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Inversion of Gamow's formula and inverse scattering

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Gamow's tunneling formula is inverted and the issue of the uniqueness of the solution is compared with the solution obtained by the method of Gel'fand and Levitan. Some insight is gained into the key differences between classical and quantum inverse scattering, which account for the fact that a potential can be uniquely determined in the latter but only to within a symmetry family in the former. © 2006 American Association of Physics Teachers.
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I. INTRODUCTION

Eugen Merzbacher has commented that¹ “among all the successes of quantum mechanics as it evolved in the third decade of the 20th century, none was more impressive than the understanding of the tunnel effect—the penetration of matter waves and the transmission of particles through a high potential barrier.” The tunnel effect provided a straightforward and remarkable explanation of the radioactive α -decay of nuclei. George Gamow was one of the protagonists in the discovery of the theory of α -decay^{1,2} and the basic formula, Eq. (14), that underlies tunneling through a potential barrier bears his name.

We will discuss how a knowledge of the tunneling behavior of a potential can be used to determine the potential. Such a procedure falls under the domain of inverse scattering. Tunneling is itself a scattering process. One considers particles incident on a localized potential (in this case, a potential barrier) and observes the frequency with which the particles are transmitted beyond the barrier or the frequency with which they are reflected. It is a scattering process limited to a certain regime—one in which the energy of the incident particles is less than the maximum of the potential barrier. Our procedure allows us to use the scattering data—the probability that the particles tunnel through the barrier—to determine information about the barrier. More generally, any procedure by which we can obtain the form of a potential based on the behavior of particles under its influence falls under the domain of inverse scattering.

In Sec. II, we consider some examples of one-dimensional inverse scattering in classical mechanics and the general one-dimensional problem in quantum mechanics. In particular, we focus on the fact that in quantum mechanics the potential can be uniquely determined, while it is only determined to within a symmetry family in classical mechanics. We revisit the quantum mechanical problem and derive a formula that determines the potential from the transmission data and demonstrate that the classical and quantum mechanical problems lead to similar solutions when similar assumptions are made. Uniqueness requires additional information which is often available in quantum mechanics from experimental data.

II. CLASSICAL VERSUS QUANTUM PROBLEM

A. The classical problem

The classical one-dimensional inverse scattering problem is exemplified by the two systems depicted in Fig. 1. Figure 1(a) depicts a particle in an attractive potential with a single minimum set to zero at $x=0$. Conservation of energy gives the period,

$$T(E) = \sqrt{2m} \int_{x_1(E)}^{x_2(E)} \frac{dx}{\sqrt{E - U(x)}}, \quad (1)$$

where $x_1(E)$ and $x_2(E)$ are the turning points for the energy E . The inverse problem is to determine the form of the potential $U(x)$ given the period as a function of energy, $T(E)$. The solution is well known and can be found in many texts on classical mechanics (see, for example, Ref. 3). If we treat x as a function of U , rather than U as a function of x , Eq. (1) may be brought to a form known as Abel's integral equation (see Ref. 4). Because $U(x)$ is not one-to-one, we need to split its domain at the origin and define the two functions $x_1(U)$ and $x_2(U)$ as in Fig. 1(a). The result is

$$x_2(U) - x_1(U) = \frac{1}{\pi\sqrt{2m}} \int_0^U \frac{T(E)dE}{\sqrt{U - E}}. \quad (2)$$

We see that Eq. (2) only determines the width Δx of the potential curve at each point of the U axis. Hence, the solution cannot be determined uniquely unless we assume that the potential is even. Let us call this unique even potential $\tilde{U}(x)$. Then, $x_2(\tilde{U}) = -x_1(\tilde{U})$. In this case,

$$x(\tilde{U}) = \frac{1}{2\pi\sqrt{2m}} \int_0^{\tilde{U}} \frac{T(E)dE}{\sqrt{\tilde{U} - E}}, \quad (3)$$

where $x(\tilde{U}) = x_2(\tilde{U}) = -x_1(\tilde{U})$.

We show in Fig. 1(b) a particle incident on a potential barrier that is confined to the interval $[0, L]$. The problem of determining the potential given the time of traversal of the potential as a function of energy has been solved by Lazenby and Griffiths.⁵ The forward and backward scattering times are

Fig. 1. (a) A particle in a potential well (attractive force field) performs oscillations. (b) A particle incident on a potential barrier (repulsive force field) will either overcome the potential if it has enough energy ($E > U_0$), or will be reflected if it does not ($E < U_0$).

$$T(E) = \sqrt{\frac{m}{2}} \int_0^L \frac{dx}{\sqrt{E - U(x)}}, \quad (E > U_0), \quad (4)$$

$$R(E) = \sqrt{\frac{m}{2}} \int_0^{x_1(E)} \frac{dx}{\sqrt{E - U(x)}}, \quad (E \leq U_0), \quad (5)$$

where $x_1(E)$ is the left turning point. $T(E)$ applies when the particle has energy exceeding U_0 , the maximum of the potential, and gives the time required for the particle to traverse the potential. $R(E)$ applies when $E \leq U_0$ and gives the time required for the particle to reach the turning point $x_1(E)$ or half the time taken for the particle to return to the origin.

The solutions to the barrier equations [Eqs. (4) and (5)] are similar to that of the previous system. The most important feature is that a class of potentials is obtained. There is, however, a unique solution with the property that it increases monotonically over the interval $[0, L]$ (and drops discontinuously to zero at L). Lazenby and Griffiths⁵ call this solution the *canonical* potential and use it to represent the class of solutions. For example, the inversion of the backward scattering data is given by

$$x(\tilde{U}) = \frac{1}{\pi} \sqrt{\frac{2}{m}} \int_0^{\tilde{U}} \frac{R(E) dE}{\sqrt{\tilde{U} - E}}, \quad (6)$$

where $\tilde{U}$ is the canonical potential.

Lazenby and Griffiths⁵ remark that it is curious that the solutions to Eqs. (4) and (5) are not determined uniquely, whereas in the quantum mechanical analogue the solution is unique. It is further remarked that “given the transmission coefficient T (the probability that the particle will surpass the barrier) as a function of energy E , (the potential) may be recovered by the method of Gel’fand and Levitan”. As will be discussed, this statement is not accurate as it stands. The transmission coefficient alone is not sufficient to determine the potential. It is the transmission amplitude, which is a complex function and which carries more information, that the method of Gel’fand and Levitan employs to uniquely identify the potential.

