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In Quest of Bernstein Inequalities Rational Functions, Askey-Wilson Operator, and Summation Identities for Entire Functions

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IN QUEST OF BERNSTEIN INEQUALITIES

RATIONAL FUNCTIONS, ASKEY-WILSON OPERATOR, AND
SUMMATION IDENTITIES FOR ENTIRE FUNCTIONS

by

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ABSTRACT

The title of the dissertation gives an indication of the material involved with the connecting thread throughout being the classical Bernstein inequality (and its variants), which provides an estimate to the size of the derivative of a given polynomial on a prescribed set in the complex plane, relative to the size of the polynomial itself on the same set.

Chapters 1 and 2 lay the foundation for the dissertation. In Chapter 1, we introduce the notations and terminology that will be used throughout. Also a brief historical recount is given on the origin of the Bernstein inequality, which dated back to the days of the discovery of the Periodic table by the Russian Chemist Dmitri Mendeleev. In Chapter 2, we narrow down the contents stated in Chapter 1 to the problems we were interested in working during the course of this dissertation. Henceforth, we present a problem formulation mainly for those results for which solutions or partial solutions are provided in the subsequent chapters.

Over the years Bernstein inequality has been generalized and extended in several directions. In Chapter 3, we establish rational analogues to some Bernstein-type inequalities for restricted zeros and prescribed poles. Our inequalities extend the results for polynomials, especially which are themselves improved versions of the classical Erdös-Lax and Turán inequalities. In working towards proving our results, we establish some auxiliary results, which may be of interest on their own.

Chapters 4 and 5 focus on the research carried out with the Askey-Wilson operator applied on polynomials and entire functions (of exponential type) respectively. In Chapter 4, we first establish a Riesz-type interpolation formula on the interval $[-1, 1]$ for the Askey-Wilson operator. In consequence, a sharp Bernstein inequality and a Markov inequality are obtained when differentiation is replaced by the Askey-Wilson operator. Moreover, an

inverse approximation theorem is proved using a Bernstein-type inequality in L^2-space. We conclude this chapter with an overconvergence result which is applied to characterize all q-differentiable functions of Brown and Ismail. Chapter 5 is devoted to an intriguing application of the Askey-Wilson operator. By applying it on the Sampling Theorem on entire functions of exponential type, we obtain a series representation formula, which is what we called an extended Boas' formula. Its power in discovering interesting summation formulas, some known and some new will be demonstrated. As another application, we are able to obtain a couple of Bernstein-type inequalities.

In the concluding chapter, we state some avenues where this research can progress.

To my parents, for everything.

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The eminent mathematician Carl Friedrich Gauss once said, "Mathematicians stand on each other's shoulders." I express my deepest gratitude to all my teachers throughout these years for inculcating knowledge in me and shaping me to become the person I am today, and also to the distinguished mathematicians whose work has inspired me throughout the course of this dissertation.

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CHAPTER 1: INTRODUCTION AND TERMINOLOGY

"Begin at the beginning," the King said, very gravely, "and go on till you come to the end: then stop."

Lewis Carroll, Alice in Wonderland

This chapter will set the tone to the dissertation in which an expository account will be given on the notations and definitions of the contents that comprise the dissertation. A brief literature review will also be provided where references are mentioned which carryout comprehensive studies of the material.

1.1 Why Inequalities ?

Inequalities pervade mathematics, arises naturally in that in many practical applications the need may occurs to bound one quantity by another. Not only being a sophisticated tool in mathematics, they also play an integral role in other disciplines as well. The theory of inequalities presents an exciting and a very much active field of research, especially in approximation theory. Two of the most fundamental inequalities are the *Bernstein inequality* and the *Markov inequality*. Apart from being ubiquitous on their own rights, these two inequalities are proven to be invaluable tools in proving inverse theorems in approximation theory.

1.1.1 Preliminaries and Notations

In conjunction with the nomenclature, let $\mathbb{N}, \mathbb{R}$, and $\mathbb{C}$ respectively denote the set of natural, real, and complex numbers. For $x, y \in \mathbb{R}$, let $z = x + iy$ denote an element of $\mathbb{C}$, whose complex conjugate is $\overline{z} = x - iy$. Let $n \in \mathbb{N}$ be fixed. In this dissertation, we will come across several spaces of functions. In particular, we will use the following special spaces.

- $\mathcal{P}_n$: space of complex algebraic polynomials of degree at most n. i.e.,

$$\mathcal{P}_n := \left\{ p : p(z) = \sum_{k=0}^{n} b_k z^k,\ b_k \in \mathbb{C} \right\}.$$

  Indeed, by the *Fundamental theorem of algebra*, if $b_n \neq 0$, there exist $z_k,\ k = 1, 2, \ldots, n$ such that

$$p(z) = b_n \prod_{k=1}^{n} (z - z_k).$$

- $\mathcal{R}_n$: space of rational functions with at most n poles among $a_1, a_2, \ldots, a_n$. Let $w(z) = \prod_{k=1}^{n}(z - a_k)$.

$$\mathcal{R}_n := \left\{ r : r = \frac{p}{w},\ p \in \mathcal{P}_n \right\}.$$

- $\mathcal{T}_n$: space of trigonometric polynomials of degree at most n. i.e.,

$$\mathcal{T}_n := \left\{ t : t(\varphi) = \sum_{k=-n}^{n} c_k e^{ik\varphi},\ c_j \in \mathbb{C} \right\}.$$

  In fact, an element of $\mathcal{T}_n$ can equivalently be written as

$$t(\varphi) = a_0 + \sum_{k=1}^{n} \{a_k \cos(k\varphi) + b_k \sin(k\varphi)\}, \quad a_k, b_k \in \mathbb{C}.$$

For a compact (closed and bounded) set S in the complex plane, by $\|\cdot\|_S$ we denote the supremum norm on S. Moreover, we use $\mathbb{T} = \{z : |z| = 1\}$ to denote the unit circle and $\mathbb{D} = \{z : |z| < 1\}$ to denote the (open) unit disk.

1.1.2 Bernstein: The protagonist

Historically, the story of (polynomial) inequalities unfolded few years after the Russian Chemist Dmitri Mendeleev's discovery of the periodic table (see [**16**], [**29**], [**48**]). While observing his results of a study of the specific gravity of a solution as a function of the percentage of the dissolved substance, Mendeleev noticed that the data could be closely approximated by quadratic arcs and wondered if the corners where the arcs (of the plotted data) joined were actually there, or was it due to error of measurement. His question, after normalization was:

If $p(x)$ is a quadratic polynomial with real coefficients and $|p(x)| \leq 1$ on $[-1, 1]$, then how large can $|p'(x)|$ be on $[-1, 1]$?.

In 1887, Mendeleev himself answered the question and showed that $|p'(x)| \leq 4$ on $[-1, 1]$, and convinced himself that the corners of the arcs were genuine. As we would've guessed, being a non-mathematician he should communicate his results with a mathematician, which is exactly what he did. He communicated his results with Andrei Markov who naturally investigated the corresponding problem in a general setting to polynomials of degree n. Few years later Markov established and proved what is now known as the *Markov's inequality** [**46**]:

If $p(x)$ is a polynomial of degree n with real coefficients and $|p(x)| \leq 1$ on $[-1, 1]$, then $|p'(x)| \leq n^2$ on $[-1, 1]$. Equality holds only at ± 1 and only when $p(x) = \pm T_n(x)$, where $T_n(x) = \cos(n \cos^{-1}(x))^\dagger$.

The next episode of this story is to look at a similar inequality for polynomials of

*The younger brother of Andrei, Vladmir Markov extended Markov's inequality to higher order derivatives in 1892.

†T_n is called Chebyshev polynomial of the first kind. We will be utilizing T_n and it's companion, U_n in Chapter 4.

degree n over the complex plane, where a typical question can be raised along the same direction [**55, Chapter 14**]:

For a given polynomial of degree at most n with some meaningful information about the kind of values it takes on a prescribed subset of the complex plane. What can we say about the size of its derivative?

The answer to this question when the subset being the unit disk was provided by Sergei Natanovich Bernstein [**13**] in 1912.

Theorem 1.1.1. *Let $p \in \mathcal{P}_n$, then*

$$\|p'\|_{\mathbb{D}} \leq n \|p\|_{\mathbb{D}} . \tag{1.1.1}$$

Equality holds for a polynomial whose zeros are at the origin, i.e., (say) $p(z) = cz^n$ for a constant c.

Theorem 1.1.1 can be stated in the following equivalent forms.

1. Since $p(z)$ is an analytic function, by the maximum modulus principle

$$\|p\|_{\mathbb{D}} = \max_{|z|\leq 1} |p(z)| = \max_{|z|=1} |p(z)| = \|p\|_{\mathbb{T}} .$$

So Theorem 1.1.1 holds on the unit circle.

Theorem 1.1.2. *Let $p \in \mathcal{P}_n$, then*

$$\|p'\|_{\mathbb{T}} \leq n \|p\|_{\mathbb{T}} . \tag{1.1.2}$$

Equality holds for $p(z) = cz^n$ for a constant c.

2. Bernstein theorem for trigonometric polynomials:

   Theorem 1.1.3. *Let $t \in \mathcal{T}_n$, then for $\varphi \in (-\pi, \pi)$*

$$|t'(\varphi)| \leq n|t(\varphi)|. \tag{1.1.3}$$

   Equality holds for $t(\varphi) = \gamma \sin(\varphi - \varphi_0)$, where $|\gamma| = 1$.

   As before a uniform version of this can also be obtained:

$$\|t'\|_{[-\pi,\pi]} \leq n \|t\|_{[-\pi,\pi]} . \tag{1.1.4}$$

3. A connection between trigonometric to algebraic polynomials can be established by taking $t(\varphi) = p(\cos(\varphi))$, for $p \in \mathcal{P}_n$ to which (1.1.3) yields:

   Theorem 1.1.4. *Let $p \in \mathcal{P}_n$, then for $x \in (-1, 1)$*

$$|p'(x)| \leq \frac{n}{\sqrt{1-x^2}} \|p\|_{[-1,1]} . \tag{1.1.5}$$

   Equality is attained at the points $x = x_j = \cos\left[\frac{(2j-1)\pi}{2n}\right]$, $1 \leq j \leq n$, if and only if $p(x) = \gamma T_n(x)$, where $|\gamma| = 1$.

   This is known in the literature as the *standard form* of the Bernstein's theorem.

The story involves with these discoveries of Bernstein is quite interesting too. When first established, in (1.1.4) Bernstein had $2n$ instead of n, which he proved using a variational method. In [**13, p.527**], Bernstein attributed his proof to Edmund Landau. Inequality (1.1.5) first appeared in a paper by Michael Fekete [**25**], who attributed his proof to Leopold Fejér [**24**]. Simpler proofs of (1.1.3) were established by Marcel Riesz [**57**], Frigyes Riesz

[56] and de la Vallée Poussin [23]. Indeed, Marcel Riesz's elegant proof of (1.1.4) uses the following interpolation formula for the derivative of a trigonometric polynomial:

$$t'(\varphi) = \frac{1}{2n}\sum_{r=1}^{2n} t(\varphi+\theta_r)\frac{(-1)^{r+1}}{2\sin^2\left(\frac{\theta_r}{2}\right)}, \tag{1.1.6}$$

where

$$\theta_r = \frac{2r-1}{2n}\pi,\ \ r = 1, 2, \ldots, 2n.$$

Over the years, Bernstein inequality‡ had been substantially generalized and extended in several directions; by restricting the zeros of the polynomials, even to date. For instances, considering different domains of interest (compact subsets of the real line, arcs of the unit circle etc.), to different classes of polynomials (L^p-norms (for $0 < p \leq \infty$), entire functions etc.) etc. Its wide applicability lies in the fact that being optimal and being the solution to the following extremal problem:

$$\max_{p\in\mathcal{P}_n}\frac{\|p'\|}{\|p\|} = A_n,\ \text{(for some } A_n > 0) \tag{1.1.7}$$

for the respective norms. Its significance is apparent as there's an area of its own in the literature named *Bernstein-type inequalities.*

1.2 Beyond the ordinary derivative: The Askey-Wilson derivative

In 1985, Richard Askey and James Wilson in [6] introduced the theory of Askey-Wilson operator in their study of a class of orthogonal polynomials, the Askey-Wilson polynomials. For a positive parameter $q(< 1)$, the *Askey-Wilson operator* or the *Askey-Wilson*

‡For the equivalent forms of the Benstein inequality and their proofs, see the couple of excellent sources in [**49, Chapter 6**] and [**55, Chapter 14**].

derivative denoted by $\mathscr{D}_q$, is defined by

$$(\mathscr{D}_q f)(x) = \frac{\breve{f}(q^{1/2}z) - \breve{f}(q^{-1/2}z)}{\breve{h}(q^{1/2}z) - \breve{h}(q^{-1/2}z)}, \quad (x \in [-1,1]), \tag{1.2.1}$$

where

$$\breve{h}(z) = \frac{1}{2}\left(z + \frac{1}{z}\right),\ \breve{f}(z) = f\left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right),\ z = e^{i\theta},\ x = \cos\theta.$$

Note that $\breve{h}(q^{1/2}z) - \breve{h}(q^{-1/2}z) = i\sin\theta \cdot (q^{1/2} - q^{-1/2})$ and thus (1.2.1) can be written as

$$(\mathscr{D}_q f)(x) = \frac{\breve{f}(q^{1/2}z) - \breve{f}(q^{-1/2}z)}{i\sin\theta \cdot (q^{1/2} - q^{-1/2})}. \tag{1.2.2}$$

Since

$$\lim_{q \to 1^-} (\mathscr{D}_q f)(x) = f'(x)$$

at any point x where $f'(x)$ exists, $\mathscr{D}_q f$ can be considered as a discrete version of the derivative[§] of f.

1.3 Entire functions of exponential type

An entire function is one which is analytic in the finite complex plane $\mathbb{C}$. Let $\mathscr{B}_\sigma$ denote the set of entire functions of exponential type σ. That is, $f \in \mathscr{B}_\sigma$ if f is an entire function and for any $\varepsilon > 0$, there is an $A_\varepsilon > 0$ such that

$$|f(z)| \le A_\varepsilon e^{(\sigma+\varepsilon)|z|}$$

for all $z \in \mathbb{C}$.

[§] A proof to justify this will be given at the beginning of Chapter 4.

For such functions, with $|f(x)| \leq M$ for all x, the *Bernstein inequality* **[12]** is:

$$|f'(x)| \leq M\tau \quad (x \in \mathbb{R}). \tag{1.3.1}$$

Equality in (1.3.1) holds for $f(z) = ae^{i\tau z} + be^{-i\tau z}$, where $|a| + |b| = M$.

For a function $f \in L^p(\mathbb{R})$, $p > 0$, we write

$$\|f\|_{L^p} := \left(\int_{-\infty}^{\infty} |f(x)|^p \right)^{1/p} .$$

Functions in class $\mathscr{B}_\sigma$ whose restriction to $\mathbb{R}$ belongs to $L^p(\mathbb{R})$ are denoted by $\mathscr{B}_\sigma^p$, for which an L^p analogue of the Bernstein inequality for $p \geq 1$ is:

$$\|f'\|_{L^p} \leq \tau \|f\|_{L^p} . \tag{1.3.2}$$

In fact, in **[54]**, Qazi Rahman and Gerhard Schmeisser proved that (1.3.2) indeed holds for $0 < p < 1$ as well.

Functions in the class $\mathscr{B}_\sigma^2$ are called *band-limited* to $[-\sigma, \sigma]$ and are characterized by the classical *Paley-Wiener theorem* (**[52]**, **[15, p.103]**):

Theorem 1.3.1. *A function f belongs to $\mathscr{B}_\sigma^2$ if and only if it can be represented in the form:*

$$f(t) = \int_{-\sigma}^{\sigma} e^{ixt} g(x) \, dx, \tag{1.3.3}$$

for $t \in \mathbb{R}$ and for some function $g \in L^2[-\sigma, \sigma]$.

1.4 The Classical Sampling theorem

Sampling theory is one of the most significant techniques in mathematics that is widely applicable in other disciplines such as Engineering and Physics. The fundamental result in sampling theory is the *sampling theorem* ([**72, Theorem 2.1, p.16**]):

Theorem 1.4.1. *If a function f is band-limited to $[-\sigma, \sigma]$, then f can be reconstructed from its samples, $f\left(\frac{k\pi}{\sigma}\right)$. The uniformly-spaced sampling points $\frac{k\pi}{\sigma}$ are located on $\mathbb{R}$. The reconstruction formula is:*

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k\pi}{\sigma}\right) \frac{\sin[\sigma t - k\pi]}{(\sigma t - k\pi)}, \tag{1.4.1}$$

for $t \in \mathbb{R}$. The series being absolutely and uniformly convergent on $\mathbb{R}$.¶

This can be proved though several approaches, the shortest one uses the convolution structure of the series in (1.4.1). Other proofs involve the use of Fourier series expansions, Parseval formula, Poisson summation formula and Cauchy's integral formula.

The series in (1.4.1) can be put in the form

$$f(t) = \sum_{k=-\infty}^{\infty} f(t_k) \frac{G(t)}{(t - t_k)G'(t_k)}, \tag{1.4.2}$$

where $t_k = k\pi/\sigma$ and

$$G(t) = \sin(\sigma t) = \sigma t \prod_{k=1}^{\infty} \left(1 - \frac{t^2}{t_k^2}\right).$$

The fact that formula (1.4.2) resembles Lagrange interpolation prompts us to call any

¶The sampling frequency σ/π is known as the *Nyquist rate*, named after Harry Nyquist [**51**]. It is the minimum rate at which a signal needs to be sampled in order to reconstruct it properly.

series of the form

$$\sum_k f(t_k) \frac{G(t)}{(t - t_k) G'(t_k)} \tag{1.4.3}$$

a *Lagrange-type interpolation series*, where $G(t)$ is an entire function whose zeros are located exactly at the points $\{t_k\}$. The points will be called the sampling points and the functions

$$G_k(t) = \frac{G(t)}{G'(t_k)(t - t_k)}, \tag{1.4.4}$$

will be called the sampling functions. The value of t_0 is often taken to be zero.

1.4.1 Historical recount

There are few names associated with the above sampling theorem. The most common ones among the mathematical community are the *Whittaker-Kotenl'nikov-Shannon* or simply *WKS/WSK* sampling theorem. The name is attributed to the two Whittakers; Edmund and (his son) John, Vladmir Kotenl'nikov[‖], and Claude Shannon. Among the Engineering community its known as *Shannon's sampling theorem* in honor of Shannon's revelationary paper **[60]** which put its mark in communications theory.

The series appeared in the sampling theorem (1.4.1) is known as the *cardinal series* or *Whittaker's interpolation series* in honor of John Whittaker, whose work in **[71]** was a refinement of the work done earlier by his father. Edmund Whittaker published his highly cited paper **[69]** on the sampling theorem in 1915. In his work, among other things, he introduced the term *cotabular functions* to refer to functions which have the same uniformly spaced samples.

[‖] In 1933, introduced the sampling theorem to the Russian literature in the setting of communication engineering.

Some absorbing accounts of the history of the sampling theorem and comprehensive studies of sampling theory can be found in the works of J. R. Higgins [**32**], A. J. Jerri [**36**], Robert J. Marks II [**47**], Ahmed I. Zayed [**72**], and references therein.

Sampling theorem has been extended and generalized in many avenues; to non-uniform sampling, sampling with non-bandlimited signals, multi-dimensional sampling etc. to name a few.

1.4.2 Significance of the sampling theorem in mathematics

Even though having already made its mark in communication engineering and information theory, the sampling theorem itself or even its equivalent forms play a unique role in several branches of mathematics, directly and indirectly (see [**20**], [**33**]). The direct impact being in the fields of combinatorics, reproducing kernel Hilbert space, frame theory, etc. The more general form of the sampling theorems are valid not only for band-limited functions, but also are shown to be equivalent to three fundamental theorems in mathematics.

1. Poisson summation formula (of Fourier analysis)
2. Cauchy's integral formula (of Complex function theory)
3. Euler-Maclaurin summation formula (of Numerical analysis)

The aforementioned equivalence is in the sense that each stated formula can be obtained from the sampling theorem by elementary methods. Because of these indirect connections, sampling theorem becomes applicable in a broad spectrum.

One of the recent developments of sampling theory is in the field of special functions, where the sampling theorem has proven to be a bona fide tool in summing infinite series.

(see [**72, Chapter 7**] and references therein)

1.5 Gosper's ingenious contribution in discovering series identities

In the early 1980s, using some computer experimentation with the package Macsyma, R. William Gosper formulated several infinite series identities involving trigonometric functions. Through indirect communications, Gosper passes the message about his identities with some interested parties. Motivated from this, in 1993, Mourad Ismail and Ruming Zhang (with Gosper himself) in [**27**] verified several of those identities utilizing techniques from Fourier transform and Mittag-Leffler expansions for meromorphic functions. In fact, Ismail and Zhang extended some of the identities from trigonometric to Bessel functions of the first kind. After this work, in the same year, A. I. Zayed [**73**] proved that some of Gosper's formulas and their generalizations by Ismail and Zhang can in fact be obtained from already known results in Sampling theory. In addition, applying those results to different types of special functions, Zayed derived some new summation formulas as well.

CHAPTER 2: FORMULATION OF THE PROBLEMS AND OUTLINE OF THE DISSERTATION

The formulation of the problem is often more essential than its solution, which may be merely a matter of mathematical or experimental skill.

Albert Einstein

In this chapter we narrow down the contents mentioned in Chapter 1 and focused on the formulation of problems of the dissertation with the main theme throughout being the *Bernstein inequality.* For the stated problems, the subsequent chapters will comprise of results which provide the solutions in most cases or partial solutions in some cases.

2.1 A brief review of some generalizations of Bernstein inequality

For $p \in \mathcal{P}_n$, recall the Bernstein inequality for the unit circle:

$$\|p'\|_{\mathbb{T}} \leq n \, \|p\|_{\mathbb{T}} \, , \tag{2.1.1}$$

which is clearly sharp and the equality holds if $p(z) = cz^n$ where c is a constant, i.e., a polynomial whose n zeros are at the origin.

So it is natural to seek for improvements, let alone generalizations and extensions of (2.1.1) by restricting the zeros of the polynomial. For starters, by considering the class of polynomials which does not vanish in $|z| \leq 1$, Paul Erdös conjectured and Peter Lax proved the following [**40**]:

$$\|p'\|_{\mathbb{T}} \leq \frac{n}{2} \, \|p\|_{\mathbb{T}} \, , \tag{2.1.2}$$

which is sharp and equality holds in (2.1.2) for the polynomial $p(z) = \alpha + \beta z^n$, where $|\alpha| = |\beta| = 1$.

A reverse inequality to (2.1.2) can be obtained by restricting the zeros of the polynomials to inside the unit disk. In **[65]**, by considering the class of polynomials which does not vanish in $|z| \geq 1$, Pál Turán proved the following:

$$\|p'\|_{\mathbb{T}} \geq \frac{n}{2} \|p\|_{\mathbb{T}}, \tag{2.1.3}$$

which is sharp and equality holds in (2.1.3) if all the zeros of $p(z)$ lie on $|z| = 1$.

The aforementioned inequalities are the cornerstone of our study. Over the years, several generalizations and extensions had been obtained. Among those results what we focused on were Bernstein-type inequalities for polynomials when zeros were restricted to disks smaller and larger than the unit disk. What follows next is a brief literature review along this direction.

First, in 1969, Mohammad Abdul Malik **[45]** established the following generalizations to (2.1.2) and (2.1.3) respectively.

Theorem 2.1.1. *For $p \in \mathcal{P}_n$ with $|p(z)| \leq 1$ on $|z| \leq 1$ and if $p(z)$ has no zero in the disk $|z| < k,\ k \geq 1,$ then*

$$|p'(z)| \leq \frac{n}{1+k}, \tag{2.1.4}$$

holds with equality for the polynomial $p(z) = \left(\dfrac{z+k}{1+k}\right)^n$.

Theorem 2.1.2. *For $p \in \mathcal{P}_n$ with all its zeros in $|z| \leq k \leq 1$,*

$$\|p'\|_{\mathbb{T}} \geq \frac{n}{1+k} \|p\|_{\mathbb{T}}, \tag{2.1.5}$$

holds with equality for the polynomial $p(z) = \left(\frac{z+k}{1+k}\right)^n$.

Few years later, in 1973, an interesting, alternative proof for (2.1.5) was presented by Narendra Kumar Govil in **[28]**. In addition, to answer to the question: *What happens to* (2.1.5) *for* $k > 1$?, he proved the following:

Theorem 2.1.3. *For* $p \in \mathcal{P}_n$ *with* $\|p\|_{\mathbb{T}} = 1$ *and if* $p(z)$ *has all its zeros in* $|z| \leq k,\ k \geq 1$, *then*

$$\|p'\|_{\mathbb{T}} \geq \frac{n}{1+k^n}, \tag{2.1.6}$$

with equality for the polynomial $p(z) = \frac{z^n + k^n}{1+k^n}$.

So for $k < 1$, the extremal polynomial is $p(z) = \left(\frac{z+k}{1+k}\right)^n$, while for $k > 1$, the extremal polynomial is $p(z) = \frac{z^n+k^n}{1+k^n}$. The critical value being 1. A smooth transition from $k < 1$ to $k > 1$ is yet to be addressed properly, which we would like to mention as a conjecture.

For $p \in \mathcal{P}_n$, by *Bernstein lemma* we refer to the following inequality (see **[53, p.158, Problem 269]**): for $R > 1$,

$$\max_{|z|=R} |p(z)| \leq R^n \|p\|_{\mathbb{T}}. \tag{2.1.7}$$

As a sharpening to (2.1.7), using the Erdos-Lax inequality, (2.1.2), in **[3]** Nesmith C. Ankeny and Theodore J. Rivlin proved the following:

Theorem 2.1.4. *For* $p \in \mathcal{P}_n$ *such that* $\|p\|_{\mathbb{T}} = 1$, *with no zeros in* $|z| \leq 1$, *then*

$$\max_{|z|=R} |p(z)| \leq \frac{1+R^n}{2}, \tag{2.1.8}$$

with equality for the polynomial $p(z) = \dfrac{\lambda + \mu z^n}{2}$, *where* $|\lambda| = |\mu| = 1$.

As a further sharpening to the Erdös-Lax inequality, (2.1.2), in 1988, Abdul Aziz and Q. M. Dawood in **[7]** proved the following:

$$\|p'\|_{\mathbb{T}} \leq \frac{n}{2}\Big\{ \|p\|_{\mathbb{T}} - \min_{|z|=1} |p(z)| \Big\}. \tag{2.1.9}$$

The inequality is sharp for $p(z) = \alpha z^n + \beta$, where $|\beta| \geq |\alpha|$. As an application to (2.1.9), they also proved the following generalization to (2.1.8).

Theorem 2.1.5. *For* $p \in \mathcal{P}_n$ *with no zeros in* $|z| < 1$*, then*

$$\max_{|z|=R} |p(z)| \leq \left(\frac{R^n + 1}{2}\right) \|p\|_{\mathbb{T}} - \left(\frac{R^n - 1}{2}\right) \min_{|z|=1} |p(z)|\,, \tag{2.1.10}$$

with equality for the polynomial $p(z) = \alpha z^n + \beta$, *where* $|\beta| \geq |\alpha|$.

In the same paper **[7]**, by considering the minimum modulus of a polynomial which does not vanish in $|z| \geq 1$, Aziz and Dawood established couple of inequalities, which can be regarded as *companion* inequalities for the Bernstein inequality (2.1.1) and Bernstein lemma (2.1.7) respectively:

$$\min_{|z|=1} |p'(z)| \geq n \min_{|z|=1} |p(z)|\,, \tag{2.1.11}$$

$$\min_{|z|=R} |p(z)| \geq R^n \min_{|z|=1} |p(z)|\,. \tag{2.1.12}$$

Both estimates are sharp with equality for the polynomial $p(z) = me^{i\alpha} z^n, \quad m > 0$.

In the dissertation, we were interested in generalizing the aforementioned inequalities, from (2.1.4) through (2.1.12) to rational functions. Chapter 3 is devoted to accomplish this task, where we considered the space of rational functions whose poles were prescribed to outside the unit disk and the zeros were restricted in conformity with the hypotheses of the corresponding polynomial problem. Our results were proved through some auxiliary results, which are pertinent in their own right.

2.2 Askey-Wilson operator on Polynomials and Entire functions

Askey-Wilson operator, $\mathscr{D}_q$ is a discretized version of the ordinary derivative operator. Bernstein inequality being the main focus, our objective was to obtain Bernstein inequality(ies) for $\mathscr{D}_q$. The motivation to our approach being the knowledge that the Bernstein inequalities are often proved through a legitimate interpolation formula, as in the cases of M. Riesz [**57**] and R. P. Boas Jr. [**14**] in establishing Bernstein inequalities for trigonometric polynomials and functions of exponential type respectively. This motivated us to work in obtaining similar interpolation formulas with the Askey-Wilson operator, which will ultimately lead us to the Bernstein inequalities we need. Once Bernstein inequalities are obtained, they can be used in related results, especially in proving inverse approximation theorems. In view of the uniform convergence of the Boas' interpolation formula by differentiating it term by term and with the use of a suitable translation to the variable leads to the (classical) sampling theorem, which can be used in deriving summation identities.

In Chapter 4, we first establish a Riesz-type interpolation formula on the interval $[-1, 1]$ for the Askey-Wilson operator. As consequences, a sharp Bernstein inequality and a Markov inequality are obtained when differentiation is replaced by the Askey-Wilson operator. Moreover, an inverse approximation theorem is proved using a Bernstein type inequality in L^2-space. We conclude chapter 4 with an overconvergence result which is applied to characterize all q-differentiable functions of Brown and Ismail.

