Terahertz Emission from the Intrinsic Josephson Junctions of High-Symmetry Thermally-Managed Bi$_2$Sr$_2$CaCu$_2$O$_{8+d}$ Annular Microstrip Antennas

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TERAHertz Emission from the Intrinsic Josephson Junctions of High-Symmetry Thermally-Managed Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ Annular Microstrip Antennas

by

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B.S. Stetson University 2016
B.A. Stetson University 2016

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ABSTRACT

The intrinsic Josephson junctions in the high transition temperature superconductor Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ (BSCCO) have shown great potential for oscillators emitting in the terahertz frequency. The radiation frequency satisfies the conditions for both the ac Josephson effect and for a mesa cavity resonance mode. The observed angular dependence of the emissions from some mesa imply that the ac Josephson effect plays the primary role in a dual source radiation mechanism. But the integrated emission power had generally been significantly below the 1 mW level suitable for many applications. This output power can be enhanced by a suitable design of an array of suitably shaped mesas that are all within a wavelength of each other so that their combined output is coherent. One such tightly packed array consists of concentric annuli. Here we calculate the angularly independent modes of thin annular microstrip antennas, with the ratio of the inner to the outer radii varying from 0.1 to 0.9. We then calculate the angular distribution of the emission power arising from the annular cavity modes and from the uniform ac Josephson current source at the frequencies of the cavity modes. We also calculate the five leading wavefunctions with the lowest order angular dependence for those annuli.
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INTRODUCTION

Electromagnetic waves in the terahertz (THz) frequency range are considered a fascinating area of research that has a multitude of applications. The interesting THz range is from about 0.3 THz to 10 THz, because this area is not well developed. This is due to the fact that having a compact and convenient solid-state source has been lacking for use in a large variety of applications, including medical detections, biosciences, high speed communications, security, etc. Not only is coherent THz radiation tunable over much or all of this range highly desired, for practical uses an output power of 1 mW or more is needed.

This thesis studies microstrip antennas that employ the intrinsic Josephson junctions of Bi$_2$Sr$_2$CaCu$_2$O$_{8+δ}$ (BSCCO) a high temperature superconductor, and coherent THz radiation emitter. (1, 2, 3, 4, 5) Thin cylinders (disks) and rectangles have already been studied greatly. Since heating effects can only be controlled by standalone mesa structures consisting of a doubly-cleaved piece of a small BSCCO crystal about 1μm thick that is sandwiched between gold layers within insulating plates, (as sketched in figure 1) the total number of junctions in one device is about one thousand. This number is not large enough to achieve the desired output power exceeding 1mW. An array of three rectangular mesas was claimed to emit with an integrated output power of 0.6 mW. (6) Since the devices in an array must be located within a wavelength of the emitted radiation for the combined output to be coherent, it is desirable to construct the array so as to occupy as little horizontal space as possible. One way to do this is to construct the array from concentric rings, or annuli. Even better would be to construct it from concentric slitted annuli, such as sketched in figure 2 This thesis will specifically investigate the
shape of an annulus (a disk with a hole in the middle, or a ring) for the THz emissions from the intrinsic Josephson junctions.

The main idea of this thesis is to investigate the frequencies for BSCCO microstrip antennas in the form of an annulus. A secondary objective is to calculate the variety of power distributions that could arise from BSCCO microstrip antennas of various ratios of the inner and outer radii.

Figure 1: Basic Mesa Structure, where the golden color is gold (Au) and the black and white is the BSCCO. With lead wires attached to the Gold.

Figure 2: concentric silted annuli (one blue, one green, one orange, the black lines are the slits)
THEORY

High performance terahertz (THz) devices have a multitude of scientific and technological uses, especially in applied research fields. THz electromagnetic waves (EMWs) are not only useful for studying the fundamental physics that is associated with the low energy excitations in molecules and solids but also for applications in chemical identifications, non-destructive evaluations, medical diagnoses, security, bioscience and biotechnologies, imaging techniques, ultra-high-speed communication, quantum computers, pharmaceutical drug development, etc. (7,8,9). The development of continuous, coherent, broadly tunable, and high-power solid-state sources that have a compact size have not been fully achieved due to fundamental and technical difficulties known as the “terahertz gap”, the 0.1-10 THz range that is important for many applications. (8,10,13) This is due to the output power values being well below 1 mW, which is the approximate desired value for many applications. (13) A great deal of effort has gone into improving the high frequency characteristics in semiconducting electronics devices, such as resonant tunneling diodes (RTDs) or improving the low frequency characteristics in laser devices, for example the quantum cascade lasers (QCLS). (8) The quantum cascade lasers are operational from 5 THz down to roughly 2 THz or below with cryogenic cooling and at the desired power values without cryogenic cooling for emission frequencies above 2.0 THz; but QCLs are limited above 5 THz due to phonons. (13) RTDs have been able to operate with sufficient power at room temperature for frequencies below 1.4 THz and at power values around 1µW, they were recently shown to emit up to 2.0 THz. (13)
The new discovery of coherent THz radiation emitted from mesas of Bi$_2$Sr$_2$CaCu$_2$O$_{8+δ}$ (BSCCO) a high temperature superconductor, have caused great excitement. (1) BSCCO devices have been shown to operate tunably, as long as the power can be sufficiently enhanced by making suitable arrays, from 1-11 THz giving them a distinct advantage over other potential sources. (20) Since BSCCO is a stack of Josephson junctions, with atomically thin superconducting layers that are separated by thicker dielectric layers it is extremely anisotropic. (15) In fact, it is so anisotropic that it behaves as a precisely regular one-dimensional array of intrinsic Josephson junctions stacked along the c-axis of a single crystal. The fundamental principle of operation of the intrinsic Josephson junction THz emitter is due to the synchronized coherent emissions that result from the dc voltage applied across the c-axis of a BSCCO crystal, straddling the many intrinsic Josephson junctions. (19) This gives rise to an ac current and photon emission at the frequency $f=2eV/(Nh)$, where $f$ is the frequency, $e$ is the electronic charge, $h$ is the Planck’s constant, $V$ is the voltage applied, and $N$ is the number of active intrinsic Josephson junctions. Since $f$ is proportional to $V$, the frequency can be varied substantially. (16)

A crystal of BSCCO with a high transition temperature of about 90 Kelvin consists of alternating double layers of superconducting CuO$_2$ and insulating Bi$_2$O$_2$ and SrO layers within a unit cell. The CuO$_2$ layers are coupled by the Josephson effect of the quantum mechanical tunneling of the Cooper pairs through the insulating SrO and Bi$_2$O$_2$ layers. There are about 670 junctions within the thickness of 1μm due to how densely packed on an atomic scale the Josephson junctions are. (19) In early experiments, rectangular mesas were made by Ar ion milling of a single crystal of BSCCO, with an Au layer covering the top of the mesa, where an electrical lead was attached, two additional electrical leads were attached to the crystal substrate of BSCCO. (15) The mesa structure acts as a geometrical electromagnetic cavity. This can cause
enhanced and synchronized combined emissions from the multitude of intrinsic Josephson
junctions due to the cavity resonance that can occur. (19)

A question regarding sample homogeneity came up when comparing the results of thin
cylindrical (disk) conventional mesa intrinsic Josephson junction emitters and thermally-
managed intrinsic Josephson junction disk microstrip antennas. (14) The strongest emissions
were observed at the lowest frequency transverse magnetic disk electromagnetic cavity mode, for
three of the conventional disk mesas. (14) The possibility of azimuthal emission anisotropy was
left unknown, since the angular dependence of the output was only measured in two
perpendicular planes normal to the mesa. (13,14) The thermally managed disk intrinsic
Josephson junction microstrip antennas are thermally much more homogenous and are therefore
more likely to exhibit symmetry in the cavity mode wavefunction. Higher frequencies that are
observed in the measured output power can cause excitations of higher frequency cavity modes.
METHODOLOGY

We start with the general form of the wave equation
\[ \nabla^2 \Psi + k^2 \Psi = 0, \]  
(1)

which in polar \((\rho, \phi)\) coordinates can be written as
\[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \Psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Psi}{\partial \phi^2} + k^2 \Psi = 0. \]  
(2)