B. The quantum mechanical problem

The quantum mechanical problem is depicted in Fig. 2. Assume a particle is incident from the left on a potential which goes to a constant (which we set to zero) as $x \rightarrow \pm\infty$. If the potential approaches zero rapidly enough, the asymptotic form of the wavefunction for $x \rightarrow \pm\infty$ is a plane wave:

$$\psi(x) \sim \begin{cases} e^{ikx} + b(k)e^{-ikx} & (x \rightarrow -\infty) \\ a(k)e^{ikx} & (x \rightarrow +\infty), \end{cases} \quad (7)$$

where the energy of the particle $E > 0$ and

$$k = \sqrt{\frac{2mE}{\hbar^2}}. \quad (8)$$

If there exists an interval over which $U(x) < 0$, then there is a discrete spectrum E_n corresponding to bound states:

Fig. 2. Quantum mechanical scattering of a right-moving particle that approaches a potential from the left. It is assumed that the potential vanishes at large distances, and therefore bound states appear only if there is a region with $U < 0$ and only for negative energies.

Fig. 3. The method of Gel'fand and Levitan (see Ref. 6) in a nutshell.

$$\psi(x) \sim \begin{cases} c_n e^{+\kappa_n x} & (x \rightarrow -\infty) \\ d_n e^{-\kappa_n x} & (x \rightarrow +\infty), \end{cases} \quad (9)$$

where $E_n < 0$ and

$$\kappa_n = \sqrt{\frac{-2mE_n}{\hbar^2}}. \quad (10)$$

The scattering data for the inverse problem is comprised of the asymptotic coefficients $b(k)$ and c_n and the discrete eigenvalues κ_n . The potential is uniquely constructed using the method of Gel'fand and Levitan⁶ as outlined in Fig. 3.

Although the derivation of the method of Gel'fand and Levitan⁶ is beyond the scope of this paper, it is not difficult to describe, at least in a general sense, the steps involved. One begins by defining an auxiliary function, $F(X)$, as

$$F(X) = \sum_n c_n^2 e^{-\kappa_n X} + \frac{1}{2\pi} \int_{-\infty}^{+\infty} b(k) e^{ikX} dk. \quad (11)$$

$F(X)$ has no particular physical significance and is merely a mathematical artifice in the procedure to recover the potential. The same goes for $K(x, z)$, a second auxiliary function that is the solution to the integral equation

$$K(x, z) + F(x+z) + \int_x^{+\infty} K(x, y) F(y+z) dy = 0. \quad (12)$$

Equation (12) is of a form known as the Marchenko equation. Its solution is nontrivial and is outlined in Ref. 7. The potential is then determined by taking the directional derivative of $K(x, z)$ along the line $z=x$:

$$U(x) = -2 \frac{d}{dx} K(x, x). \quad (13)$$

III. GAMOW'S FORMULA AND ITS INVERSION

A. Gamow's formula

We now discuss the situation illustrated in Fig. 4. A particle is incident on a potential barrier with a single maxi-

Fig. 4. A particle incident on a potential barrier has a finite probability of overcoming the barrier even if its energy E is below the maximum U_0 of the barrier.

mum, U_0 at $x=0$, with energy E less than U_0 . In quantum mechanics, there is a finite probability for the particle to surpass the barrier despite the fact that $E < U_0$.

Gamow's tunneling formula gives a good approximation to the transmission coefficient, $T(E)$, which gives the probability for the particle to cross the barrier:

$$T(E) = \exp\left(-\frac{2}{\hbar} \int_{x_1(E)}^{x_2(E)} \sqrt{2m(U(x) - E)} dx\right). \quad (14)$$

Equation (14) can be derived by considering the barrier as an infinite sum of infinitely thin rectangular barriers (see p. 219 of Ref. 8). However, although this method provides the correct result, it is mathematically inconsistent. A more careful derivation using the Jeffreys–Wentzel–Kramers–Brillouin approximation is given in Ref. 8, p. 507. In what follows, $T(E)$ will play the role analogous to the classical scattering data.

B. Inversion of Gamow's formula

Consider the task of inverting Gamow's formula, Eq. (14). By differentiating and rewriting Eq. (14) in terms of the inverse functions $x_1(U)$ and $x_2(U)$ (we again split the domain of $U(x)$ at the origin), we find

$$\begin{aligned}
\frac{\hbar}{\sqrt{2m}} \frac{1}{T(E)} \frac{dT}{dE} &= \int_{x_1(E)}^{x_2(E)} \frac{dx}{\sqrt{U-E}} \\
&= \int_E^{U_0} \left(\frac{dx_1}{dU} - \frac{dx_2}{dU} \right) \frac{dU}{\sqrt{U-E}}.
\end{aligned} \quad (15)$$

Equation (15) is nearly in the form of Abel's equation:⁴

$$\int_0^E \frac{\phi(U)}{\sqrt{E-U}} dU = f(E). \quad (16)$$

Equation (15) differs from Eq. (16) in that the position of the parameter and the variable have been switched in the square root and in the limits of integration. Consequently, the preferred approach of applying the Laplace transform to Eq. (16) and making use of the convolution theorem fails. However, Abel's equation can be solved by composition with a kernel.⁹ We can apply this procedure with some modification. We divide both sides of Eq. (15) by $\sqrt{E-\alpha}$, where $0 \leq \alpha \leq U_0$, and integrate with respect to E from α to U_0 :

Fig. 5. The integral is defined on the triangular domain $\mathcal{D}$.

$$\frac{\hbar}{\sqrt{2m}} \int_a^{U_0} \frac{dT/dE}{T(E)\sqrt{E-\alpha}} dE = \int_a^{U_0} \frac{dE}{\sqrt{E-\alpha}} \int_E^{U_0} \left(\frac{dx_1}{dU} - \frac{dx_2}{dU} \right) \frac{dU}{\sqrt{U-E}}, \quad (17a)$$

$$= \int_a^{U_0} \left(\frac{dx_1}{dU} - \frac{dx_2}{dU} \right) dU \int_a^U \frac{dE}{\sqrt{(U-E)(E-\alpha)}}, \quad (17b)$$

where we have changed the order of integration (see Fig. 5).

In the Appendix, we show that

$$\int_a^U \frac{dE}{\sqrt{(U-E)(E-\alpha)}} = \pi. \quad (18)$$

Therefore,

$$\frac{\hbar}{\sqrt{2m}} \int_a^{U_0} \frac{dT/dE}{T(E)\sqrt{E-\alpha}} dE = \pi \int_a^{U_0} \left(\frac{dx_1}{dU} - \frac{dx_2}{dU} \right) dU, \quad (19)$$

and finally,

$$x_1(U) - x_2(U) = -\frac{\hbar}{\pi\sqrt{2m}} \int_U^{U_0} \frac{dT(E)/dE}{T(E)\sqrt{E-U}} dE, \quad (20)$$

where we have used the fact that $x_1(U_0) - x_2(U_0) = 0$. We see that the solution is similar in form to the classical result [Eqs. (2) and (6)], and the solution is not unique [unless we assume the potential to be even and $x_2(U) = -x_1(U)$]. Rather, we have obtained a family of potentials that all result in the same transmission coefficient.