Chapter 5 presents an intriguing application of the Askey-Wilson operator. By applying it on the Classical Sampling Theorem, we obtain a series representation formula, which is what we called an *extended Boas' formula*. Its power in **discovering** interesting *summation formulas*, some known and some new will be demonstrated. The chapter is concluded in establishing Bernstein-type inequalities with the Askey-Wilson operator for functions of exponential type in pointwise and in uniform L^p-norm for $p \geq 1$.

2.3 Publication based on results of this dissertation

Finally, in this section, we list the papers prepared based on results established in this dissertation. Some papers have been accepted, some submitted, and some in preparation.

1. Some of the ideas appear in Chapter 3 was appeared under the title *Rational Inequalities Inspired by Rahman's Research* in [**42**].

2. The contents which comprise the material in Chapter 4 has already been submitted to the Journal of Approximation Theory under the title *A Bernstein Type Inequality for the Askey-Wilson Operator*. At the time of writing this dissertation, we have received a favorable report from the editor.

3. The subject matter in Chapter 5 except the last section appeared in [**44**], Proceedings of American Mathematical Society under the title *Askey-Wilson Operator on Entire Functions of Exponential Type*, which has already been accepted and is made available online.

4. Material that appeared in section 5.5 is in preprint under the title *Bernstein inequality for functions of exponential type with the Askey-Wilson operator*.

CHAPTER 3: SOME NEW INEQUALITIES FOR RATIONAL FUNCTIONS WITH PRESCRIBED POLES AND RESTRICTED ZEROES

"*There are three reasons for the study of inequalities: practical, theoretical, and aesthetic. As has been pointed out, beauty is in the eyes of the beholder. However, it is generally agreed certain pieces of music, art, or mathematics are beautiful. There is an elegance to inequalities that makes them very attractive.*"

Richard Bellman

3.1 Rational Functions

Recall that by $\mathcal{P}_n$, we denote the space of complex algebraic polynomials of degree at most n and by $\mathcal{R}_n$ the space of rational functions with at most n poles, $a_1, a_2, \ldots, a_n$ with a finite limit at ∞. Let $w(z) = \prod_{j=1}^{n}(z - a_j)$, and

$$\mathcal{R}_n := \left\{ r : r = \frac{p}{w},\ p \in \mathcal{P}_n \right\}.$$

For $p \in \mathcal{P}_n$, define the *inverse polynomial* of p by $p^*(z) := z^n\, \overline{p\left(\frac{1}{\overline{z}}\right)}$. So for w, we have

$$w^*(z) = \prod_{j=1}^{n}(1 - \overline{a_j} z).$$

Also, let

$$B(z) := \frac{w^*(z)}{w(z)} = \frac{z^n \overline{w(1/\overline{z})}}{w(z)} = \prod_{j=1}^{n} \frac{1 - \overline{a_j} z}{z - a_j}.$$

So $B(z)$ is a *finite Blaschke product* of degree n. In particular, $B(z) \in \mathcal{R}_n$ and $|B(z)| = 1$ when $|z| = 1$. For $r \in \mathcal{R}_n$, the *inversion* r^* of r is defined by

$$r^*(z) := B(z)\, \overline{r\left(\frac{1}{\bar{z}}\right)}.$$

If $r = p/w$, then $r^* = p^*/w$ and hence $r^* \in \mathcal{R}_n$. The derivative of $r^*(\cdot)$ will be denoted by $(r^*)'(\cdot)$. Throughout this chapter, we copiously consider the *sup-norm* on the unit circle. Henceforth we simply use $\|\cdot\|$ as oppose to $\|\cdot\|_{\mathbb{T}}$, introduced in Chapter 1.

3.2 Overview

One of the key directions in generalizing Bernstein-type inequalities for polynomials is to the space of rational functions, which, over the years has gained much interest. In this connection, Xin Li, Ram Mohaparta, Rene S. Rodriguez in [**43**] established some significant results, which were cornerstone for many authors to follow; to name a few, A. Aziz, B. A. Zarger, W. M. Shah (see [**8**], [**9**], [**10**]) obtained rational analogues by considering restricted zeros and prescribed poles.

Polynomials can be viewed as rational functions whose distinct poles are all fixed at infinity. So an inequality for rational function reduces to its polynomial counterpart in the limit as all poles approach to ∞ $(r(z) \to p(z), B(z) \to z^n, |B'(z)| \to n$ etc.$)$. To establish rational analogues for the corresponding polynomial inequalities, the proofs of the polynomial counterparts cannot just be imitated; the key observation being the presence of the poles. Also, the polynomial inequalities are often proved through as an application of a Laguerre's theorem* or Grace's theorem† or even as an equivalent form of the two, which are not readily available for rational functions. In this connection, Frank Bonsall and Morris Marden ([**17, Theorem 1**]) established a result, which states that the counting of the critical points of a rational function depends not only on its number of zeros but also on its distinct number of poles. So there is no direct extension of Laguerre's theorem and Grace's theorem to rational functions. So, we have to find alternative proofs for rational case.

In the next section, we state rational analogues to the polynomial inequalities stated in the previous chapter (Section 2.1). Although, some of our results are not sharp, they do reduce to their polynomial counterparts in the limit as all poles approach to infinity.

3.3 Statements of our results

Our first result is a rational analogue of the Ankeny-Rivlin inequality, (2.1.8).

Theorem 3.3.1. *Let $r \in \mathcal{R}_n$ with no zeros in $|z| \le 1$ and let $\hat{R} := \min_j\{|a_j|\}$. Then for $|z| = R \ge 1$,*

$$\max_{|z|=R} |r(z)| \le \|r\| + \|r\| \left\{ \frac{\|B'\|}{2}\left(\frac{R^n - 1}{n}\right) + \frac{2n}{(\hat{R}-R)}\left(\frac{R^{n+1}-1}{n+1}\right) \right\} \left(\frac{\hat{R}+\frac{1}{R}}{\hat{R}-R}\right)^n . \quad (3.3.1)$$

---

*[**39**] A *circular domain* is the image of the unit disk (open or closed) under a linear transformation. Laguerre's theorem states that: For $p \in \mathcal{P}_n$ and $p(z) \neq 0$ in a (closed or open) circular domain K, then

$$np(z) - (z-\zeta)p'(z) \neq 0 \quad \text{for } z, \zeta \in K$$

which in the case $\zeta = \infty$ means $p'(z) \neq 0$ for $z \in K$.

†[**30**] Two polynomials $p, q \in \mathcal{P}_n$ are *apolar* if

$$\sum_{k=0}^{n} (-1)^k p^k(0) g^{(n-k)}(0) = 0.$$

For such two apolar polynomials, Grace's theorem states that every circular domain that contains all the zeros of one of them contains at least one zero of the other.

Remark 3.3.2. In the limit as all poles approach to infinity, i.e., $a_j \to \infty$, for $j = 1, 2, \ldots, n$, $\hat{R} \to \infty$. So (3.3.1) is a limiting case of (2.1.8).

The next two pointwise estimates are rational analogues of Aziz and Dawoods's results, (2.1.11) and (2.1.12).

Theorem 3.3.3. *Assume that $r \in \mathcal{R}_n$ has all its zeros in $|z| \leq 1$. Then*

$$|r'(z)| \geq |B'(z)| \min_{|z|=1} |r(z)| \quad \text{for } |z| = 1. \tag{3.3.2}$$

and

$$|r(z)| \geq |B(Re^{i\theta})| \min_{|z|=1} |r(z)| \quad \text{for } |z| = R \geq 1. \tag{3.3.3}$$

Remark 3.3.4. Inequality (3.3.2) can be rewritten as

$$\frac{|B'(z)|}{\min_{|z|=1} |B(z)|} = \inf_{r \in \mathcal{R}_n^*} \frac{|r'(z)|}{\min_{|z|=1} |r(z)|}, \tag{3.3.4}$$

where $\mathcal{R}_n^*$ denotes the set of all rational functions, $\mathcal{R}_n$, with zeros in $|z| \leq 1$.

A. Aziz and W. M. Shah in **[8]** generalized Malik's inequality, (2.1.5), by proving the following:

Theorem 3.3.5. *Suppose $r \in \mathcal{R}_n$ has all its zeros in $|z| \leq k \leq 1$, then for $|z| = 1$ the following holds:*

$$|r'(z)| \geq \left(\frac{|B'(z)|}{2} + \frac{n}{2} \frac{1-k}{1+k} \right) |r(z)| \ . \tag{3.3.5}$$

The result is sharp and equality holds for $r(z) = \left(\frac{z+k}{z-a} \right)^n$, where $a > 1$, and $B(z) = \left(\frac{1-az}{z-a} \right)^n$ evaluated at $z = 1$.

A. Aziz and B. A. Zarger in **[10]** obtained a generalization of one of Malik's inequalities, (2.1.4), by proving the following:

Theorem 3.3.6. *Suppose $r \in \mathcal{R}_n$ has all its zeros in $|z| \geq K$, where $K \geq 1$, with $\|r\| = 1$, then*

$$|r'(z)| \leq \frac{|B'(z)|}{2} - \frac{n}{2}\frac{(K-1)}{K+1}\,|r(z)|^2 \text{ for } |z| = 1. \tag{3.3.6}$$

The result is sharp and equality holds for $r(z) = \left(\frac{z+K}{z-a}\right)^n$, *where* $a > 1$, *and* $B(z) = \left(\frac{1-az}{z-a}\right)^n$ *evaluated at* $z = 1$.

Our next result is a different version of (3.3.6).

Theorem 3.3.7. *Let $r \in \mathcal{R}_n$, $|r(z)| \leq 1$ for $|z| \leq 1$, with no zeros in $|z| \leq K, K \geq 1$, then*

$$|r'(z)| \leq \left(1 - \frac{|r(z)|}{2}\right)|B'(z)| - \frac{n}{2}\left(\frac{K-1}{K+1}\right)|r(z)| \quad \text{for } |z| = 1. \tag{3.3.7}$$

Remark 3.3.8. We comment on comparisons between (3.3.7) and (3.3.6).

(a) If $|B'(z)| > n\frac{K-1}{K+1}$, then their result is better for z such that

$$|r(z)| < 1. \tag{3.3.8}$$

(b) If $|B'(z)| < n\frac{K-1}{K+1}$, then our result is better for z such that

$$\frac{|B'(z)|}{n}\,\frac{K+1}{K-1} < |r(z)| < 1. \tag{3.3.9}$$

and their result is better for z such that

$$|r(z)| < \frac{|B'(z)|}{n}\,\frac{K+1}{K-1} \tag{3.3.10}$$

To illustrate this we consider the single pole case, where all poles are at $a = \tilde{r}e^{i\theta},\ \tilde{r} > 1$. Then taking the logarithmic derivative of $B(z)$,

$$\frac{zB'(z)}{B(z)} = \sum_{j=1}^{n} \frac{|a_j|^2 - 1}{|z - a_j|^2} = n\frac{\tilde{r}^2 - 1}{|z - \tilde{r}e^{i\theta}|^2},$$

which for $|z| = 1$,

$$\frac{|B'(z)|}{n} = \frac{\tilde{r}^2 - 1}{|z - \tilde{r}e^{i\theta}|^2}.$$

Using this in (3.3.9),

$$\frac{\tilde{r}^2 - 1}{|z - \tilde{r}e^{i\theta}|^2} < \frac{K - 1}{K + 1},$$

which for $|z| = 1$, yields $0 < K - r(K-1)\cos(\theta) - r^2$. This inequality holds for all θ such that $0 \le \theta < 2\pi$. So in particular, if $\theta = \pi/2$, then $r < \sqrt{K}$. So (3.3.7) is better than (3.3.6) when $r < \sqrt{K}$.

We generalized the other Malik's inequality, (2.1.5) by proving the following.

Theorem 3.3.9. *Let $r \in \mathcal{R}_n$ and assume that all its n zeros are in $|z| \le k \le 1$. Then*

$$|r'(z)| \ge \left(\frac{|B'(z)|}{2} + \frac{n}{2}\,\frac{1-k}{1+k}\right)|r(z)| \quad \text{for } |z| = 1. \tag{3.3.11}$$

Theorem 3.3.10. *Let $r \in \mathcal{R}_n$ with all its zeros in $|z| \le K,\ 1 \le K^2 < \hat{R}$, where $\hat{R} = \min_j |a_j|$. Then, for $0 \le \theta < 2\pi$, the following holds:*

$$\|r'\| \ge \frac{\min_{|z|=1} |B'(z)| - \|f\|}{1 + \max_{|z|=1} \left|\frac{B(K^2 z)}{Q(Kz)}\right|} \|r\|, \tag{3.3.12}$$

where $f(z) = \dfrac{2nK^2}{\hat{R} - R}\left|\dfrac{B(K^2 z)}{Q(Kz)}\right| + K\left|\dfrac{Q'(Kz)}{Q(Kz)}\right|$ *with* $Q(z) = \dfrac{B_K(z)}{B(z/K)} = \displaystyle\prod_{j=1}^{n} \frac{z - a_j K}{Kz - a_j}$. *Here*

$B_K(z)$ is the Blaschke product associated with the rational function $r(Kz)$ and is defined by $B_K(z) = \prod_{j=1}^{n} \frac{1 - \overline{a_j/K}z}{z - a_j/K}$.

Remark 3.3.11. Though our result is not sharp, it does reduce to its polynomial inequality, (2.1.6) when $a_j \to \infty$ for all j. The condition $K^2 < \hat{R}$ is necessary to make sure that no overlapping occurs when points on the unit circle moves to $|z| = K > 1$ and when poles shrink towards the unit circle due to the condition $|a_j|/K < 1$.

3.4 Proofs

In this section we present some auxiliary results which were used in establishing rational analogues of the results stated in the previous section. Our proofs adapt the ideas of Govil [**28**]; Li, Mohapatra and Rodriguez [**43**]; and Li [**41**].

We first prove a pointwise estimate which is a rational analogue of the *Bernstein lemma* for polynomials, (2.1.7).

Lemma 3.4.1 (Bernstein Lemma for $\mathcal{R}_n$)**.** *Let $r \in \mathcal{R}_n$, then*

$$|r(z)| \leq |B(z)| \, \|r\| \quad \textit{for } |z| \geq 1. \tag{3.4.1}$$

Indeed, a sharpened version of the above result is needed. By $\mathcal{P}_{n-m}$, $m \geq 0$, we denote the set of polynomials of degree $n - m$.

Lemma 3.4.2 (Generalized Bernstein lemma for $\mathcal{R}_n$)**.** *Let $r \in \mathcal{R}_n$ and write $r(z) = \dfrac{p(z)}{w(z)}$, where $p \in \mathcal{P}_{n-m}$ for some $m \geq 0$. Then*

$$|r(z)| \leq \left| \frac{B(z)}{z^m} \right| \|r\| \quad \textit{for } |z| \geq 1. \tag{3.4.2}$$

Proof. Note that

$$\frac{z^m r(z)}{B(z)} = \frac{z^m\ p(z)}{\prod_{k=1}^{n} 1 - \overline{a_k} z},$$

is a polynomial analytic in $|z| \geq 1$ including the point at ∞.

We have,

$$\max_{|z|=1} \left| \frac{z^m\ r(z)}{B(z)} \right| \geq \max_{|z|=R} \left| \frac{z^m\ r(z)}{B(z)} \right|$$

or, equivalently

$$|r(z)| \leq \left| \frac{B(z)}{z^m} \right| \cdot \|r\| \ \text{ for } |z| \geq 1.$$

□

Remark 3.4.3. Taking $m = 0$, we obtain Lemma 3.4.1.

Following result is due to Li et al.(see [**43, Theorem 3**]), which is a generalization of Erdös-Lax inequality for rational functions.

Lemma 3.4.4. *Let $r \in \mathcal{R}_n$ with all its zeros in $|z| \geq 1$. Then, for $|z| = 1$,*

$$|r'(z)| \leq \frac{1}{2} |B'(z)| \, \|r\| \,. \tag{3.4.3}$$

Equality in (3.4.3) *holds for $r(z) = \alpha B(z) + \beta$ with $|\alpha| = |\beta| = 1$.*

The key to our proof of Theorem 3.3.1 is the following Bernstein-type lemma for the derivative of a rational function, $r'(z)$, for $|z| > 1$.

Lemma 3.4.5. *Let $r \in \mathcal{R}_n$ with all its zeros in $|z| \geq 1$ and let $\hat{R} = \min_j\{|a_j|\}$, for $j = 1, \ldots, n$. Then for $|z| \geq 1$,*

$$|r'(z)| \leq \frac{1}{2} \, \|B'\| \, \|r\| \, \frac{|B(z)|}{|z|} + \frac{2n\|r\|}{\hat{R} - |z|} |B(z)|. \tag{3.4.4}$$

Proof. Note that

$$r'(z) = \frac{p'(z)w(z) - w'(z)p(z)}{w^2(z)} = \frac{p'(z)}{w(z)} - \frac{p(z)}{w(z)}\frac{w'(z)}{w(z)}.$$

So

$$\left|\frac{p'(z)}{w(z)}\right| = \left|r'(z) + r(z)\frac{w'(z)}{w(z)}\right| \leq |r'(z)| + |r(z)|\left|\frac{w'(z)}{w(z)}\right|. \tag{3.4.5}$$

and

$$|r'(z)| = \left|\frac{p'(z)}{w(z)} - \frac{p(z)}{w(z)}\frac{w'(z)}{w(z)}\right| \leq \left|\frac{p'(z)}{w(z)}\right| + |r(z)|\left|\frac{w'(z)}{w(z)}\right|. \tag{3.4.6}$$

To estimate $|r'(z)|$ for $|z| \geq 1$, we estimate the two terms on the right of (3.4.6) for $|z| \geq 1$. First consider $\left|\frac{p'(z)}{w(z)}\right|$. Applying Lemma 3.4.2 with $m = 1$, we get

$$\left|\frac{p'(z)}{w(z)}\right| \leq \left|\frac{B(z)}{z}\right| \ \left\|\frac{p'}{w}\right\| \quad \text{for } |z| \geq 1. \tag{3.4.7}$$

Estimating (3.4.5) on $|z| = 1$ and using it on the right hand side of (3.4.7), we get

$$\left|\frac{p'(z)}{w(z)}\right| \leq \frac{|B(z)|}{|z|} \ \left(\|r'\| + \|r\| \left\|\frac{w'}{w}\right\|\right) \text{ for } |z| \geq 1. \tag{3.4.8}$$

Now, using (3.4.8) in (3.4.6), we get

$$|r'(z)| \leq \left(\|r'\| + \|r\| \ \left\|\frac{w'}{w}\right\|\right)\frac{|B(z)|}{|z|} + |r(z)|\left|\frac{w'(z)}{w(z)}\right| \text{ for } |z| \geq 1.$$

Next, to estimate $|r(z)|$ (for $|z| \geq 1$) on the right hand side above, we use Lemma 3.4.1; which yields

$$|r'(z)| \leq \left\{\left(\|r'\| + \|r\|\left\|\frac{w'}{w}\right\|\right)\frac{1}{|z|} + \|r\|\left|\frac{w'(z)}{w(z)}\right|\right\}|B(z)|.$$

Applying Lemma 3.4.4 to estimate $|r'(z)|$ on $|z| = 1$, we get

$$|r'(z)| \leq \left\{ \left(\frac{1}{2}\|B'\|\|r\| + \|r\| \left\|\frac{w'}{w}\right\|\right) \frac{1}{|z|} + \|r\| \left|\frac{w'(z)}{w(z)}\right| \right\}|B(z)| \text{ for } |z| \geq 1. \tag{3.4.9}$$

Now, note that, for $\zeta = Re^{i\theta}$, $R \geq 1$, we have

$$\left|\frac{w'(\zeta)}{w(\zeta)}\right| = \left|\frac{w'(Re^{i\theta})}{w(Re^{i\theta})}\right| = \left|\sum_{j=1}^{n} \left(\frac{1}{Re^{i\theta} - a_j}\right)\right| \leq \sum_{j=1}^{n} \frac{1}{|a_j| - R} \leq \frac{n}{\hat{R} - R},$$

provided $R < \hat{R} \leq |a_j|$. That is, for $|\zeta| \geq 1$, we have

$$\left|\frac{w'(\zeta)}{w(\zeta)}\right| \leq \frac{n}{\hat{R} - |\zeta|}. \tag{3.4.10}$$

Since $\dfrac{w'(z)}{w(z)} = \displaystyle\sum_{j=1}^{n} \frac{1}{z - a_j}$ is analytic in $|z| \leq R$, by maximum modulus principle it follows that

$$\left|\frac{w'(z)}{w(z)}\right| \leq \max_{|z|=R} \left|\frac{w'(z)}{w(z)}\right| \text{ for } |z| \leq R.$$

In particular, with (3.4.10), it follows that

$$\max_{|z|=1} \left|\frac{w'(z)}{w(z)}\right| \leq \max_{|z|=R} \left|\frac{w'(z)}{w(z)}\right| \leq \frac{n}{\hat{R} - R}. \tag{3.4.11}$$

Note that, with (3.4.11), for $|z| = R \geq 1$, we obtain

$$\begin{aligned} \|r\| \frac{|B(z)|}{|z|} \max_{|\zeta|=1} \left|\frac{w'(\zeta)}{w(\zeta)}\right| + \|r\| \, |B(z)| \max_{|\zeta|=R} \left|\frac{w'(\zeta)}{w(\zeta)}\right| &\leq \left\{\frac{1}{R} + 1\right\} \frac{n}{\hat{R} - R} \|r\| \, |B(z)| \\ &\leq \frac{2n|B(z)|}{\hat{R} - R} \cdot \|r\|. \end{aligned}$$

Finally, using the penultimate line in (3.4.9) we obtain the desired estimate. □

Lemma 3.4.6. *Let $r \in \mathcal{R}_n$ with all its zeros in $|z| \geq 1$. Then for $|z| = R \geq 1$,*

$$\left|\left(\frac{r(z)}{B(z)}\right)'\right| \leq \left\{\frac{\|B'\|}{2} + \left\|\frac{w'}{w}\right\|\right\} \cdot \frac{\|r\|}{R} + \left|\frac{(w^*)'(z)}{w^*(z)}\right| \cdot \|r\|. \tag{3.4.12}$$

Proof. Note that

$$\begin{aligned}\left|\left(\frac{r(z)}{B(z)}\right)'\right| = \left|\left(\frac{p(z)}{w^*(z)}\right)'\right| &= \left|\frac{p'(z)}{w^*(z)} - \frac{p(z)}{w^*(z)} \cdot \frac{(w^*)'(z)}{w^*(z)}\right| \\ &\leq \left|\frac{p'(z)}{w^*(z)}\right| + \left|\frac{p(z)}{w^*(z)}\right| \cdot \left|\frac{(w^*)'(z)}{w^*(z)}\right|. \end{aligned} \tag{3.4.13}$$

Since both $zp'(z)/w^*(z)$ and $p(z)/w^*(z)$ are analytic outside $|z| = 1$ including the point at ∞, by maximum modulus principle, for $|z| > 1$ we have

$$\left|\frac{zp'(z)}{w^*(z)}\right| \leq \max_{|z|=1}\left|\frac{p'(z)}{w^*(z)}\right| \tag{3.4.14}$$

and

$$\left|\frac{p(z)}{w^*(z)}\right| \leq \max_{|z|=1}\left|\frac{p(z)}{w^*(z)}\right| \tag{3.4.15}$$

First, considering (3.4.14) on $|z| > 1$, we get

$$\begin{aligned}\left|\frac{zp'(z)}{w^*(z)}\right| \leq \left\|\frac{p'}{w^*}\right\| = \left\|\frac{p'}{w}\right\| &\leq \left\|r' + r \cdot \frac{w'}{w}\right\| \\ &\leq \|r'\| + \|r\| \cdot \left\|\frac{w'}{w}\right\| \\ &\leq \left\{\frac{\|B'\|}{2} + \left\|\frac{w'}{w}\right\|\right\} \cdot \|r\|. \end{aligned}$$

The penultimate line follows from applying Lemma 3.4.4. So for $|z| = R \geq 1$, we get

$$\left|\frac{p'(z)}{w^*(z)}\right| \leq \left\{\frac{\|B'\|}{2} + \left\|\frac{w'}{w}\right\|\right\} \cdot \frac{\|r\|}{R}. \tag{3.4.16}$$

Next, considering (3.4.15) on $|z| > 1$, we get

$$\left|\frac{p(z)}{w^*(z)}\right| \le \left\|\frac{p}{w^*}\right\| = \left\|\frac{p}{w}\right\| = \|r\| \tag{3.4.17}$$

Now, using (3.4.16) and (3.4.17) in (3.4.13), the desired result follows. □

Lemma 3.4.7. *Let $r \in \mathcal{R}_n$ with no zeros in $|z| \le 1$ and let $\beta_j = e^{-i\theta}/\overline{a_j}$, for $j = 1, \ldots, n$. Then for $|z| = R \ge 1$,*

$$\max_{|z|=R}\left|\frac{r(z)}{B(z)}\right| \le \|r\| + \left\{\frac{\|B'\|}{2} + \left\|\frac{w'}{w}\right\|\right\}\frac{\|r\|\,(R-1)}{R} + n\ \mathrm{Re}\left(\ln\left(\frac{R-\beta}{1-\beta}\right)\right), \tag{3.4.18}$$

where $\beta := \min_j |\beta_j|$.

Proof. For $0 \le \theta < 2\pi$ and $R > 1$, we have

$$\frac{r(Re^{i\theta})}{B(Re^{i\theta})} - \frac{r(e^{i\theta})}{B(e^{i\theta})} = \int_1^R e^{i\theta}\left(\frac{r(te^{i\theta})}{B(te^{i\theta})}\right)' dt.$$

Using Lemma 3.4.6 it follows that

$$\begin{aligned}\left|\frac{r(Re^{i\theta})}{B(Re^{i\theta})} - \frac{r(e^{i\theta})}{B(e^{i\theta})}\right| &\le \int_1^R \left|\left(\frac{r(te^{i\theta})}{B(te^{i\theta})}\right)'\right| dt \\ &\le \left\{\frac{\|B'\|}{2} + \left\|\frac{w'}{w}\right\|\right\} \cdot \frac{\|r\|(R-1)}{R} + \|r\| \cdot \int_1^R \left|\frac{(w^*)'(te^{i\theta})}{w^*(te^{i\theta})}\right| dt. \end{aligned} \tag{3.4.19}$$

As mentioned in the introduction, for $w(z) = \prod_{j=1}^{n}(z - a_j)$, its inverse polynomial $w^*(z) = \prod_{j=1}^{n}(1 - \overline{a_j}z)$. Taking the logarithmic derivative of $w^*(z)$, and integrating from 1 to R we get the following:

$$\begin{aligned}
\int_1^R \left|\frac{(w^*)'(te^{i\theta})}{w^*(te^{i\theta})}\right| dt &\le \int_1^R \sum_{j=1}^n \frac{1}{|t-\beta_j|} dt \quad \text{for } \beta_j = e^{-i\theta}/\overline{a_j} \\
&= \sum_{j=1}^n \int_1^R \frac{1}{|t-\beta_j|} dt \\
&= \sum_{j=1}^n Re\left(\ln\left(\frac{R-\beta_j}{1-\beta_j}\right)\right). \qquad (3.4.20)
\end{aligned}$$

Now (3.4.19) yields

$$\left|\frac{r(Re^{i\theta})}{B(Re^{i\theta})}\right| \le |r(e^{i\theta})| + \left\{\frac{\|B'\|}{2} + \left\|\frac{w'}{w}\right\|\right\} \cdot \frac{\|r\|(R-1)}{R} + \sum_{j=1}^n \mathrm{Re}\left(\ln\left(\frac{R-\beta_j}{1-\beta_j}\right)\right).$$

Thus we get

$$\max_{|z|=R}\left|\frac{r(z)}{B(z)}\right| \le \|r\| + \left\{\frac{\|B'\|}{2} + \left\|\frac{w'}{w}\right\|\right\} \cdot \frac{\|r\|\,(R-1)}{R} + \sum_{j=1}^n \mathrm{Re}\left(\ln\left(\frac{R-\beta_j}{1-\beta_j}\right)\right).$$

By letting $\beta := \min_j |\beta_j|$ we obtain the desired result. □

3.4.1 Generalization of Ankeny and Rivlin's inequality

Now we are ready to prove Theorem 3.3.1.