It is important to note that because the Josephson current, and therefore the electric field, are perpendicular to the annular plane, the transverse magnetic boundary conditions are that the normal derivative of the wave function vanishes on both the inner and outer radii. Using the separation of variables technique by setting \(\Psi(\rho, \phi) = R(\rho) \Phi(\phi)\) and requiring \(\Phi(\phi+2\pi) = \Phi(\phi)\) to be single valued upon rotations by \(2\pi\), the most general form for \(\Psi(\rho, \phi)\) may be written as
\[ \Psi(\rho, \phi) = \sum_{n=0}^{\infty} \Psi_n(\rho, \phi) = \sum_{n=0}^{\infty} [A_n J_n(k\rho) + B_n Y_n(k\rho)] \cos[n(\phi - \phi_{n,0})], \]  
(3)

where \(J_n(z)\) and \(Y_n(z)\) are the standard Bessel functions of the first and second kinds, respectively, and the \(A_n, B_n,\) and the \(\phi_{n,0}\) are the arbitrary constants. In addition, the transverse magnetic (Neumann) boundary conditions are
\[ \left. \frac{\partial \Psi_n(\rho, \phi)}{\partial \rho} \right|_{\rho=\rho_1,\rho_2} = 0. \]  
(4)

These boundary conditions lead to
\[ A_n J'_n(k \rho_1) + B_n Y'_n(k \rho_1) = 0, \]  
(5) \[ A_n J'_n(k \rho_2) + B_n Y'_n(k \rho_2) = 0 \]  
(6)

where \( J'_n(z) \) and \( Y'_n(z) \) are the first derivatives of \( J_n(z) \) and \( Y_n(z) \), respectively, and where \( \rho_1 \) and \( \rho_2 \) are the inner and outer radii, respectively. Each of these two equations leads to a different but equivalent expression for the \( B_n \) in terms of the \( A_n \), and the \( A_n \) can be determined by normalizing the \( n^{th} \) wave function. Together these two equations lead to the solutions. We are writing the wavevectors \( k_{\rho_1/\rho_2,0m} \) for the \( m^{th} \) zero of the wave function with angular quantum number \( n \) and inner and outer radii \( \rho_1 \) and \( \rho_2 > \rho_1 \), respectively, satisfy

\[
\begin{vmatrix}
J'_n\left(k_{\rho_1/\rho_2,nm} \rho_1\right) & Y'_n\left(k_{\rho_1/\rho_2,nm} \rho_1\right) \\
J'_n\left(k_{\rho_1/\rho_2,nm} \rho_2\right) & Y'_n\left(k_{\rho_1/\rho_2,nm} \rho_2\right)
\end{vmatrix} = 0.
\]  
(7)

Normalization of the \( n^{th} \) eigenfunction over the entire area of the annulus for each \( k_{nm} \) value leads to

\[
\int_0^{2\pi} d\phi \int_{\rho_1}^{\rho_2} \rho d\rho |\Psi_n(\rho, \phi)|^2 = 0.
\]  
(8)

Which allows us to obtain an expression for \( |A_n| \) for each \( k_{\rho_1/\rho_2,nm} \) value.

In this thesis, we focus on the \( n = 0 \) case, as these wave functions are rotationally invariant about the perpendicular axis passing through the center of the annulus and are therefore one-dimensional representations of the \( C_{\infty V} \) group appropriate for a disk and for an annulus. For \( n = 0 \), the \( k_{\rho_1/\rho_2,0m} \) values are found for each fixed \( \rho_1/\rho_2 \) ratio by the \( m^{th} \) zero of the above rank 2 determinant. Here we choose annuli with \( \rho_1/\rho_2 \) values ranging from 0.1 to 0.9 in increments of 0.1.
To find our \( k_{\rho_1/\rho_2,0m} \) values we took the determinant of the normal derivative of the two Bessel functions at the inner and outer radii at \( \rho_1 \) and \( \rho_2 \), respectively; Equation 7. Next it was graphed as a function of \( k\rho_2 \), and we looked for the locations where the determinant would be equal to zero, i.e. where the graph crossed the x axis. By successively enlarging the region where the function vanishes on the graph until we found the zero value to four decimals accuracy. This is our wave vector value \( (k_{\rho_1/\rho_2,0m}) \), which we need for each of the nine inner radii and for each of the first five lowest wavevectors for each radius. This then allows us to graph the wave function. In the figures below the thick black circles are the nodes in the wave function for given radii ratios and wavevectors.

Next, we wanted to find the uniform Josephson current source power distribution. This will result in a three-dimensional graph. First the \( k_m \) value, for each of the different radii and wavevectors, was divided by the index of refraction for BSSCO samples 1\( \mu \)m thick (which is \( n=4.2 \)). We then found the current relevant to the electric surface current Love equivalence principle,

\[
Current \propto J_0(\rho_2 \times k \times \sin \theta) - J_0(\rho_1 \times k \times \sin \theta);
\] (9)

where \( \rho_1 \) is the inner radius of the annulus, \( \rho_2 \) is the outer radius of the, \( k \) is the \( k_{\rho_1/\rho_2,0m} \) divided by the index of refraction 4.2, and \( J_0 \) is the zero order J Bessel function. Next to find our power distribution arising from the uniform current density source,

\[
Power \propto \sin^2 \theta \times (J_0(\rho_2 \times k \times \sin \theta) - J_0(\rho_1 \times k \times \sin \theta))^2.
\] (10)

We were then able to plot a spherical representation of the power.
Finally, we found the cavity mode Josephson current source power distribution from the Love magnetic surface current source,

\[ \text{Power} \propto \sin^2 \theta \ast (\psi_2 * J_0(\rho_2 * k * \sin \theta) - \psi_1 * J_0(\rho_1 * k * \sin \theta))^2, \]

where \( \rho_1 \) is the inner radius of the annulus, \( \rho_2 \) is the outer radius of the annulus, \( k \) is the \( k_m \) divided by the index of refraction, \( J_0 \) is the zero order J Bessel function.

\begin{align*}
\psi_1 &= a_1 \left[ J_0(\rho_1 * k) - Y_0(\rho_1 * k) \ast \left( \frac{dJ_0(x_1)}{dx_1} \right) \right],
\psi_2 &= a_2 \left[ J_0(\rho_2 * k) - Y_0(\rho_2 * k) \ast \left( \frac{dJ_0(x_2)}{dx_2} \right) \right],
\end{align*}

where \( Y_0 \) is the zero order Y Bessel function, \( x_1 = k * \rho_1, x_2 = k * \rho_2, a_1 = \frac{1}{\sqrt{b_1}}, a_2 = \frac{1}{\sqrt{b_2}} \).

\begin{align*}
b_1 &= 2\pi \int_{\rho_1}^{\rho_2} x * [J_0(x * k) + c_1 * Y_0(x * k)]^2 dx, \\
b_2 &= 2\pi \int_{\rho_1}^{\rho_2} x * [J_0(x * k) + c_2 * Y_0(x * k)]^2 dx, \\
c_1 &= -\frac{dJ_0(x_1)}{dx_1} \left/ \frac{dY_0(x_1)}{dx_1} \right., \text{and} \\
c_2 &= -\frac{dJ_0(x_2)}{dx_2} \left/ \frac{dY_0(x_2)}{dx_2} \right.
\end{align*}
This then allows us to draw the three-dimensional spherical plots. We also plot line graphs of the polar angle power distribution so that we can better see the overall shape.

We then present our data for the wave function with angular quantum number \( n = 1 \), for which the wavevectors are \( k_{\rho_1/\rho_2}m \). We show color-coded contour plots of the wave functions for \( n = 1 \) with \( m = 1 \) to \( 5 \) and \( \rho_1/\rho_2 = 0.1 \) to \( 0.9 \) in steps of \( 0.1 \).
RESULTS

For the 5 lowest wavevectors and energy, we calculated normalized n=0 wavefunctions $\Psi_o(k_{p_1/p_2,0m})$ for $p_1/p_2 = 0.1$-0.9 in steps of 0.1. In addition, for each wavefunction we calculated the uniform Josephson current source power distribution at each of the wavevectors $k_{p_1/p_2,0m}$, and the power distribution for the excited cavity mode at that wavevector. We made color coded contour plots of the wavefunction and both three dimensional plots of the full angular power distributions and linear polar angular plots of each of the power distribution functions. These plots are grouped together in increasing $p_1/p_2$ values (starting at $p_1/p_2 =0.1$) and increasing wavevector index $m=1$ to 5. Figures 3-7 are for $k_{0.1,01}$, Figures 8-12 are for $k_{0.1,02}$, etc..
Table 1: $k_{\rho_1/\rho_2,0m}$ values

