the way authentic tasks are described in the realistic and educational perspectives, in that the problem does not have to be “real in an absolute sense” (Lesh et al., 2000, p. 616). Rather, tasks need to be set in the real-world but not necessarily be a problem or situation that the modeler encounters in their daily life. The *self-assessment principle* requires the modelers to validate and be critics of their own work. Similar to the model construction principle, the *model documentation principle* requires the modelers to produce somewhat of an audit trail of their work. The *generalizability* *principle* asks the modelers to form a model that is open enough that it can be applied, or generalized, to similar situations. Lastly, the *effective prototype principle* requires the constructed model to be as simple as possible yet be a powerful mathematical tool to understand a complex situation.

Table 1

Six guiding principles of MEAs

Principle	Description of the Principle's Requirements
1. Model Construction	Explicit construction, description, explanation and justified predictions are required. The modeler should be able to quantify, coordinate, make predictions and identify patterns or trends.
2. Reality	Meaningful and relevant problems. Tasks must be based on real, or slightly modified real data. Modelers must function as scientists or engineers that are working to solve a specific problem.
3. Self-Assessment	The task should promote selection, refinement, and elaboration of a model through a clear purpose and clear criteria of when a solution is achieved. Students should be able to detect deficiencies, compare alternatives, integrate strengths and minimize weaknesses through assessing adaptions of models.
4. Model-Documentation	Students must be able to reveal their thinking processes within their solution through descriptions of their assumptions, goals and solution paths. Metacognition is supported in this principle.
5. Generalizability	The model should represent a general way of thinking instead of specific solutions for a specific context. Models should be communicated in a clear and understandable manner so that they can be used by others.
6. Effective Prototype	The model must be simple but powerful. The model should avoid the need for numerous procedures, especially those that can circumvent conceptual understanding of mathematical topics.

Lesh et al. (2000)

The modeling cycle in this perspective is iterative and similar to those discussed previously in the realistic and educational perspectives. Lesh and Doerr (2003) described this iterative process as follows: it begins in the real (or imagined) world and through description, a model is constructed. Then, through manipulation of the model, a possible solution to the original task is achieved. Through prediction, the results are translated back into the real-world, where verification takes place to confirm the solution. This process is depicted in Figure 4 below.

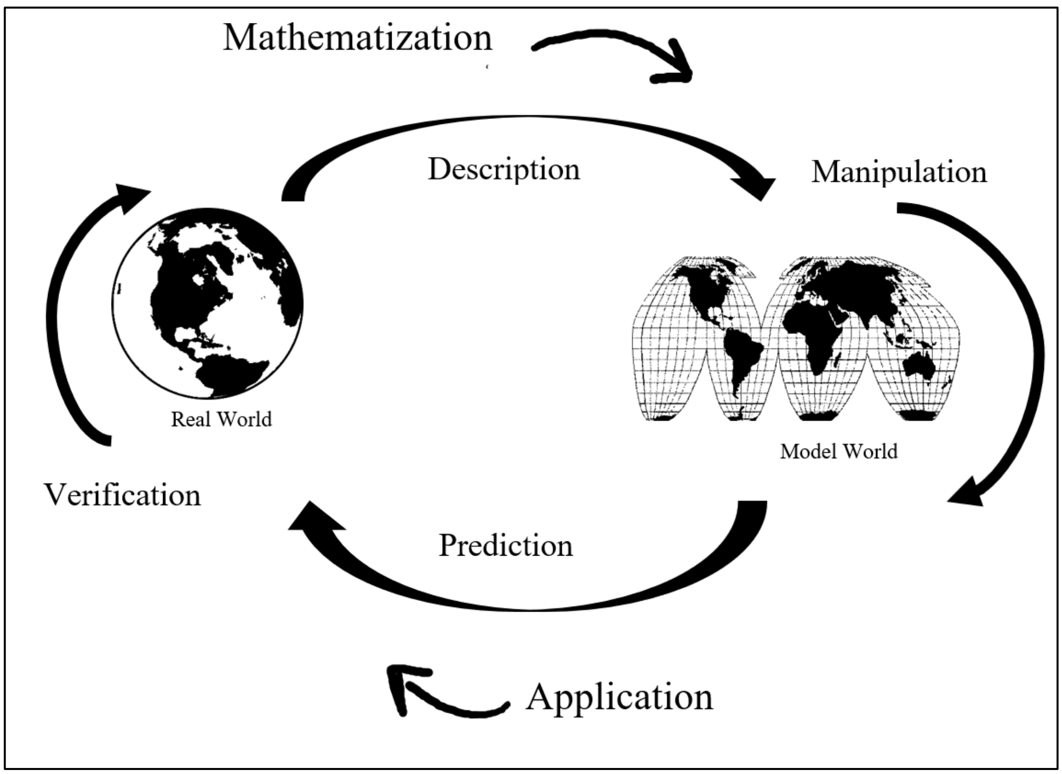


Figure 4. A Modeling Cycle from the Models and Modeling Perspective (adapted from Lesh & Doerr, 2003).

MMP in this Study

Research within the MMP requires the use of model-eliciting activities with a focus on the use of MEAs to develop and advance mathematical reasoning and understanding (Kaiser & Sriraman, 2006). Doerr and Lesh (2011) explained that the use of MEAs is critical to investigate learners' mathematical thinking. In fact, Lesh and colleagues (2000) refer to MEAs as thought-revealing activities and described them as being useful to "learn more about the nature of students' (or teachers') developing knowledge, it is productive to focus on tasks in which the products that are generated reveal significant information about the ways of thinking that produced them." (p. 592). The research focus in this perspective emphasizes mathematical reasoning and understanding, which differs from the educational perspective, because modeling competencies are not of major concern, they are only an inherent aspect of the perspective (Blomhøj, 2009).

This research study is focused on the development and advancement of PSMT' mathematical knowledge, reasoning and understanding, and specifically their MKT, through the use of MEAs. MEAs were chosen to be the form of mathematical modeling tasks incorporated in this study, because not only do MEAs help provide a foundation for the development of powerful mathematical reasoning in pre-service teachers (Lesh et al., 2008), but they also provide a way for teachers to interpret their teaching and learning of mathematics (Doerr & Lesh, 2003). In fact, Doerr and Lesh explained that MEAs can effectively guide teacher educators in helping pre-service teachers develop their teacher knowledge through interpreting mathematics (2003). MEAs also provide opportunities for PSMT to "express, test, revise, and refine their content knowledge and extend that knowledge to more powerful forms for classroom teaching" (Doerr & Lesh, 2003, p. 130). Because this research study is focused on exploring the use of MEAs in relation to pre-service teachers' MKT—a form of mathematical knowledge related to teaching, the MMP was used as the mathematical modeling lens.

Summary of the Five Perspectives of Mathematical Modeling

The mathematical modeling perspectives discussed above reveal that the mathematical modeling field has a diverse and extensive background. Even though the perspectives vary in many key aspects, nonetheless, they do have some common grounds. Table 2 displays a summary of the main features of each perspective, in terms of its goals, its definition of a mathematical model, its description of the mathematical modeling cycle, the design of the modeling task, key researchers, and its research focus. The development of the table was guided by the works of Kaiser and Sriraman (2006), and Blomhøj (2009). These perspectives are foundational to many research studies in the field of mathematical modeling, and mathematics education in general.

Table 2

Summary of the main features of the five mathematical modeling perspectives.

Perspective	Realistic	Educational	Socio-Critical	Epistemology	Models and Modeling
Goals	Develop skills to model and understand authentic, real-world, scenarios.	Develop skills to model a real-world scenario and understand mathematics.	Develop mathematical modeling skills in order to make decisions in society.	Develop formal mathematical reasoning.	Develop a deep understanding of mathematics through a modeling context.
Definition of a Mathematical Model	Mathematical objects (graphs, equations, etc.) explain the given real-world scenario.	Mathematical objects that have a relationship to the given real-world scenario.	Mathematical representation of a relevant scenario.	The result of an activity based on situations and mathematical concepts.	A conceptual system that maps the structural characteristics of a relevant-system.
Description of the Mathematical Modeling Cycle	A cyclical, multi-step process. Begins in the real-world, mathematized into the mathematical world, and ends in the real-world.	A cyclical, multi-step process that begins in the real-world, mathematized into the real-world, then ends in the real-world.	All aspects of the modeler's involvement in the exploration of a real-world problem using mathematics.	Four stages of activities. Models-of and Models-for are created to develop formal mathematical reasoning.	A cycle that begins in the real-world, once a model is developed, it goes back to the real-world. The cycle is repeated as many times as needed.
Design of Task	Authentic, messy, real-life tasks that require the use of the modeling cycle.	Authentic tasks that can be simplified to reveal specific mathematical goals.	Tasks are in societal context, but should also focus on the development of specific mathematics concepts.	No set requirements.	MEAs designed to develop a specific mathematical concept. Must be in context of a real-world problem. Must meet 6 guiding principles.
Key Researchers	Pollak	Niss, Blum, Zbiek	D'Ambrosio, Barbosa, Skovsmose	Freudenthal, Gravemeijer	Lesh, Doerr
Research Focus	Mathematical modeling competencies	Mathematics in mathematical modeling and mathematical modeling in curricula.	Learners' use of mathematics to critically understand society.	Teaching and learning of specific mathematical concepts.	The use of MEAs to teach mathematics.

Mathematical Modeling with Pre-service teachers

With the different perspectives of mathematical modeling presented above, this section of the literature review will present research on mathematical modeling with pre-service teachers, and they will be positioned within those aforementioned perspectives. Because the scope of this study is focused on pre-service teachers' (PSMT) learning in context of mathematical modeling, this section will focus on PSMT' learning of mathematical modeling competencies, as well as their mathematical learning during the mathematical modeling process. As mentioned in chapter one, after exploring the International Community of Teachers of Mathematical Modelling and Applications (ICTMMA) publications from 2007 onward, examining the contents of *Beyond Constructivism* (Lesh & Doerr, 2003), and thorough searches of academic databases and search engines using key words like *mathematical modeling* and *pre-service* (and *prospective*) *teachers,* it was evident that there are gaps in research focused on mathematical modeling and pre-service teachers. Thus, studies that took place in undergraduate mathematics courses were also included, in addition to studies that took mathematics education courses. The findings are described below, and they are organized into two categories—studies with a focus on mathematical modeling competencies (or modeling as content) and mathematical learning (or modeling as a *vehicle*). Moreover, it should also be noted that studies that focused on pre-service teachers' beliefs about mathematical modeling were excluded from this literature review because beliefs are not examined in this study.

Pre-service teachers' Mathematical Modeling Competencies

Because the research studies discussed in this section focus on mathematical modeling competencies, they are positioned within the realistic modeling perspective or the educational modeling perspective. This section is included to review PSMT' knowledge and exposure to mathematical modeling. This is important because one of the rationales for conducting this research study is to engage PSMT in MEAs, in order to gain some experience with mathematical modeling tasks and the mathematical modeling process.

Haines, Crouch, and Davis (2000) developed an instrument to examine modelers' ability to go through different sub-processes of the modeling cycle. The authors describe sub-processes as identifying and applying assumptions to simplify the given real-world problem, discerning the goal of solving the problem, developing the mathematical problem, selecting variables, developing mathematical statements, and choosing appropriate models. The instrument is composed of twelve multiple choice questions and each question has five answer choices. The instrument is divided into two parts, a pre-test and a post-test, and each part included six questions. In the instrument, each question had one preferred answer that was given a two-point rating, the other appropriate possibilities were given a one-point rating, and choices that were not appropriate were given zero points. In 2001, Haines and colleagues expanded the instrument and included a graphical representation category, and a category for connecting the real and mathematical world. The researchers used the instrument with 39 undergraduate mathematics students, and they concluded that their participants struggled in making assumptions to simplify the problem (28% scored a zero) as well as choosing appropriate models (31% scored a zero) (Haines et al., 2001).

Kaiser (2007) used the instrument with pre-service teachers in a university in Germany, where she evaluated a course in which 57 PSMT facilitated mathematical modeling tasks with secondary students. At the beginning of the course, mathematical modeling was introduced, then the pre-service teachers facilitated the modeling tasks with the students, and after, the PSMT discussed students' work, their experiences and challenges. The PSMT were given the instrument discussed above as a pre- and a post-test, and it was found that their scores significantly increased. The researcher concluded that this increase in scores demonstrated that pre-service teachers' modeling competencies improved as a result of the course. The different mathematical modeling sub processes were not discussed separately.

Kuntze (2011) created an instrument that explored teachers' perception of mathematical modeling through four-point Likert scale items. The instrument also included questions that looked at the learning potential of mathematical modeling tasks with *lower modeling requirements* and *higher modeling requirements*. Lower modeling requirements were described as tasks where only one correct model is possible, while tasks with higher modeling requirements were described as tasks with more than one possible solution and at least one *translation step* between the real-world problem and the mathematical model. Kuntze was interested in measuring teachers' views of the mathematical modeling cycle as well as their preference in the type of modeling tasks. The researcher administered the instrument in Germany and surveyed 79 teachers and 230 pre-service teachers. Kuntz found that in-service teachers favored tasks with higher modeling requirements, while pre-service teachers preferred tasks with lower modeling requirements. Additionally, the participants' scores on the four-point Likert scale items revealed that their understanding of mathematical modeling was not strong.

Similarly, Gould (2013) developed a survey with 20 questions to measure teachers' views of mathematical modeling. The questionnaire was composed of eight questions about the mathematical modeling cycle, six questions about the definition of a mathematical model, and six questions about what mathematical modeling looks like in the classroom. The researcher surveyed 274 secondary mathematics teachers (254 in-service teachers and 20 pre-service teachers) across 35 states in the U.S. The findings of Gould's study demonstrate that the majority of secondary mathematics teachers in the U.S. have misconceptions about mathematical modeling, and specifically, the participating teachers did not understand the purpose behind the *making assumptions* aspect of the mathematical modeling cycle, or the real-world context requirements behind the mathematical modeling task. In comparing the two groups of participating teachers, Gould found that experienced teachers (teachers with at least five years of teaching experience) were less likely to think that mathematical models can explain the causes of a given situation; while inexperienced teachers (pre-service teachers and teachers with less than five years of teaching experience) were more likely to have misconceptions with *validating* mathematical models.

Thomas and Hart (2010) investigated how 16 elementary school pre-service teachers explore a model-eliciting activity (MEA). The pre-service teachers worked in groups, and the collected data included their work samples, focus group discussions, as well as an audio-recording of one of the group's discussion while engaged in the task. The researchers revealed two important results: first, pre-service teachers struggled with the inherent ambiguity of the task; second, pre-service teachers' views of mathematical modeling affected their experiences with mathematical modeling. Similarly, Winter and Venkat (2013) found that pre-service teachers struggled with the *mathematizing* aspect of the modeling cycle, in their exploration of pre-service teachers in South Africa. In this study, data were collected in a methods course that entailed a three-month unit of content development and mathematical modeling.

Furthermore, Widjaja (2013) explored secondary pre-service teachers' mathematical modeling competencies by giving them a specific task called *Re-designing Parking Space Project*, in which the PSMT were asked to re-design a parking space of their own in a way that makes the best use of the space. The students were given the following steps to guide their completion of the project:

> (1) Sketch a rough plan of the parking space of your own choice. (2) List factors or variables in the problem. (3) Discuss possible designs for the parking lot. (4) Design a new parking lot that caters for an optimum parking space. (5) Derive a mathematical model from the initial design to the proposed design. (6) Identify the limitations of the proposed model for the parking space. (Widjaja, 2013, p. 587).

The groups' written reports were collected as data, and the findings revealed that PSMT struggled in clarifying the assumptions, and as a result, also struggled in identifying limitations of their models.

Tan and Ang (2013) examined how 24 secondary pre-service teachers' knowledge of mathematical modeling developed as they engage in mathematical modeling tasks. The data in this study consisted of PSMT' experiences that were recorded during a six-week mathematical modeling unit. The experiences were a collection of field notes of class discussions, PSMT' reflections, artifacts, and responses to a questionnaire. The analysis of the data was done by looking at pre-service teachers' misconceptions and areas in which they struggled. The findings of the study demonstrated that even though PSMT were engaged in the mathematical modeling cycle, their knowledge of mathematical modeling did not change.

Pre-service teachers' Mathematical Learning via Mathematical Modeling

The studies reviewed in this section describe mathematical learning through the engagement of mathematical modeling. Accordingly, the studies fall under either the models and modeling perspective, or the educational perspective, and they are organized as such below. This section is included to demonstrate the potential of incorporating mathematical modeling tasks in teacher education programs, by discussing how such tasks can help PSMT acquire a deeper understanding of various mathematical topics and develop both their subject matter knowledge as well as their pedagogical content knowledge.

Research with an Educational Modeling Perspective

Zbiek (1998) explored secondary pre-service teachers' solutions of a mathematical modeling task through the use of computing tools, like curve fitters, function graphers, and symbolic manipulators. The study focused on the types of strategies used by 13 PSMT and the tools they used to validate functions as mathematical models. The data were collected from multiple sources including interviews, class and lab observations, and PSMT' work documents. The researcher found that students can engage in tasks that address some parts of the modeling process on a regular basis, and such work can help them enhance their understanding of mathematics and revisit previously learned mathematics. She also revealed that students can be prompted both explicitly and implicitly to refine mathematical models in a way that they seek and learn new mathematical concepts and procedures.

Similarly, Zbiek and Conner (2006) explored how pre-service teachers, enrolled in a mathematical modeling course designed specifically for secondary PSMT, learn mathematics beyond modeling. The data were collected during the course through interviews with the PSMT and examination of PSMT' work documents, and the analysis focused on investigating how mathematical learning takes places in the sub-processes of the mathematical modeling cycle. The PSMT were given the following task (along with a map of a portion of the northwestern area of the United States):

> Boise (Idaho), Helena (Montana), and Salt Lake City (Utah) are three large cities in the northwestern part of the United States. While each city has medical facilities, imagine the potential of a very high-powered, high-tech, extremely modern medical facility that could be shared by the three cities and their surrounding communities! The map shows the

> locations of the three cities as well as other large cities in the area. Suppose you are hired to determine the best location for the hospital (Zbiek & Conner, 2006, p. 94).

The researchers analyzed the pre-service teachers' changes in understanding by looking at: (1) *the real-world*, which involved observing new mathematical connections and developed mathematical insights through the real-world insights; (2) *the math world*, which involved looking for new mathematics once known mathematics was no longer helpful, and when these concepts were combined and analyzed, conceptual understanding of the mathematics used took place; and (3) *working with others*, which involved aligning why a process works based on a presented solution, and reflection through communication. The results of the study demonstrated that although changes in PSMT' mathematical understanding did occur after working on the mathematical modeling task, these changes were not the same for all participants. In fact, the researchers revealed that different mathematical concepts were developed based on the different parts of the real-world scenario to which the PSMT attended. Furthermore, the researchers found that mathematical modeling can help pre-service teachers connect new contexts to known procedures, which deepens their understanding of mathematics.

Blomhøj and Kjeldsen (2013) explored undergraduate students' mathematical learning while engaged in mathematical modeling tasks. The participants were enrolled in a Differential Equations course—a course that many secondary pre-service teachers are required to take (Wagner, Speer, & Rossa, 2007). The researchers presented the students in the course with the following mathematical modeling task:

> Describe the growth of the world population in the period 1650-1960 and estimate the population in 2100. Some background for the relevance of knowing how the world population will develop, and a set of data for the world population in year 1650-1960, are provided (Blomhøj & Kjeldsen, 2013, p. 145).

Students were also given guiding questions such as: "For explosive growth, the growth rate is proportional to the square of the population size. Can the given data be described as explosive growth?", "What is your estimate of the world population in 2100?", and "Try to find newer data and discuss the model's prediction in relation to these" (Blomhøj & Kjeldsen, 2013, p. 145). The researchers used Sfard's (1991) framework of mathematical development to analyze the students' written responses. The findings of the study revealed that mathematical modeling provides a window for the instructors to get a view of their students' understanding. In fact, the researchers were able to specifically identify the reason behind some of their students' struggles. They explained that "what caused difficulties for the students is shifting between viewing a differential equation as a relation between the momentary rate of change and the actual size of a certain function (here the size of the population) and viewing it as a relation between a function and its derivative" (Blomhøj & Kjeldsen, 2013, p. 151). The researchers also explained that the students' interpretation and validation of their solutions allowed them to gain a conceptual understanding of the differential equations concepts the task entailed.

Research with a Models and Modeling Perspective

Although the research studies in this perspective are similar to the ones discussed above in that they focus on mathematical learning through the modeling process, they differ in the particular mathematical modeling task. As discussed in the prior section of this literature review, research in the MMP requires the use of specific tasks that follow explicit guidelines, called MEAs.

Carlson, Larsen, and Lesh (2003) explored pre-service elementary teachers' understanding of covariational reasoning through three MEAs. During the first MEA, the pre-service teachers were given different scenarios, and were asked to generate graphs to match the scenario, using a motion detector. The second MEA had the pre-service teachers explore an airplane flight on a distance versus time graph. The third MEA gave the pre-service teachers an image of a bottle, and they were asked to explore the volume and height. The researchers revealed that as a result of the negotiations and discussions that occurred between the PSMT, additional insights into their conceptual development and reasoning strategies were acquired.

Likewise, Lesh, Middleton, Caylor, and Gupta (2008) explored both elementary and secondary pre-service teachers' experiences fitting data using an MEA that involved students' test scores and time spent performing a drill. The researchers revealed that participants struggled in fitting the data based on a *curve of best fit* model (a deterministic view), or a model that describes unexplained variation (a probabilistic view).

Doerr and O'Neil (2011) explored undergraduate students' understanding of the rate of change concept using a model development sequence. The sequence began with an MEA in which 33 participants were asked to graph a given scenario using a motion detector (similar to the MEA developed by Carson and colleagues, described above). After being engaged in the MEA, the students explored the concept of the rate of change using various exploration activities that were designed to help students use common, or every day, language to describe and discuss the rate of change. Then, the students were encouraged to reflect back on their own conceptual models that they developed previously during the MEA. Lastly, the students were engaged in tasks that were designed to help them develop a broad understanding of rate of change, as well as promote the development of models for real-world scenarios. The researchers also developed and administered a 17 question pre- and post-test, called *a Rate of Change Concept Inventory*, and revealed that the students' understanding of rate of change was notably enhanced. Furthermore, Doerr and O'Neil analyzed students' models and written responses and found that a common misconception was that students mistook changes in a function's average rate of change with changes in the function's values.

In the book, *Beyond Constructivism: Models and Modeling perspective on Mathematics Problem Solving, Learning and Teaching*, Doerr and Lesh (2003) discussed teachers' development using the MMP framework. They explained that at the core of effective teaching is the "richness of ways in which teachers see and interpret their practice not just in the actions that they take" (p. 125-126). Furthermore, they indicate that teachers' knowledge is developed through *sophisticated models*, or ways in which teachers interpret teaching, learning, and problem-solving situations, such as identifying students' prior knowledge, recognizing how students learn important mathematical concepts, identifying possible misconceptions, using appropriate activities or tasks to support students' learning, and reflecting on their practice. Through a thorough analysis of prior research on the nature of teachers' knowledge and its development, they point out that just as K-12 students are expected to be engaged in multiple ways of interpreting a problem and have multiple paths for refining and revising their ideas in order to gain a deep understanding of the mathematics they are learning, teachers must also be able to see multiple ways of interpreting a situation, understand how ideas might be revised, and continuously refine and generalize their ways of thinking. Doerr and Lesh further asserted that this emphasis on interpretation should not only be focused on how students learn, but the interpretation of mathematics itself should also be highlighted. Additionally, they describe that in the MMP perspective, teachers' prior knowledge should be emphasized to "express, test, revise, and refine their content knowledge and extend that knowledge to more powerful forms for classroom teaching" (p. 130), rather than repairing the deficiencies in their mathematical knowledge. Lastly, it is important to mention the important idea Doerr and Lesh discussed about the instructional design principles (summarized above in Table 1). They claimed that in the same manner that MEAs can provide a foundation for the development of powerful mathematical reasoning in children, they also can effectively guide teacher educators in helping pre-service teachers develop their teacher knowledge, which is the premise of this research study. The next section of the literature review will discuss teacher preparation and teacher knowledge in greater detail.

Teacher Knowledge

Throughout the years, researchers have examined various factors in the mathematics classroom that can help students better understand mathematics and can enhance their mathematical achievements, and many researchers have found that teacher knowledge is a key factor (Baumert et al., 2010; Fennema & Franke, 1992; Lampert et al., 2010; Ma, 1999; Shulman, 1987). Such research studies have had major influences on mathematics education through the development of standards by teaching organizations (NCTM, 1989, 1991, 2000, 2014; AMTE, 2017), and national policies regarding education (No Child Left Behind, 2001; Common Core State Standards Initiative (CCSSI), 2010). Even though there is a consensus about that teachers play a key role in their students' mathematical learning, experiences, and achievement, there are different outlooks about how high-quality teaching is developed. The U.S. Department of Education (2002) reported that teachers' content knowledge is a key component of effective teaching. Shulman (1986) explained that even though content knowledge is an important aspect, it is not the only factor in teacher knowledge, rather, curriculum knowledge as well as pedagogical content knowledge play key roles in high-quality teaching. In 1987, Shulman emphasized pedagogical content knowledge due to its potential of helping teachers understand how particular mathematics concepts are organized and represented, then adapted to their students' knowledge and abilities, and then presented during instruction. Furthermore, Ball (1990) revealed the importance of pre-service (and in later work, practicing) teachers having a deep and conceptual understanding of mathematics in order to be able to properly explain and justify the mathematics they teach to students. Although researchers agree that teachers must know the content, there are some disagreements in the continuously developing area of teachers' mathematical knowledge.

Mathematical Knowledge for Teaching

During the 1980s and 1990s, as reforms in the field of mathematics education started to call for different methods for teaching mathematics to K-12 students, interest in researching teacher knowledge with respect to instruction increased (Ball, 1988; 1990; Ma, 1999; Shulman, 1986; 1987). Among the first researchers to describe a type of teacher knowledge specifically for teachers was Lee Shulman. In his work, Shulman (1986; 1987) described this type of knowledge as pedagogical content knowledge—a knowledge that goes beyond the subject matter. Researchers (e.g. Ball, 1988; Ball, 1990; Ball, Thames, & Phelps, 2008; Hill, Rowan & Ball, 2005; Hill, 2007; Rowan, Schilling, Ball & Miller, 2001) began to build on Shulman's foundational work and more details about mathematical knowledge for teaching began to develop.

Previously, it was assumed that learning mathematics in the K-12 setting, or majoring in mathematics, provided pre-service teachers with the content knowledge they need to effectively teach school mathematics. However, these assumptions were challenged by Ball (1990) in a study examining 252 pre-service elementary and secondary teachers' subject matter knowledge. In the study, the researcher used interviews and questionnaires to explore pre-service teachers' understanding, ideas, and feelings about mathematics and writing, the teaching and learning of mathematics and writing, as well as students as learners of the two subjects. The findings of the study demonstrated that pre-service teachers (both elementary and secondary) struggled with representing and conceptualizing fraction division, and the majority of the pre-service teachers based their understanding of mathematical concepts on memorized procedures and rules. Furthermore, to meet the expectations of students exploring mathematics by making conjectures and validating solutions, Ball emphasized that pre-service teachers must understand the mathematics they are teaching deeply; in ways that allow them to participate in class discourse, and justify and explain concepts to others.

After the study discussed above, Ball and her colleagues continued working on refining the domain of the mathematics knowledge needed to effectively teach mathematics, or what became to be called MKT. Along with their research group, Ball, Thames, and Phelps (2008), analyzed videos of teachers in their practice and designed measures of mathematical knowledge for teaching based on the hypotheses formulated from their analysis. These measures are described in greater detail below. The authors first describe the definition of MKT as the "mathematical knowledge that teachers need to carry out their work as teachers of mathematics" (p. 395). Their definition of MKT was purposefully framed in terms of teachers' work to emphasize the connections between the subject matter knowledge and teaching. In the study, Ball and colleagues refined the two content knowledge categories originally described by Shulman (1986), Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK), by dividing them into six subcategories. The SMK domain was subdivided into *Common Content Knowledge* (CCK), *Specialized Content Knowledge* (SCK), and *Horizon Content Knowledge* (HCK); and the PCK domain was subdivided into *Knowledge of Content and*

Students (KCS), *Knowledge of Content and Teaching* (KCT), and *Knowledge of Content and Curriculum* (KCC). These domains and subdomains were represented in what is often referred to as the *egg* diagram, which was recreated as Figure 5 below.

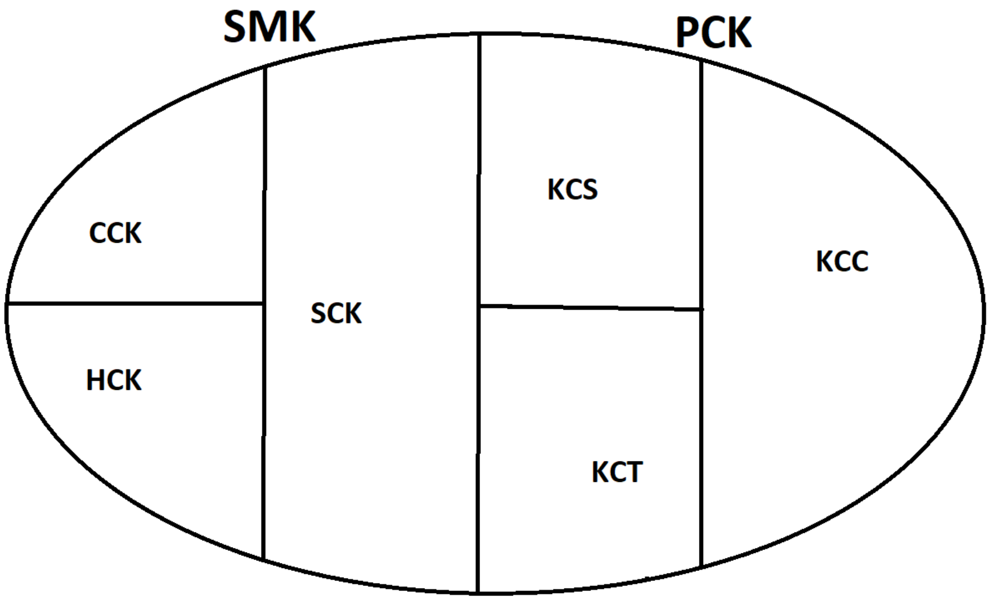


Figure 5. Domains of Mathematical Knowledge for Teaching (adapted from Ball, Thames, & Phelps, 2008).

Ball and colleagues focused their work in this article on the CCK, SCK, KCS, and KCT subdomains. They described CCK as the knowledge of mathematics that is used in settings that are not unique to teaching, this includes recognizing students' wrong answers, or textbook mistakes, or in other words, they "must be able to do the work that they assign their students" (p. 399). On the other hand, SCK is referred to as the mathematical knowledge that is not needed in professions or settings other than teaching. It requires teachers to be able to unpack mathematical concepts. For instance, being able to differentiate between the "*take-away* and the *comparison* models of subtraction, [or] the *measurement* and *partitive* models of division" (p. 400). The KCS subdomain involves teachers being able to understand students' thinking and reasoning behind different mathematical topics, and being able to anticipate when and why students' misconceptions occur. KCT refers to the knowledge of mathematics and being able to design appropriate instructional tasks and carry out instruction. The researchers gave the following example of each the four domains within a teaching context—ordering a list of decimals is CCK, while generating a list to be ordered that would reveal key mathematical issues is SCK, recognizing which decimals would cause students the most difficulty would be KCS, and deciding what to do about their difficulties can be considered KCT. Through this example, the researchers demonstrated and explicitly asserted that the "lines, or shifts, between four domains can be subtle" (p. 404).

The emergent research of teacher knowledge generated increased attention into creating ways of measuring the various aspects of it. At the beginning, such measures tended to capture teachers' knowledge either indirectly, or in ways that resembled the relationship between pedagogical knowledge and teacher performance (Rowan, Schilling, Ball, & Miller, 2001). Once such discrepancies were realized, Rowan and colleagues (2001) started developing new questions to survey teachers and measure their pedagogical knowledge in elementary mathematics, as well as reading and language arts. Furthermore, the *Study of Instructional Improvement* at the University of Michigan began to create instruments to measure the knowledge for teaching mathematics of elementary teachers (Hill, Rowan, & Ball, 2005).

Rowan and colleagues (2001) initially chose the mathematical topics of number concepts, place value, and operations to be measured, along with three identified domains of pedagogical content knowledge: content knowledge, knowledge of students' thinking, and knowledge of pedagogical strategies. The questions on the instrument represented classroom scenarios that teachers may encounter, and the solutions were based on prior research done in the area of teaching and learning with the selected mathematics content. The researchers created two forms of the instrument, and each of the multiple-choice questions was designed to measure a distinct feature of teachers' pedagogical content knowledge. Once the instrument was developed, the researchers conducted a pilot study over the course of two semesters and collected data from 104 teachers that responded to their mailed questionnaire. The developed scales had adequate reliabilities (DeVellis, 1991) ranging from 0.5 for the pedagogical content knowledge of the number concepts scale, to 0.859 for the content knowledge of multi-digit computation. Although the process was difficult, according to the researchers, these findings were influential because they revealed the plausibility of developing reliable instruments that measure parts of teachers' pedagogical content knowledge. Nonetheless, the question of developing a summative instrument of pedagogical content knowledge remained unanswered.

The work of Rowan and colleagues established a foundation for developing instruments to measure teacher knowledge. In fact, Hill, Schilling, & Ball (2004) explored the question, "Is there one construct that can be called 'mathematics knowledge for teaching' and that explains patterns of teachers' responses, or do these items represent multiple constructs and thus several distinct mathematical competencies of elementary mathematics teachers?" (p. 12), and sought to develop reliable instruments to measure such constructs. The researchers designed 138 multiple-choice items that represent the knowledge required to teach elementary school mathematics, stemming from other researchers' work. These items were organized into three mathematical topics: (1) number concepts, (2) operations, and (3) patterns, functions, and algebra; then they were crossed with two domains of teacher knowledge: (1) knowledge of content, and (2) knowledge of students and content. The items were then used in pilot forms, and analyzed through a factor analysis to identify possible patterns. The results of the statistical analysis revealed that the items that were written for the "particular ways mathematics arises in elementary classrooms" (p. 16), or what the researchers called *specialized knowledge of content* (SKC), were likely to be explained by the content knowledge domain, and thus, reinforced the researchers' premise that some content knowledge is specific to teaching. Furthermore, items that belonged to the SKC category included: "providing explanations for common mathematical rules", "showing or representing numbers or operations", and "analyzing alternative algorithms or procedures" (p. 24). When it comes to developing an instrument to measure teachers' mathematical knowledge for teaching, as one construct, the researchers clarified that this study serves as the first step in the development process.

Hill (2007) built on the work of Rowan and colleagues, with a specific focus on early secondary mathematics content. Hill explained that investigating mathematical knowledge for teaching within a middle school setting is important "in part because many view middle school—and middle school mathematics in particular—as a critical gateway to high school course taking and college enrollment" and "middle school teachers are also a unique population in that although many train specifically for these grades, many more are either former elementary or high school teachers." (p. 96). Hill looked at two content categories: *number and operation*, which included topics related to "rational number, characteristics and operations, integers, ratio, and proportion, percent, and radicals" (p. 99) and *algebra*, which included topics related to linear, quadratic, and exponential functions; algebraic expressions and simple equation solving; inequalities; and absolute value with unknowns" (p. 99). Hill and her team of researchers sent out a questionnaire to 1000 middle school teachers, and 650 of them responded. Her study revealed that teachers with strong preparation in subject matter had greater mathematical knowledge for teaching. More importantly, Hill showed that not only does a relationship exist between content courses and teacher knowledge, but "the association between mathematics methods course work and knowledge is also strong, even when controlling for pure content course work" (p. 111). This relationship is an important implication for mathematics teacher educators, and justification for conducting this study.

As research about mathematical knowledge for teaching continued, two separate mathematical constructs were identified—*Common Content Knowledge* (CCK), defined as "the mathematical content that teachers teach" (Hill, Rowan, & Ball, 2005, p. 377), and *Specialized Content Knowledge* (SCK) defined as "knowing how to represent quantities using diagrams, how to provide a mathematically careful explanation of rules, or how to appraise the mathematical validity of alternative solution methods for a problem" (Hill, Rowan, & Ball, 2005, p. 377). Once these constructs were identified, the researchers were interested in exploring the relationship between them and student achievement. Accordingly, they collected data from 115 elementary schools during a four-year period. Specifically, kindergarteners were observed through second grade, and third graders were observed through fifth grade. The students were assessed twice every year (once during the fall, and once during the spring). The data also consisted of parent interviews and teacher observations, as well as the teachers' mathematical knowledge for teaching—measured by Hill and colleagues' instrument of content knowledge for teaching mathematics (2004). The findings of the study revealed that teachers' mathematical knowledge for teaching positively predicted students' achievement gains. These results demonstrate the key role teachers' knowledge plays in mathematics education, in general. The next section focuses on the importance of teachers' knowledge of Algebra.

Teachers' Knowledge of Algebra

Research on teaching and learning algebra tends to emphasize tasks and their algebraic nature, how learners develop algebraic ideas, and the influence of technology on the learning of algebraic concepts, but the teaching and teachers' knowledge of algebra is rarely studied (Doerr, 2004; Lins & Kaput, 2004; Saul, 2008; Stacey & Chick, 2004). Nonetheless, this section of the literature review will be divided into three sections— (1) research studies that discuss teachers' content knowledge of algebra, (2) research studies that discuss teachers' conceptualization of algebra, and (3) the development of teachers' knowledge of algebra.

Teachers' Content Knowledge of Algebra

The majority of the research studies that focus on teachers' content knowledge, both subject matter knowledge and pedagogical content knowledge, focus on secondary teachers' conceptions and misconceptions of algebraic topics like functions, slope, variables, and expressions. Furthermore, there is a general consensus that the necessary subject matter knowledge for teaching algebra effectively requires much more than merely knowing the mathematical concepts (Even, 1990; Norman, 1992; Stump & Bishop, 2001).

Even (1990) explored teachers' subject matter knowledge of functions for teaching, and developed a framework with the following key features: different representations, alternative ways of approaching the concept, strength of the conceptual knowledge, basic repertoire of examples, knowledge and understanding of the concept, and knowledge of mathematics. The framework incorporated aspects that went beyond a simple knowledge of the mathematics. Relating this back to Hill and colleagues' (2008) framework of mathematical knowledge for teaching, this can be viewed as teachers' specialized content knowledge, because it is mathematical knowledge that is unique to teaching.

When it comes to the topic of functions in algebra, Norman (1992), Hitt (1994), and Thomas (2001) all studied secondary teachers and revealed that teachers tend to have inflexible views that restricted their ability to shift among representations of functions, or even identify functions in unusual contexts. Norman (1992) and Chinnappan and Thomas (2001) found that the participating teachers preferred thinking about functions graphically, and had limited ability to generate contexts for functions. Moreover, Hitt (1994) revealed that teachers struggled in constructing functions that were not continuous. Even (1993) revealed that pre-service secondary mathematics teachers tended to believe that all functions corresponded to graphs that were *well behaved*, functions can always be represented by equations, and over-emphasized the use of the vertical line test without being able to explain or justify it. Collectively, these findings demonstrate that teachers' knowledge about functions focused more on procedures and rules, rather than being relational, or conceptual due to the lack of connectedness and flexibility.

When it comes to the topic of expressions, Even, Tirosh, and Robinson (1993) compared lessons of two novice secondary teachers and an expert teacher, and explored *connectedness* when teaching equivalent algebraic expressions. The researchers found that the expert teacher was the only one that used connections between the content being taught and other mathematical topics to facilitate her lesson. The authors revealed that these findings suggest effective teaching of algebra topics like expressions has at least three components: (1) planning for connections across lessons, (2) making and capitalizing on mathematical connections, and (3) setting connectedness as a main goal of instruction.

Similarly, Doerr (2004) revealed that connections play a key role in the teaching and learning the concept of slope. She explained that when introducing slope, the concept builds on students' background knowledge of steepness and angles, encountered in previous geometry lessons, and with ratios and rates, which is also encountered throughout the upper elementary grades. Likewise, the concept of slope foreshadows important ideas students may encounter in calculus (like derivatives), and it also connects to students' developing understanding of functions.

Stump (1999) explored 39 teachers' (18 pre-service teachers and 21 in-service teachers) conceptions of slope. The researcher demonstrated that the majority of the participating teachers understood the concept of slope as a geometric ratio, most of the in-service teachers described slope as a physical property such as the slope of a ramp, only three of the pre-service teachers and four in-service teachers referred to slope as a functional concept, or as the rate of change between two variables. Furthermore, although both pre-service and practicing teachers described the importance of students attaining a conceptual understanding of slope, when asked about students' struggles with the concept, they identified difficulties that emphasized a procedural understanding, such as incorrect use of the slope formula. In a later study, Stump (2001) explored the concept of slope again, but through observations of lessons taught by three pre-service teachers in a college algebra course. The researcher found that even though pre-service teachers' knowledge "was dominated by graphs and physical situations" (p. 224), their actual teaching emphasized graphs and equations. In fact, of the three participating pre-service teachers, only one asked students to use context to interpret the meaning of slope.

Similarly, Even and Tirosh (1995) explored 162 pre-service teachers' understanding of slope through analyzing students' potential errors. The researchers showed that most of the PSMT were able to identify a common student misconception of presuming that there exists a proportional relationship between a linear function's slope and the angle that the line makes with the *x*-axis, when graphed. However, it was also shown that many of the pre-service teachers could not accurately describe the source of students' thinking. Through the findings, the researchers concluded that the PSMT did not understand the reasoning behind students' responses, and thus, subject matter knowledge is not sufficient for understanding and explaining students' thinking.

This leads to another important piece of teachers' knowledge of algebra—understanding students' conceptions and misconceptions (Doerr, 2004). Among others, Tirosh, Even and Robinson (1998) examined the four teachers' knowledge of students' understanding of simplifying expressions, more specifically their tendency to conjoin expressions like $4x + 6$ to get $10x$. The researchers explored this concept with two novice teachers and two experienced teachers, and found that during lesson planning, the novice teachers were not aware of this misconception, while the experienced teachers anticipated it from the students, and incorporated it in their lesson plans. Nonetheless, the experienced teachers could not explain why students tended to make this mistake. When it came to teaching the lessons, the novice teachers promoted the strategy of *collecting like terms*, and when their students struggled, they urged their students to adhere to the rules and steps of the procedure they were taught. On the other hand, one of the experienced teachers devoted the first part of the lesson on having students identify like terms, then proceeded to having the students collect like terms. While the other experienced teacher attempted to counter students' misconceptions by using multiple strategies, like substitution and order of operations, however, these strategies created more confusion in some students. The findings of this study demonstrate the necessity of understanding and anticipating students' mistakes in teachers' knowledge of teaching algebra, as well as understanding various approaches to teaching different concepts, and more importantly, knowing when and how to use these approaches.

Teachers' Conceptualization of Algebra

Foundational research studies on learning algebra have shown four main conceptualizations of algebra, they include: generalized arithmetic, a way to solve certain problems, a study of relationships, and structure (Bednarz, Kieran, & Lee, 1996; Kaput & Blanton, 2001; Kaput, 2008; Usiskin, 1988). Nonetheless, such conceptualizations upheld by researchers, mathematics educators, and/or curriculum developers do not always reflect conceptualizations that are sustained by teachers. In fact, through her study of pre-service teachers, Johnson (2001) revealed that the majority of the participating teachers could not explain their understandings of different topics in algebra, nor could they discuss or articulate the nature of algebra, and similar results were revealed in Menzel (2001) with participating experienced teachers. Furthermore, in his investigation of teachers' understanding of the nature of algebra, Gadanidis (2001) found that the teachers associated the use of a formula as an important aspect of algebra, during an interactive exploration of maximizing area, but the teachers did not connect graphs or the context of the problem as part of algebra.

An important part of teachers' conceptualization of algebra is the means by which they use it to solve problems and evaluate their students' work. More specifically, researchers have studied the difference between teachers' algebraic strategies in both their teaching and learning. Van Dooren, Verschaffel, and Onghena (2002) explored pre-service teachers' evaluations of student work, as well as their problem-solving strategies at the beginning of their teacher education programs and then at the end. The researchers divided their participants into two groups, elementary pre-service teachers and PSMT. They found that the majority of the elementary pre-service teachers preferred to use arithmetic strategies to solve problems, while the majority of the PSMT preferred to use algebraic strategies. Furthermore, the two groups of teachers did not have vast differences in their problem-solving behavior at the beginning and at the end of their programs, and pre-service teachers that were about done with their teacher training generally used similar strategies as their counterparts at the beginning of their training. However, it was also found that pre-service teachers in their third year of their education programs "were more skilled in applying [strategies] to complex situations" (p. 332). Accordingly, the researchers demonstrated that although teacher preparation programs may successfully improve pre-service teachers' skills, their solution strategy preferences are resistant to change.

When it came to exploring how pre-service teachers examine and evaluate students' solutions to arithmetic and algebra problems, van Dooren and colleagues found that they tended to reflect their own preferred strategies in grading their students' work. Generally, PSMT gave higher scores to algebraic solutions and elementary pre-service teachers gave higher scores to arithmetic solutions. However, it was also found that elementary pre-service teachers tended to identify the nature of the problem, as algebraic or arithmetic, and grade the students' solutions based on the appropriateness of their methods, while PSMT preferred the algebraic method regardless of the nature of the problem. Furthermore, when looking at more complex problems that required the use of algebraic strategies, the elementary pre-service teachers experienced difficulty in understanding students' solutions and reasoning, and as a result they negatively evaluated them. On the other hand, the PSMT tended to do two things: (1) view arithmetic strategies as inferior, even when they were more appropriate, and (2) use algebraic methods inflexibly, and focused on the use of rules and procedures. As a result, the researchers concluded that elementary pre-service teachers need more exposure and experiences with algebraic reasoning in order to be able to prepare their students for the transition from arithmetic to algebraic ways of thinking; while the PSMT should also be aware of the value of arithmetic strategies in order to gain insight to help their students make the transition as well.