Proof of Theorem 3.3.1. Let t be a positive real number such that $1 \le t \le |a_j|$, for $j = 1, \ldots, n$. Note that

$$|B(t)| = \prod_{j=1}^n \left|\frac{1-\overline{a_j}t}{t-a_j}\right| = t^n \prod_{j=1}^n \left|\frac{1/t-\overline{a_j}}{t-a_j}\right| = t^n \prod_{j=1}^n \left|\frac{a_j-1/t}{\overline{a_j}-t}\right|. \qquad (3.4.21)$$

Let $a = \mathfrak{r}e^{i\theta},\ \mathfrak{r} > 1,\ 0 \leq \theta < 2\pi$. Consider $\left|\dfrac{\mathfrak{r}e^{i\theta} - 1/t}{\mathfrak{r}e^{-i\theta} - t}\right|$. For $1 \leq t \leq R < \hat{R} \leq \mathfrak{r}$, we obtain the following estimates:

$$\left|\frac{\mathfrak{r}e^{i\theta} - 1/t}{\mathfrak{r}e^{-i\theta} - t}\right| \leq \frac{\mathfrak{r} + \frac{1}{t}}{\mathfrak{r} - t} \leq \frac{\hat{R} + \frac{1}{R}}{\hat{R} - R}. \tag{3.4.22}$$

Using this in (3.4.21) we get

$$|B(t)| \leq |t|^n \left(\frac{\hat{R} + \frac{1}{R}}{\hat{R} - R}\right)^n. \tag{3.4.23}$$

Now using (3.4.23) in Lemma 4 with $z = te^{i\alpha},\ 1 \leq t \leq R < \hat{R} \leq r$, we get

$$|r'(te^{i\alpha})| \leq \frac{1}{2}\|B'\|\ \|r\|\ t^{n-1} \left(\frac{\hat{R} + \frac{1}{R}}{\hat{R} - R}\right)^n + \frac{2n\,\|r\|}{\hat{R} - t}\ t^n \left(\frac{\hat{R} + \frac{1}{R}}{\hat{R} - R}\right)^n.$$

Since $1 \leq t \leq R < \hat{R},\ \hat{R} - t \geq \hat{R} - R$, so the right hand side of the above inequality is less than or equals to

$$\frac{1}{2}\|B'\|\ \|r\|\ t^{n-1} \left(\frac{\hat{R} + \frac{1}{R}}{\hat{R} - R}\right)^n + \frac{2n\,\|r\|}{\hat{R} - R}\ t^n \left(\frac{\hat{R} + \frac{1}{R}}{\hat{R} - R}\right)^n.$$

i.e.,

$$|r'(te^{i\alpha})| \leq \frac{1}{2}\|B'\|\ \|r\|\ t^{n-1} \left(\frac{\hat{R} + \frac{1}{R}}{\hat{R} - R}\right)^n + \frac{2n\,\|r\|}{\hat{R} - R}\ t^n \left(\frac{\hat{R} + \frac{1}{R}}{\hat{R} - R}\right)^n. \tag{3.4.24}$$

Note that, for $0 \leq \alpha < 2\pi$ and $R > 1$, we have

$$r(Re^{i\alpha}) - r(e^{i\alpha}) = \int_1^R e^{i\alpha} r'(te^{i\alpha})dt.$$

In accordance with (3.4.24), for $0 \leq \theta < 2\pi$ and $R > 1$, it follows that

$$\begin{aligned}
|r(Re^{i\alpha}) - r(e^{i\alpha})| &\leq \int_1^R |r'(te^{i\alpha})|dt \\
&\leq \frac{1}{2}\|B'\|\ \|r\| \left(\frac{\hat{R}+\frac{1}{R}}{\hat{R}-R}\right)^n \int_1^R t^{n-1}dt + \frac{2n\|r\|}{\hat{R}-R}\left(\frac{\hat{R}+\frac{1}{R}}{\hat{R}-R}\right)^n \int_1^R t^n dt \\
&= \left\{\frac{\|B'\|\ \|r\|}{2}\left(\frac{R^n-1}{n}\right) + \frac{2n\|r\|}{(\hat{R}-R)}\left(\frac{R^{n+1}-1}{n+1}\right)\right\}\left(\frac{\hat{R}+\frac{1}{R}}{\hat{R}-R}\right)^n.
\end{aligned}$$

Thus

$$|r(Re^{i\alpha})| \leq |r(e^{i\alpha})| + \left\{\frac{\|B'\|\ \|r\|}{2}\left(\frac{R^n-1}{n}\right) + \frac{2n\|r\|}{(\hat{R}-R)}\left(\frac{R^{n+1}-1}{n+1}\right)\right\}\left(\frac{\hat{R}+\frac{1}{R}}{\hat{R}-R}\right)^n,$$

for $0 \leq \theta < 2\pi$ and $R > 1$. Consequently we obtain

$$\max_{|z|=R}|r(z)| \leq \|r\| + \left\{\frac{\|B'\|\ \|r\|)}{2}\left(\frac{R^n-1}{n}\right) + \frac{2n\|r\|}{(\hat{R}-R)}\left(\frac{R^{n+1}-1}{n+1}\right)\right\}\left(\frac{\hat{R}+\frac{1}{R}}{\hat{R}-R}\right)^n,$$

as desired. □

3.4.2 Generalization of Aziz and Dawood's inequalities

To prove Theorem 3.3.3 we first need the following result of Li ([**41, Theorem 3.1**]).

Lemma 3.4.8. *Let $r, s \in \mathcal{R}_n$ and assume that s has all its n zeros in $|z| \leq 1$, and $|r(z)| \leq |s(z)|$ for $|z| = 1$. Then $|r'(z)| \leq |s'(z)|$ for $|z| = 1$.*

Proof of Theorem 3.3.3. Let $\tilde{m} := \min_{|z|=1} |r(z)|$. For any complex number α such that $|\alpha| < 1$, we have

$$|\alpha \cdot \tilde{m} \cdot B(z)| = |\alpha| \cdot \tilde{m} < \tilde{m} \leq |r(z)| \quad \text{for } |z| = 1.$$

Since r has all its zeros in $|z| \leq 1$, by Lemma 3.4.8, it follows that

$$|\alpha \tilde{m} \cdot B'(z)| \leq |r'(z)| \;\text{ for } |z| = 1.$$

Letting $\alpha \to 1$, we obtain $\tilde{m}\ |B'(z)| \leq |r'(z)|$, for $|z| = 1$, which proves the desired result, (3.3.2).

Now to prove (3.3.3), consider $r^*(z) = B(z)\overline{r(1/\overline{z})}$. It is clear that $\tilde{m} \leq |r(z)| = |r^*(z)|$, for $|z| = 1$. Also, since r has all zeros in $|z| \leq 1$, $r^*(z)$ has all its zeros in $|z| \geq 1$. Assume that $r^*(z)$ has no zeros on $|z| = 1$. Then $\tilde{m}/r^*(z)$ is analytic in $|z| \leq 1$, and hence by the maximum modulus principle we have

$$\tilde{m} \leq |r^*(z)| \;\text{ for } |z| < 1. \tag{3.4.25}$$

Now, by replacing z with $1/\overline{z}$, (3.4.25) yields

$$\tilde{m} \cdot |B(z)| \leq |r(z)| \;\text{ for } |z| > 1. \tag{3.4.26}$$

In particular, for $z = Re^{i\theta}$, $R > 1$, and $0 \leq \theta < 2\pi$, we have $\tilde{m} \cdot |B(Re^{i\theta})| \leq |r(z)|$, which proves the desired result. Finally, using the continuity of the zeros of $r^*(z)$ we can obtain the inequality when some zeros of $r^*(z)$ lie on $|z| = 1$ as well. □

3.4.3 Generalization of Malik's inequalities

It turns out that it is easier to establish the rational version of Theorem 3.3.5 and use it to prove Theorem 3.3.7. We first prove Theorem 3.3.9, which is a modification of a proof of Govil in **[28, p.543]**.

Proof of Theorem 3.3.9. If $b_1, b_2, \ldots, b_n$ are all the zeros of $r(z)$ and if all are in $|z| \le k \le 1$, then

$$\begin{aligned}\left|\frac{r'(e^{i\theta})}{r(e^{i\theta})}\right| &\ge \operatorname{Re}\left(e^{i\theta}\frac{r'(e^{i\theta})}{r(e^{i\theta})}\right)\\ &= \sum_{j=1}^{n}\operatorname{Re}\left(\frac{e^{i\theta}}{e^{i\theta}-b_j}\right) - \sum_{j=1}^{n}\operatorname{Re}\left(\frac{e^{i\theta}}{e^{i\theta}-a_j}\right)\\ &\ge \frac{n}{1+k} - \sum_{j=1}^{n}\operatorname{Re}\left(\frac{e^{i\theta}}{e^{i\theta}-a_j}\right).\end{aligned}$$

Note that

$$\frac{n}{2} - \sum_{j=1}^{n}\operatorname{Re}\left(\frac{e^{i\theta}}{e^{i\theta}-a_j}\right) = \frac{|B'(e^{i\theta})|}{2}.$$

Thus we obtain

$$\left|\frac{r'(e^{i\theta})}{r(e^{i\theta})}\right| \ge \frac{n}{2}\left(\frac{1-k}{1+k}\right) + \frac{|B'(e^{i\theta})|}{2},$$

which implies the desired result. □

Proof of Theorem 3.3.7. For any $\alpha \in \mathbb{R}$, define $R(z) = r(z) - e^{i\alpha}$. Then by the Maximum Modulus Principle, $R(z)$ has no zeros in the disk $|z| < 1$. So

$$S(z) = R^*(z) = B(z)\overline{R\left(\frac{1}{\bar{z}}\right)} = r^*(z) - B(z)e^{-i\alpha},$$

would have no zeros in $|z| > 1$. By Lemma 3.4.8, for $|z| = 1$, $|R'(z)| \le |S'(z)|$. Thus, for $|z| = 1$,

$$|r'(z)| \le |(r^*)'(z) - B'(z)e^{-i\alpha}|. \tag{3.4.27}$$

Choose α such that $|(r^*)'(z) - B'(z)e^{-i\alpha}| = |B'(z)| - |(r^*)'(z)|$, and use it in (3.4.27) to get

$$|r'(z)| \le |B'(z)| - |(r^*)'(z)| \text{ for } |z| = 1. \tag{3.4.28}$$

Since r has all its zeros in $|z| > K,\ K \geq 1$, $r^*(z)$ has all its zeros in $|z| < \frac{1}{K},\ K \geq 1$. So by Theorem 3.3.9,

$$|(r^*)'(z)| \geq \left\{\frac{n}{2}\frac{(1-1/K)}{(1+1/K)} + \frac{|B'(z)|}{2}\right\}|r(z)| \text{ for } |z| = 1. \tag{3.4.29}$$

By (3.4.28) and (3.4.29), for $|z| = 1$, we obtain

$$|r'(z)| \leq |B'(z)| - \left\{\frac{n}{2}\left(\frac{K-1}{K+1}\right) + \frac{|B'(z)|}{2}\right\}|r(z)|,$$

as desired. □

3.4.4 Generalization of Govil's inequality

Govil proved his result, (2.1.6) with the help of few auxiliary results. Our proof for a rational analogue is a modification of Govil's method. The rational analogue, Theorem 3.3.10 of Govil's inequality, (2.1.6) requires the restriction $K^2 < \hat{R}$, which is to be understood as a necessity for no overlapping to occur when points on the unit circle moves to a bigger circle, $|z| = K > 1$ and when poles shrink towards the unit circle due to the condition $|a_j|/K < 1$.

Lemma 3.4.9. *Let $r \in \mathcal{R}_n$ with all its zeros in $|z| \leq K,\ 1 \leq K^2 < \hat{R}$, where $\hat{R} = \min_j\{|a_j|\}$, for $j = 1, \ldots, n$. Let $B_K(z)$ be the Blaschke product associated with the rational function $r(Kz)$ and be defined by $B_K(z) = \prod_{j=1}^{n} \frac{1 - \overline{a_j}/Kz}{z - a_j/K}$. Then for $0 \leq \theta < 2\pi$,*

$$|r'(K^2e^{i\theta})| \geq \frac{|Q(Ke^{i\theta})|}{K^2}\ |(r^*)'(e^{i\theta})| - K\left|\frac{Q'(Ke^{i\theta})}{Q(Ke^{i\theta})}\right||r(e^{i\theta})|, \tag{3.4.30}$$

where $Q(z) = \dfrac{B_K(z)}{B(z/K)} = \prod_{j=1}^{n} \dfrac{z - a_jK}{Kz - a_j}$.

Proof. Define $R(z) = r(Kz)$ for $K \geq 1$. Since zeros of r are in $|z| \leq K$, zeros of R are in $|z| \leq 1$. Let $s(z) = B_K(z)\overline{R(1/\overline{z})}$. So the zeros of $s(z)$ are in $|z| \geq 1$. Since $|s(z)| = |R(z)|$ on $|z| = 1$ and the function $s(z)/R(z)$ is analytic in $|z| \geq 1$ including the point at ∞, from the maximum modulus principle $|s(z)| \leq |R(z)|$ for $|z| \geq 1$. This implies that $|s(Kz)| \leq |R(Kz)|$ for $|z| = 1$ for any $K > 1$. It follows that $|s'(Kz)| \leq |R'(Kz)|$ for $|z| = 1$ for any $K > 1$, or equivalently for $|z| \geq 1$,

$$|s'(z)| \leq |R'(z)|. \tag{3.4.31}$$

Now, note that

$$\begin{aligned} s(z) &= \frac{B_K(z)}{B(z/K)} \left[B\left(\frac{z}{K}\right) \overline{r\left(\frac{1}{\overline{z/K}}\right)} \right] \\ &= \frac{B_K(z)}{B(z/K)}\, r^*\left(\frac{z}{K}\right) \\ &= Q(z)\, r^*\left(\frac{z}{K}\right), \text{ where } Q(z) = \frac{B_K(z)}{B(z/K)}. \end{aligned}$$

In accordance with this, for $z = Ke^{i\theta}$ with $K \geq 1$ and $0 \leq \theta < 2\pi$, from (3.4.31) it follows that

$$\frac{|Q(Ke^{i\theta})|}{K^2}|(r^*)'(e^{i\theta})| \leq |r'(K^2e^{i\theta})| + \frac{|Q'(Ke^{i\theta})|}{K}|r^*(e^{i\theta})|,$$

which implies the desired result. □

Using Lemma 3.4.9 we prove the following rational analogue for Lemma 2 in **[28]**.

Lemma 3.4.10. *Under the same assumptions as in Lemma 3.4.9, for* $0 \leq \theta < 2\pi$,

$$|(r^*)'(e^{i\theta})| \leq \left|\frac{B(K^2e^{i\theta})}{Q(Ke^{i\theta})}\right| \|r'\| + \frac{2nK^2}{\hat{R} - R}\left|\frac{B(K^2e^{i\theta})}{Q(Ke^{i\theta})}\right| \|r\| + K\left|\frac{Q'(Ke^{i\theta})}{Q(Ke^{i\theta})}\right| |r(e^{i\theta})|, \tag{3.4.32}$$

holds.

Proof. By Lemma 3.4.9,

$$|(r^*)'(e^{i\theta})| \leq \frac{K^2}{|Q(Ke^{i\theta})|}\ |r'(K^2e^{i\theta})| + K\left|\frac{Q'(Ke^{i\theta})}{Q(Ke^{i\theta})}\right| |r(e^{i\theta})|. \tag{3.4.33}$$

Applying Lemma 3.4.5 with $z = K^2e^{i\theta}$ we obtain

$$|r'(K^2e^{i\theta})| \leq \frac{|B(K^2e^{i\theta}))|}{K^2}\ \|r'\| + \frac{2n|B(K^2e^{i\theta}))|}{\hat{R}-R}\ \|r\|\,. \tag{3.4.34}$$

By (3.4.33) and (3.4.34), the desired result follows. □

Remark 3.4.11. Note that a uniform version of (3.4.32) can also be established.

Let

$$f(z) = \frac{2nK^2}{\hat{R}-R}\left|\frac{B(K^2z)}{Q(Kz)}\right| + K\left|\frac{Q'(Kz)}{Q(Kz)}\right|.$$

Then, from (3.4.32) it follows that

$$\|(r^*)'\| \leq \max_{|z|=1}\left|\frac{B(K^2z)}{Q(Kz)}\right| \|r'\| + \|f\|\,\|r\|\,. \tag{3.4.35}$$

In **[43]**, Li et al. proved the following result which is analogous to Lemma 3 in **[28]**.

Lemma 3.4.12. *Let $r \in \mathcal{R}_n$, then*

$$|(r^*)'(z)| + |r'(z)| \leq |B'(z)|\,\|r\| \quad \text{for } |z| = 1, \tag{3.4.36}$$

and equality holds for $r(z) = uB(z)$ with $|u| = 1$.

Our next result establishes an inequality that reverses (3.4.36) when $\|r\|$ is replaced by $|r(z)|$. A rational function $r \in \mathcal{R}_n$ is called *self-inversive* if $r^*(z) = \lambda r(z)$ for some $|\lambda| = 1$.

Lemma 3.4.13. *Let $r \in \mathcal{R}_n$ and $r(z) \equiv r^*(z)$, i.e., r is self-inversive. Then*

$$\left|\frac{r'(z)}{B'(z)}\right| \geq \frac{1}{2}\ |r(z)| \quad \text{for } |z| = 1, \tag{3.4.37}$$

and equality holds for $r(z) = B(z) + 1$.

Proof. Write $\overline{r}(z) = \overline{r(\overline{z})}$. Then $r^*(z) = B(z)\overline{r(1/\overline{z})} = B(z)\ \overline{r}(1/z)$, and so

$$(r^*)'(z) = B'(z)\ \overline{r}\left(\frac{1}{z}\right) - \frac{B(z)}{z^2}\ \overline{r}'\left(\frac{1}{z}\right).$$

Note that for $|z| = 1$,

$$|(r^*)'(z)| = \left|B'(z)\overline{r}\left(\frac{1}{z}\right) - \frac{B(z)}{z^2}\overline{r}'\left(\frac{1}{z}\right)\right| = \left|\left(\frac{zB'(z)}{B(z)}\right)\overline{r(z)} - \overline{z\ r'(z)}\right|. \tag{3.4.38}$$

Taking the logarithmic derivative of $B(z)$, for $|z| = 1$, we get

$$\frac{zB'(z)}{B(z)} = \sum_{j=1}^{n} \frac{|a_j|^2 - 1}{|z - a_j|^2}. \tag{3.4.39}$$

That is, $zB'(z)/B(z)$ is positive. Now, from (3.4.38), for $|z| = 1$, it follows that

$$\begin{aligned} |(r^*)'(z)| &= |B'(z)r(z) - r'(z)B(z)| \\ &\geq |B'(z)||r(z)| - |r'(z)|. \end{aligned}$$

The last inequality can be rewritten as

$$\left|\frac{(r^*)'(z)}{B'(z)}\right| + \left|\frac{r'(z)}{B'(z)}\right| \geq |r(z)| \quad \text{for } |z| = 1. \tag{3.4.40}$$

Since $r(z) \equiv r^*(z)$, (3.4.40) yields the desired result. □

Now, we are ready to prove the rational analogue of Govil's inequality.

Proof of Theorem 3.3.10. For any ϵ such that $|\epsilon| = 1$, define

$$\tilde{R}(z) = \frac{1}{2}(r(z) + \epsilon \cdot r^*(z)). \tag{3.4.41}$$

Note that $\tilde{R}^*(z) \equiv B(z)\tilde{R}(z)$ and $\max_{|z|=1} |\tilde{R}(z)| = \|r\|$. So applying Lemma 3.4.13 yields

$$\max_{|z|=1} |r'(z) + \epsilon \cdot (r^*)'(z)| \geq |r(z)||B'(z)|, \text{ for } |z| = 1. \tag{3.4.42}$$

Equivalently

$$\|r'\| + \|(r^*)'\| \geq |r(z)||B(z)|, \text{ for } |z| = 1.$$

Using (3.4.35) to estimate from above yields for $|z| = 1$,

$$\|r'\| + \max_{|z|=1} \left| \frac{B(K^2 z)}{Q(Kz)} \right| \|r'\| + \|f\| \, \|r\| \geq \|r\| \min_{|z|=1} |B'(z)|,$$

from which it follows that

$$\|r'\| \geq \frac{\min_{|z|=1} |B'(z)| - \|f\|}{1 + \max_{|z|=1} \left| \frac{B(K^2 z)}{Q(Kz)} \right|} \|r\| .$$

□

CHAPTER 4: ACTION OF THE ASKEY-WILSON OPERATOR ON POLYNOMIALS

If things are nice there is probably a good reason why they are nice: and if you do not know at least one reason for this good fortune, then you still have work to do.

Richard Askey

4.1 Introduction

In 1985, Richard Askey and James Wilson introduced the theory of Askey-Wilson operator in their study of a class of orthogonal polynomials, the Askey-Wilson polynomials. (see [**6**])

Definition 4.1.1. Given a polynomial p, we set $\breve{p}(e^{i\theta}) := p(x),\ x = \cos\theta$, that is

$$\breve{p}(z) = p\left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right), \quad z = e^{i\theta}. \tag{4.1.1}$$

The *Askey-Wilson operator*, $\mathscr{D}_q$, is defined by

$$(\mathscr{D}_q p)(x) = \frac{\breve{p}(q^{1/2}e^{i\theta}) - \breve{p}(q^{-1/2}e^{i\theta})}{\breve{e}(q^{1/2}e^{i\theta}) - \breve{e}(q^{-1/2}e^{i\theta})}, \tag{4.1.2}$$

with $e(x) = x$. A straightforward calculation shows

$$\breve{e}(q^{1/2}e^{i\theta}) - \breve{e}(q^{-1/2}e^{i\theta}) = i\sin\theta \cdot (q^{1/2} - q^{-1/2}),$$

which reduces (4.1.2) to

$$(\mathscr{D}_q p)(x) = \frac{\breve{p}(q^{1/2}e^{i\theta}) - \breve{p}(q^{-1/2}e^{i\theta})}{i\sin\theta \cdot (q^{1/2} - q^{-1/2})}. \tag{4.1.3}$$

In the theory of the Askey-Wilson polynomials (see [**6**]), $\mathscr{D}_q$ plays an analogous role to that of differentiation in the theory of Jacobi, Hermite, and Laguerre polynomials. Note that $\mathscr{D}_q$ is a linear operator, and $\mathscr{D}_q = \mathscr{D}_{1/q}$.

Since $\lim_{q\to 1^-} (\mathscr{D}_q p)(x) = p'(x)$ at any point x where $p'(x)$ exists, $\mathscr{D}_q p$ can be considered as a discrete version of the derivative of p. To illustrate this fact we consider an application. Recall that the *Chebyshev polynomials* (see [**58**])

$$T_n(x) = \cos(n\theta), \quad x = \cos\theta, \tag{4.1.4}$$

and

$$U_n(x) = \frac{\sin((n+1)\theta)}{\sin(\theta)}, \quad x = \cos\theta. \tag{4.1.5}$$

From a typical orthogonality argument it can be easily shown that $\{T_n\}_{n=0}^{\infty}$ and $\{U_n\}_{n=0}^{\infty}$ form an orthogonal basis for $L^2(1/\sqrt{1-x^2})$ and $L^2(\sqrt{1-x^2})$, respectively. In fact

$$\int_{-1}^{1} T_m(x)T_n(x)\ \frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0 & m \neq n, \\ \pi & m = n = 0, \\ \frac{\pi}{2} & m = n \neq 0. \end{cases} \tag{4.1.6}$$

and

$$\int_{-1}^{1} U_m(x)U_n(x)\ \sqrt{1-x^2}\ dx = \begin{cases} 0 & m \neq n, \\ \frac{\pi}{2} & m = n. \end{cases} \tag{4.1.7}$$

Now let us apply $\mathscr{D}_q$ on T_n using the definition (4.1.3) itself:

$$(\mathscr{D}_q T_n)(x) = \frac{\breve{T}_n(q^{1/2}z) - \breve{T}_n(q^{-1/2}z)}{(q^{1/2} - q^{-1/2}) \cdot i \sin\theta}.$$

Note that, from

$$\breve{T}_n(z) = T_n\left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right), \quad z = e^{i\theta}.$$

$$\breve{T}_n(z) = T_n\left(\cos(\theta)\right) = \cos(n\theta) = \frac{1}{2}\left(z^n + z^{-n}\right),$$

it follows from the uniqueness theorem the above holds for all z and so,

$$\breve{T}_n(q^{1/2}z) = \frac{1}{2}\left(q^{n/2}z^n + q^{-n/2}z^{-n}\right),$$

and

$$\breve{T}_n(q^{-1/2}z) = \frac{1}{2}\left(q^{-n/2}z^n + q^{n/2}z^{-n}\right).$$

So

$$\begin{aligned}
\breve{T}_n(q^{1/2}z) - \breve{T}_n(q^{-1/2}z) &= \frac{1}{2}\Big\{\left(q^{n/2}z^n + q^{-n/2}z^{-n}\right) - \left(q^{-n/2}z^n + q^{n/2}z^{-n}\right)\Big\} \\
&= (q^{n/2} - q^{-n/2}) \cdot \frac{1}{2}\left(z^n - z^{-n}\right) \\
&= (q^{n/2} - q^{-n/2}) \cdot \sin(n\theta).
\end{aligned}$$

From which we obtain

$$(\mathscr{D}_q T_n)(x) = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} \cdot \frac{\sin(n\theta)}{\sin(\theta)} \quad x = \cos(\theta),$$

or equivalently

$$(\mathscr{D}_q T_n)(x) = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} U_{n-1}(x). \tag{4.1.8}$$

Now in the limit as $q \to 1^-$, $(\mathscr{D}_q T_n)(x) \to nU_{n-1}(x)$. Therefore

$$\lim_{q \to 1^-} (\mathscr{D}_q p)(x) = p'(x),$$

holds for $p \equiv T_n$. Indeed, since $\{T_n(x)\}$ forms a basis for $\mathcal{P}_n$ and $\mathscr{D}_q$ is a linear operator, the above relation holds for all polynomials as well.

In this chapter, we will establish some new results using the Askey-Wilson operator.

4.2 Interpolation Formulas and Bernstein Inequalities

In this section, a brief overview will be given on the interpolation formulas for the derivative of functions in the classes $\mathcal{P}_n$ and $\mathcal{T}_n$. Recall that $t \in \mathcal{T}_n$ if

$$t(\varphi) = a_0 + \sum_{k=1}^{n} \{a_k \cos(k\varphi) + b_k \sin(k\varphi)\}. \tag{4.2.1}$$

For $p \in \mathcal{P}_n$, we have the following Bernstein inequality:

$$|p'(x)| \leq \frac{n}{\sqrt{1-x^2}} \|p\|_{[-1,1]}, \quad x \in (-1,1). \tag{4.2.2}$$

This pointwise inequality eventually leads to a uniform estimate, the Markov inequality:

$$\|p'\|_{[-1,1]} \leq n^2 \|p\|_{[-1,1]}, \tag{4.2.3}$$

which plays a fundamental role in proving the inverse theorems in the theory of approximation. (4.2.3) can be established from (4.2.2) with the help of the Schur's inequality ([**18, Theorem 5.1.9, p.233**]).

One common way to establish (4.2.2) is by deriving it from a Bernstein inequality for trigonometric polynomials:

$$\|t'\|_{[0,2\pi]} \le n \|t\|_{[0,2\pi]} . \tag{4.2.4}$$

Bernstein claimed (4.2.4) with n replaced by $2n$ and M. Riesz in **[57]** gave a very elegant proof of (4.2.4) with the sharp constant "n" through an interpolating formula for $t'(\varphi)$:

$$t'(\varphi) = \frac{1}{2n}\sum_{r=1}^{2n} t(\varphi+\theta_r)\frac{(-1)^{r+1}}{2\sin^2\left(\frac{\theta_r}{2}\right)}, \tag{4.2.5}$$

where

$$\theta_r = \frac{2r-1}{2n}\pi,\ \ r = 1, 2, \ldots, 2n. \tag{4.2.6}$$

Let $p \in \mathcal{P}_n$. Consider the trigonometric polynomial $t(\varphi) = p(\cos\varphi)$. By (4.2.5), we have

$$p'(\cos\varphi)\cdot(\sin\varphi) = \frac{1}{2n}\sum_{r=1}^{2n} p(\cos(\varphi+\theta_r))\frac{(-1)^{r}}{2\sin^2\left(\frac{\theta_r}{2}\right)}$$

or, equivalently,

$$p'(x) = \frac{1}{2n\sqrt{1-x^2}}\sum_{r=1}^{2n} p(\cos(\varphi+\theta_r))\frac{(-1)^{r}}{2\sin^2\left(\frac{\theta_r}{2}\right)}. \tag{4.2.7}$$

This is the Riesz's formula (4.2.5) adapted for algebraic polynomials. Similarly, it is easy to verify the following consequence of Riesz's formula on the unit circle. For $p \in \mathcal{P}_n$ with $|z| = 1$, we have

$$p'(z) = \frac{1}{2inz}\sum_{r=1}^{2n} p(ze^{i\theta_r})\frac{(-1)^{r+1}}{2\sin^2\left(\frac{\theta_r}{2}\right)}. \tag{4.2.8}$$

In 1974, André Giroux and Qazi Rahman (**[26, Lemma 1]**) established a discrete version of (4.2.8): for $z = e^{i\varphi}$, and $R > 1$, we have

$$p(Rz) = p(z) + \frac{1}{2n}\sum_{k=1}^{2n}(-1)^k A_k p(e^{i(\varphi+k\pi/n)}), \tag{4.2.9}$$

where

$$A_k = (R^n - 1) + 2\sum_{j=1}^{n-1}(R^{n-j} - 1)\cos\left(\frac{jk\pi}{n}\right). \tag{4.2.10}$$

The coefficients A_k are positive and

$$\frac{1}{2n}\sum_{k=1}^{2n} A_k = R^n - 1. \tag{4.2.11}$$

For $t \in \mathcal{T}_n$ as in (4.2.1), let $\tilde{t}$ denote its *conjugate (trigonometric) polynomial:

$$\tilde{t}(\varphi) = \sum_{k=1}^{n}\{a_k \sin(k\varphi) - b_k \cos(k\varphi)\}. \tag{4.2.12}$$

In 1928, Gábor Szegö **[61]** extended Riesz's formula (4.2.5) to the following:

$$\cos(\alpha)\cdot\tilde{t}'(\varphi) - \sin(\alpha)\cdot t'(\varphi) = \frac{1}{2n}\sum_{r=1}^{2n}(-1)^r t(\varphi+\theta_{r,\alpha})\frac{1-(-1)^r\cos(\alpha)}{1-\cos(\theta_{r,\alpha})}, \tag{4.2.13}$$

where

$$\theta_{r,\alpha} = \frac{r\pi}{n} - \frac{\alpha}{n},\ r = 1, 2, \ldots, 2n.$$

Remark 4.2.1. When $\alpha = \frac{\pi}{2}$, $\theta_{r,\alpha} = \theta_r$. Accordingly

$$1 - \cos(\theta_{r,\alpha}) = 1 - \cos(\theta_r) = 2\sin^2\left(\frac{\theta_r}{2}\right).$$

So Szegö's formula (4.2.13) reduces to Riesz's formula (4.2.5).