<table>
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<tr>
<th>$\rho_1/\rho_2$</th>
<th>m=1</th>
<th>m=2</th>
<th>m=3</th>
<th>m=4</th>
<th>m=5</th>
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<tr>
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<td>8.05536</td>
<td>11.92658</td>
<td>15.82104</td>
<td>19.72706</td>
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<tr>
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<td>9.10423</td>
<td>13.55317</td>
<td>18.01998</td>
<td>22.49481</td>
</tr>
<tr>
<td>0.4</td>
<td>5.39118</td>
<td>10.55773</td>
<td>15.76646</td>
<td>20.98819</td>
<td>26.21548</td>
</tr>
<tr>
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<td>6.39316</td>
<td>12.62470</td>
<td>18.88893</td>
<td>25.16240</td>
<td>31.43910</td>
</tr>
<tr>
<td>0.6</td>
<td>7.93009</td>
<td>15.74728</td>
<td>23.58833</td>
<td>31.43576</td>
<td>39.28579</td>
</tr>
<tr>
<td>0.7</td>
<td>10.52203</td>
<td>20.96939</td>
<td>31.43294</td>
<td>41.90067</td>
<td>52.37010</td>
</tr>
<tr>
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<td>31.43801</td>
<td>47.13383</td>
<td>62.83931</td>
<td>78.54578</td>
</tr>
<tr>
<td>0.9</td>
<td>31.42916</td>
<td>62.83848</td>
<td>94.25220</td>
<td>125.66702</td>
<td>157.08228</td>
</tr>
</tbody>
</table>

The table shows the $k_{\rho_1/\rho_2,0m}$ values for our different ratios (0.1-0.9) of the inner and outer radii and for the first five lowest wavevectors (m).
Ratio of Inner to Outer Radii—0.1

Lowest Wavevector

Figure 3: Contour plot of the wave function for the lowest wavevector, $k_{0.1,0.1}$ of an annular disk with $\rho_1/\rho_2 = 0.1$. The black circle is a node.
Figure 4: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.1,0.1}$ of an annular disk with $\rho_1/\rho_2 = 0.1$.

Figure 5: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.1,0.1}$ of an annular disk with $\rho_1/\rho_2 = 0.1$. 
Figure 6: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.1,01}$ of an annular disk with $\rho_1/\rho_2 = 0.1$.

Figure 7: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.1,01}$ of an annular disk with $\rho_1/\rho_2 = 0.1$. 
Figure 8: Contour plot of the wave function for the lowest wavevector, $k_{0,1,02}$ of an annular disk with $\rho_1/\rho_2 = 0.1$. The black circles are nodes.
Figure 9: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.1,0.2}$ of an annular disk with $\rho_1/\rho_2 = 0.1$.

Figure 10: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.1,0.2}$ of an annular disk with $\rho_1/\rho_2 = 0.1$. 
Figure 11: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0,1,02}$ of an annular disk with $\rho_1/\rho_2 = 0.1$.

Figure 12: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0,1,02}$ of an annular disk with $\rho_1/\rho_2 = 0.1$. 
Figure 13: Contour plot of the wave function for the lowest wavevector, $k_{0.1,0.3}$ of an annular disk with $\rho_1/\rho_2 = 0.1$. The black circles are nodes.
Figure 14: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0,1,03}$ of an annular disk with $\rho_1/\rho_2 = 0.1$.

Figure 15: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0,1,03}$ of an annular disk with $\rho_1/\rho_2 = 0.1$. 

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Figure 16: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.1, 0.3}$ of an annular disk with $\rho_1/\rho_2 = 0.1$.

Figure 17: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.1, 0.3}$ of an annular disk with $\rho_1/\rho_2 = 0.1$. 
Figure 18: Contour plot of the wave function for the lowest wavevector, $k_{0.1,04}$ of an annular disk with $\rho_1/\rho_2 = 0.1$. The black circles are nodes.
Figure 19: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.1,0.4}$ of an annular disk with $\rho_1/\rho_2 = 0.1$.

Figure 20: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.1,0.4}$ of an annular disk with $\rho_1/\rho_2 = 0.1$. 
Figure 21: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.1,0.4}$ of an annular disk with $\rho_1/\rho_2 = 0.1$.

Figure 22: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.1,0.4}$ of an annular disk with $\rho_1/\rho_2 = 0.1$. 
Figure 23: Contour plot of the wave function for the lowest wavevector, $k_{0.1,0.05}$ of an annular disk with $\rho_1/\rho_2 = 0.1$. The black circles are nodes.
Figure 24: Uniform Josephson current source power distribution for the lowest wavevector, \( k_{0.1,0.5} \) of an annular disk with \( \rho_1/\rho_2 = 0.1 \).

Figure 25: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, \( k_{0.1,0.5} \) of an annular disk with \( \rho_1/\rho_2 = 0.1 \).
Figure 26: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.1,0.5}$ of an annular disk with $\rho_1/\rho_2 = 0.1$.

Figure 27: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.1,0.5}$ of an annular disk with $\rho_1/\rho_2 = 0.1$. 
Figure 28: Contour plot of the wave function for the lowest wavevector, $k_{0.2,0.1}$ of an annular disk with $\rho_1/\rho_2 = 0.2$. The black circle is a node.
Figure 29: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0,2,01}$ of an annular disk with $\rho_1/\rho_2 = 0.2$.

Figure 30: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0,2,01}$ of an annular disk with $\rho_1/\rho_2 = 0.2$. 
Figure 31: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.2,01}$ of an annular disk with $\rho_1/\rho_2 = 0.2$.

Figure 32: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.2,01}$ of an annular disk with $\rho_1/\rho_2 = 0.2$. 

Figure 33: Contour plot of the wave function for the lowest wavevector, \( k_{0.02} \) of an annular disk with \( \rho_1/\rho_2 = 0.2 \). The black circles are nodes.
Figure 34: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.2.02}$ of an annular disk with $\rho_1/\rho_2 = 0.2$.

Figure 35: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.2.02}$ of an annular disk with $\rho_1/\rho_2 = 0.2$. 
Figure 36: Cavity mode Josephson current source power distribution for the lowest wavevector, \( k_{0.2,02} \) of an annular disk with \( \rho_1/\rho_2 = 0.2 \).

Figure 37: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, \( k_{0.2,02} \) of an annular disk with \( \rho_1/\rho_2 = 0.2 \).
Figure 38: Contour plot of the wave function for the lowest wavevector, $k_{0.2,03}$ of an annular disk with $\rho_1/\rho_2 = 0.2$. The black circles are nodes.
Figure 39: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.2,0.3}$ of an annular disk with $\rho_1/\rho_2 = 0.2$.

Figure 40: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.2,0.3}$ of an annular disk with $\rho_1/\rho_2 = 0.2$. 
Figure 41: Cavity mode Josephson current source power distribution for the lowest wavevector, 
$k_{0.2,03}$ of an annular disk with $\rho_1/\rho_2 = 0.2$.

Figure 42: Plot of the polar angle dependence of the cavity mode Josephson current source 
power distribution for the lowest wavevector, $k_{0.2,03}$ of an annular disk with $\rho_1/\rho_2 = 0.2$. 

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Figure 43: Contour plot of the wave function for the lowest wavevector, $k_{0.2,0.4}$ of an annular disk with $\rho_1/\rho_2 = 0.2$. The black circles are nodes.
Figure 44: Uniform Josephson current source power distribution for the lowest wavevector, \( k_{0.2,04} \) of an annular disk with \( \rho_1/\rho_2 = 0.2 \).

Figure 45: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, \( k_{0.2,04} \) of an annular disk with \( \rho_1/\rho_2 = 0.2 \).
Figure 46: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.2,04}$ of an annular disk with $\rho_1/\rho_2 = 0.2$.

Figure 47: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.2,04}$ of an annular disk with $\rho_1/\rho_2 = 0.2$. 
Figure 48: Contour plot of the wave function for the lowest wavevector, $k_{0.2,0.5}$ of an annular disk with $\rho_1/\rho_2 = 0.2$. The black circles are nodes.
Figure 49: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.2,0.5}$ of an annular disk with $\rho_1/\rho_2 = 0.2$.

Figure 50: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.2,0.5}$ of an annular disk with $\rho_1/\rho_2 = 0.2$. 
Figure 51: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.2,05}$ of an annular disk with $\rho_1/\rho_2 = 0.2$.

Figure 52: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.2,05}$ of an annular disk with $\rho_1/\rho_2 = 0.2$. 
Ratio of Inner to Outer Radii—0.3

Lowest Wavevector

Figure 53: Contour plot of the wave function for the lowest wavevector, $k_{0.3,01}$ of an annular disk with $\rho_1/\rho_2 = 0.3$. The black circle is a node.
Figure 54: Uniform Josephson current source power distribution for the lowest wavevector, \( k_{0,3,01} \) of an annular disk with \( \rho_1/\rho_2 = 0.3 \).