The Development of Teachers' Knowledge of Algebra

Several researchers have focused on exploring different experiences that may help develop teachers' knowledge for teaching algebra. Most researchers have focused on developing teachers' subject matter knowledge and their pedagogical content knowledge of teaching algebra. The following discussion of research studies describe teachers that were a part of a group engaged in at least a year long professional development program. Agudelo-Valderrama (2008) explored affective factors of seventh and eighth grade algebra teachers in relation to the development of their teaching knowledge of *variables.* The researcher found that some affective factors, like teachers' conceptions of themselves as mathematics teachers, played a role in their reflections of their instructional practices and led some teachers to improving their practices, which positively affected student performance. Brown and Smith (1997) developed a professional development model that focused on addressing teachers' learning through planning, instruction, and reflection. This brought about positive changes in the practice of an experienced algebra teacher, such as the effective use of multiple representations, student discussions, and questioning techniques that promoted explorations of concepts. Similarly, Miller (1992) explored the continuous use of writing in algebra classes. The researcher revealed that writing allowed teachers to examine their students' understanding of mathematical topics more thoroughly, which allowed the teachers to adjust their instructional practices accordingly.

More recently, McCrory, Floden, Ferrini-Mundy, Reckase, and Senk (2012) examined the specific teacher knowledge needed to teach algebra successfully. They designed a framework of knowledge and practices relating to the knowledge of algebra for teaching (KAT). The development of the framework was based on prior research (Ball & Bass, 2003; Brophy, 2006; Doerr, 2006; Lampert, 2001; NCTM, 1991; Shulman, 1986), analysis of videos, interviews with teachers, and analysis of textbooks. Their analysis lead to the creation of the KAT framework, in which the researchers defined categories of knowledge and practices of teaching, for understanding and assessing teachers' knowledge for teaching algebra. The three categories of knowledge include *school, advanced, and teaching* knowledge. School knowledge is defined as "knowing what [teachers] will teach" (McCrory et al., 2012, p. 595), advanced knowledge is defined as "knowing more advanced mathematics that is relevant to what they will teach" (p. 595), and teaching knowledge was described as "knowing mathematics that is particularly relevant for teaching and would not typically be taught in undergraduate mathematics courses" (p. 595). They also described *mathematics-for-teaching knowledge* as mathematical knowledge "that is intuitively useful for teaching, and that is unlikely to be taught explicitly, except to teachers" (p. 598). The researchers also related their *teaching knowledge* to Hill and colleagues' definition of MKT (2008). The three categories of teaching practices—which address mathematical uses of specific teaching knowledge, include *trimming, bridging, and decompressing*. Bridging is described as making connections across topics, assignments, representations and domains. Trimming is described as removing complexity while retaining integrity; meaning the ability to make the mathematics accessible to students while retaining the rigor of the mathematical ideas, and getting to the mathematical essence of a real-world problem. Decompressing is defined as unpacking the complexities found in mathematical ideas in ways that make the content comprehensible, and specifically knowing the difficulties entailed in students' understandings of particular aspects of algebra content. This framework will be used in the data analysis phase of this study and will be discussed more in the following chapters.

Importance of Introductory Algebra Content to the Preparation of Secondary Teachers

Introductory algebra topics, such as proportional, linear, and quadratic reasoning and relationships, are typically explored in introductory algebra courses (Hill, 2007). Such topics are taught in both middle and high school grades (CCSSM, 2010), and they are foundational to learning upper and post-secondary mathematics (Cai & Moyer 2008; Carraher et al., 2006; Greenes et al., 2001). These concepts tend to be introduced in the middle grades but are usually revisited or extended throughout high school mathematics classes.

When it comes to developing PSMT' knowledge of introductory algebra concepts, it has been shown that the typical symbol-manipulation approach to learning these topics is unlikely to provide a solid foundation on which to build pre-service teachers' MKT (Wilkie, 2013; Usiskin, 2001; Silverman and Thompson, 2008). Rather, teachers need to have a conceptual, connected and flexible knowledge of algebra in order to be able to teach it effectively (Evan, 1995; Doerr, 2003). In fact, *The Mathematical Education of Teachers II* (Conference Board of the Mathematical Sciences [CBMS], 2012) report stated that "the mathematical topics in courses for pre-service high school teachers and in professional development for practicing teachers should be tailored to the work of teaching, examining connections between middle grades and high school mathematics" (p.54). Furthermore, PSMT can build a strong conceptual foundation to major ideas in upper secondary mathematics by forming connections of both introductory and advanced mathematical concepts through explorations of MEAs (Carlson et al., 2003; Lesh et al., 2008). Therefore, the introductory algebra topics applied in this study are important for PSMT to know deeply, flexibly and coherently.

Conceptual Framework

This study aims to examine the nature of the relationship between engagement in MEAs and PSMT' MKT algebra. Based on the analysis of the literature described above, Figure 6 is included to illustrate the conceptual framework for this study, along with Table 3 which outlines the relationship between the concepts and the supporting literature on which that relationship is founded. The framework includes the two main concepts of interest: *MEAs* and pre-service teachers' *MKT Algebra,* and connected to them are supporting concepts *SMK*, *MKT*, and *Reflections*.

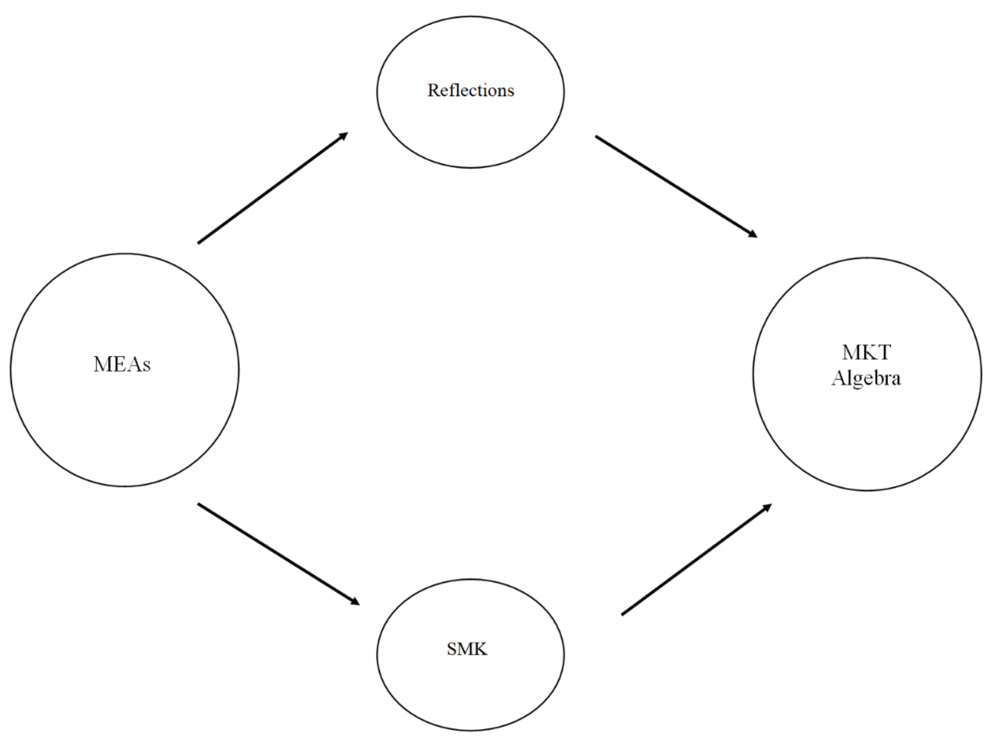


Figure 6. Conceptual Framework.

In the discussion of research with PSMT within the MMP above, the relationship between MEAs and teacher reflections was reviewed by Doerr and Lesh (2003). The researchers explained how MEAs can be used by teachers, in the same manner that students use them, to reflect on, and develop, their own practice and subject matter knowledge. Furthermore, Carlson and colleagues (2003) along with Lesh and colleagues (2008), found that MEAs help PSMT acquire a deeper understanding of various mathematical topics, which is the basis for the relationship between *MEAs* and *SMK*. The relationship between *Reflections* and *MKT Algebra* was based on the findings of Miller (1992) and Brown and Smith (1997), where writing and reflecting were found to allow teachers to examine their students' understanding of mathematical topics more thoroughly and adjust their instructional practices accordingly. The mutual relationship between *SMK* and *MKT* was described by Hill (2007) and Ball, Thames, and Phelps (2008). The researchers found that teachers with strong preparation in subject matter have higher MKT, and reciprocally, teachers with high MKT also had a high SMK. Lastly, McCrory and colleagues (2012) described the relationship between *SMK* and *MKT Algebra* (or what they called KAT), as well as the mutual relationship between *MKT* and *MKT Algebra*. Their KAT framework was founded on Ball, Hill and colleagues' work with general *MKT*. *MKT Algebra* was described as the equivalent of MKT, but with an emphasis on algebra within a secondary context. Furthermore, the researchers explained that teachers' SMK (or what they called *school* and *advanced* knowledge of algebra) underlies the three practices of teaching knowledge— *trimming, bridging,* and *decompressing*. Accordingly, this study uses those three practices to analyze PSMT' *MKT Algebra* before, during and after engaging in *MEAs*, in order to examine the nature of the relationship between engagement in *MEAs* and *MKT Algebra*.

The table above outlines the relationship between the fundamental concepts in this research study, and the supporting literature on which that relationship is founded. A *rightwards* arrow represents a one-way connection, and a *left right* arrow represents a mutual connection between the concepts in the conceptual framework. For example, Lesh and colleagues (2008) demonstrated how MEAs related to PSMT' subject matter knowledge, and Hill (2007) discussed the direct relationship between the two concepts of teachers' SMK and MKT. This study aimed to explore the nature of the relationship between engagement in MEAs and MKT Algebra.

Table 3

Connections between concepts in the conceptual framework and supporting literature.

Relationship between Concepts	Supporting Literature
MEAs→ Reflections	Doerr & Lesh (2003)
MEAs → SMK	Carlson et al. (2003); Lesh et al. (2008)
Reflections → MKT Algebra	Brown & Smith, (1997); Miller (1992)
SMK → MKT Algebra	McCrory et al. (2012)

Conclusion

In summary, the research studies discussed in Chapter two have revealed important takeaways that support the relevance of this study. To begin, the rich and diverse background of mathematical modeling provides a thorough context for this study. Although results showcase that pre-service teachers' experiences with mathematical modeling tasks and MEAs have the potential to help PSMT acquire a deeper understanding of various mathematical topics (Lesh et al., 2008; Zbiek, 1998; Zbiek & Conner, 2006) and enhance their modeling competencies (Kaiser, 2007; Kuntze, 2011), pre-service teachers' knowledge and exposure to mathematical modeling is limited (Gould, 2013; Hart & Thomas, 2010; Kuntze, 2011; Widjaja, 2013; Winter & Venkat, 2013). Moreover, it was revealed that pre-service teachers' mathematical learning, thinking, reasoning and teacher knowledge can be thoroughly explored via MEAs (Carlson et al., 2003, Doerr & Lesh, 2003; Lesh et al., 2000, 2008). For that reason, the MMP was chosen as the mathematical modeling lens for this research study. The literature review also demonstrated that teachers' mathematical knowledge for teaching is an important factor to continuously explore and develop (Ball, 1990; Hill, Rowan, & Ball, 2004; 2005; Rowan, Schilling, Ball, & Miller, 2001). When it comes to algebra, in particular, many studies found that in order for effective algebra teaching to take place, the necessary subject matter knowledge requires much more than merely being familiar with the mathematical concepts, rather, it requires knowing the content in a thorough, flexible, and connected way (Doerr, 2004; Even, Tirosh, & Robinson 1993; Stump, 1999; Van Dooren, Verschaffel & Onghena, 2002).

With the relevance of this study embedded in the literature, this study is well-positioned to address a gap in the research base. Through the reviewed research studies in this chapter, the importance of studying PSMT' mathematical knowledge for teaching, and particularly, mathematical knowledge for teaching algebra, was made clear. However, this important domain has yet to be explored in relation to mathematical modeling, another pertinent domain in mathematics education. The potential of mathematical modeling supporting the development of pre-service teachers' content knowledge were revealed in some of the reviewed studies (Lesh et al., 2008; Zbiek, 1998; Zbiek & Conner, 2006), but there is a gap in the literature when it comes to investigating the relationship between mathematical modeling tasks and PSMT' mathematical knowledge for teaching. Nevertheless, researchers within both the MKT field (Bass, 2005; Hill, Dean, & Goffney, 2007) and the MEA field (Doerr & Lesh, 2003) have called for exploring and eliciting teachers' knowledge through tasks that apply mathematics to a context of interpreting, analyzing, and revising. The current study aims to address this gap by answering the question:

What is the nature of the relationship between engagement in MEAs and pre-service secondary mathematics teachers' (PSMT') MKT algebra?

CHAPTER THREE: RESEARCH DESIGN AND METHODOLOGY

Introduction

In this chapter, the research question is stated, and the research design is described in relation to the purpose and conceptual framework of the study. The population and sampling procedures are discussed, selection of cases, along with the research setting. Additionally, data collection procedures are explained, along with the instrumentation, which includes the Learning Mathematics for Teaching (LMT) instrument for measuring Mathematical Knowledge for Teaching (MKT) and the Teacher Knowledge Assessment System (TKAS). The Model-Eliciting Activities (MEAs) used in this study along with the use of the Knowledge of Algebra Teaching (KAT) framework are also shared; followed by a discussion about the conducted interviews, and data analysis procedures.

Research Question

The conceptual framework discussed in Chapter two guided the formation of the research question: *What is the nature of the relationship between engagement in MEAs and pre-service secondary mathematics teachers' MKT algebra*? Figure 7 below illustrates the research question in relation to the conceptual framework.

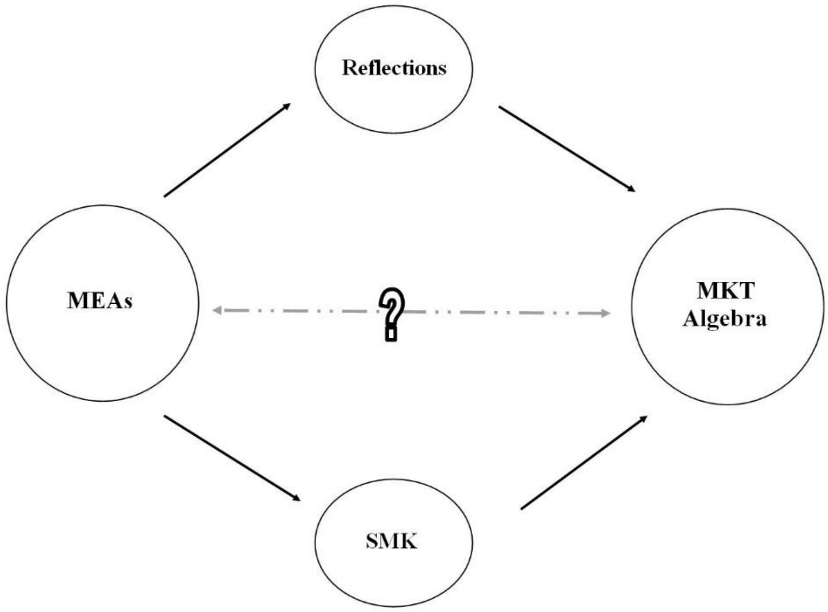


Figure 7. A Visualization of the Research Question in relation to the Conceptual Framework.

Research Design

The purpose of this research study was to explore the nature of the relationship between engagement in MEAs and pre-service teachers' MKT algebra. Because mathematical modeling, and MEAs in particular, have not been extensively studied in relation to pre-service teachers' MKT, a qualitative research methodology is used. According to Creswell (2013), if the topic under study is new and has not been previously examined, existing theories do not apply, or if a concept needs to be understood because little research has been conducted on the topic, then it merits a qualitative methodology.

Specifically, this research study was designed as a multiple case study (also called collective case study) as described by Yin (1994), Merriam (1998), and Creswell (2013). The multiple case study design was selected because it enables the researcher to attain an in-depth understanding of the topic being studied (Merriam, 1998). A multiple case study design is appropriate for research that focuses on providing an in-depth understanding of cases and a comparison of several cases within a particular context to illustrate the topic being examined (Creswell, 2013; Merriam, 1998; Stake, 2006; Yin, 1994). This study aimed to provide an understanding of pre-service teachers' MKT algebra in relation to MEAs through in-depth descriptions and comparison of three cases of PSMT.

Yin's (1994) replication approach for designing multiple-case studies was used. Yin recommended that the initial step in designing the study must consist of theory development. Then, the case selection and definition of specific measures contribute to the design and data collection process. Yin then recommended that each individual case should consist of a *whole* study and connecting evidence should be sought for the conclusions of each case. Next, Yin suggested that "both individual cases and the multiple-case results should be the focus of the summary report" (1994, p. 49).

For this study, the researcher first reviewed the relevant literature thoroughly regarding the case under study—PSMT' mathematical knowledge for teaching algebra and mathematical modeling, prior to conducting any data collection. The review of the literature guided the method for selection of cases and the data collection protocol (Creswell, 2013; Safi, 2009). A maximum variation method was used for selecting the cases. This method consists of determining criteria that differentiates the participants, and selecting participants that vary based on the criteria (Creswell, 2013). This was chosen in order to show different perspectives for the topic being examined. The sampling procedures will be discussed in great detail in the *Selection of Cases* section below. The data collection protocol a pre- and a post-test, recordings of PSMT' engagement in the MEAs, interviews, participant artifacts, and reflections. The use of multiple sources of information was recommended by Yin (1994) and Creswell (2013) in order to capture the cases in their complexity and entirety. Once the cases were selected, and the data collection protocol was developed, the data for the three case studies were gathered. During the data collection process, three principles recommended by Yin (1994) were followed: (a) *multiple sources of evidence* to converge the same findings, (b) a *case study database* to formally assemble evidence distinctly from the final case study report to manage data properly, and (c) *a chain of evidence* to explicitly link the questions asked, the collected data, and the drawn conclusions. These principles were followed to maximize the quality of the inquiry (Yin, 1994).

When it came to analyzing the data, the researcher conducted a within-case analysis, in which each case was examined in their entirety, and descriptions, themes, and interpretations were written and reported. Furthermore, a *within-case* analysis was conducted where each case was analyzed for themes, and then a *cross-case* analysis was conducted where themes were examined across cases in order to discern themes that are common and different to all cases (Creswell, 2013). The *Data Analysis* section below discusses the analysis of the data in greater detail. The design of the study is summarized in the following figure.

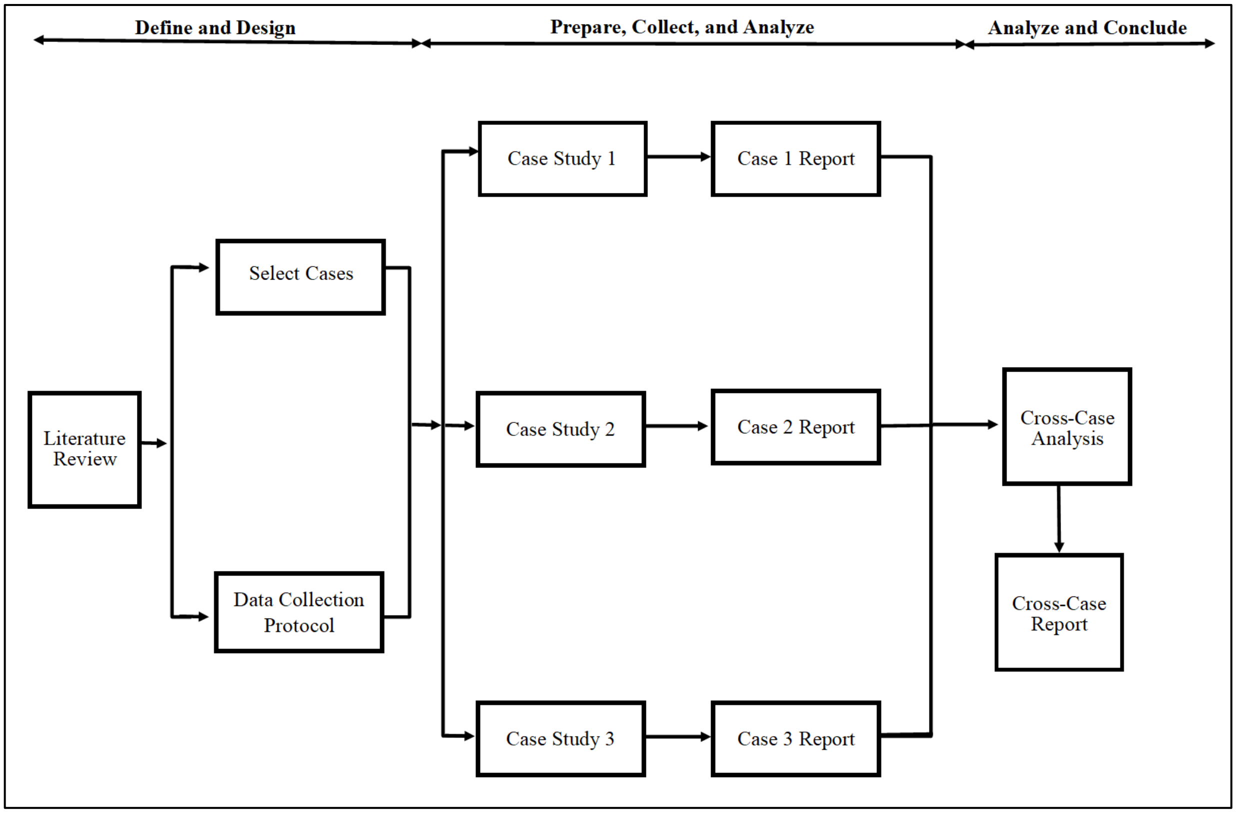


Figure 8. Research Design (adapted from Yin, 1994).

Population and Sampling

The population for this research study consists of PSMT, defined as pre-service Grade 6-12 mathematics teachers, as this aligns to the grade band certification structure at the university where the study was conducted. Because this research study is qualitative in nature, purposeful sampling was used to select the examined cases. The researcher selected the site and participants of the study because they can purposefully inform an understanding of the research problem (Creswell, 2013). Moreover, a *maximum variation* method was used to select the three cases in order to document diverse variations of the pre-service teachers' engagement in MEAs. This is discussed in greater detail in the *Selection of Cases* section.

Setting of the Study

The course selected for which the sample was drawn for this study was an undergraduate mathematics education course that took place during the Spring 2018 semester. The course was required for all PSMT in the college, and it was offered once during the academic year, during the time of the research study. The total number of PSMT enrolled in the course was 30 undergraduate students, of which 19 were majoring in secondary mathematics education, nine had secondary mathematics education as a minor, and two were pending admission into the secondary mathematics education program.

This research study took place in a secondary mathematics education content course that focused on mathematics content for teaching middle and high school, in a college of education at a university in the southeastern United States of America. Through the use of the description contained in the university catalogue and conversations with the course instructor, this course aims to provide a foundation in content knowledge for teaching mathematics appropriate for the middle and high school settings with an emphasis on algebra and geometry. The class met once a week during a 170-minute time block, for 15 weeks. The MEA exploration took place over the course of four weeks, for 90 minutes each week. During the time of the study, after the explorations of the MEAs, the course instructor facilitated discussions with the PSMT that focused on whole number operations and fractions. More information regarding both the MEA explorations and interviews is presented in the following sections.

The PSMT explored the MEAs in heterogeneous groups of three or four, selected by the researcher based on their MKT pre-test scores, with the intention of having varied perspectives and different approaches in exploring the MEA. The PSMT were given 45 minutes to work on the modeling task together, and then they presented their solutions to the rest of the class. Laptop computers were available for the groups to use while exploring the MEAs. The specific individuals who were selected for the case study are discussed in greater detail in the following section.

Selection of Cases

All the PSMT enrolled in the class that the research study took place in were given the LMT pre-and post-assessments, and all of them explored the three MEAs. However, only a subgroup of three participants were selected to be video- and audio-recorded while exploring the MEAs, interviewed twice after the first and third MEAs, and they wrote a reflection about their exploration of the second MEA. The three participants are the cases being examined in this research study. A maximum variation sampling method was applied to select the PSMT based on their scores on the LMT pre-assessment. As stated previously, the researcher used TKAS to conduct the pre-and post-assessments measuring the participants' MKT middle school algebra. The TKAS system reported the participants' IRT scores upon the participants' completion of the test. Prior to discussing how the participants were grouped and the cases were selected, a brief discussion of IRT scores is given below. The presented information is established from module four of the LMT training.

An item characteristic curve (ICC) is shown in Figure 9, and it is considered the building block of IRT theory. The *x*-axis represents the underlying ability being measured by the assessment, in this case it is the PSMT' MKT middle school algebra. This ability is denoted as the symbol theta. This ability spectrum is measured in standard deviation units with zero being average. Higher abilities are represented by positive values and lower abilities are represented by negative values. The *y*-axis is the probability of answering an item correctly, and the solid line represents one item. The plotted curve represents the probability of getting this item correct at different levels of the ability spectrum. For example, for this sample item, individuals with ability two standard deviations below average, or -2 on the *x*-axis, have only about a 5% probability of getting this item correct. Individuals with ability two standard deviations above average, have a greater than 70% probability of getting this item correct.

The difficulty of an item is the point at which the item characteristic curve has the maximum slope. This coincides with the point at which a person of that ability has a 50% chance of getting the item correct. In Figure 9, the difficulty of the item is one, or one standard deviation above the mean in the population, which means that a person with an estimated ability of one has a 50% chance of getting this item correct.

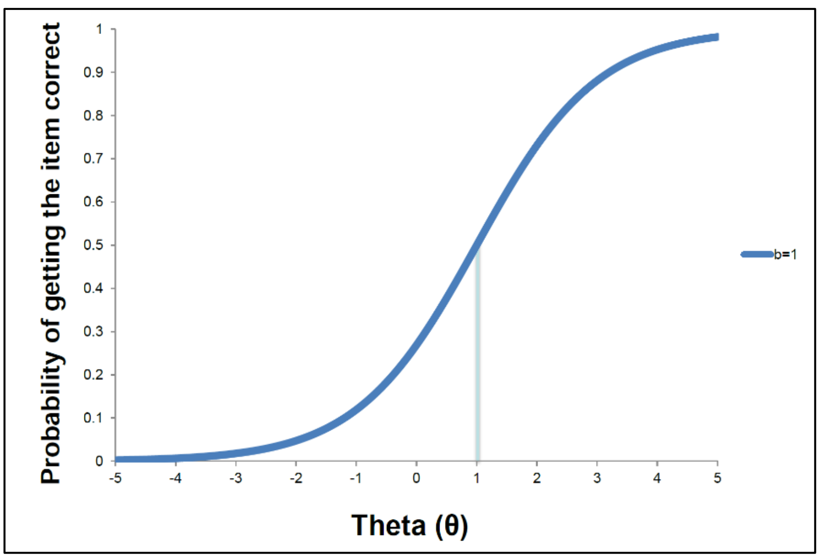


Figure 9. Item Characteristic Curve (LMT training, Module 4, September 2017).

The researcher used PSMT' IRT scores to divide them into heterogeneous groups based on their abilities. Once all the PSMT' IRT scores were received, the researcher sorted the scores from smallest to largest, and used percentiles to reorganize all of the 30 PSMT into three groups: (1) participants with less than or equal to 25^{th} percentile, (2) participants with scores greater than

25th percentile and less than or equal to 50th percentile, and (3) participants with scores greater than 50th percentile. The researcher then used randomly generated numbers to pick participants from each category and organized them into eight groups. The three selected cases were then assigned to be in a group together, and the rest of the class was also grouped heterogeneously with a PSMT from each category in the group. This was done to ensure varied cases are represented, in order to be able to describe multiple perspectives of PSMT. This purposeful sampling method is known as the *maximum variation* method, which Creswell (2013) described as the following:

> This approach consists of determining in advance some criteria that differentiate the sites or participants, and then selecting sites or participants that are quite different on the criteria. This approach is often selected because when a researcher maximizes differences at the beginning of the study, it increases the likelihood that the findings will reflect differences or different perspectives— an ideal in qualitative research (p. 157).

Accordingly, the maximum variation method was applied to select the cases and document diverse variations of the pre-service teachers’ engagement in MEAs, and MKT algebra.

The three selected cases were Oriana— a PSMT with high MKT; Bianca—a PSMT with medium MKT; and Helaine— a PSMT with low MKT. All participant names are pseudonyms. All three PSMT are females, and were in their third or fourth year of a *Secondary Mathematics Education* program. A more detailed description of each participant is included in the *Description of Participant* presented in Chapter five.

It is important to note that Helaine was not originally chosen to be one of the cases, however, upon consulting with the instructor of the course in which the data were collected, all of the groups needed to be reconfigured after the first MEA, and Helaine was then selected to be one of the cases. As a result of the rearrangement of the groups, Helaine was not recorded as she worked on the *Summer Jobs* MEA. However, her experience with the MEA was described and examined through other forms of data, including artifacts the first interview, and recorded presentation of solution to the class.

Data Collection and Procedures

Prior to data collection, the approval of the Institutional Review Board (IRB) was obtained by the researcher (Approval Letter is included in Appendix A), along with permission from the instructor of the course. Additionally, the researcher explained the nature and goals of the study to the PSMT enrolled in the course and obtained their permission to participate in the study that took place in their course during the first-class session (consent form is attached in Appendix A). It was explained to the pre-service teachers that although the activities and assignments they would take part in for the study are required by their instructor, if they wish to not participate in the study, their work would not be included in the data for the study. The procedures of the study were also explained to the pre-service teachers prior to the data collection.

The data collection consisted of a pre- and a post-test, recordings of PSMT' engagement in the MEAs, interviews, participant artifacts, and reflections. Table 4 below provides a summary of the data collected for each of the three cases, and Figure 10 provides a timetable of the data collection process.

Table 4

Data collected from selected cases.

Selected Case	Pre-Assessment	MEA 1 Artifacts	Interview 1	MEA 2 Artifacts	Reflection	MEA 3 Artifacts	Interview	Post-Assessment
Oriana	✓	✓	✓	✓	✓	✓	✓	✓
Bianca	✓	✓	✓	✓	✓	✓	✓	✓
Helaine	✓	–	✓	✓	✓	✓	✓	✓

	Mon	Tue	Wed	Thu	Fri	Sat/Sun
	Holiday Break					6/7
Week 1	Pre-Data Collection Introduction to the study and modeling. January 8	9	10	11	Pre-Assessment Due on TKAS 12	13/14
Week 2	HOLIDAY-MLK 15	16	17	18	19	20/21
Week 3	MEA 1: Summer Jobs Task 22 Interview 1 with Bianca	23	Interview 1 with Helaine 24	25	Interview 1 with Oriana 26	27/28
Week 4	MEA 2: Phone Plans Task 29		Written Reflections Due 31	February 1	2	3/4
Week 5	MEA 3: Historic Hotel Task 5 Interview 2 with Bianca	6	Interview 2 with Helaine 7 Interview 2 with Oriana	8	Post-Assessment Due on TKAS 9	10/11

Figure 10. Timetable of Data Collection Process.

According to the literature reviewed in chapter two, many pre-service teachers have a limited understanding of the definition and process of mathematical modeling (Gould, 2013; Hart & Thomas, 2010; Winter & Venkat, 2013). Yet, researchers have demonstrated that a comprehensive understanding of the modeling process and overall modeling competencies is needed to allow modelers to work on a modeling problem successfully (Vorholter & Kaiser, 2016). Therefore, prior to the first stage of the data collection, the researcher used the first class during week one of the course to introduce the PSMT to mathematical modeling, because they had no prior knowledge of (or experience with) mathematical modeling. Moreover, metacognitive skills play a key role in the modeling process; and a lack, or a low level, of metacognitive knowledge about the modeling process can result in considerable problems when dealing with modeling tasks (Stillman, 2011). Therefore, the first session of the data collection period was reserved to introduce and discuss important aspects of mathematical modeling—what the process entails, how it may be carried out in the classroom, and its potential merits in learning mathematics. Because this study applies the models and modeling perspective (MMP), the works of Lesh, Doerr, and colleagues were used as a guide for defining and describing mathematical modeling, its process, and value (Lesh et al., 2000; Lesh & Doerr, 2003; Lesh et al., 2008). An outline of the *Pre-Data Collection* session is included in Appendix B.

Stage one of the data collection consisted of measuring pre-service teachers' MKT levels using a Learning Mathematics for Teaching (LMT) instrument (which will be discussed in greater detail in the upcoming *LMT Measures* section), prior to their engagement in MEAs. This was completed during the second session via the online system for administering LMT measures, called Teacher Knowledge Assessment System (TKAS) (also discussed in greater detail in the LMT section). This part of the data collection took place during the second week of the semester, which happened to be a national holiday, and the class did not meet face-to-face during that week. However, the PSMT were instructed on how to access, complete, and submit the pre-test on TKAS during the pre-data collection session. The scores attained from this pre-test were used to rearrange students in heterogeneous groups to work together during stage two of the data collection (the group formation is discussed in greater detail in the *selection of cases* section).

Stage two of the data collection took place during weeks three, four, and five of the course This stage involved PSMT' exploration of three MEAs, and individual interviews with the researcher following the group exploration of the first and third MEA. These tasks were explored during the first 90 minutes of the 170-minute block of class. During week three, PSMT were given the *Summer Jobs* task, week four they were given *Phone Plans* task, and week five they were given the *Historic Hotels* task. The order in which the tasks were given to the PSMT was intentional, based on the complexity of the potential CCSM big ideas that could be used to explore each MEA. The *Summer Jobs* MEA was given first because it engaged the participants in investigating ratios and proportional relationships while exploring the task. The *Phone Plans* task was given next because ratios and proportional relationships could also be used, along with expressions, functions, and quantities. Lastly, the *Historic Hotel* MEA was given, because all the previous mathematical ideas could be investigated, along with equations and inequalities. These MEAs can be found in Appendices D, E, and F, respectively.

Moreover, these specific tasks were chosen purposefully because they have been field tested with students and teachers (Chamberlin, 2005). The MEA section below describes the tasks in greater detail. The pre-service teachers were audio-and video-recorded as they engaged in exploring the MEAs in groups of three to four. Furthermore, during the MEA exploration, the researcher displayed a list of questions created by Bleiler-Baxter, Barlow, and Stephens (2016) on the board, to help support the PSMT in making modeling decisions, as well as a diagram of the modeling cycle from the MMP perspective (shown in Figure 4 above). This was done because prior research has demonstrated that modelers often struggle with the inherent ambiguity of modeling tasks, and they tend to have difficulties explicitly stating and clarifying their assumptions, identifying important variables or factors in the given context, and interpreting their mathematical models (Bleiler-Baxter et al., 2016; Thomas & Hart, 2010; Widjaja, 2013; Winter &Venkat, 2013). The researcher wanted the participants to have clarity regarding the modeling cycle in order to focus on the mathematical content. Participants were encouraged to look at the list of questions (shown in Figure 11) as well as the modeling cycle (Figure 4) throughout their exploration of the tasks.

• What information is (was) most relevant? • What information is most influential? • Can I discard any information? • Can I make any assumptions and/ or approximations to simplify the situation? • Does any information behave "nearly" like something that is very simple to explain?	• What relationships and/ or patterns exist in the information? • What tool(s) will help to discover relationships and patterns? • What tool(s) would be best to use in order to express known relationships and patterns? • Is there another way to represent this situation that might uncover new patterns?	• Does my model describe the situation in a way that agrees with known information? • Does my model make predictions that are reasonable? • Does my model explain the situation in a way that is helpful for a novice to understand it more easily? • Would my model adapt well if small changes were made to the original information?

Figure 11. Questions to help PSMT in Making Modeling Decisions (adapted from Bleiler-Baxter et al., 2016, p. 56).

The group of the selected three cases were video- and audio-recorded as they worked together on the tasks, and their written work was collected at the end of each session. Furthermore, the PSMT were also interviewed after the first MEA, and third MEA. This was done in order to gain an understanding of PSMT' experiences with MEAs, their MKT, and potential impact the engagement in MEAs may have had on the PSMT. The first and second interviews were conducted outside of class. These interviews are discussed in more details in the *Interviews* section below. After the second MEA, the three selected participants were asked to write a reflection, guided by eight questions (shown in Table 6).

Stage three of the data collection process involved measuring the PSMT' MKT scores following their engagement with the MEAs. This stage took place after the third and final MEA task. The PSMT were sent the post-assessment via the TKAS system and were asked to complete the assignment by the following class session.

Learning Mathematics for Teaching (LMT) Measures

Prior to conducting the study, during September 2017, the researcher completed an online training developed by the LMT project at the University of Michigan, School of Education and Institute for Social Research. A screenshot of the confirmation email that was sent after the training was completed is included in Appendix C. This training is required for researchers interested in obtaining and administering the LMT measures. The online training included a total of six modules, three of which are required and the other three are optional. The required modules included information on instrument background and terms of use (module one), designing an assessment plan (module two) and administering assessments and interpreting results in TKAS (module three). The optional modules included information on MKT validity argument (module four), designing a new assessment using selected LMT questions (module five), and using and scoring paper and pencil administrations of the LMT assessment (module six). The researcher completed all six modules prior to conducting the study. Once the required modules were completed, the researcher received access to all the forms and scales of the instruments, along with access to TKAS. The information presented in this section was retrieved from the training modules, unless otherwise stated. Although this section will include some aspects of the history and methods involved in developing the LMT measures, a more detailed account of the instrument was discussed in Chapter two, the literature review.

The purpose for using the LMT scale in this study was primarily for the selection of cases. According to the LMT modules, the purpose of the measures is to capture teachers' MKT, not *common* content knowledge alone, rather the knowledge that is specific to teaching students mathematics. It was used to select three PSMT with varying MKT scores to describe multiple perspectives on engagement with MEAs and MKT algebra.

The LMT instrument is based on Ball, Thames and Phelps' (2008) MKT framework. Generally, the MKT assessments cover teachers' Common Content Knowledge, Specialized Content Knowledge, Knowledge of Curriculum and Students, and Knowledge of Curriculum and Teaching, but the test forms are particularly strong in the two subject matter knowledge domains—Common Content Knowledge and Specialized Content Knowledge (LMT training, Module 1, September 2017). The mathematical content areas the instruments cover include: number concepts and operations; geometry; rational numbers; patterns, functions and algebra; proportional reasoning; and data, probability and statistics. The middle school algebra scale, which is the scale that was used in this research study, includes items that measure "teachers' knowledge of linear, quadratic, and exponential functions; algebraic expressions and simple equation solving; inequalities; and absolute value with unknowns" (Hill, 2007, p. 99). It includes 17 CCK questions and 31 SCK questions (Hill, 2007). Overall, the instruments are concentrated on grades K-8, and provide separate testing modules and teacher scores for a variety of mathematical topics, as shown in Figure 12 below. For almost all of these areas, there are parallel test forms that can be used as pre-and post-measures.

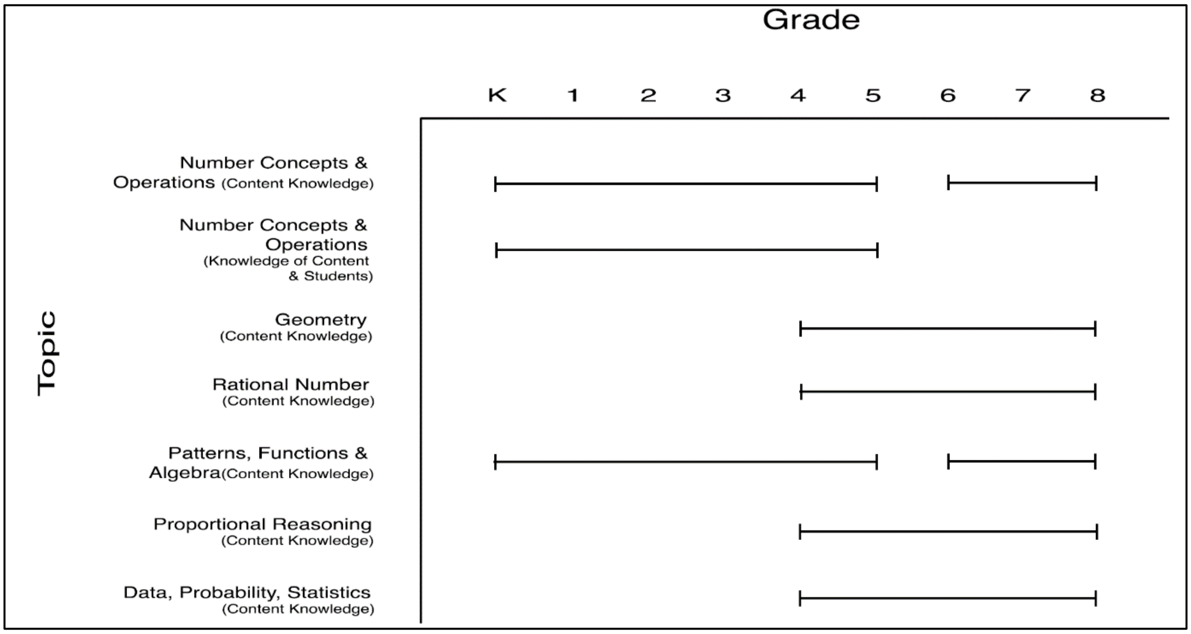


Figure 12. MKT Content Areas and Grade Bands Assessments (LMT training, Module 1, September 2017).

As noted earlier, the purpose of the instrument is to measure teachers' MKT, which is defined as the mathematics which teachers use in teaching, not just what they teach. A related idea is that the items in the instrument were created to reflect the realities of how mathematics instruction plays out in classrooms (LMT training, Module 1, September 2017). This means that the items are embedded in teaching contexts, a feature that adds to the length of some items. Similarly, in some instances, mathematically ambiguous statements are included in the items, in order to portray the kinds of statements teachers and students actually say in classrooms. While the more informal language may be ambiguous at times, this approach is intentionally done as teachers are expected to interpret the students' comments that may not be stated using formal mathematical language. Additionally, it was found by the researchers on the LMT project that such items perform better in psychometric analyses, compared to items that contain advanced mathematical terminology. A sample released item is included in Figure 13. Although this specific question was not a part of the instrument used in this research study, it is included below to show the format of the questions. The questions used in this study were not a part of the released item pool, which is why they are not shown.

34. Ms. Hurlburt was teaching a lesson on solving problems with an inequality in them. She assigned the following problem.

$$-x < 9$$

Marcie solved this problem by reversing the inequality sign when dividing by – 1, so that $x > -9$. Another student asked why one reverses the inequality when dividing by a negative number; Ms. Hurlburt asked the other students to explain. Which student gave the best explanation of why this method works? (Mark ONE answer.)

a) Because the opposite of x is less than 9.

b) Because to solve this, you add a positive x to both sides of the inequality.

c) Because $-x < 9$ cannot be graphed on a number line, we divide by the negative sign and reverse the inequality.

d) Because this method is a shortcut for moving both the x and 9 across the inequality. This gives the same answer as Marcie's, but in different form: $-9 < x$.

Figure 13. Sample Middle School LMT Released Item (Hill, Schilling, & Ball, 2004).

The items were oriented around situations that teachers might face when teaching mathematics. First, the items are intended for broad grade bands but not to specific grades. For instance, the forms are intended for use with middle school teachers, but not for more narrow ranges like 6th grade and 8th grade teachers. Additionally, the items do not represent any single view of teaching, like *reform* instruction (LMT training, Module 4, September 2017). Meaning, there are no items that test teachers' use of manipulatives or engage students in discussion. Instead, the items cover the mathematics that would be needed to use manipulatives with students, or to engage students in a productive discussion. Likewise, the instrument was created to provide situations that could occur using a wide range of curriculum materials.

Furthermore, the items were written to distinguish among teachers, in the sense that scores provide information about how teachers compare with one another in mathematical knowledge for teaching. To achieve such an instrument, the LMT research team followed the following strategies: (1) they wrote a mix of easy, medium, and difficult items. The difficult items allow for the measurement of high-knowledge teachers accurately; (2) the assessment was designed in a way that the average [*sic*] teacher answers 50% of items correctly, this was done to ensure the measurement of the middle of the MKT spectrum well; and (3) the various scales of the instrument were purposefully designed so that each construct could be completed by most teachers within 30 minutes, which as a result, lowered the reliabilities of the measures. The reliabilities ranged from 0.75-0.85 for the different measures. The scale that was used in this study is the middle school algebra measure, and its reliability score is 0.93 (Hill, 2007).

The developers of the LMT measures designed the items as multiple-choice items because their initial studies included several hundred teachers, effectively ruling out other methods for measuring MKT. Some of the response choices include *I'm not sure* as an answer, which is never the correct answer; instead, it is placed there to decrease the possibility of guessing. Guessing results in items that do not discriminate well, in the sense that low-performing teachers often can get those items correct just by chance. Furthermore, the forms composed after 2004 were reviewed by mathematicians for accuracy in order to improve the face validity and statistical performance of items (LMT training, Module 5, September 2017).

When it comes to the results of the LMT measures, it is emphasized that there is no reason to think that teachers must answer any or all of the items correctly to be effective in the classroom. Moreover, the MKT instruments are not criterion referenced, meaning they cannot be used to compare how teachers perform to a gold standard, or to make conclusions about individuals' competency. Instead, they are designed to provide norm-referenced comparisons, either to teachers in the creators' norm-referenced samples or among teachers in the study's sample. In this research study, the instrument is used to compare PSMT in the sample among each other.

In this research study, the scale measuring teachers' MKT middle school algebra was used to select the three cases and compare PSMT to themselves and each other, which is an anticipated use of the instrument by the LMT researchers (LMT training, Module 1, September 2017). As discussed in the previous section, this was implemented in the form of a pre/post comparison of scores before and after PSMT' exploration of MEAs to describe the selected cases. The researcher also looked at how PSMT with different MKT scores explore MEAs differently, and if MEAs provide a context for the PSMT to elicit MKT.

The Teacher Knowledge Assessment System (TKAS)

TKAS is the online system for administering the MKT assessment. It was developed by Geoffrey Phelps and Steve Schilling, at the University of Michigan's Learning Mathematics for Teaching project (LMT training, Module 3, September 2017). Participants taking the assessment are able to log on to this system and take the assessment via a web-based interface. Teachers' data are recorded, scored, and output to the assessment administrators. The use of the system was chosen by the researcher because there is no need to enter the responses manually, which eliminates data entry error. TKAS also balances alternative forms of the assessment at pre-test and post-test via random assignment, so that each teacher gets different forms at administration of the test, and at each administration time point half the sample is assigned to a particular form.

These forms are equated to one another, which means that the scores are equivalent no matter what version of the test a teacher takes at a given time point.