C. An example: Cold emission

Although the photoelectric effect is the most popular example of emission of electrons by metals, it is not the only such phenomenon. Electrons can be emitted by metals at room temperature by the application of an external electric field $\mathcal{E}$. To contrast to the emission of electrons when a metal is heated, this phenomenon is termed *cold emission* (see Fig. 6).¹⁰

When an external field is applied, an electron in the metal sees a potential

$$U(x) = U_0 - e\mathcal{E}x, \quad (21)$$

where x is the distance from the surface of the metal. This potential does not incorporate the fact that a positive image charge will appear at the surface of the metal as the electron is removed, and thus an additional Coulomb attraction will appear. For an electron of energy E we find, using Eq. (14),

Fig. 6. Model potential for cold emission. E is typically taken as the Fermi energy. W is the work function of the metal.

$$T(E) = e^{-a(U_0 - E)^{3/2}}, \quad (22)$$

with

$$a = \frac{4\sqrt{2m}}{3e\mathcal{E}\hbar}. \quad (23)$$

Equation (22) is the Fowler–Nordheim equation. The quantity $U_0 - E$ is the work function.

We shall now assume that $T(E)$ is known, say from experimental data. Can we find the potential that reproduces it? According to Eq. (20),

$$x_1(U) - x_2(U) = -\frac{2}{\pi e\mathcal{E}} \int_U^{U_0} \sqrt{\frac{U_0 - E}{E - U}} dE. \quad (24)$$

The integral in Eq. (24) is elementary, and we calculate it in the Appendix. The result is

$$x_1(U) - x_2(U) = -\frac{1}{e\mathcal{E}}(U_0 - U). \quad (25)$$

The reader might believe that we have recovered the potential (21). Unfortunately, this is not the case as any two functions $x_1(U)$ and $x_2(U)$ that differ by the above amount are solutions of the inverse problem. The cold emission potential is recovered if we assume that $x_1(U) = 0$.

D. The issue of uniqueness

At this point, the reader might believe that the apparent conflict of our result with what would have been obtained by the method of Gel'fand–Levitan is due to the approximations used to produce Eq. (14). However, this apparent conflict cannot be so, because the mathematical statement of the problem is independent of the underlying physics.

The answer to this puzzle is simple. The method of Gel'fand and Levitan makes use of the amplitude $b(k)$ (in the present case, there is no bound spectrum) which is a complex quantity. However, we made use of T which is a real quantity; $T = |a(k)|^2$ with $|a(k)|^2 + |b(k)|^2 = 1$. Thus, we have lost information about the phase.

Given the transmission coefficient $T(E)$, $b(k)$ can have the form

$$b(k) = \sqrt{1 - T(E(k))} e^{if(k)}, \quad (26)$$

where $f(k)$ is a real-valued function. Each distinct potential among the family of our solutions corresponds to a different choice of $f(k)$.

IV. DISCUSSION AND CONCLUSION

We have found that $T(E)$, the probability for transmission and the analog of the classical scattering data, does not uniquely determine the potential, just as in classical mechanics. However, quantum mechanics gives us an additional set of data, the phase difference $f(k)$, which corresponds to measurements of time delay (Ref. 11, p. 138). It is only with both of these sets of data that we can uniquely determine the potential.

The approximate nature of Gamow's formula is an irrelevant feature of the problem we have studied. However, a different kind of question may be asked which makes this feature relevant. If the potential barrier is even and a unique solution $U(x)$ can be found, what is the error in determining the potential? That is, how close is the solution $U(x)$ to the real potential that gave the experimental data $T(E)$? This question remains open.

We have succeeded in solving a modified version of Abel's equation:¹² Given the integral equation,

$$\int_y^a \frac{\phi(x)}{\sqrt{x-y}} dx = f(y), \quad (27)$$

where $f(y)$ is a known function, $\phi(x)$ is unknown, and a is a constant. We have shown that the solution is given by

$$\phi(x) = -\frac{1}{\pi} \frac{d}{dx} \int_x^a \frac{f(y)}{\sqrt{y-x}} dy. \quad (28)$$

A final comment is in order. Cole and Good obtained a relation equivalent to Eq. (20) [see Eq. (2.2b) of Ref. 13]. However, instead of using Gamow's penetrability factor as their starting point, they began with results obtained from the use of the JWKB method, which reduces to Eq. (14) when $T(E) \ll 1$. The work of Cole and Good¹³ was motivated by the inversion procedure of Rydberg, Klein, and Rees, which determines the interparticle interaction from scattering, transport, and thermodynamic data.¹⁴ The work of Cole and Good was subsequently used in Ref. 15, which discusses an inversion formula for the internucleus potential using the subbarrier fusion cross section.

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APPENDIX A: INTEGRATION OF TWO IRRATIONAL FUNCTIONS

The integrals

$$I = \int \frac{dE}{\sqrt{(\beta-E)(E-\alpha)}}, \quad (A1a)$$

$$J = \int \frac{\sqrt{\beta-E}}{\sqrt{E-\alpha}} dE, \quad (A1b)$$

are elementary, but their calculation is lengthy. We introduce the substitution

$$u^2 = \sqrt{\frac{\beta-E}{E-\alpha}} \quad (A2)$$

in J and rewrite it as

$$J = -2(\beta-\alpha) \int \frac{u^2}{(1+u^2)^2} du = -2(\beta-\alpha) \left[\int \frac{1}{1+u^2} du - \int \frac{1}{(1+u^2)^2} du \right]. \quad (A3)$$

We have

$$\frac{1}{\lambda} \tan^{-1} \frac{u}{\lambda} = \int \frac{1}{\lambda^2 + u^2}, \quad (A4)$$

and therefore,

$$J = -2(\beta-\alpha) \left[\tan^{-1} u - \frac{1}{2} \frac{\partial}{\partial \lambda} \left(\frac{1}{\lambda} \tan^{-1} \frac{u}{\lambda} \right) \right]_{\lambda=1}, \quad (A5)$$

and

$$J = \sqrt{(E-\alpha)(\beta-E)} - (\beta-\alpha) \tan^{-1} \sqrt{\frac{\beta-E}{E-\alpha}}. \quad (A6)$$

Hence,

$$\int_{\alpha}^{\beta} \sqrt{\frac{\beta-E}{E-\alpha}} dE = (\beta-\alpha) \frac{\pi}{2}. \quad (A7)$$

We can also obtain

$$I = \int_{\alpha}^{\beta} \frac{dE}{\sqrt{(\beta-E)(E-\alpha)}} = 2 \frac{\partial J}{\partial \beta} = \pi. \quad (A8)$$

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Popper's experiment and communication

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We comment on an analysis by Qureshi of an experiment proposed by Popper and show that an analysis based solely on conventional nonrelativistic quantum mechanics is sufficient to exclude the possibility of subluminal or superluminal communication. That is, local operations cannot be employed to transmit information. © 2006 American Association of Physics Teachers.