Figure 55: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, \( k_{0,3,01} \) of an annular disk with \( \rho_1/\rho_2 = 0.3 \).
Figure 56: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.3,0.1}$ of an annular disk with $\rho_1/\rho_2 = 0.3$.

Figure 57: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.3,0.1}$ of an annular disk with $\rho_1/\rho_2 = 0.3$. 

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Figure 58: Contour plot of the wave function for the lowest wavevector, \( k_{0.3,02} \) of an annular disk with \( p_1/p_2 = 0.3 \). The black circles are nodes.
Figure 59: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.3,0.2}$ of an annular disk with $\rho_1/\rho_2 = 0.3$.

Figure 60: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.3,0.2}$ of an annular disk with $\rho_1/\rho_2 = 0.3$. 
Figure 61: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.3,0.2}$ of an annular disk with $\rho_1/\rho_2 = 0.3$.

Figure 62: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.3,0.2}$ of an annular disk with $\rho_1/\rho_2 = 0.3$. 
Third Lowest Wavevector

Figure 63: Contour plot of the wave function for the lowest wavevector, \( k_{0.3,0.3} \) of an annular disk with \( \rho_1/\rho_2 = 0.3 \). The black circles are nodes.
Figure 64: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.3,0.3}$ of an annular disk with $\rho_1/\rho_2 = 0.3$.

Figure 65: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.3,0.3}$ of an annular disk with $\rho_1/\rho_2 = 0.3$. 
Figure 66: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.3,0.3}$ of an annular disk with $\rho_1/\rho_2 = 0.3$.

Figure 67: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.3,0.3}$ of an annular disk with $\rho_1/\rho_2 = 0.3$. 

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Fourth Lowest Wavevector

Figure 68: Contour plot of the wave function for the lowest wavevector, $k_{0.3,04}$ of an annular disk with $\rho_1/\rho_2 = 0.3$. The black circles are nodes.
Figure 69: Uniform Josephson current source power distribution for the lowest wavevector, \(k_{0.3,0.4}\) of an annular disk with \(\rho_1/\rho_2 = 0.3\).

Figure 70: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, \(k_{0.3,0.4}\) of an annular disk with \(\rho_1/\rho_2 = 0.3\).
Figure 71: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.3,0.4}$ of an annular disk with $\rho_1/\rho_2 = 0.3$.

Figure 72: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.3,0.4}$ of an annular disk with $\rho_1/\rho_2 = 0.3$. 
Figure 73: Contour plot of the wave function for the lowest wavevector, $k_{0.3,0.5}$ of an annular disk with $\rho_1/\rho_2 = 0.3$. The black circles are nodes.
Figure 74: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.3,0.5}$ of an annular disk with $\rho_1/\rho_2 = 0.3$.

Figure 75: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.3,0.5}$ of an annular disk with $\rho_1/\rho_2 = 0.3$. 
Figure 76: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.3,0.5}$, of an annular disk with $\rho_1/\rho_2 = 0.3$.

Figure 77: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.3,0.5}$, of an annular disk with $\rho_1/\rho_2 = 0.3$. 
Ratio of Inner to Outer Radii—0.4

Lowest Wavevector

Figure 78: Contour plot of the wave function for the lowest wavevector, $k_{0.4,01}$ of an annular disk with $\rho_1/\rho_2 = 0.4$. The black circle is a node.
Figure 79: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.4,0.1}$ of an annular disk with $\rho_1/\rho_2 = 0.4$.

Figure 80: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.4,0.1}$ of an annular disk with $\rho_1/\rho_2 = 0.4$. 

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Figure 81: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.4,01}$ of an annular disk with $\rho_1/\rho_2 = 0.4$.

Figure 82: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.4,01}$ of an annular disk with $\rho_1/\rho_2 = 0.4$. 
Figure 83: Contour plot of the wave function for the lowest wavevector, $k_{0.4,02}$ of an annular disk with $\rho_1/\rho_2 = 0.4$. The black circles are nodes.
Figure 84: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.4,0.2}$ of an annular disk with $\rho_1/\rho_2 = 0.4$.

Figure 85: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.4,0.2}$ of an annular disk with $\rho_1/\rho_2 = 0.4$. 
Figure 86: Cavity mode Josephson current source power distribution for the lowest wavevector, \( k_{0.4,0.2} \) of an annular disk with \( \rho_1/\rho_2 = 0.4 \).

Figure 87: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, \( k_{0.4,0.2} \) of an annular disk with \( \rho_1/\rho_2 = 0.4 \).
Third Lowest Wavevector

Figure 88: Contour plot of the wave function for the lowest wavevector, \( k_{0.4,0.3} \) of an annular disk with \( \rho_1/\rho_2 = 0.4 \). The black circles are nodes.
Figure 89: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.4,0.3}$ of an annular disk with $\rho_1/\rho_2 = 0.4$.

Figure 90: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.4,0.3}$ of an annular disk with $\rho_1/\rho_2 = 0.4$. 
Figure 91: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.4,0.3}$ of an annular disk with $\rho_1/\rho_2 = 0.4$.

Figure 92: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.4,0.3}$ of an annular disk with $\rho_1/\rho_2 = 0.4$. 
Fourth Lowest Wavevector

Figure 93: Contour plot of the wave function for the lowest wavevector, $k_{0.4,0.4}$ of an annular disk with $\rho_1/\rho_2 = 0.4$. The black circles are nodes.
Figure 94: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.4,04}$ of an annular disk with $\rho_1/\rho_2 = 0.4$.

Figure 95: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.4,04}$ of an annular disk with $\rho_1/\rho_2 = 0.4$. 
Figure 96: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.4,0.4}$ of an annular disk with $\rho_1/\rho_2 = 0.4$.

Figure 97: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.4,0.4}$ of an annular disk with $\rho_1/\rho_2 = 0.4$. 
Figure 98: Contour plot of the wave function for the lowest wavevector, \( k_{0.4,0.5} \) of an annular disk with \( \rho_1/\rho_2 = 0.4 \). The black circles are nodes.
Figure 99: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.4,0.5}$ of an annular disk with $\rho_1/\rho_2 = 0.4$.

Figure 100: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.4,0.5}$ of an annular disk with $\rho_1/\rho_2 = 0.4$. 
Figure 101: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.4,0.5}$ of an annular disk with $\rho_1/\rho_2 = 0.4$.

Figure 102: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.4,0.5}$ of an annular disk with $\rho_1/\rho_2 = 0.4$. 

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Ratio of Inner to Outer Radii—0.5

Lowest Wavevector

Figure 103: Contour plot of the wave function for the lowest wavevector, $k_{0.5,0.1}$ of an annular disk with $\rho_1/\rho_2 = 0.5$. The black circle is a node.
Figure 104: Uniform Josephson current source power distribution for the lowest wavevector, \( k_{0.5,0.1} \) of an annular disk with \( \rho_1/\rho_2 = 0.5 \).

Figure 105: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, \( k_{0.5,0.1} \) of an annular disk with \( \rho_1/\rho_2 = 0.5 \).
Figure 106: Cavity mode Josephson current source power distribution for the lowest wavevector, \( k_{0.5,01} \) of an annular disk with \( \rho_1/\rho_2 = 0.5 \).

Figure 107: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, \( k_{0.5,01} \) of an annular disk with \( \rho_1/\rho_2 = 0.5 \).
Figure 108: Contour plot of the wave function for the lowest wavevector, $k_{0.5,0.2}$ of an annular disk with $\rho_1/\rho_2 = 0.5$. The black circles are nodes.
Figure 109: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.5,0.2}$ of an annular disk with $\rho_1/\rho_2 = 0.5$.

Figure 110: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.5,0.2}$ of an annular disk with $\rho_1/\rho_2 = 0.5$. 
Figure 111: Cavity mode Josephson current source power distribution for the lowest wavevector, \( k_{0.5,0.2} \) of an annular disk with \( \rho_1/\rho_2 = 0.5 \).

Figure 112: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, \( k_{0.5,0.2} \) of an annular disk with \( \rho_1/\rho_2 = 0.5 \).
Figure 113: Contour plot of the wave function for the lowest wavevector, $k_{0.5,0.3}$ of an annular disk with $\rho_1/\rho_2 = 0.5$. The black circles are nodes.
Figure 114: Uniform Josephson current source power distribution for the lowest wavevector, \( k_{0.5,0.3} \) of an annular disk with \( \rho_1/\rho_2 = 0.5 \).