The basic design of TKAS includes two separate sites, the administrator site and the teacher participant site. The administrator site is only accessible to trained TKAS administrators, or researchers, and there, they can create and manage the administration of assessments and view and download teacher data, along with simple statistical analyses provided by the system. An access code is required to enter the site, which the test administrators receive after completing the online LMT training. The teacher site is where participants go to take the assessments. Participants access this site using a URL and program code given to them by the TKAS administrator.

There are three basic steps in the TKAS system to administer the pre- and post-assessment. First, the researcher created an assessment plan by selecting the middle school algebra form from the list of available assessment forms, and the pre/post assessment administration option. Then, the researcher chose the first teacher login option, which entailed identifying participants by their full name, but the researcher has the option to hide the names in data reports. The second step involves the participants taking the pre-assessment and survey within a specified date window. When the assessment plan was created, the researcher was sent a URL for the participant site. This URL was sent to the PSMT via the course's online *Canvas* learning management system page. After the PSMT explore the MEAs, step three took place, which is taking the post-assessment.

It was recommended by the LMT project personnel to not have the participants use tablets or smart phones to complete the assessment, since the system was not tested for these devices and there would a possibility for the PSMT' data to not be saved if they use these devices. Accordingly, the researcher urged the participants to use a computer to complete the assessments, and to make a note of the email address they used in setting up their TKAS account to be able to login to take the post-assessment.

The assessment data on TKAS can be viewed four ways. The first way is an overall score generated by the system using Item Response Theory (IRT) scoring methods. The system only provides IRT scores for PSMT who have completed the assessment (a general overview of IRT scores was described in the *selection of cases* section above). The second way to view the data is by actual response, which includes the raw responses given by the participants. The data are recorded as digits, which means that if a participant answered option *a* for a particular question, that response is recorded in the report as a one. Similarly, *b* is recorded as two, *c* is recorded as three and so on. The third way to view the data is by scored response, which indicates for every question whether the participant answered it correctly, recorded as a one, or incorrectly, and recorded as a two. For questions that are on a form not given to the participants at an administration, it is recorded with n/a, which indicates that the participant is not presented with this question. The final way to view the data is in a summary analysis format. Because this study is qualitative in nature, the data gathered from the LMT measures were used to describe the PSMT' MKT, rather than generalize beyond the cases.

Model-Eliciting Activities (MEAs)

Research within the MMP requires the use of model-eliciting activities with a focus on the use of MEAs to develop and advance mathematical reasoning and understanding (Kaiser & Sriraman, 2006). Lesh and Doerr (2011) explained that the use of MEAs is valuable to investigate learners' thinking and mathematical knowledge as it develops. Furthermore, MEAs are deliberately designed to support students' development of conceptual foundations of mathematical topics (Lesh et al., 2000). Because the purpose of this research study is to explore the nature of the relationship between PSMT' mathematical knowledge for teaching with engagement in mathematical modeling tasks, the MMP perspective was chosen to guide the researcher's decisions about the definition of mathematical modeling, description of the modeling cycle, and selection of MEAs.

When it came to selecting MEAs, the researcher based her decision on the following important ideas: first, all three tasks have been field-tested in the middle-school setting and revised accordingly (Chamberlin, 2005). This is important because the particular MKT scale being used is the middle-school algebra measure. Which is related to the second motive of selecting MEAs could be interpreted with a variety of algebraic concepts (Chamberlin, 2005). Third, the selected MEAs were known for evoking learners' thinking in different ways (Lesh et al., 2000). Moreover, the three selected MEAs were acquired from the MEA list developed by the Small Group Mathematical Modeling (SGMM) and School Mathematics and Science Center (SMSC) at Purdue University, under the direction of Richard Lesh and Lyn English. The copyrights for the selected MEAs are held by the SMSC at Purdue University, and permission to use them was granted for classroom use and research.

The three MEAs that were used in the study are called *Summer Jobs*, *Phone Plans,* and *Historic Hotels.* The tasks can be found in Appendices D, E, and F respectively. Also included in the appendices, is an overview of potential mathematics content that can be used in solving the tasks from the CCSSM. In Table 5 below, descriptions of the MEAs are given, along with potential algebra concepts applied when engaged in the task. A more detailed description of potential algebra content that can be used when engaged in this task is included in the respective appendices. While the algebra topics embedded within the chosen MEAs and MKT instrument pertain to introductory algebra, it is important for pre-service secondary mathematics teachers to explore such content because early algebra topics are considered foundational to learning upper and post-secondary mathematics (Cai & Moyer 2008; Carraher et al., 2006; Greenes et al., 2001). Furthermore, these topics are generally reviewed and constantly used throughout high school mathematics classes. As such, secondary teachers need a firm understanding of introductory algebra concepts. This was discussed in greater detail in the previous chapter.

Table 5

MEA descriptions and potential algebra concepts.

MEA Name	MEA Description	Potential Algebra Concepts
Summer Jobs	Participants are asked to write a description that explains why six out of nine employees should be rehired by their employer based on given information about the number of hours worked and money made during the previous summer.	Proportional relationships and reasoning.
Phone Plans	Participants are asked to develop a strategy for determining the cheapest phone plan based on given information about several phone plans and phone usage.	Linear relationships and reasoning
Historic Hotels	Participants are asked to propose a method for determining a hotel room rate that produces the maximum profit. The given information included: (a) when the rate is $60 per room, all 80 rooms in the hotel get occupied, (b) for every $1 increase in the price, one less room is used, and (c) each room has a $4 maintenance cost.	Quadratic relationships and reasoning.

The researcher created a protocol for the administration of the three MEAs. The 90 minutes were divided into three parts. The first five minutes were used to discuss the group (small and whole-class) norms while exploring the MEAs. The next 35 to 40 minutes were spent on the exploration of the tasks. The last part of the session was the group presentations. A more detailed description of the protocol is included in Appendix G. The researcher collected audio-and video-recordings of the groups exploring the MEAs to capture their discussions, mathematical reasoning, and engagement in the modeling cycle. The researcher also collected their artifacts and reflections to gain an understanding of the PSMT' experiences with MEAs and their MKT. The written reflections were done immediately following the second MEA, and they were written by the same selected PSMT that were asked to be interviewed after the first and third MEA. The reflections were guided by questions shown in Table 6. During the exploration of the tasks, the PSMT had laptop computers available, and they were encouraged to use them.

Table 6

Reflection questions.

Reflection Questions
1. How did you make sense of today's MEA?
2. What mathematical concepts did you use to solve it?
3. How are these concepts connected/related?
4. What assumptions did you make? What factors did you consider? What did you ignore?
5. How did the context of the problem play a part in solving the MEA?
6. What are some possible mathematical misconceptions students may have solving this task?

Knowledge of Algebra Teaching (KAT) Framework

The KAT framework, which was discussed in Chapter two, was used in this research study in two ways. First, it was used as a guide in describing the *MKT algebra* concept of this study. McCrory and colleagues' (2012) definitions of *teaching knowledge, trimming, bridging,* and *decompressing* were used to characterize MKT algebra. In particular, the degree to which PSMT were able to (1) make connections across topics, assignments, representations and domains (*bridging*), (2) remove complexity while retaining integrity (*trimming*), and (3) unpack the complexities found in mathematical ideas in ways that make the content comprehensible and anticipate the difficulties entailed in students' understandings of particular aspects of algebra content (*decompressing*) represented their knowledge of the mathematics that is particularly relevant for teaching (*mathematics-for-teaching knowledge*). Table 7 provides an example of the three categories of teaching knowledge of slope. The KAT framework was also used in this study as a coding guide for the collected data during the data analysis phase of the study. This will be discussed in the *Data Analysis* section of this chapter.

Table 7

An example of teaching knowledge of slope (adapted from McCrory et al., 2012).

	Trimming	Bridging	Decompressing
Teaching Knowledge of Slope	Knowing the different places slope is used in algebra, and attending to appropriate definitions	Knowing methods of graphing rates of change and slopes; knowing the applications and properties of the different definitions and representations of slope.	Knowing that describing slope as *rise over run* may lead to misconceptions and limits the concept to linear functions

Interviews

Interviews were chosen to be a part of the data collection in this study for two main reasons: (1) to understand their experiences with engaging in MEAs and (2) to gain an understanding of their MKT. The researcher used Creswell's (2013) approach for collecting data through interviews, which involves the following steps. The first step was identifying the purpose of the interviews. In this study, the purpose for conducting the interviews was to gain a deeper understanding of PSMT' experiences with and views of MEAs, and their MKT. The second step involved identifying interviewees. A maximum variation sampling method was used to select the participants to be interviewed, based on their MKT scores during the pre-assessment. The third step involved determining the type of interview. In this study, semi-structured, one-on-one interviews were conducted to gain an understanding of individual PSMT' experiences with MEAs, and their MKT, without the influence of other participants. The next step was determining the recording procedures of the interview. The researcher followed Creswell's recommendation of audio and video recording the interviews using proper equipment.

The interview protocol is included in Appendix H and contains the questions the PSMT were asked and the purpose behind asking them. The researcher asked reflection questions to gain information about the participants' experiences with, and views of, MEAs, from both a learner perspective as well as a teacher perspective. Content questions were also included to elicit information related to *trimming, bridging,* and *decompressing* from the KAT framework. Furthermore, the last two questions were *think-aloud* questions, and they were chosen from the LMT assessment. They assessed participants' knowledge of modeling functions to a given scenario. The two LMT questions are not included in the protocol in the appendix because they were not part of the items released by the LMT project. However, a general description of the questions is given in the following paragraph. The purpose for including those two questions was to gain an understanding of any changes in the way PSMT explain and justify their solutions to the questions before and after their engagement with the MEAs. Three items from the LMT assessment were chosen, the first question was on both forms of the test (question 21 on both forms), and the other two were on either form A or form B (question 31 on form A, and question 35 on form B). Those three specific items were chosen from the LMT instrument because they required the PSMT to apply their knowledge of algebra topics to solve problems within a context of a given situation.

For the first interview, which was conducted after the pre-assessment and the first MEA, the participants were given the two questions that were on the form of the test that they were randomly assigned to take for the pre-assessment. They were asked to explain their thinking as they solved the problem. For the second interview - conducted after the third MEA, but before the post-assessment - the participants were given the same first question (the one that was included on both forms). However, for the second question, they were asked to solve the one pertaining to the form that they have not taken yet. The think-aloud questions examined PSMT' understanding of a variety of algebraic topics including proportional, linear, and quadratic thinking and reasoning.

The first question (question 21 on both forms) asked the PSMT to choose a story problem that does *not* represent linear behavior in the context of the four given story problems. The first choice was a story problem that could be represented by an exponential growth function, in which an initial amount doubles every passing day, and the question asks for the total amount on the day x. The second choice is a story problem with a proportion, and could be represented by a linear function in the form of $y = mx$. The third story problem could be represented by a linear function of the form $y = mx + b$. The fourth choice was a story problem that could be represented by a linear function of the form $y= mx + b$, and m is a negative slope. The last choice stated that all the stories represent linear behavior. Similarly, the second question (question 31 on form A) asked the PSMT to decide whether three story problems can be modeled by a given linear equation of the form $y = mx + b$. The PSMT could circle *yes, no*, or *I'm not sure* for each of the three-story problems. The first story problem could *not* be modeled by the given linear equation, rather it could be represented by an exponential growth function. The second story problem could *not* be modeled by the given equation, although it could be represented by a different linear equation. The last story problem could be represented by the given linear equation. The third question (question 35 on form B) gave the PSMT three situations and asked them to choose the situation(s) that could be used as an example for teaching about linear functions. The first situation could be represented using a linear function of the form $y = mx$. The second situation could be represented with a linear equation of the form $y = mx + b$. The third situation could be represented using a *quadratic* function rather than a linear function. Thus, situations one and two are the only ones that could be used as examples for teaching about linear functions.

Next, the researcher chose the location for conducting the interviews, and as Creswell recommended, it was a quiet setting free from distractions. After arriving at the interview site, the researcher explained the purpose of the study, and plans for using the results from the interview to the participants. The interviews lasted between 15 and 30 minutes.

Data Analysis

The participants' individual MKT scores were first analyzed using descriptive statistics and histograms for the pre-assessment and the post-assessment separately. The researcher generated scatter plots of PSMT' pre-test scores and post-test scores and included descriptive statistics and histograms to describe and compare the cases, not to generalize results.

The PSMT' experiences with the MEAs were examined through their recorded explorations of the three MEAs, written artifacts, the two conducted interviews, and their written reflections; and the Knowledge of Algebra for Teaching (KAT) framework served as a guide for coding. More specifically, the framework was used was used to examine the PSMT' MKT algebra PSMT' by looking at the degree to which the PSMT were able to:

> (1) make connections across topics, assignments, representations and domains (*bridging*);
> (2) remove complexity while retaining integrity, and getting to the mathematical essence of a real-world problem (*trimming*), and;
> (3) unpack the complexities found in mathematical ideas in ways that make the content comprehensible, anticipate the difficulties entailed in students' understandings of particular aspects of algebra content, help students understand the usefulness and importance of algebra, and decompress rote procedures in order to grasp the logic behind them (*decompressing*) (McCrory et al., 2012).

The degrees to which the PSMT were able to demonstrate evidence of *trimming, bridging,* and *decompressing* was described as *high* or *strong evidence, medium* or *some evidence*, and *low* or *no evidence.* High or strong evidence were used if the PSMT was able to talk about and explain the different ideas corresponding to each category. Medium or some evidence were used if the PSMT only mentions the different ideas relating to each category, without explanation. Low or no evidence were used if the PSMT either does not mention, or incorrectly mentions, the different ideas pertaining to each category. For example, if a PSMT discussed the connections between the vertex formula of a quadratic, the first derivative and second derivative test, and finding the maximum of a parabola, and explained how these mathematical ideas are related, then it was determined that the PSMT demonstrates high or strong evidence of *bridging*. If a PSMT mentions that there are connections between the three mathematical ideas, without further discussion, then it was categorized as medium or some evidence of *bridging.* If a PSMT does not mention those connections, or incorrectly describes how they are related, then it was considered as low or no evidence of *bridging*.

The researcher began by applying open coding method, which involved taking the data and segmenting them into four categories: MEA experiences, trimming, bridging, and decompressing. Then, the researcher looked for common themes within and across the cases since multiple cases are chosen to be analyzed (Creswell, 2013). The researcher first conducted a within-case analysis, in which detailed descriptions were written and themes were derived within each case. This is followed by a cross-case analysis, or a thematic analysis across the cases. In order to saturate the themes, the researcher noted any repetitive instances that produce a theme and continued looking for such instances until no other themes were represented. Furthermore, the researcher identified sub-themes that might arise from the main themes. Lastly, the detected themes were reviewed with reference to the supporting quantitative data gathered from the pre- and post-assessments.

To ensure trustworthiness, the researcher followed guidelines recommended by Creswell (2013). First, triangulation was applied to provide validity to the findings. More specifically, multiple and different sources of data were used to provide corroborating evidence to investigate the research question. Additionally, the researcher frequently arranged *debriefing sessions* with research and content experts, peers, and the course instructor to provide an external check of the research process. Lastly, *thick and rich* descriptions were used to describe the cases when writing about themes in order to allow the readers to make decisions regarding transferability.

Summary

In this chapter, the research design, and the population and sampling procedures were discussed. The data collection procedures and the research setting were explained, along with the instrumentation, which included the LMT instrument for measuring pre-service teachers' MKT middle school algebra, and MEAs that were chosen. The KAT framework was described, along with the interviews, selection of cases, and data analysis procedures.

The next three chapters will present the findings for the three cases of the research study. They are organized as follows: chapter four describes the selected cases' explorations (as a group) of each of the three MEAs; chapter five presents findings from each of the three cases in terms of their individual experiences with the MEAs and their MKT algebra; and lastly, chapter six provides a cross-case analysis to discern themes that are common and different to all cases.

CHAPTER FOUR: GROUP RESPONSES TO MEAS

The purpose of this study was to explore the nature of the relationship between engagement in model-eliciting activities (MEAs) and pre-service secondary mathematics teachers' (PSMT') mathematical knowledge for teaching (MKT) algebra. Participants were given the LMT measure to assess their MKT algebra prior to their exploration of the first MEA. The participants' MKT scores were then used to calculate percentile ranges and create three categories: (a) low MKT— included participants with scores greater than or equal to zero percentile and less than or equal to 25^{th} percentile, (2) medium MKT— included participants with scores greater than 25 percentile and less than or equal to 50th percentile, and (3) high MKT— included participants with scores greater than 50^{th} percentile. A PSMT was selected from each category for an in-depth exploration of their experiences with the MEAs and their MKT algebra providing three cases. The three cases are: Oriana— a PSMT with high MKT; Bianca—a PSMT with medium MKT; and Helaine— a PSMT with low MKT. All participant names are pseudonyms. All three PSMT are females, in their third or fourth year of a Secondary Mathematics Education program. A more detailed description of each participant is included in the Description of Participant in the following chapter.

This chapter presents the PSMT' exploration of the three MEAs, as a group. The group consisted of Oriana and Bianca for MEA 1 (*Summer Jobs),* and all three PSMT for MEA 2 (*Phone Plans)* and MEA 3 (*Historic Hotel)*. These findings were gathered from audio and video recordings of the group while exploring the MEAs, as well as participants' artifacts.

MEA 1: Summer Jobs MEA

The *Summer Jobs* MEA asked the PSMT to help Maya, a concession business owner, decide on which six of the nine vendors she should rehire, based on their selling records from the previous year. Their records included the total amount of hours worked and money collected when the park had high, medium, and low attendance (also referred to as busy, steady and slow times). See *Appendix D* for the vendor records. The PSMT were asked to write a letter to Maya giving the results, and in the letter, they were asked to describe how they evaluated the vendors. They were to give enough details and a clear explanation, so Maya can check their recommendations and decide whether their method is a good one for her to use. The task is shown in Figure 14.

Problem Statement:

Last summer Maya started a concession business at Wild Days Amusement Park. Her vendors carry popcorn and drinks around the park, selling wherever they can find customers. Maya needs your help deciding which workers to rehire next summer.

Last year Maya had nine vendors. This summer, she can have only six – three full-time and three half-time. She wants to rehire the vendors who will make the most money for her. She doesn't know how to compare them because they worked different numbers of hours. Also, when they worked makes a big difference. After all, it is easier to sell more on a crowded Friday night than a rainy afternoon.

Maya reviewed her records from last year. For each vendor, she totaled the number of hours worked and the money collected – when business in the park was busy (high attendance), steady (medium attendance), and slow (low attendance). (See the tables.) Please evaluate how well the different vendors did last year for the business and decide which three she should rehire full-time and which three she should rehire half-time.

Write a letter to Maya giving your results. In your letter describe how you evaluated the vendors. Give details so Maya can check your work and give a clear explanation, so she can decide whether your method is a good one for her to use.

Figure 14. Summer Jobs MEA Problem Statement.

Oriana and Bianca worked on this task together. After talking about different strategies—using graphs or writing equations; and discussing the different assumptions related to all the possible solutions (all of which will be discussed in great detail in the *single-case analysis* section), the PSMT agreed to use *unit rates* and *averages* to compare the different vendors, based on both the amount of money they made, and the number of hours they worked. They calculated the total amount of money each vendor earned during each of the busy, steady, and slow times at the park, then they calculated total number of hours the corresponding vendors worked. Next, the PSMT used those sums to calculate three unit rates for each vendor, one for when the park attendance was high, medium, and low. Then, the PSMT calculated the average amount of money per hour each vendor makes by adding up the unit rates and dividing by three. The PSMT then used a table to organize their results, which is shown in Figure 15. It is important to note that although there was a third PSMT in the group with Oriana and Bianca, he did not participate in the exploration of the MEA, nor did he contribute to the final solution. However, Bianca and Oriana explained during their presentation to the class that if they had more time to calculate the unit rates and averages for the rest of the vendors, their answer might be different, but the process and logic behind the process still hold.

	Busy	Steady	Slow	Average
Maria	62.20	62.29	44.70	56.40
Kim	87.94	49.21	29.27	55.47
Terry	67.92	36.43	20.25	41.53
Willy	75	42.5	55.5	50
Tony	31	103.5	160	56
Robin	135	127.5	35.5	43

Figure 15. Table of Unit Rates and Averages for Summer Jobs MEA.

Oriana and Bianca then used the table to compare the six vendors. Because they only had six to choose from, they could not eliminate any, rather they made recommendations to Maya for the type of employment each vendor should have. For the part time positions, they recommended Kim to be hired during the busy times, and Terry and Willy during the steady and slow times. Their reasoning was that each vendor should work at the specific time during which he or she made the most amount of money; they also explained that the vendors that had higher averages, should be hired full time. The excerpt below is the explanation behind those recommendations that Oriana and Bianca presented to the class:

Oriana: So, Robin and Tony, we said they should both work, but schedule them at different times, so Robin should work mostly during the busy season and the steady season, and Tony would work mostly in the steady season and the slow season, because they both have huge…outliers, where they made a bunch more money than anybody else during those times, so still hire them but like schedule them during those times.

Bianca: The reason we used some of those part timers, because they have huge outliers. They are the most efficient during those time periods. So, for instance, Kim, I think she was most busy in the steady and slow season, which was extremely different than any other case that we saw, so we believe that she would be best for a part time position. and Terry and Willy kind of had similar results.

Oriana: And Maria we chose for full time, because over all she was really steady. All of her averages for the busy, for the steady and for the slow, were all in the $60 an hour range, which was different than the rest of them, a lot of them had a lot more variation. But she was pretty steady, so she would be the one working full time, because she’s pretty reliable, and you know what to expect.

They recommended that Maya should hire Maria and Tony full time and schedule them to work during the steady and slow times and hire Robin full time but schedule her during the busy and steady times.

In summary, as a group, Oriana and Bianca found unit rates of how much money they made per hour, for each of the vendors during the busy, steady, and slow times at the park. Then, they found the average of the three unit rates. They explained that the number of hours each vendor worked at the park, as well as the time each of them worked— whether they worked during the busy, slow, or steady times of the park, are all important factors to consider. Thus, they concluded that simply looking only at the amount of money each vendor made is not as informative as using unit rates to compare them.

MEA 2: Phone Plans MEA

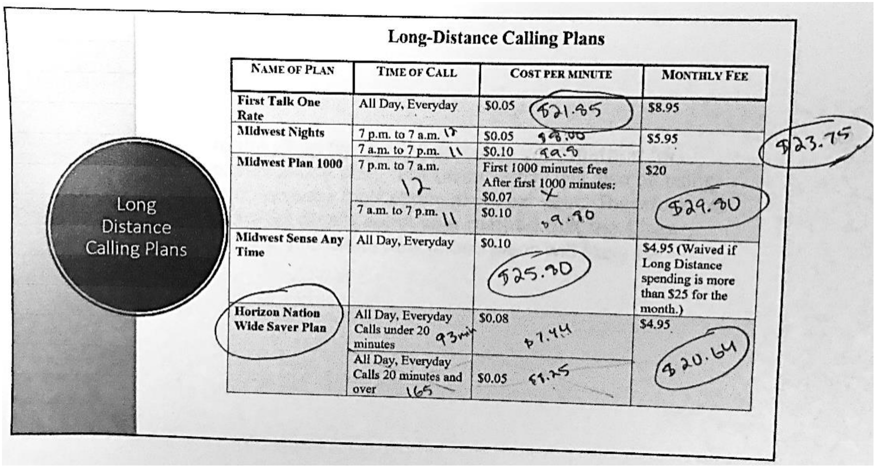
The *Phone Plans MEA* asked the PSMT to help the Olsen family save money by choosing a new long-distance phone plan. They were given a list of phone plans with details about the cost per minute, monthly fees and constraints about the time and duration of the call. The family's long-distance call log for the month of June was also provided. See *Appendix E* for the call logs and the different phone plans. The PSMT were asked to write a letter to the Olsen family and help them select the cheapest plan and describe their method for finding the cheapest plan so they can use it to re-evaluate their phone plan again next year. The problem statement for the task is shown in Figure 16.

> Problem Statement:
>
> The Olsen family just discovered that they were *slammed* and thus, had received a higher phone bill. Instead of simply switching back to their old plan, they decided to look into the various long-distance calling plans that are available to see if another plan would save them even more money. Included in the following pages are a list of the current long-distance calling plans available and a list of the long-distance calls that the Olsen family made in June.
>
> Furthermore, Mr. and Mrs. Olsen realized that the long-distance plans that are available often change over time. So, to take advantage of any new calling plans that might become available, they decided it would be a good idea to review their long-distance calls and their long-distance calling plan once a year to determine if a different plan might save them more money.
>
> The Olsen family needs your help in selecting the cheapest long-distance calling plan for their family. Write a letter to them that describes your method for finding the cheapest phone plan. The Olsen Family will also be using your method to re-evaluate their phone plan next year. Therefore, be sure to include enough details about your method that the Olsen Family can use it next year when new phone plans will likely be available.

Figure 16. Phone Plans MEA Problem Statement.

Oriana, Bianca and Helaine worked together on this MEA, with a fourth PSMT. First, they began by finding the total amount of minutes the Olsen family talked, between 7PM and 7AM and between 7AM and 7PM. They explained that this had to be done because two out of the five phone plan choices had a different price per minute for each time interval. They also found the total amount of minutes for which the family was on the phone over than 20 minutes, and under 20 minutes, because they noticed that one of the plans had a different price for the cost per minute when the phone call duration is more than 20 minutes or less than 20 minutes. They discussed graphing linear equations but abandoned the idea because they agreed that calculating the total price by multiplying the number of minutes by the cost per minute, and adding the monthly fee when applicable, is easier.

The PSMT originally wanted to go with the *First Talk One Rate* plan because the cost per minute was $0.05, all day, every day, and to them this *looked* like it would be the cheapest plan, however after Oriana insisted that they should calculate the prices for all of the plans, the group was surprised to find out that the *Horizon Nation Wide Saver Plan* was in fact cheaper. Helaine and Bianca thought that it would be more expensive because the cost per minute was $0.08 if the call duration was under 20 minutes and $0.05 if the duration was over 20 minutes. Helaine and Bianca were convinced that this plan would be not be the selected plan because of the cost per minute was more expensive than the *First Talk One Rate* plan when the calls are under 20 minutes. However, Oriana pointed out that the monthly rate was “almost half of the first one” and the cost per minute was the same if the call duration was over 20 minutes (Oriana, MEA 2 exploration, January 29). Once they calculated the cost for each plan, they decided to go with the *Horizon Nation Wide Saver* plan because it was the cheapest. Figure 17 shows the prices that the PSMT calculated for each plan.



Long-Distance Calling Plans

Name of Plan	Time of Call	Cost per Minute	Monthly Fee
First Talk One Rate	All Day, Everyday	$0.05	$8.95
Midwest Nights	7 p.m. to 7 a.m.	$0.05	$5.95
	7 a.m. to 7 p.m.	$0.10	
Midwest Plan 1000	7 p.m. to 7 a.m.	First 1000 minutes free After first 1000 minutes: $0.07	$20
	7 a.m. to 7 p.m.	$0.10	
Midwest Sense Any Time	All Day, Everyday	$0.10	$4.95 (Waived if Long Distance spending is more than $25 for the month.)
Horizon Nation Wide Saver Plan	All Day, Everyday Calls under 20 minutes	$0.08	$4.95
	All Day, Everyday Calls 20 minutes and over	$0.05	

Figure 17. Calculated Costs for each Phone Plan in MEA 2.

After the PSMT selected the phone plan that they wanted to recommend to the Olsen family, Oriana, Bianca and Helaine began writing their letter. Initially, the PSMT struggled with composing the letter. Helaine and Oriana recapped the process they went through, orally, for finding the solution:

Helaine: The way we started this before you write it down. Basically, just added their minutes, then labeled them to the times based on the plans available.

Oriana: Whether they were like over 20 or under 20.

Helaine: Over 20 under 20. And umm... then basically we compared each phone plan to find the cheapest call price.

Oriana: For June.

Helaine: For June okay.

Oriana: So, if you want to say that as I write it down.

Helaine: I'm going to try like the best way to explain it, based on the data given…

The PSMT discussed the letter for approximately ten minutes, then Oriana suggested to make an outline prior writing the letter. The outline (shown in Figure 22) consisted of three bullet points describing the steps the group took when exploring the MEA. Once they had some ideas written down, the fourth PSMT in the group began writing the letter, while Bianca, Helaine and Oriana narrated. Below is a small portion of the discussion:

Helaine: So, we added.

Oriana: Minutes, for the monthly minutes.

Helaine: We then labeled them like we separated them depending on time.

Oriana: So 7AM to 7PM and 7PM to 7AM.

Helaine: Then, we multiplied … by the corresponding cost per minute to get. We multiplied it and then added a monthly fee, if applicable, to get the total cost per month, and... and then we can just say, we chose the Horizon Nationwide Saver Plan because...

Oriana: So, we first added up the total minutes spent on the phone, put more detail. We … first added up the total amount of minutes called, for the month, and after that

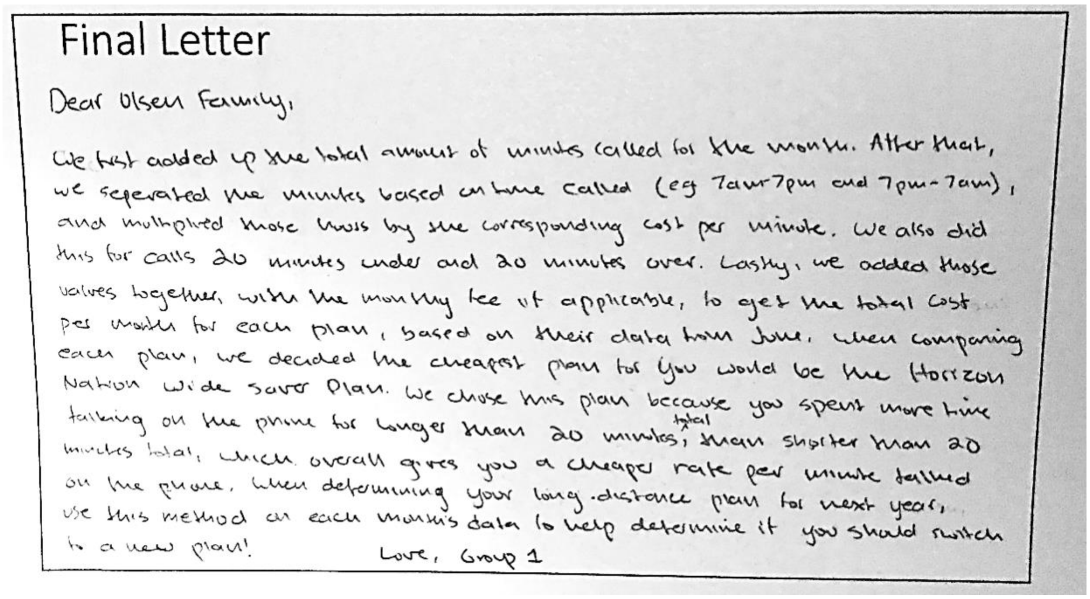
Helaine: …we separated the minutes based on time called and you can put in print this…. Like e.g. 7am-7pm and 7pm-7am.

Oriana: Based on the time called and… multiplied those hours by the

Helaine: Corresponding calls per minute. Also, because we compared the last plan, we also compared how many minutes were spent over 20 and under 20. We have to put that in there I don’t know the best way to word it.

Oriana: Yeah, because that was the plan that we chose too. Maybe like if we grouped them according to whether they were above 20 or under 20.

The PSMT continued writing for the next, approximately, six more minutes. The final letter is shown in Figure 18.



Final Letter

Dear Olsen Family,

We first added up the total amount of minutes called for the month. After that, we seperated the minutes based on time called (eg 7am-7pm and 7pm-7am), and multiplied those hours by the corresponding cost per minute. We also did this for calls 20 minutes under and 20 minutes over. Lastly, we added those values together, with the monthly fee if applicable, to get the total cost per month for each plan, based on their data from June. When comparing each plan, we decided the cheapest plan for you would be the Horizon Nation Wide Saver Plan. We chose this plan because you spent more time talking on the phone for longer than 20 minutes total, than shorter than 20 minutes total, which overall gives you a cheaper rate per minute talked on the phone. When determining your long-distance plan for next year, use this method on each month's data to help determine if you should switch to a new plan!

Love, Group 1

Figure 18. Group’s Final Letter for Phone Plans MEA.

Once they completed composing the letter, the PSMT began reflecting on the process they went through to get to their final recommendation. The participants discussed what they could have done differently and the assumptions they considered and ignored in the development of their solution. For instance, Oriana explained that the only thing the group could have done differently is use graphs to visualize how the plans change over time. However, Helaine explained that would not be accurate because they only have data for the month of June. The conversation below illustrates the PSMT’ reflection on their strategy and why it worked.

Oriana: Is there anything we possibly could have messed up?

Helaine: I don’t think so.

Bianca: Yeah.

Oriana: The only thing we could have done differently is like make graphs to see how it changes over time and how they compare.

Helaine: But we can’t predict what they’re going to call in July.

Oriana: Exactly.

Bianca: If we had more information, may be.

Helaine: Maybe they added somebody to the plan.

I mean, what, they called three times on June 15^{th}, what if they called seventeen times? We don't know. They have to be able to tell any time of the day, whether it would be over 20 or under 20 that they can get the best rate.

Oriana: Yeah, I think this is the right way to do it.

Throughout the exploration of the MEA, Oriana put forward the strategy of using graphs to model the task. Nonetheless, the other PSMT did not see how graphing the cost of each plan would be beneficial. However, Oriana explains why creating linear equations and graphing them may be beneficial in her written reflection, which will be discussed in the *single-case analysis* section.

Not only did the PSMT reflect on the *Phone Plans* MEA after they were done exploring the task, they also reflected and discussed the *Summer Jobs* MEA from the previous week. Oriana and Helaine compared the two tasks and agreed that they enjoyed the *Phone Plans* MEA more, because there were less factors to consider. The following is a part of the discussion between the two PSMT:

Helaine: I liked this activity way better.

Oriana: Than the last one?

Helaine: Yeah, it's just … [For the first MEA], Willy didn't work that day but every other day he kicked butt, it's like what? The thing is, how do you determine if he should be full time or part-time? Because you can keep him part-time because he's good at part-time. Or you can make him full time and work him more.

Oriana: Yeah, there were so many variables to consider. Like for the people that didn't work for like a month, that might have just been like circumstantial and like that summer they might be able to work better.

Helaine: Then you have a college kid one month and then next they're trying to move and it's like, "How are you going to be available." It's like you call Willy,

> "hey buddy you want to come work?" and he's like "Oh no Fam, I have a job at Dunkin' Donuts. It's over."

In summary, as a group, the PSMT began by calculating the total cost for each phone plan, by finding the total amount of minutes the family talked between 7AM and 7PM, 7PM and 7AM, as well as the total amount of minutes the family's phone-calls had a duration of less than 20 minutes and those that lasted 20 minutes or more. They did that because the given information about the costs for each of the phone plans were organized in a similar manner. Then, the PSMT multiplied the sums of minutes in each category with the corresponding costs per minute, and added the monthly fee, when applicable, and that's how they calculated how much each phone plan will cost the family.

The PSMT emphasized that their recommendation is based solely on the family's long-distance calls for the month of June, and if their call patterns change, i.e., they have shorter phone-calls—less than 20 minutes, then a different plan may be cheaper. Once they had a recommendation for the family, the PSMT spent about 20 minutes discussing writing the letter. Although they were able to explain their method orally to each other, to the researcher, and to their classmates, the PSMT agreed that it was harder to write down their explanation than to talk about it. After the PSMT wrote the letter to the Olsen family, they began reflecting on the strategies they used, different strategies, and the previous week's MEA.

<u>MEA 3: Historic Hotel MEA</u>

The third MEA was called the *Historic Hotel MEA.* It asked the PSMT to help a hotel owner choose a daily rate that maximizes profit, given that: (a) all 80 rooms of the hotel are occupied when the rate is $80, and (b) every time the price is increased by one dollar, one less room is occupied. The PSMT were asked to write a letter to the hotel owner with their recommendation and include an explanation of their procedure that the owner can use in the future, even if prices and costs change. The problem statement of the task is shown in Figure 19.

> Problem Statement:
>
> Mr. Frank Graham, from Elkhart District in Indiana, has just inherited a historic hotel. He would like to keep the hotel, but he has little experience in hotel management. The hotel has 80 rooms, and Mr. Graham was told by the previous owner that all of the rooms are occupied when the daily rate is $60 per room. He was also told that for every dollar increase in the daily $60 rate, one less room is rented. So, for example, if he charged $61 dollars per room, only 79 rooms would be occupied. If he charged $62, only 78 rooms would be occupied. Each occupied room has a $4 cost for service and maintenance per day.
>
> Mr. Graham would like to know how much he should charge per room in order to maximize his profit and what his profit would be at that rate. Also, he would like to have a procedure for finding the daily rate that would maximize his profit in the future even if the hotel prices and the maintenance costs change. Write a letter to Mr. Graham telling him what price to charge for the rooms to maximize his profit and include your procedure for him to use in the future.

Figure 19. Historic Hotel Problem Statement.

Helaine, Bianca, Oriana, and a fourth PSMT worked on this MEA together. They began by discussing the goal of the task, which they all agreed is maximizing the hotel owner's profit. Then, they began suggesting different strategies. The first strategy that the PSMT discussed was graphing, first suggested by Oriana, which then led to the discussion about creating a function to model the scenario. The PSMT first assumed that the situation can be modeled using a linear equation. This can be seen in the following excerpt of their exchange:

Oriana: I think definitely graph is probably going to be the easiest way to do it.

Helaine: Oh! Yeah for this one because of the function and stuff like that and some kind of slope or something. So, 80 rooms for sure, okay. So, when all are occupied daily rate is.

Bianca: Are empty.

Oriana: We could start off with the intercept, that maybe the easiest to do. I could say, where would it be where no rooms would be occupied, we would have to add $80.

Helaine: Yeah, because if 79, yes, so it would be 80 plus 60.

Oriana: So, 140, so, if he charges $140 no rooms would be rented.

Helaine: Yeah, that would be so.

Helaine: Well, first we need to find how much like the total is, so 60 times 80 will get us the total and then we need to subtract 60 minus, or 60x and then x will be the number of rooms not occupied.

Then, the group shifted their focus to the maintenance cost. The PSMT deliberated whether the maintenance was a cost to the hotel owner, or to the customers. They went back and forth over whether the daily $4 cost would be added or subtracted from the total amount of money the hotel acquires. Oriana proposed for the service and maintenance fee to be subtracted, while Bianca suggested that it should be added. Bianca explained that the fee should be added because the customers would have to pay the fee, so it should be added to the total amount of money the customer pays the hotel. Helaine first agreed with Bianca's reasoning. Then, Oriana explained that this is a cost to the hotel owner, so it would have to be subtracted from the amount of money Mr. Graham receives. She further clarified that it's not an added charge because:

> It would probably just have been included in the rate if it was being charged to the people. It probably would just be like a $64 daily rate. But this way would be like you know, that hotel is actually having to pay that, so it's taken away (Oriana, MEA 3 Exploration, February 5).

Once Helaine heard Oriana's explanation, she then agreed that the maintenance fee should be subtracted, and Bianca and the other PSMT went along with the subtraction of the cost strategy.

Next, the group deliberated how to calculate the best rate per room, to recommend for the hotel owner. First, they calculated the profit if all 80 rooms in the hotel were to be occupied. They calculated the amount of money acquired by the hotel, which they found to be $4800 by multiplying $60 by 80, and then subtracted the total cost for maintenance which they found to be $320, by multiplying $4 by 80. Therefore, they found that the profit if all the room were occupied would be $4480. As they were discussing how to move on to calculate the profit for a different rate and amount of rooms occupied, Oriana suggested creating a graph profit vs. number of rooms occupied graph (shown in Figure 25 below). Helaine and Bianca on the other hand wanted to derive a function that can represent the given scenario. The following is the dialogue that took place between the PSMT:

Oriana: Okay, so if 80 rooms rented, the cost would be $4480. Oh no, that's profit, not cost.

Helaine: Yeah, profit. All right, so when there is for example zero rooms.

Oriana: I'm putting it on here too [she said pointing to the graph she created, which is shown in Figure 25 below]. The cost per room, I mean profit, and then the number of rooms occupied. Now we can see how that changes.

Helaine: Okay, so like, what's the formula we can write for this?

Bianca: Can we come up with a solid formula because if the rate is changing then that would be a separate line every time?

Helaine: Yeah, that's what I want to do is try to come up with a formula that works because I mean your *x* is going to be the amount of rooms. So, if it's…

Oriana: We could try to do, you know, this version and then also like no rooms being occupied and then that can give us like *intercept* maybe and then we can try and make a line off of that.

Helaine: I'm trying to think of the actual question, because you're going to have 80 rooms, so, then if that's your x it's going to fluctuate. If its $60 a room that's the max you can go, so you're going to do 60 plus, what? One, or not one, but every time you lose a room is going to be…

Oriana: Yeah, one, it increases by a dollar, then you lose one room.

Helaine: So, it's going to be 60 plus something times x, right?

Bianca: Like if we made x the other rooms with 60 times minus the one…

Helaine: Plus, one, wait, yeah, because it goes up.

Bianca: Yeah, plus one.

Oriana: Wait, if we do like 60 plus the number of rooms, so, when it goes up by, wait no, it should be like 80 minus the number of rooms, right?

Helaine: Wait, yeah, because you do 80 minus whatever rooms are unoccupied.

Oriana: Let's figure out what the price per room is going to be, and then what the number of rooms is going to be, then we can multiply them.

Bianca: Say that again.

Oriana: I feel like if we figure out what the price per room is going to be, considering it might be changing and then what the number of rooms is going to be considering that might change too, then if we multiply the price times the number of rooms, and that is going to give us the total. Then we have to subtract the $4 cost and then we will also figure out the cost. So, it can be like, *price times number of rooms minus cos*t. So, let's trying and figure out each part.

At this point, the PSMT go back and forth trying to represent "each part" of the "*price times number of rooms minus cost*" that Oriana mentioned. Helaine suggested to represent the *number of rooms* term as 80 – *n,* where *n* is the number of rooms, and then she suggested the *price* term be represented by 60 + *n*. Helaine explained that this works because "if we do it as in like you're losing rooms, so if *n* is one room, you're losing so you're going to have 79 rooms which should be 61", which led Oriana to say that "this would be 60 plus 80 minus *n,* because if we have 79 rooms occupied, that means one room is left when we subtract 80 – *n,* so then we can add that difference to 60 and make it 61, which would be the price if 79 rooms are occupied." After Oriana spent a few more minutes clarifying why this works, Helaine explained that she thought about *n* as the number of unoccupied rooms, she said "so if there are 79 occupied rooms, then *n* would be 1, so 80 – *n* is 79 and 60 + *n* is 61." Then, Oriana concluded that what Helaine came up with will help them with calculating the "total", not just the price. At this point, *Bianca* asked clarification from her group members about what the variable *n* represents, to which *Helaine* clarified, “*n* is equal to the rooms that are not occupied, so when one room is unoccupied, you’re going to have 79 rooms occupied.”

The PSMT continued by trying to represent the *cost* term, which Oriana and Helaine agreed would be represented by 80 – *n* times 4, because only the occupied rooms cost the hotel $4 a room, for maintenance. The function the group came up with is shown in Figure 20.

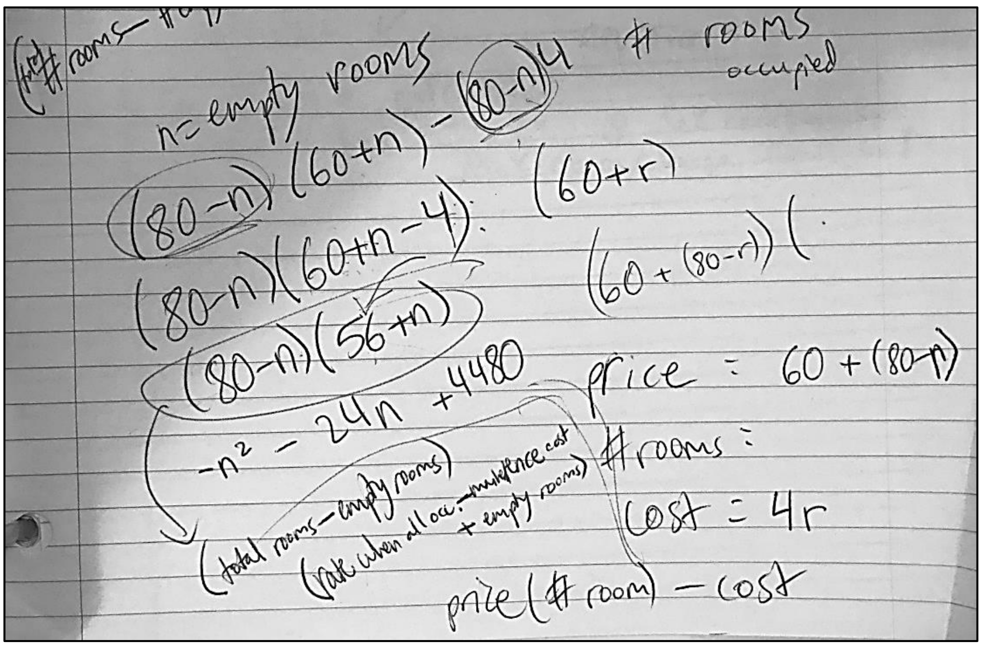


Figure 20. Derived Profit Function for Historic Hotel MEA.

At this point, Oriana asked Bianca if she’s making sense of what the group was doing, but since Bianca hesitated to answer that question, Oriana explained to her the process they went through to come up with their function. The PSMT then checked if their function worked by substituting different values of *n*, more specifically, Helaine substituted *n* = 5, and then *n* = 1, and the group agreed that their function works. Bianca was not sure what the total number they were getting represented, to which Helaine clarified it is the profit that the hotel is making.

Oriana continued trying to simplify the function, which she did by factoring out the common term of $(80 - n)$.

Next, the PSMT began working on finding the price that would maximize the profit, and right away, Oriana suggested to use a graph, because the function would be represented graphically by a "*downward parabola*" and it would be "*maximized on the middle point*". The PSMT then used an online graphing calculator, to graph their profit function. After the function was graphed, the group discussed adjusting the scale for the *y*- and the *x-axes*. Helaine suggested that the *x-axis* should be between zero and 80, Oriana agreed, and justified that choice by explaining that this works because it represents the number of rooms, and the most amount that can be is 80, and the smallest number that can be is zero. When it came to the *y-axis* Oriana suggested that it should be from zero to 5000, "because we don't obviously want to look at the negative profit, and the profit is not going to be more than $5000". The graph that the PSMT plotted is shown in Figure 21.