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I. INTRODUCTION

In a recent paper,¹ Tabish Qureshi analyzed an experiment proposed by Karl Popper^{2,3} to test the standard interpretation of quantum theory. We describe Popper's experiment and, because Qureshi's analysis might leave the reader with some misconceptions, we comment on his analysis.¹ In Sec. II, we show that in the situation envisioned by Popper, even conventional nonrelativistic quantum mechanics suffices to exclude the possibility of communication, whether superluminal (faster than the speed of light) or subluminal. The Appendix gives the details of the analysis on which the conclusions in Sec. II rely. Some brief remarks are given in Sec. III.

Popper^{2,3} and Qureshi¹ consider a source S that emits non-interacting pairs of nonidentical particles 1 and 2 moving predominantly along the x direction (horizontal), but with small components of momentum along the y direction (vertical) and with zero components along the z direction (perpendicular to x and y) (see Ref. 1, Fig. 1). Figure 1 in Ref. 1 lies in the x, y plane. The total momentum of each pair is zero. Also, any distribution of the components of the momentum along the x direction is inconsequential, so that we are concerned solely with the momenta $p_1 = -p_2$ of particles 1 and 2 along the y direction. We assume, as Qureshi does in effect, that: (a) The source at $x=0$ emits a negligible number of particles with vertical momenta outside the range $|p_1| \leq p_{\max}$; (b) $p_{\max} > 0$ is much larger than that required by the Heisenberg uncertainty principle to limit the spread of the beam along y in the region between A and B (see Ref. 1, Fig. 1); (c) the beam of particles 1 moves to the left and encounters a screen at $x=-X$ with a narrow slit centered at $y=0$ (slit A of Ref. 1, Fig. 1); and (d) this slit introduces a momentum spread along the y axis that is much larger than $p_{\max}$, with the

result that after passing through Slit A, the particle 1 beam spreads much more broadly along y than it did before encountering slit A.

The question discussed by Popper and Qureshi is the following. The particle 2 beam moves to the right but does not encounter a slit. Does conventional quantum mechanics (what Qureshi calls the Copenhagen interpretation) predict that this beam will also be spread much more broadly along y at horizontal distances $x > X$ because of the entanglement between particles 1 and 2 embodied in the requirement that $p_1 = -p_2$ when emitted?

The unequivocal answer to this question, without the need to do a calculation, is no. Indeed, the observable effects of the beam on the screen behind B (for example, darkening as a function of y) must in every respect be completely independent of the size of the slit encountered at A. Otherwise, the observer at A (conventionally named Alice) could instantaneously transmit messages to her counterpart observer (conventionally named Bob) viewing the screen behind B, now placed at a very long distance $x \gg X$. In particular, if what Bob observes depends on the size of the slit, Alice (using a code on which she and Bob had previously agreed) can send Bob a message simply by widening and narrowing Slit A. Such superluminal (faster than light speed) communication of information is impossible.⁴ Furthermore, as Peres⁵ has emphasized, conventional quantum mechanics implies that it is impossible for Alice, by solely local operations, to transmit *any* information to Bob at any speed. Alice's control of the slit size at A, without performing any operations whatsoever at any points between A and the screen behind B, is a "local operation" by definition.

Unfortunately Sec. IV of Ref. 1 can be read to imply that Alice, by detecting the passage of particles 1 through slit A as she controls the width of the slit, can affect the spread of

the beam on the screen behind B (see, in particular, the text immediately following Eq. (13) of Ref. 1). Qureshi has assured us that this reading is not his intention.⁶ Rather, his Sec. IV is concerned with coincidence measurements, on particle 1 at Slit A and on particle 2 at the screen behind B, performed on a pair of particles that were originally simultaneously emitted from the source S . In fact, Qureshi, in an analysis⁷ of Popper's experiment written only a few months before Ref. 1 was submitted, explicitly stated that in the absence of such coincidence measurements the observable effects of the particle 2 beam on the screen behind B will be independent of the width of Slit A. The clarifications in this paragraph concerning the implications of Sec. IV of Ref. 1 (which does not explicitly mention coincidences), and in the preceding paragraph concerning the predictions of conventional quantum mechanics respecting Popper's experiment, are among the goals of this paper.

II. QUANTUM MECHANICAL ANALYSIS

It is useful to present a simple derivation demonstrating that the application of conventional quantum mechanics to Popper's experiment predicts that the observable effects of the beam on the screen behind B must be completely independent of the size of the slit encountered at A, or of any other local operations at A. In particular, we show that in Popper's experiment, conventional quantum mechanics precludes *any and all* local operations on particles 1 from changing the probability distributions in the beam of particles 2, despite the entanglement between simultaneously emitted particles 1 and 2. Moreover, manipulations of the measurement equipment at A, for example, switching on electromagnetic fields in the vicinity of A, are included in the local operations for which our derivation holds (see Sec. II A), as is the performance of actual measurements at A (see Sec. II B). That is, we show that conventional quantum mechanics prevents information about the manipulations of Slit A or measurements performed by Alice from being transmitted via the beams at any speed.

Our analysis, although certainly generalizable to many other experimental circumstances, is confined to local operations in Popper's experiment. Our derivation has the complication that measurements, which collapse the wave function (Sec. II B), require a different treatment than do all other local operations, which affect the wave function via force terms generated in the Schrodinger equation (Sec. II A). Nevertheless our derivation is easily grasped. We know of no other readily grasped demonstration that in Popper's experiment local operations cannot be employed to transmit information; certainly no such derivation is to be found in Ref. 1 or in Qureshi's earlier paper.⁷ More general proofs not restricted to Popper's experiment, demonstrating that information cannot be transmitted by local operations exist (see, for example, Refs. 4 and 8), but are difficult for nonexperts.