Figure 115: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, \( k_{0.5,0.3} \) of an annular disk with \( \rho_1/\rho_2 = 0.5 \).
Figure 116: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.5,0.3}$ of an annular disk with $\rho_1/\rho_2 = 0.5$.

Figure 117: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.5,0.3}$ of an annular disk with $\rho_1/\rho_2 = 0.5$. 
Fourth Lowest Wavevector

Figure 118: Contour plot of the wave function for the lowest wavevector, \( k_{0.5,0.4} \) of an annular disk with \( \rho_{1}/\rho_{2} = 0.5 \). The black circles are nodes.
Figure 119: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.5,0.4}$ of an annular disk with $\rho_1/\rho_2 = 0.5$.

Figure 120: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.5,0.4}$ of an annular disk with $\rho_1/\rho_2 = 0.5$. 
Figure 121: Cavity mode Josephson current source power distribution for the lowest wavevector, \( k_{0.5,04} \) of an annular disk with \( \rho_1/\rho_2 = 0.5 \).

Figure 122: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, \( k_{0.5,04} \) of an annular disk with \( \rho_1/\rho_2 = 0.5 \).
Fifth Lowest Wavevector

Figure 123: Contour plot of the wave function for the lowest wavevector, $k_{0.5,0.5}$ of an annular disk with $\rho_1/\rho_2 = 0.5$. The black circles are nodes.
Figure 124: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.5,0.5}$ of an annular disk with $\rho_1/\rho_2 = 0.5$.

Figure 125: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.5,0.5}$ of an annular disk with $\rho_1/\rho_2 = 0.5$. 
Figure 126: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.5,0.5}$ of an annular disk with $\rho_1/\rho_2 = 0.5$.

Figure 127: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.5,0.5}$ of an annular disk with $\rho_1/\rho_2 = 0.5$. 
Ratio of Inner to Outer Radii—0.6

Lowest Wavevector

Figure 128: Contour plot of the wave function for the lowest wavevector, $k_{0.6,0.1}$ of an annular disk with $\rho_1/\rho_2 = 0.6$. The black circle is a node.
Figure 129: Uniform Josephson current source power distribution for the lowest wavevector, \( k_{0.6,01} \) of an annular disk with \( \rho_1/\rho_2 = 0.6 \).

Figure 130: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, \( k_{0.6,01} \) of an annular disk with \( \rho_1/\rho_2 = 0.6 \).
Figure 131: Cavity mode Josephson current source power distribution for the lowest wavevector, \( k_{0.6,0.1} \) of an annular disk with \( \rho_1/\rho_2 = 0.6 \).

Figure 132: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, \( k_{0.6,0.1} \) of an annular disk with \( \rho_1/\rho_2 = 0.6 \).
Figure 133: Contour plot of the wave function for the lowest wavevector, $k_{0.6,0.2}$ of an annular disk with $\rho_1/\rho_2 = 0.6$. The black circles are nodes.
Figure 134: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.6,0.2}$ of an annular disk with $\rho_1/\rho_2 = 0.6$.

Figure 135: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.6,0.2}$ of an annular disk with $\rho_1/\rho_2 = 0.6$. 
Figure 136: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.6,0.2}$ of an annular disk with $\rho_1/\rho_2 = 0.6$.

Figure 137: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.6,0.2}$ of an annular disk with $\rho_1/\rho_2 = 0.6$. 

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Third Lowest Wavevector

Figure 138: Contour plot of the wave function for the lowest wavevector, \( k_{0.6,0.3} \) of an annular disk with \( \rho_1/\rho_2 = 0.6 \). The black circles are nodes.
Figure 139: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.6,0.3}$ of an annular disk with $\rho_1/\rho_2 = 0.6$.

Figure 140: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.6,0.3}$ of an annular disk with $\rho_1/\rho_2 = 0.6$. 
Figure 141: Cavity mode Josephson current source power distribution for the lowest wavevector, \( k_{0.6,0.3} \) of an annular disk with \( \rho_1/\rho_2 = 0.6 \).

Figure 142: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, \( k_{0.6,0.3} \) of an annular disk with \( \rho_1/\rho_2 = 0.6 \).
Fourth Lowest Wavevector

Figure 143: Contour plot of the wave function for the lowest wavevector, $k_{0.6,0.4}$ of an annular disk with $\rho_1/\rho_2 = 0.6$. The black circles are nodes.
Figure 144: Uniform Josephson current source power distribution for the lowest wavevector, \( k_{0.6,0.4} \) of an annular disk with \( \rho_1/\rho_2 = 0.6 \).

Figure 145: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, \( k_{0.6,0.4} \) of an annular disk with \( \rho_1/\rho_2 = 0.6 \).
Figure 146: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.6,0.4}$ of an annular disk with $\rho_1/\rho_2 = 0.6$.

Figure 147: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.6,0.4}$ of an annular disk with $\rho_1/\rho_2 = 0.6$. 
Figure 148: Contour plot of the wave function for the lowest wavevector, $k_{0.6,05}$ of an annular disk with $\rho_1/\rho_2 = 0.6$. The black circles are nodes.
Figure 149: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.6,0.5}$ of an annular disk with $\rho_1/\rho_2 = 0.6$.

Figure 150: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.6,0.5}$ of an annular disk with $\rho_1/\rho_2 = 0.6$. 
Figure 151: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.6,0.05}$ of an annular disk with $\rho_1/\rho_2 = 0.6$.

Figure 152: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.6,0.05}$ of an annular disk with $\rho_1/\rho_2 = 0.6$. 
Figure 153: Contour plot of the wave function for the lowest wavevector, $k_{0.7,0.1}$ of an annular disk with $\rho_1/\rho_2 = 0.7$. The black circle is a node.
Figure 154: A zoomed in view of the previous plot to help better see the node (black circle).

Figure 155: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.7,0.1}$ of an annular disk with $\rho_1/\rho_2 = 0.7$. 
Figure 156: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.7,01}$ of an annular disk with $\rho_1/\rho_2 = 0.7$.

Figure 157: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.7,01}$ of an annular disk with $\rho_1/\rho_2 = 0.7$. 
Figure 158: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, \( k_{0.7,0.1} \) of an annular disk with \( \rho_1/\rho_2 = 0.7 \).
Second Lowest Wavevector

Figure 159: Contour plot of the wave function for the lowest wavevector, $k_{0.7,0.2}$ of an annular disk with $\rho_1/\rho_2 = 0.7$. The black circles are nodes.
Figure 160: A zoomed in view of the previous plot to help better see the 2 nodes (black circles).

Figure 161: Uniform Josephson current source power distribution for the lowest wavevector, \( k_{0.702} \) of an annular disk with \( \rho_1/\rho_2 = 0.7 \).
Figure 162: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.7,0.2}$ of an annular disk with $\rho_1/\rho_2 = 0.7$.

Figure 163: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.7,0.2}$ of an annular disk with $\rho_1/\rho_2 = 0.7$. 
Figure 164: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, \( k_{0.7.02} \) of an annular disk with \( \rho_1/\rho_2 = 0.7 \).
Figure 165: Contour plot of the wave function for the lowest wavevector, $k_{0.7.03}$ of an annular disk with $\rho_1/\rho_2 = 0.7$. The black circles are nodes.
Figure 166: A zoomed in view of the previous plot to help better see the 3 nodes (black circles).

Figure 167: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.7,0.3}$ of an annular disk with $\rho_1/\rho_2 = 0.7$. 
Figure 168: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.7,0.3}$ of an annular disk with $\rho_1/\rho_2 = 0.7$.

Figure 169: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.7,0.3}$ of an annular disk with $\rho_1/\rho_2 = 0.7$. 
Figure 170: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.7,0.3}$ of an annular disk with $\rho_1/\rho_2 = 0.7$. 
Figure 171: Contour plot of the wave function for the lowest wavevector, $k_{0.7,0.4}$ of an annular disk with $\rho_1/\rho_2 = 0.7$. The black circles are nodes.
Figure 172: A zoomed in view of the previous plot to help better see the 4 nodes (black circles).

Figure 173: Uniform Josephson current source power distribution for the lowest wavevector, \( k_{0.7.04} \) of an annular disk with \( \rho_1/\rho_2 = 0.7 \).
Figure 174: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.7,04}$ of an annular disk with $\rho_1/\rho_2 = 0.7$.