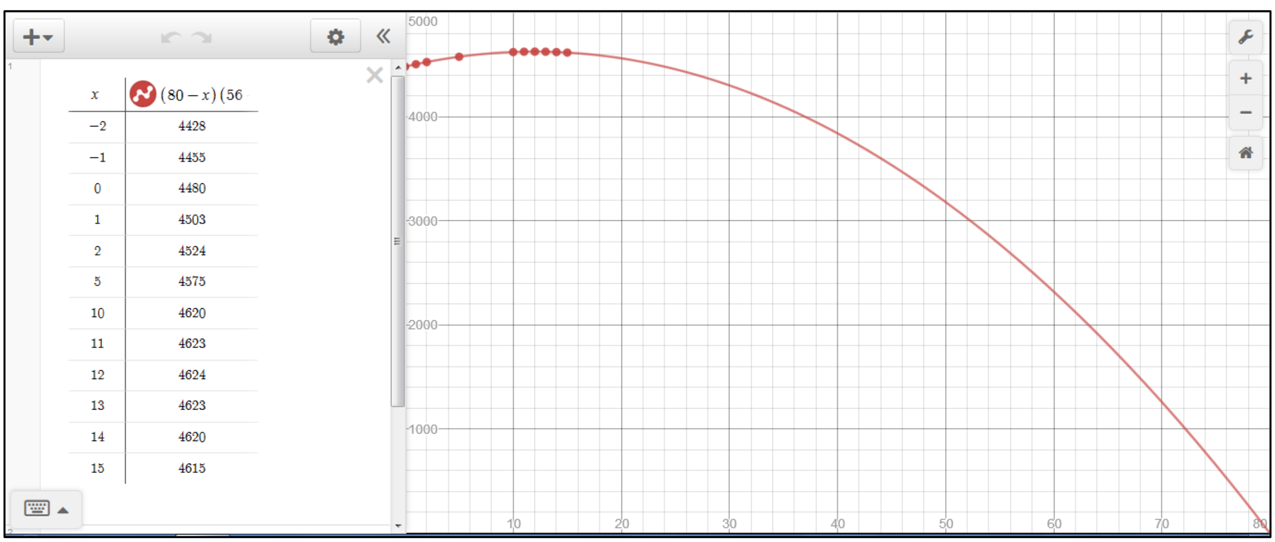


Figure 21. Graph of Profit Function for the Historic Hotel MEA.

After graphing the profit function, the PSMT used their graphing calculators to find the maximum point, because the online graphing calculator did not have a feature to give the maximum point on a graph. Although the table they generated on the online graphing calculator gave them an idea of where the maximum point on the graph was, they wanted to get an accurate maximum and not want to estimate. Oriana used the *calculate maximum* command on her graphing calculator and found the maximum occurs when x (or n, using previous notation) is 12. Then, the PSMT discussed what that actually meant in terms of the problem. Their dialogue is included below:

Oriana: So, that's 12 unoccupied here, so that would be 68 rooms, this is max profit.

Bianca: Equals 4624, so that's in the calculator.

Helaine: We can show, I mean we'll have the graph here so that way they can look at it.

Bianca: How did you get this?

Oriana: The way that he can change it for his prices or whatever is he would just change the 80 to produce x amount.

Helaine: He just always should have it as 80 yeah, like 68, yeah, $68 per room and then that will be max profit, right?

Oriana: Was it?

Helaine: Yeah, equals 12.

Oriana: But no 72, this is rate per room, $72, he needed that and then that would be 68 rooms available, occupied.

Helaine: 68 plus 56 plus, what would it be?

Oriana: 80 minus 12, 68, so there will be 68 rooms occupied.

Helaine: That make sense because that's when they're both 68.

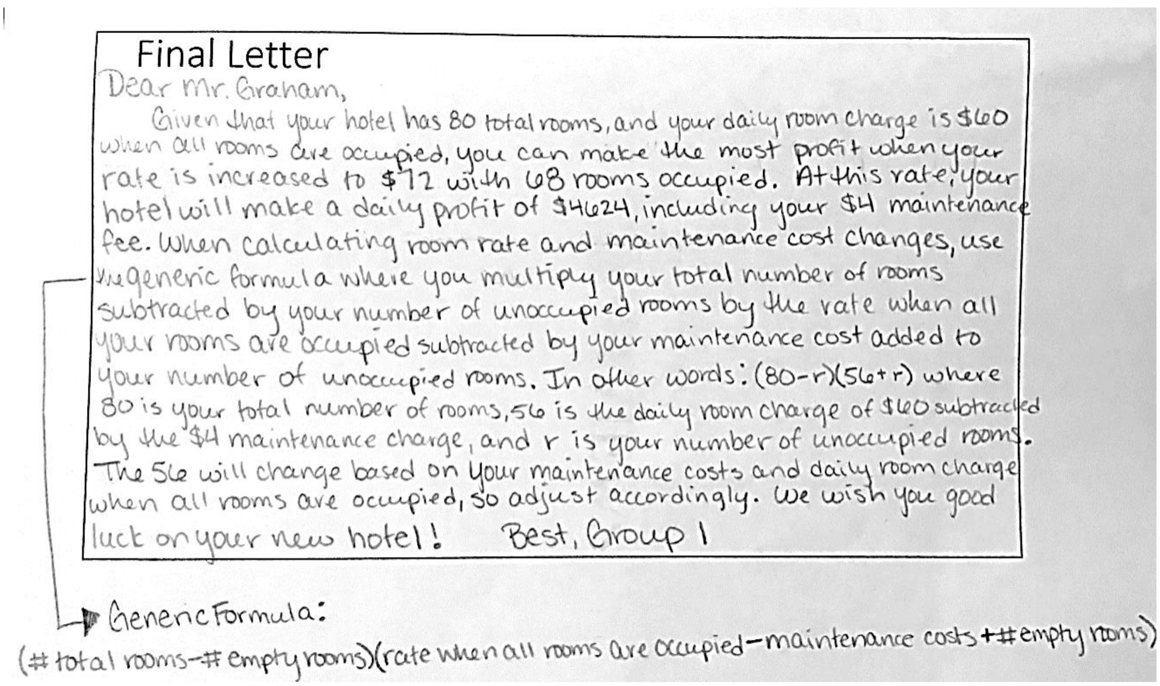
Oriana: Yeah.

Helaine: 2624, so, $68 per room.

Oriana: Wait, it would be, would it be 72 per room.

Helaine: $72 because yeah, we have to do that… got it.

Once the PSMT agreed that the maximum profit occurs when the hotel owner charges $72 per room, the group began working on writing the letter for Mr. Graham. They began by generalizing their profit formula, because, Oriana explained, "*everything changes, like price, how much in fluctuates, and maintenance cost.*" The generalized formula is included in the letter, which is shown in Figure 22 below. The group spent about twelve minutes writing the letter and debating how specific they should be with their generalized formula and description of their procedure.



Final Letter

Dear Mr. Graham,

Given that your hotel has 80 total rooms, and your daily room charge is $60 when all rooms are occupied, you can make the most profit when your rate is increased to $72 with 68 rooms occupied. At this rate, your hotel will make a daily profit of $4624, including your $4 maintenance fee. When calculating room rate and maintenance cost changes, use the generic formula where you multiply your total number of rooms subtracted by your number of unoccupied rooms by the rate when all your rooms are occupied subtracted by your maintenance cost added to your number of unoccupied rooms. In other words: (80-r)(56+r) where 80 is your total number of rooms, 56 is the daily room charge of $60 subtracted by the $4 maintenance charge, and r is your number of unoccupied rooms. The 56 will change based on your maintenance costs and daily room charge when all rooms are occupied, so adjust accordingly. We wish you good luck on your new hotel! Best, Group 1

Generic Formula:

(# total rooms - # empty rooms)(rate when all rooms are occupied - maintenance costs + # empty rooms)

Figure 22. Group's Final Letter for Historic Hotel MEA.

As the PSMT presented their work to the class, they were asked about why the four in their non-simplified formula was negative, and Oriana and Helaine both explained that they assumed that the hotel has to pay the maintenance cost, so they would subtract it from the amount of money the hotel receives, to calculate the profit.

In summary, the PSMT derived a function to model the profit earned based on the pattern the previous owner shared with Mr. Graham. They used the general idea of *price times number of rooms minus cos*t to help them develop their profit function. After discussing the pattern between the number of occupied rooms and the rate for each room, the PSMT modeled the given scenario by the function $y = (80 - n)(60 + n) - 4(80 - n)$, where y represented profit, and n represented the number of unoccupied rooms. They further simplified the profit function as $y = (80 - n)(56 + n)$. They used their function to find the price that would maximize the profit for Mr. Graham by graphing it and using their graphing calculator to help find the maximum point, which was at (12, 4624). After discussing what this point meant in terms of the context of the problem, the PSMT concluded that the maximum profit occurs when the room rate is $72 and there are 68 rooms occupied. Lastly, the PSMT wrote the letter for Mr. Graham, and created a *generic formula* to help him repeat their procedure when the prices or costs change. The generic formula was described as the following: (number of total rooms – number of empty rooms) (daily rate when all rooms are occupied – total number of empty rooms) – (maintenance sot of occupied rooms) × (total number of rooms – total number of empty rooms), which they then simplified again to: (number of total rooms – number of empty rooms) × (daily rate when all the rooms are occupied – maintenance costs + number of empty rooms).

Chapter Four Summary

This chapter presented the PSMT' exploration of the three MEAs, as a group. The group consisted of Oriana and Bianca for MEA 1 (*Summer Jobs)*, and all three PSMT for MEA 2 (*Phone Plans)* and MEA 3 (*Historic Hotel)*. These findings were gathered from audio and video recordings of the group while exploring the MEAs, as well as participants' artifacts. The results discussed in this chapter show the process in which Oriana, Bianca and Helaine explored the tasks, and demonstrate how the MEAs provided a context for planning, analyzing, writing, forming connections, and reflecting on the mathematical content and processes.

CHAPTER FIVE: SINGLE-CASE ANALYSIS

This section describes Oriana's, Bianca's, and Helaine's experiences with the three MEAs and their MKT algebra. Their experiences with the MEAs were examined through their recorded explorations of the three MEAs, written artifacts, the two conducted interviews, and their written reflections. Furthermore, the Knowledge of Algebra for Teaching (KAT) framework was used to explore the PSMT' MKT algebra, by looking at the degree to which the PSMT were able to:

> (1) make connections across topics, assignments, representations and domains (*bridging*);
> (2) remove complexity while retaining integrity, and getting to the mathematical essence of a real-world problem (*trimming*), and;
> (3) unpack the complexities found in mathematical ideas in ways that make the content comprehensible, anticipate the difficulties entailed in students' understandings of particular aspects of algebra content, help students understand the usefulness and importance of algebra, and decompress rote procedures in order to grasp the logic behind them (*decompressing*) (McCrory et al., 2012).

The degrees to which the PSMT were able to demonstrate evidence of *trimming*, *bridging,* and *decompressing* was described as *high* or *strong evidence*, *medium* or *some evidence,* and *low* or *no evidence.* High or strong evidence were used if the PSMT was able to talk about and explain the different ideas corresponding to each category. Medium or some evidence were used if the PSMT only mentions the different ideas relating to each category, without explanation. Low or no evidence were used if the PSMT either does not mention, or incorrectly mentions, the different ideas pertaining to each category. For example, if a PSMT discussed the connections between the vertex formula of a quadratic, the first derivative and second derivative test, and finding the maximum of a parabola, and explained how these mathematical ideas are related, then it was determined that the PSMT demonstrates high or strong evidence of *bridging*. If a PSMT mentions that there are connections between the three mathematical ideas, without further discussion, then it was categorized as medium or some evidence of *bridging.* If a PSMT does not mention those connections, or incorrectly describes how they are related, then it was considered as low or no evidence of *bridging*.

Moreover, during each interview, the PSMT were given two content questions from the LMT assessments and were asked to solve them using the think-aloud strategy in order to gain an understanding of any changes in the way the PSMT explain and justify their solutions to the questions, after their engagement in one MEA, and then again after exploring the third MEA. The PSMT' responses are discussed in the *Interview* section of each case. Lastly, the PSMT' performance on the LMT assessments, that were administered through TKAS, are also discussed for each case.

For each of the three cases, a description of the participant is presented, then the PSMT' exploration of the first MEA is discussed, followed by a discussion of the first interview. Next, the PSMT' exploration of the second MEA is presented, followed by the written reflection. Lastly, the PSMT' exploration of the third MEA is examined, followed by a discussion of the second interview, as well as the LMT assessments. A summary of each of the cases is also included.

Case 1: Oriana

"I love Math! I don't think I will ever get tired of doing it."

Oriana, January 2018.

Description of Participant

Oriana was selected from the *high MKT* category— which included participants with scores greater than 50th percentile and less than or equal to 90th percentile on the LMT pre-assessment. Oriana's score on the LMT pre-assessment was in the 90th percentile. Oriana was in her fourth year of the secondary mathematics education program. She has always enjoyed her mathematics classes. She explained that she has always loved learning mathematics because she views the content as a puzzle, and it always *"comes pretty easy and quickly"* to her (Oriana, First Interview, January 19). Throughout her high school career, Oriana tutored her peers in mathematics and after graduating high school she began working for a mathematics tutoring company where she had been a lead instructor for six months at the time of the first interview. She explained that she loves her job and it has helped her gain a better understanding and view of mathematics, because of having to explain content to her students in multiple different ways.

During the semester that this research was conducted, Oriana was enrolled in a *number theory* course and a *statistics* course. When asked to give an overview of the content being taught in those classes, she said:

> In number theory right now, we just started congruences and then we just went over divisibility and primes. In statistics we just started ANOVA, and we just finished other things with different variances like confidence intervals. (Oriana, Second Interview, February 7)

After graduating, Oriana's goals were to teach high school mathematics and later perhaps attend graduate school.

MEA 1-Exploration of the Task

As previously stated, Oriana explored the *Summer Jobs* MEA with Bianca, and another PSMT who was not a part of this study. The task was shown in Figure 14 above, and *Appendix D*. From the beginning of the exploration, Oriana suggested to her group that using *unit rates* and *averages* to compare the different vendors is key. She explained her strategy to the group members:

> I think a good way to go about this is thinking of amount of money and hours made for each time of the day, so for busy, slow and steady, and then compare the unit rates for each one. I feel like that's the best way because it takes into account the numbers of hours that they worked.

Using McCrory and colleagues' (2012) description of *trimming*, one can see that Oriana realized the *mathematical essence* of the MEA was the use of proportions, or what she identified as unit rates.

As her group members discussed different strategies, Oriana was open to exploring the task using the various methods, but she was persistent in relating the method to the context of the problem. First, Bianca suggested to look at the total amount of money each vendor made, and then choose the vendors that made the business the most amount of money. To that, Oriana replied:

> But it depends because they are working different amounts of hours. Like if you look at Kim's, she only worked 5.5 hours when it was busy, but she worked 15.5 hours when it was slow. It's going to depend per person.

When another PSMT brought up that calculating all the unit rates and averages for all the vendors may take a long time, and perhaps using *estimation* may be easier, Oriana defended her strategy by discussing the context of the problem:

> I think we will still have to do the same amount of calculations regardless, but it will just depend on whether we use a calculator or not, so might as well make it exact. Because if you're in that type of business situation you would want it to be pretty accurate.

Another strategy discussed in the group was using graphs to represent the given data and compare the vendors visually. Once more, Oriana insisted that they should consider all the data they were given, not just the amount of money each vendor made. The group was unable to create a graph that represented both the amount of money made and the hours worked, using an online graphing calculator. After their attempt to create a graph, she said:

> What is the best way to graph this? Should we do a graph for busy, slow and steady, or a graph for each person with like three lines, busy steady and slow… And it's weird because I don't know how to combine that with the hours worked. Because if we just graph how much money they made it's not going to tell us anything really… And if we do just one and not the other, the graph isn't really going to tell us anything, because it's not taking into account the time of day. Also, we will be relying on what we see vs. the actual numbers.

The last strategy the group members discussed prior to calculating the unit rates and averages was using the given data to create equations. At this point, Oriana was convinced that unit rates and averages were the best method to explore the MEA. She had already begun calculating some of the unit rates for the vendors. She, once again, used the given context of the MEA to make a case for her strategy. She related equations with making predictions and explained that's not the what their task asked them to do. She said:

> I don't think we need to make an equation... Because for this as long as we can tell what they are doing for each [attendance category], it will be fine. Because she's not asking us to like figure out…it's more like we don't have to predict it, we just have to see who is the most efficient, and then suggest which of those people we should hire.

Once again, Oriana is showing strong evidence for *trimming* by describing the mathematical essence of the given task in explaining why specific strategies may not be beneficial to use.

After the others agreed to use unit rates and averages, Oriana repeatedly explained to her group members how to do the calculations. The following is Oriana's explanation for finding the unit rate for each attendance category:

> Adding up all the money when it was busy and then dividing by the total number of hours she worked when it was busy. So, like for Maria, you can do 690 +699+ 788, and divide by the total number of hours that she worked. So…35. So that makes it about… $62.20 per hour when it was.

Oriana calculated the unit rates and averages for Maria, Kim and Kerry. Her work is shown in Figure 23 below. Once the unit rates and averages were calculated for the vendors, Oriana suggested to organize the results in a table (shown in Figure 15 above), to make it easier to compare all the vendors based on the average amount of money they made per hour for each attendance category.

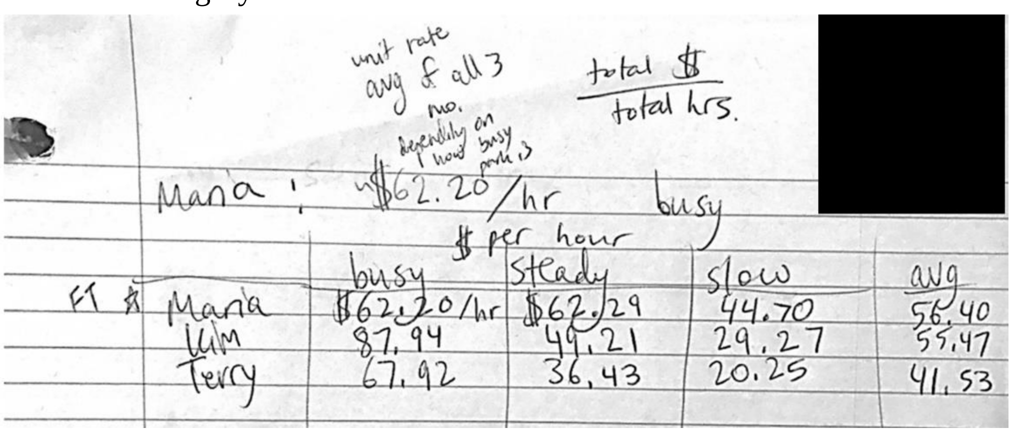


Figure 23. Oriana's Unit Rate Calculations for the Summer Jobs MEA.

Oriana presented the solution to the rest of the class. She discussed what the group did—calculated unit rates and averages; how they did that—by finding the total amount of money each vendor made for each attendance category and dividing by the total amount of hours the vendors worked for the corresponding category; and why the group chose their method— to take into account all the variables in the given scenario (hours worked, and amount of money made for each attendance category). Oriana discussed that they came up with their final recommendations based on how each vendor did during the *busy, steady,* and *slow* times at the park. She discussed the reason behind choosing some employees to be full time and others were chosen to be part time, based on their unit rates during each attendance category. She explained:

> So like Robin and Tony, we said they should both work, but schedule them at different times, so like Robin should work mostly during the busy season and the steady season, and Tony would work mostly in the steady season and the slow season, because they both have huge outliers, where they made a bunch more money than anybody else during those times, so still hire them but schedule them during those times… Maria we chose for full time, because over all she was really steady. All of her averages for the busy, for the steady and for the slow, were all in the $60 an hour range, which was different than the rest of them, a lot of them had a lot more variation. But she was pretty steady, so she would be the one working full time, because she's pretty reliable, and you know what to expect.

More of Oriana's reflections on, and experiences with the *Summer Jobs MEA* are described in the following section.

First Interview

The first interview with Oriana took place four days after the exploration of the MEA. After answering the ice breaker questions (included in the interview protocol in *Appendix H*), the researcher started asking Oriana questions about the *Summer Jobs* MEA she had explored a few days prior, and the LMT pre-assessment she had taken the previous week.

Experiences and Reflections related to MEA 1

During the first part of the interview, Oriana discussed her experiences with the *Summer Jobs* MEA. When she was explaining how she made sense of the MEA, she started by detailing the assumptions that she made, more specifically, the factors that she thought were most relevant and influential, and factors she discarded. She said:

> What I figured is that, we didn't really have to worry about June versus July versus August because they had already sorted it when it was busy, when it was slow, when it was normal. I kind of combined all of that data, based on busy, slow and [steady]… I

> combined those and then tried to figure out, the dollars per hour that each person made depending on, which time of day it was, whether it was busy or whatever. To me that made the most sense because it basically impossible to try and compare it just money wise, since you don't know how many hours they worked. Somebody worked way more hours, they could be making more money even if they are not as efficient as someone else. That's kind of how I tried to make sense of it, was getting them all the same types of units per thing… dollars per hour for each person.

Accordingly, strong evidence of *trimming* was evident in the above explanation because she was able to discuss the mathematical essence of the given real-world problem.

She then talked about the specific mathematics content that helped her make sense of the MEA. Oriana once again described calculating the unit rates for each vendor and then finding their average or mean. She then discussed comparing the averages and ordering them from least to greatest. She also explained that if the group had more time, they would have also found the median to compare the vendors. When asked about how these mathematical ideas are connected, Oriana stated:

> Okay. I feel like kind of comparing kind of connects them because we needed the unit rates to be able to compare fairly, then mean and median are fairly similar, they just depend on your data, which one is going to be better to use. That's kind of how I would do it, rates are kind of connected by it (Oriana, First Interview, January 19).

Although Oriana said that these mathematics concepts are connected, she was not specific in identifying which concepts are connected and was not able to explain how they were connected. Thus, there was low evidence *bridging* from this part of the interview. She later described the difference in using the mean and the median when reflecting about the use of MEAs in an algebra classroom and possible student misconceptions, which are discussed next.

As Oriana began reflecting on her experience with the *Summer Jobs* MEA, she discussed how going through the modeling process helped her see the connections between unit rates, means, and medians, as well as identifying the best strategy to solve the open-ended task. She stated:

> I hadn’t even thought of these things being super connected before but doing it in this kind of context made them seem more, grouped to me almost… It’s neat because these things can be grouped for this problem, and then you use them to completely separately for different things.
>
> I also feel like it kind of put in a new light for me trying to understand, which would be the best strategy to use… It helped me with figuring out the best strategy because I realized we could take the mean of a million different things, we could do average of the money, there are so, many different pathways we could take… so thinking of when the right time is to use different types of strategies, it definitely helped me see that.

When reflecting on how this task can be used in an Algebra class, Oriana discussed the overall advantages of modeling, not specific to this task. She explained:

> I think it would really help… like make them kind of hungry to figure out more, which is awesome, because that’s kind of my goal for my all my students to realize that Math isn't just a bunch of numbers… I feel like that could be really helpful in a classroom kind of with getting them to think critically. Like giving them something where they feel, “Oh! Well, I'm actually using this, and I would use it in the future, this makes more sense.” Also, interesting with this, it kind of shows them how people think so differently, because what one group comes up with is, it’s going to be totally different from what another group comes up with and that’s okay.

Similarly, she discussed how open-ended tasks similar to this MEA would allow teachers to focus on various mathematics content being taught to the students. She gave the following example:

> Say you wanted them to model something with… if they had a business and their cost versus their profit or something, they would probably end up with a graph drawing a line and drawing a linear equation, coming up with it without even realizing it. Then you can kind of break down how they came up with that equation. That will kind of help them see why they should know this [mathematical content] and how it is actually used.

Although not specific to the *Summer Jobs* MEA, Oriana's reflections demonstrate strong evidence of *decompressing* in explaining how to help students understand the usefulness and importance of algebra.

Lastly, Oriana discussed possible misconceptions students may have with solving the *Summer Jobs* MEA. She discussed mean and median and outliers. She explained that "some students may not know the difference between a mean or a median or realize that outliers affect which one should be used". This reveals some of *decompressing* by anticipating the difficulties entailed in students' understandings.

Content Questions

The next part of the interview entailed the two content questions from the LMT assessments. As previously stated, the questions were not part of the items released by the LMT project, and thus, the description of the findings in this section do not have specific details of the context of the questions.

The first question, Oriana was asked to choose a story problem that does *not* represent linear behavior in the context of the four given story problems. A description of each choice was given in chapter three. When thinking about this question, Oriana correctly explained that the first choice would be exponential and not linear, because the initial amount gets doubled every day, rather than "change by the same amount every time, it will [change] by varying amounts". The second choice represented linear behavior and could be modeled using a proportion and the equation y= *m*x. Oriana explained, "this one works for me… it makes more sense as a linear equation because it's keeping the same ratio every time, which means the slope is constant". The third choice, Oriana explained also works by identifying the y-intercept, and the slope. Similarly, the last choice she said represented linear behavior because the initial amount was "changing by the same amount every time". Thus, Oriana concluded that the first choice was the only one that did not represent linear behavior. Oriana answered the question correctly, and showed strong evidence of *trimming*, and *bridging* by getting to the mathematical essence of each of the four real-world problems (*trimming*) and making the connection between ratios and slope of the linear equation in explaining the second choice (*bridging)*, but there was low evidence for *decompressing*, because she focused on rote procedures of writing linear equations.

The second question was question 35 on form B (which was described in chapter three). Oriana was asked to choose the situation that could be used as an example for teaching linear functions. The first situation could be modeled using a linear function of the form $y=mx$. Oriana struggled with determining if the situation represents a linear function. She said that "it would not be much of a whole line, it's more like a line segment". She left it alone and came back to it after looking at the other two situations. Oriana explained that the second situation worked because "starting with an initial amount and increasing by the same amount every time" would represent a linear function. The third situation Oriana was sure would not represent a linear function because "it goes up by 2, then 4, then 6, then 8, so it's not going to be a straight line". Then, she went back to the first situation and started substituting different values for the variable to check if represents a linear function. By doing this procedure, Oriana was able to recognize that it in fact does represent a linear function. Therefore, Oriana explained that situations one and two could be used as examples of linear functions, but the third situation could not.

When answering the second question, Oriana relied on the use of a procedure to check if the first situation represented a linear function, thus there is no evidence of *decompressing*. Although Oriana made sense of the second and third situations in terms of the context of the problem, some evidence of *trimming* can be seen by recognizing the mathematical essence of the real-world scenarios. However, she did not make connections across the different topics, representations or domains, thus there was no evidence of *bridging.*

MEA 2- Exploration of the Task

Oriana worked on this MEA with Bianca, Helaine, and a fourth PSMT that was not a part of this research study. The task was shown in Figure 16 above, and *Appendix E*. As the group began discussing the assumptions behind the task, Oriana brought up the following factors to consider: the duration of each of the calls the family made, and when the calls were made (between 7AM and 7PM, or between 7PM and 7AM). Once the group began comparing phone plans, Oriana right away suggested a graphing strategy. She stated:

> Maybe we could like… just graph the lines for this [phone plan] and this, this and this. Because then we can see where the break-even point is for each one. So, like which one could be the most and then we could add up how many minutes they did to kind of see where each one is at the total minutes (Oriana, MEA 2 Exploration, January 29).

However, Oriana's recommendation of graphing got pushed aside as Helaine began to calculate the total cost for each plan. Oriana continued to work with the other PSMT and assisted in calculating the cost for each plan by multiplying the total number of minutes and the cost per minute and adding the monthly fee, when applicable. After calculating the cost for the *Midwest Sense Any Time* phone plan—which offered a $0.10 cost per minute, all day, every day, with a $4.95 monthly fee that could be waived if the family spent more than $25 for the month. The PSMT found that the family's bill would be $25.80 using this plan. It's important to note that none of the PSMT suggested that $4.95 be subtracted from the total cost, because it was over $25, making the total cost $21.80 instead. At this point, Bianca and Helaine were satisfied with the cost, however, Oriana suggested that they should check the other phone plans because they might be cheaper.

As Helaine and the fourth PSMT were calculating the cost for each phone plan using a calculator, Oriana was observed and confirmed the calculations were correct. She repeatedly questioned them on which numbers were being adding, if they added the applicable monthly fee, or if they multiplied by the correct rate. After calculating the costs for all the plans, the PSMT were all surprised that the *Horizon Nation Wide Saver Plan* was the cheapest. Then Oriana made sense of why that is the case by explaining that, “if you think about it, the majority the minutes, even if they don’t do that many [of them], are over 20 minutes. That builds up a lot of minutes at the cheaper rate”. Once more, she was able to get to the mathematical essence of a real-world problem, and thus, strong evidence of *trimming* is seen.

Next, the PSMT began writing the letter to the Olsen family. Oriana suggested to make an outline prior to writing the letter to “stay organized and not forget any of the details”, she explained. A picture of the outline Oriana created is included in Figure 24 below. Oriana, along with Helaine, were recalling the steps that the group took prior to choosing the phone plan and narrating the process, as the fourth PSMT wrote them down in the letter. Oriana then suggested to include that

> this recommendation is based on the data from [the month of June], and if this wasn’t a normal month for them, then they need to reevaluate. We could even say too because …like they’ll do it again next year, they will have the data from like every month, so they can do it more precisely to see which plan to choose.

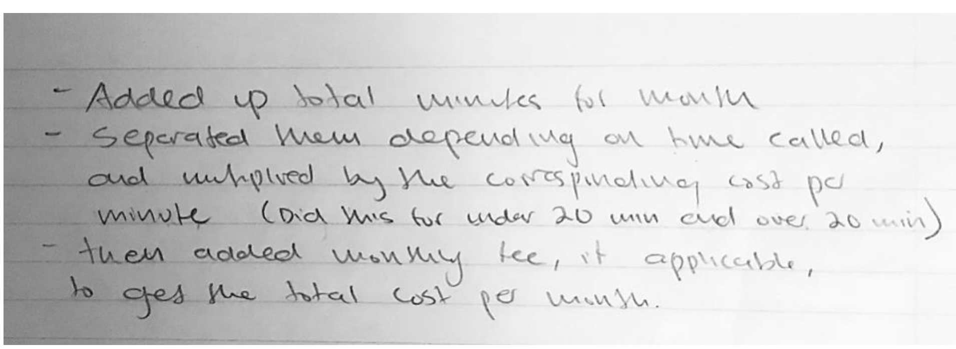

- Added up total minutes for month
- Separated them depending on time called, and multiplied by the corresponding cost per minute (Did this for under 20 min and over 20 min)
- then added monthly fee, if applicable, to get the total cost per month.

Figure 24. Oriana’s outline for Phone Plans MEA Letter.

As the group continued writing the letter, Oriana kept justifying the choices they made as they explored the MEA. For example, when they wrote that the cheapest plan is the *Horizon Nationwide Saver Plan*, Oriana once again clarified that "this makes sense because like they can't benefit from the lower rates from the other ones, because [the calls] are like split 50-50, for the times they are being made" referring to the phone plans that offer cheaper rates when the calls are made between 7pm and 7 am.

Once the group finished writing the letter, Oriana once again brought up using graphs to help explain the solution they came up with. She said, "[Graphing] could help us see what would happen in the future, like if they talked longer than this or whatever. We were just kind of assuming this is an average month, but it may not be". However, the rest of the group members did not want to do more work and suggested that they can just explain it out loud if someone brought it up during their presentation to the class.

Lastly, as the group was waiting for the rest of their class to finish, they began reflecting on the *Phone Plans* MEA and the *Summer Jobs* MEA. Oriana, for the third time, brought up the graphing strategy and explained that "the only thing that we could have done differently is like make graphs to see how it changes over time and how they compare, but I think what we did is the right way to do it". Because Oriana made connections across representations, strong evidence of *bridging* was demonstrated. The PSMT also discussed the MEA that they explored the previous week, and they all agreed that they preferred exploring the *Phone Plans* MEA over the *Summer Jobs* MEA. Oriana explained that she liked this MEA more because there were not as many variables to consider as the previous one. She added that "even though it was so intimidating the first time you see it, but it wasn't that bad, I liked these calculations better".

Reflections on MEA 2

Oriana wrote a reflection about her exploration of MEA 2 and sent it to the researcher a few days after she explored the *Phone Plans* task. First, Oriana discussed her experiences with the modeling task. She explained that the group focused on finding the Olsen family the cheapest plan and worked towards the goal by looking at the given data and calculating how much the family would have to pay, for the given month, under each given plan. She also described the assumptions the group made in relation to the context of the problem. She explained, "we ignored the number of people in the family, the days of the month on which they called, and the numbers they were calling…we ignored them because we knew all the family cared about was getting the cheapest plan".

Then, Oriana talked about the mathematical concepts the group used while exploring the MEA. She said,

> We used the concept of linear equations without actually forming any equations. Essentially, we plugged given values in (substitution) to each of the phone plans which all had the form of a base value (y-intercept) and a consistent rate per minute (slope) depending on the type of minute. These all made equations, but we chose to look at the individual vales [for each plan] instead.

Oriana then continued talking about how these mathematical concepts are connected. She explained,

> Since lines are just an infinite set of points, the values we found for each plan can be seen as just a point on that plan's line. These points can also be viewed in the form of a table, such as a simplified version of the one given. Points, tables, linear equations, and graphs of lines are all different ways of viewing the same data.

Oriana's discussion of mathematical concepts illustrates evidence of *trimming* because she was able to get to the mathematical essence of a real-world problem, as well as *bridging*, because she was able to make connections across topics and representations.

Next, Oriana discussed the possible misconceptions students may have solving this task. She described an error with the use of systems of equations,

> Students could think that the cheapest plan this month will always be the cheapest. This would suggest they did not understand the concept that we were dealing with a system of more than two linear equations, which would intersect each other at different places, altering which was cheapest according to the number of minutes used.

By anticipating the difficulties entailed in students' understandings of systems of equations and decompressing rote procedures to grasp the logic behind them, Oriana's description of possible student misconceptions demonstrates strong evidence of *decompressing.*

MEA 3-Exploration of the Task

Oriana worked on the *Historic Hotel* MEA with the same group she worked with for MEA 2. The task was shown in Figure 19 above. The group began by comparing the MEA with the *Phone Plans* MEA they explored the prior week to check if they could use a similar strategy. Oriana explained that it's different from the prior MEA because the rates fluctuate. Then, the group discussed different strategies they can use. Oriana suggested that graphing may be "the easiest way to do it", and it can be done by "starting off with the intercept, because that would be where no rooms are occupied… he would charge *80 plus 60,* so if he charges $140 no rooms would be rented". Then the group wanted to calculate the total amount of money the owner would get if all 60 rooms were occupied. They multiplied the room rate ($60) by the amount of occupied rooms that corresponded to that price (80), and then added the maintenance fee per room ($4 times 80). That was when the discussion about maintenance cost emerged. The PSMT discussed whether the maintenance cost would be added to or subtracted from the total amount of money the owner gets. Oriana asked her groupmates, "the $4 cost, are they charging it or are they saying like that's a cost to them that they have to pay?" Oriana suggested that they should subtract the fee rather than add it, because “it’s costing the hotel that much, it’s not added profit”. Bianca and the fourth PSMT in the group advocated for adding the maintenance cost, because the customer pays the fee along with the rate, so it’s added to the total. After deliberating and talking about the meaning of cost, and how the money acquired from the fees would be used, the group decided the hotel owner must pay for it, and therefore, it should be subtracted from the total amount of money he would receive.

Once the group recalculated the total amount of money the owner receives from 80 rooms being occupied, Oriana realized that they calculated was profit, “so if 80 rooms are rented, the total would be… Oh! Wait a minute! That’s the profit… we are calculating the profit.” She relabeled the graph she was creating on her paper and plotted the corresponding point. The graph shown in Figure 25 below.

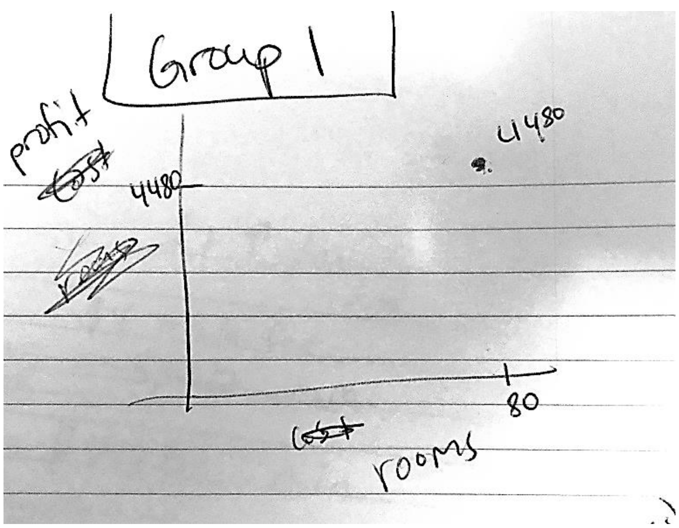


Figure 25. Oriana’s graph for Historic Hotel MEA.

Next, the group discussed deriving a formula for the scenario. Oriana first suggested using *slope-intercept* form of a linear equation, “we could try like using this point on the graph, and no rooms being occupied, that would give us like the intercept and maybe we can try to make a line off of that". Then Helaine suggested using the context to help them. The rest of the group agreed.

As the group discussed what the variable would be in the context and how they could represent the pattern of the rate and the corresponding rooms being occupied, Oriana suggested,

> I feel like if we figure out what the price per room is going to be, considering it like changing, and then what the number of rooms is going to be, considering like you know, that changes, then if we *multiply the price times the number of rooms* is going to give us the total. Then we have to *subtract the $4 cost* and then we will also figure out the cost. So, it can be like, *price times number of room*s minus *cost*. So, let's trying and figure out each part.

This demonstrates evidence for *trimming* because Oriana was able to get to the mathematical essence of a real-world problem.

Then, the group worked to "figure out each part", as Oriana suggested. The first part was recognizing the fluctuation in the price, in relationship to the number of occupied rooms. Oriana proposed,

> Okay, so what I was thinking is like if we do *80 minus the number of rooms*, right, the like I said there are 79 rooms back then just one room left when we subtract. Then if we add that to the 60 that makes it 61 which is the price it will be really 79 rooms, does that kind of make sense?

When asked whether the variable would represent the number of rooms *occupied* or *empty*, Oriana explained it would be the number of rooms occupied. However, Helaine then suggested to make the variable, n, represent the number of empty rooms, so when there is one room empty, $80-n$ would be 80-1, or 79 rooms occupied; and the price would be $60 + n$, in this case 60 +1, or $61. Oriana agreed with the use of this strategy, and then suggested that the cost would be "80 minus n times 4, because it's the occupied rooms times 4." She then justified her suggestion by describing the meaning of the expression, "if we do 80 minus n it's going to give us the number of occupied rooms, right, and then if we multiply that by 4 it's going to give us their cost for the service".

As the group finished writing the equation that represented the real-world scenario, Oriana suggested simplifying the final product, by factoring. She said, "you can actually make it just be $80 - n$ times $56 + n$ and it works, you get the same thing every time, just like factoring this part of it" pointing to the $80 - n$ in the equation. Bianca and Helaine asked her to clarify where the 56 came from, and she replied, "you factor out the $80 - n$ from both parts of the equation, and then it becomes ($60 + n - 4$) because they both had that in common, and that would be $56 + n$". As Oriana thought about the derived equation, she elaborated, "this makes so much sense because it's going to be a downward parabola. So, it will be maximized on, like, the middle point" referring to the vertex. Because Oriana was able to make connections across representations, strong evidence of *bridging* is shown.

Now that the group had the equation, they wanted to find the rate that maximized profit. The group agreed with Oriana's suggestion of graphing the derived equation and find the maximum on the graph. First, the group attempted to graph it using an online graphing calculator, so they can use an image of the graph during their presentation to the class. Once the equation was inputted, Oriana suggested changing the window for the graph to be more visible. She suggested that the x-axis have a minimum value of *zero* and a maximum value of *80* because it represented the number of rooms; and the y-axis have a minimum value of *zero*, because "a negative profit is not going to be needed, in this scenario" and a maximum value of 5000, because "the profit won't be higher". The group then struggled to find the maximum using the online tool, so, Oriana used her graphing calculator and the *calculate maximum* command to locate the maximum on the graph. She found that the maximum was at $x=12$. At this point, the group tried to make sense of what that meant in terms of the context of the MEA. Oriana reminded her groupmates that the variable represented the number of unoccupied rooms, so 12 would be the number of unoccupied rooms, she continued explaining that "the maximum profit occurs when 68 rooms are occupied". The rest of the group continuously asked Oriana to clarify, and she repeatedly explained what each number meant—12 represented the number of empty rooms, 68 represented the number of occupied rooms, "because 80 – *n* was the number of occupied rooms", and $72 is the rate that maximizes profit, "because 60 + 12 was the rate". This demonstrated strong evidence of *trimming* because Oriana was able to get to the mathematical essence of a real-world problem, as well as *decompressing,* because she was able to decompress rote procedures to grasp the logic behind them.

Next, the group discussed writing the letter for the owner of the hotel. As with the previous MEA, Oriana narrated, as another PSMT in the group wrote what was being dictated. Oriana suggested that they should "generalize the equation in terms of *number of rooms*, *empty rooms*, and *cost*, like general terms" to accommodate for the possibility of the pattern changing. Oriana used the equation to label what each represented. The group began with *(80-n),* and all agreed that represented the total number of occupied rooms, and to generalize it, Oriana described it as "total amount of rooms in the hotel *minus* the number of empty rooms". Next was the *(60+n),* which Oriana described as "the rate when all the rooms are occupied plus the number of empty rooms". Lastly, the *(80−n)* x *4*, which Oriana described as "the total rooms *minus* the number of empty rooms, *multiplied* by the maintenance cost". Then, Oriana suggested that they should also describe what 56 meant in the simplified equation and described it as "the rate when all rooms are occupied minus the maintenance cost". Again, this demonstrated strong evidence of *trimming* because Oriana was able to get to the mathematical essence of a real-world problem, as well as *decompressing,* because she was able to decompress rote procedures of writing expressions and simplifying them, in order to grasp the logic behind them.

Once the group derived the specific equation, and generalized it, they began writing the letter. The PSMT recalled the process that they took and explained the equations and how to use them in the letter. During the group's presentation of their solution to the MEA, a classmate asked why the maintenance fee was subtracted, rather than added. Oriana explained,

> At first, we were not sure on that too. But then we figured it, if the people have to pay the fee, then it would be included in the rate. So, I guess that was an assumption our group made, that the hotel owner is paying out this fee. But even if the customers are paying it, that money isn't profit, because then the hotel owner has to use it to pay for the cleaning… so either way that money is not going to the owner".

Once more, evidence of *trimming* is demonstrated because Oriana was able to get to the mathematical essence of a real-world problem, as well as *decompressing,* because she was able to decompress rote procedures in order to grasp the logic behind them.

Second Interview

The second interview with Oriana took place two days after the exploration of the *Historic Hotel* MEA. The researcher asked Oriana questions (included in *Appendix H*) about the MEA and two content questions from the LMT assessment.

Experiences and Reflections related to MEA 3

During the first part of the interview, Oriana discussed her experiences with the *Historic Hotel* MEA. When she was explaining how she made sense of the task, she started by discussing the profit equation the group created and explained what each part of the equation meant. She said:

> For us what we did was we broke it down for each specific thing, number of rooms where we said that would be 80 minus the number of unoccupied rooms. Then we did that also for the price of the room that was like base price of $60 *plus* the number of unoccupied

> rooms. Then we went from there combining that so, to get the total *revenue* that would be number of rooms' times' price per room then we subtracted the service fee.

Oriana then discussed the maintenance cost, and how the group decided to represent it in the profit equation:

> Because that was the one thing that we were kind of like on the fence of. We started off being kind of unsure whether that was an added cost to the person who was running the hotel or to the person who was staying at the hotel. That was one of the things that we were confused about at first. But we kind figured if it was the cost for the person who was staying at the hotel, it would just be included in the room rate. So, we did that as a subtraction like a cost for the person running the hotel. We got our *profit* by doing that *revenue* minus the *cost* to him.

Strong evidence of *trimming* can be seen in the above explanation because she was able to discuss the mathematical essence of the given real-world problem.

She then talked about the specific mathematics content that helped her make sense of the MEA. Oriana once again described the profit equation, and finding the maximum using the graph of the equation. She explained that the equation they wrote was *quadratic*, and that the group could have found the maximum using different methods. She continued, "But because we didn't have that much time, we just did it using a graphing calculator to find the maximum". Oriana then discussed using a table of values, "because you can't have a part of a number of rooms occupied, so it has to be within a whole number value". She then discussed how the expressions they wrote for the number of rooms and rate per room were linear, but the overall profit equation was quadratic, "what was so interesting to me is that both the 80- r and the 60 + r in the profit equation were both linear, and when we multiplied them, we got the quadratic equation".

At this point, the researcher asked Oriana about different methods that were presented by other groups, such has the first and second derivative test, as well as finding the x-value of the vertex of the parabola using the formula $x = -\frac{b}{2a}$, and how these methods related to the strategy they used in their group. She said,

> Those all are kind of the same thing, it's just that we used technology to do it for us, but that's what the technology is doing to get it. So, first derivative test gives you and equation of the line which gives you the slope at any point, so, when you solve that equation through the line it gives you $-\frac{b}{2a}$ because that's all that's left in the equation so, those are both kind of the same thing.
>
> It's just normally they teach the $-\frac{b}{2a}$ part before you know how to take a derivative, so, you can still solve it, but those give you the x value of the vertex, that would tell you how many rooms, it wouldn't tell you the max profit, you would have to plug that in to the equation to get the profit. But they are all like different ways of doing the same thing.

This demonstrated strong evidence of *trimming*, *decompressing*, and *bridging* by: (1) removing complexity while retaining integrity when describing the derivation of the vertex formula of a quadratic (*trimming*), (2) making the connection between vertex formula and first derivative (*bridging)*, and (3) *decompressing* rote procedures of finding a vertex of a quadratic and focusing on the logic behind the vertex formula and the first derivative test.