Remark 4.2.2. For $p \in \mathcal{P}_n$ with real coefficients, if we let $t(\varphi) = \mathrm{Re}(p(z))$, with $z = e^{i\varphi}$,

*When a_k and b_k are all real numbers, $t(\varphi)$ and $\tilde{t}(\varphi)$ are respectively the real and imaginary parts of $a_0 + \sum_{k=1}^{n}(a_k - ib_k)z^k$ for $z = e^{i\varphi}$.

then $\tilde{t}(\varphi) = \text{Im}(p(z))$. So, (4.2.13) gives, for $|z| = 1$,

$$\text{Re}(e^{-i\alpha} z p'(z)) = \frac{1}{2n} \sum_{r=1}^{2n} (-1)^r \text{Re}\left(p(ze^{i\theta_{r,\alpha}})\right) \frac{1 - (-1)^r \cos(\alpha)}{2\sin^2\left(\frac{\theta_{r,\alpha}}{2}\right)}. \tag{4.2.14}$$

4.3 Bernstein Inequalities for the Askey-Wilson Operator

In this section, we first state our main result, Theorem 4.3.1 from which a Bernstein inequality for the Askey-Wilson operator is obtained in pointwise norm followed up with Bernstein inequalities for the Askey-Wilson operator in uniform and integral norms as well. Immediately after each statement of our results, we show that they are indeed a limiting case of the corresponding classical result. The proofs of the results in this section will be given in the subsection 4.4.2.

Theorem 4.3.1. *For $p \in \mathcal{P}_n$, with $x = \cos\varphi$, the following holds:*

$$(\mathscr{D}_q p)(x) = \frac{1}{2n \cdot \sqrt{1-x^2}} \sum_{r=1}^{2n} (-1)^r A_r \, p(\cos(\varphi + \theta_r)), \tag{4.3.1}$$

where $\theta_r = \dfrac{2r-1}{2n}\pi$, $r = 1, 2, \ldots, 2n$, are as before (4.2.6). *The coefficients A_r, $r = 1, 2, \ldots, 2n$, are non-negative constants that are independent of p and x satisfying*

$$A_r = \frac{q^{n/2} + q^{-n/2}}{q^{1/2} - 2\cos(\theta_r) + q^{-1/2}} \ , \tag{4.3.2}$$

and

$$\frac{1}{2n} \sum_{r=1}^{2n} A_r = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} \ . \tag{4.3.3}$$

Remark 4.3.2. As $q \to 1^-$, (4.3.1) reduces to

$$p'(x) = \frac{2n}{\sqrt{1-x^2}} \cdot \sum_{r=1}^{2n} (-1)^r A_r p(\cos(\varphi + \theta_r)), \tag{4.3.1$'$}$$

with the coefficients satisfying

$$A_r = \frac{1}{2\sin^2(\frac{\theta_r}{2})} \quad \text{and} \quad \frac{1}{2n}\sum_{r=1}^{2n} A_r = n \ .$$

This is the Riesz interpolation formula (4.2.5) adapted for algebraic polynomials, (4.2.7).

From (4.3.1), we obtain the following Bernstein-type inequality, which is an analogue of (4.2.2).

Theorem 4.3.3. *For $p \in \mathcal{P}_n$, and for $x \in (-1, 1)$, the following pointwise estimation holds:*

$$|(\mathscr{D}_q p)(x)| \leq \frac{1}{\sqrt{1-x^2}} \cdot \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} \cdot \|p\|_{[-1,1]} \, . \tag{4.3.4}$$

Equality in (4.3.4) *holds for $p(x) = cT_n(x)$, where $T_n(x)$ is the Chebyshev polynomial of the first kind and c is an arbitrary constant.*

Remark 4.3.4. As $q \to 1^-$, (4.3.4) yields

$$|p'(x)| \leq \frac{n}{\sqrt{1-x^2}} \cdot \|p\|_{[-1,1]} \, , \qquad (x \in (-1, 1)) \tag{4.3.4$'$}$$

which is the *Bernstein inequality* for $p \in \mathcal{P}_n$ on $[-1, 1]$.

A uniform version of (4.3.4) can also be established, which is a generalization of the Markov inequality (4.2.3).

Theorem 4.3.5. *For $p \in \mathcal{P}_n$, the following uniform estimation holds:*

$$
\begin{aligned}
\|\mathscr{D}_q p\|_{[-1,1]} &\leq \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} \max\left\{n, \frac{1}{\sqrt{1-x^2}}\right\} \cdot \|p\|_{[-1,1]} \\
&\leq n \, \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} \|p\|_{[-1,1]} \, .
\end{aligned}
\tag{4.3.5}
$$

Remark 4.3.6. As $q \to 1^-$, of (4.3.5) yields

$$
\|p'\| \leq n^2 \|p\| \, , \tag{4.3.5$'$}
$$

which is the *Markov inequality* for $p \in \mathcal{P}_n$.

Our next result is an integral form of the Bernstein-type inequality.

Theorem 4.3.7. *For $p \in \mathcal{P}_n$, the following estimation holds:*

$$
\|\mathscr{D}_q p\|_{L^2\left(\sqrt{1-x^2}\right)} \leq \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} \|p\|_{L^2\left(1/\sqrt{1-x^2}\right)} \, . \tag{4.3.6}
$$

Remark 4.3.8. As $q \to 1^-$, (4.3.6) yields

$$
\|p'\|_{L^2\left(\sqrt{1-x^2}\right)} \leq n \|p\|_{L^2\left(\sqrt{1-x^2}\right)} \, , \tag{4.3.6$'$}
$$

which is the *Bernstein inequality* in L^2-norm for $p \in \mathcal{P}_n$.

4.4 Proofs

4.4.1 Szegö's multiplier version

Our proof of the Riesz-type interpolation formula, (4.3.1) is based on the ideas of Szegö **[61]**. More precisely, we need the following identity:

Lemma 4.4.1. *Let $t \in \mathcal{T}_n$ and $\tilde{t}$ be its conjugate trigonometric polynomial as in* (4.2.1) *and* (4.2.12)*. Let $\lambda_0, \ldots, \lambda_{n-1}$ be given real numbers and define*

$$\Lambda(t)(\varphi) = \sum_{k=1}^{n} \lambda_{n-k}\{b_k \cos(k\varphi) - a_k \sin(k\varphi)\}, \tag{4.4.1}$$

and

$$\Lambda(\tilde{t})(\varphi) = \sum_{k=1}^{n} \lambda_{n-k}\{a_k \cos(k\varphi) + b_k \sin(k\varphi)\}. \tag{4.4.2}$$

Then

$$\begin{aligned} &\cos(\alpha) \cdot \Lambda(\tilde{t})(\varphi) - \sin(\alpha) \cdot \Lambda(t)(\varphi) \\ &\quad = \frac{1}{2n} \sum_{r=1}^{2n} (-1)^r t(\varphi + \theta_{r,\alpha}) \left(\lambda_0 + 2 \sum_{k=1}^{n-1} \lambda_k \cos(k\theta_{r,\alpha}) \right), \end{aligned} \tag{4.4.3}$$

where

$$\theta_{r,\alpha} = \frac{r\pi}{n} - \frac{\alpha}{n}, \ r = 1, 2, \ldots, 2n.$$

Remark 4.4.2. The identity (4.4.3) was not explicitly given by Szegö in the above general form but it was implied in a handwritten note of Dr. N. K. Govil by following Szegö's argument of deriving (4.2.13) in **[61]**. We are grateful to Dr. Ram Mohapatra for making the note accessible to us. For the convenience of the reader, we give a complete proof here.

Proof. From $t(\varphi)$ we can compute its Fourier coefficients:

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} t(\theta)\cdot\cos(k\theta)\ d\theta$$

and

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} t(\theta)\cdot\sin(k\theta)\ d\theta,$$

for $k = 1, 2, \ldots, n$.

Note that

$$\begin{aligned}
\Lambda(\tilde{t})(\varphi) &= \sum_{k=1}^{n} \lambda_{n-k}\left(a_k\cos(k\varphi) + b_k\sin(k\varphi)\right) \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} t(\theta)\cdot\left\{\sum_{k=1}^{n}\lambda_{n-k}\cdot\left(\cos(k\varphi)\cos(k\theta) + \sin(k\varphi)\sin(k\theta)\right)\right\} d\theta \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} t(\theta)\cdot\left\{\sum_{k=1}^{n}\lambda_{n-k}\cdot\cos(k(\theta-\varphi))\right\} d\theta \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} t(\theta+\varphi)\sum_{k=1}^{n}\lambda_{n-k}\cos(k\theta)\ d\theta,
\end{aligned}$$

and

$$\begin{aligned}
\Lambda(t)(\varphi) &= \sum_{k=1}^{n} \lambda_{n-k}\left(b_k\cos(k\varphi) - a_k\sin(k\varphi)\right) \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} t(\theta)\cdot\left\{\sum_{k=1}^{n}\lambda_{n-k}\cdot\left(\cos(k\varphi)\sin(k\theta) - \sin(k\varphi)\cos(k\theta)\right)\right\} d\theta \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} t(\theta)\cdot\left\{\sum_{k=1}^{n}\lambda_{n-k}\cdot\sin(k(\theta-\varphi))\right\} d\theta \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} t(\theta+\varphi)\sum_{k=1}^{n}\lambda_{n-k}\sin(k\theta)\ d\theta.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\cos(\alpha)\cdot\Lambda(\tilde{t})(\varphi)-\sin(\alpha)\cdot\Lambda(t)(\varphi)\\
&=\frac{1}{\pi}\int_{-\pi}^{\pi} t(\theta+\varphi)\sum_{k=1}^{n}\lambda_{n-k}\big\{\cos(k\theta)\cos(\alpha)-\sin(k\theta)\sin(\alpha)\big\}\;d\theta\\
&=\frac{1}{\pi}\int_{-\pi}^{\pi} t(\theta+\varphi)\sum_{k=1}^{n}\lambda_{n-k}\cos(k\theta+\alpha)\;d\theta.
\end{aligned}$$

Since $t(\theta+\varphi)$ is a trigonometric polynomial of degree n, from orthogonality relations we can add $\cos((2n-k)\theta+\alpha))$ terms to the sum above without changing the value of the integral:

$$\begin{aligned}
&\cos(\alpha)\cdot\Lambda(\tilde{t})(\varphi)-\sin(\alpha)\cdot\Lambda(t)(\varphi)\\
&=\frac{1}{\pi}\int_{-\pi}^{\pi} t(\theta+\varphi)\,\left\{\lambda_0\cos(n\theta+\alpha)+\sum_{k=1}^{n-1}\lambda_{n-k}\big[\cos(k\theta+\alpha)+\cos((2n-k)\theta+\alpha))\big]\right\}\,d\theta\\
&=\frac{1}{\pi}\int_{-\pi}^{\pi} t(\theta+\varphi)\,\left\{\lambda_0\cos(n\theta+\alpha)+\sum_{k=1}^{n-1}2\lambda_{n-k}\cos(n\theta+\alpha)\cdot\cos((n-k)\theta)\right\}\,d\theta\\
&=\frac{1}{\pi}\int_{-\pi}^{\pi} t(\theta+\varphi)\;\cos(n\theta+\alpha)\,\left\{\lambda_0+2\sum_{k=1}^{n}\lambda_k\cos(k\theta)\right\}\,d\theta.
\end{aligned}$$

Now, we replace $\cos(n\theta+\alpha)$ by $h(n\theta+\alpha)$, where $h(\theta)$ is a continuous, periodic function with period 2π having a Fourier series of the form

$$h(\theta)\sim\cos(\theta+\alpha)+c_2\sin(2\theta+\alpha)+d_2\cos(2\theta+\alpha)+\dots.$$

For $0<r<1$, we take

$$\begin{aligned}
h(n\theta+\alpha)&=\frac{1}{4r}\left(\frac{1-r^2}{1-2r\cos(n\theta+\alpha)+r^2}-\frac{1-r^2}{1+2r\cos(n\theta+\alpha)+r^2}\right)\\
&=\cos(\theta+\alpha)-r^2\cos(3\theta+\alpha)+r^4\cos(\theta+\alpha)+\dots.
\end{aligned}$$

In view of the uniform convergence of the right hand side above, we have

$$\begin{aligned} &\cos(\alpha)\cdot\Lambda(\tilde{t})(\varphi)-\sin(\alpha)\cdot\Lambda(t)(\varphi) \\ &=\lim_{r\to 1^-}\frac{1}{4\pi r}\int_{-\pi}^{\pi} t(\varphi+\theta)\left\{\lambda_0+2\sum_{k=1}^{n}\lambda_k\cos(k\theta)\right\} \\ &\qquad\qquad\times\left\{\frac{1-r^2}{1-2r\cos(n\theta+\alpha)+r^2}-\frac{1-r^2}{1+2r\cos(n\theta+\alpha)+r^2}\right\}d\theta. \end{aligned} \tag{4.4.4}$$

Using the following well-known property of the Poisson kernel: if F is continuous periodic with period 2π, then

$$\begin{aligned} &\lim_{r\to 1^-}\frac{1}{4\pi r}\int_{-\pi}^{\pi} F(\theta)\left(\frac{1-r^2}{1-2r\cos(n\theta+\alpha)+r^2}-\frac{1-r^2}{1+2r\cos(n\theta+\alpha)+r^2}\right)d\theta \\ &=\frac{1}{2n}\sum_{\substack{r=1\\ r\text{ even}}}^{2n}F(\theta_{r,\alpha})-\frac{1}{2n}\sum_{\substack{r=1\\ r\text{ odd}}}^{2n}F(\theta_{r,\alpha}),\ \text{where } \theta_{r,\alpha}=-\frac{\alpha}{n}+\frac{r\pi}{n},\quad r=1,2,\ldots,2n \\ &=\frac{1}{2n}\sum_{r=1}^{2n}(-1)^r F(\theta_{r,\alpha}), \end{aligned}$$

with

$$F(\theta)=t(\varphi+\theta)\left\{\lambda_0+2\sum_{k=1}^{n}\lambda_k\cos(k\theta)\right\};$$

(4.4.4) yields

$$\cos(\alpha)\cdot\Lambda(\tilde{t})(\varphi)-\sin(\alpha)\cdot\Lambda(t)(\varphi)=\frac{1}{2n}\sum_{r=1}^{2n}(-1)^r t(\varphi+\theta_{r,\alpha})\left(\lambda_0+2\sum_{k=1}^{n-1}\lambda_k\cos(k\theta_{r,\alpha})\right),$$

as desired. □

As an application of Lemma 4.4.1 let us give an alternative proof to Giroux and Rahman's result, (4.2.9) ([**26, Lemma 1**]), independent of theirs.

Proof. Let $t(\varphi) = P_n(e^{i\varphi})$. For $R > 1$, we have

$$\begin{aligned} P_n(Re^{i\varphi}) - P_n(e^{i\varphi}) &= \sum_{k=1}^{n} c_k e^{ik\varphi}(R^k - 1) \\ &= \sum_{k=1}^{n} (R^k - 1)\left(c_k \cos(k\varphi) + ic_k \sin(k\varphi)\right) \\ &= \sum_{k=1}^{n} (R^k - 1)\left(a_k \cos(k\varphi) + b_k \sin(k\varphi)\right), \qquad (a_k = c_k,\ b_k = ic_k). \end{aligned}$$

With the notations as in Lemma (4.4.1), the right hand side of the above is $\Lambda(\tilde{t})(\varphi)$ with $\lambda_{n-k} = R^k - 1$. So applying (4.4.3) with $\alpha = 0$ yields

$$P_n(Re^{i\varphi}) - P_n(e^{i\varphi}) = \frac{1}{2n}\sum_{r=1}^{2n}(-1)^r P_n(e^{\varphi + \frac{r\pi}{n}})\left\{(R^n - 1) + 2\sum_{k=1}^{n-1}(R^{n-k} - 1)\cos\left(k\frac{r\pi}{n}\right)\right\},$$

which is Lemma 1 in [**26**]. □

4.4.2 Proofs of the Results in Section 4.3

Before proving Theorem 4.3.1, we first establish the following auxiliary result.

Lemma 4.4.3. *For* $0 < q < 1$ *and* $0 \le \theta < 2\pi$*, the following holds:*

$$\frac{(q^{n/2} - q^{-n/2})}{(q^{1/2} - q^{-1/2})} + 2\sum_{k=1}^{n-1}\frac{(q^{k/2} - q^{-k/2})}{(q^{1/2} - q^{-1/2})}\cos\left((n-k)\theta\right) = \frac{q^{n/2} - 2\cos(n\theta) + q^{-n/2}}{q^{1/2} - 2\cos(\theta) + q^{-1/2}}.$$

Proof. We shall prove

$$\begin{aligned} &(q^{1/2} - q^{-1/2}) \cdot (q^{n/2} - 2\cos(n\theta) + q^{-n/2}) \\ &= (q^{1/2} - 2\cos(\theta) + q^{-1/2})\left\{(q^{n/2} - q^{-n/2}) + 2\sum_{k=1}^{n-1}(q^{k/2} - q^{-k/2})\cos((n-k)\theta)\right\}, \end{aligned} \tag{4.4.5}$$

from which the desired result clearly follows. Indeed, we show that the two cosine trigonometric polynomials in (4.4.5) agree on their Fourier coefficients. Let

$$f(\theta) = (q^{1/2} - 2\cos(\theta) + q^{-1/2})\left\{(q^{n/2} - q^{-n/2}) + 2\sum_{k=1}^{n-1}(q^{(n-k)/2} - q^{-(n-k)/2})\cos(k\theta)\right\}.$$

We claim the followings:

(a) $\displaystyle\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\ d\theta = (q^{1/2} - q^{-1/2})\cdot(q^{n/2} + q^{-n/2})$.

(b) $\displaystyle\int_{-\pi}^{\pi} f(\theta)\cdot\cos(k\theta)\ d\theta = 0$ for $k = 1, 2, \ldots, n-1$.

(c) $\displaystyle\frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cdot\cos(n\theta)\ d\theta = -2(q^{1/2} - q^{-1/2})$.

To prove these, we make use of couple of standard orthogonality relations:

$$\int_{-\pi}^{\pi}\cos(m\theta)\cdot\cos(n\theta)\ d\theta = \pi\delta_{m,n} \quad\text{and}\quad \int_{-\pi}^{\pi}\cos(m\theta)\ d\theta = 0,$$

where $\delta_{m,n}$ is the Kronecker delta. In view of these, it is apparent that (b) holds. Proof of (a) follows from the following computation:

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\ d\theta &= (q^{1/2} + q^{-1/2})(q^{n/2} - q^{-n/2}) - \frac{1}{\pi}(q^{(n-1)/2} - q^{-(n-1)/2})\int_{-\pi}^{\pi} 2\cos^2(\theta)\ d\theta \\
&= q^{(n+1)/2} - q^{-(n-1)/2} + q^{(n-1)/2} - q^{-(n+1)/2} - 2q^{(n-1)/2} + 2q^{-(n-1)/2} \\
&= \left(q^{(n+1)/2} - q^{-(n+1)/2}\right) - \left(q^{(n-1)/2} - q^{-(n-1)/2}\right) \\
&= (q^{1/2} - q^{-1/2})\cdot(q^{n/2} + q^{-n/2}).
\end{aligned}$$

To prove (c), multiply $f(\theta)$ by $\cos(n\theta)$ and integrate from $-\pi$ to π. From the orthogonality relations, the only surviving term is, $k = n-1$:

$$\begin{aligned}
&\frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cdot\cos(n\theta)\ d\theta\\
&= -2\sum_{k=1}^{n-1}(q^{(n-k)/2}-q^{-(n-k)/2})\Big\{\frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\Big[2\cos(k\theta)\cdot\cos(\theta)\Big]\cdot\cos(n\theta)\ d\theta\Big\}\\
&= -2\sum_{k=1}^{n-1}(q^{(n-k)/2}-q^{-(n-k)/2})\Big\{\frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\Big[\cos((k+1)\theta)+\cos((k-1)\theta)\Big]\cdot\cos(n\theta)\ d\theta\Big\}\\
&= -(q^{1/2}-q^{-1/2})\cdot\frac{1}{\pi}\int_{-\pi}^{\pi} 2\cos^2(n\theta)\ d\theta\\
&= -2(q^{1/2}-q^{-1/2}).
\end{aligned}$$

So all three items (a), (b), and (c) hold and consequently the two sides of (4.4.5) are equal.

□

Proof of Theorem 4.3.1. Let $p\in\mathcal{P}_n$ and write

$$p(x)=\sum_{k=0}^{n} a_kT_k(x),$$

where T_k is the Chebyshev polynomial of the first kind. Applying the Askey-Wilson operator $\mathscr{D}_q$ to $p(x)$ and using (4.1.8) we get

$$(\mathscr{D}_qp)(x)=\sum_{k=0}^{n} a_k(\mathscr{D}_qT_k)(x)=\sum_{k=0}^{n} a_k\frac{q^{k/2}-q^{-k/2}}{q^{1/2}-q^{-1/2}}U_{k-1}(x). \tag{4.4.6}$$

Let

$$\lambda_{n-k}=\frac{q^{k/2}-q^{-k/2}}{q^{1/2}-q^{-1/2}},\ x=\cos k\varphi,\quad\text{and}\quad t(\varphi)=p(\cos(\varphi))=\sum_{k=0}^{n} a_k\cos(k\varphi).$$

Then (4.4.6) implies

$$\sin(\varphi)\cdot(\mathscr{D}_qp)(\cos(\varphi)=\sum_{k=0}^{n}\lambda_{n-k}a_k\sin(k\varphi)=:-\Lambda(t)(\varphi). \tag{4.4.7}$$

Now applying (4.4.3) for λ_k and t as in here with $\alpha = \dfrac{\pi}{2}$ (so from (4.2.6), $\theta_{r,\frac{\pi}{2}} = \theta_r$) yields

$$\sum_{k=1}^{n} \lambda_{n-k} a_k \sin(k\varphi) = \frac{1}{2n} \sum_{r=1}^{2n} (-1)^r \; t\left(\varphi + \theta_r\right) \times \left\{ \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} + 2 \sum_{j=1}^{n-1} \frac{q^{(n-j)/2} - q^{-(n-j)/2}}{q^{1/2} - q^{-1/2}} \cos(j\theta_r) \right\} .$$

From (4.4.7) with $x = \cos\varphi$, it follows that

$$(\mathscr{D}_q p)(x) = \frac{1}{2n \cdot \sqrt{1-x^2}} \sum_{r=1}^{2n} (-1)^r A_r \; p\left(\cos\left(\varphi + \theta_r\right)\right),$$

where

$$A_r = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} + 2 \sum_{j=1}^{n-1} \frac{q^{(n-j)/2} - q^{-(n-j)/2}}{q^{1/2} - q^{-1/2}} \cos\left(j\theta_r\right),$$

which in view of Lemma 4.4.3 and (4.2.6), equals to

$$\frac{q^{n/2} - 2\cos n\theta_r + q^{-n/2}}{q^{1/2} - 2\cos\theta_r + q^{-1/2}} = \frac{q^{n/2} + q^{-n/2}}{q^{1/2} - 2\cos\theta_r + q^{-1/2}} \; ,$$

which is clearly positive. This establishes (4.3.2) and hence (4.3.1).

Now, we shall verify (4.3.3). Applying (4.3.1) to $p(x) = T_n(x)$ and using (4.1.8), we get

$$\frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} U_{n-1}(x) = \frac{1}{2n \sin\varphi} \sum_{r=1}^{2n} (-1)^r A_r \; T_n(\cos(\varphi + \theta_r)).$$

This implies

$$\sin(n\varphi) \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} = \frac{1}{2n} \sum_{r=1}^{2n} (-1)^r A_r \; \cos(n\varphi + n\theta_r). \tag{4.4.8}$$

Choosing $\varphi = \dfrac{\pi}{2n}$ above, we get

$$\frac{1}{2n}\sum_{r=1}^{2n} A_r = \frac{q^{n/2}-q^{-n/2}}{q^{1/2}-q^{-1/2}},$$

which is (4.3.3). □

Proof of Theorem 4.3.3. Taking the modulus of both sides of (4.3.1) in Theorem 4.3.1 and using the triangle inequality, we obtain

$$\begin{aligned}|(\mathscr{D}_q p)(x)| &\le \frac{1}{2n\cdot\sqrt{1-x^2}}\ \sum_{r=1}^{2n} A_r\,|p\,(\cos(\varphi+\theta_r))| \\ &\le \frac{1}{2n\cdot\sqrt{1-x^2}}\cdot\|p\|_{[-1,1]}\cdot\sum_{r=1}^{2n} A_r.\end{aligned}$$

Using (4.3.3), the right hand side equals to

$$\frac{1}{\sqrt{1-x^2}}\cdot\frac{q^{n/2}-q^{-n/2}}{q^{1/2}-q^{-1/2}}\cdot\|p\|_{[-1,1]}\,,$$

which is the desired result. □

To prove Theorem 4.3.5 we need the following well-known inequality of Schur's (**[18, Theorem 5.1.9, p.233]**).

Lemma 4.4.4. *For $p\in\mathcal{P}_{n-1}$, the following holds:*

$$\|p\|_{[-1,1]} \le n\cdot\left\|p(x)\cdot\sqrt{1-x^2}\right\|_{[-1,1]}. \tag{4.4.9}$$

Proof of Theorem 4.3.5. For $p\in\mathcal{P}_n$, $\mathscr{D}_q p\in\mathcal{P}_{n-1}$. So, by Schur's inequality (4.4.9), we have

$$\|\mathscr{D}_q p\|_{[-1,1]} \le n \cdot \left\| (\mathscr{D}_q p)(x) \cdot \sqrt{1-x^2} \right\|_{[-1,1]},$$

which is bounded by

$$n \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} \|p\|_{[-1,1]},$$

according to Theorem 4.3.3. □

Proof of Theorem 4.3.7. Using (4.3.1), with $x = \cos\varphi$, we obtain

$$\begin{aligned}
&\int_0^{2\pi} |(\mathscr{D}_q p)(\cos(\varphi))|^2 \sin^2(\varphi)\ d\varphi \\
&= \frac{1}{4n^2} \int_0^{2\pi} \left| \sum_{r=1}^{2n} (-1)^r A_r\ p\left(\cos\left(\varphi + \theta_r\right)\right) \right|^2 d\varphi.
\end{aligned}$$

So, using triangle inequality,

$$\left\{ \int_0^{2\pi} |(\mathscr{D}_q p)(\cos(\varphi))|^2 \sin^2(\varphi)\ d\varphi \right\}^{1/2} \le \frac{1}{2n} \sum_{r=1}^{2n} A_r \left\{ \int_0^{2\pi} \ |\ p\left(\cos\left(\varphi + \theta_r\right)\right)|^2 d\varphi \right\}^{1/2}. \tag{4.4.10}$$

The integral on the right of (4.4.10) can be written as

$$\begin{aligned}
\int_0^{2\pi} \ |\ p\left(\cos\left(\varphi + \theta_r\right)\right)|^2\ d\varphi &= \int_0^{2\pi} \ |\ p\left(\cos\left(\varphi\right)\right)|^2\ d\varphi \\
&= 2 \int_{-1}^{1} \ |\ p\left(x\right)|^2\ \frac{dx}{\sqrt{1-x^2}}.
\end{aligned}$$

Putting this in (4.4.10) and using (4.3.3), we get

$$\begin{aligned}
\left\{ \int_0^{2\pi} |(\mathscr{D}_q p)(\cos\varphi)|^2 \sin^2(\varphi)\ d\varphi \right\}^{1/2} &\le \frac{1}{2n} \sum_{k=1}^{2n} A_k \cdot \left\{ 2 \int_{-1}^{1} \ |\ p\left(x\right)|^2\ \frac{dx}{\sqrt{1-x^2}} \right\}^{1/2} \\
&= \sqrt{2}\ \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} \|p\|_{L^2\left(1/\sqrt{1-x^2}\right)},
\end{aligned}$$

or equivalently

$$\|\mathscr{D}_q p\|_{L^2\left(\sqrt{1-x^2}\right)} \leq \frac{q^{n/2}-q^{-n/2}}{q^{1/2}-q^{-1/2}} \, \|p\|_{L^2\left(1/\sqrt{1-x^2}\right)} \, .$$

$\square$

4.4.3 A second proof for Theorem 4.3.1

An alternative proof of Theorem 4.3.1. Take $p \in \mathcal{P}_n$ to be $p(x) = \sum_{j=0}^{n} a_j x^j$. Then

$$\breve{p}(z) = p\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right) = a_0 + \frac{a_1}{2^1}\left(z+\frac{1}{z}\right) + \frac{a_2}{2^2}\left(z+\frac{1}{z}\right)^2 + \cdots + \frac{a_n}{2^n}\left(z+\frac{1}{z}\right)^n$$

$$= \sum_{j=-n}^{n} c_j z^j \text{ with } c_j = c_{-j} \text{ for } j = 1, 2, \ldots, n \, .$$

With $x = \cos(\varphi)$, we have $p(\cos\varphi) = \breve{p}(e^{i\varphi})$. So

$$p(\cos\varphi) = \sum_{j=-n}^{n} c_j e^{ij\varphi} = c_0 + 2\sum_{j=1}^{n} c_j \cos(j\varphi) \ ,$$

from which we obtain

$$c_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} p(\cos\theta)\cdot\cos(j\theta) \; d\theta, \text{ for } j = 0, 1, 2, \ldots, n.$$

Now we shall compute the difference $\breve{p}(q^{1/2}z) - \breve{p}(q^{-1/2}z)$.