We emphasize that our paper is concerned solely with particle 2 observations made by Bob without any knowledge of coincident particle 1 observations made by Alice. Therefore, our results do not contradict those of Ref. 1, once it is recognized that Ref. 1 is concerned solely with coincident measurements by Alice and Bob. Because the particles leave the source in pairs with equal and opposite momenta, the position (or momentum) of a particle 2 observed by Bob will be correlated with the position (or momentum) of the paired particle 1 coincidentally observed by Alice. Correspondingly,

because the position (or momentum) range of particles 1 that Alice is able to observe behind the slit is affected by the width of the slit, an apparent correlation between the width of the slit and the position (or momentum) range of particles 2 observed by Bob in coincidence measurements is understandable. But any such coincidence correlations observable by Bob are not inconsistent with our conclusion that Alice's local operations at A cannot affect Bob's noncoincident observations at B. In particular, our results imply that when Bob performs his noncoincident position (or momentum) measurements on the collection of particles 2 reaching his screen, the results of those measurements will not depend on whether Alice did or did not make coincident measurements on the particles 1 paired with the particles 2 in this collection, or on the position (or momentum) values Alice might or might not have found in these coincident measurements, or even on whether the bulk of those paired Type 1 particles actually passed through Alice's slit so that Alice might be able to observe them.

A. Local nonmeasurement operations by Alice

We assume here, consistent with our description of Popper's experiment in Sec. I, that we need to be concerned solely with particle motions along the y direction. On the basis of this assumption (which is relaxed in the Appendix), for any given pair of particles 1 and 2 that simultaneously leave the source S , the unnormalized wave function expressing their entanglement at the instant they leave the source is

$$\Psi(y_1, y_2) = \int dK W(K) e^{-iKy_1} e^{iKy_2}, \quad (1)$$

where the plane waves have momenta $p_2 = -p_1 = \hbar K$. The integral (like all integrals in this paper) runs from $-\infty$ to ∞ ; $W(K)$ describes the particle momentum distribution along the y direction; $|W(K)|^2$ is negligible for $|\hbar K| \geq p_{\max}$; the initial presumably random phase $e^{i\phi(K)}$ multiplying each plane-wave pair $e^{-iKy_1} e^{iKy_2}$ has been absorbed into $W(K)$. Because every entangled particle pair moves independently of every other such pair, $\Psi(t)$, the function into which $\Psi(y_1, y_2) \equiv \Psi(0)$ evolves as time t increases, predicts the evolution of the probability distribution of all particle 2 trajectories toward the screen behind B (see Ref. 1, Fig. 1) even though Ψ depends on the coordinates of only a single pair of particles.

We can assume that $W(K)$ in Eq. (1) has been normalized so that $\int dK |W(K)|^2 = 1$. In this case, $|W(k_2)|^2 dk_2$ can be interpreted as the probability that when the source emits a particle pair, the wave number of particle 2 will lie between k_2 and $k_2 + dk_2$ (still considering motion only along the y direction). In the context of Popper's experiment, any burst of particles 2 can be assumed to move freely, with no changes in momentum, until the corresponding burst of particles 1 encounters the local operations being performed by Alice in the vicinity of Slit A. Peres⁹ has proved that when the individual particles are represented by wave packets, the paired particles 1 and 2 emitted with opposite momenta move in opposite directions along the same straight line. Consequently, because the paired particles are emitted with opposite momenta along x as well as along y , until any particles 1 reach the vicinity of Slit A, the distribution as a function of y of

any darkening or other observable effects produced by particles 2 on any screen intercepting the particle 2 beam is determined entirely by $|W(k_2)|^2$.

The components $\Phi(k_1, k_2)$ of $\Psi(y_1, y_2)$ in wave number space are

$$\begin{aligned}\Phi(k_1, k_2) &= (1/2\pi) \int dy_1 dy_2 e^{-ik_1 y_1} e^{-ik_2 y_2} \Psi(y_1, y_2) \\ &= 2\pi \int dK W(K) \delta(K + k_1) \delta(K - k_2) \\ &= 2\pi W(k_2) \delta(k_1 + k_2),\end{aligned}\quad (2)$$

where we have used the result $(1/2\pi) \int e^{-is(u-v)} ds = \delta(u-v)$. At the source, the number of particles 2 with wave numbers between k_2 and $k+dk_2$ must be proportional to $D(k_2)dk_2$, where the particle 2 wave number distribution function is

$$\begin{aligned}D(k_2) &= \int dk_1 |\Phi(k_1, k_2)|^2 \\ &= (2\pi)^2 \int dk_1 |W(k_2)|^2 [\delta(k_1 + k_2)]^2 \\ &= (2\pi)^2 \delta(0) |W(k_2)|^2.\end{aligned}\quad (3)$$

Thus, until any particles 1 reach the vicinity of A, the function $D(k_2)$, which is proportional to $|W(k_2)|^2$, also completely determines the distribution as a function of y of any darkening or other observable effects produced by the corresponding burst of particles 2 on any screen intercepting the particle 2 beam. In addition, we prove in the Appendix that no matter what local operations are performed by Alice, provided these operations do not involve measurements, the particle 2 wave number distribution function $D(k_2; t)$ obeys $D(k_2; t) = D(k_2) \equiv D(k_2; 0)$ at all times $t > 0$, where $D(k_2; t)$ is defined in terms of $\Psi(t)$ in the same way as was $D(k_2)$ in terms of $\Psi \equiv \Psi(0)$. We conclude (as elaborated in the Appendix) that, irrespective of local nonmeasurement operations by Alice, the observable effects of the particle 2 beam on the screen behind B remain precisely what they would have been had the particle 1 beam moved totally freely after leaving source S. That is, we have proved that neither modifying the size of Slit A, nor performing any other local nonmeasurement operations, permits Alice to send messages to Bob.

The $\delta(0)$ factor on the right side of Eq. (3) reflects the fact that Ψ of Eq. (1) is unnormalizable, not merely unnormalized. Indeed,

$$\begin{aligned}\Psi^\dagger \Psi &= \Phi^\dagger \Phi = \int dk_1 dk_2 |\Phi(k_1, k_2)|^2 = \int dk_2 D(k_2) \\ &= (2\pi)^2 \delta(0) \int dk_2 |W(k_2)|^2 = (2\pi)^2 \delta(0),\end{aligned}\quad (4)$$

using the fact that W is normalized; the dagger $\dagger$ denotes the adjoint. Because the Schrodinger equation is linear, multiplying $\Psi(0)$ by any constant factor C causes $\Psi(t)$, $D(k_2)$, and $D(k_2; t)$ to be multiplied by the same C [recall the definitions of $D(k_2)$ and $D(k_2; t)$]. Therefore (as elaborated in the Appendix), our proof of the equality $D(k_2; t) = D(k_2)$, and the important conclusion stated at the end of the preceding paragraph, are not invalidated by employing an unnormalized Ψ .