Figure 175: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.7,04}$ of an annular disk with $\rho_1/\rho_2 = 0.7$. 
Figure 176: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.7.04}$ of an annular disk with $\rho_1/\rho_2 = 0.7$. 
Figure 177: Contour plot of the wave function for the lowest wavevector, $k_{0.7,0.5}$ of an annular disk with $\rho_1/\rho_2 = 0.7$. The black circles are nodes.
Figure 178: A zoomed in view of the previous plot to help better see the 5 nodes (black circles).

Figure 179: Uniform Josephson current source power distribution for the lowest wavevector, \( k_{0.7,0.05} \) of an annular disk with \( \rho_1/\rho_2 = 0.7 \).
Figure 180: Plot of the radial dependence of the uniform Josephson current source power
distribution for the lowest wavevector, \(k_{0.7,05}\) of an annular disk with \(\rho_1/\rho_2 = 0.7\).

Figure 181: Cavity mode Josephson current source power distribution for the lowest wavevector,
\(k_{0.7,05}\) of an annular disk with \(\rho_1/\rho_2 = 0.7\).
Figure 182: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.7,0.5}$ of an annular disk with $\rho_1/\rho_2 = 0.7$. 
Figure 183: Contour plot of the wave function for the lowest wavevector, $k_{0.8,0.1}$ of an annular disk with $\rho_1/\rho_2 = 0.8$. The black circle is a node.
Figure 184: A zoomed in view of the previous plot to help better see the node (black circle).

Figure 185: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.8,0.1}$ of an annular disk with $\rho_1/\rho_2 = 0.8$. 
Figure 186: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.8,0.1}$ of an annular disk with $\rho_1/\rho_2 = 0.8$.

Figure 187: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.8,0.1}$ of an annular disk with $\rho_1/\rho_2 = 0.8$. 

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Figure 188: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.8,0.1}$ of an annular disk with $\rho_1/\rho_2 = 0.8$. 
Figure 189: Contour plot of the wave function for the lowest wavevector, $k_{0,8,02}$ of an annular disk with $\rho_1/\rho_2 = 0.8$. The black circles are nodes.
Figure 190: A zoomed in view of the previous plot to help better see the 2 nodes (black circles).

Figure 191: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.8,0.2}$ of an annular disk with $\rho_1/\rho_2 = 0.8$. 
Figure 192: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.8,0.2}$ of an annular disk with $\rho_1/\rho_2 = 0.8$.

Figure 193: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.8,0.2}$ of an annular disk with $\rho_1/\rho_2 = 0.8$. 
Figure 194: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.8,0.2}$ of an annular disk with $\rho_1/\rho_2 = 0.8$. 
Third Lowest Wavevector

Figure 195: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, \( k_{0.8,0.2} \) of an annular disk with \( \rho_1/\rho_2 = 0.8 \).
Figure 196: A zoomed in view of the previous plot to help better see the 3 nodes (black circles).

Figure 197: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.8,03}$ of an annular disk with $\rho_1/\rho_2 = 0.8$. 
Figure 198: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.8,0.3}$ of an annular disk with $\rho_1/\rho_2 = 0.8$.

Figure 199: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.8,0.3}$ of an annular disk with $\rho_1/\rho_2 = 0.8$. 
Figure 200: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.8,0.3}$ of an annular disk with $\rho_1/\rho_2 = 0.8$. 
Fourth Lowest Wavevector

Figure 201: Contour plot of the wave function for the lowest wavevector, $k_{0.8,0.4}$ of an annular disk with $\rho_1/\rho_2 = 0.8$. The black circles are nodes.
Figure 202: A zoomed in view of the previous plot to help better see the 4 nodes (black circles).

Figure 203: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.8.04}$ of an annular disk with $\rho_1/\rho_2 = 0.8$. 
Figure 204: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.8,0.4}$ of an annular disk with $\rho_1/\rho_2 = 0.8$.

Figure 205: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.8,0.4}$ of an annular disk with $\rho_1/\rho_2 = 0.8$. 
Figure 206: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.8,0.4}$ of an annular disk with $\rho_1/\rho_2 = 0.8$. 
Figure 207: Contour plot of the wave function for the lowest wavevector, $k_{0.8,0.5}$ of an annular disk with $\rho_1/\rho_2 = 0.8$. The black circles are nodes.
Figure 208: A zoomed-in view of the previous plot to help better see the 5 nodes (black circles).

Figure 209: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.8,05}$ of an annular disk with $\rho_1/\rho_2 = 0.8$. 
Figure 210: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.8,0.5}$ of an annular disk with $\rho_1/\rho_2 = 0.8$.

Figure 211: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.8,0.5}$ of an annular disk with $\rho_1/\rho_2 = 0.8$. 
Figure 212: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.8,0.5}$ of an annular disk with $\rho_1/\rho_2 = 0.8$. 
Figure 213: Contour plot of the wave function for the lowest wavevector, $k_{0.9,0.1}$ of an annular disk with $\rho_1/\rho_2 = 0.9$. The black circle is a node.
Figure 214: A zoomed in view of the previous plot to help better see the node (black circle).

Figure 215: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.9,0.1}$ of an annular disk with $\rho_1/\rho_2 = 0.9$. 
Figure 216: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.9,01}$ of an annular disk with $\rho_1/\rho_2 = 0.9$.

Figure 217: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.9,01}$ of an annular disk with $\rho_1/\rho_2 = 0.9$. 
Figure 218: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.9,0.1}$ of an annular disk with $\rho_1/\rho_2 = 0.9$. 
Second Lowest Wavevector

Figure 219: Contour plot of the wave function for the lowest wavevector, $k_{0.9,0.2}$ of an annular disk with $\rho_1/\rho_2 = 0.9$. The black circles are nodes.
Figure 220: A zoomed in view of the previous plot to help better see the 2 nodes (black circles).

Figure 221: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.9,0.2}$ of an annular disk with $\rho_1/\rho_2 = 0.9$. 
Figure 222: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.9,0.2}$ of an annular disk with $\rho_1/\rho_2 = 0.9$.

Figure 223: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.9,0.2}$ of an annular disk with $\rho_1/\rho_2 = 0.9$. 
Figure 224: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.9,0.2}$ of an annular disk with $\rho_1/\rho_2 = 0.9$. 
Figure 225: Contour plot of the wave function for the lowest wavevector, $k_{0.9,0.3}$ of an annular disk with $\rho_1/\rho_2 = 0.9$. The black circles are nodes.
Figure 226: A zoomed in view of the previous plot to help better see the 3 nodes (black circles).

Figure 227: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.9,0.3}$ of an annular disk with $\rho_1/\rho_2 = 0.9$. 
Figure 228: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.9,0.3}$ of an annular disk with $\rho_1/\rho_2 = 0.9$.

Figure 229: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.9,0.3}$ of an annular disk with $\rho_1/\rho_2 = 0.9$. 
Figure 230: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.9,0.3}$ of an annular disk with $\rho_1/\rho_2 = 0.9$. 
Fourth Lowest Wavevector

Figure 231: Contour plot of the wave function for the lowest wavevector, $k_{0.9,0.4}$ of an annular disk with $\rho_1/\rho_2 = 0.9$. The black circles are nodes.
Figure 232: A zoomed in view of the previous plot to help better see the 4 nodes (black circles).

Figure 233: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.9,0.04}$ of an annular disk with $\rho_1/\rho_2 = 0.9$. 
Figure 234: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.9,0.4}$ of an annular disk with $\rho_1/\rho_2 = 0.9$.

Figure 235: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.9,0.4}$ of an annular disk with $\rho_1/\rho_2 = 0.9$. 
Figure 236: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.9,0.4}$ of an annular disk with $\rho_1/\rho_2 = 0.9$. 
Figure 237: Contour plot of the wave function for the lowest wavevector, $k_{0.9,0.5}$ of an annular disk with $\rho_1/\rho_2 = 0.9$. The black circles are nodes.
Figure 238: A zoomed in view of the previous plot to help better see the 5 nodes (black circles).

Figure 239: Uniform Josephson current source power distribution for the lowest wavevector, $k_{0.9,0.5}$ of an annular disk with $\rho_1/\rho_2 = 0.9$. 
Figure 240: Plot of the radial dependence of the uniform Josephson current source power distribution for the lowest wavevector, $k_{0.9,0.5}$ of an annular disk with $\rho_1/\rho_2 = 0.9$.

Figure 241: Cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.9,0.5}$ of an annular disk with $\rho_1/\rho_2 = 0.9$. 
Figure 242: Plot of the polar angle dependence of the cavity mode Josephson current source power distribution for the lowest wavevector, $k_{0.9,0.5}$ of an annular disk with $\rho_1/\rho_2 = 0.9$.