$$\breve{p}(q^{1/2}z) - \breve{p}(q^{-1/2}z) = \sum_{j=-n}^{n} c_j \cdot (q^{j/2} - q^{-j/2}) \cdot z^j$$

$$= \sum_{j=1}^{n} c_j \cdot (q^{-j/2} - q^{j/2}) \cdot (z^{-j} - z^j) \ .$$

Using $z = e^{i\varphi}$ the right hand side equals to

$$
\begin{aligned}
&2i \sum_{j=1}^{n} c_j \cdot (q^{j/2} - q^{-j/2}) \cdot \sin(j\varphi) \\
&= 2i \sum_{j=1}^{n} (q^{j/2} - q^{-j/2}) \cdot \sin(j\varphi) \cdot \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} p(\cos\theta)\cos(j\theta)\ d\theta \right\} \\
&= \frac{i}{\pi} \int_{-\pi}^{\pi} p(\cos\theta) \cdot \sum_{j=1}^{n} (q^{j/2} - q^{-j/2}) \cos(j\theta)\sin(j\varphi)\ d\theta.
\end{aligned}
$$

Since p is a *cosine* polynomial, by orthogonality relations we can write

$$
\begin{aligned}
\breve{p}(q^{1/2}z) - \breve{p}(q^{-1/2}z) &= \frac{i}{\pi} \int_{-\pi}^{\pi} p(\cos\theta) \sum_{j=1}^{n} (q^{j/2} - q^{-j/2}) \\
&\qquad\qquad \times [\ \sin(j\varphi)\cos(j\theta) - \sin(j\theta)\cos(j\varphi)\ ]\ d\theta \\
&= -\frac{i}{\pi} \int_{-\pi}^{\pi} p(\cos\theta) \cdot \sum_{j=1}^{n} (q^{j/2} - q^{-j/2}) \cdot \sin(j(\theta - \varphi))\ d\theta.
\end{aligned}
$$

Replacing θ by $\theta + \varphi$, the right hand side above equals to

$$
-\frac{i}{\pi} \int_{-\pi}^{\pi} p(\cos(\theta + \varphi)) \cdot \sum_{j=1}^{n} (q^{j/2} - q^{-j/2}) \cdot \sin(j\theta)\ d\theta.
$$

As before we use orthogonality relations to get the following:

$$
\begin{aligned}
&\breve{p}(q^{1/2}z) - \breve{p}(q^{-1/2}z) \\
&= -\frac{i}{\pi} \int_{-\pi}^{\pi} p(\cos(\theta + \varphi)) \Bigg\{ (q^{n/2} - q^{-n/2}) \sin(n\theta) \\
&\qquad\qquad + \sum_{j=1}^{n-1} (q^{j/2} - q^{-j/2}) \left[\sin(j\theta) + \sin(2n - j)\theta)\right] \Bigg\}\ d\theta \\
&= -\frac{i}{\pi} \int_{-\pi}^{\pi} p(\cos(\theta + \varphi))\ \sin(n\theta) \left\{ (q^{n/2} - q^{-n/2}) + 2\sum_{j=1}^{n-1} (q^{j/2} - q^{-j/2})\ \cos((n - j)\theta) \right\}\ d\theta.
\end{aligned}
$$

From a similar argument as in the proof of Lemma 4.4.1, for $0 < r < 1$, we take

$$\begin{aligned} h(\theta) &= \frac{1}{4r}\left(\frac{1-r^2}{1-2r\sin\theta+r^2} - \frac{1-r^2}{1+2r\sin\theta+r^2}\right) \\ &= \sin\theta - r^2\sin 3\theta + r^4\sin 5\theta + \dots . \end{aligned}$$

The series on the penultimate line converges uniformly. Thus we have

$$\begin{aligned} &\breve{p}(q^{1/2}e^{i\varphi}) - \breve{p}(q^{-1/2}e^{i\varphi}) \\ &= \lim_{r\to 1^-} \frac{-i}{4\pi r}\int_{-\pi}^{\pi} p(\cos(\theta+\varphi)) \cdot \left\{(q^{n/2}-q^{-n/2}) + 2\sum_{j=1}^{n-1}(q^{j/2}-q^{-j/2})\cdot\cos((n-j)\theta)\right\} \\ &\qquad \times \left\{\frac{1-r^2}{1-2r\cos\left(n\theta-\frac{\pi}{2}\right)+r^2} - \frac{1-r^2}{1+2r\cos\left(n\theta-\frac{\pi}{2}\right)+r^2}\right\} d\theta. \end{aligned}$$

As before in the proof of Lemma 4.4.1: if F is continuous periodic with period 2π, then

$$\begin{aligned} &\lim_{r\to 1^-} \frac{1}{4\pi r}\int_{-\pi}^{\pi} F(\theta)\left\{\frac{1-r^2}{1-2r\cos(n\theta+\alpha)+r^2} - \frac{1-r^2}{1+2r\cos(n\theta+\alpha)+r^2}\right\} d\theta \\ &= \frac{1}{2n}\sum_{r=1}^{2n}(-1)^r F(\theta_{r,\alpha}), \text{ where } \theta_{r,\alpha} = \frac{r\pi}{n} - \frac{\alpha}{n}, \quad r = 1, 2, \dots, 2n. \end{aligned}$$

Apply this result with $\alpha = -\frac{\pi}{2}$ and

$$F(\theta_{r,-\frac{\pi}{2}}) = p(\cos(\theta_{r,-\frac{\pi}{2}}+\varphi))\left\{(q^{n/2}-q^{-n/2}) + 2\sum_{j=1}^{n-1}(q^{j/2}-q^{-j/2})\cdot\cos((n-j)\theta_{r,-\frac{\pi}{2}})\right\}$$

to get

$$\begin{aligned} (\mathscr{D}_q p)(x) &= \frac{\breve{p}(q^{1/2}e^{i\varphi}) - \breve{p}(q^{-1/2}e^{i\varphi})}{i\sin(\varphi)\cdot(q^{1/2}-q^{-1/2})} \\ &= \frac{1}{i\sin(\varphi)}\left\{\frac{i}{2n}\sum_{r=1}^{2n}(-1)^{r+1}\, p\left(\cos\left((2r+1)\frac{\pi}{2n}+\varphi\right)\right)\right. \\ &\qquad \left.\times\left\{(q^{n/2}-q^{-n/2}) + 2\sum_{j=1}^{n-1}(q^{j/2}-q^{-j/2})\cdot\cos\left((n-j)(2r+1)\frac{\pi}{2n}\right)\right\}\right\}. \end{aligned}$$

Let

$$A_r = \frac{(q^{n/2} - q^{-n/2})}{(q^{1/2} - q^{-1/2})} + 2\sum_{j=1}^{n-1} \frac{(q^{(n-j)/2} - q^{-(n-j)/2})}{(q^{1/2} - q^{-1/2})} \cos\left((2r+1)\frac{j\pi}{2n}\right) ,$$

so we have

$$(\mathscr{D}_q p)(x) = \frac{1}{2n\sqrt{1-x^2}} \sum_{r=1}^{2n} (-1)^{r+1} A_r \, p\left(\cos\left((2r+1)\frac{\pi}{2n} + \varphi\right)\right) .$$

To show the non-negativity of the coefficients, A_r, we shall use the following result due to Rogosinski and Szegö ([**59**], p.75): If $\lambda_n \geq 0$, $\lambda_{n-1} - 2\lambda_n \geq 0$ and $\lambda_{j-1} - 2\lambda_j + \lambda_{j+1} \geq 0$ for $j = 1, 2, \ldots, n-1$, then

$$\lambda_0 + 2\sum_{j=1}^{n} \lambda_j \cos j\theta \geq 0,$$

holds for all θ. So we shall choose a λ_j to satisfy the hypothesis of the above result. Let

$$\lambda_j = \frac{q^{(n-j)/2} - q^{-(n-j)/2}}{q^{1/2} - q^{-1/2}} , \quad j = 0, 1, 2, \ldots, n.$$

Note that $\lambda_n = 0$, $\lambda_{n-1} - 2\lambda_n = 1 > 0$, and

$$\begin{aligned} \lambda_{j-1} - 2\lambda_j + \lambda_{j+1} &= q^{(n-j)/2} \cdot \frac{q^{1/2} - 2 + q^{-1/2}}{q^{1/2} - q^{-1/2}} - q^{-(n-j)/2} \cdot \frac{q^{-1/2} - 2 + q^{1/2}}{q^{1/2} - q^{-1/2}} \\ &= \frac{(1 - q^{1/2}) \cdot \left(1 - q^{(n-j)}\right)}{q^{(n-j)/2} \cdot (q^{1/2} + 1)} > 0. \end{aligned}$$

Sum of the coefficients follows as before in the proof of Theorem 4.3.1. □

Remark 4.4.5. The difference between the first proof and the second is merely a phase angle of $\frac{\pi}{n}$ in the interpolation formula.

4.5 On q-differentiability of Brown and Ismail

In this section, we first use our integral form of Bernstein inequality, (4.3.6) to study the concept of q-differentiability introduced in **[19]** by Malcolm Brown and Mourad Ismail. It has a flavor of inverse theorems in approximation. The definition of $\mathscr{D}_q$ given in **[6]** uses values of f at points in the complex plane except $[-1, 1]$. To make it applicable to more general classes of functions other than polynomials, Brown and Ismail **[19]** defined $\mathscr{D}_q$ on a dense subset of $L^2(1/\sqrt{1-x^2})$:

Definition 4.5.1. Let $f \in L^2(1/\sqrt{1-x^2})$, and then f has a *Fourier-Chebyshev expansion* in $L^2(1/\sqrt{1-x^2})$:

$$f(x) = \sum_{n=0}^{\infty} f_n T_n(x). \tag{4.5.1}$$

Let the *Fourier-Chebyshev coefficients* $\{f_n\}$ satisfy

$$\sum_{n=0}^{\infty} |(1-q^n) q^{-n/2} f_n|^2 < \infty. \tag{4.5.2}$$

Then $\mathscr{D}_q f$ is defined as the unique function with the following Fourier-Chebyshev expansion in $L^2(\sqrt{1-x^2})$:

$$(\mathscr{D}_q f)(x) = \sum_{n=1}^{\infty} \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} f_n U_{n-1}(x). \tag{4.5.3}$$

Such functions f are called q-differentiable.

Remark 4.5.2. In Definition 4.5.1, (4.5.2) is satisfied on a dense subset of $L^2(1/\sqrt{1-x^2})$, namely the set of all polynomials. Also $\mathscr{D}_q$ maps polynomials into polynomials which form a dense subset of $L^2(\sqrt{1-x^2})$.

4.5.1 Sufficient condition for a function to have a Askey-Wilson derivative

Our next result provides a sufficient condition for a function f to have a *continuous* Askey-Wilson derivative $\mathscr{D}_q f$.

Theorem 4.5.3. *Let $f \in L^2(1/\sqrt{1-x^2})$. If, for some $\alpha > 1/2$, and for some $p_n \in \mathcal{P}_n$, $n = 1, 2, 3, ...$, we have*

$$\|f - p_n\|_{L^2(1/\sqrt{1-x^2})} = \mathcal{O}\left(q^{\alpha n}\right),$$

as $n \to \infty$, then

(i) *f is q-differentiable and $\mathscr{D}_q f$ is continuous on $[-1, 1]$;*

(ii) $\|\mathscr{D}_q f - \mathscr{D}_q p_n\|_{L^2(\sqrt{1-x^2})} = \mathcal{O}\left(q^{(\alpha-1/2)n}\right)$, *as $n \to \infty$.*

Proof of Theorem 4.5.3. From the hypothesis, for some constant $C > 0$, we have, for $n = 1, 2, 3, \ldots,$

$$\|f - p_n\|_{L^2(1/\sqrt{1-x^2})} \le Cq^{\alpha n}. \tag{4.5.4}$$

Let $\sum_{n=1}^{\infty} f_n T_n(x)$ be the Fourier-Chebyshev expansion of f. Then, for $N = 0, 1, 2, ...,$ by the best approximation property of the partial sums of the Fourier-Chebyshev expansion,

$$\left\| f - \sum_{n=0}^{N} f_n T_n \right\|_{L^2(1/\sqrt{1-x^2})} \le \|f - p_N\|_{L^2(1/\sqrt{1-x^2})} \le Cq^{\alpha N}. \tag{4.5.5}$$

From the orthogonality relations of T_N (see (4.1.6)) and using the triangle inequality yields

$$\begin{aligned} |f_N|\sqrt{\frac{\pi}{2}} = \|f_N T_N\|_{L^2(1/\sqrt{1-x^2})} &= \left\| \left(f - \sum_{n=0}^{N} f_n T_n\right) - \left(f - \sum_{n=0}^{N-1} f_n T_n\right) \right\|_{L^2(1/\sqrt{1-x^2})} \\ &\le \left\| f - \sum_{n=0}^{N} f_n T_n \right\|_{L^2(1/\sqrt{1-x^2})} + \left\| f - \sum_{n=0}^{N-1} f_n T_n \right\|_{L^2(1/\sqrt{1-x^2})}. \end{aligned}$$

Using (4.5.5) to estimate each norm term above yields

$$|f_N|\sqrt{\frac{\pi}{2}} \leq Cq^{\alpha N} + Cq^{\alpha(N-1)} = C_1 q^{\alpha N} \ ,$$

where $C_1 = C(1 + q^{-\alpha})$.

i.e.,

$$|f_N|\sqrt{\frac{\pi}{2}} \leq C_1 q^{\alpha N}. \tag{4.5.6}$$

Thus, (4.5.2) is satisfied and hence f is q-differentiable and (4.5.3) holds. Furthermore, (4.5.6) implies that the series in both (4.5.1) and (4.5.3) converge uniformly for $x \in [-1, 1]$ since we have

$$|T_N(x)| \leq 1 \quad \text{and} \quad |U_N(x)| \leq N + 1$$

there, and thus, both $f(x)$ and $(\mathscr{D}_q f)(x)$ are continuous functions on $[-1, 1]$. This completes the proof of (i). To verify (ii), it suffices to show that

$$\left\| \sum_{n=0}^{N} f_n \mathscr{D}_q T_n - \mathscr{D}_q p_N \right\|_{L^2(\sqrt{1-x^2})} = \mathcal{O}(q^{(\alpha-1/2)N}). \tag{4.5.7}$$

Before we verify (4.5.7), let us observe how it can help us to prove (ii). Indeed, assume that (4.5.7) holds. Then, in view of (4.1.7) and (4.1.8), from (4.5.6) it follows that

$$\left\| \sum_{n=0}^{N} f_n \mathscr{D}_q T_n - \mathscr{D}_q f \right\|^2_{L^2(\sqrt{1-x^2})} = \sum_{n=N+1}^{\infty} \left| f_n \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} \right|^2 \frac{\pi}{2} = \mathcal{O}(q^{2(\alpha-1/2)N}).$$

This and (4.5.7) will verify (ii). To prove (4.5.7), we apply the integral form of Bernstein inequality (4.3.6) to the polynomial $\sum_{n=0}^{N} f_n T_n - p_N \in \mathcal{P}_N$ to obtain

$$\left\| \sum_{n=0}^{N} f_n \mathscr{D}_q T_n - \mathscr{D}_q p_N \right\|_{L^2(\sqrt{1-x^2})} \leq \frac{q^{N/2} - q^{-N/2}}{q^{1/2} - q^{-1/2}} \left\| \sum_{n=0}^{N} f_n T_n - p_N \right\|_{L^2(1/\sqrt{1-x^2})}$$

which, by (4.5.4), is no larger than

$$\frac{q^{N/2}-q^{-N/2}}{q^{1/2}-q^{-1/2}}Cq^{\alpha N}=\mathcal{O}(q^{(\alpha-1/2)N}).$$

This verifies (4.5.7). □

4.5.2 Smoothness of q-differentiable functions

As it turns out, the continuity established in Theorem 4.5.3 is not a special case. Our final result will address the smoothness of any q-differentiable function. So, we turn to the following natural question: *How smooth must* f *be to ensure that* (4.5.2) *is true* ? In this connection, we will give a precise description of all functions f that are q-differentiable in terms of the analytic continuation of f. We do so by casting Definition 4.5.1 in the setting of *overconvergence.*

Analytic continuation or analytic extension refers to the process of extending the domain over which a complex function is defined to a larger one. By the uniqueness theorem any such continuation of a function is uniquely determined. The phenomenon of overconvergence (in a different setting than our consideration) was discovered by M. B. Porter in 1906. It was rediscovered by Robert Jentzsch in 1914 and again by Alexander Ostrowski in 1921. The main ideas for our study come purely from Joseph L. Walsh whose contribution is quite significant on the subject matter (see **[66]**, **[67]**).

Theorem 4.5.4. *Let* K *be a compact set of the* z*-plane, whose complementary set with respect to the entire plane is simply connected. Let* $w=\phi(z)$ *be a function which maps the exterior of* K *conformally onto the exterior of the unit circle in the* w*-plane so that the points at infinity correspond to each other. Let* C_R *denote the curve* $|\phi(z)|=R>1$*, that is, the* *transform in the z-plane of the circle $|w| = R$.*

If the function $f(z)$ is analytic on and within C_R, then there exist polynomials $p_n(z)$ of respective degrees $n = 0, 1, 2, \ldots,$ such that

$$|f(z) - p_n(z)| \leq \frac{M}{R^n} \quad \text{for every } z \in K, \tag{4.5.8}$$

holds, where M is independent of n and z.

Theorem 4.5.5. *Let K be a compact set whose complement is simply connected. Let the polynomials $p_n(z)$ satisfy inequality (4.5.8) for some $R > 1$, where M is independent of n and z. Then the sequence $\{p_n(z)\}$ converges for z interior to C_R, uniformly on any closed point set interior to C_R. The function $f(z)$ can be extended from K along paths interior to C_R so as to be single-valued and analytic at every point interior to C_R.*

The phenomenon illustrated in Theorem 4.5.5 is known as the *Overconvergence*. In formal terms it describes the cases where a sequence of polynomials of best approximation to an analytic function in a given region $\mathcal{G}$ converges to that function (or its analytic continuation) not merely in $\mathcal{G}$ but in a larger region containing $\mathcal{G}$ in its interior. This is achievable provided the rate of convergence of (4.5.8) is fast enough.

Now we are ready to present our final result of this Chapter.

Theorem 4.5.6. *A function f is q-differentiable on $[-1, 1]$ for some $q \in (0, 1)$ if and only if f is analytic over an open set containing the interval $[-1, 1]$ in the complex plane.*

***Proof*.** Suppose f is q-differentiable on $[-1, 1]$ for some $q \in (0, 1)$. In view of Definition 4.5.1,

$$f(x) = \sum_{n=0}^{\infty} f_n T_n(x) \text{ in } L^2(1/\sqrt{1 - x^2}) \tag{4.5.9}$$

and

$$\sum_{n=0}^{\infty} |(1-q^n)q^{-n/2} f_n|^2 < \infty.$$

Consequently, we have $|q^{-n/2} f_n| \le C$ for some $C > 0$ or equivalently $|f_n| \le C q^{n/2}$. Take $R = 1/q^{1/2}$, so that $R > 1$ and

$$|f_n| \le \frac{C}{R^n}, \quad n = 0, 1, 2, \cdots . \tag{4.5.10}$$

Thus the series defined by

$$\tilde{f}(x) := \sum_{n=0}^{\infty} f_n T_n(x) \tag{4.5.11}$$

is uniformly convergent for $x \in [-1, 1]$. Moreover,

$$\left| \tilde{f}(x) - \sum_{n=0}^{N} f_n T_n(x) \right| = \left| \sum_{n=N+1}^{\infty} f_n T_n(x) \right| \le \sum_{n=N+1}^{\infty} |f_n||T_n(x)| \le \frac{C'}{R^N},$$

where $C' = C/(R-1)$.

Therefore, by Theorem 4.5.5, $\tilde{f}$ is analytic on $E_{R'} := \{x = (z + 1/z)/2 \mid 1 \le |z| \le R'\}$ for all $R' \in (1, R)$. Note that

$$\int_{-1}^{1} \left| \tilde{f}(x) - \sum_{n=0}^{N} f_n T_n(x) \right|^2 \frac{dx}{\sqrt{1-x^2}} = \sum_{n=N+1}^{\infty} \frac{\pi}{2} |f_n|^2 \to 0 \ ,$$

as $N \to \infty$.

This, together with (4.5.9), implies that

$$\tilde{f}(x) = \sum_{n=0}^{\infty} f_n T_n(x) = f(x) \text{ a.e. for } x \in [-1, 1]. \tag{4.5.12}$$

Thus, f has an analytic continuation to the interior of E_R which contains $[-1, 1]$.

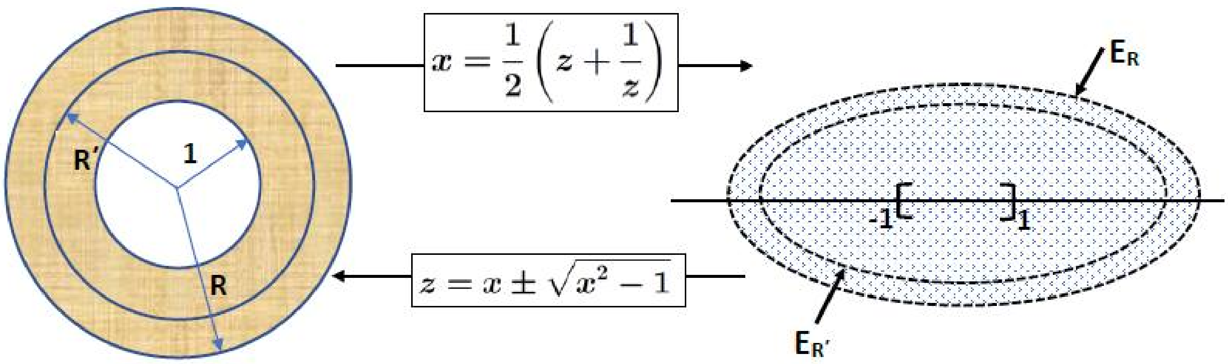

Figure 4.1
Joukowsky transformation

Conversely, suppose that a function f is analytic over an open set G. Choose $R > 1$ such that $E_R \subseteq G$. Then, by Theorem 4.5.4, there exist $M > 0$ and a sequence of polynomials $\{P_N(z)\}$ such that

$$|f(z) - P_N(z)| \leq \frac{M}{R^N}, \quad z \text{ on } E_R. \tag{4.5.13}$$

Consider the Fourier-Chebyshev expansion

$$f(x) = \sum_{n=0}^{\infty} c_n T_n(x). \tag{4.5.14}$$

We have

$$\begin{aligned} \left\| f(x) - \sum_{n=0}^{N} c_n T_n(x) \right\|_{L^2(1/\sqrt{1-x^2})} &:= \min_{p_N \in \mathcal{P}_N} \|f(x) - p_N(x)\|_{L^2(1/\sqrt{1-x^2})} \\ &\leq \|f(x) - P_N(x)\|_{L^2(1/\sqrt{1-x^2})} . \end{aligned}$$

By (4.5.13), the right hand side is less than or equals to

$$\frac{M}{R^N} \left\{ \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} \right\}^{1/2} = \frac{\sqrt{\pi} M}{R^N}.$$

Using this we can estimate $|c_n|$:

$$\begin{aligned}
&\|c_N T_N\|_{L^2(1/\sqrt{1-x^2})} \\
&\leq \left\| f(x) - \sum_{n=0}^{N} c_n T_n \right\|_{L^2(1/\sqrt{1-x^2})} + \left\| f(x) - \sum_{n=0}^{N-1} c_n T_n \right\|_{L^2(1/\sqrt{1-x^2})} \\
&\leq \frac{\sqrt{\pi} M}{R^N} + \frac{\sqrt{\pi} M}{R^{N-1}} = \frac{M'}{R^N},
\end{aligned}$$

where $M' = \sqrt{\pi} M(R+1)$. Since $\|T_N\|_{L^2(1/\sqrt{1-x^2})} = \sqrt{\pi/2}$, we get, with $M'' = \sqrt{2} M(R+1)$,

$$|c_N| \leq \frac{M''}{R^N}.$$

Then, if $q \in (R^{-2}, 1)$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \left|(1-q^n) q^{-n/2} \cdot c_n\right|^2 &\leq \sum_{n=0}^{\infty} \left|(1-q^n) q^{-n/2} \cdot \frac{M''}{R^n}\right|^2 \\
&= M'' \sum_{n=0}^{\infty} (1-q^n)^2 \left(\frac{1}{qR^2}\right)^n < \infty.
\end{aligned}$$

So f is q-differentiable for all $q \in \big(R^{-2}, 1\big)$. $\square$

Remark 4.5.7. The proof of Theorem 4.5.6 above indeed shows that

(i) if f is analytic on $E_{1/q^{1/2}}$, then f is q-differentiable;

(ii) if f is q-differentiable, then f is analytic on $\mathring{E}_{1/q^{1/2}}$, the interior of $E_{1/q^{1/2}}$.

CHAPTER 5: ACTION OF THE ASKEY-WILSON OPERATOR ON ENTIRE FUNCTIONS.

Somebody came up to me after a talk I had given, and say, "You make mathematics seem like fun." I was inspired to reply, "If it isn't fun, why do it?"

Ralph Boas Jr.

5.1 Definitions

In Chapter 4 we have already introduced the Askey-wilson operator in the setting of polynomials. In this chapter, we consider only entire functions; complex-valued functions which are analytic in the finite complex plane, $\mathbb{C}$.

Definition 5.1.1 (**Entire Functions of exponential type**)**.** Let $\mathscr{B}_\sigma$ denote the set of entire functions of exponential type σ. That is, $f \in \mathscr{B}_\sigma$ if f is an entire function and for any $\varepsilon > 0$, there is an $A_\varepsilon > 0$ such that $|f(z)| \leq A_\varepsilon e^{(\sigma+\varepsilon)|z|}$, for all $z \in \mathbb{C}$.

Next we state few examples and elementary properties of entire functions of exponential type (see [**15**], [**22**]):

- Functions of exponential type $\sigma > 1$ include all functions of type less than or equal to σ, type 1, and functions of type less than 1.
- Rational entire functions are of exponential type zero. In particular polynomials $p(z)$ are of exp. type zero, as for large values of $z \in \mathbb{C}$, $|p(z)|$ is dominated by $e^{\epsilon|z|}$, for any $\epsilon > 0$.

- If a is a constant, the function e^{az} is of exponential type. $|a|$.
- If $f(x)$ is of exponential type σ, then $f(ax+b)$ is of exp. type $|a|\sigma$, when a and b are constants.
- If $f_1(x)$ and $f_2(x)$ are of exponential types σ_1 and σ_2, respectively, then the product $f_1(x)\cdot f_2(x)$ is of exponential type not exceeding $\sigma_1+\sigma_2$.

Ralph Boas Jr (see **[14]**, **[15, pp.210-211]**), in providing a simpler and elegant proof of a Bernstein's inequality given in **[12]**, established an interesting interpolating formula for the derivatives of functions in $\mathscr{B}_\sigma$, known as Boas' formula, which is a generalization of an interpolation formula of Marcel Riesz **[57]** for trigonometric polynomials:

Theorem 5.1.2. *If $f\in\mathscr{B}_\sigma$ and is bounded on the real line $\mathbb{R}$, then the following holds:*

$$f'(x)=\frac{4\sigma}{\pi^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{(2n+1)^2}\,f\left(x+\frac{\pi}{2\sigma}+\frac{n\pi}{\sigma}\right). \tag{5.1.1}$$

In this chapter, we extend Boas' formula by replacing the differentiation with the Askey-Wilson operator, and then show its power in discovering summation formulas.

For the sake of completeness we re-state the definition of the Askey-Wilson operator in the setting of entire functions.

Definition 5.1.3. Let f be an entire function and $q\in(0,1)$. Then from (4.1.2) we have the following equivalent form of (4.1.3):

$$(\mathscr{D}_q f)(x)=\frac{\breve{f}(q^{1/2}z)-\breve{f}(q^{-1/2}z)}{i\sin\theta\cdot(q^{1/2}-q^{-1/2})}\ , \tag{5.1.2}$$

where

$$\breve{f}(z) = f\left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right), \; z = e^{i\theta}, \; x = \cos\theta.$$

Recall that the definition of $\mathscr{D}_q$ given in **[6]** was mainly used to act on polynomials f and it uses values of f at points in $\mathbb{C} \setminus [-1, 1]$. To extend the domain of the operator to act on more general classes of functions, Brown and Ismail **[19]** proposed an approach to define $\mathscr{D}_q$ on a dense subset of $L^2[(1 - x^2)^{-1/2}, [-1, 1]]$. (see our discussion on this in section 4.5) In this chapter, we consider only entire functions so $\mathscr{D}_q$ is well-defined as in (5.1.2), even for $x \in \mathbb{C}$.

5.2 Main result: Generalized Boas' formula

To set the stage to our main result in this chapter, we first introduced a convenient way of writing the Askey-Wilson operator. The novelty of our method lies in the following two parameter family: with $x = \cos\theta$, write

$$\alpha := \frac{1}{2}(q^{1/2} + q^{-1/2}) \; \cos(\theta) \text{ and } \beta := (q^{1/2} - q^{-1/2}) \; \sin(\theta). \tag{5.2.1}$$

Note that, when $(x, q) \in [-1, 1] \times (0, 1)$, we have $\alpha, \beta \in \mathbb{R}$. Now, we are ready to state our main result.

Theorem 5.2.1. *Assume that $f \in \mathscr{B}_\sigma$ and the restriction of f on $\mathbb{R}$ is bounded. Then, for $x \in [-1, 1]$,*

$$\left(\mathscr{D}_q f\right)(x) = \frac{4}{\sigma} \sum_{k=-\infty}^{\infty} f\left(\alpha + \frac{\pi}{2\sigma}(2k+1)\right) \; \frac{(-1)^k \cosh\left(\frac{\sigma}{2}\beta\right)}{\beta^2 + (2k+1)^2 \pi^2/\sigma^2}. \tag{5.2.2}$$

Remark 5.2.2. When $q \to 1^-$, we have $\alpha \to x$ and $\beta \to 0$, and thus, the limiting case of

(5.2.2) becomes the classical *Boas' formula* (5.1.1).