B. Measurements by Alice

Appendix A and Sec. II A are not relevant to local operations at A involving the performance of actual measurements. To see that measurements at A also cannot enable Alice to send messages to Bob, let us examine the consequences of a decision by Alice to make wave number measurements of her own on the particle 1 beam, before Bob has a chance to make his measurements. We want to show that our conclusion in Sec. II A, namely that $D(k_2)$ of Eq. (3) determines the distribution of wave numbers k_2 observed by Bob, irrespective of Alice's local operations on Type 1 particles, remains valid when Alice's local operations include measurements. For this purpose, it is desirable to first examine an experimental situation that is not complicated by the facts that Ψ of Eqs. (1) or (3) is unnormalizable, and that the unit basis vectors

$w(y, k) = (1/\sqrt{2\pi})e^{iky}$ in wave number space lie in the continuum. Assume that we again have entangled pairs of particles 1 and 2, with Alice and Bob capable, respectively, of making local measurement observations on particles 1 at A and on particles 2 at B. Assume further that at some instant, the wave function describing the state of a representative entangled pair 1 and 2 now is

$$\Psi = \sum_{i,j} a_{ij} \alpha_i \beta_j. \quad (5)$$

In Eq. (5), the α_i are an orthonormal set of eigenstates for the measurement operation Alice plans to make; the β_j are similarly defined for Bob; Ψ is normalized, implying that the coefficients a_{ij} satisfy $\sum_{i,j} |a_{ij}|^2 = 1$. At this instant, for any given a_{ij} the quantity $|a_{ij}|^2$ is the probability that measurements on the particle pair will find particle 1 in the eigenstate α_i and particle 2 in the eigenstate β_j . Correspondingly, if we sum $|a_{ij}|^2$ over all possible states i in which the particle 1 paired with this particle 2 might have been found, we obtain the actual probability ρ_{2j} of finding particle 2 in the eigenstate β_j , namely $\rho_{2j} = \sum_i |a_{ij}|^2$. If many independently moving entangled pairs 1 and 2 are being observed, the number of particles 2 in the various different states β_j actually observed by Bob cannot but be proportional to their respective probabilities ρ_{2j} .

We have not specified whether or not Alice actually has performed measurement observations on particle 1. Because nothing has been said about any collapse of Ψ induced by Alice's measurements, we might infer that the preceding paragraph presumed that Alice had not made any actual measurements before Bob made his measurements. The important point, which we will demonstrate, is that whether or not Alice did her measuring before Bob is irrelevant to the validity of our interpretations of $|a_{ij}|^2$ and ρ_{2j} . In particular, suppose that Alice, before Bob makes any measurements on particle 2, observes that the paired particle 1 is in the state α_i . According to the conventional understanding of measurements in quantum mechanics, this measurement immediately collapses Ψ of Eq. (5) to the new wave function¹⁰

$$\Psi_{ci} = \alpha_i \left[\sum_k |a_{ik}|^2 \right]^{-1/2} \sum_j a_{ij} \beta_j. \quad (6)$$

Except for the factor $[\sum_k |a_{ik}|^2]^{-1/2}$, Ψ_{ci} has plucked from Ψ of Eq. (5) all terms containing α_i and only those terms, as we expect for the collapsed wave function after observing particle 1 in the state α_i . The factor $[\sum_k |a_{ik}|^2]^{-1/2}$, which is con-

sistent with the Born rule,¹⁰ is required in order that Ψ_{ci} be normalized, that is, $\Psi_{ci}^\dagger \Psi_{ci} = 1$, as any wave function describing an actual physical situation should be. According to Eq. (6), the probability $\rho_{2j|1i}$ of observing particle 2 in state β_j , knowing that particle 1 has been observed in the state α_i , is $\rho_{2j|1i} = |a_{ij}|^2 [\sum_k |a_{ik}|^2]^{-1}$. But, consistent with the preceding paragraph, the probability ρ_{1i} that Alice has observed particle 1 in the state α_i must be $\rho_{1i} = \sum_j |a_{ij}|^2$.

Thus, the probability of Alice *first* observing particle 1 in the state α_i and Bob *only then* observing the paired particle 2 in the state β_j must be $\rho_{1i}\rho_{2j|1i} = |a_{ij}|^2$, exactly the probability given in the first paragraph of this Sec. II B for finding particle 1 in the eigenstate α_i and the paired particle 2 in the eigenstate β_j without a specified temporal order of making the measurements on the two particles. Correspondingly, because Alice had to find her particle 1 in some α_i , the actual probability that Bob will find the paired particle 2 in the state β_j after Alice made her measurement again will be the probability $\rho_{2j} = \sum_i |a_{ij}|^2$ obtained in the penultimate paragraph. We conclude that when many independently moving entangled pairs are being observed (as in Popper's experiment), the numbers of particles 2 in the various different states β_j actually observed by Bob will be proportional to the same respective probabilities ρ_{2j} whether or not Bob has made his observations after measurements by Alice. This conclusion does not depend on the nature of the states α_i and β_j , that is, it does not depend on the kinds of measurements Alice (on particles 1 only) and Bob (on particles 2 only) have chosen to perform. Of course, it is assumed that the measurements are performed independently, meaning that Bob receives no communications from Alice that could enable him to modify his measurements depending on Alice's measurement results. Therefore, we have proved that when the experimental situation involves many pairs of independently moving pairs of entangled particles 1 and 2, and when the state of any representative entangled pair is described by the wave function Ψ of Eq. (5), Alice cannot employ her local measurement observations on particles 1 at A to send messages to Bob at B, because the nature of her measurements and whether or not she performs them will not in any way alter Bob's observations of the particles 2 at B.

The proof in the preceding paragraph is generally valid for particle pair systems described by Eq. (5), wherein Ψ is normalized and is defined by a discrete sum; for example, this proof is valid for the commonly discussed case of observations on a large number of similarly entangled qubit pairs. Therefore, as discussed further in the Appendix, this proof is valid if Ψ of Eq. (5) is the discrete sum normalized wave function which replaces Ψ of Eq. (1) when the wave function is required to satisfy suitable boundary conditions at the interior surface of a large but finite volume. With this replacement, the complications listed at the beginning of Sec. II B are avoided.

III. CONCLUDING REMARKS

We have shown that the application of conventional quantum mechanics to Popper's experiment predicts that the observable effects of the beam on the screen behind B must be completely independent of the size of the slit encountered at A, or of any other local operations at A. Our derivation is not fully mathematically rigorous, but we believe it captures, in a fashion accessible to nonexperts, the essence of the physics involved in Popper's experiment when the particles involved

are not photons. Our demonstration is not convincing for a Popper-type experiment with pairs of photons, which do not obey the usual Schrodinger equation and can be destroyed in the course of detection, a possibility that our derivation does not include. The possibility of establishing (though not necessarily doing so simply) theorems for photons similar to those derived in this paper follows from our general remarks in the third paragraph of Sec. I.