Table 2: Calculated Width Ratios for a Long Rectangle

<table>
<thead>
<tr>
<th>cross</th>
<th>first</th>
<th>second</th>
<th>third</th>
<th>fourth</th>
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</tr>
</thead>
<tbody>
<tr>
<td>m=1</td>
<td>0.5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>m=2</td>
<td>0.25</td>
<td>0.75</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>m=3</td>
<td>0.16667</td>
<td>0.5</td>
<td>0.833333</td>
<td></td>
<td></td>
</tr>
<tr>
<td>m=4</td>
<td>0.125</td>
<td>0.375</td>
<td>0.625</td>
<td>0.875</td>
<td></td>
</tr>
<tr>
<td>m=5</td>
<td>0.1</td>
<td>0.3</td>
<td>0.5</td>
<td>0.7</td>
<td>0.9</td>
</tr>
</tbody>
</table>

This is what we would expect to see for a long rectangle of width 1
Table 3: Expected Ratios for 0.1 in the Long Rectangle Model

<table>
<thead>
<tr>
<th>cross</th>
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<th>fifth</th>
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</thead>
<tbody>
<tr>
<td>m=1</td>
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<tr>
<td>m=2</td>
<td>0.325</td>
<td>0.775</td>
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<tr>
<td>m=3</td>
<td>0.25</td>
<td>0.55</td>
<td>0.85</td>
<td></td>
<td></td>
</tr>
<tr>
<td>m=4</td>
<td>0.2125</td>
<td>0.4375</td>
<td>0.6625</td>
<td>0.8875</td>
<td></td>
</tr>
<tr>
<td>m=5</td>
<td>0.19</td>
<td>0.37</td>
<td>0.55</td>
<td>0.73</td>
<td>0.91</td>
</tr>
</tbody>
</table>

The predicted values for $\rho_1/\rho_2 = 0.1$ are shown here. They were obtained by multiplying the known values for the long rectangle by the width of the annulus and then adding the inner radius 0.1.

Table 4: Actual Ratios for 0.1

<table>
<thead>
<tr>
<th>cross</th>
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<th>second</th>
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</tr>
</thead>
<tbody>
<tr>
<td>m=1</td>
<td>0.63683</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>m=2</td>
<td>0.37003</td>
<td>0.79553</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>m=3</td>
<td>0.27626</td>
<td>0.56669</td>
<td>0.85834</td>
<td></td>
<td></td>
</tr>
<tr>
<td>m=4</td>
<td>0.2295</td>
<td>0.44968</td>
<td>0.67066</td>
<td>0.89184</td>
<td></td>
</tr>
<tr>
<td>m=5</td>
<td>0.20185</td>
<td>0.37904</td>
<td>0.55677</td>
<td>0.73466</td>
<td>0.91261</td>
</tr>
</tbody>
</table>

The actual values for $\rho_1/\rho_2 = 0.1$ are shown here.
Table 5: Expected Ratios for 0.9 in the Long Rectangle Model

<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>m=1</td>
<td>0.95</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>m=2</td>
<td>0.925</td>
<td>0.975</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>m=3</td>
<td>0.91667</td>
<td>0.95</td>
<td>0.98333</td>
<td></td>
<td></td>
</tr>
<tr>
<td>m=4</td>
<td>0.9125</td>
<td>0.9375</td>
<td>0.9625</td>
<td>0.9875</td>
<td></td>
</tr>
<tr>
<td>m=5</td>
<td>0.91</td>
<td>0.93</td>
<td>0.95</td>
<td>0.97</td>
<td>0.99</td>
</tr>
</tbody>
</table>

The predicted values for \( \rho_1/\rho_2 = 0.9 \) are shown here. They were obtained by multiplying the known values for the long rectangle by the width of the annulus and then adding the inner radius 0.9.

Table 6 Actual Ratios for 0.9

<table>
<thead>
<tr>
<th>cross</th>
<th>first</th>
<th>second</th>
<th>third</th>
<th>fourth</th>
<th>fifth</th>
</tr>
</thead>
<tbody>
<tr>
<td>m=1</td>
<td>0.95053</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>m=2</td>
<td>0.92513</td>
<td>0.97513</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>m=3</td>
<td>0.91672</td>
<td>0.95006</td>
<td>0.98339</td>
<td></td>
<td></td>
</tr>
<tr>
<td>m=4</td>
<td>0.91253</td>
<td>0.93753</td>
<td>0.96253</td>
<td>0.98753</td>
<td></td>
</tr>
<tr>
<td>m=5</td>
<td>0.91002</td>
<td>0.93002</td>
<td>0.95002</td>
<td>0.97002</td>
<td>0.99002</td>
</tr>
</tbody>
</table>

The actual values for \( \rho_1/\rho_2 = 0.1 \) are shown here.

For \( n=1 \), we chose \( \Psi_{10} = \pi/2 \) so that \( \cos[n(\Psi-\Psi_{n0})]=\sin(\Psi) \). We then generated plots of the wavefunctions \( \Psi_{\rho}, \psi_{k_{\rho}1/\rho_2^m} \) for \( \rho_1/\rho_2 = 0.1 \) to 0.9 and \( m=1 \) to 5, as for the \( n=0 \) cases studied and shown above. These plots are displayed in figures 243 to 302.
Figure 243: Contour plot of the $\Psi_{k_{0.1,11}}$ wavefunction for $n=1$, $m=1$, and $\rho_1/\rho_2=0.1$. 

Ratio of Inner to Outer Radii—0.1, $n=1$

Lowest Wavevector
Second Lowest Wavevector

Figure 244: Contour plot of the $\Psi_{k_{0.1,12}}$ wavefunction for $n=1, m=2, \text{ and } \rho_1/\rho_2=0.1$. 
Third Lowest Wavevector

Figure 245: Contour plot of the $\Psi_{k_{0.1,13}}$ wavefunction for $n=1$, $m=3$, and $\rho_1/\rho_2=0.1$. 
Fourth Lowest Wavevector

Figure 246: Contour plot of the $\Psi_{k_{0,1,4}}$ wavefunction for $n=1$, $m=4$, and $\rho_1/\rho_2=0.1$. 
Figure 247: Contour plot of the $\Psi_{k_{0.1,15}}$ wavefunction for $n=1$, $m=5$, and $\rho_1/\rho_2=0.1$. 
Ratio of Inner to Outer Radii—0.2, n=1

Lowest Wavevector

Figure 248: Contour plot of the $\Psi_{k_{0.2,11}}$ wavefunction for n=1, m=1, and $\rho_1/\rho_2=0.2$. 
Second Lowest Wavevector

Figure 249: Contour plot of the $\Psi_{k_{0.2,12}}$ wavefunction for $n=1$, $m=2$, and $\rho_1/\rho_2=0.2$. 
Third Lowest Wavevector

Figure 250: Contour plot of the $\Psi_{k_{0,13}}$ wavefunction for $n=1$, $m=3$, and $\rho_1/\rho_2=0.2$. 
Figure 251: Contour plot of the $\Psi_{k_{0.2,14}}$ wavefunction for $n=1$, $m=4$, and $\rho_1/\rho_2=0.2$. 
Fifth Lowest Wavevector

Figure 252: Contour plot of the $\Psi_{k_{0.2,15}}$ wavefunction for $n=1$, $m=5$, and $\rho_1/\rho_2=0.2$. 
Ratio of Inner to Outer Radii—0.3, n=1

Lowest Wavevector

Figure 253: Contour plot of the $\Psi_{k_{0.3,11}}$ wavefunction for $n=1$, $m=1$, and $\rho_1/\rho_2=0.3$. 
Second Lowest Wavevector

Figure 254: Contour plot of the $\Psi_{k_{0.3,12}}$ wavefunction for $n=1$, $m=2$, and $\rho_1/\rho_2=0.3$. 
Third Lowest Wavevector

Figure 255: Contour plot of the $\Psi_{k_{0.3,13}}$ wavefunction for $n=1$, $m=3$, and $\rho_1/\rho_2=0.3$. 
Fourth Lowest Wavevector

Figure 256: Contour plot of the $\Psi_{k_{0.3,14}}$ wavefunction for $n=1$, $m=4$, and $\rho_1/\rho_2=0.3$. 
Figure 257: Contour plot of the $\Psi_{k_{0.3,15}}$ wavefunction for $n=1$, $m=5$, and $\rho_1/\rho_2=0.3$. 
Ratio of Inner to Outer Radii—0.4, n=1