When asked about how all the mathematical ideas she just discussed were connected, Oriana said:

> Okay, I think the quadratics have a lot to do with $-\frac{b}{2a}$ and 1^{st} derivative test, because they are different ways of solving for the vertex, so, the thing that kind of connects quadratics to those is the vertex or the maximum. That vertex ended up being a whole number because of the context of the problem, so with other things where you can have part of a whole it would not necessarily have to be that.
>
> Then linear equation is kind of connected to quadratics by the $-\frac{b}{2a}$ and 1^{st} derivative test because those create linear equations to solve the quadratic. We also ended up creating the quadratic by multiplying two linear equations.

The figure below was made by Oriana, as she discussed the connections between the different mathematical topics used in exploring the MEA. Accordingly, strong evidence of *bridging* was demonstrated because she was able to make connections across topics, representations and domains.

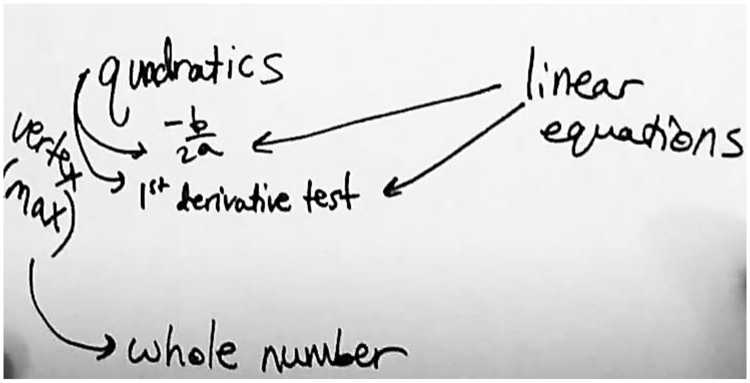


Figure 26. Oriana's Mathematical Connections for MEA 3.

As Oriana began reflecting on her experience with the *Historic Hotel* MEA, she discussed how the context of the task helped her make sense of the mathematics the group used:

> Even though I am pretty familiar with the math that we used in this task, more so than the other [MEAs], I feel like it helped me see how the context kind of dictates how the problem is solved. So, like we had to ignore the decimal values for the x part of the vertex, because it doesn't make sense to not have a whole number…This helped me have a deeper understanding of that because every time I use something in a real situation it helps me understand it more than just doing like a vague computation problem.

She continued by discussing the connections between *linear* and *quadratic* equations and her opinion of the task. She said:

> It also helped me see the relationship between all of [the mathematics] used. I never really thought about the fact that when you factor a quadratic like each of the factors are linear, I don't know why I never thought about it. It was something interesting to me because we kind of created the quadratic from those linear factors. I think this was definitely my favorite one out of the three that we did, it was fun to do. I liked it.

When reflecting on how this task can be used in an Algebra class, Oriana discussed how the context can help students make sense of the mathematics. She explained,

> I think this one is, especially out of all of them would be really useful just because I think this is something that so, many students struggle with at first. It’s just like the concept of what a quadratic is, what the shape it is and that kind of a thing.
> This context of it makes a lot of sense, when you charge more for the room you are getting more money, but you are also less people are buying it, and so, it kind of balances out to where, if you go to either of the extremes you are not making very much but you have to go somewhere in the middle. So, a parabola shape to represent it is makes so much sense.

Oriana then reflected about the usefulness of quadratics, and specifically the vertex or maximum of a parabola:

> I think this would help solidify the concept of why we need to find maximum sometimes, what you are actually doing to find the maximum. I do think that’s something that’s so confusing to a lot of students at first is like.
>
> Also, they make say like “why do we care what this value is?” So, this would really help to actually show how it can be used like in the context of something instead of just random equations. But it really can help them be like, “Wow! That’s actually a practical use of math, I can see doing in real life.” I think that would be a really good thing, that’s the kind of thing I would definitely consider using in my own classroom.

Oriana’s reflections above demonstrated strong evidence for *decompressing*, because she discussed how the context of this MEA would help students understand the usefulness and importance of algebra.

Lastly, Oriana talked possible misconceptions students may have with solving the *Historic Hotels* MEA. She described general misconceptions about parabolas, not specific to the task. She explained that some students assume that a vertex of a parabola is always the maximum without realizing that it could be the minimum, if the parabola opens up, or “if [the parabola] is flipped the other way around, which makes the quadratic not a function, if they open left or right, then the vertex is not the maximum or the minimum”. This reveals evidence of *decompressing* by anticipating the difficulties entailed in students’ understandings of vertices of parabolas, but they were not specific to the mathematics explored in the *Historic Hotel* MEA.

Content Questions

The next part of the interview entailed the two content questions from the LMT assessments. As previously stated, the questions were not part of the items released by the LMT project, and thus, the description of the findings in this section will not have specific details of the context of the questions.

The first question, Oriana was asked to choose a story problem that does *not* represent linear behavior in the context of the four given story problems. A description of each choice was given in chapter three. When thinking about this question, Oriana provided a similar explanation to the one she gave during the first interview, when she was presented with the same question. She explained that the first choice would be exponential and not linear, because the initial amount gets doubled every day, “it’s not a constant amount that’s being added every time”. The second choice represented a linear behavior and could be modeled using a proportion and the equation $y = m$x. Oriana explained, “the amount is being changed by a constant rate every time, so it works”. The third choice, Oriana explained also worked by identifying the y-intercept, and the slope. Similarly, the last choice she said would have a negative slope, so the initial amount was “decreasing by the same amount every time” and concluded it would be linear. Thus, Oriana determined that the first choice is the only one that does not represent linear behavior. Oriana answered the question correctly, and showed evidence of *trimming*, and *decompressing* by: getting to the mathematical essence of each of the four real-world problems (*trimming*) and *decompressing* rote procedures of writing equations to focus on the logic behind what the equation would mean in terms of the context of each story problem.

The second question was question 31 on form A (which was described in chapter three). Oriana was asked to decide whether three story problems could be modeled by a given linear equation of the form $y = mx + b$, by saying *yes, no,* or *I'm not sure* for each of the three-story problems. The first story problem could *not* be modeled by the given linear equation, rather it could be represented by an exponential growth function because the initial amount was being doubled each month. When Oriana examined the first choice, she said "it's more of an exponential function because you're doubling the amount each time, not increasing by a certain amount, so that would be a *no*". She then connected the reasoning behind the elimination of this choice with one of the choices she discussed in the previous question. The second story problem could *not* be modeled by the given equation, although it could be represented by a different linear equation. Oriana explained, "this is linear, but the slope is different, so *no*". The last story problem was the only one could be represented by the given linear equation. Oriana explained, "you start with the [same initial amount] and then we are adding a certain amount every week, it's not being doubled, which is what we are looking at, so this one is a *yes*". When answering the question, evidence of *trimming* could be seen because she was able to get to the to the mathematical essence of the three real-world problems, and *bridging,* because she was able to make connections across questions.

LMT Assessments

As stated previous, Oriana was selected from the *high MKT* category— which included participants with scores greater than 50^{th} percentile on the LMT pre-assessment. Oriana's score on the LMT pre-assessment was in the 90^{th} percentile. As discussed in chapter three, IRT scores represent the underlying ability being measured by the assessment, in this case it is the PSMT' MKT algebra. This ability is denoted as the symbol theta. This ability spectrum is measured in standard deviation units with zero being average. Higher abilities are represented by positive values and lower abilities are represented by negative values. Oriana's overall IRT score on the pre-assessment was θ =1.145, and standard error (*SE*)= 0.416. On the post-assessment, Oriana had the highest score among the 27 PSMT that completed the measure. Her overall IRT score on the post-assessment was θ = 2.193, and *SE*= 0.543. The box and whisker plots below illustrate Oriana's IRT scores in relation to her classmates on the pre-assessment (n = 30) and post-assessment (n = 27). These results demonstrate that Oriana's MKT algebra was relatively high compared to other PSMT in her class, throughout the study.

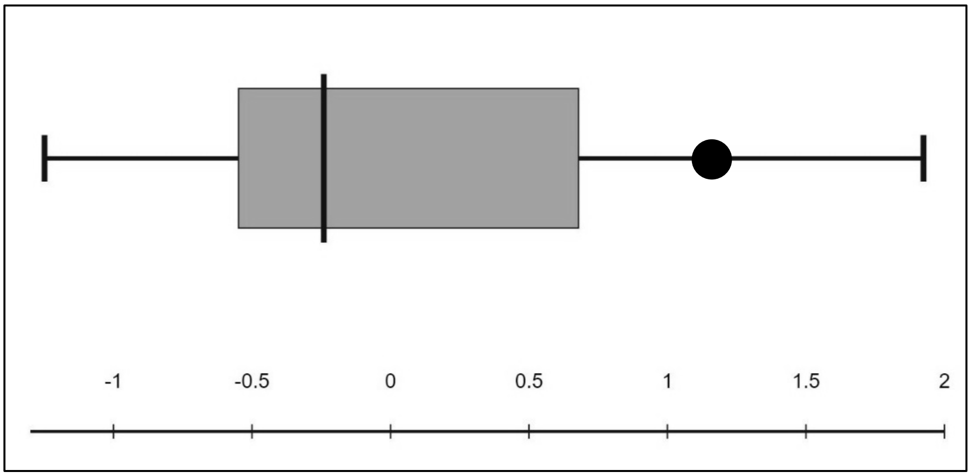


Figure 27. Oriana's Pre-Assessment Score on a Box Plot

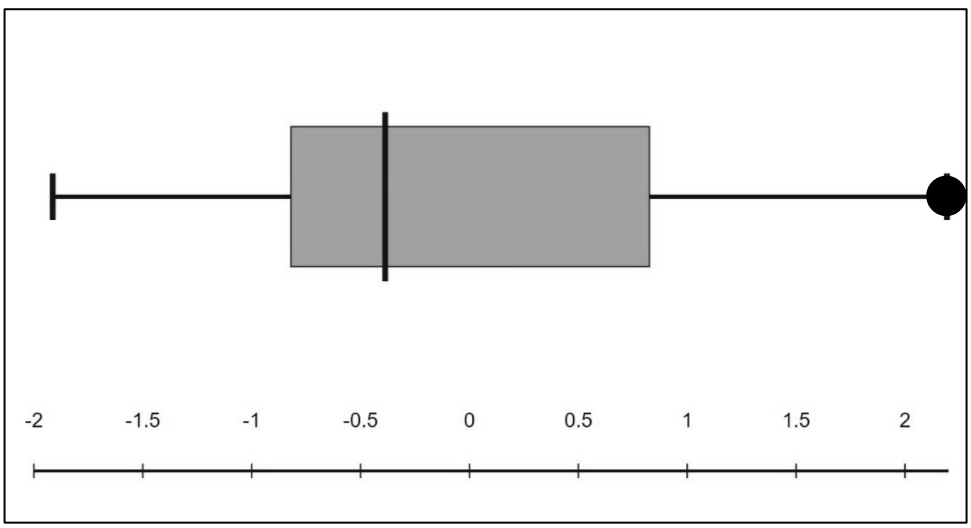


Figure 28. Oriana's Post-Assessment Score on a Box Plot.

Discussion of Case 1

Oriana was selected from the high MKT category. Her score on the LMT pre-assessment (θ =1.145, *SE* = 0.416) was in the 90th percentile of the participating PSMT in her class (*n* = 30). Her score on the LMT post-assessment (θ = 2.193, *SE*= 0.543) was the highest among the participating PSMT in her class (*n*=27), and thus, she remained in the high MKT category throughout the research study.

During the first MEA, Oriana worked with Bianca and she led the exploration of the task by recommending and explaining the use of unit rates to compare and evaluate different vendors. Throughout the exploration Oriana demonstrated *strong evidence for trimming* by describing the mathematical essence of the given task in explaining why unit rates and proportions were useful to make sense of the given scenario, while other strategies, like graphing the amount of money made, or creating equations, were not helpful in selecting the vendors to be employed. There was *no evidence for bridging or decompressing* during the exploration of the Summer Jobs MEA.

During the first interview, Oriana described her experiences with the Summer Jobs MEA by discussing the mathematics she used that helped her make sense of the task. She explained that by calculating the unit rates and averages, she was able to take into consideration both the amount of money each vendor made, and the number of hours the vendor worked, to make her recommendation of which vendor should be hired again. She discussed how mean and median could have also been used to compare the different vendors. She stated that these concepts are connected in that they help make decisions about the data, but she did not explain how these mathematical ideas are related or connected. Thus, there was strong evidence for *trimming,* but some evidence for *bridging.*

Next, when discussing how the task could be implemented in an algebra class, Oriana explained how tasks like the Summer Jobs MEA, or a similar one that may require students to create equations and graphs to model a real-world scenario, would help students understand the mathematics as well as help them understand the usefulness and importance of algebra. Oriana also anticipated possible misconceptions in using the mean, median, and mentioned that outliers in the data could influence the appropriate use of each measure of central tendency, but she did not explain how outliers affect the mean or explain why the median would be a more appropriate measure to use when outliers are present. Therefore, only some evidence of decompressing was demonstrated.

When it came to answering the content questions in the first interview, Oriana answered both questions correctly, and as she thought about both questions aloud, she was able to get to the mathematical essence of each of the four real-world problems, and make the connection between proportions, ratios, and slope in linear equations. Thus, strong evidence for *trimming and bridging* was demonstrated, however there was low evidence for *decompressing*, because she focused on rote procedures for writing linear equations, without explaining the logic behind those procedures.

During the second MEA, Oriana thoroughly contributed to the exploration of the task. She suggested various strategies for comparing the different phone plans, including writing linear equations for each plan, graphing to compare the plans visually, and creating a table of values. Oriana demonstrated strong evidence of *trimming* because she was able to get to the mathematical essence of the real-world problem by consistently describing the mathematics that helped them compare the different phone plans based on the given information. Oriana discussed making connections across representations of linear equations (table of values, and graphing), which she further explained on her written reflection, demonstrating strong evidence of *bridging.* During the exploration of the MEA, there was no evidence of decompressing.

On her written reflection, once again, Oriana demonstrated strong evidence for *trimming* by describing and explaining how linear relationships were used to help make sense of the MEA. She talked about creating a table of values, and even though they did not write equations, they could have identified the y-intercept and slope for each phone plan, by looking at the rate per minute and the monthly fees. She then proceeded to explain how these concepts are connected by discussing the different ways linear relationships could be represented, demonstrating strong evidence for *bridging.* Next, Oriana reflected on possible misconceptions students may have while solving the task. She discussed the logic behind creating different equations for the various phone plans and anticipated that students may have difficulties interpreting the solution of the systems of equations in context of the task. As a result, there was strong evidence for decompressing.

During the third MEA, Oriana contributed to the group's exploration of the Historic Hotel task extensively, once again. She suggested various strategies, such as graphing and writing expressions and equations to model the given real-world scenario. She was able to get to the mathematical essence of the given real-world problem by using linear expressions to represent the number of rooms occupied at the hotel, the price per room, and the maintenance cost, and then used her knowledge of quadratic equations to represent the profit and find the rate per room that maximized the profit for the hotel owner. Oriana also helped her groupmates understand the mathematics in context of the problem, when they found aspects of the task challenging. Moreover, Oriana was able to make connections across representations of quadratic equations and explain to her groupmates how each representation could help them explore the task. Therefore, strong evidence for trimming, bridging and decompressing was demonstrated.

During the second interview, Oriana described her experiences with the Historic Hotel MEA by discussing the mathematics she used that helped her make sense of the MEA. She explained that using linear expressions to represent the number of rooms occupied, price per room, and maintenance cost guided the group in writing the quadratic equation that represented profit, which then led her and her groupmates to graph the downward-facing parabola, and then find the maximum using a graphing calculator. She then discussed other strategies used by different groups, such as the vertex formula for finding the x-value of a parabola, $x = -\frac{b}{2a}$, and the first and second derivative tests. She explained how all these methods for finding the maximum are connected by describing the logic behind each procedure. Thus, there was strong evidence for trimming, bridging and decompressing.

Next, when discussing how the task could be implemented in an algebra class, Oriana explained the Historic Hotel MEA and similar tasks would help students understand the usefulness and importance of quadratic equations and linear expressions. Oriana also anticipated and explained possible misconceptions about the vertex of a parabola and finding the maximum and minimum of quadratic equations, demonstrating strong evidence for decompressing.

When it came to answering the content questions in the second interview, Oriana answered both questions correctly again, and as she explained the reasoning behind her answers, she was able to get to the mathematical essence of each of the four real-world problems, and decompress rote procedures of writing equations to focus on the logic behind what the written equations meant in terms of the context of each problem. Thus, strong evidence for *trimming and* *decompressing was* demonstrated. She was also able to make connections in her reasoning across different questions, demonstrating strong evidence for *bridging.*

The findings above are summarized in the following table and figure. Table 8 shows the different levels of evidence—high (H), medium (M), or low (L) in terms of the three KAT categories, *trimming*, *bridging* and *decompressing,* for each of the collected data sources.

Table 8

Oriana's levels of evidence for trimming, bridging and decompressing.

	MEA 1		MEA 2		MEA 3	
KAT Categories	Exploration of Summer Jobs MEA	First Interview	Exploration of Phone Plans MEA	Written Reflection	Exploration of Historic Hotel MEA	Second Interview
Trimming	H	H	H	H	H	H
Bridging	L	M	H	H	H	H
Decompressing	L	M	L	H	H	H

The figure below illustrates the overall level of evidence for each of the two factors of interest—MEA contribution, and MKT. The plot consisted of the three KAT categories placed on a continuum for each factor. An overall rating of high, medium, or low was given for each of the three MEAs, based on the table above. An overall high rating was attained if a high was given on both the exploration of the MEA and the interview or written reflection that followed. An overall medium rating was given for a medium-medium combination, high-low combination, and a high-medium combination. An overall low rating was given for a low-low combination, and a low-medium combination.

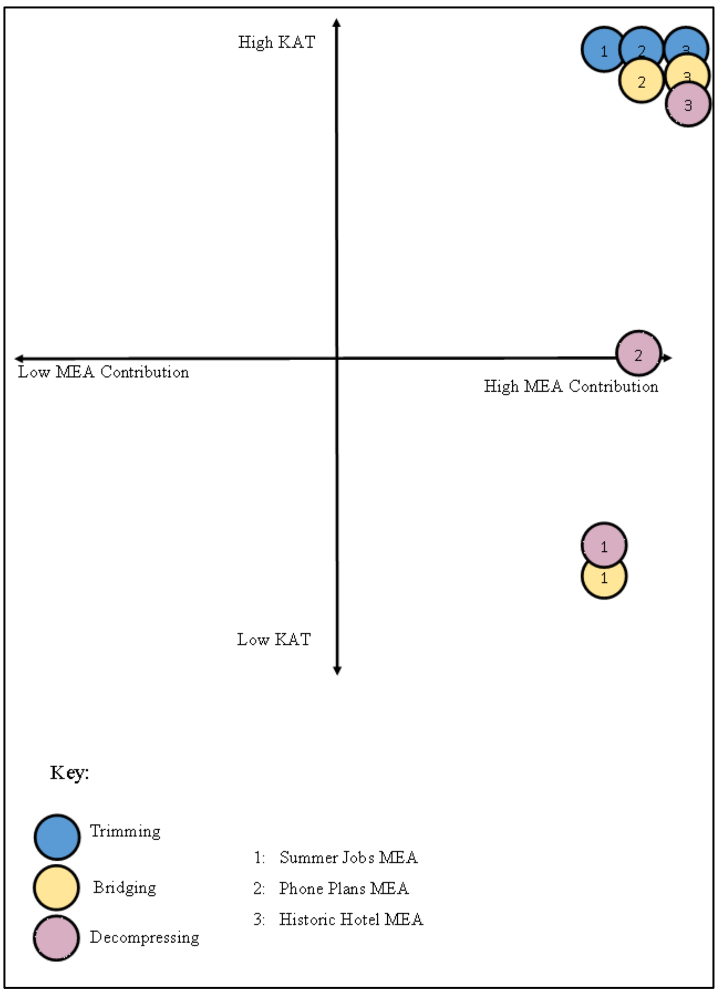


Figure 29. A Plot of Oriana's Overall Level of Evidence for MEA Contribution and KAT.

The plot above illustrates the following:

(1) Oriana's overall level of evidence for the Summer Jobs MEA and first interview was *high* for trimming, *medium* for bridging, and *medium* for decompressing, while her contribution to the exploration of the task was high.

(2) Oriana's overall level of evidence for the Phone Plans MEA and the written reflection that followed was *high* for trimming, *high* for bridging, and *medium* for decompressing, while her contribution to the exploration of the task was high.

(3) Oriana's overall level of evidence for the Historic Hotel and the second interview was *high* for trimming, bridging, and decompressing, while her contribution to the exploration of the task was high.

Connecting this back to McCrory and colleagues' definition of *trimming, bridging, and decompressing*, it was evident that:

(1) For the first MEA, Oriana was highly involved in exploring the task, and she was able to she was able to able to talk about and explain the mathematics used to explore real-world problem (*trimming*), she was able to make some connections across mathematical topics she discussed (*bridging*), and she was able to anticipate the difficulties entailed in students' understandings of particular aspects of algebra content, and decompress rote procedures in order to grasp the logic behind them (*decompressing).*

(2) For the second MEA, Oriana was highly involved in the exploration of the task, and she was able to make connections across mathematical topics and representations(*bridging*), and she was able to discuss and explain the mathematics used to explore real-world problem (*trimming*), and she was to anticipate some difficulties entailed in students'

understandings of particular aspects of algebra content, explain how to help students understand the usefulness and importance of algebra, and decompress some rote procedures in order to grasp the logic behind them (*decompressing).*

(3) For the third MEA, Oriana was highly involved in the exploration of the task, and she was able to make connections across mathematical topics (*bridging*), and she was able to discuss and explain the mathematics used to explore real-world problem (*trimming*), and she was able to anticipate the difficulties entailed in students' understandings of particular aspects of algebra content, explain how to help students understand the usefulness and importance of algebra, and decompress rote procedures in order to grasp the logic behind them (*decompressing).*

Case 2: Bianca

"I have always been a Math person, Math has always come naturally to me"

Bianca, January 2018

Description of Participant

Bianca was selected from the *medium MKT* category— which included participants with scores greater than 25th percentile and less than or equal to 50th percentile on the LMT pre-assessment. Bianca's score on the LMT pre-assessment was in the 50th percentile. Bianca was in her fourth year of the secondary mathematics education program. She expressed that she has always been a *"math person"* since she was young (Bianca, First Interview, January 22). Although she had to repeat third grade due to not passing a high-stakes standardized test, she explained that relearning the content made her like mathematics even more. Throughout school, Bianca explained that she always helped her classmates with mathematics, because it came naturally to her.

During the semester that this research was conducted, Bianca was enrolled in a *differential equations* course and a *statistics* course. When asked to give an overview of the content being taught in those classes, she said:

> With stats we just are reviewing basically what stat 1 was all about, that's hypothesis testing and all that. For differential equations, not really sure what it's called, I know we just started chapter 4 but I'm not really sure…Homogenous equations basically (Bianca, Second Interview, February 5).

After graduation, Bianca's goal was to teach mathematics at the high school level and attend graduate school after working for a few years.

MEA 1-Exploration of the Task

Bianca explored the *Summer Jobs* MEA with Oriana and another PSMT that was not a part of this study. The task was shown in Figure 14 above, and *Appendix D*. Bianca first suggested that the group could explore the task by looking at the total amount of money each vendor makes. Although the MEA specifically stated to compare the vendors by also considering the number of hours they worked, and which attendance category these hours were in, at the beginning, Bianca only wanted to compare the total amount of money each vendor made. Below is the exchange between Bianca and Oriana:

> *Oriana*: Like that there's more people coming in to the park. So that would be the busy time of the day. So, this is slow, and like they will be making less money if there are less people in the park.
>
> *Bianca*: Theoretically it doesn't look like that. Based on the correlation, they are still making more money.

Bianca refers to the relationship between the amount of money made and the numbers hours worked as *correlation*, which she explains during the first interview. Once Oriana made the case for why it is important to look at both the hours worked, and the amount of money made, Bianca suggested using a graph to help them compare the vendors. When Oriana asked how that can be done, Bianca was not able to create a graph that showed both the two variables of interest. The following is the discussion about graphs between Bianca and Oriana:

Bianca: What about like a graph? Where we can compare the correlation.

Oriana: Is there a way we can do unit rates like that though?

Bianca: Yeah. Depending how you want to do it. We can graph it. We can only graph those numbers according to what we want.

Oriana: So how would be the best way to graph this? Should we do a graph for busy, slow and steady, or a graph for each person with like three lines, busy steady and slow.

Bianca: I mean there are two different graphs we are comparing. We don't have to do it all at once.

Oriana: But if we do just one and not the other, the graph isn't really going to tell us anything, because it's not taking into account the time of day. Also, we will be relying on what we see vs. the actual numbers.

Bianca was not able to justify how and why a graph can be used to explore the MEA, and thus, the method was not used, and the group decided to use unit rates and averages, as Oriana suggested. Bianca calculated the unit rates for Willy, Tony and Robin, her work is shown in Figure 30.

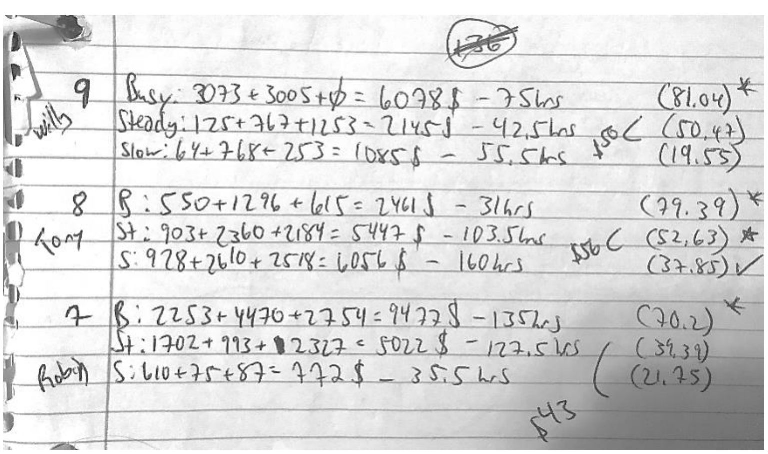


Figure 30. Bianca's Unit Rate Calculations for the Summer Jobs MEA.

Once Bianca was done with her calculations, she gave the answers to Oriana to input them in the shared table. During the presentation of solutions to the class, Bianca echoed Oriana's reasoning for choosing specific people for being part time employees while others were chosen to be full time employees. She explained that some vendors had "huge outliers" during specific time periods, and they were recommended to work during those times.

More of Bianca's reflections on, and experiences with the *Summer Jobs MEA* are described in the following section.

First Interview

The first interview with Bianca took place an hour after the exploration of the MEA. After answering the ice breaker questions (included in the interview protocol in *Appendix H*), the researcher started asking Bianca questions about the *Summer Jobs* MEA she had just explored, and the LMT pre-assessment she had taken the previous week.

Experiences and Reflections related to MEA 1

During the first part of the interview, Bianca discussed her experiences with the *Summer Jobs* MEA. When she was explaining how she made sense of the MEA, she spoke about graphing, and representing the data visually to compare the different vendors. She explained:

> First of all, I looked at it, I was like okay well, it's basically hours and money comparison. In my head, I was thinking graph, but as a group, we all came to doing the average of everything instead. Because I'm more of a visual person, so I'd have probably found a way to compare each of the graphs, and then seeing if there was a high slope, or low slope, and those people would've been hired.

After being asked to elaborate on how she would graph the given data, she said that they could have used *Excel* to create a line graph that represented the given data, and from the graph the slope could be identified along with the "progress of each worker". When asked what the slope would mean in terms of the MEA, Bianca said, “they would tell me if they were making progress regards to making more money or working less hours. Then comparing the two correlations”. Bianca was then asked to clarify what she meant by the two correlations, to which she replied, “how much money and the number of hours”.

Next, Bianca discussed the specific mathematics content that helped her make sense of the MEA. She spoke about slope, averages, finding the sum, creating expressions. She said:

> We did slope, averages of time of day, also averages of hours, or money Also we did like, we basically added all the total of hours, actually, I think it was money. And then we divided by like the total hours. Within that number, we had the 3 different types of time of day, so I just put 3X for time of day. We divided that by 3, to find our main average for each person… Because there was 3 times of the day that we did, like the busy, steady and the slow.

Bianca was not able to explain how the graphing strategy would help address the task they were given in the MEA, nor was she able to precisely describe how she calculated the averages and unit rates. In other words, she was not able to talk about the mathematical essence of the given real-world problem, therefore *trimming* is not evident.

Next, Bianca was asked about how the mathematical ideas she talked about are connected, she said averages and slope are connected, but those are “a little different” from the expression $\frac{3x}{3}$, that she previously discussed. When asked how averages and slope are connected, she was not able to explain why mathematically, but rather she talked about the context of the MEA and said, “they both tell about the progress of the workers, like the averages of the amount of money and averages of how many hours they worked, and then slope combines the two, and we can use them to compare the workers.” Although Bianca said that the applied mathematics concepts are connected (like slopes and averages), she was not able to explain and justify the reasoning behind the connection. Thus, only some evidence of *bridging* can be seen from this part of the interview.

As Bianca began reflecting on her experience with the *Summer Jobs* MEA, she spoke about her understanding of *averages,* and her interpretation of why the group ultimately chose the use of averages as the strategy to explore the MEA. She stated:

> So, we used the averages and stuff. But I think I understand averages, it is mostly just adding and then dividing by the total amount that you had. Averages in general like, we do them so often, within school, so I feel like that's why we decided to do that for the modeling activity, because we know that role and there are numbers. We are comparing things to try to see who will be best for full time and part time.

Because Bianca is not demonstrating the ability to decompress rote procedures to grasp the logic behind the specific mathematics topics she discussed, *decompressing* is not evident.

When reflecting on how this task could be used in an Algebra class, Bianca said that she would use this task with high school students and shorten the amount of data given to the students. She discussed how students may learn mathematics content through such tasks; she said, "they will take whatever they may know from that time, and try to figure it out that way, and then learn new ways from their peers as well."

Lastly, when Bianca was asked about possible misconceptions students may have with solving the *Summer Jobs* MEA, instead of talking about misconceptions specifically related to mathematics content, she discussed the difficulty of the task. She explained that "there were a lot of numbers, so students might like get confused, and start adding things together without realizing what they are adding." Because Bianca was not able to anticipate the difficulties entailed in students' understandings of specific mathematical ideas, *decompressing* is not demonstrated.

Content Questions

The next part of the interview entailed the two content questions from the LMT assessments. As previously stated, the questions were not part of the items released by the LMT project, and thus, the description of the findings in this section will not have specific details of the context of the questions.

The first question, Bianca was asked to choose a story problem that does *not* represent linear behavior in the context of the four given story problems. A description of each choice was given in chapter three. When thinking about this question, Bianca explained that the first choice represented linear behavior because "it's going day by day, and we end with the n^{th} day, so that's a linear pattern." She did not talk about rate of change by which the initial amount increased—because the initial amount was doubling for each passing day, the first choice represented exponential behavior rather than linear. For the second choice, Bianca said that it was similar to the first scenario because it's also asking for the n^{th} unit. Bianca continued with this reasoning strategy for the third and fourth choices and explained that "all these [choices] considered the n^{th} of something, and to me that represents linear behavior, so I picked that [all four choices] represent linear behavior." Bianca answered the question incorrectly, and the reasoning behind her choices did not demonstrate evidence for *trimming*, *decompressing*, or *bridging*, because she was not able to: (1) get to the mathematical essence of each of the four real-world problems (*trimming*); (2) make connection across the different mathematical topics integrated in the question (*bridging);* and (3) *decompress* rote procedures of writing equations to focus on the logic behind what the equation would mean in terms of the context of each story problem.

The second question was question 31 on form A (which was described in chapter three). Bianca was asked to decide whether three story problems could be modeled by a given linear equation of the form $y=mx + b$, by saying *yes, no,* or *I'm not sure* for each of the three-story problems. Bianca used the same type of reasoning that she had used for the first question. The first story problem could *not* be modeled by the given linear equation, rather it could be represented by an exponential growth function because the initial amount was doubling each month. However, Bianca said that the given linear equation could in fact be used to model the first scenario because "we are changing by 2 for each *x* month, and we started off at [the initial amount], so it matches, so it's a yes". The second story problem could *not* be modeled by the given equation, although it could be represented by a different linear equation. Bianca underlined all the numbers that were given in the story problem and said that they "match up" with the given linear equation, so "it would be a yes too". The last story problem was the only one could be represented by the given linear equation. Bianca used the same reasoning as before, by explaining that the initial amount changes by a specific rate each *x* amount of days, "so yes it works".

When answering the second question, Bianca relied on the use of an inaccurate procedure to incorrectly answer the question. For all three choices, she first underlined the numbers written in the problem. If they matched the numbers that were given in the equation, she said the story problem could be represented by the equation. Accordingly, there was no evidence of *trimming, bridging,* or *decompressing*, because she was not able to (1) get to the mathematical essence of each of the three real-world problems (*trimming*); (2) make connection across the different mathematical topics integrated in the question (*bridging);* and (3) *decompress* rote procedures of writing equations to focus on the logic behind what the equation would mean in terms of the context of each story problem.

MEA 2-Exploration of the Task

Bianca worked on this task with Oriana, Helaine and a fourth PSMT who was not a part of this research study. The task was shown in Figure 16 above, and *Appendix E*. Bianca's participation in the exploration of the task with the group was limited. When her group was discussing the assumptions of the task, Bianca pointed out that sometimes the family calls more than once during the same days, while other days they don't make any phone calls. However, this observation was not used in making sense of the task, after it was stated. As the group continued discussing different strategies they could use, Bianca stayed quiet.

When the PSMT began calculating the different costs for each phone plan, Bianca advocated for the recommendation of *Midwest Sense Any Time* plan because the family would be able to "call any day no matter what". But then she also brought up the condition of the monthly fee being waived only if the total cost was over $25. She explained that if the family "went under [the $25 cost] they would have to pay extra". Then, her group calculated the total cost for the other phone plans.

As Helaine and the fourth PSMT were calculating the total cost for the rest of the phone plans, Bianca was monitoring their work. Occasionally, Bianca would find a calculation error and point it out. For instance, as Helaine was finding the total amount of minutes called between 7PM to 7AM, Bianca pointed out that the minutes used at 7:40 am would not be counted for this group of calls. A few minutes later, she reminded Helaine to add the monthly fee to the total cost, because the $5.95 applied to the *Midwest Nights* plan. As the group calculated the cost for the *Horizon Nation Wide Saver* plan, Bianca reminded Helaine that she did not have to recalculate the amount of minutes in each group, but Helaine pointed out that what they previously calculated was the amount of minutes for the two different intervals of times of day, rather than the duration of the call, which was the condition of this plan. Once the group made the decision to recommend the *Horizon Nation Wide Saver* plan to the Olsen family, Bianca pointed out that the data that they were provided with was "only like for 15 days of one month." eluding to the idea that the rest of the month, and year, may look different, which she discusses in her written reflection.

Once the group was done making their recommendation and writing the letter for the Olsen family, they began reflecting on the process. Bianca acknowledged her limited contribution by saying, "I stayed quiet for the most part". Lastly, when the PSMT were discussing the amount of data given, and how some found it intimidating when they first read the task, Bianca agreed and revealed that when she saw the data she "automatically wanted to add, and basically just find the common things between the numbers."

As stated previously, Bianca's participation in the exploration of the MEA was limited; and the few times she did collaborate with her groupmates, evidence of *trimming, bridging,* or *decompressing* were not demonstrated, because she was not able to (1) get to the mathematical essence of the real-world problem (*trimming*); (2) make connection across the different mathematical topics integrated in the task (*bridging);* and (3) *decompress* rote procedures of linear reasoning to focus on the logic behind what the process would mean in terms of the context of the MEA.

Reflections on MEA 2

Bianca wrote a reflection about her exploration of MEA 2 and sent it to the researcher a few days after she explored the *Phone Plans* task. First, Bianca discussed her experiences with the modeling task. She explained that she made sense of the MEA by "calculating and comparing rates before choosing a phone plan" She described the given data as being "easy to read quickly because it was already [organized] in small sections." She then explained that her group wanted to choose the least expensive plan, and "once we had the data, this was easy." Bianca continued by describing the assumptions the group made. She pointed out that "since we were only given one month, which happened to be at the end of school, or beginning of summer, we factored out that this month may be [unusual]". Although Bianca discussed important factors the group considered and ignored when exploring the task, she did not describe the mathematics behind their exploration. Accordingly, evidence of *trimming* is not demonstrated, because she was not able to get to the mathematical essence of a real-world problem.

Then, Bianca talked about the mathematical concepts the group used while exploring the MEA. She said,

> The mathematical concepts we used were mainly the average concept, where we would find the total amount of minutes the family used in the previous month of June and calculate the average on an entire scale as well as the ranges of times depending on the type of minute. We calculated what the charge would be under each plan offered. Through systems of equations we established an average total for each plan leading to our cheapest plan.

Bianca then continued talking about how these mathematical concepts are connected. She explained that the concepts are related because "we could not compare rates without the averages of each time frame and we could not make a decision without knowing which plans were more efficient for the family." Bianca's discussion of mathematical concepts did not show evidence for *trimming,* because she was not able to get to the mathematical essence of a real-world problem, nor *bridging*, because she was not able to make connections across topics and representations.

Next, Bianca discussed the possible misconceptions students may have solving this task. She explained that students may "just look at the plans and add the rate per minute with the possible fee" or "use the sum of minutes for all plans, not realizing the differences in three of the plans that change the rate per minute based on time of call and length of call." Bianca did not describe specific difficulties entailed in students' understandings of particular aspects of algebra content, thus, there was no evidence of *decompressing*.

MEA 3-Exploration of the Task

Bianca worked on the *Historic Hotel* MEA with the same group that she had worked with on *Phone Plans* MEA. The task was shown in Figure 19 above and *Appendix F*. Similar to the exploration of the *Phone Plans* MEA, Bianca's engagement with this MEA was not as frequent as Helaine's and Oriana's. As the PSMT began exploring the total amount of profit earned for different variations of room rates, Bianca promptly brought up the maintenance fee. She asked her groupmates, "what about the maintenance fee? Do we subtract the four or add the" four?" Bianca suggested they should add the total cost, without explaining or justifying her reasoning. Although the other PSMT initially agreed, after the discussion continued about calculating the profit for different room rates, Oriana asked her group a similar question about the cost—is the hotel charging an extra $4 per room per day, or does the hotel have to pay?" At this point Oriana advocated for subtracting the maintenance cost from the total money earned, however, Bianca was not convinced. She said, "I would think that they *add it* because it's service." The fourth PSMT in the group agreed with Bianca. Nonetheless, Oriana continued making the case for why the maintenance fee should be subtracted, and until the rest of the group agreed.

Next, as the PSMT discussed different strategies for finding the maximum, and Oriana plotted points on the *number of rooms occupied vs. profit* graph, shown in Figure 25 above, Bianca suggested that the group should derive an equation to represent the given scenario. She said, "can we come up with a solid formula because if the rate is changing then that would be a separate line every time." The group agreed and used the context of the MEA to help them write an equation. Bianca suggested to make the variable the number rooms. She said, "like if we made x the number of rooms, with 60 times x, then minus one…" To that, Helaine replied, "I think it would be plus one, because it goes up" and Bianca agreed. As the group continued their discussion on finding an equation that represented the given scenario, Bianca asked a few clarification questions. First, she asked her groupmates about the variable, and whether it represents the number of rooms occupied or not occupied, to which Helaine explained that it would represent the number of empty rooms and showed her an example by substituting values into the equation. Then, Bianca asked about the maintenance cost, once more, why it would be subtracted and not added, to which Oriana responded by explaining the context of problem and the meaning of profit and cost.

As the group finished writing the equation, they continued justifying the different parts of it, and worked on finding the rate that maximized profit. At this point, Bianca was doing work on her own, and asked Oriana to tell her what the final equation the group wrote, she said, "so can you tell me what the final equation was, so I can make sure what I came up with was right?" Oriana shared the final equation they wrote, which was $(80-n)\ (56+n)$ and explained to her how she simplified the original equation by factoring. Bianca said she understood and that the equation she wrote was similar. The figure below shows Bianca's individual work.

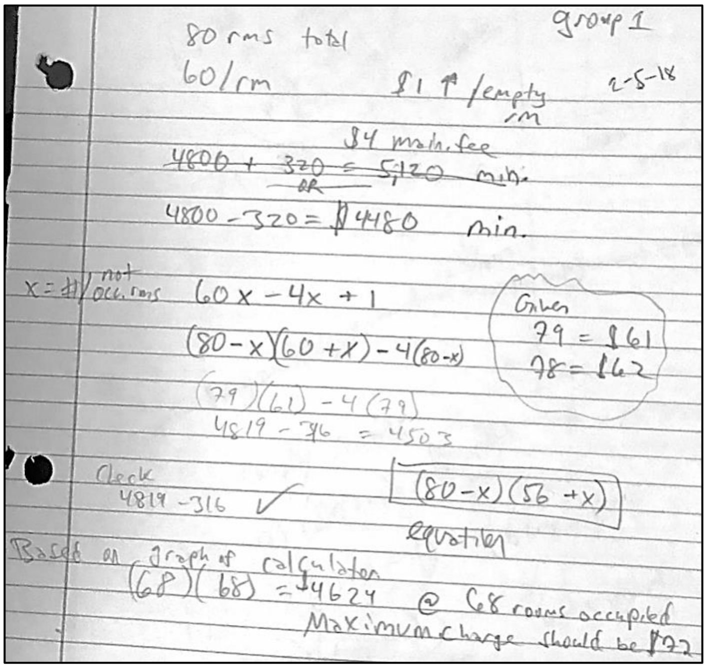


Figure 31. Bianca's Individual Work *Historic Hotel* MEA.

To find the maximum profit, the group then graphed their equation using an online graphing calculator. Once they graphed the parabola that corresponded to their equation, Bianca estimated that the maximum would be 13, by the way the graph looks. However, Oriana used her handheld graphing calculator, and found the exact maximum at x =12, rather than estimate. Immediately, Bianca took the value of the variable, and substituted it into the profit equation, and calculated that the maximum profit would be $4624. However, the other PSMT in the group did not focus on the value of the maximum profit, they were discussing the meaning of the 12, in terms of the context. Helaine and Oriana were discussing how to use 12 to find the number of rooms occupied, and the corresponding rate. Oriana suggested that because 12 was the number of empty rooms, then the maximum profit would take place when there are 68 rooms occupied, because “that would be 80 – 12”, and the maximum would be $72 because “it would be 60 + 12”, Oriana explained as she pointed to each term in the equation they wrote.

As the group composed the letter for the hotel owner, Bianca suggested that they should be “as specific as possible, in order for the process to make sense”. As Helaine and Oriana narrated the process, and the fourth PSMT teacher in the group wrote the letter, Bianca was listening to what was being discussed and looked at what was being written. Once the group was discussing writing the general form of the equation, Bianca asked, “did you guys say how we got the total to be 68 and 72?” Oriana then explained that the rate goes up from $60 for each unoccupied room, and because the maximum occurred when there were 12 unoccupied rooms, it would be 12 added to the 60, which was $72. Once the group was done writing the letter, Bianca, once again, asked Oriana about what 72 represents and how she found it. Below is the discussion between the two PSMT,

Bianca: Can you remind me again why its 72? I’m sorry.

Oriana: Yeah, because, no you’re fine. So, like here, the empty rooms were 12.

Bianca: Yes.

Oriana: So basically, like it was going, every time there was an empty room the price went up by one.

So, like if the price started at 60 and then there were 12 empty rooms it’s going up by 1 for every empty room so 60 plus 12 and that’s how we got the 72.

Bianca: Okay, just wrote that, I was making sure, because I was just like that’s totally where you can get 72.

Oriana: Yeah.

Bianca: Then like unless we are just having 60 plus 4 [instead of 1] but no.

Oriana: Yeah, it’s because of like the rule that they gave us, was like every time that the price goes up by 1, one less room is occupied, so yeah. Then that would be like why we got the 68 also because it’s like there were 80 rooms but 12 were empty.

Bianca: Yeah. Just kind of funny how you guys came up with that equation though. I was just like, doodling everything on my paper, okay… this looks nice. And then you

> guys are talking together, it's like, oh! You came up with the same thing, okay, I just need to adapt. Not too bad. Then I checked, after you got the equation I was like, let's plug in 72, and it works!

Lastly, as the group discussed their equation, graph, and letter, Bianca said, "I wonder if hotel people have people they hire to like help do what we did, or they just randomly like pull up the person, and go like, this is what we charge. Evidence of *trimming, bridging,* or *decompressing* was not shown because Bianca was not able to (1) get to the mathematical essence of the MEA (*trimming*); (2) make connection across the different mathematical topics integrated in the task (*bridging);* and (3) *decompress* rote procedures of maximizing profit, to focus on the logic behind what the equation, and calculated numbers, would mean in terms of the context of the MEA.

Second interview

The second interview with Bianca took place an hour after the exploration of the MEA. The researcher asked Bianca questions (included in *Appendix H*) about the MEA and two content questions from the LMT assessment.

Experiences and Reflections related to MEA 3

During the first part of the interview, Bianca discussed her experiences with the *Historic Hotel* MEA. When she was explaining how she made sense of the MEA, she spoke about having to read the task multiple times to be able to understand it. She spoke about how her groupmates and her all had similar strategies by wanting to represent the given scenario using an equation. More specifically, Bianca said they had to represent "the amount of rooms rented, amount of money the hotel earns, and the maintenance fee". Bianca did not specifically talk about the strategies the group used to write the expressions that represented the different aspects of the profit equation, nor did she explain the specific process she went through to write the equation.

Next, Bianca discussed the specific mathematics content that helped her make sense of the MEA. She spoke about the use of equations, she said:

> First, I thought of using systems of equations, because first we had the $60 per day and then we had the maintenance fee per day as well. So, at first, I was like we have two equations, but eventually I added the two because it was in the same day so, I might as well just put it in one equation instead.

When asked to elaborate about what she means by system of equations and how she used the equations to solve the task, Bianca was not able to explain how the group derived the profit equation. She explained that the group used their graphing calculators to input the equation and use the calculator's *find maximum* command to find the maximum. She then simply described how the equation looks like, she said. "we left it with two parentheses, that type of equation. I mean technically if we multiplied it out, it would have been a *polynomial*. I don't know what it would look like as a polynomial though." The researcher then gave Bianca the group's written work and asked her to clarify what she meant by *systems of equations.* Bianca replied,

> well see, we had this representing price per room, and we multiplied it by the number of rooms, and because these have two variables in line, you think of a system. So, again system of equation doesn't necessarily have to be two different equations, it could just be one equation.