Remark 5.2.3. In view of our two parameter family, (5.2.1), we can write

$$\begin{aligned} q^{1/2}z + \frac{1}{q^{1/2}z} &= q^{1/2}e^{i\theta} + q^{-1/2}e^{-i\theta} \\ &= q^{1/2} \cdot (\cos(\theta) + i\sin(\theta)) + q^{-1/2} \cdot (\cos(\theta) - i\sin(\theta)) \\ &= (q^{1/2} + q^{-1/2})\cos(\theta) + i\sin(\theta)(q^{1/2} - q^{-1/2}) \\ &= 2\alpha + i\beta \ , \end{aligned}$$

and

$$\begin{aligned} q^{-1/2}z + \frac{1}{q^{-1/2}z} &= q^{-1/2}e^{i\theta} + q^{1/2}e^{-i\theta} \\ &= q^{-1/2} \cdot (\cos(\theta) + i\sin(\theta)) + q^{1/2} \cdot (\cos(\theta) - i\sin(\theta)) \\ &= (q^{1/2} + q^{-1/2})\cos(\theta) - i\sin(\theta)(q^{1/2} - q^{-1/2}) \\ &= 2\alpha - i\beta \ . \end{aligned}$$

So we can write the numerator term in (5.1.2) as

$$\begin{aligned} \breve{f}(q^{1/2}z) - \breve{f}(q^{-1/2}z) &= f\left(\frac{1}{2}\left(q^{1/2}z + \frac{1}{q^{1/2}z}\right)\right) - f\left(\frac{1}{2}\left(q^{-1/2}z + \frac{1}{q^{-1/2}z}\right)\right) \\ &= f\left(\alpha + \frac{i\beta}{2}\right) - f\left(\alpha - \frac{i\beta}{2}\right) \ . \end{aligned}$$

Accordingly, now we can write (5.1.2) as:

$$(\mathscr{D}_q f)(x) = \frac{f\left(\alpha + \frac{i\beta}{2}\right) - f\left(\alpha - \frac{i\beta}{2}\right)}{i\beta} \ . \tag{5.2.3}$$

Observe that the left hand side of (5.2.3) depends on x and q while the right hand side depends on α and β. So we will introduce a new notation, $(\mathscr{D}f)(\alpha, \beta)$ to denote the right-hand side to emphasize the dependence of $(\mathscr{D}_q f)(x)$ on α and β.

Remark 5.2.4. Theorem 5.2.1 holds under the restriction on $(x, q) \in [-1, 1] \times (0, 1)$. Indeed, the theorem holds for complex x and q. Our next result is such an example by using variables α and β.

Corollary 5.2.5. *Under the same assumptions on f as in Theorem 5.2.1, we have*

$$(\mathscr{D}f)(\alpha, \beta) = \frac{4}{\sigma} \sum_{k=-\infty}^{\infty} f\left(\alpha + \frac{\pi}{2\sigma}(2k+1)\right) \frac{(-1)^k \cosh\left(\frac{\sigma}{2}\beta\right)}{\beta^2 + (2k+1)^2 \pi^2/\sigma^2}, \tag{5.2.4}$$

holds for all $\alpha, \beta \in \mathbb{C}$ and the convergence is locally uniform for $\alpha, \beta \in \mathbb{C}$.

5.2.1 Classical sampling theorem

The proof of Theorem 5.2.1 is based on the well-celebrated WSK sampling theorem, which was named after Edmound Whittaker, John Whittaker, Vladmir Kotelnikov and Claude Shannon. Among the four, the latter gets much of the recognition solely for him being the pioneer of information theory where it is heavily used. In layman's terms, the theorem states that a signal (a function) can be reconstructed from its samples, evaluated at uniformly spaced points on the real line. For comprehensive studies of the sampling theorem see [**36**], [**60**], [**72**], and references therein.

There are many variations of the sampling theorem. The form we use here is due to Paul Butzer, Gerhard Schmeisser, and Rudolf Stens as used in [**21**].

Theorem 5.2.6. *If $f \in \mathscr{B}_\sigma$ and f is bounded on $\mathbb{R}$, then*

$$f(x) = \sum_{k=-\infty}^{+\infty} f\left(\frac{k\pi}{\sigma}\right) \operatorname{sinc}\left[\frac{\sigma}{\pi}\left(x - \frac{k\pi}{\sigma}\right)\right] \quad (x \in \mathbb{C}), \tag{5.2.5}$$

the convergence being absolute and uniform on compact subsets of $\mathbb{C}$.

Definition 5.2.7. The sinc function is defined as:

$$\operatorname{sinc} x := \begin{cases} \dfrac{\sin(\pi x)}{\pi x} & \text{for } x \in \mathbb{C}\backslash\{0\}, \\ 1 & \text{for } x = 0. \end{cases}$$

The Boas' formula (5.1.1) can be obtained from the Sampling theorem in the following manner [?]. Starting from (5.2.5), apply it to the function $g_y(x) = f(x+y)$ to get

$$g_y(x) = \sum_{k=-\infty}^{+\infty} f\left(y + \frac{k\pi}{\sigma}\right) \operatorname{sinc}\left[\frac{\sigma}{\pi}\left(x - \frac{k\pi}{\sigma}\right)\right] \quad (x \in \mathbb{C}),$$

from which we can recover f:

$$f(x) = g_y(x-y) = \sum_{k=-\infty}^{+\infty} f\left(y + \frac{k\pi}{\sigma}\right) \operatorname{sinc}\left[\frac{\sigma}{\pi}\left(x - y - \frac{k\pi}{\sigma}\right)\right] .$$

In view of the absolute and uniform convergence on compact subsets of $\mathbb{C}$ of the series above, we can differentiate term by term to get

$$f'(x) = \sum_{k=-\infty}^{\infty} f\left(y + \frac{k\pi}{\sigma}\right) \frac{d}{dx}\left\{\frac{\sin\left[\sigma\left(x - y - \frac{k\pi}{\sigma}\right)\right]}{\sigma\left(x - y - \frac{k\pi}{\sigma}\right)}\right\}$$

$$= \sum_{k=-\infty}^{\infty} f\left(y + \frac{k\pi}{\sigma}\right)\left\{\frac{\cos\left[\sigma\left(x - y - \frac{k\pi}{\sigma}\right)\right]\cdot\sigma\cdot\sigma\left(x - y - \frac{k\pi}{\sigma}\right) - \sin\left[\sigma\left(x - y - \frac{k\pi}{\sigma}\right)\right]\cdot\sigma}{\sigma^2\left(x - y - \frac{k\pi}{\sigma}\right)^2}\right\}.$$

Let $y = x + \dfrac{\pi}{2\sigma}$, to get

$$f'(x) = \sum_{k=-\infty}^{\infty} f\left(x + \frac{(2k+1)\pi}{2\sigma}\right) \frac{-\cos\left[(2k+1)\frac{\pi}{2}\right]\cdot(\sigma)\cdot(2k+1)\frac{\pi}{2} + \sigma\sin\left[(2k+1)\frac{\pi}{2}\right]}{(2k+1)^2\frac{\pi^2}{4}},$$

which equals to

$$f'(x) = \frac{4\sigma}{\pi^2} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(2k+1)^2}\, f\left(x + \frac{(2k+1)\pi}{2\sigma}\right),$$

which is the Boas' formula, (5.1.1).

5.3 Proof of the main result

This prime purpose of this section is to present the proof of Theorem 5.2.1, which will be accomplished by applying the Askey-Wilson operator on the Classical Sampling theorem, Theorem 5.2.6. We first establish a lemma that gives us the action of the Askey-Wilson operator on the sinc function. To indicate that the operator $\mathscr{D}_q$ is applied with respect to x, we will use the notation $\mathscr{D}_{q,x}$.

Lemma 5.3.1. *Let $x \in \mathbb{C}$ and k be any integer. Then the following holds:*

$$\left(\mathscr{D}_{q,x}\left(\operatorname{sinc}\left[\frac{\sigma}{\pi}(x - y - \frac{k\pi}{\sigma})\right]\right)\right)\Bigg|_{y=\alpha+\pi/(2\sigma)} = \frac{4}{\sigma} \cdot \frac{(-1)^k \cdot \cosh\left(\frac{\sigma}{2}\beta\right)}{\beta^2 + (2k+1)^2 \frac{\pi^2}{\sigma^2}}. \tag{5.3.1}$$

Remark 5.3.2. The key feature of the lemma is the evaluation of y at a point $\alpha + \frac{\pi}{2\sigma}$ that is independent of k.

Proof. Let $g(x) := \operatorname{sinc}\left[\frac{\sigma}{\pi}(x - y - \frac{k\pi}{\sigma})\right]$. Then, by the definition of sinc function, we have

$$g(x) = \frac{\sin\left(\sigma(x - y - \frac{k\pi}{\sigma})\right)}{\sigma(x - y - \frac{k\pi}{\sigma})}.$$

Note that

$$\breve{g}(q^{1/2}z) = \frac{\sin\left(\frac{\sigma}{2}\cdot\left(q^{1/2}z + q^{-1/2}z^{-1}\right) - (\sigma y + k\pi)\right)}{\frac{\sigma}{2}\cdot(q^{1/2}z + q^{-1/2}z^{-1}) - (\sigma y + k\pi)}, \tag{5.3.2}$$

and

$$\breve{g}(q^{-1/2}z) = \frac{\sin\left(\frac{\sigma}{2}\cdot\left(q^{-1/2}z+q^{1/2}z^{-1}\right)-(\sigma y+k\pi)\right)}{\frac{\sigma}{2}\cdot(q^{-1/2}z+q^{1/2}z^{-1})-(\sigma y+k\pi)}. \tag{5.3.3}$$

So, we have

$$\left(\mathscr{D}_{q,x}\left(\operatorname{sinc}\left[\frac{\sigma}{\pi}(x-y-\frac{k\pi}{\sigma})\right]\right)\right)(x) = \frac{\breve{g}(q^{1/2}z)-\breve{g}(q^{-1/2}z)}{i\sin\theta\cdot(q^{1/2}-q^{-1/2})} \tag{5.3.4}$$

with

$$\begin{aligned}
&\breve{g}(q^{1/2}z)-\breve{g}(q^{-1/2}z)\\
&= \frac{\sin\left(\frac{\sigma}{2}\left(q^{1/2}z+\frac{1}{q^{1/2}z}\right)-\sigma y-k\pi\right)}{\frac{\sigma}{2}\left(q^{1/2}z+\frac{1}{q^{1/2}z}\right)-\sigma y-k\pi} - \frac{\sin\left(\frac{\sigma}{2}\left(\frac{z}{q^{1/2}}+\frac{q^{1/2}}{z}\right)-\sigma y-k\pi\right)}{\frac{\sigma}{2}\left(\frac{z}{q^{1/2}}+\frac{q^{1/2}}{z}\right)-\sigma y-k\pi}\\
&= (-1)^k\left[\frac{\sin\left(-\frac{\sigma}{2}\left(q^{1/2}z+\frac{1}{q^{1/2}z}\right)+\sigma y\right)}{-\frac{\sigma}{2}\left(q^{1/2}z+\frac{1}{q^{1/2}z}\right)+\sigma y+k\pi} - \frac{\sin\left(\frac{\sigma}{2}\left(\frac{z}{q^{1/2}}+\frac{q^{1/2}}{z}\right)-\sigma y\right)}{\frac{\sigma}{2}\left(\frac{z}{q^{1/2}}+\frac{q^{1/2}}{z}\right)-\sigma y-k\pi}\right].
\end{aligned}$$

Note that, when $y=\alpha+\frac{\pi}{2\sigma}$, we have $y=\frac{1}{4}(q^{1/2}+q^{-1/2})(z+z^{-1})+\frac{\pi}{2\sigma}$ and

$$-\frac{1}{2}\left(q^{1/2}z+\frac{1}{q^{1/2}z}\right)+y-\frac{\pi}{2\sigma} = \frac{1}{2}\left(\frac{z}{q^{1/2}}+\frac{q^{1/2}}{z}\right)-y+\frac{\pi}{2\sigma}. \tag{5.3.5}$$

Let the common value of the two sides of (5.3.5) be w. Then

$$w = \frac{1}{4}(q^{1/2}-q^{-1/2})(z^{-1}-z) = \frac{1}{4}(q^{1/2}-q^{-1/2})(-2i)\sin\theta = -\frac{1}{2}i\beta. \tag{5.3.6}$$

Thus, we can write

$$\begin{aligned}
\breve{g}(q^{1/2}z)-\breve{g}(q^{-1/2}z) &= (-1)^k\left\{\frac{\sin\left(\sigma w+\frac{\pi}{2}\right)}{\sigma w+\frac{\pi}{2}+k\pi} - \frac{\sin\left(\sigma w-\frac{\pi}{2}\right)}{\sigma w-\frac{\pi}{2}-k\pi}\right\}\\
&= \frac{(-1)^k\cdot 2w\cos(\sigma w)}{\sigma\cdot\left(w^2-\left(k+\frac{1}{2}\right)^2\frac{\pi^2}{\sigma^2}\right)}.
\end{aligned}$$

From this and (5.3.6), (5.3.4) yields

$$\left(\mathscr{D}_{q,x}(\operatorname{sinc}\left[\frac{\sigma}{\pi}(x-y-\frac{k\pi}{\sigma})\right])\right)\Big|_{y=\alpha+\pi/(2\sigma)} = \frac{4}{\sigma}\cdot\frac{(-1)^k\cdot\cosh\left(\frac{\sigma}{2}\beta\right)}{\beta^2+(2k+1)^2\,\frac{\pi^2}{\sigma^2}},$$

which is the desired result, (5.3.1). □

Proof of Theorem 5.2.1. We start by introducing a translation parameter in the *Sampling Theorem*, Theorem 5.2.6: Fix $y\in\mathbb{R}$ and apply Theorem 5.2.6 to $g_y(x) := f(x+y)$ to obtain

$$g_y(x) = \sum_{k=-\infty}^{\infty} f(y+\frac{k\pi}{\sigma})\operatorname{sinc}\left[\frac{\sigma}{\pi}(x-\frac{k\pi}{\sigma})\right] .$$

Then

$$f(x) = g_y(x-y) = \sum_{k=-\infty}^{\infty} f(y+\frac{k\pi}{\sigma})\operatorname{sinc}\left[\frac{\sigma}{\pi}(x-y-\frac{k\pi}{\sigma})\right]. \tag{5.3.7}$$

Now, apply $\mathscr{D}_q$ with respect to x on both sides of (5.3.7) to obtain

$$\left(\mathscr{D}_q f\right)(x) = \sum_{k=-\infty}^{\infty} f(y+\frac{k\pi}{\sigma})\ \mathscr{D}_{q,x}\left(\operatorname{sinc}\left[\frac{\sigma}{\pi}(x-y-\frac{k\pi}{\sigma})\right]\right).$$

The left-hand side is independent of y and so we can take a special value of y on the right-hand side. Letting $y=\alpha+\frac{\pi}{2\sigma}$ and using (5.3.1) of Lemma 5.3.1, for $x\in[-1,1]$, we get

$$\left(\mathscr{D}_q f\right)(x) = \sum_{k=-\infty}^{\infty} f\left(\alpha+(k+\frac{1}{2})\frac{\pi}{\sigma}\right)\ \frac{4}{\sigma}\cdot\frac{(-1)^k\cdot\cosh\left(\frac{\sigma}{2}\beta\right)}{\beta^2+(2k+1)^2\,\frac{\pi^2}{\sigma^2}},$$

which is (5.2.2). □

To prove Corollary 5.2.5 we employ the following uniqueness (or identity) theorem.

Theorem 5.3.3. ([**1**]) *Let each of the two functions $f(z)$ and $g(z)$ be analytic in a common domain D. If $f(z)$ and $g(z)$ coincide in some sub-portion $D'\subset D$ or on a curve Γ interior* *to D, then $f(z) = g(z)$ everywhere in D.*

Proof of Corollary 5.2.5. Let $(\alpha, \beta) \in K$, where K is a compact subset in $\mathbb{C}^2$. Note first that both sides of (5.2.4) are entire functions of α and of β, of exponential types σ and $\sigma/2$ respectively. Since as (x, q) runs through $[-1, 1] \times (0, 1)$, $\alpha, \beta \in \mathbb{R}$, both sides of (5.2.4) are equal by Theorem 5.2.1. Thus, by the Identity Theorem, (5.2.4) holds for all $\alpha, \beta \in \mathbb{C}$. Finally, we shall prove the local uniform convergence of the series in (5.2.4). To this end, we need an estimate of Boas (**[15, p.84]**): for $\varepsilon > 0$, there is an $A_\varepsilon > 0$ such that

$$|f(z)| \leq A_\varepsilon e^{(\sigma+\varepsilon)|\mathrm{Im}(z)|} \quad \text{for } z \in \mathbb{C}.$$

Applying this to the series in (5.2.4), yields

$$\begin{aligned}
|(\mathscr{D}f)(\alpha,\beta)| &= \left| \frac{4}{\sigma} \sum_{k=-\infty}^{\infty} f\left(\alpha + \frac{\pi}{2\sigma}(2k+1)\right) \frac{(-1)^k \cosh\left(\frac{\sigma}{2}\beta\right)}{\beta^2 + (2k+1)^2 \pi^2/\sigma^2} \right| \\
&\leq \frac{4}{\sigma} \sum_{k=-\infty}^{\infty} \left| f\left(\alpha + \frac{\pi}{2\sigma}(2k+1)\right) \right| \frac{\left|\cosh\left(\frac{\sigma}{2}\beta\right)\right|}{|\beta^2 + (2k+1)^2 \pi^2/\sigma^2|} \\
&\leq \frac{4}{\sigma} \sum_{k=-\infty}^{\infty} A_\varepsilon e^{(\sigma+\varepsilon)\cdot\left|\mathrm{Im}(\alpha + \frac{\pi}{2\sigma}(2k+1))\right|} \frac{\left|\cosh\left(\frac{\sigma}{2}\beta\right)\right|}{|\beta^2 + (2k+1)^2 \pi^2/\sigma^2|} \\
&= \frac{4}{\sigma} \cdot A_\varepsilon e^{(\sigma+\varepsilon)\cdot|\mathrm{Im}(\alpha)|} \cdot \left|\cosh\left(\frac{\sigma}{2}\beta\right)\right| \sum_{k=-\infty}^{\infty} \frac{1}{|\beta^2 + (2k+1)^2 \pi^2/\sigma^2|} \, .
\end{aligned}$$

Note that $e^{(\sigma+\varepsilon)\cdot|\mathrm{Im}(\alpha)|}$ and $\left|\cosh\left(\frac{\sigma}{2}\beta\right)\right|$ are both bounded in K. Now, $|\beta^2 + (2k+1)^2 \pi^2/\sigma^2| \geq (2k+1)^2 \pi^2/\sigma^2 - |\beta|^2$ and if $|k| > |\beta|$,

$$\frac{1}{|\beta^2 + (2k+1)^2 \pi^2/\sigma^2|} \leq \frac{1}{(2k+1)^2 \pi^2/\sigma^2 - |\beta|^2} \, .$$

Thus by the *Weierstrass M-test*, the series in (5.2.4) is locally uniformly convergent for all $\beta \in \mathbb{C} \setminus \{\pm(2k+1)\frac{i\pi}{\sigma}\}_{-\infty}^{\infty}$. $\square$

5.4 Identities of infinite series

5.4.1 Overview

To elucidate the capability of the extended Boas' formula (Theorem 5.2.1) and its Corollary 5.2.3, we derive identities on infinite series, some new and some known. Several other series identities can be established as a by product. In essence, we pick few candidates that satisfy the hypotheses of the aforementioned and apply either (5.2.2) or (5.2.4) accordingly. We begin with two general remarks.

(i) As a direct consequence from the locally uniform convergence in (5.2.4), convergence in series below is locally uniform in α and β.

(i) The extra parameter q introduced in the Askey-Wilson operator (5.2.2), which is not available in Boas' formula, will be seen as a desirable feature.

5.4.2 Applications of the main result

First, we apply (5.2.2) for $f(x) = 1$. It can be easily seen that $\mathscr{D}_q f(x) = 0$. So we obtain

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{\beta^2 + (2k+1)^2} = 0, \tag{5.4.1}$$

which can be directly verified. Next, take the function $f(x) = \sin(\sigma x)$. Applying (5.2.3) yields

$$\begin{aligned}(\mathscr{D} f)\sin(\sigma x) &= \frac{\sin\left[\sigma\left(\alpha + \frac{i\beta}{2}\right)\right] - \sin\left[\sigma\left(\alpha - \frac{i\beta}{2}\right)\right]}{i\beta} \\ &= \frac{2\cos(\sigma\alpha)\sin(\frac{i\sigma\beta}{2})}{i\beta}.\end{aligned}$$

Since $\sin(i\Theta) = i\sinh(\Theta)$ for any Θ, it follows that

$$(\mathscr{D}f)\sin(\sigma x) = \frac{2\cos(\sigma\alpha)\sinh(\frac{\sigma}{2}\beta)}{\beta}. \tag{5.4.2}$$

This is the left hand side of (5.2.4). The right hand side of (5.2.4) is

$$\begin{aligned}
&\frac{4}{\sigma}\cosh\left(\frac{\sigma}{2}\beta\right)\sum_{k=-\infty}^{\infty}\sin\left[\sigma(\alpha+\frac{\pi}{2\sigma}(2k+1))\right]\frac{(-1)^k}{\beta^2+(2k+1)^2\pi^2/\sigma^2}\\
&=\frac{4}{\sigma}\cosh\left(\frac{\sigma}{2}\beta\right)\sum_{k=-\infty}^{\infty}\cos(\sigma\alpha)\frac{1}{\beta^2+(2k+1)^2\pi^2/\sigma^2}.
\end{aligned}$$

Combining this with (5.4.2), we get

$$\frac{\sigma\tanh(\frac{\sigma}{2}\beta)}{2\beta} = \sum_{k=-\infty}^{\infty}\frac{1}{\beta^2+(2k+1)^2\pi^2/\sigma^2}. \tag{5.4.3}$$

Note that, by Corollary 5.2.5, (5.4.3) holds for all $\beta \in \mathbb{C}\backslash\{\pm(2k+1)\frac{i\pi}{\sigma}\}_{-\infty}^{\infty}$. This is equivalent to a known result, see, e.g., **[70, p. 136]** or **[31, 1.421.2]**.

Here is one more identity that can be derived directly from (5.2.4): Let $f(x) = \operatorname{sinc} x$. Note that

$$(\mathscr{D}f)(\alpha,\beta) = \frac{-4\beta\sin(\pi\alpha)\cosh(\frac{\pi}{2}\beta)+8\alpha\cos(\pi\alpha)\sinh(\frac{\pi}{2}\beta)}{\pi\beta(4\alpha^2+\beta^2)}.$$

Using this in (5.2.2), we have

$$\begin{aligned}
&\frac{-4\beta\sin(\pi\alpha)\cosh(\frac{\pi}{2}\beta)+8\alpha\cos(\pi\alpha)\sinh(\frac{\pi}{2}\beta)}{\pi\beta(4\alpha^2+\beta^2)}\\
&=\frac{4}{\pi}\cosh\left(\frac{\pi}{2}\beta\right)\sum_{k=-\infty}^{\infty}\frac{(-1)^k\sin(\pi(\alpha+k+\frac{1}{2}))}{\pi(\alpha+k+\frac{1}{2})(\beta^2+(2k+1)^2)}\\
&=\frac{4}{\pi}\cosh\left(\frac{\pi}{2}\beta\right)\sum_{k=-\infty}^{\infty}\frac{\cos(\pi\alpha)}{\pi(\alpha+k+\frac{1}{2})(\beta^2+(2k+1)^2)}.
\end{aligned}$$

So, dividing both sides of the above by $\frac{4}{\pi}\cosh(\frac{\pi}{2}\beta)\cos(\pi\alpha)$, we get

$$\frac{-\pi\beta\tan(\pi\alpha)+2\pi\alpha\tanh(\frac{\pi}{2}\beta)}{\beta(4\alpha^2+\beta^2)}=\sum_{k=-\infty}^{\infty}\frac{1}{(\alpha+k+\frac{1}{2})(\beta^2+(2k+1)^2)}, \tag{5.4.4}$$

which implies several known identities as special cases. For example, writing (5.4.4) as

$$\frac{-\pi\tan(\pi\alpha)}{4\alpha^2+\beta^2}+\frac{2\pi\alpha}{4\alpha^2+\beta^2}\cdot\frac{\tanh(\frac{\pi}{2}\beta)}{\beta}=\sum_{k=-\infty}^{\infty}\frac{1}{(\alpha+k+\frac{1}{2})(\beta^2+(2k+1)^2)},$$

and letting $\beta\to 0$ yields

$$\frac{-\pi\tan(\pi\alpha)}{4\alpha^2}+\frac{\pi^2}{4\alpha}=\sum_{k=-\infty}^{\infty}\frac{1}{(\alpha+k+\frac{1}{2})(2k+1)^2},\qquad \alpha\neq 0.$$

Set $\alpha=1$ in the above to get

$$\frac{\pi^2}{8}=\sum_{k=-\infty}^{\infty}\frac{1}{(2k+3)(2k+1)^2}.$$

Now, we will drift a bit from the functions considered so far and take $f(x)=x$, which is an entire function of type 0. So $f\in\mathscr{B}_0\subset\mathscr{B}_\pi$. Since $\mathscr{D}_q f(x)=1$, it is tempting to let $f(x)=x$ in (5.2.2) with $\sigma=\pi$ to get

$$1=\frac{4}{\pi}\cosh\left(\frac{\pi}{2}\beta\right)\sum_{k=-\infty}^{\infty}\left(\alpha+\frac{1}{2}(2k+1)\right)\frac{(-1)^k}{\beta^2+(2k+1)^2}. \tag{5.4.5}$$

But there is a major issue here. For $f(x)=x$, the assumption that f being bounded on $\mathbb{R}$ is not satisfied, and so, we could not apply Theorem 5.2.1 directly to $f(x)=x$. Fortunately, we can prove (5.4.5) through a limiting process motivated by the ones used in [**15, p. 211**] and [**33, Lemmas 1 and 2**]. One of the key ideas involved in our proof is the following *Abel's partial summation formula.*

Theorem 5.4.1 (**[38]**). *Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be sequences in $\mathbb{C}$ and write*

(i) $A_{-1} = 0$ *and for* $n \geq 0$, $A_n = \sum_{k=0}^{n} a_k$.

Then for any integers $q > p \geq -1$, we have

(ii) $\sum_{n=p+1}^{q} a_n b_n = \sum_{n=p+1}^{q} A_n(b_n - b_{n+1}) + A_q b_{q+1} - A_p b_{p+1}.$

Proof of (5.4.5). For $\delta \in (0, \frac{1}{2})$, define $g_\delta(x) = \sin(\delta x)$. Then $g_\delta \in \mathscr{B}_\delta \subseteq \mathscr{B}_\pi$ and g_δ is also bounded on $\mathbb{R}$. Note that

$$(\mathscr{D}_q g_\delta)(x) = \frac{2\cos(\delta\alpha)\sinh(\frac{\delta}{2}\beta)}{\beta}.$$

Applying Theorem 5.2.1 to g_δ with $\sigma = \pi$ yields

$$\frac{2\cos(\delta\alpha)\sinh(\frac{\delta}{2}\beta)}{\delta\beta} = \frac{4}{\pi}\cosh\left(\frac{\pi}{2}\beta\right)\sum_{k=-\infty}^{\infty}(-1)^k\frac{\sin(\delta(\alpha+k+\frac{1}{2}))}{\delta(\beta^2+(2k+1)^2)}. \tag{5.4.6}$$

Claim. The partial sums $\sum_{k=-K}^{K}(-1)^k\frac{\sin(\delta(\alpha+k+\frac{1}{2}))}{\delta}$ is $\mathcal{O}(K)$ uniformly in $\delta > 0$.

Let $S_K = \sum_{k=0}^{K}(-1)^k\frac{\sin\left(\delta\left(\alpha+k+\frac{1}{2}\right)\right)}{\delta}$. First assume that K is odd. Then

$$|S_K| = \left|\frac{\sin\left(\delta\left(\alpha+0+\frac{1}{2}\right)\right)}{\delta} - \frac{\sin\left(\delta\left(\alpha+1+\frac{1}{2}\right)\right)}{\delta} + \frac{\sin\left(\delta\left(\alpha+2+\frac{1}{2}\right)\right)}{\delta}\right.$$
$$\left. - \frac{\sin\left(\delta\left(\alpha+3+\frac{1}{2}\right)\right)}{\delta} + \cdots + (-1)^K\frac{\sin\left(\delta\left(\alpha+K+\frac{1}{2}\right)\right)}{\delta}\right|.$$

Let

$$f(t) = \frac{\sin\left(\delta\left(\alpha + 0 + \frac{1}{2} + t\right)\right)}{\delta} + \frac{\sin\left(\delta\left(\alpha + 2 + \frac{1}{2} + t\right)\right)}{\delta} + \dots \\ + \frac{\sin\left(\delta\left(\alpha + K - 1 + \frac{1}{2} + t\right)\right)}{\delta} \ .$$

So

$$\begin{aligned} |f'(t)| &= \left|\cos\left(\delta\left(\alpha + 0 + \frac{1}{2} + t\right)\right) + \cdots + \cos\left(\delta\left(\alpha + K - 1 + \frac{1}{2} + t\right)\right)\right| \\ &\leq \left|\cos\left(\delta\left(\alpha + 0 + \frac{1}{2} + t\right)\right)\right| + \cdots + \left|\cos\left(\delta\left(\alpha + K - 1 + \frac{1}{2} + t\right)\right)\right| \\ &\leq \frac{K+1}{2}. \end{aligned}$$

Similarly, for K is even, we obtain $|f'(t)| \leq \dfrac{K}{2} + 1$.