Finally, we remark that when (as in Sec. II A) we can ignore particle motions along directions other than the vertical y direction, then if $|W(k_2)|^2$ in Eq. (3) is zero for $k_2 > p_{\max}/\hbar$, our results imply no wave function collapsing measurements on particles 1, or any other local operations on these particles for that matter, can result in *any* particles 2 arriving at screen B with vertical momenta greater than $p_{\max}$ in magnitude. From this feature *alone*, we can conclude that in Popper's experiment inserting a vertical narrow slit in the path of particles 1 will *not* cause an increased spread in the angular trajectories of particles 2.

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APPENDIX A: THE EFFECT OF LOCAL OPERATIONS

Unless explicitly stated otherwise, the local operations considered in this Appendix, like the local operations considered in Sec. II A, do not involve measurements. Suppose that as a consequence of some such local operation at A, the particles 1 no longer move freely once they reach the vicinity of A, but the particles 2 continue to move freely. Also assume, until stated otherwise, that this local operation permits us to concentrate solely on motions along y as we have done, and as would be the case if the local operation were the interruption of the particle 1 beam by a narrow horizontal slit at A. With these assumptions, the Hamiltonian $\mathbf{H}$ governing the particle motions can be written in the form $\mathbf{H} = \mathbf{H}_1 + \mathbf{T}_2$, where $\mathbf{T}_2$ is the kinetic energy operator for particle 2; because motion along y and z is ignored, $\mathbf{T}_2 = p_2^2/2m_2$, with $p_2 = -i\hbar \partial/\partial y_2$; $\mathbf{H}_1$ depends on the particular local operation, but because the operation is local, $\mathbf{H}_1$ is independent of y_2 and p_2 (as well as any other particle 2 coordinates and momentum components). Then, if $\mathbf{H}_1$ is time independent,¹¹

$$\Psi(t) \equiv \Psi(y_1, y_2; t) = e^{-i\mathbf{H}t/\hbar} \Psi(0) = \int dK W(K) u(y_1, K; t) \times [e^{-i\hbar K^2 t/2m_2} e^{iKy_2}], \quad (\text{A1})$$

where, as before, $\Psi(0)$ is the wave function specified by Eq. (1), and we define $u(y_1, K; t)$ as the y_1 component of the function $e^{-i\mathbf{H}_1 t/\hbar} e^{-iKy_1}$, that is

$$u(y_1, K; t) \equiv [e^{-i\mathbf{H}_1 t/\hbar} e^{-iKy_1}]_{y_1} = \int dy U(y_1, y; t) e^{-iKy}, \quad (\text{A2})$$

with U the unitary operator $e^{-i\mathbf{H}_1 t/\hbar}$. If $\mathbf{H}_1$ is time dependent, as it would be if Alice were to change the slit width at A while the beam of particles 1 is impinging on A, it is necessary only¹¹ to replace $\mathbf{H}_1 t$ in $e^{-i\mathbf{H}_1 t/\hbar}$ by the appropriately

time ordered integral $\int_0^t dt' \mathbf{H}_1(t')$. The key point is that the right side of Eq. (A2) remains a valid relation for $u(y_1, K; t)$ in Eq. (A1), with $\mathbf{U}$ still a unitary operator. Thus, whatever the local operations, time independent or time dependent, the components $\Phi(t) \equiv \Phi(k_1, k_2, t)$ of the wave function in wave number space now are, recalling Eq. (2) and using Eq. (A2),

$$\begin{aligned}\Phi(t) &= (1/2\pi) \int dy_1 dy_2 e^{-ik_1 y_1} e^{-ik_2 y_2} \Psi(t) \\ &= W(k_2)^{-i\hbar k_2^2 t / 2m_2} \int dy_1 e^{-ik_1 y_1} u(y_1, k_2; t).\end{aligned}\quad (\text{A3})$$

Consequently, recalling Eq. (3), once particles 1 have reached A the number of particles 2 with wave numbers between k_2 and $k_2 + dk_2$ becomes proportional to $D(k_2; t) dk_2$, where the particle 2 wave number distribution function now is

$$\begin{aligned}D(k_2; t) &= \int dk_1 |\Phi(t)|^2 \\ &= |W(k_2)|^2 \int dk_1 \int dy_1 e^{-ik_1 y_1} u(y_1, k_2; t) \\ &\quad \times \int dy'_1 e^{ik_1 y'_1} u^*(y'_1, k_2; t) \\ &= (2\pi) |W(k_2)|^2 \int dy_1 \int dy'_1 \\ &\quad \times \delta(y_1 - y'_1) u(y_1, k_2) u^*(y'_1, k_2) \\ &= (2\pi) |W(k_2)|^2 \int dy_1 |u(y_1, k_2, t)|^2.\end{aligned}\quad (\text{A4})$$

In Eq. (A4) recall Eq. (A2) and use the fact that $\mathbf{U}$ is unitary. Then

$$\begin{aligned}\int dy_1 |u(y_1, k_2, t)|^2 &= u(t)^\dagger u(t) = [\mathbf{U} e^{-ik_2 y_1}]^\dagger \mathbf{U} e^{-ik_2 y_1} \\ &= [e^{-ik_2 y_1}]^\dagger \mathbf{U}^\dagger \mathbf{U} e^{-ik_2 y_1} \\ &= \int dy_1 |e^{-ik_2 y_1}|^2 = (2\pi) \delta(0),\end{aligned}\quad (\text{A5})$$

where we have employed standard matrix manipulations.

Equations (A4) and (A5) make $D(k_2, t)$ identical with $D(k_2)$ from Eq. (3). In other words, we have shown that, no matter what local operations Alice performs on the particle 1 beam, the momentum distribution of the particles 2 reaching screen B is exactly the same as would have been observed had no local operations been performed. Because $D(k_2, t)$ involves only the amplitude of each $\Phi(k_1, k_2; t)$ but not its phase [recall Eq. (A4)], this result suggests, but does not prove, that the motion of particles 2 from the source to the Screen B is independent of the local operations performed on particles 1. The proof of this independence is readily demonstrated. Because $\mathbf{H}_1$ does not depend on the position or momentum of particle 2, $\mathbf{H}_1$ commutes with any operator Θ_2 that acts on particle 2 but is independent of particle 1. If Θ_2 depends only on y_2 and p_2 , a straightforward generalization of the previous derivation of $D(k_2, t) = D(k_2)$ shows that the

value of $\int dy_1 dy_2 \Psi(t)^\dagger \Theta_2 \Psi(t)$, though possibly time dependent, is independent of $\mathbf{H}_1$, that is, is the same as if $\mathbf{H}_1 = \mathbf{T}_1$, where $\mathbf{T}_1$ is the particle 1 kinetic energy operator. Were Ψ normalized, $\int dy_1 dy_2 \Psi(t)^\dagger \Theta_2 \Psi(t)$ would yield the expectation value of Θ_2 as a function of time. These expectation values encompass the results of all possible observations of the particle 2 beam (still assuming that motion along directions other than the y directions can be ignored).