This demonstrated that Bianca was not able to get to the mathematical essence of the given real-world problem, therefore *trimming* was not evident.

Next, Bianca was asked about how the mathematical ideas she talked about—graphing, systems of equations, and polynomials are connected, she replied that "plotting would be used to solve the system of equation because you will get a point, and when you multiply the system itself the result is a polynomial". Bianca's responses contained many misconceptions and imprecise use of mathematical terms; moreover, no evidence of *bridging* was demonstrated from this part of the interview.

At this point, the researcher asked Bianca about different methods that were presented by other groups, such has the first and second derivative test, as well as finding the *x*-value of the vertex of the parabola using the formula $x = -\frac{b}{2a}$, and how these methods related to the strategy that they used in their group. Bianca replied "well, $-\frac{b}{2a}$, they used the term to find the slope, and when we graphed our equation, we also had a slope, so we both had 12." The researcher then asked Bianca what she meant by *slope,* to which she responded,

> our graph had the variable *n* which represented the number of empty rooms, and we looked at [the table of values] and found that the maximum was at 12, so that's when the number of empty rooms was 12. That was just basically the *midpoint,* so $-\frac{b}{2a}$ I think kind of relates to the slope and the point. I can't remember exactly where the formula but it's familiar.

Bianca's answer again contained misconceptions and imprecise use of mathematical terms; moreover, no evidence of *bridging, decompressing, or trimming* was evident. When the researcher asked Bianca about the first and second derivative test, Bianca acknowledged that although she was *intrigued* by the idea, she did not know why it works and how it connected to their method of exploring the MEA.

As Bianca reflected on her experience with the *Historic Hotel* MEA, she spoke about the context of the task. She stated:

> It made it seem more useful, because it's like in the real world you don't think of people thinking about this. Because we don't really see it daily, we know about it in school, but it's different whenever you are given a problem and you can see the context and use the context in that way.

Although Bianca broadly about the usefulness and importance of algebra, she did not explain the specific mathematical concepts students would benefit from, and she was not able to decompress rote procedures to grasp the logic behind them, thus, *decompressing* was not evident.

When reflecting on how this task can be used in an Algebra class, Bianca said that prior to giving the task to the students, she would "make sure the students knew about systems of equations first, and then how to convert the problem into one and interpret the system of equations." She explained that a task like the *Historic Hotel* MEA can help students "use what they know and then listen to what other people know as well".

Lastly, when Bianca was asked about possible misconceptions students may have with solving the *Historic Hotel* MEA, she talked about forgetting to incorporate the maintenance fee into the equation. Because Bianca was not able to anticipate the difficulties entailed in students' understandings of specific mathematical ideas, *decompressing* was not demonstrated.

Content Questions

The next part of the interview entailed the two content questions from the LMT assessments. As previously stated, the questions were not part of the items released by the LMT project, and thus, the description of the findings in this section will not have specific details of the context of the questions.

The first question, Bianca was asked to choose a story problem that did *not* represent linear behavior in the context of the four given story problems. A description of each choice was given in chapter three. When thinking about this question, Bianca explained that the first choice represented linear behavior because "it's asking about the n^{th} day, which means the next day, and the next day, meaning like a line, so that's a linear." She did not talk about rate of change by which the initial amount was increasing. Because the initial amount was doubling for each passing day, the first choice represented exponential behavior rather than linear. For the second choice, Bianca described that it was similar to the first scenario because it's also asking for the "n^{th}" unit. Bianca continued using this reasoning strategy for the third and fourth choices, and explained that, "all these stories represent linear behavior, that would be my answer." Bianca answered the question incorrectly, and the reasoning behind her choices did not demonstrate evidence for *trimming*, *decompressing*, or *bridging*, because she was not able to: (1) get to the mathematical essence of each of the four real-world problems (*trimming*); (2) make connection across the different mathematical topics integrated in the question (*bridging);* and (3) *decompress* rote procedures of writing equations to focus on the logic behind what the equation would mean in terms of the context of each story problem.

The second question was question 35 on form B (which was described in chapter three). Bianca was asked to choose the situation that could be used as an example for teaching linear functions. The first situation could be modeled using a linear function of the form $y=mx$. Bianca struggled with determining if the situation represented a linear function. She said that "this is not really going to go past the initial amount, we can just multiply [the rate by the amount that we want]." She left it alone and came back to it after looking at the other two situations. Bianca explained that the second and third situations would represent linear behavior because they "have a variable." Then, she went back to the first situation and said "Oh! But we don't know how many [units] they want, so there is a variable, so this one works too!" Bianca concluded that all three situations represent linear behavior. Bianca answered the question incorrectly, because the third situation could be represented using a *quadratic* function rather than a linear function, while the first two situations were in fact linear.

When answering the second question, Bianca relied on the use of an inaccurate procedure to incorrectly answer the question. For all three choices, she looked at what the question was asking, and if she could use a variable to represent a part of the context, then she concluded that the story problem could be modeled with a linear equation. The reasoning behind her answer did not demonstrate evidence of *trimming, bridging,* or *decompressing,* because she was not able to (1) get to the mathematical essence of each of the three real-world problems (*trimming*); (2) make connection across the different mathematical topics integrated in the question (*bridging);* and (3) *decompress* rote procedures of writing equations to focus on the logic behind what the equation would mean in terms of the context of each story problem.

LMT Assessments

As stated previously, Bianca was selected from the *medium MKT* category— which included participants with scores greater than 25th percentile and less than or equal to 50th percentile on the LMT pre-assessment. Bianca's score on the LMT pre-assessment was in the 50th percentile. As discussed in chapter three, IRT scores represent the underlying ability being measured by the assessment, in this case it is the PSMT'MKT middle school algebra. This ability is denoted as the symbol theta. This ability spectrum is measured in standard deviation units with zero being average. Higher abilities are represented by positive values and lower abilities are represented by negative values. Bianca's overall IRT score on the pre-assessment was $\theta = -0.241$, and $SE = 0.391$ On the post-assessment, Bianca's overall IRT score was $\theta = -0.683$, and $SE = 0.374$, which was in the 35th percentile. The box and whisker plots below illustrate Bianca's IRT scores in relation to her classmates on the pre-assessment (n = 30) and post-assessment (n = 27). These results demonstrate that Bianca's MKT algebra was in the middle (or second) quartile on the pre-assessment, and in the lower half of the scores on the post-assessment, in comparison to her classmates.

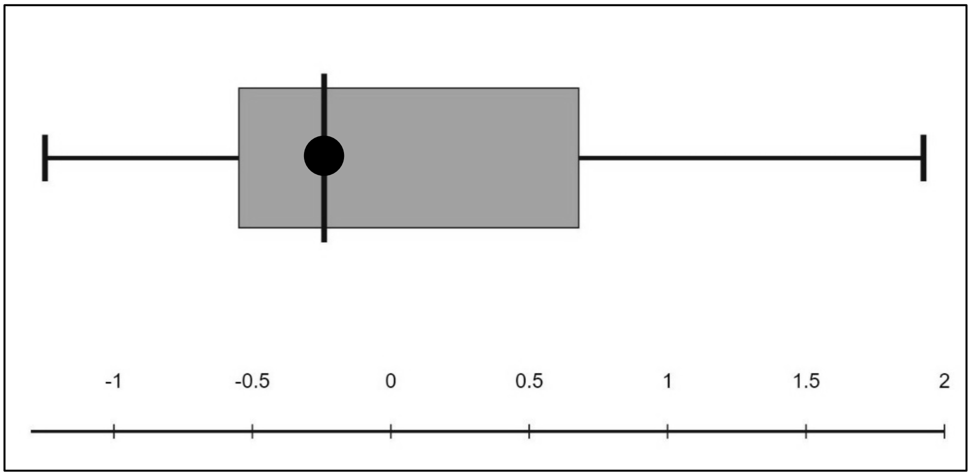


Figure 32. Bianca's Pre-Assessment Score on a Box Plot.

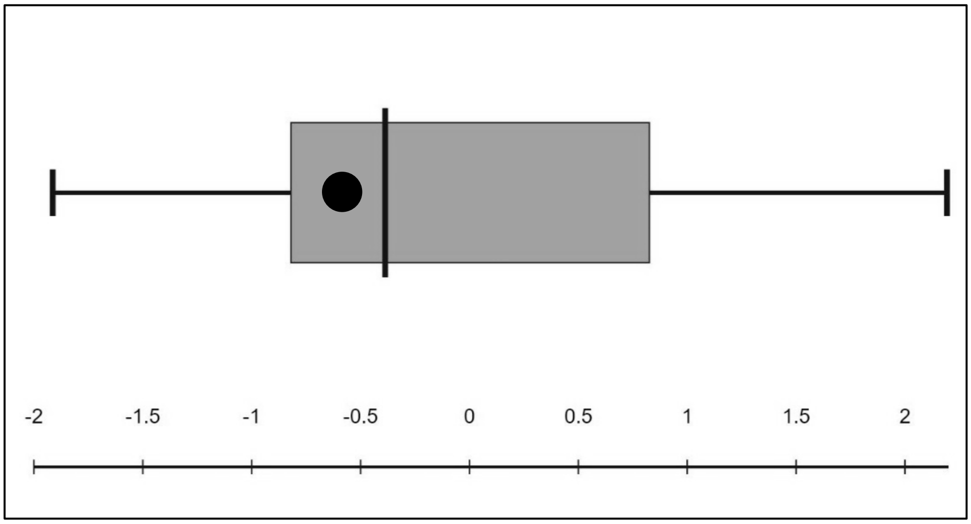


Figure 33. Bianca's Post-Assessment Score on a Box Plot.

Discussion of Case 2

Bianca was selected from the medium MKT category. Her score on the LMT pre-assessment ($\theta = -0.241$, $SE = 0.391$) was in the 50th percentile of the participating PSMT scores in her class ($n = 30$). Her score on the LMT post-assessment decreased ($\theta = -0.683$, $SE = 0.374$) and was in the 35th percentile of the participating PSMT' scores in her class ($n = 27$). Although her score decreased, and changed positions in view of score quartiles, her MKT category did not change because her score was still greater than 25th percentile and less than or equal to 50th percentile, which was the designated range for a medium MKT.

During the first MEA, Bianca worked with Oriana, and she suggested a few strategies to explore the *Summer Jobs* MEA, such as comparing the amount of money each vendor made, and graphing. However, when Oriana questioned the usefulness of the strategies in helping them make recommendations based on all of the given data, Bianca could not explain or justify her suggested methods. Nonetheless when Oriana suggested and explained the use of unit rates and averages, Bianca complied, and calculated the unit rates and averages for some vendors. Thus, some evidence of *trimming* was demonstrated, because Bianca was able to talk about the mathematical essence of the given task after Oriana explained to the group why unit rates and proportions were useful in making sense of the given scenario. Nonetheless, Bianca was not able to explain why other strategies, like graphing the amount of money made, or creating equations, were not helpful in selecting the vendors to be employed. There was *no evidence for bridging or decompressing* during the exploration of the Summer Jobs MEA.

During the first interview, Bianca described her experiences with the Summer Jobs MEA by discussing the mathematics that helped her make sense of the task. She mentioned topics such as slope, averages, graphing, and creating expressions however, Bianca was not able to explain how the graphing strategy would help address the given task, nor was she able to precisely describe how she calculated the averages and unit rates. In other words, she was not able to talk about the mathematical essence of the given real-world problem, therefore *trimming* was not evident. Similarly, there was no evidence of *bridging* because she was not able to correctly identify which mathematical concepts were connected and how. Likewise, there was no evidence of decompressing, because she focused on describing the rote procedures of averages and unit rates and did not explain the logic behind the specific mathematics topics she discussed.

Next, when discussing how the task could be implemented in an algebra class, Bianca broadly discussed how groupwork can help students better understand mathematical concepts, but she did not specifically discuss the MEA. Furthermore, Bianca was not able to anticipate possible student misconceptions, rather she talked about the difficulty of the task. Therefore, decompressing was not evident, once again.

When it came to answering the content questions in the first interview, Bianca answered both questions incorrectly. She used the same reasoning to answer both questions. For each of the given story problems, she looked at the given initial amount, identified the variable for each scenario, but did not reason with the context of each scenario. Thus, there was no evidence for *trimming, bridging* or *decompressing* shown.

During the second MEA, Bianca's contribution to the exploration of the task was limited. She shared her opinion about each of the phone plans without mathematical justifications. She monitored the calculations being done by the other PSMT and pointed out any errors that were made. She acknowledged her limited contribution to the exploration of the task when the group was finished working on the task. Evidence of *trimming, bridging,* or *decompressing* were not demonstrated, because she was not able to (1) get to the mathematical essence of the real-world problem (*trimming*); (2) make connection across the different mathematical topics integrated in the task (*bridging);* and (3) *decompress* rote procedures of linear reasoning to focus on the logic behind what the what the process would mean in terms of the context of the MEA.

On her written reflection, Bianca discussed important factors the group considered and ignored when exploring the task, such as the time of day the phone calls were made, and the duration of the calls. However, she did not correctly describe the mathematical topics used during. She talked about usin*g* averages, which the group did not do. Furthermore, she was not able to make connections across the mathematical topics she discussed. When discussing possible student misconceptions, Bianca did not explain specific difficulties entailed in students' understandings of particular aspects of algebra content, thus, there was no evidence of decompressing.

During the third MEA, Bianca's contribution to the exploration of the task was not as frequent as her groupmates. Bianca suggested to the group that writing an equation to represent the given scenario and identified that they would have to use the 80 (which was the total number of rooms in the hotel) and 60 (which was the rate for each room, if all the rooms were occupied) in their equation. However, Bianca was not able to make sense of the profit equation her groupmates derived, she was not able to explain how it was derived and what it meant, during the interview. Evidence of trimming, bridging, or decompressing were not shown because Bianca was not able to (1) get to the mathematical essence of the MEA (trimming); (2) make connection across the different mathematical topics integrated in the task (bridging); and (3) decompress rote procedures of maximizing profit, to focus on the logic behind what the equation, and calculated numbers, would mean in terms of the context of the MEA.

During the second interview, Bianca described her experiences with the Historic Hotel MEA by discussing the mathematics she used that helped her make sense of the MEA. She stated that her group used systems of equations but did not clarify how the use of a system of equations helped them explore the task. Accordingly, no there was no evidence for *trimming.* Bianca was not able to correctly identify any connections between the mathematical topics she discussed, and the researcher asked about, nor was she able to anticipate possible student misconceptions. Thus, there was no evidence for *bridging*, or *decompressing*.

Next, when discussing how the task could be implemented in an algebra class, Bianca broadly described how tasks like the Historic Hotel MEA, would help students understand the usefulness and importance of algebra. However, she did not specify what algebra content, or how students might see the usefulness of such topics. Accordingly, there was low evidence of decompressing in her response.

When it came to answering the content questions in the second, Bianca answered both questions incorrectly again. She used the same reasoning to answer both questions, and she provided the same type of justifications for her responses as those in the first interview. For each of the given story problems, she looked at the given initial amount, identified the variable for each scenario, and but did not reason with the given context. Thus, there was no evidence for *trimming*, *bridging* or *decompressing* shown.

The findings above are summarized in the following table and figure. Table 9 shows the different levels of evidence—high (H), medium (M), or low (L) in terms of the three KAT categories *trimming*, *bridging* and *decompressing*, for each of the collected data sources.

Table 9

Bianca's levels of evidence for trimming, bridging and decompressing.

	MEA 1		MEA 2		MEA 3	
KAT Categories	Exploration of Summer Jobs MEA	First Interview	Exploration of Phone Plans MEA	Written Reflection	Exploration of Historic Hotel MEA	Second Interview
Trimming	M	L	M	L	L	L
Bridging	L	M	L	L	L	L
Decompressing	L	L	L	L	L	L

The figure below illustrates the overall level of evidence for each of the two factors of interest—MEA contribution, and MKT. The plot consisted of the three KAT categories placed on a continuum for each factor. An overall rating of high, medium, or low was given for each of the three MEAs, based on the table above. An overall high rating was attained if a high was given on both the exploration of the MEA and the interview or written reflection that followed. An overall medium rating was given for a medium-medium combination, high-low combination, and a high-medium combination. An overall low rating was given for a low-low combination, and a low-medium combination.

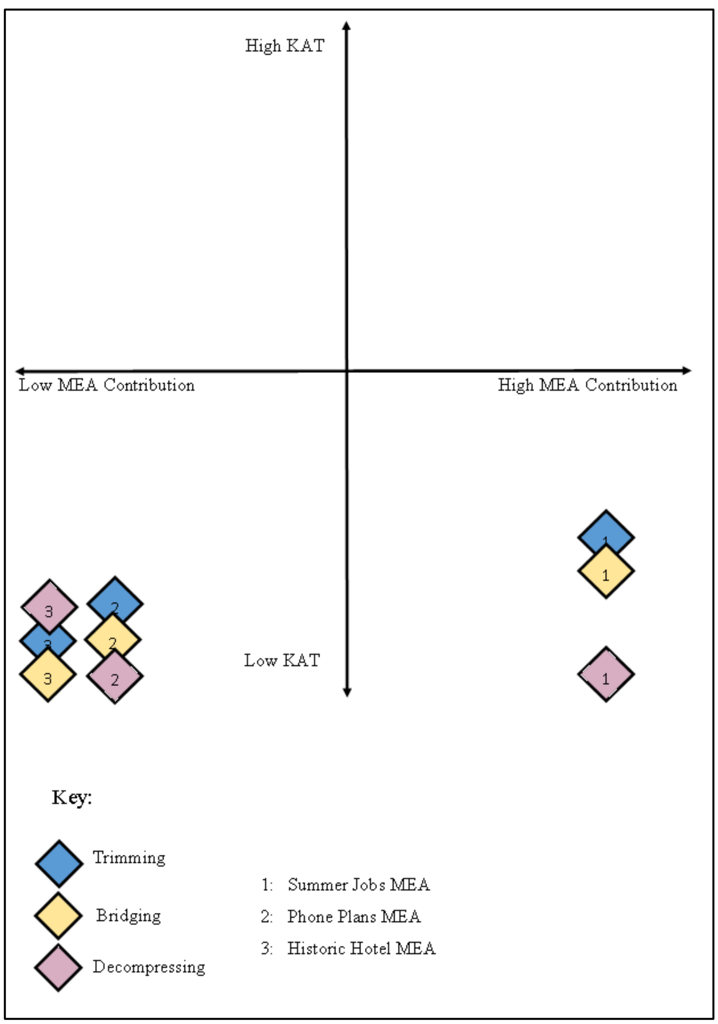


Figure 34. A Plot of Bianca's Overall Level of Evidence for MEA Contribution and KAT.

The plot above illustrates the following:

(1) Bianca's overall level of evidence for the Summer Jobs MEA and first interview was *low* for trimming and bridging, and *no* evidence of decompressing, while her contribution to the exploration of the task was medium-high.

(2) Bianca's overall level of evidence for the Phone Plans MEA and the written reflection that followed was *low* for trimming, bridging decompressing, while her contribution to the exploration of the task was medium-low.

(3) Bianca's overall level of evidence for the Historic Hotel and the second interview was *low* for trimming, bridging, and decompressing, while her contribution to the exploration of the task was low.

Connecting this back to McCrory and colleagues' definition of *trimming, bridging, and decompressing*, it was evident that:

(1) For the first MEA, Bianca was involved in exploring the task, she was able to make some connections across mathematical topics (*bridging*), and she was also able to talk about some of the mathematics used to explore real-world problem (*trimming*), however she was not able to anticipate the difficulties entailed in students' understandings of particular aspects of algebra content, explain how to help students understand the usefulness and importance of algebra, and decompress rote procedures in order to grasp the logic behind them (*decompressing).*

(2) For the second MEA, Bianca was not fully involved in the exploration of the task, she was not able to make connections across mathematical topics (*bridging*), she was not able

to discuss the mathematics used to explore real-world problem (*trimming*), and she was not able to anticipate the difficulties entailed in students' understandings of particular aspects of algebra content, explain how to help students understand the usefulness and importance of algebra, and decompress rote procedures in order to grasp the logic behind them (*decompressing).*

(3) For the third MEA, Bianca was not fully involved in the exploration of the task, she was not able to make connections across mathematical topics (*bridging*), she was not able to discuss the mathematics used to explore real-world problem (*trimming*), and she was not able to anticipate the difficulties entailed in students' understandings of particular aspects of algebra content, explain how to help students understand the usefulness and importance of algebra, and decompress rote procedures in order to grasp the logic behind them (*decompressing).*

Case 3: Helaine

"Anything Math, I am all for!"

Helaine, January 2018

Description of participant

Helaine was selected from the *low MKT* category— which included participants with scores less than or equal to 25th percentile on the LMT pre-assessment. Helaine's score on the LMT pre-assessment was in the 25th percentile. Helaine was in her third year of the secondary mathematics education program. She expressed that she enjoys taking mathematics classes. In high school she enrolled in non-required mathematics classes as electives and she has enjoyed all of her mathematics classes at the college level. Helaine explained that even when she doesn't do well in a mathematics class, she still enjoys learning the content. Throughout middle school and high school, Helaine always tutored her peers in mathematics, so when it came to choose a major in college, she thought "why not teach math" (Helaine, First Interview, January 24).

During the semester that this research was conducted, Helaine was enrolled in a mathematics course called *Logic and Proofs*. When asked to give an overview of the content being taught in the class, she said:

> Right now, what we are doing is, we basically are given a proof, and we have to logically come up with an explanation on why this proof is the way that it is. We started out with statements and we are okay, are these statements true or false…Today we had to determine which one [statement] was tetralogy or contradiction. To determine between the two, you can create truth table, and say, okay this is tetralogy because everything is true, once you do all the fancy work, and this one is a contradiction because it is false, and stuff like that (Helaine, Second Interview, February 7)

After graduation, Helaine hoped to teach mathematics at the high school level and she planned on attending graduate school after teaching for a few years.

First Interview

The first interview with Helaine took place two days after the exploration of the MEA. After answering the ice breaker questions (included in the interview protocol in *Appendix H*), the researcher asked Helaine questions about the *Summer Jobs* MEA she had explored two days prior, and the LMT pre-assessment she had taken the previous week.

Experiences and Reflections related to MEA 1

During the first part of the interview, Helaine discussed her experiences with the *Summer Jobs* MEA. When she explained how she made sense of the MEA, she shared the challenges she faced while exploring the task. She explained, “It was hard to think of different numerous ways to do it... I only had one other person in my group because our third group member didn’t show up.” She then discussed the strategies her groupmate recommended they use. First, she discussed looking at the vendors that made the most amount of money during the *slow* attendance times at the park because “if they can work during the slow hours and make more money, then they should be able to make more money as fulltime employees”. She explained that that strategy made sense to her because “for a duration of time they made more profit, so for a longer duration of time, the bigger the profit, so it would be proportional”. Then, Helaine explained that because the strategy didn’t work for every vendor, her groupmate suggested a different strategy. Helaine said:

> She decided to take a different approach with looking at how many hours they work during the different times. Actually, I think it was more like, we look at the slow and we took the two highest people who can make the most money during this. Then the steady hours, who made the highest during those and then the busy. Then we thought, okay, they can make the most money during this time, So, if you keep them during a busier time, then they may be able to make more money and stuff like that. It made sense but at the same time it didn’t.

Then, Helaine mentioned the presentations of solutions to the MEA that she heard from the other groups, and acknowledged that "their methods make more sense, and what we tried to do made like half sense."

When asked about how she would approach the problem now, Helaine said that she would use averages. Specifically, she explained that she would find the average amount of hours the vendors work for each month, and compare them to the average amount of money they made during those same months; whoever made the most amount of money and worked the most amount of hours would be hired full time, and the vendors that worked less hours but still made a good amount of money would be hired full time.

Next, Helaine talked about the specific mathematics content that helped her make sense of the MEA. Helaine mentioned *proportional reasoning*, "because if you can make this much money in this amount of time, if you were to double it, you should be able to double your profits based on doubling your time." She also spoke about *averages* by discussing using the average number of hours worked per month and the average amount of money made per month to compare the different vendors. Some evidence of *trimming* can be seen in the explanations discussed above because she was able to discuss the mathematical essence of the given real-world problem.

When asked about how these mathematical ideas are connected, Helaine stated:

> When you're working with the average, you're taking numbers and you're dividing that by how many of that quantity that you have, to come up with a generalized idea. Then when you are looking at it in a proportion-wise, you're using your average to basically compare it to something bigger or smaller than it, to where you can like, I'm trying to think of the right terms to use.... But you have your averages and you can relate them to be like, if you do it on a bigger scale or a smaller scale, you can proportionalize *[sic]* it to be like A goes to B and C goes to D.

Although Helaine said that proportions and averages are connected, she was not able to explain how. Thus, only some evidence of *bridging* was seen from this part of the interview. Furthermore, by emphasizing the procedure of finding average, rather than the meaning of it, she was not able to decompress rote procedures in order to grasp the logic behind them, and thus *decompressing* was also not evident in this part of the interview.

As Helaine reflected on her experience with the *Summer Jobs* MEA, she discussed how exploring the task helped her manage a large amount of data. She said:

> When looking at the numbers, it makes it easier to compare whenever you break it down in two and get yourself smaller amounts of data to work with, like the averages of hours worked and money made… This helped me get knowledge of how I can work with bigger amounts of data to get the average that I need to help me find the right answer.

When reflecting on how this task could be used in an algebra class, Helaine discussed the context of the problem and it could help students see the importance of algebra in real-life, and the use of different methods. She explained:

> It basically gives you a real-life idea of how this stuff actually works. It's funny because a lot of kids are like, "Teacher, why do we have to learn this. When are we ever going to use this?" The modelling task that we did gives you a real-life example of this is what people have to go through to figure out, okay, who do I want to hire and why based on data. What do you think, do you think we should base them on how many hours they worked or based on how much money they make?
>
> Even when they get an answer, they should ask themselves, does that make sense or is there some other kind of method to get to it?

Helaine also discussed altering the problem to a new context that students can relate to, like "how many hours should you study a week for the upcoming test that we're going to have?" Although not specific to the specific MEA, Helaine's reflections demonstrate evidence of *decompressing* in explaining how to help students understand the usefulness and importance of algebra.

Lastly, when asked about possible misconceptions students may have with solving the *Summer Jobs* MEA, she broadly discussed the different backgrounds students have, as well as their knowledge of mathematical terms. She said,

> The misconceptions and challenges that they could face besides different ideas coming from different people, it can all come from their past experiences. They might have had to deal with the situation like our modelling task in class and already know how to [approach] it.
>
> Or they may not have ever heard of it before and they may be thinking that they don't understand. We're talking about the average or the proportions or ratios. They might not have any idea what that even means and that's why knowing Math vocabulary is so important especially in situations and tasks like these.

Although Helaine alluded to the importance of using precise mathematical terms, she did not specifically explain how that can help or hinder students' understanding of mathematics or the exploration of the MEA. Therefore, evidence of *decompressing* is not observed in this part of the interview because Helaine was not able to anticipate the difficulties entailed in students' understanding.

Content Questions

The next part of the interview entailed the two content questions from the LMT assessments. As previously stated, the questions were not part of the items released by the LMT project, and thus, the description of the findings in this section will not have specific details of the context of the questions.

The first question, Helaine was asked to choose a story problem that does *not* represent linear behavior in the context of the four given story problems. A description of each choice was given in chapter three. Prior to discussing the question, Helaine began by sharing her experience with this question when she first encountered it on the pre-assessment. She said, "It's funny when I viewed this question before, because there were previous questions before it, they kind of helped me with this one". Then, she started discussing the meaning of a linear equation. She said, "So, $y = mx + b$, it's a typical linear function, you have m for your slope, x for your x-coordinate and b for your *y-intercept…*So, $y = mx + b$ would be an example of a linear function or more like just a line."

Next, Helaine looked at the first choice. She explained that the first choice represents linear behavior, because the initial amount gets doubled every day, so "that would be $2n$" and n represented time. She modeled the story problem by the linear equation $y = 2n + 1$. She then substituted values for n, "on day 90, it would be 90 x 2 + 1, that would be the amount you're going to have on that day. So, this would be linear." However, the first choice represented exponential behavior, because the initial amount was being doubled every n days. The second choice represented a linear behavior and could be modeled using the equation $y = mx$. Helaine said that this choice could not represent linear behavior, she said:

> It seems like it should be linear, but I feel like it's not [linear], only because…like we started with two [units of measure] if we go up to four [units of measure], that would be 2 x 2 which is four. So, [the total] x 2 which is [double the amount pertaining to 2] and so on and so forth, it's not as much linear as it is proportional. That's why I assume that B wasn't linear. So, A is linear, B is not.

Then, Helaine moved on to the third choice. She explained that it could represent linear behavior, but she came up with the wrong equation to model it. She spoke about the initial amount as being the y-intercept, but she did not make sense of the rate of change. Similarly, the last choice she said represents linear behavior, but the equation she used to model the context was incorrect. Although her initial amount, or the y-intercept was correct, the rate of change was negative, but the equation she came up with for the scenario had a positive rate of change. Helaine concluded that "B is the only one that's not linear, so B would be the answer here." For this question, Helaine did not demonstrate evidence of *trimming, decompressing,* and *bridging* because she was not able to: (1) get to the mathematical essence of each of the four real-world problems (*trimming*), (2) make the connection between proportions and linear equation in the second choice (*bridging),* and (3) *decompress* rote procedures of writing equations to focus on the logic behind what the equation would mean in terms of the context of each story problem.

The second question was question 35 on form B (which was described in chapter three). Helaine was asked to choose the situation that could be used as an example for teaching linear functions. The first situation could be modeled using a linear function of the form $y = mx$. Helaine connected this problem to the second choice of the previous problem, she said, "this is another one of those proportional type things because you're going to have a [unit of measurement] and the rate for each unit." However, as she began talking about it, she changed her mind. She continued, "wait a minute…you're going to have a [unit of measurement] where it [increases by the rate of change]. If we multiply the two we will get the total amount." She then gave the equation that would represent the situation, correctly, in the form $y = mx$, and concluded "that's pretty linear, you don't have a *y*-intercept, but you don't need one, it's 0 so this one works."

Helaine then moved on to the second given situation. Right after reading it, she claimed that "this one makes more sense because it's more straightforward than the first one" and quickly came up with the correct linear equation to represent the initial amount, as the *y*-intercept, and the rate of change as the slope, and multiplied it by the variable, *n.* After reading the third given scenario, Helaine quickly concluded that the situation "is not linear as much as it is curvature. It's like a parabola type of thing, because you're going to start with 0, then go up to 4, and it's going to curve up." She continued her explanation by relating it to a different topic, she said,

> I know that because I remember having these problems in calculus. The only reason I can make that type of connection. The total number you have if [the initial amount] you're given is two more than what you were given the previous day. So, if you have one the first day, the next day you're going to have is 2 and then 4 and then 6, 8 etc.

Therefore, she concluded that the first and second situations can be used as examples of linear behavior, which was correct.

When answering the second question, Helaine showed evidence of *trimming*, *decompressing*, and *bridging* because she was able to (1) get to the mathematical essence of each of the four real-world problems (*trimming*); (2) make the connection between the first choice in this problem, and the second choice of the previous problem, because they both could be solved using proportion— although she did not change her solution to the first problem, based on her new realization, she was also able connect the third scenario to problems that she had done in a different mathematical topics and domains (*bridging);* and she was able to (3) *decompress* rote procedures of writing equations to focus on the logic behind what the equation would mean in terms of the context of each story problem.

MEA 2-Exploration of the Task

Helaine worked on this MEA with Oriana, Bianca, and a fourth PSMT that was not a part of this study. The task was shown in Figure 16 above, and *Appendix E*. She initiated the group discussion by going through the information given in the task. She then suggested the factors that may influence their choice of phone plan. Helaine explained, "I don't know the best way to go about this, but my biggest focus would be, *how often do they call, what time, and for how long*?" Along with the rest of her groupmates, Helaine then discussed the advantages and disadvantages of the various plans. She first talked about the plans that charge different rates for the calls made during the day or at night. She said that these plans may not be suited for the Olsen family because "they do it half, or about 50-50, percent of the time, they're one or the other. So, if you get an all-day every day plan, they don't have to stress about the time they call." By half and 50-50 she was referring to the number of calls being made from 7AM to 7PM being about the same as the number of calls being made from 7PM to 7AM. Prior to making that statement, the group counted how many calls were in each category, and found that 12 calls were made between 7PM and 7AM and 11 calls were made between 7AM and 7PM. She further explained that if they chose an *all day, every day* plan, "the cost per minute would be the same no matter what time you call…but the number of minutes they spend during these calls could also play into how much they would pay."

At this point, Oriana suggested a graphing strategy to help them compare the costs of each of the phone plans. Helaine agreed that strategy "could work" but calculating the total cost by "adding all [the minutes] and multiplying them by [the respective rate]" will give them the total cost of each plan. This demonstrates some evidence of *trimming* because she was able to get the mathematical essence of a real-world problem.

Then, Helaine proceeded to use this strategy to calculate the cost of each phone plan using a calculator. As she was doing the calculations for the *Midwest Sense Any Time*, Helaine brought up the assumption that

> This is all for the month of June. So, it's not like it goes into any other month. So, I'd say if this is like the average amount that they usually talk per month, then, this plan is a pretty solid plan.

She then continued calculating the total cost for *First Talk One Rate* plan, the *Midwest Nights* plan, and the *Midwest Plan 1000*. The cost of the *Horizon Nation Wide Saver Plan* was the last to be calculated. Helaine was hesitant to calculate the cost, because "I just don't see how this will be cheaper. It doesn't make sense… like how can it be that the less time you're on the phone, the more you pay? I don't get it." However, the rest of the PSMT in the group were persistent in calculating the cost, to stay consistent with applying the strategy they have been using. Accordingly, Helaine calculated the total amount of minutes for calls that lasted under 20 minutes and multiped by the corresponding rate ($0.08), then she calculated the product of the total amount of minutes for the class that lasted 20 minutes or over and the corresponding rate ($0.05), next she added the results of both calculations and the monthly fee for the plan, and thus, she determined that the cost of the plan would be $20.64. To Helaine's surprise, this was the cheapest cost out of the five values they calculated, she said, "I'm shocked! Let's make sure we did this right." She recalculated the cost, and obtained the same answer, and said, "I can't believe this is the cheapest plan! Man, how does this work? You think that something is wrong, and then it's not!" Helaine's emphasis on rote procedures, without grasping the logic behind them reveals a lack of evidence for *decompressing.*

Then, Helaine, along with Oriana, began narrating the letter as the fourth PSMT in the group wrote it. After struggling in organizing their ideas, Oriana suggested to create an outline, and Helaine helped her recall the steps they took to make their final recommendation. As the group grappled with writing the letter, Helaine helped put their thoughts into sentences. Below is an example:

Helaine: You spent more minutes.

Oriana: Yeah, the long.

Helaine: You spent more time talking over 20 minutes.

Oriana: Yeah than under.

Helaine: Than under 20 which gives you an overall cheaper rate per minute.

Oriana: Okay, how should we end it?

Helaine: When determining your long-term plan for next year, use this method for each month to determine what would be the best plan for you.

When the group finished composing the letter, Helaine read it out loud for the group, and commended their work, "I like it, it makes a lot of sense!" Her groupmates agreed and reflected on their work, solution, and reasoning. As Oriana suggested that they should create a graph to visualize their reasoning, Helaine responded, "We could put it in graph form, but it doesn't make sense. We should probably just do the letter. We can explain it better…We also can't predict what they're going to call in July." The rest of the group members agreed. This demonstrates a lack of evidence for *bridging*, because she was not able to connections across linear representations.

Lastly, the PSMT reflected on the MEA from the previous week. Helaine explained that the *Summer Jobs* MEA had too many variables to consider and acknowledged that she "liked this activity way better!" She continued,

> So, Willy didn't work that day but every other day he kicked butt, it's like what? The thing is, how you determine if he should fulltime or part-time because you can keep him part-time because he's good at part-time. Or you can make him full time and work him more.

Helaine then discussed the *Phone Plans* MEA they had just explored, and reflected on her understanding of the task,

> It's just weird for me like when I first was reading the introduction and looking at this, I was like, I have absolutely no freaking idea of doing it. And now if you look at it like step by step it's like, that wasn't hard.

Once the rest of the groups in the class were done exploring the MEA, each group presented their recommendation and reasoning to the class. Although all the PSMT in this group stood in front of the class, Helaine presented their solution by herself. She explained why they recommend the

Horizon Nation Wide Saver Plan by repeating the ideas and assumptions the group discussed when they wrote the letter.

Reflections on MEA 2

Helaine wrote a reflection about her exploration of MEA 2 and sent it to the researcher a few days after she explored the *Phone Plans* task. First, Helaine discussed her experiences with the modeling task. She described her initial thoughts of the MEA, “At first, looking at the *two graphs* of information was a lot to take in, but as a group we separated the information to fit the different plans.” Helaine used imprecise terms to describe the *two tables* that show the family’s phone records, and information about the phone plan (shown in *Appendix E*). She continued by explaining that the group focused on finding the Olsen family the cheapest plan and worked towards the goal by looking at the given data and calculating the total cost by multiplying the number of minutes used by the cost per minute and adding the monthly fee when appropriate. She also described the assumptions the group made in relation to the context of the problem. She explained, “we focused more on the monetary value rather that if their calls would fluctuate over the next few months or had even changed before the month of June.”

Then, Helaine discussed the mathematical concepts the group used while exploring the MEA. She said,

> Without even realizing it, it was using slope, average, and slope-intercept form. The average includes taking a set of data and adding it all up, then dividing by how many data sets you had. This gives you a general range of what it would be like, rather than assessing every individual piece. This correlates to slope as when you do an equation, your slope determines both the rate of change and sign of your average line, such as, if it is positive or negative and how steep it is. The slope gives you the average on a graph, instead of in just number.

Helaine's discussion of mathematical concepts illustrates some evidence of *trimming* because she was able to identify some mathematical concepts that are manifested in the MEA. However, there was no evidence of bridging, because she was not able to make connections across topics and representations.

When discussing possible mathematical misconceptions students may have when solving the task, Helaine said it would be "not realizing how slope would play a factor in the solution." She did not anticipate the difficulties entailed in students' understandings of particular aspects of algebra, thus evidence of *decompressing* is not demonstrated.

MEA 3-Exploration of the Task

Helaine explored the *Historic Hotel* MEA with the same group she explored the *Phone Plans* MEA with the previous week. The task is shown in Figure 19 above. Helaine began the exploration of the task by comparing the two MEAs. She predicted that the group will have to write an equation to represent the given scenario. She explained,

> This is going to be tricky, just looking at it in comparison to the last time, it's like you have to come up with some kind of formula in order to be like, okay, this is the rate of change and you have to find the price that even all the rooms are rented that he could still make good money. This is going to be a little bit different than last week.

Her groupmates agreed with her claim, and they brainstormed ideas for approaching the task.

Oriana suggested a graphing strategy, to which Helaine replied "Oh yeah! For this one because of the function and stuff like that, and some kind of slope or something." Helaine calculated the profit when all the rooms are occupied by multiplying the number of rooms (80) by the corresponding nightly rate ($60), and after agreeing with Bianca's suggestion to add the maintenance cost, Helaine then added $320 (80 multiplied by four dollars). However, Oriana brought up the idea that perhaps they should subtract the $320 rather than add it because it was a cost to the hotel. Helaine agreed and recalculated the profit and found that it would be $4480 when all 80 rooms are occupied. Bianca and the fourth PSMT in the group asked about the change in calculation, and Helaine responded, "Because the 4 dollars is cost for service maintenance per day for each room that's occupied, that's going to be the price that it's going to cost the hotel owner."

Helaine then suggested to come up with an equation to represent the given scenario. She explained, "I want to try to come up with a formula that works, because I mean your *x* will be the amount of rooms…" Oriana suggested to find a point other than (80, 4480) on the *number of rooms occupied* vs. *profit* graph and use *slope-intercept* form of a linear-equation to represent the given scenario. However, Helaine wanted to use the context of the question to derive the equation. She explained,

> I'm trying to think of the actual question, because you're going to have 80 rooms, so, then if that's your *x* it's going to fluctuate. If its $60 a room that's the max you can go…so you're going to do 60 plus, what? One or not one but every time you lose a room is going to be… 60 plus something times *x*, right?

This initiated the discussion for coming up with the profit equation in terms of *price* times *the number of rooms,* minus *cost,* and the group began discussing and representing each part in terms of the context of the given scenario. Starting with the number of rooms and corresponding prices, Helaine explained,

> we can do like 80 minus *n* because *n* can be the number of rooms or something, and it's going to be times 60 plus *n*. Because if we do it as in like you're losing rooms, so if like, if n is one room, you're losing so you're going to have 79 rooms which should be $61.

This demonstrates evidence of *trimming*, because she was able to get to the mathematical essence of the real-world problem.

Then, the PSMT began debating what the variable, *n*, would represent—the number of occupied rooms or unoccupied rooms. Helaine explained that it would be the number of empty rooms, and she gave an example when *n* is one. She explained, "if you use one, because we have 80 minus *n* times 60 plus *n*, that would be 80 minus 1, which is 79, times 60 plus one which is 61, which is what we want." The other PSMT in the group agreed with Helaine.

The next term the group discussed was representing *cost* in context of the given scenario. Oriana explained that it would be "80 minus *n* times 4, because it's the occupied rooms times 4" and Helaine agreed by saying, "yes, that makes so much sense." Once all the terms of the profit equation were represented by an expression, Helaine proceeded to check if what they had was correct by substituting values into the equation. She started by substituting $n=5$ into their equation, and calculated the profit to be $4,575, and again by substituting $n=1$, and calculated the profit to be $4,503. The group then agreed that their profit equation was correct.

Next, the PSMT began working on finding the rate that would maximize profit for the hotel owner. They created a graph using an online graphing calculator. Helaine worked with Oriana on creating the graph and setting the minimum and maximum values for the *x*- and *y*-coordinates. Helaine suggested that the *x*-axis should have a minimum value of zero and a maximum value of 80 because it represents the number of rooms. Once the group graphed their profit function, they discussed how to find the maximum. By inspecting the graph, Helaine suggested that the maximum "looks like it's about like a 12…I'm going to go with 12 to 13." Once Oriana used her graphing calculator to find that the maximum profit occurred when $x= 12$, Helaine began the discussion within the group of what that number means in terms of the context of the problem. She asked her groupmates, "The owner will always have 80, so then 68, right? At $68 per room, that will give the maximum profit, right?" Referring to the number of rooms at the hotel (80), and not realizing that the 68 would be the amount of occupied rooms that would produce the maximum profit, not the daily rate. Oriana then explained to the group that it would not be 68, it would be 72, because in the profit function they wrote, the *price term* was represented by 60 + *n*, not 80 – *n*. Helaine then calculated the maximum profit, and replied, "that makes sense, so the maximum profit would be $4626, at $68 per room". Once again, Oriana explained that it would be $72 per room by referring back to the equation and talking about what each term meant in terms of the problem. Helaine then realized the error she was making by saying, "Oh! I was doing the number of rooms, not the rate… of course. Okay."

Next, the group wrote the generalized formula for the profit equation, Helaine listened and agreed with the suggestions that Oriana and the other PSMT were making. Once the group had the generalized formula, Helaine and Oriana began narrating the letter, as the fourth PSMT wrote what they were saying. Oriana repeated the process they went through as a group to come up with their recommendation, Helaine was filling in miss words and details. She was persistent in explaining what each term of their equation meant in context of the MEA, she said, "this is going to be really confusing if we don't like relate it back to what we were given in the question. Because it doesn't make sense to just give numbers without saying what they represent." This demonstrated evidence for *decompressing*, because Helaine was able to decompress rote procedures of writing a profit equation in order to grasp the logic behind it.

Once the PSMT finished writing the letter, the presentations of the groups' solutions to the class began. Helaine led the group's presentation by first explaining how they came up with the specific profit equation, which she wrote on the board. Right away, a PSMT from a different group asked, "why is the four negative?" To which she replied, "the four is negative because it's the maintenance cost and it's costing the hotel. This is the total profit, this is the profit that you're gaining from charging out your rooms," she said pointing to the *revenue* term of the equation, "and this is the maintenance fee that is costing the hotel in order to rent out the rooms."

Then, Helaine discussed how the group came up with the maximum profit, she explained, "we found the maximum by plugging in a few numbers, and we found that at 12, the maximum profit is $4,624, and when he has that, it means that he should charge $72 per room". She then wrote the generalized profit equation on the board and discussed that if the owner wanted to "adjust the prices, or if the maintenance cost changed, or the daily rate per room changes, Mr. Graham can adjust accordingly."

Second interview

The second interview with Helaine took place two days after the exploration of the *Historic Hotel* MEA. The researcher asked Helaine questions (included *in Appendix H*) about the MEA and two content questions from the LMT assessment.

Experiences and Reflections related to MEA 3

During the first part of the interview, Helaine discussed her experiences with the *Historic Hotel* MEA. When she was explaining how she made sense of the MEA, she started describing the task—what the real-world scenario looked like, what they were given, and what they were asked to do. Helaine explained that to find the rate that maximizes profit, her group wrote a function, which helped them make the recommendation for the owner of the hotel. She then described the equation the group wrote,

> We had $80 - x$ where x is the amount of the unoccupied room. If you had one unoccupied room that's means you have 79 available room and so, on and so, forth depending on what your x variable was; then we multiply that by $60 + x$. And 60 being the amount of money it cost to have all the rooms rented for a day and then you add x, based on how many empty rooms you have. Minus 4 which is the maintenance fee and we subtracted it

> because that's what's going to take away from the profit that the hotel will make times 80 − x because that gives you how many rooms you going to have to pay for.

Helaine continued describing the steps the group took to find the maximum profit. She explained that after writing the equation, the group then graphed it using an online graphing calculator and noticed that the graph looked like a parabola and used the graph to find the maximum profit. She explained,

> it makes sense that it was a parabola, because if you have more rooms [occupied] it's going to cost more to clean and the rate will be higher, so profit won't be as high, and same thing if you have less rooms rented out, then obviously you won't make money.
>
> We then looked at the height of the parabola, we looked at the highest point that it came to, we were like okay, that's when you going to make the most money, because the higher up you are with your profit being on the left margin then your number of rooms being on the bottom, you can say, okay, here is our net profit.
>
> Then, we tested points from 10 to 15, I think it was 12, yeah that was the point, and we said okay, we are going to make the most money when we have 12 unoccupied rooms. We have 68 rooms occupied and then that was $72 a night.