Since

$$f(t) = \frac{\sin\left(\delta\left(\alpha + 1 + \frac{1}{2} + t\right)\right)}{\delta} + \frac{\sin\left(\delta\left(\alpha + 3 + \frac{1}{2} + t\right)\right)}{\delta} + \dots \\ + \frac{\sin\left(\delta\left(\alpha + K + \frac{1}{2} + t\right)\right)}{\delta} \ .$$

by Mean value theorem, it follows that

$$|S_K| = |f(t) - f(t+1)| = |f'(\xi)| \leq \begin{cases} \frac{K+1}{2} & \text{if } K \text{ is odd,} \\ \frac{K}{2} + 1 & \text{if } K \text{ is even.} \end{cases}$$

In either of the cases above, we can bound $|S_K|$ by $3(K+1)$. So for $\delta > 0$,

$$\sum_{k=-K}^{K} (-1)^k \frac{\sin(\delta(\alpha + k + \frac{1}{2}))}{\delta} \leq 3(K+1). \tag{5.4.7}$$

Apply Theorem 5.4.1 with $n = k, p + 1 = L, q = M, A_n = S_k$ and

$$a_k = (-1)^k \frac{\sin\left[\delta\left(\alpha + k + \frac{1}{2}\right)\right]}{\delta}, \quad b_k = \frac{1}{\beta^2 + (2k+1)^2}$$

and compute:

$$\sum_{k=L}^{M} (-1)^k \frac{\sin\left[\delta\left(\alpha + k + \frac{1}{2}\right)\right]}{\delta \cdot (\beta^2 + (2k+1)^2)}$$
$$= \sum_{k=L}^{M} S_k \left[\frac{1}{\beta^2 + (2k+1)^2} - \frac{1}{\beta^2 + (2k+3)^2}\right] + \frac{S_M}{\beta^2 + (2M+3)^2} - \frac{S_{L-1}}{\beta^2 + (2L+1)^2}.$$

Consequently,

$$\left|\sum_{k=L}^{M} (-1)^k \frac{\sin\left[\delta\left(\alpha + k + \frac{1}{2}\right)\right]}{\delta \cdot (\beta^2 + (2k+1)^2)}\right|$$
$$\leq \left|\sum_{k=L}^{M} S_k \left[\frac{1}{\beta^2 + (2k+1)^2} - \frac{1}{\beta^2 + (2k+3)^2}\right]\right| + \left|\frac{S_{L-1}}{\beta^2 + (2L+1)^2}\right| + \left|\frac{S_M}{\beta^2 + (2M+3)^2}\right|$$

$$\leq \left|\sum_{k=L}^{M-1} S_k \left[\frac{8k+8}{\left\{\beta^2 + (2k+1)^2\right\} \cdot \left\{\beta^2 + (2k+3)^2\right\}}\right]\right| + \left|\frac{S_{L-1}}{\beta^2 + (2L+1)^2}\right| + \left|\frac{S_M}{\beta^2 + (2M+3)^2}\right|$$
$$= \sum_{k=L}^{M-1} |S_k| \left|\left[\frac{8k+8}{\left\{\beta^2 + (2k+1)^2\right\} \cdot \left\{\beta^2 + (2k+3)^2\right\}}\right]\right| + \frac{|S_{L-1}|}{\beta^2 + (2L+1)^2} + \frac{|S_M|}{\beta^2 + (2M+3)^2}.$$

Note that, for sufficiently large k, using (5.4.7) we have

$$|S_k| \left|\left[\frac{8k+8}{\left\{\beta^2 + (2k+1)^2\right\} \cdot \left\{\beta^2 + (2k+3)^2\right\}}\right]\right| \leq \frac{3(k+1) \cdot 8(k+1)}{4k^2 \cdot 4k^2} \leq \frac{2}{k^2}.$$

Also when $|L| \to \infty$ and $|M| \to \infty$,

$$\frac{|S_{L-1}|}{\beta^2 + (2L+1)^2}, \ \frac{|S_M|}{\beta^2 + (2M+3)^2} \to 0.$$

Thus, for all $\epsilon > 0$, there exists $N(\epsilon) > 0$, independent of $\delta > 0$, such that

$$\left| \sum_{k=L}^{M} (-1)^k \frac{\sin\left[\delta\left(\alpha + k + \frac{1}{2}\right)\right]}{\delta \cdot (\beta^2 + (2k+1)^2)} \right| < \epsilon,$$

whenever $|M|, |L| > N(\epsilon)$. Hence, by the *Cauchy criterion*, $\sum_{k=-\infty}^{\infty} (-1)^k \frac{\sin\left[\delta\left(\alpha + k + \frac{1}{2}\right)\right]}{\delta \cdot (\beta^2 + (2k+1)^2)}$ is uniformly convergent for $\delta \in \left(0, \frac{1}{2}\right]$. So, by using the Abel's partial summation formula, it is not hard to verify that the series on the right-hand side of (5.4.6) converges uniformly in $\delta > 0$. Thus, by taking limits as $\delta \to 0^+$ on both sides of (5.4.6), we obtain

$$1 = \frac{4}{\pi} \cosh\left(\frac{\pi}{2}\beta\right) \sum_{k=-\infty}^{\infty} (\alpha + k + \frac{1}{2}) \frac{(-1)^k}{\beta^2 + (2k+1)^2}.$$

which is (5.4.5). □

Now, (5.4.5) being established, let's put it in the equivalent form:

$$\frac{\pi}{4\cosh(\frac{\pi}{2}\beta)} = \sum_{k=-\infty}^{\infty} \left(\alpha + \frac{1}{2}(2k+1)\right) \frac{(-1)^k}{\beta^2 + (2k+1)^2}. \tag{5.4.8}$$

Using (5.4.1) (or setting $\alpha = 0$), we get, for $\beta \in \mathbb{C}$,

$$\frac{\pi}{2\cosh\left(\frac{\pi}{2}\beta\right)} = \sum_{k=-\infty}^{\infty} \frac{(-1)^k(2k+1)}{\beta^2 + (2k+1)^2}. \tag{5.4.9}$$

This is another known identity (see [**70, p. 136**] or [**31**]). Note that the series in (5.4.9) converges locally uniformly for $\beta \in \mathbb{C} \setminus \{\pm(2k+1)i\}_{-\infty}^{\infty}$. So by integrating both sides of (5.4.9) from $\beta = 0$ to $\beta = x$, with term-by-term integration on the right-hand side, Berndt used this identity to obtain an identity of Ramanujan ([**11, p. 457**]).

We can apply the same idea to obtain extensions to yet another known identity due to Gosper, Ismail, and Zhang [**27, (1.3)**]. For $b \in \mathbb{R}$, consider the function

$$f_{\delta,b}(x) = \frac{\sin(\delta x)\sin(\sqrt{b^2 + (\pi - \delta)^2 x^2}\,)}{\sqrt{b^2 + (\pi - \delta)^2 x^2}}.$$

Then $f_{\delta,b} \in \mathscr{B}_\pi$ and $f_{\delta,b}$ is bounded on the real line. So, we can apply (5.2.2) to $f_{\delta,b}$ to get:

$$\frac{\pi(\mathscr{D}_q f_{\delta,b})(x)}{4\delta\cosh(\frac{\pi}{2}\beta)} = \sum_{k=-\infty}^{\infty} \frac{(-1)^k\ \sin(\delta(\alpha + k + \frac{1}{2}))\ \sin\left(\sqrt{b^2 + (\pi - \delta)^2(\alpha + k + \frac{1}{2})^2}\ \right)}{\delta(\beta^2 + (2k+1)^2)\sqrt{b^2 + (\pi - \delta)^2(\alpha + k + \frac{1}{2})^2}}.$$

As we did in the proof of (5.4.5), it can be verified that the series on the right-hand side above is uniformly convergent in $\delta \in (0, \frac{1}{2})$. So taking the limit as $\delta \to 0^+$ on both sides above and bringing the limit inside the sum yields

$$\lim_{\delta\to 0^+} \frac{\pi(\mathscr{D}_q f_{\delta,b})(x)}{4\delta\cosh(\frac{\pi}{2}\beta)} = \sum_{k=-\infty}^{\infty} \frac{(-1)^k(\alpha + k + \frac{1}{2})\ \sin\left(\sqrt{b^2 + \pi^2(\alpha + k + \frac{1}{2})^2}\ \right)}{(\beta^2 + (2k+1)^2)\sqrt{b^2 + \pi^2(\alpha + k + \frac{1}{2})^2}}. \tag{5.4.10}$$

Now, using (5.2.3) on $f_{\delta,b}$ yields

$$\begin{aligned}(\mathscr{D}_q f_{\delta,b})(x) = &\frac{\sin(\alpha + \frac{1}{2}i\beta)\ \sin\left(\sqrt{b^2 + (\pi - \delta)^2(\alpha + \frac{i}{2}\beta)^2}\ \right)}{i\beta\cosh(\frac{\pi}{2}\beta)\sqrt{b^2 + (\pi - \delta)^2(\alpha + \frac{i}{2}\beta)^2}} \\ &- \frac{\sin(\alpha - \frac{1}{2}i\beta)\ \sin\left(\sqrt{b^2 + (\pi - \delta)^2(\alpha - \frac{i}{2}\beta)^2}\ \right)}{i\beta\cosh(\frac{\pi}{2}\beta)\sqrt{b^2 + (\pi - \delta)^2(\alpha - \frac{i}{2}\beta)^2}}.\end{aligned} \tag{5.4.11}$$

Divide both sides above by δ and take the limit as $\delta \to 0^+$ to get

$$\begin{aligned}&\lim_{\delta\to 0^+} \frac{(\mathscr{D}_q f_{\delta,b})(x)}{\delta} \\ &= \frac{(\alpha + \frac{1}{2}i\beta)\sin\left(\sqrt{b^2 + \pi^2(\alpha + \frac{i}{2}\beta)^2}\ \right)}{i\beta\sqrt{b^2 + \pi^2(\alpha + \frac{i}{2}\beta)^2}} - \frac{(\alpha - \frac{1}{2}i\beta)\sin\left(\sqrt{b^2 + \pi^2(\alpha - \frac{i}{2}\beta)^2}\ \right)}{i\beta\sqrt{b^2 + \pi^2(\alpha - \frac{i}{2}\beta)^2}}.\end{aligned}$$

Using this on the left hand side in (5.4.10), we obtain the following new identity: For any $\alpha, \beta \in \mathbb{C}$,

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k(\alpha+k+\frac{1}{2})\ \sin\left(\sqrt{b^2+\pi^2(\alpha+k+\frac{1}{2})^2}\ \right)}{(\beta^2+(2k+1)^2)\sqrt{b^2+\pi^2(\alpha+k+\frac{1}{2})^2}} \tag{5.4.12}$$
$$= \frac{\pi(\alpha+\frac{1}{2}i\beta)\sin\left(\sqrt{b^2+\pi^2(\alpha+\frac{i}{2}\beta)^2}\ \right)}{4i\beta\cosh(\frac{\pi}{2}\beta)\sqrt{b^2+\pi^2(\alpha+\frac{i}{2}\beta)^2}} - \frac{\pi(\alpha-\frac{1}{2}i\beta)\sin\left(\sqrt{b^2+\pi^2(\alpha-\frac{i}{2}\beta)^2}\ \right)}{4i\beta\cosh(\frac{\pi}{2}\beta)\sqrt{b^2+\pi^2(\alpha-\frac{i}{2}\beta)^2}}.$$

If we let $\alpha = 0$, then the above identity becomes

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k(2k+1)\sin\left(\sqrt{b^2+\pi^2(k+\frac{1}{2})^2}\ \right)}{(\beta^2+(2k+1)^2)\sqrt{b^2+\pi^2(k+\frac{1}{2})^2}} = \frac{\pi\sin\left(\sqrt{b^2-\frac{\pi^2\beta^2}{4}}\ \right)}{2\cosh(\frac{\pi}{2}\beta)\sqrt{b^2-\frac{\pi^2\beta^2}{4}}}. \tag{5.4.13}$$

By taking β to be a purely imaginary number, say $\beta = i\gamma$, for $\gamma \in \mathbb{R}$ we can recover identity (1.10) of Gosper, Ismail, and Zhang in [**27**]:

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k(2k+1)\sin\left(\sqrt{b^2+\pi^2(k+\frac{1}{2})^2}\ \right)}{((2k+1)^2-\gamma^2)\sqrt{b^2+\pi^2(k+\frac{1}{2})^2}} = \frac{\pi\sin\left(\sqrt{b^2+\frac{\pi^2\gamma^2}{4}}\ \right)}{2\cos(\frac{\pi}{2}\gamma)\sqrt{b^2+\frac{\pi^2\gamma^2}{4}}}.$$

If we further take $\beta = 0$, then (5.4.13) reduces to

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k\sin\left(\sqrt{b^2+\pi^2(k+\frac{1}{2})^2}\ \right)}{(2k+1)\sqrt{b^2+\pi^2(k+\frac{1}{2})^2}} = \frac{\pi\sin b}{2b}, \tag{5.4.14}$$

which is the identity (1.3) of Gosper, Ismail, and Zhang in [**27**]. Thus, identity (5.4.12) provides a two-parameter family extension of these identities of Gosper, Ismail, and Zhang.

In fact the above argument really applies to a much more general family of functions as indicated by our next result, Theorem 5.4.2 whose statement involves not only the Askey-

Wilson operator but also a companion operator to it called the *Average operator*.

The average operator, $\mathscr{A}_q$ is defined by ([**35, p. 301**]):

$$(\mathscr{A}_q f)(x) = \frac{1}{2}\left\{\breve{f}(q^{1/2}z) + \breve{f}(q^{-1/2}z)\right\}. \tag{5.4.15}$$

In terms of our new notations, α and β introduced in (5.2.1), we can write (5.4.15) as:

$$(\mathscr{A}_q f)(x) = \frac{1}{2}\left\{f\left(\alpha + \frac{i\beta}{2}\right) + f\left(\alpha - \frac{i\beta}{2}\right)\right\}. \tag{5.4.16}$$

We will use the notation $(\mathscr{A} f)(\alpha, \beta)$ to denote the right-hand side of (5.4.16) to emphasize the dependence of $(\mathscr{A}_q f)(x)$ on α and β.

Theorem 5.4.2. *Let g be an entire function of exponential type π that is bounded on $\mathbb{R}$. Then, for $\alpha, \beta \in \mathbb{C}$,*

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k(\alpha + k + \frac{1}{2})g(\alpha + k + \frac{1}{2})}{\beta^2 + (2k+1)^2} \tag{5.4.17}$$

$$= \frac{\pi\alpha\left[g(\alpha + \frac{i}{2}\beta) - g(\alpha - \frac{i}{2}\beta)\right] + \frac{i}{2}\pi\beta\left[g(\alpha + \frac{i}{2}\beta) + g(\alpha - \frac{i}{2}\beta)\right]}{4i\beta\cosh(\frac{\pi}{2}\beta)}$$

$$= \frac{\pi}{4\cosh(\frac{\pi}{2}\beta)}\left\{\alpha\left(\mathscr{D}_q g\right)(x) + \left(\mathscr{A}_q g\right)(x)\right\}. \tag{5.4.18}$$

Proof. Let $g_\delta(x) = g\left(\dfrac{\pi - \delta}{\pi}x\right)$, for $\delta \in \left(0, \frac{1}{2}\right)$. First, we shall apply Theorem 5.2.1 to the function $\tilde{g}(x) = \sin(\delta x)g_\delta(x)$ with $\sigma = \pi$ to get

$$\frac{\left(\mathscr{D}_q \tilde{g}\right)(x)}{\delta} = \frac{4}{\pi}\cosh\left(\frac{\pi}{2}\beta\right)\sum_{k=-\infty}^{\infty}(-1)^k\,\frac{\sin\left(\delta\left(\alpha + k + \frac{1}{2}\right)\right)g_\delta\left(\alpha + k + \frac{1}{2}\right)}{\delta(\beta^2 + (2k+1)^2)}.$$

Again, as in the proof of (5.4.5), we can show that the series is uniformly convergent in $\delta \in \left(0, \frac{1}{2}\right)$. So, we can take the limit as $\delta \to 0^+$ of both sides of above with taking limit inside the sum to get

$$\lim_{\delta\to 0^+}\frac{(\mathscr{D}_q\tilde{g})(x)}{\delta}=\frac{4}{\pi}\cosh\left(\frac{\pi}{2}\beta\right)\sum_{k=-\infty}^{\infty}\frac{(-1)^k(\alpha+k+\frac{1}{2})g(\alpha+k+\frac{1}{2})}{\beta^2+(2k+1)^2}. \tag{5.4.19}$$

Now we shall directly compute $\lim_{\delta\to 0^+}\frac{(\mathscr{D}_q\tilde{g})(x)}{\delta}$ with the use of (5.2.3).

$$\begin{aligned}\lim_{\delta\to 0^+}\frac{(\mathscr{D}_q\tilde{g})(x)}{\delta}&=\frac{\left(\alpha+\frac{i}{2}\beta\right)g\left(\alpha+\frac{i}{2}\beta\right)-\left(\alpha-\frac{i}{2}\beta\right)g\left(\alpha-\frac{i}{2}\beta\right)}{i\beta}\\&=\frac{\alpha\Big[g\left(\alpha+\frac{i}{2}\beta\right)-g\left(\alpha-\frac{i}{2}\beta\right)\Big]+\frac{i\beta}{2}\Big[g\left(\alpha+\frac{i}{2}\beta\right)+g\left(\alpha-\frac{i}{2}\beta\right)\Big]}{i\beta}.\end{aligned} \tag{5.4.20}$$

Equating the two sides of (5.4.19) and (5.4.20) yields (5.4.17). Finally, (5.4.18) follows from (5.4.16) with f replaced by g. □

Remark 5.4.3. When $g(x)=\frac{\sin\left[\sqrt{b^2+\pi^2x^2}\right]}{\sqrt{b^2+\pi^2x^2}}$, (5.4.17) implies (5.4.12).

As applications of Theorem 5.4.2, we illustrate additional examples of entire functions of exponential type π that are also bounded on $\mathbb{R}$. Applying Theorem 5.4.2 to these functions will verify extensions of (5.4.14), (1.6) of **[27]**, and identities of Zayed **[73, p.702]**.

Definition 5.4.4. A Bessel function* of the first kind, $J_\nu(z)$ is defined by

$$J_\nu(z):=\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{z}{2}\right)^{\nu+2n}}{n!\,\Gamma(n+\nu+1)}. \tag{5.4.21}$$

The Bessel functions of order $\nu=\frac{1}{2}$ and $\nu=-\frac{1}{2}$ are defined as

$$J_{1/2}(z)=\sqrt{\frac{2}{\pi z}}\sin(z)\quad\text{and}\quad J_{-1/2}(z)=\sqrt{\frac{2}{\pi z}}\cos(z).$$

*see the classical text **[68]** for a comprehensive study of Bessel functions.

The relationship between $J_\nu(z)$ and $J_{-\nu}(z)$ for an integer ν is

$$J_{-\nu}(z) = \sum_{n=0}^{\infty} \frac{(-1)^{(n+m)}}{2^{2n+m}\ n!\ (n+m)!} x^{2n+m}$$
$$= (-1)^{\nu}\ J_\nu(z).$$

Example 5.4.5. Consider the function $\dfrac{J_\nu(\sqrt{b^2+\pi^2 x^2})}{\left(\sqrt{b^2+\pi^2x^2}\right)^{\nu}}$, for $b > 0$ and $\mathrm{Re}(\nu) > -\frac{1}{2}$. Applying Theorem 5.4.2 to this function, we get the following identity: with $\alpha_k := \alpha + k + \frac{1}{2}$,

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k \cdot \alpha_k J_\nu(\sqrt{b^2+\pi^2(\alpha_k)^2})}{\left(\sqrt{b^2+\pi^2(\alpha_k)^2}\right)^{\nu} \beta^2 + (2k+1)^2}$$
$$= \pi\alpha \left\{ \frac{J_\nu\left(\sqrt{b^2+\pi^2\left(\alpha+\frac{i\beta}{2}\right)^2}\right) \Big/ \left(\sqrt{b^2+\pi^2\left(\alpha+\frac{i\beta}{2}\right)^2}\right)^{\nu}}{4i\beta\cosh(\frac{\pi}{2}\beta)} - \frac{J_\nu\left(\sqrt{b^2+\pi^2\left(\alpha-\frac{i\beta}{2}\right)^2}\right) \Big/ \left(\sqrt{b^2+\pi^2\left(\alpha-\frac{i\beta}{2}\right)^2}\right)^{\nu}}{4i\beta\cosh(\frac{\pi}{2}\beta)} \right\}$$
$$+ \frac{i\pi\beta}{2} \left\{ \frac{J_\nu\left(\sqrt{b^2+\pi^2\left(\alpha+\frac{i\beta}{2}\right)^2}\right) \Big/ \left(\sqrt{b^2+\pi^2\left(\alpha+\frac{i\beta}{2}\right)^2}\right)^{\nu}}{4i\beta\cosh(\frac{\pi}{2}\beta)} + \frac{J_\nu\left(\sqrt{b^2+\pi^2\left(\alpha-\frac{i\beta}{2}\right)^2}\right) \Big/ \left(\sqrt{b^2+\pi^2\left(\alpha-\frac{i\beta}{2}\right)^2}\right)^{\nu}}{4i\beta\cosh(\frac{\pi}{2}\beta)} \right\},$$

which extends (1.6) in **[27]** for two parameters α and β. Also, with $v = 1,\ \alpha = 0$, and with $\beta \to 0$, (1.3) of **[27]** can be obtained. Taking $\alpha = 0$ and letting $\beta \to 0$, we get

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k \cdot J_\nu\left(\sqrt{b^2+\pi^2(k+\frac{1}{2})^2}\ \right)}{\left(\sqrt{b^2+\pi^2(k+\frac{1}{2})^2}\ \right)^{\nu} (k+\frac{1}{2})} = \frac{\pi J_\nu(b)}{b^\nu},$$

which is the identity (1.6) in **[27]**.

Example 5.4.6. Consider the function $J_\nu\left[\frac{\pi}{2}(\sqrt{b^2+x^2}+x)\right] J_\nu\left[\frac{\pi}{2}(\sqrt{b^2+x^2}-x)\right]$. Applying Theorem 5.4.2 to this function, we get the following identity: with $\alpha_k := \alpha + k + \frac{1}{2}$,

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k \alpha_k J_\nu[\frac{\pi}{2}(\sqrt{b^2+\alpha_k^2}+\alpha_k)] J_\nu[\frac{\pi}{2}(\sqrt{b^2+\alpha_k^2}-\alpha_k)]}{\beta^2+(2k+1)^2}$$

$$= \pi\alpha \left\{ \frac{J_\nu[\frac{\pi}{2}(\sqrt{b^2+(\alpha+\frac{i\beta}{2})^2}+(\alpha+\frac{i\beta}{2}))] J_\nu[\frac{\pi}{2}(\sqrt{b^2+(\alpha+\frac{i\beta}{2})^2}-(\alpha+\frac{i\beta}{2}))]}{4i\beta\cosh(\frac{\pi}{2}\beta)} \right.$$
$$\left. - \frac{J_\nu[\frac{\pi}{2}(\sqrt{b^2+(\alpha-\frac{i\beta}{2})^2}+(\alpha-\frac{i\beta}{2}))] J_\nu[\frac{\pi}{2}(\sqrt{b^2+(\alpha-\frac{i\beta}{2})^2}-\left(\alpha-\frac{i\beta}{2}\right)\Big)\Big]}{4i\beta\cosh(\frac{\pi}{2}\beta)} \right\}$$

$$+ \frac{i\pi\beta}{2} \left\{ \frac{J_\nu[\frac{\pi}{2}\Big(\sqrt{b^2+\left(\alpha+\frac{i\beta}{2}\right)^2}+\left(\alpha+\frac{i\beta}{2}\right))] J_\nu[\frac{\pi}{2}(\sqrt{b^2+\left(\alpha+\frac{i\beta}{2}\right)^2}-\left(\alpha+\frac{i\beta}{2}\right))]}{4i\beta\cosh(\frac{\pi}{2}\beta)} \right.$$
$$\left. + \frac{J_\nu[\frac{\pi}{2}(\sqrt{b^2+\left(\alpha-\frac{i\beta}{2}\right)^2}+\left(\alpha-\frac{i\beta}{2}\right)\Big)\Big] J_\nu[\frac{\pi}{2}(\sqrt{b^2+(\alpha-\frac{i\beta}{2})^2}-(\alpha-\frac{i\beta}{2}))]}{4i\beta\cosh(\frac{\pi}{2}\beta)} \right\}.$$

Taking $\alpha = 0$ and letting $\beta \to 0$ above, we get

$$\left[J_\nu\left(\frac{\pi b}{2}\right)\right]^2 = 2\sum_{k=0}^{\infty} J_\nu\left[\frac{\pi}{2}\left(\sqrt{b^2+\left(k+\frac{1}{2}\right)^2}+\left(k+\frac{1}{2}\right)\right)\right]$$
$$\times J_\nu\left[\frac{\pi}{2}\left(\sqrt{b^2+\left(k+\frac{1}{2}\right)^2}-\left(k+\frac{1}{2}\right)\right)\right]\frac{(-1)^k}{\pi(k+1/2)},$$

which is an identity of Zayed [**73, p.702**].

Example 5.4.7. Let $g(x) = J_{\nu+x}(b) \cdot J_{\nu-x}(b)$. with $\alpha_k := \alpha + k + \frac{1}{2}$, we have

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k \alpha_k \ J_{\nu+\alpha_k}(b) J_{\nu-\alpha_k}(b)}{\beta^2 + (2k+1)^2}$$
$$= \pi\alpha \left\{ \frac{J_{\nu+(\alpha+\frac{i\beta}{2})}(b) J_{\nu-(\alpha+\frac{i\beta}{2})}(b) - J_{\nu+(\alpha-\frac{i\beta}{2})}(b) J_{\nu-(\alpha-\frac{i\beta}{2})}(b)}{4i\beta \cosh(\frac{\pi}{2}\beta)} \right\}$$
$$+ \frac{i\pi\beta}{2} \left\{ \frac{J_{\nu+(\alpha+\frac{i\beta}{2})}(b) J_{\nu-(\alpha+\frac{i\beta}{2})}(b) + J_{\nu+(\alpha-\frac{i\beta}{2})}(b) J_{\nu-(\alpha-\frac{i\beta}{2})}(b)}{4i\beta \cosh(\frac{\pi}{2}\beta)} \right\}.$$

Taking $\alpha = 0$ yields

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k (k+1/2) \ J_{\nu+(k+1/2)}(b) J_{\nu-(k+1/2)}(b)}{\beta^2 + (2k+1)^2}$$
$$= \pi\alpha \left\{ \frac{J_{\nu+\frac{i\beta}{2}}(b) J_{\nu-\frac{i\beta}{2}}(b) - J_{\nu-\frac{i\beta}{2}}(b) J_{\nu+\frac{i\beta}{2}}(b)}{4i\beta \cosh(\frac{\pi}{2}\beta)} \right\}$$
$$+ \frac{i\pi\beta}{2} \left\{ \frac{J_{\nu+\frac{i\beta}{2}}(b) J_{\nu-\frac{i\beta}{2}}(b) + J_{\nu-\frac{i\beta}{2}}(b) J_{\nu+\frac{i\beta}{2}}(b)}{4i\beta \cosh(\frac{\pi}{2}\beta)} \right\}.$$

Letting $\beta \to 0$ yields

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k \ J_{\nu+(k+1/2)}(b) J_{\nu-(k+1/2)}(b)}{(2k+1)} = \frac{\pi}{2} [J_\nu(b)]^2. \tag{5.4.22}$$

This is true for any $\nu > -\frac{1}{4}$. So in particular letting $\nu = \frac{1}{2}$, we get

$$b \sum_{k=-\infty}^{\infty} \frac{(-1)^k \ J_{k+1}(b) J_k(b)}{(2k+1)} = (\sin(b))^2 \ , \tag{5.4.23}$$

which is another identity of Zayed [**73, p.703**].

5.5 Generalized Bernstein inequality

In this section we establish two types of generalizations of Bernstein inequality for the Askey-Wilson operator:

- In pointwise norm,
- In uniform L^p-norm for $p \geq 1$.

The main ingredient for our results is the extended Boas formula, Theorem 5.2.1.

5.5.1 Generalized Bernstein inequality for entire functions

For the derivative of an entire function of exponential type σ, Bernstein **[12]** proved, for $x \in \mathbb{R}$,

$$|f'(x)| \leq \sigma \sup_{x \in \mathbb{R}} |f(x)|. \tag{5.5.1}$$

Boas (see **[14]**, **[15, pp.210-211]**) used his interpolating formula (5.1.1) to give a simpler proof of (5.5.1). Now, following Boas and using our extended Boas' formula for the Askey-Wilson operator, we obtain the following result.