We now drop the assumption that we are concerned solely with local operations which permit us to ignore the motions of particles 1 along the x or z directions. In general, local operations on particles 1 may be expected to mix momenta along x and y , as well as to deflect particles 1 out of the x, y plane in which we have assumed they move. Certainly, such deflection is likely to occur if the local operation involves electromagnetic interactions. It follows that, for the purpose of determining the time dependence of the particle motions when particles 1 are subject to actual local operations in the vicinity of slit A, Eq. (1), with its neglect of all particle coordinates other than y_1 and y_2 , generally is no longer useful. Instead it is necessary to start from

$$\Psi(0) \equiv \Psi(\mathbf{r}_1, \mathbf{r}_2) = \int d\mathbf{K} W(\mathbf{K}) e^{-i\mathbf{K} \cdot \mathbf{r}_1} e^{i\mathbf{K} \cdot \mathbf{r}_2}, \quad (\text{A6})$$

where $d\mathbf{K} = dK_x dK_y dK_z$ and $\int d\mathbf{K} |W(\mathbf{K})|^2 = 1$; the notation should otherwise be obvious. It still is true that $\mathbf{H} = \mathbf{H}_1 + \mathbf{T}_2$, where $\mathbf{T}_2$ is the kinetic energy operator for particle 2 and $\mathbf{H}_1$ is independent of any particle 2 coordinates.

It now is seen that the above derivation of $D(k_2, t) = D(k_2)$ starting from Eq. (1) is parallel to a derivation, starting from Eq. (A6), that yields $D(\mathbf{k}_2, t) = D(\mathbf{k}_2)$, where $D(\mathbf{k}_2)$ is the particle 2 wave number distribution function at the source S for arbitrary wave number vector $\mathbf{k}_2$. [When starting from the unnormalizable three-dimensional $\Psi(\mathbf{r}_1, \mathbf{r}_2)$ of Eq. (A6), the equations corresponding to Eqs. (2)–(4) and Eqs. (A1)–(A5) contain three-dimensional delta functions rather than one-dimensional delta functions. For instance, the equation corresponding to Eq. (3) is

$$D(\mathbf{k}_2) = (2\pi)^6 \delta(\mathbf{0}) |W(\mathbf{k}_2)|^2, \quad (\text{A7})$$

where $\delta(\mathbf{K}) = \delta(K_x) \delta(K_y) \delta(K_z)$ is the three-dimensional Dirac delta function.] Note that $D(\mathbf{k}_2)$ is proportional to $|W(\mathbf{k}_2)|^2$, just as $D(k_2)$ is proportional to $|W(k_2)|^2$. Similarly it can be shown that $\int d\mathbf{r}_1 d\mathbf{r}_2 \Psi(t)^\dagger \Theta_2 \Psi(t)$ remains independent of $\mathbf{H}_1$ when the operator Θ_2 is independent of particle 1. Thus even when the particles can move along all three directions, conventional quantum mechanics permits the conclusion that irrespective of local nonmeasurement operations by Alice, the observable effects of the particle 2 beam on the screen behind B remain the same as if the particle 1 beam had moved totally freely after leaving the source S.

The Schrodinger equation for freely moving particles 1 impinging on a slit screen at $x = -X$ (as Popper's experiment envisages) usually would be solved by imposing some appropriate boundary condition at points on the plane $x = -X$. In this formulation, the equation $\Psi(t) = e^{-i\mathbf{H}t/\hbar} \Psi(0)$ of Eq. (A1), with $\mathbf{H}$ as the usual free particle Hamiltonian for both particles, will not yield the correct $\Psi(t)$ at times t after particles 1 have reached $x = -X$. We have assumed the relevant physics of particles impinging on a screen can be adequately reproduced by the replacement of the boundary condition with suitable forces. Such forces must exist, because otherwise the

particles would penetrate the screen. We do not doubt the validity of our assumption, but believe we should make it explicit. Note that merely postulating the existence of such forces describable by a Hamiltonian is sufficient for our purpose; the preceding analysis in this Appendix depends only on the existence of such an $\mathbf{H}$ (whose details we need not know). A similar assumption must be made for any other conceivable local operation at A that Alice might impose and that at first sight is describable by a boundary condition, not by forces.

In Sec. II A it is argued that because the Schrodinger equation is linear, equalities like $D(k_2, t) = D(k_2)$ are not invalidated by having been derived using an unnormalized Ψ ; the result that $\int dy_1 dy_2 \Psi(t)^\dagger \Theta_2 \Psi(t)$ is independent of $\mathbf{H}_1$ similarly remains unaltered when $\Psi(0)$ is multiplied by a constant factor C . Nevertheless, we will defend our use of unnormalized wave functions. The wave function Ψ of Eq. (A6) [or of Eq. (1)] can be made normalizable by the device of confining Popper's experiment to the interior of the large volume V formed by the distant planes $x = \pm L$, $y = \pm L$, and $z = \pm L$. At these planes, the wave number eigenfunctions in the expansion of Ψ are required to satisfy periodic (or other suitable) boundary conditions, a requirement that limits the allowed values of the particle wave numbers to a discrete (though infinite) set. In this fashion, the extensions of all our earlier results can be derived straightforwardly using normalized wave functions only. In this case, coordinate integrals over all space are replaced by integrals over the interior of V , and integrals over all wave numbers are replaced by sums over the allowed wave number values.

We have not employed such normalized wave functions in the Appendix because the sums would obfuscate the transparency of our analysis. We were forced to employ such normalized wave functions in Sec. II B for reasons explained there. There is little doubt that for arbitrarily large L it should be possible to represent the physics of a spatially confined experiment, such as Popper's, with arbitrarily high precision,

even though the allowed particle wave numbers are limited to a discrete set. Physicists have used discretized wave expansions and box normalization ever since the dawn of quantum mechanics.¹² To put it differently, because the allowed discrete wave numbers are very close to each other for large L and change as L changes, it is unreasonable to think that our proof in Sec. II B does not carry over to all wave numbers (where we now at last are including local measurement operations).

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¹¹K. Gottfried, *Quantum Mechanics* (W.A. Benjamin, New York, 1966), pp. 235–240.

¹²See, for example, W. Heitler, *The Quantum Theory of Radiation* (Clarendon Press, Oxford, 1954), pp. 38 ff.

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