Lowest Wavevector

Figure 258: Contour plot of the $\Psi_{k_{0.4,11}}$ wavefunction for $n=1$, $m=1$, and $\rho_1/\rho_2=0.4$. 
Figure 259: Contour plot of the $\Psi_{k_{0.4,12}}$ wavefunction for $n=1$, $m=2$, and $\rho_1/\rho_2=0.4$. 
Figure 260: Contour plot of the $\Psi_{k_{0.4,13}}$ wavefunction for $n=1$, $m=3$, and $\rho_1/\rho_2=0.4$. 
Fourth Lowest Wavevector

Figure 261: Contour plot of the $\Psi_{k_{0.4,14}}$ wavefunction for $n=1$, $m=4$, and $\rho_1/\rho_2=0.4$. 
Figure 262: Contour plot of the $\Psi_{k0.4,15}$ wavefunction for $n=1$, $m=5$, and $\rho_1/\rho_2=0.4$. 
Ratio of Inner to Outer Radii—0.5, n=1

Lowest Wavevector

Figure 263: Contour plot of the $\Psi_{k0.5,11}$ wavefunction for n=1, m=1, and $\rho_1/\rho_2=0.5$. 
Second Lowest Wavevector

Figure 264: Contour plot of the $\Psi_{k_{0.5,12}}$ wavefunction for $n=1$, $m=2$, and $\rho_1/\rho_2=0.5$. 
Third Lowest Wavevector

Figure 265: Contour plot of the $\Psi_{k_{0,13}}$ wavefunction for $n=1$, $m=3$, and $\rho_1/\rho_2=0.5$. 
Fourth Lowest Wavevector

Figure 266: Contour plot of the $\Psi k_{0.5,14}$ wavefunction for $n=1$, $m=4$, and $\rho_1/\rho_2=0.5$. 
Figure 267: Contour plot of the $\Psi_{k0.5,15}$ wavefunction for $n=1$, $m=5$, and $\rho_1/\rho_2=0.5$. 

Fifth Lowest Wavevector
Ratio of Inner to Outer Radii—0.6, \(n=1\)

Lowest Wavevector

Figure 268: Contour plot of the \(\Psi_{k_{0.6,11}}\) wavefunction for \(n=1, m=1,\) and \(\rho_1/\rho_2=0.6.\)
Second Lowest Wavevector

Figure 269: Contour plot of the $\Psi_{k_{0.6,12}}$ wavefunction for $n=1$, $m=2$, and $\rho_1/\rho_2=0.6$. 
Figure 270: Contour plot of the $\Psi_{k_{0.6.13}}$ wavefunction for $n=1$, $m=3$, and $\rho_1/\rho_2=0.6$. 
Fourth Lowest Wavevector

Figure 271: Contour plot of the $\Psi_{k_{0.6,14}}$ wavefunction for $n=1$, $m=4$, and $\rho_1/\rho_2=0.6$. 
Figure 272: Contour plot of the $\Psi_{k_{0.6,15}}$ wavefunction for n=1, m=5, and $\rho_1/\rho_2=0.6$. 

Fifth Lowest Wavevector
Ratio of Inner to Outer Radii—0.7, $n=1$

Lowest Wavevector

Figure 273: Contour plot of the $\Psi_{k_{0.7},11}$ wavefunction for $n=1$, $m=1$, and $\rho_1/\rho_2=0.7$. 

195
Figure 274: A zoomed in view of the previous plot to help better see the node (black circle).
Second Lowest Wavevector

Figure 275: Contour plot of the $\Psi_{k_{0.7,12}}$ wavefunction for $n=1$, $m=2$, and $\rho_1/\rho_2=0.7$. 
Figure 276: A zoomed in view of the previous plot to help better see the 2 nodes (black circles).
Figure 277: Contour plot of the $\Psi_{0.7,13}$ wavefunction for $n=1$, $m=3$, and $p_1/p_2=0.7$. 
Figure 278: A zoomed in view of the previous plot to help better see the 3 nodes (black circles).
Fourth Lowest Wavevector

Figure 279: Contour plot of the $\Psi_{k_{0.7,14}}$ wavefunction for $n=1$, $m=4$, and $\rho_1/\rho_2=0.7$. 
Figure 280: A zoomed in view of the previous plot to help better see the 4 nodes (black circles).
Figure 281: Contour plot of the $\Psi_{k_{0.7,15}}$ wavefunction for $n=1$, $m=5$, and $\rho_1/\rho_2=0.7$. 
Figure 282: A zoomed in view of the previous plot to help better see the 5 nodes (black circles).
Figure 283: Contour plot of the $\Psi_k^{0.8,11}$ wavefunction for $n=1$, $m=1$, and $\rho_1/\rho_2=0.8$. 
Figure 284: A zoomed in view of the previous plot to help better see the node (black circle).
Figure 285: Contour plot of the $\Psi_{k_0.8,12}$ wavefunction for $n=1$, $m=2$, and $\rho_1/\rho_2=0.8$. 
Figure 286: A zoomed in view of the previous plot to help better see the 2 nodes (black circles).
Figure 287: Contour plot of the $\Psi_{k_{0.8,13}}$ wavefunction for $n=1$, $m=3$, and $\rho_1/\rho_2=0.8$. 
Figure 288: A zoomed in view of the previous plot to help better see the 3 nodes (black circles).
Figure 289: Contour plot of the $\Psi_{k_{0,8,14}}$ wavefunction for $n=1$, $m=4$, and $\rho_1/\rho_2=0.8$. 
Figure 290: A zoomed in view of the previous plot to help better see the 4 nodes (black circles).
Fifth Lowest Wavevector

Figure 291: Contour plot of the $\Psi_{k_{0.8,1.5}}$ wavefunction for $n=1$, $m=5$, and $\rho_1/\rho_2=0.8$. 
Figure 292: A zoomed in view of the previous plot to help better see the 5 nodes (black circles).
Ratio of Inner to Outer Radii—0.9, n=1

Lowest Wavevector

Figure 293: Contour plot of the $\Psi_{k_{0.9,11}}$ wavefunction for n=1, m=1, and $\rho_1/\rho_2=0.9$. 
Figure 294: A zoomed in view of the previous plot to help better see the node (black circle).
Second Lowest Wavevector

Figure 295: Contour plot of the $\Psi_{k_{0.9,12}}$ wavefunction for $n=1$, $m=2$, and $\rho_1/\rho_2=0.9$. 
Figure 296: A zoomed in view of the previous plot to help better see the 2 nodes (black circles).
Figure 297: Contour plot of the $\Psi_{k_{0.9,13}}$ wavefunction for $n=1$, $m=3$, and $\rho_1/\rho_2=0.9$. 
Figure 298: A zoomed in view of the previous plot to help better see the 3 nodes (black circles).
Fourth Lowest Wavevector

Figure 299: Contour plot of the $\Psi_{k_{0.9,14}}$ wavefunction for $n=1$, $m=4$, and $\rho_1/\rho_2=0.9$. 
Figure 300: A zoomed in view of the previous plot to help better see the 4 nodes (black circles).
Figure 301: Contour plot of the $\Psi_{k_{0.9,15}}$ wavefunction for $n=1$, $m=5$, and $\rho_1/\rho_2=0.9$. 
Figure 302: A zoomed in view of the previous plot to help better see the 5 nodes (black circles).
SUMMARY

After calculating the 5 lowest wavevectors and energy normalized n=0 wavefunctions for $\rho_1/\rho_2 = 0.1-0.9$ in steps of 0.1, it is easy to see that as you increase the m value you get more corresponding nodes (for n=0) and that these wavefunctions are invariant. Each of these is a one-dimensional representation of the $C_{\infty v}$ point group. As the ratio of $\rho_1/\rho_2$ increases, decreasing the overall width of the annulus, they become more similar to a long rectangle (which can be seen in Tables 2-6), this means that the nodes become almost evenly spaced across the radial width. In addition, for each wavefunction we calculated the uniform Josephson current source power distribution at each of the wavevectors $k_{\rho_1/\rho_2,0m}$, and the power distribution for the excited cavity mode at that wavevector. This allowed us to see the “doughnut” like spherical plots that one would expect. Finally, we calculated and presented color-coded contour plots of the wavefunctions $\Psi_{\rho_1/\rho_2,1m}(\rho,\phi)$ for $\rho_1/\rho_2$ from 0.1 to 0.9 and for m = 1 to 5, as for the n = 0 wavefunctions.
REFERENCES


[20] E. A. Borodianskyi and V. M. Krasnov, Josephson Emission with Frequency Span 1-11 THz from Small Bi$_2$Sr$_2$CaCu$_2$O$_{8+δ}$ Mesa Structures, Nature Communications 8, 1742 (2017)