Theorem 5.5.1. *If $f \in \mathscr{B}_\sigma$ with $|f(x)| \leq M$ for $x \in \mathbb{R}$, then for*

$$|x| < \sqrt{\frac{\pi^2}{\sigma^2 (q^{1/2} - q^{-1/2})^2} + 1}\ , \tag{5.5.2}$$

the following inequality holds:

$$|\mathscr{D}_q f(x)| \leq \frac{2M \sinh\left(\frac{\sigma}{2}(q^{1/2} - q^{-1/2})\ \sqrt{1-x^2}\right)}{(q^{1/2} - q^{-1/2})\ \sqrt{1-x^2}}\ . \tag{5.5.3}$$

The branch of $\sqrt{z}$ *is chosen such that the function is analytic in* $\mathbb{C}\backslash\{z : z = ia,\ a \geq 0\}$. *Note that for* $x \in \mathbb{R}$, *the right hand side of* (5.5.3) *is always positive.*

Remark 5.5.2. Recall that $z = e^{i\theta}$ and $x = \cos\theta$. Note that

$$\sqrt{z} = e^{\frac{1}{2}\log(z)} = e^{\frac{1}{2}(\ln(|z|)+i\ \text{Arg}(z))}\ .$$

1. If we choose the *principal branch* of the logarithm to be $-\frac{\pi}{2} \leq \text{Arg}(z) \leq \frac{3\pi}{2}$, then $\text{Arg}(z) = 0$ for $z > 0$ and $\text{Arg}(z) = \pi$ for $z < 0$. Consequently $\sqrt{z} = e^{\frac{1}{2}(\ln(|z|)+i\pi)} = i\sqrt{z}$.

2. If we choose $-\frac{3\pi}{2} \leq \text{Arg}(z) \leq \frac{\pi}{2}$, then $\text{Arg}(z) = 0$ for $z > 0$ and $\text{Arg}(z) = -\pi$ for $z < 0$. Consequently $\sqrt{z} = e^{\frac{1}{2}(\ln(|z|)-i\pi)} = -i\sqrt{z}$.

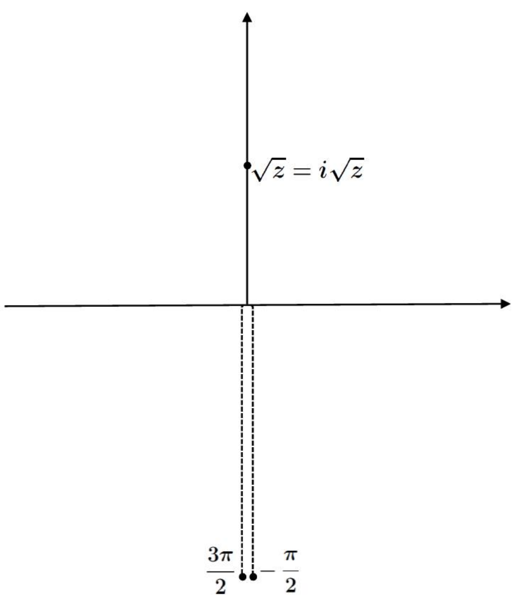

Figure 5.1
Branch cut for $-\frac{\pi}{2} \leq \text{Arg}(z) \leq \frac{3\pi}{2}$

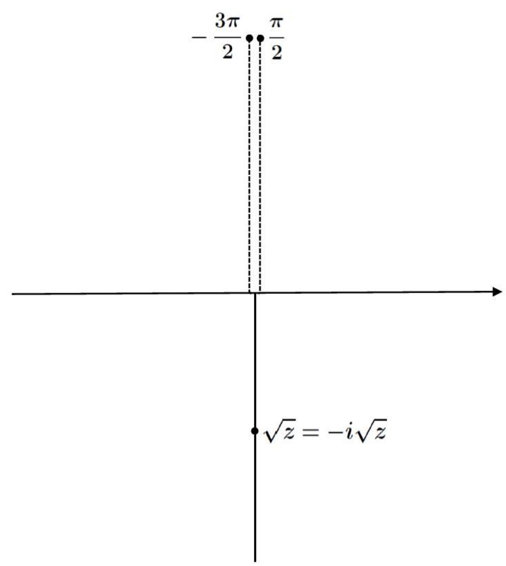

Figure 5.2
Branch cut for $-\frac{3\pi}{2} \leq \text{Arg}(z) \leq \frac{\pi}{2}$

Proof of Theorem 5.5.1. Recall first that $q \in (0, 1)$. For $x = \pm 1$, the right hand side of (5.5.3) equals to $\sigma M > 0$. For $|x| < 1$, $1 - x^2$ is real and positive and thus the right hand side of (5.5.3) is real and positive. For $|x| > 1$, $\sqrt{1-x^2} = i\sqrt{x^2-1}$, and thus the right hand side of (5.5.3) equals to

$$\frac{2M \sin\left(\frac{\sigma}{2}(q^{1/2} - q^{-1/2})\ \sqrt{x^2-1}\ \right)}{(q^{1/2} - q^{-1/2})\ \sqrt{x^2-1}}\ .$$

Now, for $|x| < \sqrt{\frac{\pi^2}{\sigma^2 (q^{1/2} - q^{-1/2})^2} + 1}$,

$$0 < \sqrt{x^2-1} < \frac{\pi}{\sigma \cdot (q^{1/2} - q^{-1/2})}\ .$$

It follows that

$$0 < \frac{\sigma}{2}(q^{1/2} - q^{-1/2})\ \sqrt{x^2-1} < \frac{\pi}{2}\ .$$

Since sin is increasing on $\left(0, \frac{\pi}{2}\right)$, It follows that

$$0 < \sin\left(\frac{\sigma}{2}(q^{1/2} - q^{-1/2})\ \sqrt{x^2-1}\ \right) < 1\ .$$

Next, we shall establish the desired inequality in view of two cases; $|x| \le 1$ and $|x| > 1$.

Case 1: First, suppose that $|x| \le 1$. From (5.2.2), we have, for $|x| \le 1$,

$$|\left(\mathscr{D}_q f\right)(x)| \le \frac{4}{\sigma} \left|\cosh\left(\frac{\sigma}{2}\beta\right)\right| \sum_{k=-\infty}^{\infty} \left| f\left(\alpha + \frac{\sigma}{2\pi}(2k+1)\right)\right|\ \frac{1}{|\beta^2 + (2k+1)^2\, \pi^2/\sigma^2|},$$

which is less than or equal to

$$\frac{4M}{\sigma} \cosh\left(\frac{\sigma}{2}\beta\right) \sum_{k=-\infty}^{\infty} \frac{1}{\beta^2 + (2k+1)^2\, \pi^2/\sigma^2}$$

which, by (5.4.3), equals to

$$\frac{2M \sinh(\frac{\sigma}{2}\beta)}{\beta}\ .$$

Case 2: Now, suppose that $x \in (-\infty, -1) \cup (1, \infty)$. So β is purely imaginary, and so $\beta^2 = -|\beta|^2$. Thus, from (5.2.2) we obtain

$$\begin{aligned}
|(\mathscr{D}_q f)(x)| &\leq \frac{4M}{\sigma} \left|\cosh\left(\frac{\sigma}{2}\beta\right)\right| \sum_{k=-\infty}^{\infty} \frac{1}{\left|-|\beta|^2 + (2k+1)^2 \pi^2/\sigma^2\right|} \\
&= \frac{4M}{\sigma} \left|\cosh\left(\frac{\sigma}{2}\beta\right)\right| \sum_{|\beta|<|2k+1|\frac{\pi}{\sigma}} \frac{1}{-|\beta|^2 + (2k+1)^2 \pi^2/\sigma^2} \\
&\quad + \frac{4M}{\sigma} \left|\cosh\left(\frac{\sigma}{2}\beta\right)\right| \sum_{|\beta|>|2k+1|\frac{\pi}{\sigma}} \frac{1}{|\beta|^2 - (2k+1)^2 \pi^2/\sigma^2}.
\end{aligned}$$

For $|x| < \sqrt{\frac{\pi^2}{\sigma^2(q^{1/2}-q^{-1/2})^2} + 1}$, from (5.2.1) we have

$$1 + \frac{|\beta|^2}{(q^{1/2} - q^{-1/2})^2} < \frac{\pi^2}{\sigma^2(q^{1/2} - q^{-1/2})^2} + 1 \ .$$

Solving the inequality for β yields $|\beta| < \frac{\pi}{\sigma}$. So the second sum above is empty (and thus have value zero), and hence,

$$\begin{aligned}
|(\mathscr{D}_q f)(x)| &\leq \frac{4M}{\sigma} \left|\cosh\left(\frac{\sigma}{2}\beta\right)\right| \sum_{k=-\infty}^{\infty} \frac{1}{-|\beta|^2 + (2k+1)^2 \pi^2/\sigma^2} \\
&= \frac{2M \sinh(\frac{\sigma}{2}\beta)}{\beta} \ .
\end{aligned}$$

The desired result follows from (5.2.1) with $x = \cos\theta$. □

Remark 5.5.3. Note that, in the limit as $q \to 1^-$, the finite interval

$$\left(-\sqrt{\frac{\pi^2}{\sigma^2(q^{1/2} - q^{-1/2})^2} + 1}, \sqrt{\frac{\pi^2}{\sigma^2(q^{1/2} - q^{-1/2})^2} + 1}\right)$$

expands to the whole real line and

$$\frac{2\sinh\left(\frac{\sigma}{2}(q^{1/2} - q^{-1/2})\ \sqrt{1-x^2}\ \right)}{(q^{1/2} - q^{-1/2})\ \sqrt{1-x^2}} \to \sigma \ .$$

So the Bernstein inequality is a limiting case of Theorem 5.5.1.

Now, we establish the same inequality in terms of only α and β.

Theorem 5.5.4. *If $f \in \mathscr{B}_\sigma$ with $|f(x)| \leq M$ for $x \in \mathbb{R}$, then for*

$$\beta = u + iv \in \mathbb{R} \cup \left\{ it : |t| < \frac{\pi}{\sigma} \right\} \ , u, v \in \mathbb{R}$$

the following inequality holds:

$$|(\mathscr{D}f)(\alpha, \beta)| \leq 2M \cdot \left| \cosh\left(\frac{\sigma}{2}\beta\right) \right| \cdot \frac{\tanh\left(\frac{\sigma}{2}\sqrt{u^2 - v^2}\right)}{\sqrt{u^2 - v^2}} \ . \tag{5.5.4}$$

Proof. For any $\alpha, \beta \in \mathbb{C}$, from (5.2.4) we have

$$|\left(\mathscr{D}f\right)(\alpha, \beta)| \leq \frac{4}{\sigma} \left| \cosh\left(\frac{\sigma}{2}\beta\right) \right| \sum_{k=-\infty}^{\infty} \left| f\left(\alpha + \frac{\sigma}{2\pi}(2k+1)\right) \right| \ \frac{1}{\left|\beta^2 + (2k+1)^2 \pi^2/\sigma^2\right|} \ ,$$

which is less than or equals to

$$\frac{4M}{\sigma} \left| \cosh\left(\frac{\sigma}{2}\beta\right) \right| \sum_{k=-\infty}^{\infty} \frac{1}{\left|\beta^2 + (2k+1)^2 \pi^2/\sigma^2\right|} \ .$$

For $\beta = u + iv$, note that

$$\begin{aligned}
\left| \beta^2 + (2k+1)^2 \frac{\pi^2}{\sigma^2} \right|^2 &= \left| (u+iv)^2 + (2k+1)^2 \frac{\pi^2}{\sigma^2} \right|^2 \\
&= \left| u^2 - v^2 + (2k+1)^2 \frac{\pi^2}{\sigma^2} + 2iuv \right|^2 \\
&= u^4 + v^4 - 2u^2v^2 + 2(u^2 - v^2) \cdot (2k+1)^2 \frac{\pi^2}{\sigma^2} + (2k+1)^4 \frac{\pi^4}{\sigma^4} + 16u^4v^4 \\
&\geq u^4 + v^4 - 2u^2v^2 + 2(u^2 - v^2) \cdot (2k+1)^2 \frac{\pi^2}{\sigma^2} + (2k+1)^4 \frac{\pi^4}{\sigma^4} \\
&= \left(u^2 - v^2 + (2k+1)^2 \frac{\pi^2}{\sigma^2} \right)^2 \ .
\end{aligned}$$

Thus we have

$$\begin{aligned}|(\mathscr{D}f)(\alpha,\beta)| &\leq \frac{4M}{\sigma}\left|\cosh\left(\frac{\sigma}{2}\beta\right)\right|\sum_{k=-\infty}^{\infty}\frac{1}{(u^2-v^2+(2k+1)^2\pi^2/\sigma^2)}\\ &= \frac{4M}{\sigma}\left|\cosh\left(\frac{\sigma}{2}\beta\right)\right|\sum_{k=-\infty}^{\infty}\frac{1}{\left((\sqrt{u^2-v^2})^2+(2k+1)^2\pi^2/\sigma^2\right)}\ .\end{aligned}$$

By (5.4.3) the right hand side above equals to

$$\frac{4M}{\sigma}\left|\cosh\left(\frac{\sigma}{2}\beta\right)\right|\cdot\frac{\sigma\tanh\left(\frac{\sigma}{2}\sqrt{u^2-v^2}\right)}{2\sqrt{u^2-v^2}} = 2M\cdot\left|\cosh\left(\frac{\sigma}{2}\beta\right)\right|\cdot\frac{\tanh\left(\frac{\sigma}{2}\sqrt{u^2-v^2}\right)}{\sqrt{u^2-v^2}}\ .$$

Note that if $u^2-v^2+\frac{\pi^2}{\sigma^2}>0$, β is in the region bounded by the hyperbola $v=-\sqrt{u^2+\frac{\pi^2}{\sigma^2}}$ and $v=\sqrt{u^2+\frac{\pi^2}{\sigma^2}}$ in the complex plane, where $u=\mathrm{Re}(\beta)$ and $v=\mathrm{Im}(\beta)$. □

Remark 5.5.5. Note that, in the limit as $q\to 1^-$, $\beta\to 0$, and hence

$$2M\cdot\left|\cosh\left(\frac{\sigma}{2}\beta\right)\right|\cdot\frac{\tanh\left(\frac{\sigma}{2}\sqrt{u^2-v^2}\right)}{\sqrt{u^2-v^2}}\to\sigma M\ .$$

So the Bernstein inequality is a limiting case of Theorem 5.5.4.

5.5.2 Generalized Bernstein inequality in L^p-norm for entire functions

The following theorem is an L^p analogue of the Bernstein inequality for entire functions of exponential type.

Theorem 5.5.6. *Let f be an entire function of exponential type τ belonging to $L^p(\mathbb{R})$, i.e., $f\in\mathscr{B}^p_\sigma$. Then*

$$\|f'\|_{L^p}\leq\tau\,\|f\|_{L^p}\,,\qquad (p\geq 1). \tag{5.5.5}$$

For the proof of our result we need the following classical inequality of Holder's:

Theorem 5.5.7 (Hölder's inequality). *Let* $\frac{1}{p}+\frac{1}{p'}=1$. *Then*

$$\sum_{k=-\infty}^{\infty}|a_kb_k| \le \left\{\sum_{k=-\infty}^{\infty}|a_k|^p\right\}^{1/p}\left\{\sum_{k=-\infty}^{\infty}|a_k|^{p'}\right\}^{1/p'}. \tag{5.5.6}$$

The case when $p=p'=2$, *is called the Cauchy-Schwarz inequality.*

Now we will state and prove our result for the generalization of Theorem 5.5.6 for the Askey-Wilson operator.

Theorem 5.5.8. *Let* $f\in\mathscr{B}^p_\sigma$. *Assume that* $\alpha,\beta\in\mathbb{R}$. *Then, for* $p\ge 1$, *the following holds:*

$$\|\mathscr{D}f\|_{L^p}\le\sigma\,\|f\|_{L^p}\,\left\|\frac{\sinh\left(\frac{\sigma}{2}\beta\right)}{\frac{\sigma}{2}\beta}\right\|_{L^p}. \tag{5.5.7}$$

Here

$$\|\mathscr{D}f\|_{L^p}=\left(\int_{\beta=-\infty}^{\infty}\int_{\alpha=-\infty}^{\infty}|(\mathscr{D}f)(\alpha,\beta)|^p\,d\alpha\,d\beta\right)^{1/p}, \tag{5.5.8}$$

and $\mathscr{D}f$ *is as defined in*(5.2.3).

Proof. Assume that $\alpha,\beta\in\mathbb{R}$. We will start with our main result and integrate both sides of (5.2.2) with respect to α and β from $-\infty$ to ∞ to get

$$\int_{\beta=-\infty}^{\infty}\int_{\alpha=-\infty}^{\infty}|(\mathscr{D}f)(\alpha,\beta)|^p\,d\alpha\,d\beta$$
$$=\frac{4^p}{\sigma^p}\int_{\beta=-\infty}^{\infty}\cosh^p\left(\frac{\sigma}{2}\beta\right)\left\{\int_{\alpha=-\infty}^{\infty}\left|\sum_{k=-\infty}^{\infty}f\left(\alpha+\frac{\pi}{2\sigma}(2k+1)\right)\frac{(-1)^k}{\beta^2+(2k+1)^2\pi^2/\sigma^2}\right|^p d\alpha\right\}d\beta. \tag{5.5.9}$$

Using the *Hölder's inequality* yields

$$\left|\sum_{k=-\infty}^{\infty} f\left(\alpha+\frac{\pi}{2\sigma}(2k+1)\right)\ \frac{(-1)^k}{\beta^2+(2k+1)^2\pi^2/\sigma^2}\right|$$

$$=\left|\sum_{k=-\infty}^{\infty} f\left(\alpha+\frac{\pi}{2\sigma}(2k+1)\right)\ \left\{\frac{(-1)^k}{\beta^2+(2k+1)^2\pi^2/\sigma^2}\right\}^{1/p}\cdot\left\{\frac{(-1)^k}{\beta^2+(2k+1)^2\pi^2/\sigma^2}\right\}^{1/p'}\right|$$

$$\leq\left[\sum_{k=-\infty}^{\infty}\left|f\left(\alpha+\frac{\pi}{2\sigma}(2k+1)\right)\right|^p\ \left\{\frac{1}{\beta^2+(2k+1)^2\pi^2/\sigma^2}\right\}\right]^{1/p}\left[\sum_{k=-\infty}^{\infty}\frac{1}{\beta^2+(2k+1)^2\pi^2/\sigma^2}\right]^{1/p'}$$

$$=\left[\sum_{k=-\infty}^{\infty}\left|f\left(\alpha+\frac{\pi}{2\sigma}(2k+1)\right)\right|^p\ \left\{\frac{1}{\beta^2+(2k+1)^2\pi^2/\sigma^2}\right\}\right]^{1/p}\left\{\frac{\sigma\tanh(\frac{\sigma}{2}\beta)}{2\beta}\right\}^{1/p'}.$$

The penultimate line follows from (5.4.3). Using this in (5.5.9) yields

$$\int_{\beta=-\infty}^{\infty}\int_{\alpha=-\infty}^{\infty}|(\mathscr{D}f)(\alpha,\beta)|^p\ d\alpha\ d\beta$$

$$\leq\frac{4^p}{\sigma^p}\int_{\beta=-\infty}^{\infty}\cosh^p\left(\frac{\sigma}{2}\beta\right)\left[\int_{\alpha=-\infty}^{\infty}\left\{\sum_{k=-\infty}^{\infty}\left|f\left(\alpha+\frac{\pi}{2\sigma}(2k+1)\right)\right|^p\ \left\{\frac{1}{\beta^2+(2k+1)^2\pi^2/\sigma^2}\right\}\right\}\right.$$
$$\left.\times\left\{\frac{\sigma\tanh(\frac{\sigma}{2}\beta)}{2\beta}\right\}^{p/p'}d\alpha\right]\ d\beta.$$

Isolating the integrals with respect to α and β, the right hand side of the above inequality equals to

$$\frac{4^p}{\sigma^p}\int_{\beta=-\infty}^{\infty}\cosh^p\left(\frac{\sigma}{2}\beta\right)\cdot\left\{\frac{\sigma\tanh(\frac{\sigma}{2}\beta)}{2\beta}\right\}^{p/p'}\left[\sum_{k=-\infty}^{\infty}\left\{\int_{\alpha=-\infty}^{\infty}\left|f\left(\alpha+\frac{\pi}{2\sigma}(2k+1)\right)\right|^p\ d\alpha\right\}\right.$$
$$\left.\times\left\{\frac{1}{\beta^2+(2k+1)^2\pi^2/\sigma^2}\right\}\right]\ d\beta.$$

From $\frac{1}{p}+\frac{1}{p'}=1$, we have $\frac{p}{p'}=p-1$, so the above equals to

$$\frac{4^p}{\sigma^p}\|f\|_{L^p}^p\int_{\beta=-\infty}^{\infty}\cosh^p\left(\frac{\sigma}{2}\beta\right)\cdot\left\{\frac{\sigma\tanh(\frac{\sigma}{2}\beta)}{2\beta}\right\}^{p-1}\left[\sum_{k=-\infty}^{\infty}\frac{1}{\beta^2+(2k+1)^2\pi^2/\sigma^2}\right]\ d\beta\ ,$$

which in view of (5.4.3) equals to

$$\begin{aligned}
&\frac{4^p}{\sigma^p}\left\|f\right\|_{L^p}^p \int_{\beta=-\infty}^{\infty} \cosh^p\left(\frac{\sigma}{2}\beta\right)\cdot\left\{\frac{\sigma\tanh(\frac{\sigma}{2}\beta)}{2\beta}\right\}^p\,d\beta \\
&= \sigma^p\left\|f\right\|_{L^p}^p \int_{\beta=-\infty}^{\infty}\left\{\frac{\sinh(\frac{\sigma}{2}\beta)}{\frac{\sigma}{2}\beta}\right\}^p\,d\beta \\
&= \sigma^p\left\|f\right\|_{L^p}^p \left\|\frac{\sinh(\frac{\sigma}{2}\beta)}{\frac{\sigma}{2}\beta}\right\|_{L^p}^p .
\end{aligned}$$

i.e.,

$$\int_{\beta=-\infty}^{\infty}\int_{\alpha=-\infty}^{\infty}\left|(\mathscr{D}f)(\alpha,\beta)\right|^p\,d\alpha\,d\beta \leq \sigma^p\left\|f\right\|_{L^p}^p\left\|\frac{\sinh(\frac{\sigma}{2}\beta)}{\frac{\sigma}{2}\beta}\right\|_{L^p}^p .$$

Taking the pth root on both sides yields

$$\left\|\mathscr{D}f\right\|_{L^p} \leq \sigma^p\left\|f\right\|_{L^p}\cdot\left\|\frac{\sinh\left(\frac{\sigma}{2}\beta\right)}{\frac{\sigma}{2}\beta}\right\|_{L^p} .$$

$\square$

Remark 5.5.9. In the limit as $q \to 1^-$,

$$\frac{\sinh\left(\frac{\sigma}{2}\beta\right)}{\frac{\sigma}{2}\beta} \to 1.$$

Hence Theorem 5.5.8 is a limiting case of Theorem 5.5.6.

CHAPTER 6: CONCLUSION

It's fine to work on any problem, so long as it generates interesting mathematics along the way - even if you don't solve it at the end of the day.

Andrew Wiles

As the title of the dissertations states, this was indeed an adventure seeking for Bernstein inequalities in which along the way we explore several avenues; Rational functions, Askey-Wilson operator, Interpolation formulae and Summation identities for entire functions. The journey has not come to a destination, rather it has opened several paths for me to continue and explore this exciting branch of mathematics. Following I will state some possible directions that the current work in the dissertation can be carried out.

6.1 Generalizing and extending polynomial inequalities to rational functions

In accordance with Chapter 3, I want to continue with the inequalities we already have and to improve the final forms if possible, especially in the rational analogue of the Govil's inequality, Theorem 3.3.10, and to work on the *conjecture* stated on page 15.

In establishing rational analogues to polynomial inequalities, the main focus in the dissertation was Bernstein-type inequalities associated with the unit disk; generalizations and extensions. While exploring more along this direction, I would also like to look not only at Bernstein inequalities, but also at Markov inequalities in different domains, for instance smooth and non-smooth Jordan curves, arcs, and intervals on the real line etc. as in the recent works of Vilmos Totik and his collaborators (see **[37]**, **[50]**, **[62]**, **[63]**, and **[64]**).

6.2 A systematic way of looking at summation identities

There are diverse collections of identities, even to the scale of encyclopedic nature stated in references such as **[27], [73]**, Ramanujan's notebooks, and **[31]**. Our plan is to discover as many identities we can with our two parameter extended Boas' formula. We are optimistic that our frame work will pave the way to accomplish this task. Also, we are looking at not only working with the Askey-Wilson operator alone but combining it and the Average operator.

During one of the discussions we had, Dr. Mourad Ismail suggested to us that our expansions should extend to *theta* functions and the proper setting to the kind of expansions that we're looking at is not in terms of *sine* and *cosine* but in terms of *theta* functions. We believe that a clear understanding of this procedure will aid us in establishing a systematic method to derive summation identities starting from our framework. Dr. Ismail also explained to us that the way we write things is similar to the way of that of the *Wilson* operator [**6**]:

$$(\mathscr{W} f)(x) := \frac{\breve{f}\left(y+\frac{i}{2}\right)-\breve{f}\left(y-\frac{i}{2}\right)}{2yi}, \qquad y=\sqrt{x},\ f(x)=\breve{f}(y).$$

So by an appropriate change of variables, followed by a limit, our results should yield results related to Wilson's operator.

As a sequel to his paper [**73**], A. I. Zayed in [**74**], yet again by borrowing techniques from *sampling theory*, derived summation formulas for doubly infinite series involving trigonometric functions and Bessel functions of the first kind. A higher dimensional/order version of the Askey-Wilson operator is yet unknown. Once such an operator is established, our framework can be used in **discovering** multivariate summation identities as well.

6.3 A general setting for interpolation formulas

It is well known in the folklore that once we have the "correct" interpolation formula we can get the "corresponding" Bernstein inequality(ies), which was what we accomplished through our generalized Riesz-type interpolation formula (**Theorem 4.3.1**) and the extended Boas' formula (**Theorem 5.2.1)**. More precisely, the former led us in establishing (4.3.4); the latter in establishing (5.5.3), (5.5.4), and (5.5.7).

In [**2, p.144, Theorem 3**] Naum I. Achieser showed that Bernstein inequality for functions of exponential type is in fact a special case of each of the two theorems below:

Theorem 6.3.1. *If $f \in \mathscr{B}_\sigma$, then the inequality*

$$\sup_{-\infty<x<\infty} |\sin(\alpha) \cdot f'(x) - \sigma \cos(\alpha) \cdot f(x)| \le \sigma \sup_{-\infty<x<\infty} |f(x)| \tag{6.3.1}$$

and

$$\sup_{-\infty<x<\infty} |\sin(\alpha) \cdot f'(x) + \cos(\alpha) \cdot (\tilde{f})'(x)| \le \sigma \sup_{-\infty<x<\infty} |f(x)| \ . \tag{6.3.2}$$

are satisfied for every $\alpha \in \mathbb{R}$. The equality holds if and only if $f(z) = ae^{i\sigma z} + be^{-i\sigma z}$.

To prove (6.3.1) and (6.3.2), Achieser used the following two *interpolation formulas* respectively for the functions in $\mathscr{B}_\sigma$:

$$\sin(\alpha) \cdot f'(x) - \sigma \cos(\alpha) \cdot f(x) = \sigma \sum_{k=-\infty}^{\infty} (-1)^{k-1} \frac{\sin^2(\alpha)}{(\alpha - k\pi^2)} \, f\left(\frac{k\pi - \alpha}{\sigma} + x\right) \tag{6.3.3}$$

and

$$\sin(\alpha) \cdot f'(x) + \cos(\alpha) \cdot (\tilde{f})'(x) = \sigma \sum_{k=-\infty}^{\infty} (-1)^{k-1} \, \frac{2\sin^2\left(\frac{\alpha-k\pi}{2}\right)}{(\alpha - k\pi)^2} \, f\left(\frac{k\pi - \alpha}{\sigma} + x\right) \ . \tag{6.3.4}$$

Stimulated by Achieser's work, our objective is to find interpolation formulas of the following types where $p \in \mathcal{P}_n$, $f \in \mathscr{B}_\sigma$, and $t \in \mathcal{T}_n$.

1. $\sin(\alpha) \cdot (\mathscr{D}_q f)(x) - \sigma \cos(\alpha) \cdot f(x) = \ldots\ldots?$

2. $\sin(\alpha) \cdot (\mathscr{D}_q p)(x) - \sigma \cos(\alpha) \cdot p(x) = \ldots\ldots?$

For the following items, $(\Lambda p)(\cdot)$ is called the *Szegö composition** of the polynomials Λ and p.

3. $\sin(\alpha) \cdot (\Lambda t)(\varphi) - \cos(\alpha) \cdot t(\varphi) = \ldots\ldots?$

4. $\sin(\alpha) \cdot (\Lambda \tilde{t})(\varphi) - \cos(\alpha) \cdot \tilde{t}(\varphi) = \ldots\ldots?$ etc.

6.4 Bernstein inequality in L^p-norm for $0 < p < 1$

Q. I. Rahman and G. Schmeisser in [**54**] proved that a Bernstein inequality for L^p-norm indeed holds for $0 < p < 1$ as well. They claimed their result with the use of couple of significant results of Vitalii Arestov [**4**] and Lars Hörmander [**34**] with the use of a function ϕ given by $\phi(t) := \psi(\log(t))$, where ψ is a nonnegative, nondecreasing convex function defined on $\mathbb{R}$. Following the methods of Rahman and Schmeisser, I would like to work on generalizing such a Bernstein inequality for the Askey-Wilson operator.

*[**5**] More precisely, for the polynomials

$$\Lambda(z) = \sum_{k=0}^{n} \lambda_k \binom{n}{k} z^k, \quad p(z) = \sum_{k=0}^{n} a_k \binom{n}{k} z^k ,$$

the polynomial

$$(\Lambda p)(z) = \sum_{k=0}^{n} \lambda_k a_k \binom{n}{k} z^k$$

is called the *Szegö composition of the polynomials* Λ *and* p.

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