Although Helaine's explanation contained the use of imprecise mathematical terms, she was able to get to the mathematical essence of a real-world problem, demonstrating evidence for *trimming*.

Next, Helaine discussed the specific mathematical content that helped her make sense of the MEA. Helaine talked about *slope-intercept form, parabolas, quadratics*. She explained,

> It was like *slope-intercept form*, but basically it was a like *a rate of* change type thing, it was using a function to come up with, a *rate of change*, so, whenever you have a certain amount of rooms available or unavailable you have to look at the rate of change, if you have this many rooms occupied compared to this many rooms unoccupied, then it's going to be this price compared to this price.
>
> You have to be able to compare them and see where you could come up with the highest profit. Looking at *parabola*, if you would have multiplied $(80-x)\ (60+x)$ you would get an x^2, that gives you your parabola type shape, but because it was negative. It's an upside-down parabola.

Once again, although Helaine's explanation contained the use of imprecise mathematical terms, she was able to get to the mathematical essence of a real-world problem, demonstrating evidence for *trimming*.

At this point, the researcher asked Helaine about different methods that were presented by other groups, such has the first and second derivative tests, as well as finding the *x*-value of the vertex of the parabola using the formula $x = -\frac{b}{2a}$, and how these methods related to the strategy they used in their group. She said,

> We were all trying to reach the same goal, but they used a different way. What the other groups did when they did the first and second derivative tests, it is the same thing because when you take the first derivative test and you set something to zero, it will help get you the point that you need, the point we got on the graph. The second derivative test is taking the derivative of that, so, if you are to take the derivative of something squared, which is a parabola it would change into a line, and then the second derivative would be the slope. So, it's like your rate of change on the line, I guess, it's hard to explain it but that's like the generalized way.
>
> The $-\frac{b}{2a}$, I don't know how to explain that one, I can connect the 1st derivative and the 2nd derivative test but the $-\frac{b}{2a}$ it's just like, that one didn't make sense to me.

When asked about how these mathematical ideas are connected, Helaine explained that a parabola is the graph of a quadratic function, and the quadratic function is the product of two equations that have a slope and a rate of change, she then explained that using those can help find the maximum of a function, which is connected the first and second derivative test, because those also can help find the maximum point. Helaine then acknowledged that she doesn't know how $-\frac{b}{2a}$ is connected to the rest of the mathematical ideas. This demonstrates some evidence for *bridging* because she was able to make some connections across topics and representations, As Helaine reflected on her experience with the *Historic Hotel* MEA, she discussed how exploring the task helped her understand the mathematics they used in their group by making connections, talking through the process, and writing explanations down. She explained,

> Before I didn't really know how to graph powers higher than x, like I wouldn't be able to have a general idea of what the graph of like x^2 or x^3 is. But after doing this, it like makes sense to kind of look at the function and see how many x's there are, so like with our profit function we had two x's so right away I know that the graph will be a parabola, without multiplying it out. Because now I see that we have two x's and one of them is negative, so that means we will have a downward facing parabola. I would have never been able to make that connection without multiplying it through before. Like it makes a lot more sense especially if you think about the context of the activity we did.

This demonstrates evidence for *decompressing* because Helaine was able to decompress rote procedures in graphing quadratics in order to grasp the logic behind them.

Helaine continued by talking about how the general modeling process helped her make sense of the mathematics. She said,

> Thinking about it in a different way, also helped me get a better understanding of the problem and the math we used. Definitely through talking it out with other people and especially writing it down in the letter, at the end, also helps. When we talked about the math in our group, it helped me understand it more and make connections between the different ideas, and like writing it also forced us to think more about it because it was easier when we said it to each other, but when we tried to write it, it was so much harder.

When reflecting on how this task can be used in an algebra class, Helaine discussed the context of the problem and it can help students see the importance of algebra in real-life, and the use of different methods. She explained:

> This gives them a great, real-life problem of finding the maximum profit, by graphing an equation with an x^2 and it's a great example of finding the maximum value. This problem can help them better demonstrate how to write a formula and simplify it and graph it and like if the answer makes sense relating it back to the question. I think high school this would help a lot.

Helaine's reflections demonstrate evidence of *decompressing* in explaining how to help students understand the usefulness and importance of algebra.

Lastly, when asked about possible misconceptions students may have with solving the *Historic Hotel* MEA, she discussed the maintenance fee. She explains that her group, along with others in the class, were not sure if the fee would be added or subtracted in the profit equation, so students may add the fee instead of subtracting it, and that would affect the way the problem is solved. This demonstrates some evidence of *decompressing* because although Helaine was able to anticipate some student difficulties in solving the MEA, she did not describe how these challenges were specific to students' understanding of algebra.

Content Questions

The next part of the interview entailed the two content questions from the LMT assessments. As previously stated, the questions were not part of the items released by the LMT project, and thus, the description of the findings in this section will not have specific details of the context of the questions.

The first question, Helaine was asked to choose a story problem that does *not* represent linear behavior in the context of the four given story problems. A description of each choice was given in chapter three. Helaine started by discussing the meaning of a linear equation. She said, "I'm pretty sure a linear equation is $y = mx + b$, and if you graph it, it's a line."

Next, Helaine looked at the first choice, which represented exponential behavior rather than linear, because the initial amount was being doubled. Helaine began substituting different values into the context to see how they change. She noticed that the rate of change is different, and was not sure how to explain that, so she said, "I want to come back to that one, because it's being doubled, it's not really changing by the same amount." Helaine then moved on to the second choice, which represented a linear behavior and could be modeled using the equation y = *m*x. Helaine first promptly said that she feels “like this wouldn’t be linear, because it’s a direct correlation” as she continued explaining the context of the problem, she realized that “actually, it would be linear because it’s going up by [the same] measurement every time”. She then concluded that this choice would be linear.

Then, Helaine moved on to the third choice, which could be represented using a linear equation of the form $y= mx + b$. She explained that “this one is much easier” because there is an initial amount, which would be thee y-intercept, and she also describes the rate of change, and she wrote the correct linear equation that represents the story problem. She concluded that the third choice does represent linear behavior. Similarly, she said that the last choice represents linear behavior, and she explained what the *y-intercept* and *slope* would be in context of the story problem and wrote the correct equation to represent the given scenario. She concluded that the last choice also represents linear behavior.

Helaine went back to the first choice, and concluded that it would not represent linear behavior, because the amounts change by a different rate, and “you can’t just use a different slope every time for $y = mx +b$”. Helaine answered the question correctly, and showed evidence of *trimming*, and *decompressing* by: getting to the mathematical essence of each of the four real-world problems (*trimming*) and *decompressing* rote procedures of writing equations to focus on the logic behind what the equation would mean in terms of the context of each story problem.

The second question was question 31 on form A (which was described in chapter three). Helaine was asked to decide whether three story problems can be modeled by a given linear equation of the form $y = mx + b$, by saying yes, no, or I’m not sure for each of the three-story problems. The first story problem could not be modeled by the given linear equation, rather it could be represented by an exponential growth function because the initial amount was doubling each month. Helaine connected this choice with the first choice from the previous question, she explained, "this is just like the other one, it won't work because again, [the initial amount] is being doubled, so it's not linear." She concluded that the first story problem cannot be modeled by the given linear equation. She then moved on to the second story problem which could *not* be modeled by the given equation, although it could be represented by a different linear equation. She promptly said that "this one works because this would be the *y*-intercept, and this would be the slope" pointing to the numbers in the problem, and not relating what the problems actually mean in terms of the context. The last story problem was the only one could be represented by the given linear equation. Helaine used similar reasoning as the previous story problem, and explained that this worked because of the initial amount, which would represent the *y-intercept* and the rate of change, which would represent the *slope*, so she concluded that the last story problem "is a yes too". When answering the question, some evidence of (*trimming*) can be seen because Helaine was able to get to the to the mathematical essence of two of the three real-world problems.

LMT Assessment

As stated previous, Helaine was selected from the low MKT category— which included participants with scores greater than or equal to zero percentile and less than or equal to 25th percentile on the LMT pre-assessment. Helaine's score on the LMT pre-assessment was in the 25^{th} percentile. As discussed in chapter three, IRT scores represent the underlying ability being measured by the assessment, in this case it is the PSMT' MKT middle school algebra. This ability is denoted as the symbol theta. This ability spectrum is measured in standard deviation units with zero being average. Higher abilities are represented by positive values and lower abilities are represented by negative values. Helaine's overall IRT score on the pre-assessment was $\theta = -0.549$, and *SE* = 0.372. On the post-assessment, Helaine's overall IRT score was $\theta = -0.093$, and *SE* = 0.394, which was in the 60th percentile. The box and whisker plots below illustrate Helaine's IRT scores in relation to her classmates on the pre-assessment (*n* = 30) and post-assessment (*n* = 27). These results demonstrate that Helaine's MKT algebra was in the lower half of the scores on the pre-assessment, and upper half of the scores on the post-assessment, in comparison to her classmates.

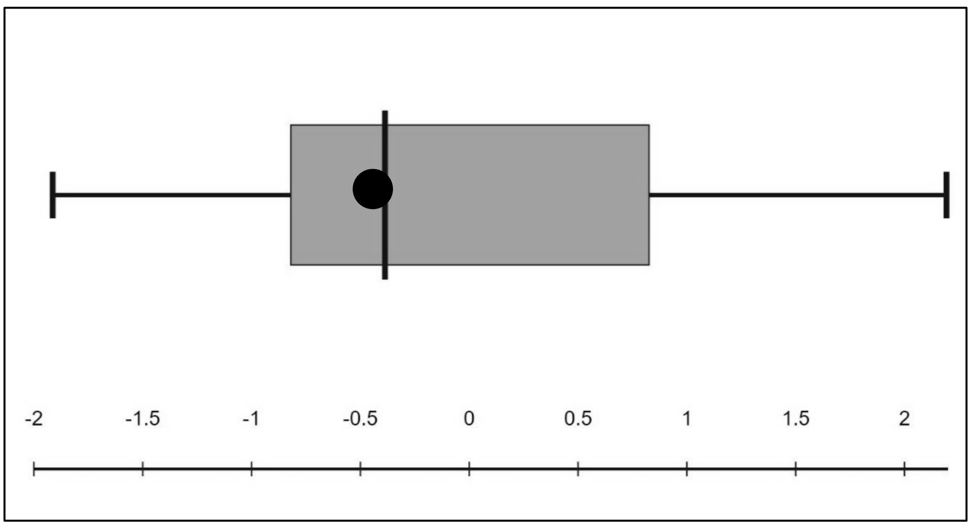


Figure 35. Helaine's Pre-Assessment Score on a Box Plot.

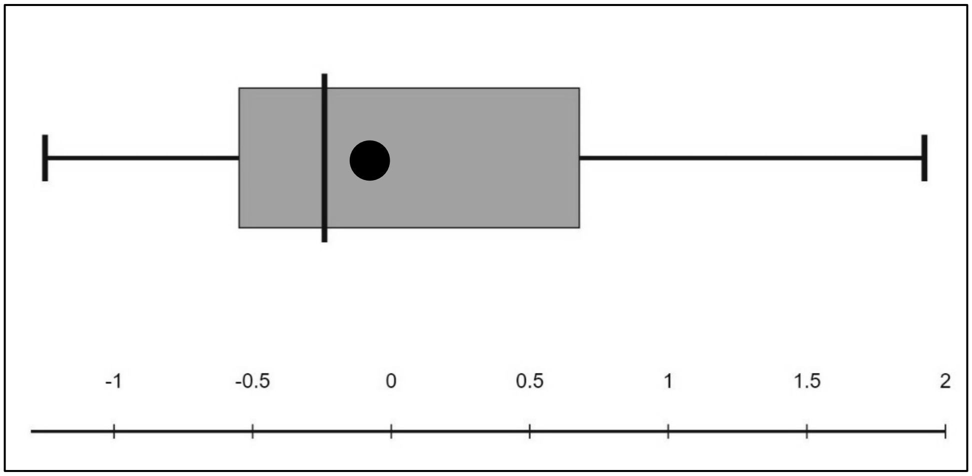


Figure 36. Helaine's Post-Assessment Score on a Box Plot.

Discussion of Case 3

Helaine was selected from the low MKT category. Her score on the LMT pre-assessment ($\theta = -0.549$, $SE = 0.372$) was in the 25th percentile of the participating PSMT' scores in her class ($n = 30$). Her score on the LMT post-assessment increased ($\theta = -0.093$, $SE = 0.394$) and was in the 60th percentile of the participating PSMT' scores in her class ($n = 27$). Moreover, her MKT category changed from low MKT to high MKT because her post-assessment score was greater than 50th percentile and less than or equal to 100th percentile, which was the designated range for a high MKT

During the first interview, Helaine described her experiences with the Summer Jobs MEA, which she explored with a PSMT that was not a part of the study. She discussed the challenges she faced considering the several factors that affected her group's recommendation of which vendors to rehire. She explained that after hearing different groups' presentations, she was able to make more sense of the task. When discussing the specific mathematical content that helped her make sense of the task, she explained that proportional reasoning and averages could be used to explore the MEA. She explained how each topic related to the context of the given scenario, and thus, she was able to get to the mathematical essence of the given real-world problem. Accordingly, there was strong evidence for *trimming*. Helaine then stated that the two ideas (proportions and averages) are connected but was not able to explain and justify her reasoning. Thus, only some evidence for bridging was demonstrated.

Next, when reflecting on how the task could be implemented in an algebra class, Helaine discussed the context of the problem and how it can help students see the importance of algebra in real-life. She also explained the importance of using different methods to verify the solution. Nonetheless, Helaine was not able to anticipate possible student misconceptions, rather she talked about the difficulty of the task. Therefore, Helaine's reflections demonstrate some evidence of *decompressing* in explaining how to help students understand the usefulness and importance of algebra.

When it came to answering the content questions in the first interview, Helaine answered the first question incorrectly, but she answered the second question correctly. For the first question, Helaine focused on using rote procedures to write equations to represent the given scenarios, and she had difficulties reasoning with the context of each story problem. However, Helaine answered the second question correctly, even though it required similar type of content knowledge and reasoning to answer as the first question. Nonetheless, Helaine was able to explain the equation she derived for each story-problem in context, and she was able to (1) get to the mathematical essence of each of the four real-world problems, (2) make connections between proportions and linear equations in one of the choices, and (3) decompress rote procedures of writing equations to focus on the logic behind what the equation meant in context of the story problem. Accordingly, evidence of trimming, bridging and decompressing was demonstrated in her explanation to the second content question.

During the second MEA, Helaine thoroughly contributed to the exploration of the task. She suggested calculating the total cost for each phone plan based on the family's given records and explained how to do it to the other PSMT, and why it works. Therefore, strong evidence of *trimming* was demonstrated during the exploration of the Phone Plans MEA. However, there was low evidence of *bridging*, because Helaine was not able to make connections across different representations of linear equations, and *decompressing* because she focused on rote procedures of calculating the total cost, without grasping the logic behind the process.

On her written reflection, Helaine discussed important factors the group considered and ignored when exploring the task, such as focusing on which plan would be the cheapest for the given records and ignoring any possible changes that may occur to their records at different time periods. She then described using linear reasoning to explore the MEA and demonstrated strong evidence for *trimming* by explaining the mathematical essence of the real-world problem. There was some evidence of bridging in her reflection, because although she described that there were connections between *slope, averages, graphs* and *linear equations*, she was not able to explain how these topics are related. Helaine was not able to anticipate possible mathematical misconceptions students may have when solving this task, or possible difficulties entailed in students' understandings of aspects of algebra, thus evidence of decompressing was not demonstrated.

During the third MEA, Helaine contributed to the group's exploration of the Historic Hotel task extensively, once again. She used the context of the problem to derive the expressions used to represent the number of rooms occupied in the hotel, price per room, and maintenance cost. She was able to explain and justify her reasoning to the rest of the PSMT in the group, and thus, strong evidence of trimming was shown. Additionally, some evidence of bridging was shown because Helaine was able to discuss the different representations (graphs, equations, and table-of-values) with her groupmates. Furthermore, when writing the letter describing the group's recommendation to the hotel owner, Helaine was persistent in reminding her groupmates to relate the procedure and written equation back to the context of the given scenario, demonstrating strong evidence for decompressing.

During the second interview, Helaine described her experiences with the Historic Hotel MEA by discussing the mathematics she used that helped her make sense of the MEA. She explained the usefulness of linear expressions, quadratic equations, the graph of parabola to explore the task, demonstrating strong evidence for *trimming*. When describing possible connections between these mathematical topics, and others used by different groups and classmates, Helaine was able to explain that a parabola is the graph of a quadratic equation, and the derived quadratic equation was the product of two linear equations, she then explained that using the graph of the parabola can help find the maximum, which she connected to the first and second derivative test. Thus, there was strong evidence for *bridging.*

Next, when discussing how the task could be implemented in an algebra class, Helaine explained how this task can help students understand the usefulness and importance of quadratic equations, writing expressions, and graphing. She also discussed the modeling process and how it can be beneficial in understanding mathematical topics better, by discussing the mathematics aloud with others, and writing explanations. Helaine's reflections demonstrated strong evidence for decompressing.

When it came to answering the content questions, Helaine answered both questions correctly. Unlike the first interview, she kept referring to the context of the given scenarios and she was able to correctly explain and justify her reasoning for the first question. Her explanations for both questions demonstrated strong evidence of *trimming* and *decompressing* by, getting to the mathematical essence of each of the four real-world problems (*trimming*), and *decompressing* rote procedures of writing equations to focus on the logic behind what the equation would mean in terms of the context of each story problem.

The findings above are summarized in the following table and figure. Table 10 shows the different levels of evidence—high (H), medium (M), or low (L) in terms of the three KAT categories *trimming, bridging* and *decompressing,* for each of the collected data sources.

Table 10

Helaine's levels of evidence for trimming, bridging and decompressing.

	MEA 1		MEA 2		MEA 3	
KAT Categories	Exploration of Summer Jobs MEA	First Interview	Exploration of Phone Plans MEA	Written Reflection	Exploration of Historic Hotel MEA	Second Interview
Trimming	-	H	H	H	H	H
Bridging	-	M	L	M	M	H
Decompressing	-	L	L	L	H	H

The figure below illustrates the overall level of evidence for each of the two factors of interest—MEA contribution, and MKT. The plot consisted of the three KAT categories placed on a continuum for each factor. An overall rating of high, medium, or low was given for each of the three MEAs, based on the table above. It is important to note that because there was no data collected during Helaine's exploration of MEA 1, the data included in the plot for MEA 1 was solely based on her first interview responses. During the first interview, Helaine admitted to contributing to her group's exploration of the task, but rather she went along with her groupmate's suggestions and strategies and helped calculate "the total amount of money made, and hours worked" (Helaine, First Interview, January 24). Thus, her contribution for MEA 1 was placed on the medium-low end of the *MEA Contribution* continuum, and her evidence levels for the three KAT categories were based on her interview responses alone. For MEA 2 and MEA 3, an overall high rating was attained if a high was given on both the exploration of the MEA and the interview or written reflection that followed. An overall medium rating was given for a medium-medium combination, high-low combination, and a high-medium combination. An overall low rating was given for a low-low combination, and a low-medium combination.

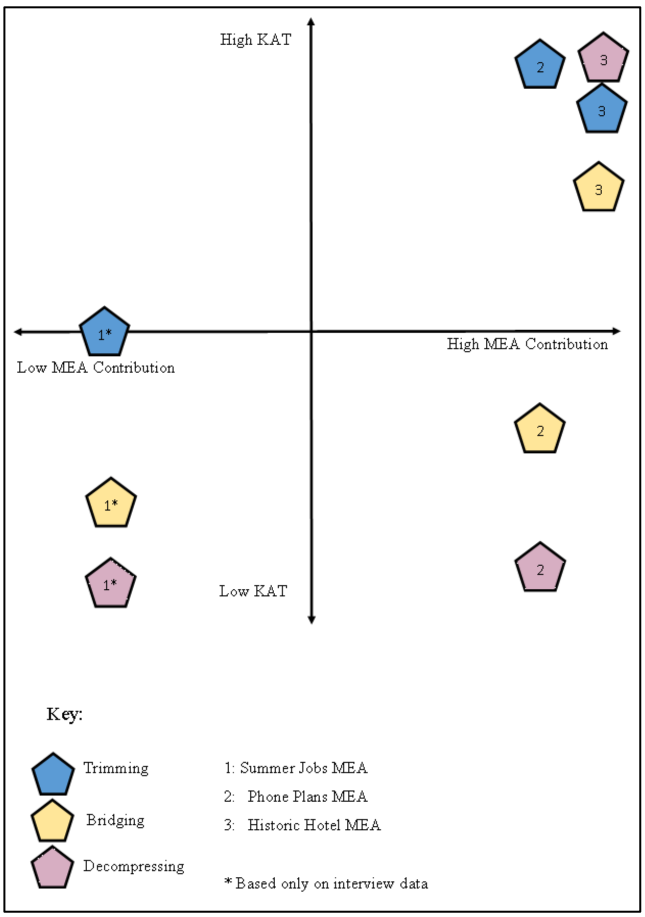


Figure 37. A Plot of Helaine’s Overall Level of Evidence for MEA Contribution and KAT.

The plot above illustrates the following:

(1) Helaine's overall level of evidence for the first interview was *high* for trimming, and *medium* evidence for bridging and decompressing, while her contribution to the exploration of the task was low, based on what she stated in the interview.

(2) Helaine's overall level of evidence for the Phone Plans MEA and the written reflection that followed was *high* for trimming, *medium* for bridging, and *low* decompressing; and her contribution to the exploration of the task was high.

(3) Helaine's overall level of evidence for the Historic Hotel MEA and the second interview was high for trimming, bridging, and decompressing; and her contribution to the exploration of the task was high.

Connecting this back to McCrory and colleagues' definition of *trimming, bridging, and decompressing*, it was evident that:

(1) For the first MEA, Helaine had low contribution to the task exploration, she was able to talk about the mathematics used to explore real-world problem (*trimming*), and she was able to make some connections across mathematical topics (*bridging*), however she was not able to anticipate some difficulties entailed in students' understandings of particular aspects of algebra content, identify ways to help students understand the usefulness and importance of algebra, or decompress some rote procedures in order to grasp the logic behind them (*decompressing).*
(2) For the second MEA, Helaine was highly involved in the exploration of the task, she was able to discuss and explain the mathematics used to explore real-world problem

(*trimming)*, she was able to make connections across mathematical topics (*bridging*), but she was not able to anticipate the difficulties entailed in students' understandings of particular aspects of algebra content, explain how to help students understand the usefulness and importance of algebra, or decompress rote procedures in order to grasp the logic behind them (*decompressing).*

(3) For the third MEA, Oriana was highly involved in the exploration of the task, and she was able to make connections across mathematical topics (*bridging*), she was able to discuss and explain the mathematics used to explore real-world problem (*trimming*), and she was able to anticipate the difficulties entailed in students' understandings of particular aspects of algebra content, explain how to help students understand the usefulness and importance of algebra, and decompress rote procedures in order to grasp the logic behind them (*decompressing).*

CHAPTER SIX: CROSS-CASE ANALYSIS

The purpose of a cross-case analysis is to "build abstractions across cases" (Merriam, 1998, p. 195). Accordingly, the researcher used observations across cases to answer the research question—what is the nature of the relationship between engagement in model-eliciting activities (MEAs) and pre-service secondary mathematics teachers' (PSMT') mathematical knowledge for teaching (MKT) algebra? Because the question revolves around *MEAs* and *MKT algebra,* in the cross-case synthesis, the researcher both describes common themes across the relationships between PSMT' experiences with MEAs and their MKT algebra, and compares qualitative differences between the three selected cases. This was done by analyzing the data collected from various sources, such as the group's exploration of the three MEAs and the corresponding participant artifacts, the two conducted interviews, and written reflections. This section includes the synthesis of findings across the cases of Oriana Bianca and Helaine. First, the LMT Pre and Post Assessments are analyzed, then the three KAT categories are discussed, followed by a discussion on the interview think-aloud questions; and lastly an analysis of the PSMT' experiences with the MEAs and MKT algebra is included.

LMT Pre/Post-Assessments

Oriana was selected from the high MKT category, while Helaine was selected from the low MKT category, based on their scores on the LMT pre-assessment. Oriana's score on the pre-assessment (θ =1.145, SE = 0.416) was in the 90th percentile, while Helaine's ($\theta = -0.549$, SE= 0.372) was in the 25th percentile of the participating PSMT in their class (n=30). Their scores on the LMT post-assessment were both greater than 50th percentile which placed both PSMT in the high MKT relative to their classmates. Helaine's score ($\theta = -0.093$, SE= 0.394) was in the 60th

percentile, while Oriana's score (θ = 2.193, SE= 0.543) was the highest among the participating PSMT in their class (n=27). On the other hand, Bianca was initially selected from the medium MKT category. Her score on the LMT pre-assessment (θ= − 0.241, SE = 0.391) was in the 50^{th} percentile of the participating PSMT' scores in her class (n=30). Her score on the LMT post-assessment decreased (θ = −0.683, SE= 0.374) and was in the 35^{th} percentile of the participating PSMT' scores in her class (n=27). Although her score decreased, and changed positions in view of score quartiles, her MKT category did not change because her score was still greater than 25^{th} percentile and less than or equal to 50^{th} percentile, which was the designated range for a medium MKT. These results were corroborated by the levels of evidence for *trimming, bridging and decompressing* demonstrated in the sources of collected data, which are discussed below.

Trimming

Oriana and Helaine demonstrated strong evidence of trimming for all the collected data sources. Nonetheless, Helaine acknowledged that she was not able to make sense of the Summer Jobs MEA with her group, she was only able to make sense of it after other groups presented their solutions. Bianca on the other hand was only able to show some evidence of trimming during the first MEA and demonstrated low evidence of trimming for the rest of the collected data sources. These findings reveal that Oriana and Helaine were able to get to the mathematical essence of a real-world problem by explaining and justifying how mathematics was used to make sense of each of the given MEAs; but Bianca was not able to discuss, explain, or justify the mathematics used to explore the three tasks. These findings are summarized in the table below.

Table 11

Oriana's, Bianca's, and Helaine's levels of evidence for trimming.

	MEA 1		MEA 2		MEA 3	
PSMT	Exploration of Summer Jobs MEA	First Interview	Exploration of Phone Plans MEA	Written Reflection	Exploration of Historic Hotel MEA	Second Interview
Oriana	H	H	H	H	H	H
Bianca	M	L	M	L	L	L
Helaine	L*	H	H	H	H	H

*Based on Helaine's interview responses, not her recorded exploration.

<u>Bridging</u>

Oriana, Bianca, and Helaine all started with a low rating for bridging during the exploration of the first MEA. Similarly, during the first interview, all three were able to identify connections between various mathematical topics and explanations, but none of the PSMT was able to explain those connections. During the second MEA Oriana demonstrated high evidence for bridging on both the exploration of the task, as well as her written reflection. On the other hand, Helaine and Bianca demonstrated low evidence during the exploration of the task, Helaine demonstrated some evidence on her written reflection, while Bianca demonstrated low on her written reflection. Similarly, during the exploration of the third MEA, Oriana once again demonstrated high evidence of bridging, for both the task and the second interview. Helaine was able to make some connections across topics during the exploration of the task, and strong evidence of bridging was demonstrated during the second interview. While no evidence of bridging was shown in Bianca's case. These findings reveal that although all three PSMT initially were not able to explain how different mathematical topics were connected on the initial task and interview, by the last interview both Oriana and Helaine were able to make connections across topics, assignments, representations or domains; while Bianca was still not able to do so. These findings are summarized in the table below.

Table 12

Oriana's, Bianca's, and Helaine's levels of evidence for bridging.

	MEA 1		MEA 2		MEA 3	
PSMT	Exploration of Summer Jobs MEA	First Interview	Exploration of Phone Plans MEA	Written Reflection	Exploration of Historic Hotel MEA	Second Interview
Oriana	L	M	H	H	H	H
Bianca	L	M	L	L	L	L
Helaine	L*	M	L	M	M	H

*Based on Helaine's interview responses, not her recorded exploration.

Decompressing

Oriana, Bianca, and Helaine all started with a low rating for decompressing during the exploration of the first MEA. However, during the first interview, Oriana was able to anticipate some student misconceptions, but was not able to explain them; while Bianca and Helaine were not able to anticipate any student misconceptions. During the second MEA all three PSMT demonstrated low evidence for *decompressing* because they focused on rote procedures of calculating the total cost, without grasping the logic behind the process. On the written reflection, Oriana was able to anticipate student misconceptions, while Bianca and Helaine could not. During the exploration of the third MEA and the second interview, Oriana and Helaine both showed high evidence of decompressing because they were both able to decompress rote procedures of writing linear expressions and quadratic equations, and focused on the logic behind them, while Bianca was not able to do so. These findings reveal that although the three PSMT were not able to explain the logic behind rote procedures on the first MEA, by the last interview, Oriana and Helaine were able to anticipate the difficulties entailed in students' understandings of particular aspects of algebra content, discuss how to help students understand the usefulness and importance of algebra, and/or decompress rote procedures in order to grasp the logic behind them, while Bianca was still not able to do so. These findings are summarized in the table below.

Table 13

Oriana's, Bianca's, and Helaine's levels of evidence for decompressing.

	MEA 1		MEA 2		MEA 3	
PSMT	Exploration of Summer Jobs MEA	First Interview	Exploration of Phone Plans MEA	Written Reflection	Exploration of Historic Hotel MEA	Second Interview
Oriana	L	M	L	H	H	H
Bianca	L	L	L	L	L	L
Helaine	L*	L	L	L	H	H

*Based on Helaine's interview responses, not her recorded exploration.

LMT Interview Think-Aloud Questions

Comparing the PSMT' levels of evidence for the three KAT categories one can see some change in the quality of explanations and justifications for Oriana and Helaine. Although Oriana answered both questions correctly in the first interview, she was not able to make connections across topics and representations related to linear equations when answering the second question, nor was she able to decompress rote procedures for writing linear equations to explain the logic behind them, on either question. However, by the second interview, Oriana explained and justified the reasoning behind her answers in a way that demonstrated strong evidence for *trimming, bridging* and *decompressing* , because she was able to get to the mathematical essence of each of the real-world problems, decompress rote procedures of writing equations to focus on the logic behind what the written equations mean in terms of the context of each problem, and make connections in her reasoning across different questions, topics and representations.

Furthermore, during the second interview, Helaine answered both questions correctly, and she was also able to improve the quality of her explanation and justification by consistently referring to the context of the given scenarios to validate her reasoning, which she did not do during her first interview. By the second interview, Helaine's levels of evidence increased to high for three KAT categories on the first question, and remained at high on the second question. Nonetheless, Bianca used the same reasoning to incorrectly answer both questions during both interviews, and maintained a low rating on all three categories; her explanations and justifications did not change from the first interview to the second. These findings are summarized in the two tables below.

Table 14

Oriana's, Bianca's, and Helaine's levels of evidence for the three KAT categories for the LMT content questions asked in the first interview.

	Oriana		Bianca		Helaine	
KAT Categories	Interview 1 Question 1	Interview 1 Question 2	Interview 1 Question 1	Interview 1 Question 2	Interview 1 Question 1	Interview 1 Question 2
Trimming	H	H	L	L	L	H
Bridging	H	L	L	L	L	H
Decompressing	L	L	L	L	L	H

Table 15

Oriana's, Bianca's, and Helaine's levels of evidence for the three KAT categories for the LMT content questions asked in the second interview

	Oriana		Bianca		Helaine	
KAT Categories	Interview 2 Question 1	Interview 2 Question 2	Interview 2 Question 1	Interview 2 Question 2	Interview 2 Question 1	Interview 2 Question 2
Trimming	H	H	L	L	H	H
Bridging	H	H	L	L	H	H
Decompressing	H	H	L	L	H	H

MEA Experiences and MKT Algebra

The figure below is provided to illustrate the overall level of evidence for each of the two factors of interest—MEA contribution, and MKT algebra, for Oriana, Bianca and Helaine. The plots shown in the *Single-Case Analysis* section were merged to compare the three PSMT.

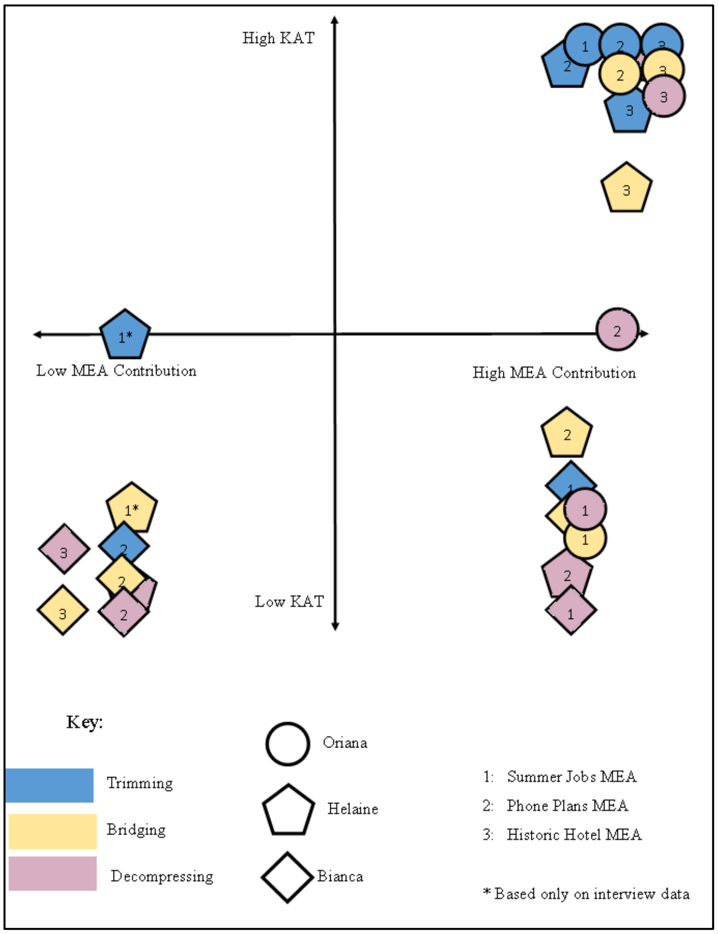


Figure 38. A Plot of Oriana's, Bianca's, and Helaine's Overall Level of Evidence for MEA Contribution and KAT.

The plot above illustrates the findings that Oriana and Bianca started in the high MEA contribution quadrants, while Helaine was in the low contribution for the first MEA. More specifically, Oriana had high MEA contribution, high trimming, medium/low bridging and decompressing levels; Bianca had high MEA contribution, medium/low trimming and bridging and low decompressing, and Helaine had low MEA contribution, medium trimming, and medium/low bridging and low decompressing.

For the second MEA, Oriana had high MEA contribution, and she maintained the high level of trimming, her level of bridging increased from medium/low to high, and her level of decompressing also increased from medium/low to medium. Similarly, Helaine's level of trimming increased from medium to high, while her levels of bridging and decompressing were maintained at the same levels of medium/low and low, respectively. On the other hand, Bianca's contribution to the exploration of the second task decreased considerably, along with her levels of trimming, bridging and decompressing, which decreased from a combination of mediums and lows on the first MEA, to all lows and one medium on the second MEA.

By the third MEA, Bianca's contribution to the exploration of the task was low, along with her ratings for trimming, bridging and decompressing. On the other hand, both Helaine and Oriana maintained their high MEA contribution rating for the third MEA, and Oriana had high levels of evidence for all three KAT categories, while Helaine had a medium/high rating for bridging, and high levels of evidence for trimming and decompressing.

Additionally, all three PSMT reflected on their experiences with the MEAs after exploring each of the tasks. They reflected on their solutions, their assumptions, and the methods and mathematics they applied. Furthermore, Helaine and Oriana both deliberated the rigor in writing the letters for the MEAs, and by having to re-explain and justify their strategies to each other numerous times. Moreover, all three PSMT discussed the importance of using different methods to explore the MEAs and reflected on how students would benefit from the multiple strategies that emerge through the mathematical modeling process.

The following chapter contains a summary and discussion of the findings. Implications for educational policy and practice are also discussed. The limitations of the study will be identified, and the potential contributions and recommendations for future research are presented.

CHAPTER SEVEN: SUMMARY, DISCUSSION, AND RECOMMENDATIONS

Introduction

This chapter contains a summary and discussion of the findings. Implications for educational policy and practice are then discussed. Furthermore, the limitations of this study are identified and the potential contributions and recommendations for future research are presented.

Summary and Discussion of Findings

The purpose of this multiple case study was to explore the nature of the relationship between engagement in model-eliciting activities (MEAs) and pre-service secondary mathematics teachers' (PSMT') mathematical knowledge for teaching (MKT) algebra. *Maximum variation purposeful* sampling method was used to select three cases in order to document diverse variations of the pre-service teachers' engagement in MEAs. Participants were given the Learning Mathematics for Teaching (LMT) measure to assess their MKT algebra prior to their exploration the MEAs. The participants' MKT scores were then used to calculate percentile ranges and create three categories: (a) *low MKT*— included participants with scores greater than or equal to zero percentile and less than or equal to 25th percentile, (2) *medium MKT*— included participants with scores greater than 25 percentile and less than or equal to 50th percentile, and (3) *high MKT*— included participants with scores greater than 50th percentile. A PSMT was selected from each category for an in-depth exploration of their experiences with the MEAs and their MKT algebra. The three cases were: Oriana— a PSMT with high MKT; Bianca—a PSMT with medium MKT; and Helaine— a PSMT with low MKT.

The three PSMT' experiences with MEAs were examined through their recorded explorations of the *Summer Jobs* MEA, *Phone Plans* MEA, and *Historic Hotel* MEA, along with their written artifacts, the two conducted interviews, and written reflections. A discussion about the three PSMT' experiences with the MEAs is presented first, then their MKT algebra is discussed.

MEA Engagement and Experiences

The PSMT' experiences with the three MEAs varied. For the first MEA, Oriana was able to make sense of the given scenario promptly and was able to discuss and explain different strategies that worked in exploring the task (such as finding unit rates and averages) and those that did not work (such as comparing the total amount of money made by each vendor). On the other hand, Helaine struggled in exploring the task. Helaine relied on other PSMT' strategies in making sense of the given scenario and was not able to justify why different strategies could or could not work. Although Bianca made sense of Oriana's strategy during the exploration of the task, she was not able to explain why other strategies, like graphing the amount of money made, or creating equations, were not helpful in selecting the vendors to be employed. Nonetheless, a common theme among all three PSMT was addressing the importance of using of multiple strategies to explore the MEA. This is consistent with the findings of Carlson and colleagues (2003), in which the PSMT in the study were able to anticipate and describe the various ways that a student might think about *covariation,* using the different justifications presented when solving the MEAs.

For the second MEA, Oriana and Helaine thoroughly contributed to the group's discussion and development of the final product, while Bianca's involvement and contribution were limited. Oriana and Helaine suggested various strategies for comparing the different costs for each of the phone plans, including writing linear equations for each plan, graphing to compare the plans visually, and creating a table of values. Once the group decided to use a table of values to compare the different phone plans, Bianca focused on the procedures of calculating the costs for each of the phone plan. Helaine and Oriana worked together to compose the letter explaining their recommendation to the family. Both PSMT professed the difficulty of explaining their method in writing, and they had to re-explain and justify their strategy to each other numerous times in the process. It was evident that this process helped the PSMT involved in the discussion better understand the mathematics used explore the MEA through the verbalization, justification, negotiation, and validation of their mathematical models and responses. In fact, during the second interview, Helaine recognized that composing the letter helped her make more sense of the mathematical topics used through the discussions she had with her group members. This is consistent with the findings of Miller (1992), in which the researcher found that writing in algebra allows the thorough examination of mathematical understandings. Furthermore, after the letter was written, all three PSMT reflected on their experiences with the *Phone Plans* MEA in comparison to the *Summer Jobs* they explored the previous week. The PSMT compared the strategies used and discussed the similarities and differences in the mathematical content related to the assumptions they made for each context. This is consistent with the recommendations of Doerr and Lesh (2003), where the researchers discussed how MEAs could be used by teachers to reflect on, and develop, their own practice and subject matter knowledge.

For the third task, *Historic Hotel* MEA, once again Oriana and Helaine both lead the exploration, while Bianca’s contribution was limited. Oriana suggested various strategies, such as graphing and writing expressions and equations, Helaine was persistent about writing an equation to model the given real-world scenario. Both Oriana and Helaine worked together to derive the profit equation by writing expressions that represent the number of rooms occupied in the hotel, price per room, and maintenance cost. On the other hand, Bianca was not able to make sense of the derived equation and expressions. Oriana and Helaine took turns explaining to Bianca the meaning of the different expressions in context of the given scenario. Like the second MEA, when composing the recommendation letter to the hotel owner, both Oriana and Helaine acknowledged the difficulty of explaining their method in writing, and they had to re-explain and justify their strategy to each other numerous times in the process. It was evident that this process helped the PSMT involved in the discussion better understand the mathematics used explore the MEA. Also similar to the second MEA, after the letter was written, all three PSMT reflected on their experiences with the *Historic Hotel* MEA in comparison to the *Phone Plans,* and *Summer Jobs* MEAs, *they* explored the previous weeks. The PSMT compared the solutions, the mathematical content, assumptions and strategies used. Additionally, the PSMT also reflected on the modeling process, and discussed how going back and forth to explore the task can help modelers understand the content better and see the usefulness and importance of different representations of quadratic equations. This is consistent with findings from Carlson and colleagues (2003), and Lesh and colleagues (2008), where the researchers discussed the process that the PSMT go through while solving MEAs—includes negotiations, discussions and reflections, which provides additional insights into the development and reasoning of strategies used. This is also consistent with Doerr and Lesh (2003), where they discussed how MEAs could be used by teachers to reflect on, and develop, their own practice and subject matter knowledge.

As previously stated, Oriana was selected from the high MKT category, Bianca was selected from the medium MKT category, and Helaine was selected from the low MKT category; these categories were formed based on the PSMT' scores on the LMT pre-assessment. Accordingly, diverse variations of their experiences with the three MEAs were documented, which was the goal of applying the *maximum variation* method to select the three cases. Based on a synthesis of the results discussed in the previous chapter, the following observations were perceived. First, Oriana made sense of context of the MEAs using the mathematics she knew. She started by suggesting various mathematical topics to apply to the given scenarios. She was confident about her knowledge of mathematics, and she used the context of the tasks to justify her strategies and validate her models. On the other hand, Helaine made sense of the scenarios using the context of the MEA, and then tried to find specific mathematical content that could apply to the given tasks. This was important during the exploration of the Historic Hotel MEA, because Helaine's understanding of the context of the task and reasoning, was the guide that helped the group look for variables, linear expressions and quadratic equations to represent the scenario. This is consistent with the work of Lesh and colleagues (1993, 2000), in which the researchers found that often students with low content knowledge tend to first recognize the need for mathematical constructs using the context of the task, and then investigate and develop these constructs, which was the case with Helaine. On the other hand, Bianca was not able to connect the mathematics used during the exploration of the MEAs to the context of the tasks. During the interviews, she often stated that she "just knew to do it" (Bianca, Interview 1, January 22), and she often discussed mathematical constructs without using the context of the MEA for validation or justification. It could be argued that Bianca's limited engagement and contribution to the exploration of the tasks were the reason behind this inconsistency. Nonetheless, the explorations of the three MEAs were useful for investigating the nature of the PSMT' MKT algebra, which is discussed next.

MKT Algebra

The PSMT’ MKT algebra was examined using McCrory and colleagues’ (2012) Knowledge of Algebra for Teaching (KAT) framework, by looking at the degree to which the PSMT were able to:

> (1) remove complexity while retaining integrity, and getting to the mathematical essence of a real-world problem [*trimming]*,
> (2) make connections across topics, assignments, representations and domains [*bridging]*, and
> (3) unpack the complexities found in mathematical ideas in ways that make the content comprehensible, anticipate the difficulties entailed in students’ understandings of particular aspects of algebra content, help students understand the usefulness and importance of algebra, and decompress rote procedures in order to grasp the logic behind them [*decompressing]* (McCrory et al., 2012).

More specifically, the degrees to which each PSMT was able to demonstrate evidence of *trimming, bridging,* and *decompressing* were described as *high* or *strong evidence*, *medium* or *some evidence*, and *low* or *no evidence.* High or strong evidence were used if the PSMT was able to talk about and explain the different ideas corresponding to each category. Medium or some evidence were used if the PSMT only mentioned the different ideas relating to each category, without explanation. Low or no evidence were used if the PSMT either did not identify, or incorrectly identified, the different ideas pertaining to each category.

When it came to the PSMT’ ability to explain “the mathematical essence” of the given real-world tasks, Oriana demonstrated strong evidence of *trimming* for all the sources of collected data—all three MEA explorations, the two conducted interviews, and the written reflection. Although Helaine acknowledged that she was not able to make sense of the first MEA with her group, and that she was only able to make sense of it after other groups presented their solutions, Helaine demonstrated strong evidence of trimming during the explorations of the Phone Plans and Historic Hotel MEAs, the two conducted interviews, as well as her written reflection. Bianca, on the other hand, was only able to show some evidence of trimming during the first MEA and demonstrated low evidence of trimming for the rest of the collected data sources. These findings reveal that Oriana and Helaine were able to get to the mathematical essence of a real-world problem by explaining and justifying how mathematical topics were used to make sense of each of the given MEAs; but Bianca was not able to discuss, explain, or justify the mathematics used to explore the three tasks. Lastly, it was discerned that during the explorations of the MEAs, strategies and mathematical constructs were implemented by the group based on how well the PSMT was able to explain, justify, and validate their use. More specifically, the PSMT went through a process of negotiating their mathematical models by verbalizing, justifying and validating their responses in context of the given MEA, and to compare their own ideas to others suggested by the group. Such discussions elicited PSMT' ability to "trim" the mathematical content in a way that matched their groupmates' current level of comprehension of both the mathematics being deliberated, as well as the given real-world scenarios.

When it came to the PSMT' ability to make connections across topics, assignments, representations and/or domains, Oriana, Bianca, and Helaine all started with a low rating for *bridging* during the exploration of the first MEA. Similarly, during the first interview, all three were able to identify connections between various mathematical topics and explanations, but none of the PSMT was able to explain those connections. During the second MEA Oriana demonstrated high evidence for bridging on both the exploration of the task, as well as her written reflection. On the other hand, Helaine and Bianca demonstrated low evidence during the exploration of the task, Helaine demonstrated some evidence on her written reflection, while Bianca demonstrated low on her written reflection. Similarly, during the exploration of the third

MEA, Oriana once again demonstrated high evidence of bridging, for both the task and the second interview. Helaine was able to make some connections across topics during the exploration of the task, and strong evidence of bridging was demonstrated during the second interview. While no evidence of bridging was shown in Bianca's case. These findings reveal that although all three PSMT initially were not able to explain how different mathematical topics were connected on the initial task and interview, by the last interview both Oriana and Helaine were able to make connections across topics, assignments, representations or domains; while Bianca was still not able to do so. Lastly, based on a synthesis of the results for the *bridging* KAT category discussed in the previous chapter, the following observation was perceived. The PSMT that were deeply engaged in the MEA were able to provide their groupmates (during the group explorations, and the researcher (during the individual interviews) with the big picture of the mathematical constructs applied, making explicit connections across topics, keeping a range of ideas in play and presenting mathematics as a coherent, connected endeavor. This is key feature of bridging described by McCrory and colleagues (2012).

When it came to decompressing, or what McCrory and colleagues (2012) described as unpacking the complexities found in mathematical ideas in ways that make the content comprehensible, anticipating the difficulties entailed in students' understandings of particular aspects of algebra content, helping students understand the usefulness and importance of algebra, and decompressing rote procedures in order to grasp the logic behind them, Oriana, Bianca, and Helaine all started with a low rating during the exploration of the first MEA. However, during the first interview, Oriana was able to anticipate some student misconceptions, but was not able to explain them; while Bianca and Helaine were not able to anticipate any student misconceptions. During the second MEA all three PSMT demonstrated low evidence for *decompressing* because they focused on rote procedures of calculating the total cost, without grasping the logic behind the process. On the written reflection, Oriana was able to anticipate student misconceptions, while Bianca and Helaine could not. During the exploration of the third MEA and the second interview, Oriana and Helaine both showed high evidence of decompressing because they were both able to decompress rote procedures of writing linear expressions and quadratic equations, and focused on the logic behind them, while Bianca was not able to do so. These findings reveal that although the three PSMT were not able to explain the logic behind rote procedures on the first MEA, by the last interview, Oriana and Helaine were able to anticipate the difficulties entailed in students' understandings of particular aspects of algebra content, discuss how to help students understand the usefulness and importance of algebra, and/or decompress rote procedures in order to grasp the logic behind them, while Bianca was still not able to do so. It was observed that during the individual interviews, the PSMT that were deeply engaged with the MEAs were able to anticipate student misconceptions based on the difficulties, error, and mistakes perceived during the exploration of the task. It was evident that MEAs elicited the PSMT' abilities to decompress the rote procedures of the mathematical constructs being used, and focus on explaining and justifying the logic, usefulness and importance behind them.

Lastly, during each interview, the PSMT were given two content questions from the LMT assessments and were asked to solve them using the think-aloud strategy to gain an understanding of any changes in the way the PSMT explained and justified their solutions to the questions, after their engagement with one MEA, and then again after exploring the third MEA. These findings are discussed next.

The three KAT categories, trimming, bridging, and decompressing were also used to analyze and compare the PSMT' explanations and justifications to the given content questions. Some changes observed in the quality of the explanations and justifications provided by Oriana and Helaine. Although Oriana answered both questions correctly in the first interview, she was not able to make connections across topics and representations related to linear equations when answering the second question, nor was she able to decompress rote procedures for writing linear equations to explain the logic behind them, on either question. However, by the second interview, Oriana explained and justified the reasoning behind her answers in a way that demonstrated strong evidence for *trimming, bridging* and *decompressing* , because she was able to get to the mathematical essence of each of the real-world problems, decompress rote procedures of writing equations to focus on the logic behind what the written equations mean in terms of the context of each problem, and make connections in her reasoning across different questions, topics and representations. Furthermore, during the second interview, Helaine answered both questions correctly, and she was also able to improve the quality of her explanation and justification by consistently referring to the context of the given scenarios to validate her reasoning, which she did not do during her first interview. By the second interview, Helaine's levels of evidence increased to high for three KAT categories on the first question, and remained at high on the second question. Nonetheless, Bianca used the same reasoning to incorrectly answer both questions during both interviews and maintained a low rating on all three categories; her explanations and justifications did not change from the first interview to the second.

The findings discussed above were not only corroborated by the PSMT' elicited knowledge from their experiences with the MEAs, which were discussed previously, but they were also supported by their results on the LMT pre- and post-assessments. On the pre-assessment Oriana's score (θ =1.145, SE = 0.416) was in the 90th percentile, while Helaine's (θ = −0.549, SE= 0.372) was in the 25th percentile of the participating PSMT in their class (n=30). Their scores on the LMT post-assessment were both greater than 50th percentile which placed both PSMT in the high MKT relative to their classmates. Helaine's score (θ = −0.093, SE= 0.394) was in the 60th percentile, while Oriana's score (θ = 2.193, SE= 0.543) was the highest among the participating PSMT in their class *(n*=27). On the other hand, Bianca was initially selected from the medium MKT category. Her score on the LMT pre-assessment (θ= − 0.241, SE = 0.391) was in the 50th percentile of the participating PSMT' scores in her class (n=30). Her score on the LMT post-assessment decreased (θ = −0.683, SE= 0.374) and was in the 35th percentile of the participating PSMT' scores in her class (n=27). This demonstrates that the ratings for the PSMT' MKT algebra using the KAT framework during the exploration of the MEAs and reflections during the interviews, were consistent with their explanations and justifications of the LMT content questions during the think-aloud portions of the interviews, as well as the LMT assessments.

In conclusion, the results of this study suggest that being engaged in MEAs can elicit PSMT' MKT algebra, and in some cases (like Oriana and Helaine) such tasks can be beneficial to their *trimming, bridging,* and *decompressing* abilities. Exploring MEAs immersed the PSMT in generating descriptions, explanations, and constructions, that helped reveal how they interpreted mathematical situations that they encountered. This occurred because by design, as Lesh and colleagues (2000) explained, because MEAs disclose how such situations are mathematized, and interpreted. Accordingly, the problems served as useful tools for PSMT to have deep discussions and productive discourse on various algebra topics and make many different mathematical connections in the process.

Based on the results discussed above, and in the previous chapters, the nature of the relationship between engagement in MEAs and PSMT' MKT Algebra is illustrated in the figure below. It demonstrates how engagement in MEAs elicited PSMT' MKT Algebra through the probing of their ability to (1) distill and clarify the mathematics used in their explanations and validation of reasoning, (2) make explicit connections across different mathematical topics, and (3) unpack rote procedures and focus on the logic and usefulness behind them. The implications for this research study are discussed in the next section.

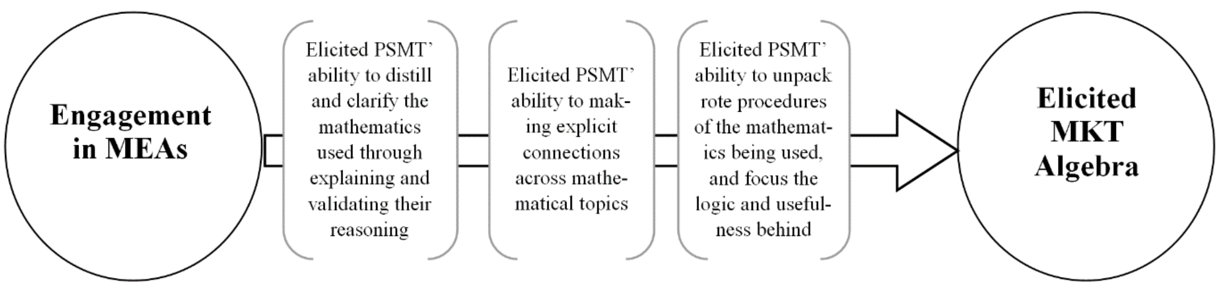


Figure 39. Summary Illustration of Results.

Implications

At the beginning of Chapter One, a question that mathematics teachers often hear from their students in the classroom was posed— "Why do I have to learn this?" This important question was recently addressed in the book *Catalyzing Change in High School Mathematics* (NCTM, 2018), in the discussion of *the purposes of school mathematics*. A key recommendation was endorsed that "each and every student should learn [mathematics] in order to expand professional opportunities, understand and critique the world, and experience the joy, wonder, and beauty of mathematics" (NCTM, 2018, p. 9). These ideas can be closely related to the mathematical modeling process, because it affords modelers the opportunity to explore how different events, phenomena, or scenarios might unfold under specific assumptions or constrains. This process can empower modelers to be productive and informed decision makers, which directly supports their ability to both critique and make sense of the world. Accordingly, mathematical modeling and the modeling cycle was one of the three essential mathematical practices and processes included in the *Catalyzing Change* manuscript. Moreover, the task force argued that the mathematical modeling process "provides opportunities for teachers to employ each of the eight Mathematics Teaching Practices [recommended by NCTM (2014)] and their connection with equitable teaching" (NCTM, 2018, p. 41).

An important implication of the findings of this research study is the observation that through working together and posing purposeful questions, connecting different representations, and experience productive struggle and discourse, the mathematical modeling process provides pre-service teachers the opportunity to practice and apply their mathematical knowledge for teaching and effective teaching practices. This is consistent with propositions made by researchers in both the Mathematical Knowledge for Teaching field, as well as the Models and Modeling Perspective field. More specifically, researchers have previously argued that because MKT is a type of mathematical knowledge that can be applied and assessed, it can also be learned through increased experiences with tasks that focus on the application of mathematical knowledge to the work of teaching (Bass, 2005; Stylianides & Stylianides, 2014); and it has also been suggested that particular mathematical tasks have the potential of eliciting various types of knowledge by teachers (Hill, Dean, & Goffney, 2007). In the models and modeling perspective, Doerr and Lesh (2003) described MEAs as mathematical tasks that require the application of mathematical knowledge to a context of interpretations and analyzing, resembling the work of a teacher.

In the same manner that MEAs can provide a foundation for the development of powerful mathematical reasoning in children (Lesh et al., 1993; 2000; 2003; McLean & Doerr, 2016) they also can effectively guide teacher educators in helping pre-service teachers develop their mathematical teacher knowledge, which was the premise of this research study. The thought-revealing nature of MEAs is particularly important to mathematics teacher educators because introducing such tasks in content and methods courses, not only demonstrates PSMT' MKT, but pre-service teachers may also be encouraged to use similar activities to gain access to the developing understandings and reasoning patterns of their future students. Therefore, intentional experiences in sequences of coursework and field experiences with MEAs would be valuable.

When it comes to algebra specifically, the modeling cycle can provide pre-service teachers the opportunity to deepen their understanding of representing relationships between quantities through forming mathematical connections between constructs and their symbolic, tabular, and graphic descriptions. By investigating scenarios such as maximizing profit for a hotel, recommending the most affordable phone plan, or selecting employees for a job, PSMT are able to make sense of real-world situations by identifying relevant quantities, defining assumptions, generalizing mathematical ideas, examining the fit of a model to its context, comparing multiple models, and reflecting on the content and process. Such practices can lead to further opportunities to discuss the value of different representations of algebraic content and can bring about a positive influence in the future practices of pre-service algebra teachers.

It has been contended that one of the most effective ways to help teachers enhance their teaching practices is to help them become more aware and familiar with their students' developing ways of thinking about different mathematical ideas (Carpenter & Fennema, 1992; NCTM, 2014). By design, MEAs are purposefully created to reveal students' ways of thinking to teachers (Lesh et al., 2000). Such tasks also engage learners in the modeling process, which can develop and deepen their mathematical knowledge. In order for PSMT to recognize the importance, and value the benefits of mathematical modeling, positive experiences with mathematical modeling are needed in their teacher preparation program (Beswick, 2012). Furthermore, the CCSSM (2010) describes mathematical modeling as "best interpreted not as a collection of isolated topics but in relation to other standards" (p.57). The PSMT in this research study experienced authentic mathematical modeling through the exploration of three MEAs, made sense of various mathematical constructs by *trimming, bridging* and *decompressing,* and reflected on their experiences with modeling and mathematical knowledge, as learners and future teachers. Consequently, the findings demonstrate that MEAs can be used to elicit PSMT' MKT algebra through the implicit characteristics of planning, analyzing, writing, forming connections, and reflecting on the mathematical content and processes.

Limitations

This research study incurred a few potential limitations. First, the researcher chose the three examined cases from a small sample size, and thus, the three selected PSMT may not be representative of other pre-service secondary mathematics teachers with *low, medium* and *high* MKT algebra. Furthermore, although the purpose of the study was not focused on measuring differences of participants' MKT, the progress of their mathematical knowledge in terms of *bridging, trimming,* and *decompressing,* was discussed. Accordingly, a threat to internal validity could have been posed because pre-existing group differences may have contributed to any changes in the PSMT' development of *bridging, trimming* and *decompressing* of algebra knowledge. Such differences could have encompassed their prior experience working with students, a stronger mathematics background, or group interactions.

Prolonged engagement with participants is a recommended validation strategy for case studies (Creswell, 2013), however, the researcher was only able to spend four class sessions with the participants, due to time constraints. Persistent observations and engagements in more MEAs throughout the entire semester would have benefited the study in various ways, including building trust with participants, learning the culture and interactions within the group of PSMT, and checking for misinformation that stems from distortions introduced by the researcher, the instructor of the course, or other sources.

The researcher facilitated the MEAs, conducted the interviews, and analyzed the data solely. Accordingly, the multiple roles played may have introduced biases that could have shaped the interpretation and approach of the study. Having more researchers involved in the data collection and data analysis aspects of the study would have increased the confirmability of the research. This would have ensured more objectivity in evaluating the results, and in describing how well the research findings were supported by the actual data collected. By having other researchers examine the data to corroborate, confirm, or perhaps disagree with the findings, then less biases may have impacted the data analysis.

Additionally, the researcher was not able to administer the *member-checking* strategy recommended by Creswell (2013), to seek participants' views of the credibility of the findings and interpretations. Taking the data, analyses, interpretations, and conclusions of the study back to the participants to judge the accuracy and credibility of the account not only would have increased the validity of the study, but this strategy could have also helped the researcher acquire a deeper insight into the PSMT' experiences with the MEAs as well as their MKT.

Recommendations for Future Research

The purpose of this research study was to examine the relationship between engagement in MEAs and PSMT' MKT algebra. One definition for MKT is the mathematical knowledge that is needed by teachers to effectively teach mathematics (Ball, Thames and Phelps, 2008). An important role in algebra teachers' everyday work is anticipating and knowing the difficulties entailed in students' understandings (McCrory et al. 2012). Although this research study asked the PSMT' to anticipate student difficulties during the interviews, it would be interesting to examine what Schorr and Lesh (2003) called *teacher level thought-revealing activities* with the PSMT. The authors described these activities as contexts in which teachers can express their current ways of thinking about mathematics and teaching mathematics. These tasks could entail having the groups of PSMT come up with possible student solutions, analyzing authentic student work, and creating concept maps for the MEAs. It would be interesting to investigate if such tasks can elicit PSMT' mathematical knowledge in relation to students' understandings and instructional practices. A relevant question to study could be, what is the relationship between engagement in *teacher level thought-revealing activities* and PSMT' MKT? Additionally, because this study intentionally selected the three MEAs to be given in a specific order, it would be interesting to examine if the order of the given tasks had any effect on the PSMT' MEA explorations and MKT.

Furthermore, various researchers have recommended the infusion of mathematical modeling throughout teacher education preparation programs, rather than to confine it to a single unit or course (Lingefjärd, 2007; Zbiek 2016). Thus, a recommendation for future research is to examine the development of MKT via MEAs through prolonged investigations, to ascertain the level and depth of differences or changes in the PSMT' knowledge over time. A related question to examine could be, how do PSMT' understanding of teaching and learning of mathematical modeling co-develop with the understanding of teaching and learning of mathematical content?

Lastly, based on the individual analyses, it was evident that the interactions between the group members affected the PSMT' contributions and understandings of the MEAs. In context of the study, during the first MEA Helaine was not able to make sense of the *Summer Jobs* MEA with her group, however, once she heard other groups' solution, she was able to explain and justify the various mathematical ideas used to explore the task. Furthermore, once Helaine (who was selected from the low MKT category) joined a different group for the second and third MEAs, and she worked with Oriana (who was selected from the high MKT category), she was able to make sense of the tasks in deeper and more meaningful ways. Therefore, a possible recommendation for future research is to connect "group work" research with the MKT development of PSMT, and examine whether in fact what took place during this research study was an isolated incident, or part of a greater trend. A related question to study could be, how would this research connect with the Zone of Proximal Development as it relates to the way PSMT learn with or without the presence of a guiding figure?

Summary

This section discussed results from the study, implications, limitations, potential contributions, and recommendations for future research. This research indicates the need for more mathematical modeling tasks and activities to be incorporated in teacher education programs to help PSMT experience authentic mathematical modeling and learn mathematics through the process. The findings in this study suggest that through working together and posing purposeful questions, connecting different representations, and experiencing productive struggle and discourse, the mathematical modeling process provided the PSMT the opportunity to practice and apply their mathematical knowledge for teaching. As the *Catalyzing Change in High School Mathematics* (NCTM, 2018) suggested, mathematical modeling not only affords modelers (both students and teachers) the opportunity to explore how different events, phenomena, or scenarios unfold under specific assumptions or constrains, and as a result, empowers them to be productive and informed decision makers; but the mathematical modeling process can also provide opportunities for pre-service (and in-service) teachers to practice and employ effective and equitable teaching strategies.

Furthermore, the rising awareness of the value of mathematical modeling — as an essential part of the mathematical experiences for all students — requires focused attention on what is needed in terms of teacher preparation. One of the major motivations for conducting this research was to engage PSMT in mathematical modeling tasks, and the mathematical modeling process, in order to gain some pedagogical and subject matter insights of mathematical modeling. The emphasis on mathematical modeling in the K-16 mathematics curriculum is relatively new, which indicates that most teachers (both pre-service and in-service) who are expected to teach mathematical modeling, or teach mathematics through modeling, have not themselves explored the various educational aspects involved in solving modeling tasks in a systematic way (Phillips, 2016), this demonstrates the importance of continued research in this field.

APPENDIX A: INSTITUTIONAL REVIEW BOARD FORMS

IRB Approval Letter

University of Central Florida

University of Central Florida Institutional Review Board
Office of Research & Commercialization
12201 Research Parkway, Suite 501
Orlando, Florida 32826-3246
Telephone: 407-823-2901 or 407-882-2276
www.research.ucf.edu/compliance/irb.html

Approval of Human Research

From: **UCF Institutional Review Board #1**
FWA00000351, IRB00001138

To: **Aline Zghayyar Abassian** and Co-PI: **Dr. Farshid Safi**

Date: **February 01, 2018**

Dear Researcher:

On 02/01/2018 the IRB approved the following modifications to human participant research until 12/19/2018 inclusive:

Type of Review:	IRB Addendum and Modification Request Form Expedited Review based on 45 CFR 46.110(b) (2) Minor Changes to Previously Approved Research.
Modification Type:	Title change. Revised Study Application was attached. Revised Consent and Protocol were uploaded.
Project Title:	A Case Study Exploring the Relationship between Model-Eliciting Activities and Prospective Secondary Teachers' Mathematical Knowledge for Teaching Algebra Topics
Investigator:	Aline Zghayyar Abassian
IRB Number:	SBE-17-13605
Funding Agency:	
Grant Title:	
Research ID:	N/A

The scientific merit of the research was considered during the IRB review. Continuing Review Application must be submitted 30days prior to the expiration date for studies that were previously expedited, and 60 days prior to the expiration date for research that was previously reviewed at a convened meeting. Do not make changes to the study (i.e., protocol, methodology, consent form, personnel, site, etc.) before obtaining IRB approval. A Modification Form **cannot** be used to extend the approval period of a study. All forms may be completed and submitted online at https://iris.research.ucf.edu .

If continuing review approval is not granted before the expiration date of 12/19/2018, approval of this research expires on that date. When you have completed your research, please submit a Study Closure request in iRIS so that IRB records will be accurate.

Use of the approved, stamped consent document(s) is required. The new form supersedes all previous versions, which are now invalid for further use. Only approved investigators (or other approved key study personnel) may solicit consent for research participation. Participants or their representatives must receive a copy of the consent form(s).

All data, including signed consent forms if applicable, must be retained and secured per protocol for a minimum of five years (six if HIPAA applies) past the completion of this research. Any links to the identification of participants should be maintained and secured per protocol. Additional requirements may be imposed by your funding agency, your department, or other entities. Access to data is limited to authorized individuals listed as key study personnel.

In the conduct of this research, you are responsible to follow the requirements of the Investigator Manual.

This letter is signed by:

Signature applied by Jennifer Neal-Jimenez on 02/01/2018 01:23:20 PM EST

Designated Reviewer

Informed Consent Form

A Case Study Exploring the Relationship between Model-Eliciting Activities and Prospective Secondary Teachers' Mathematical Knowledge for Teaching Algebra Topics

Informed Consent

Principal Investigator: Aline Abassian, Doctoral Candidate

Faculty Advisor: Farshid Safi, PhD

Investigational Site(s): [redacted]

Introduction: Researchers at the University of Central Florida (UCF) study many topics. To do this we need the help of people who agree to take part in a research study. You are being invited to take part in a research study which will include your [redacted] class during Spring 2018 semester. You have been asked to take part in this research study because you are a prospective secondary mathematics education teacher. You must be 18 years of age or older to be included in the research study.

The person doing this research is Aline Abassian [redacted]. Because the researcher is a doctoral candidate, she is being guided by Dr. Farshid Safi, a UCF faculty advisor in the [redacted]

What you should know about a research study:

- Someone will explain this research study to you.
- A research study is something you volunteer for.
- Whether or not you take part is up to you.
- You should take part in this study only because you want to.
- You can choose not to take part in the research study.
- You can agree to take part now and later change your mind.
- Whatever you decide it will not be held against you.
- Feel free to ask all the questions you want before you decide.

Purpose of the research study:

The purpose of this research is to investigate the nature of the relationship between model-eliciting activities and prospective secondary teachers' mathematical knowledge for teaching Algebra topics.

For years, mathematical modeling has been shown to support developing mathematical competencies in students that help in solving problems that arise in everyday life, society and the workplace. With the recent implementation of the *Common Core State Standards for Mathematics* (CCSSM), mathematical modeling is expected to be integrated in K-12 mathematics classrooms, and teachers are expected to play a major role in their students' mathematical modeling experiences. However, teachers have not themselves experienced mathematical modeling in their training. Although there exists some research that begin to illuminate how learners engage in mathematical modeling and how mathematics is supported by engaging in modeling tasks, there is a gap in looking at how prospective teachers' mathematical knowledge for teaching (MKT) is supported by such tasks. It is important to explore modeling in relation to mathematical knowledge for teaching, because many prior studies have linked MKT to the effectiveness of mathematics instruction.

Therefore, this study aims to fill the gap by exploring the relationship between mathematical modeling and prospective secondary teachers' MKT of algebra and functions.

What you will be asked to do in the study:

This research study will take place in your [redacted] course. You will be asked to:

- Complete two assignments online (a pre and a post assessment) outside of class.
- You will be given participation points for completing the two assignments (you will NOT be graded for accuracy on these assessments), and you may opt out of doing them if you wish to do so.
- You will be asked to complete tasks during class with a group of your peers and your work will be collected.
- These tasks will be facilitated by the researcher and the faculty advisor, while the instructor of the course is present.
- Some groups will be audio and video recorded as you are engaged in the tasks so the researcher can look at how you explored the task thoroughly.
- Some participants will be asked to be interviewed before or after class, in order to ask follow-up questions and gain a deepr understanding of your solutions to the tasks.
- The researcher will be conducting the interviews without the presence of the instructor or the faculty advisor.
- The two out-of-class assignments will take you approximately 25-30 minutes to complete.
- The tasks will take place during the first 90 minutes of your class.
- If you are asked to be interviewed, the interviews will last between 15-30 minutes, outside of class.

Location: The study will take place during your [redacted] The interviews will take place in [redacted] to insure confidentiality.

Time required: We expect that you will be in this research study for about 5 weeks.

- A tentative timeline for the research study is below:
  - **Week 1 (January 8):** Introduction and discussion related to the modeling process: what it is, how it looks in the classroom, and why modeling is important.
  - **Week 2 (Week of January 15)**: *MLK Holiday, no class*—You will be sent the pre-assessment online and asked to complete it before next class.
  - **Week 3-Week 5 (January 22, 29 and February 5):** Tasks will be explored during the first 90 minutes of class.
  - **Week 5 (Week of February 5):** Post-assessment will be sent online and you will be asked to complete before next class.
- The majority of the research study will take place during your class.
  - The tasks will take place during the first 90 minutes of your class.
  - If you are asked to be interviewed, the interviews will last between 15-30 minutes, outside of class, right before or after your class (during week 3-week 5)
  - The two out-of-class assignments will take you approximately 25-30 minutes to complete.

Audio or video taping:

You will be audio taped during this study. If you do not want to be audio taped, you can still be able to be in the study. Discuss this with the researcher or a research team member. If you are audio taped, the recording will be kept in a locked, safe place. The recording will be erased or destroyed when the research study is concluded.

You will be video taped during this study. If you do not want to be video taped, you can still be able to be in the study. Discuss this with the researcher or a research team member. If you are video taped, the recording will be kept in a locked, safe place. The tape will be erased or destroyed when the research study is concluded.

Risks: There are no reasonably foreseeable risks or discomforts involved in taking part in this study.

Benefits:

We cannot promise any benefits to you or others from your taking part in this research. However, possible benefits include gaining knowledge about the mathematical modeling process (a practice you are expected to be aware of when you teach in k-12 mathematics classrooms), as well as possibly gaining a deeper understanding of mathematical concepts relating to Algebra and Functions.

Compensation or payment:

There is no compensation, payment or extra credit for taking part in this study.

Confidentiality: We will limit your personal data collected in this study to people who have a need to review this information. We cannot promise complete secrecy. Organizations that may inspect and copy your information include the IRB and other representatives of UCF.

Anonymous research: This research study will not be anonymous, meaning the researcher will know your identity from the data that are to be collected. .

Study contact for questions about the study or to report a problem: If you have questions, concerns, or complaints, or think the research has hurt you, talk to

Aline Abassian— Doctoral Candidate in the Matheamtics Education track of the Education PhD program [redacted]

Dr. Farshid Safi— Faculty Supervisor, [redacted]

IRB contact about your rights in the study or to report a complaint: Research at the University of Central Florida involving human participants is carried out under the oversight of the Institutional Review Board (UCF IRB). This research has been reviewed and approved by the IRB. For information about the rights of people who take part in research, please contact: Institutional Review Board, University of Central Florida, Office of Research & Commercialization, 12201 Research Parkway, Suite 501, Orlando, FL 32826-3246 or by telephone at (407) 823-2901. You may also talk to them for any of the following:

- Your questions, concerns, or complaints are not being answered by the research team.
- You cannot reach the research team.
- You want to talk to someone besides the research team.
- You want to get information or provide input about this research.

Withdrawing from the study:
If you decide to leave the research without informing the researcher or the instructor of the course, you may lose participation points . If you decide to leave the study, contact the investigator so that the investigator can inform the instructor of the course and you will not lose any participation points.

Your signature below indicates your permission to take part in this research.

DO NOT SIGN THIS FORM AFTER THE IRB EXPIRATION DATE BELOW

____
Name of participant

______________________________ ______________
Signature of participant Date

______________________________ ______________
Signature of person obtaining consent Date

____
Printed name of person obtaining consent

UCF University of Central Florida IRB
IRB NUMBER: SBE-17-13605
IRB APPROVAL DATE: 02/01/2018
IRB EXPIRATION DATE: 12/19/2018

APPENDIX B: PRE-DATA COLLECTION OUTLINE

Pre-Data Collection Outline

1 Goals for this session

- Introduce the research study that will take place in your class.
- Discuss mathematical modeling—the what, how, and why.
- Explore a Model-Eliciting Activity (MEA).

2 Explain the Research Study

- Who?
  - Aline Abassian-doctoral candidate pursuing a Ph.D. in Mathematics Education
- What?
  - Purpose: explore the nature of the relationship between engaging in MEAs and Mathematical Knowledge for Teaching (MKT) Algebra.
- Why?
  - Research has shown that engaging in mathematical modeling tasks can help understand mathematics in deeper and more meaningful ways.
- How?
  -Over the next 4 weeks you will explore MEAs individually and in groups
  -Some of you will be asked to be interviewed outside of class.
  -You will be asked to take a pre/post assessment online

3 Relevance of mathematical modeling

"When are we going to use this?"

- What are some goals for teaching mathematics in school?
- How Old Is the Shepherd? Video by Robert Kaplinsky

4 Mathematical Modeling— the What

- General idea: Using mathematics to solve *rea*l world problems.
- Different definitions of mathematical modeling and descriptions of "real" world problems.
- *Modeling as a vehicle:* where modeling is used to develop mathematical modeling
- *Modeling as content:* where modeling is used to develop the skills of modeling.

- Modeling in the Common Core
  - Some confusion (Koestler, 2016)
  - Emphasizes "linking classroom mathematics and statistics to everyday life, work, and decision- making" for all grade levels with SMP 4
  - The High School Modeling standards are integrated into the content standards as opposed to listed separately in their own cluster.
    - A star symbol (★) is placed next to each standard to designate that this content standard is more easily modeled.
    - If all the standards in a domain are modeling standards, then only the domain will have an asterisk (not a star) by the name [such as High

School: Number and Quantity: Quantities]

For our purposes:

- Mathematical modeling will be defined as both content and a vehicle. However, there is a stronger emphasis on the learning of mathematics.
- Modeling tasks we will explore are called Model-Eliciting Activities (MEAs).

5 Mathematical Modeling—the How

- Mathematical modeling involves going through an iterative cycle

--Show picture of cycle (MMP perspective)

- Mathematical modeling involves the use of modeling tasks
- We will use MEAs
  - Specifically tailored to elicit and reveal students' thinking and understanding of mathematical concepts.
  - Must follow six design principles:
    - Model Construction
    - Reality
    - Self-Assessment
    - Model-Documentation
    - Generalizability
    - Effective Prototype

6 Mathematical Modeling—An MEA Example

- The Bottle Example
- Sample Solution
- ➨ Sample Transcript

7 Mathematical Modeling—the Why

- Helps students realize the relevance of mathematics.
- Helps students learn mathematics in deeper, more meaningful, and connected ways.
- Helps teachers identify students' strengths and weaknesses
- Helps teachers learn the mathematics better themselves.

8 Relate back to the Research Study

Goals as you explore the MEAs:

- Gain experience with mathematical modeling.
- Gain a deeper understanding of Algebra topics.
- Gain some insight into common classroom scenarios

9 Pre-Assessment Directions

- You will be emailed a link and directions for the assessment.
- Make sure you save your login information somewhere (email and password) to use for the post- assessment
- <u>PRE-ASSESSMENT DUE FRIDAY 2/19 11:59 PM</u>

10 Contact Information

- Aline Abassian– [redacted]
- Farshid Safi – [redacted](Faculty Advisor/Committee Chair)

APPENDIX C: LMT TRAINING CONFIRMATION

Teacher Knowledge Assessment System (TKAS) - Trained User Information

noreply@qualtrics.com
Tue 10/3, 5:05 PM
[redacted]

Reply all

Label: UCF Delete after 7 Years (7 years) Expires: 10/1/2024 5:04 PM

Dear Aline Abassian,

You are receiving this email as confirmation of your completion of the mandatory modules for the LMT/TKAS online training.

To access the measures, please go to the TKAS website at https:[redacted] From TKAS, you can access the measures to download and use as paper and pencil forms, or you may choose instead to create an assessment plan for administration through TKAS.

Your access code is [redacted]

Your email address is [redacted]

Your password is the one you were given when you were approved for training or whatever you reset it to, if you reset it.

Please save this information in a safe place for future reference.

APPENDIX D: SUMMER JOBS MEA AND ALGEBRA STANDARDS

Problem Statement:

Last summer Maya started a concession business at Wild Days Amusement Park. Her vendors carry popcorn and drinks around the park, selling wherever they can find customers. Maya needs your help deciding which workers to rehire next summer.

Last year Maya had nine vendors. This summer, she can have only six – three full-time and three half-time. She wants to rehire the vendors who will make the most money for her. She doesn't know how to compare them because they worked different numbers of hours. Also, when they worked makes a big difference. After all, it is easier to sell more on a crowded Friday night than a rainy afternoon.

Maya reviewed her records from last year. For each vendor, she totaled the number of hours worked and the money collected – when business in the park was busy (high attendance), steady (medium attendance), and slow (low attendance). (See the tables.) Please evaluate how well the different vendors did last year for the business and decide which three she should rehire full-time and which three she should rehire half-time.

Write a letter to Maya giving your results. In your letter describe how you evaluated the vendors. Give details so Maya can check your work and give a clear explanation, so she can decide whether your method is a good one for her to use.

	HOURS WORKED LAST SUMMER								
	June			July			August		
	Busy	*Steady*	*Slow*	*Busy*	*Steady*	*Slow*	*Busy*	*Steady*	*Slow*
MARIA	12.5	15	9	10	14	17.5	12.5	33.5	35
KIM	5.5	22	15.5	53.5	40	15.5	50	14	23.5
TERRY	12	17	14.5	20	25	21.5	19.5	20.5	24.5
JOSE	19.5	30.5	34	20	31	14	22	19.5	36
CHAD	19.5	26	0	36	15.5	27	30	24	4.5
CHERI	13	4.5	12	33.5	37.5	6.5	16	24	16.5
ROBIN	26.5	43.5	27	67	26	3	41.5	58	5.5
TONY	7.5	16	25	16	45.5	51	7.5	42	84
WILLY	0	3	4.5	38	17.5	39	37	22	12

	MONEY COLLECTED LAST SUMMER								
	June			July			August		
	Busy	*Steady*	*Slow*	*Busy*	*Steady*	*Slow*	*Busy*	*Steady*	*Slow*
MARIA	690	780	452	699	758	835	788	1732	1462
KIM	474	874	406	4612	2032	477	4500	834	712
TERRY	1047	667	284	1389	804	450	1062	806	491
JOSE	1263	1188	765	1584	1668	449	1822	1276	1358
CHAD	1264	1172	0	2477	681	548	1923	1130	89
CHERI	1115	278	574	2972	2399	231	1322	1594	577
ROBIN	2253	1702	610	4470	993	75	2754	2327	87
TONY	550	903	928	1296	2360	2610	615	2184	2518
WILLY	0	125	64	3073	767	768	3005	1253	253

Figures are given for times when park attendance was high (busy), medium (steady), and low (slow).

Potential Algebra Standards for Summer Jobs MEA:

Grade 6 Common Core Big Ideas

Ratios and Proportional Relationships: Understand ratio concepts and use ratio reasoning to solve problems.

Expressions and Equations: Represent and analyze quantitative relationships between dependent and independent variables.

Grade 7 Common Core Big Ideas

Ratios and Proportional Relationships: Analyze proportional relationships and use them to solve real-world and mathematical problems.

Expressions and Equations: Solve real-life and mathematical problems using numerical and algebraic expressions and equations.

Grade 8 Common Core Big Ideas

Expressions and Equations: Understand the connections between proportional relationships, lines, and linear

High School Common Core Big Ideas

Number and Quantity

Quantities: Reason quantitatively and use units to solve problems

The MEA above was developed by the Small Group Mathematical Modeling (SGMM) and School Mathematics and Science Center (SMSC) at Purdue University, under the direction of Richard Lesh and Lyn English.

Retrieved from: https://engineering.purdue.edu/ENE/Research/SGMM/CASESTUDIESKIDSWEB/CASESTUDIESKIDSWEB/case_studies_table.htm

The big ideas were retrieved form the CCSSM document.

APPENDIX E: PHONE PLANS MEA AND ALGEBRA STANDARDS

Problem Statement:

The Olsen family just discovered that they were *slammed* and thus, had received a higher phone bill. Instead of simply switching back to their old plan, they decided to look into the various long-distance calling plans that are available to see if another plan would save them even more money. Included in the following pages are a list of the current long-distance calling plans available and a list of the long-distance calls that the Olsen family made in June.

Furthermore, Mr. and Mrs. Olsen realized that the long-distance plans that are available often change over time. So, to take advantage of any new calling plans that might become available, they decided it would be a good idea to review their long-distance calls and their long-distance calling plan once a year to determine if a different plan might save them more money.

The Olsen family needs your help in selecting the cheapest long-distance calling plan for their family. Write a letter to them that describes your method for finding the cheapest phone plan. The Olsen Family will also be using your method to re-evaluate their phone plan next year. Therefore, be sure to include enough details about your method that the Olsen Family can use it next year when new phone plans will likely be available.

Long-Distance Calling Plans:

NAME OF PLAN	TIME OF CALL	COST PER MINUTE	MONTHLY FEE
First Talk One Rate	All Day, Everyday	$0.05	$8.95
Midwest Nights	7 p. m. to 7 a.m.	$0.05	$5.95
	7 a.m. to 7 p. m.	$0.10	
Midwest Plan 1000	7 p. m. to 7 a.m.	First 1000 minutes free After first 1000 minutes: $0.07	$20
	7 a.m. to 7 p. m.	$0.10	
Midwest Sense Any Time	All Day, Everyday	$0.10	$4.95 (Waived if Long Distance spending is more than $25 for the month.)
Horizon Nation Wide Saver Plan	All Day, Everyday Calls under 20 minutes	$0.08	$4.95
	All Day, Everyday Calls 20 minutes and over	$0.05	

The Olsen's Long-Distance Calls

Call Number	Date	Time	Place Called	Number Called	Minutes
1	June 1	9:18 pm	Monroe, MI	735 289-2293	19.0
2	June 3	7:40 am	Monroe, MI	735 289-2293	8.0
3	June 3	8:55 pm	Cleveland, OH	216 371-4092	23.0
4	June 3	9:18 pm	Arlington, VA	703 979-2902	1.0
5	June 4	8:43 pm	Crescent Ville, OH	513 942-6531	27.0
6	June 6	7:15 pm	Monroe, MI	735 289-2293	10.0
7	June 6	7:24 pm	Monroe, MI	735 289-2293	17.0
8	June 7	11:55 am	Monticello, IN	219 583-7690	1.0
9	June 7	11:57 am	Logansport, IN	219 722-1039	1.0
10	June 9	6:25 pm	Monroe, MI	735 289-2293	20.0
11	June 10	9:26 pm	Yuma, CO	970 876-3270	33.0
12	June 15	9:14 am	Monroe, MI	735 289-2293	2.0
13	June 15	8:32 pm	Ft. Collins, CO	970 493-2876	1.0
14	June 15	7:58 pm	Ft. Collins, CO	970 493-2876	2.0
15	June16	7:29 pm	Ft. Collins, CO	970 493-2876	16.0
16	June 16	7:59 pm	Monroe, MI	734 289-2345	3.0
17	June 20	9:01 am	Toledo, OH	419 474-3546	1.0
18	June 21	9:03 pm	Belleville, MI	734 289-2897	1.0
19	June 22	9:15 am	Toledo, OH	419 474-3546	29.0
20	June 22	7:09 pm	Monroe, MI	735 289-2293	7.0
21	June 23	10:17 am	Yuma, CO	970 876-3270	33.0
22	June 25	2:05 pm	Trenton, MI	734 675-1152	2.0
23	June 28	3:24 pm	Monroe, MI	735 289-2293	1.0

Potential Middle School Algebra Standards for Phone Plans MEA:

Grade 6 Common Core Big Ideas

Expressions and Equations: Represent and analyze quantitative relationships between dependent and independent variables.

Grade 7 Common Core Big Ideas

Ratios and Proportional Relationships: Analyze proportional relationships and use them to solve real-world and mathematical problems.

Expressions and Equations: Solve real-life and mathematical problems using numerical and algebraic expressions and equations.

Grade 8 Common Core Big Ideas:
Expressions and Equations: Understand the connections between proportional relationships, lines, and linear equations.
Expressions and Equations: Analyze and solve linear equations and pairs of simultaneous linear equations
Functions: Define, evaluate, and compare functions.
Functions: Use functions to model relationships between quantities.

High School Common Core Big Ideas:
Quantities: Reason quantitatively and use units to solve problems.
Interpreting Functions: Understand the concept of a function and use function notation
Interpreting Functions: Interpret functions that arise in applications in terms of the context.
Interpreting Functions: Analyze functions using different representations.
Building Functions: Build a function that models a relationship between two quantities
Creating Equations: Create equations that describe numbers or relationships.
Reasoning with Equations and Inequalities: Understand solving equations as a process of reasoning and explain the reasoning.
Quantities: Reason quantitatively and use units to solve problems.

Center (SMSC) at Purdue University, under the direction of Richard Lesh and Lyn English.

Retrieved from:
https://engineering.purdue.edu/ENE/Research/SGMM/CASESTUDIESKIDSWEB/CASESTUDIESKIDSWEB/case_studies_table.htm

The big ideas were retrieved form the CCSSM document.

APPENDIX F: HISTORIC HOTEL MEA AND ALGEBRA STANDARDS

Problem Statement:

Mr. Frank Graham, from Elkhart District in Indiana, has just inherited a historic hotel. He would like to keep the hotel, but he has little experience in hotel management. The hotel has 80 rooms, and Mr. Graham was told by the previous owner that all of the rooms are occupied when the daily rate is $60 per room. He was also told that for every dollar increase in the daily $60 rate, one less room is rented. So, for example, if he charged $61 dollars per room, only 79 rooms would be occupied. If he charged $62, only 78 rooms would be occupied. Each occupied room has a $4 cost for service and maintenance per day.

Mr. Graham would like to know how much he should charge per room in order to maximize his profit and what his profit would be at that rate. Also, he would like to have a procedure for finding the daily rate that would maximize his profit in the future even if the hotel prices and the maintenance costs change. Write a letter to Mr. Graham telling him what price to charge for the rooms to maximize his profit and include your procedure for him to use in the future.

Potential Middle School Algebra Standards for Historic Hotel MEA:

Grade 6 Common Core Big Ideas

Expressions and Equations: Represent and analyze quantitative relationships between dependent and independent variables.

Grade 7 Common Core Big Ideas

Ratios and Proportional Relationships: Analyze proportional relationships and use them to solve real-world and mathematical problems.

Expressions and Equations: Solve real-life and mathematical problems using numerical and algebraic expressions and equations.

Grade 8 Common Core Big Ideas:

Expressions and Equations: Understand the connections between proportional relationships, lines, and linear equations.

Expressions and Equations: Analyze and solve linear equations and pairs of simultaneous linear equations

Functions: Define, evaluate, and compare functions.

Functions: Use functions to model relationships between quantities.

High School Common Core Big Ideas:

Interpreting Functions: Understand the concept of a function and use function notation

Interpreting Functions: Interpret functions that arise in applications in terms of the context

Interpreting Functions: Analyze functions using different representations

Building Functions: Build a function that models a relationship between two quantities

Linear Quadratic Exponential Models: Interpret expressions for functions in terms of the situation they model

Seeing Structure in Expressions: Interpret the structure of expressions

Seeing Structure in Expressions: Write expressions in equivalent forms to solve problems

Creating Equations: Create equations that describe numbers or relationships

Reasoning with Equations and Inequalities: Understand solving equations as a process of reasoning and explain the reasoning

Reasoning with Equations and Inequalities: Represent and solve equations and inequalities graphically.

Retrieved from:
https://engineering.purdue.edu/ENE/Research/SGMM/CASESTUDIESKIDSWEB/CASESTUDIESKIDSWEB/case_studies_table.htm

APPENDIX G: MEA PROTOCOL

MEA Protocol

Part 1: 5 minutes

A. Remind PSMT about the following descriptions of MEAs
   a. Work on the problem together, and gradually revise solution over the period of 45 minutes
   b. There is more than one solution and/or way to think about the problem.
   c. Groups will be presenting solutions to the class (prepare 3-5-minute presentation).
   d. Facilitator will not be answering questions like "is this right? Are we done yet?" or any guiding questions.
   e. If stuck, look at the provided scaffolding questions and modeling cycle
   f. Return to the problem statement to verify your solutions.

Part 2: 35-40 minutes

B. Have PSMT read the problem statement and discuss the following questions as a group:
   a. Who are you working for?
   b. What do you need to create?
   c. How will you provide this information?

C. Have PSMT work together
   a. Remind participants to work as a team, not individually.
   b. Once the team has a solution that all members agree on, create a short presentation of solution with explanations and justifications (could be done on power point or on chart paper)
   c. Turn in ALL work (presentations, written work, etc.)

Part 3: 35-40 minutes

D. Have PSMT present their solutions to the class (30-35 minutes)
   a. Presentations should take 5 minutes or less.
   b. Encourage groups to (1) make sense of the other teams' solutions; (2) analyze solutions in terms of strengths and weaknesses (3) ask questions.

APPENDIX H: INTERVIEW PROTOCOL

Interview Protocol	
Time of the Interview:	
Date:	
Participant:	
Questions Related to the MEA	Rationale
1. Tell me a little about yourself and your experiences in mathematics. 2. What are your goals after you graduate?	-Icebreaker question. -Used to get the participant more comfortable. -Gain background information.
3. How did you make sense of the MEA?	-Elicit answers related to *Trimming.*
4. When solving the MEA, what mathematical ideas and skills did you use? 5. Why does it make sense to use them?	-Gain information about the mathematical topics that adhered after completing the MEA. -Elicit answers related to *Trimming.*
6. Think about how these mathematical ideas and skills relate in your solution. Would you draw a diagram of the ideas and skills you used, such that they are arranged in a way that those ideas that are related are close to each other and those that don't relate are far apart?	-Elicit answers related to *Bridging.*
7. After engaging in the MEA, describe your understanding of the mathematical ideas you just stated. 8. Why do you feel that way?	-Gain information about participants' self-assessment regarding mathematical topics.
9. How do you think MEAs can be used in an algebra classroom?	-Gain information about participants' teaching perspective.
10. How do you think students would learn mathematics through MEAs?	-Gain information about participants' teaching perspective.
11. What possible mathematical misconceptions do you see students having with this MEA?	-Elicit answers related to *Decompressing.*

Content Questions

Ask the PSMT to explain their solution to this problem from the LMT pre-assessment (#21 on LMT Pretest forms A and B)

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

If PSMT had form A of the pretest, ask them to explain how they solved this problem (Form A#31)

[redacted]

[redacted]

	[redacted]	[redacted]	[redacted]
[redacted]	[redacted]	[redacted]	[redacted]
[redacted]	[redacted]	[redacted]	[redacted]

[redacted]

If PSMT had form B of the pretest, ask them to explain how they solved this problem (Form B#35)

[redacted]

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